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Spectrum Auction Framework for Access Allocation in Cognitive Radio Networks

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Keywords
algorithms, cognitive radio networks, spectrum auctions

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Spectrum Auction Framework for Access Allocation in Cognitive Radio Networks

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Abstract—**In cognitive radio networks, there are two categories of networks on different channels: primary networks, which have high-priority access, and secondary networks, which have low-priority access. We develop an auction-based framework that allows networks to bid for primary and secondary access based on their utilities and traffic demands. The bids are used to solve the access allocation problem, which is that of selecting the primary and secondary networks on each channel either to maximize the auctioneer's revenue or to maximize the social welfare of the bidding networks, while enforcing incentive compatibility. We first consider the case when the bids of a network depend on which other networks it will share channels with. When there is only one secondary network on each channel, we design an optimal polynomial-time algorithm for the access allocation problem based on reduction to a maximum matching problem in weighted graphs. When there can be two or more secondary networks on a channel, we show that the optimal access allocation problem is NP-complete. Next, we consider the case when the bids of a network are independent of which other networks it will share channels with. We design a polynomial-time dynamic programming algorithm to optimally solve the access allocation problem when the number of possible cardinalities of the set of secondary networks on a channel is upper-bounded. Finally, we design a polynomial-time algorithm that approximates the access allocation problem within a factor of 2 when the above upper bound does not exist.**

Index Terms—**Algorithms, cognitive radio networks, spectrum auctions.**

I. Introduction

With the proliferation of different wireless network technologies like cellular networks, wireless local area networks, wireless metropolitan area networks, etc., demand for radio spectrum is increasing. Currently, spectrum is regulated by a government agency like the Federal Communications Commission (FCC), and it allocates spectrum by assigning exclusive licenses to users to operate their networks in different geographical regions.

There is a widespread belief that radio spectrum is becoming increasingly crowded. However, spectrum measurements indicate that the allocated spectrum is underutilized, i.e., at any given time and location, much of the spectrum is unused [2]. Cognitive radio networks are emerging as a promising solution to this dilemma. In these networks, there are two levels of networks on a channel: *primary networks* and *secondary networks*. A primary or secondary network is a network of multiple wireless devices, which we call primary and secondary nodes, respectively. A primary node has prioritized access to the channel, i.e., it can transmit on the channel regardless of the transmissions of secondary nodes. On the other hand, a secondary node can transmit on the channel provided primary nodes are not transmitting. Thus, whenever a secondary node wants to transmit on the channel, it senses the channel to check for ongoing transmissions. It initiates a transmission only if a primary node is not transmitting. Cognitive radio technology [4] allows secondary nodes to detect which channel is not being used by primary nodes, share this channel with other nodes, and vacate the channel when a primary node is detected. Surveys on cognitive radio networks can be found in [1] and [4].

An important question faced by a spectrum regulator is how to allocate the rights to be primary and secondary networks on its channels. Different networks may attach different value to being primary and secondary. A network may wish to mainly transmit delay-sensitive traffic like voice or video. Such a network will attach a high value to the rights to be primary. On the other hand, a network may be mainly interested in transmitting delay-insensitive traffic like e-mail or file transfer. Such a network would not need primary rights and would prefer secondary rights since the latter would be priced lower than the former. Also, a network whose traffic is a mixture of delay-sensitive and delay-insensitive traffic would want primary rights on some channels and secondary rights on some channels.

Auctions are suitable for selling the rights to be primary and secondary on the channels. Since the regulator need not know the values that bidders attach to primary and secondary rights, auctions provide a mechanism for the regulator to get a higher revenue than that obtainable through static pricing [6]. Auctions are also beneficial for the bidders since in general they assign goods to the bidders who value them most [6]. The FCC has been conducting spectrum auctions since 1994 to allocate licenses for radio spectrum [3] (however, so far no auctions have been conducted for cognitive radio networks).

Spectrum auctions have been studied in [6]–[9] and [22]. In [6], a framework is developed to distribute spectrum in real-time to a set of wireless users. Channel allocation is done under interference constraints, in which the same channel cannot be allocated to two or more users whose transmissions interfere with each other. In [7], there is a set of bidders and multiple chunks of spectrum. The paper investigates sequential and concurrent auction mechanisms to allocate the chunks of spectrum to the bidders such that each bidder is allocated at most one chunk. In [8], a set of spread spectrum users is considered, who share the spectrum with the owner of the spectrum. The goal is to design auctions to allocate the transmit power to each user subject to a limit on the interference at a measurement location. In [9],

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there are multiple primary users who own the licenses to channels in a region and multiple secondary users who are interested in leasing the unused portions of the channels of the primaries. The paper proposes a double-auction mechanism with multiple sellers (the primaries) and multiple buyers (the secondaries). In [22], a knapsack-based auction model is proposed to allocate spectrum to providers while maximizing revenue and spectrum usage.

We now explain how our work differs from previous work. In some of the existing work on spectrum auctions [6], [7], [22], each channel is assigned to a single network, i.e., there is no notion of primary and secondary networks on a single channel. We consider the case when there is a primary network and one or more secondary networks on each channel. Now, there are two possibilities for allocating secondary rights on the channels [10]. In one possibility, which we refer to as two-step allocation, the regulator allocates channels to primary networks and the primary networks independently allocate unused portions on their channels to the secondary networks. Auctions have been designed for this scenario in [8] and [9]. In the other possibility, which we refer to as one-step allocation, the regulator allocates the rights to be the primary and secondary networks on the channels in a single allocation [10]. To the best of our knowledge, no work has been done in designing auctions for this scenario. In this paper, we develop a comprehensive auction framework for the one-step allocation scenario, using that which a regulator can simultaneously allocate the rights to be primary and secondary on the channels. One-step allocation may be more desirable than its two-step counterpart in certain cases. For example, one-step allocation gives a greater degree of control to the regulator. In particular, it allows the regulator to choose a "socially beneficial" channel allocation that maximizes the social welfare. Note that in one-step allocation, a network can bid for, and can even be granted, primary and secondary access to more than one or even all channels. Also, the allocation resulting from two-step allocation may indeed turn out to be that for one-step allocation, but only when it is the most socially beneficial allocation.

We consider a scenario in which the regulator conducts an auction to sell the rights to be primary and secondary networks on a set of channels. Networks can bid for these rights based on their utilities and traffic demands. The regulator uses these bids to solve the *access allocation* problem, i.e., the problem of deciding which networks will be the primary and secondary networks on each channel. The goal of the regulator may be either to maximize its revenue or to maximize the social welfare of the bidding networks. Now, networks can have utilities or valuations that are functions of the number of channels on which they get primary and secondary rights, how many and which other networks they share these channels with, etc. The number of valuations of a network may be large, and an exponential amount of space may be required to express a bid for each valuation. Therefore, we design bidding languages, that is, compact formats for networks to express bids for their valuations. For different bidding languages, we design algorithms for the access allocation problem.

The paper is organized as follows. We describe the system model in Section II. In Section III, we describe how the bidding languages and algorithms that we design in the paper can be used to maximize the auctioneer's revenue or to maximize social welfare. In Section IV, we describe a model in which the bids of a network depend on which other networks it shares a channel with. In Section IV-A, we design an optimal algorithm for the access allocation problem for a simple case with only one secondary network on each channel. We show the intractability (NP-competeness of the access allocation problem or exponential size of bids) of the extensions of this simple case in Section IV-B. In Section V, we consider the case in which the bids of a network are independent of which networks it shares a channel with and provide an optimal dynamic programming algorithm for the access allocation problem in Section VI. The algorithm is polynomial-time when the number of possible cardinalities of the set of secondary networks on a channel is upper-bounded. In Section VII, we describe a bidding language that can be used for the independent bids case for an arbitrary number of cardinalities of the set of secondary networks on a channel and provide a greedy 2-approximation algorithm for the access allocation problem. In Section VIII, using simulations, we show that the above approximation algorithm in fact performs optimally in a variety of scenarios.

II. System Model

We consider a scenario in which there are M identical orthogonal channels in a region. A regulator conducts an auction to sell the rights to be the primary and secondary networks on the channels. N bidders participate in the auction. Each bidder is an independent network of multiple wireless nodes. Each bidding network submits bids to the regulator, and based on the bids, the latter allocates the rights to be the primary and secondary networks on the channels.

A primary network on a channel must have prioritized access to the channel. If two or more independent networks were to be the primary networks on a single channel, then the access of each one of them would be constrained by the transmissions of the other primary networks, which would transmit at the same priority level. To avoid this, we assume that there is exactly one primary network on each channel. However, we allow multiple networks to have secondary rights on a channel.

We assume that all the secondary networks on a channel have equal rights on the channel. This is because complicated multiple access protocols [5] would be required to grant access at different priority levels to different secondary networks on a channel (with all of them getting lower priority access than the primary network). On the other hand, simple multiple access protocols would suffice if all secondary networks have equal rights on the channel.

Now, since a primary network has prioritized access on a channel, the average delay of its traffic is low. On the other hand, the average delay of a secondary network's traffic is high. Hence, primary rights (respectively secondary rights) are suitable for communicating delay-sensitive (respectively delay-insensitive) traffic. We assume that each network has two kinds of traffic: 1) delay-sensitive traffic like voice, video, etc.; and 2) delay-insensitive or elastic traffic like e-mail, file transfer, etc. A network uses its primary rights to transmit its delay-sensitive traffic, and its secondary rights to transmit its elastic traffic.

A single network i may be both the primary network and one of the secondary networks on a channel. In this case, we assume that it transmits its delay-sensitive traffic as a primary network, i.e., with high priority, and when it does not have any delay-sensitive traffic to transmit, it transmits its elastic traffic as a secondary network. Also, the other secondary networks on the channel can transmit whenever network i is not transmitting its delay-sensitive traffic.

Let K be the set of all possible ways in which the M channels can be allocated to the N bidders. For example, consider the simple case in which $M = 3$, $N = 9$, and there can be at most four secondary networks on a channel. An example of an allocation of the channels is one in which network 1 becomes the primary network on channels 1 and 2; network 2 becomes primary on channel 3; network 3 becomes the sole secondary network on channel 1; networks 4 and 5 become secondary networks on channel 2; networks 1, 4, 6, and 7 become secondary networks on channel 3; and networks 8 and 9 do not become primary or secondary networks on any channel.

Let $x_i(k)$ be network i's *valuation* or *utility* from the channel allocation $k \in K$, i.e., the value that it *conjectures or expects* to derive from the allocation k *when it submits the bids*. Note that since network i will share channels with other networks in the allocation k, the *actual* utility that network i will derive from an allocation k after the networks start using the allocated channels depends on the transmission patterns of the other networks that are not completely known to network i when it submits the bids. Therefore, a network can only submit bids based on the expected utilities $x_i(k)$, which reflect its expectations about the actual utilities that it will eventually get. Henceforth, we use the terms valuation or utility for $x_i(k)$, but they should be understood to mean the conjectured utility or valuation of network i for the channel allocation k.

The valuations $x_i(.)$ of network i for the allocations in K depend on its traffic demands, i.e., the volumes of delay-sensitive and elastic traffic that it wants to transmit. Now, for given traffic demands, the valuation of a network i for a channel allocation $k \in K$ may depend upon the number of channels on which network i has primary and secondary rights in the allocation k, how many and which other networks have rights on each of the channels on which network i has primary or secondary rights, etc. For example, a network that wants to transmit a lot of delay-sensitive traffic will ascribe a high valuation to an allocation in which it is primary on several channels. Note that network i may have the same valuation for different allocations $k \in K$.

Network i's net utility is of the form

$$u_i(k, \tau_i, x_i) = x_i(k) - \tau_i \tag{1}$$

where τ_i is the payment that network i makes to the auctioneer. The auctioneer determines the channel allocation and the payment τ_i that each network i makes to the auctioneer. The *social welfare* of an allocation k is defined to be the quantity

$$\sum_{i=1}^{N} x_i(k).$$

Thus, the social welfare is the sum of utilities of all bidders from the allocation k.

Now, there could be two goals for designing the auction: revenue maximization and maximizing social welfare. In the first goal, based on its valuations, each network submits a set of bids to the auctioneer. Let $z_i(k)$ be the bid of network i for the allocation $k \in K$, i.e., the amount of money it is willing to pay if the allocation $k \in K$ is chosen. Let k^* be the channel allocation that maximizes the revenue of the auctioneer, given the bids $z_i(.)$ for bidders $1, \ldots, N$. That is, k^* satisfies

$$\sum_{i=1}^{N} z_i(k^*) \geq \sum_{i=1}^{N} z_i(k) \forall k \in K. \tag{2}$$

In the second goal of maximizing social welfare, $z_i(.)$ are not the bids of the networks, but have a different interpretation: They are the *declared valuations* of the networks (explained in Section III). In this case, the channel allocation that maximizes the social welfare of the N networks can again be found by finding the k^* satisfying (2).

For both goals, the *access allocation* problem is to determine the channel allocation k^* satisfying (2). Depending on the interpretation of $z_i(.)$, this allocation k^* either maximizes the auctioneer's revenue or the social welfare of the N networks.

Now, the set K of possible channel allocations may be exponential in size. Hence, the total number of different valuations of network i may be exponential in general. However, it is not computationally tractable to communicate a bid for each valuation in this large set. Therefore, we introduce *bidding languages* for the auction models that we consider. A bidding language [12] is a format to compactly encode the bid information of a bidder. When there are an exponential number of valuations, a bidding language expresses the bids approximately, not exactly.

We now remark on some implementation issues: 1) One way in which the regulator can implement the auction is by deploying a central controller in the region, which would periodically collect bids that are sent by the bidding networks over a common control channel, compute the channel allocation and payments, and send them to the bidders over the control channel. 2) The frequency at which auctions are conducted is determined by the following tradeoff: The higher the frequency, the more responsive is the channel allocation to changes in traffic demands and the higher is the spectrum utilization, but the overhead is also higher. Hence, the interval between successive auctions is chosen to be as small as possible while ensuring that the overhead is below an acceptable limit.

III. Solution Framework

As stated earlier, an auction could be designed for two different objectives. In our context, the first objective is to choose the channel allocation that maximizes the regulator's revenue for a given set of bids of the bidders. This can be done by choosing the allocation k^* satisfying (2) when $z_i(k)$ is the bid of network i for the channel allocation k.

The second possible objective for the auction could be to achieve efficiency, that is, to choose the allocation that maximizes social welfare. To this end, each bidder is asked to declare its valuation function $x_i(.)$. With an abuse of notation, let $z_i(k)$ denote the *declared* valuation of network i for the allocation k, which may be different from $x_i(k)$ if bidder i believes that falsely declaring its valuations will improve its net utility. *Truth-telling* is said to be a *weakly dominant strategy* [18] for network i if, for any possible declarations of networks other than i, the net utility of network i is maximized when it sets $z_i(k) = x_i(k)\ \forall k \in K$. It follows from the revelation principle

[18] that to maximize social welfare, it is sufficient to consider mechanisms in which the payments τ_i are chosen such that for each bidder i, truth-telling is a weakly dominant strategy. Such a mechanism is called *incentive compatible*.

To date, the Vickrey–Clarke–Groves (VCG) mechanism [18] is the only known general incentive compatible mechanism that can be used to maximize social welfare. Under this mechanism, given the declared valuation functions $z_i(.)$ of the bidders, the allocation k^* satisfying (2) is chosen. Let k^*_{-i} be the allocation that would have maximized the social welfare if network i did not participate in the auction. That is, k^*_{-i} satisfies

$$\sum_{j=1, j\neq i}^{N} z_j(k^*_{-i}) \geq \sum_{j=1, j\neq i}^{N} z_j(k) \forall k \in K. \tag{3}$$

Under the VCG mechanism, the payment made by network i to the auctioneer is given by

$$\tau_i = \sum_{j=1, j\neq i}^{N} z_j(k^*_{-i}) - \sum_{j=1, j\neq i}^{N} z_j(k^*). \tag{4}$$

The key to implementing the VCG mechanism is to find the allocations k^* and k^*_{-i}, $i = 1, \ldots, N$. Now, k^* can be found using an algorithm for the access allocation problem (2), and k^*_{-i} can be found by running the same algorithm on the set of bidders $\{1, \ldots, N\} \backslash i$.

Now, in general, the set of different valuations of a bidding network is exponential in size. First, we consider the special case when the number of different valuations of each bidding network is of poynomial space complexity. However, K can still be exponential in size. This is because a bidder may have the same valuation for two or more allocations in K. Even in this case, it is sometimes computationally intractable to devise an algorithm to find the optimal allocation k^* satisfying (2), possibly because this is NP-hard, but instead an approximation algorithm for the access allocation problem can be devised. In this case, if the payments are chosen according to the VCG formula (4) with suboptimal allocations instead of k^* and k^*_{-i}, then truth-telling is no longer a weakly dominant strategy for the bidders. To address this problem, Nisan and Ronen [20] devised the *second-chance mechanism* under which the auctioneer publishes the suboptimal algorithm that it will use for the access allocation problem. Each bidder submits its (declared) valuations $z_i(.)$ and a so-called *appeal function* (see [20]) to the auctioneer. Each bidder optimizes the valuations and the appeal functions to submit so as to maximize its own utility. The auctioneer specifies a time limit by which the valuations and appeal functions must be submitted. The auctioneer uses the suboptimal algorithm for the access allocation problem to find the channel allocation using the submitted valuations *and appeal functions*. The VCG formula (4) is used to determine the payment that each bidder will make. Now, the *strategic knowledge* of a bidder i is a function that, for a set of valuations submitted by the other bidders, gives the valuation that bidder i must declare so as to get the maximum utility. It is shown in [20] that when there is a bound on the time each bidder i can take to compute its strategic knowledge, and when the time limit allowed to each bidder to compute the valuations and appeal functions to submit is at least as much as this bound, then truthfully declaring the valuation function is a dominant strategy for each bidder under the second-chance mechanism. Moreover, the social welfare attained by the second-chance mechanism is at least as good as the social welfare of the suboptimal algorithm used for the access allocation problem.

Now, in some cases, the set of valuations of a bidder takes an exponential amount of space, and hence bidders have to use incomplete bidding languages (see Section II) to convey their valuations. In this case as well, incentive compatibility does not hold if the VCG formula (4) is used for payments. As a solution to this problem, Ronen [21] devised the *extended second-chance mechanism*. In these mechanisms, each bidder submits a description of its set of valuations in some bidding language, an appeal function, and an *oracle* [21], which is a program that can be queried by the auctioneer for the bidder's valuation. The auctioneer determines an allocation based on the above submitted quantities using a (possibly suboptimal) algorithm for the access allocation problem. It is shown in [21] that under reasonable assumptions (see [21]), truth-telling is a dominant strategy for the bidders under the extended second-chance mechanism.

Note that in addition to incentive compatibility, the VCG, second-chance, and extended second-chance mechanisms have the desirable property of *individual rationality* [18], i.e., bidders get a nonnegative utility when they participate in the auction.

In this paper, we propose several spectrum auction models and design bidding languages and algorithms for the access allocation problem. These can be used for the objective of maximizing the revenue of the auctioneer or for maximizing the social welfare of the bidders in conjunction with the VCG, second-chance, or extended second-chance mechanism, as appropriate. In particular, in Section IV-A, we describe an auction model that allows networks to completely express their bids under certain assumptions (Assumptions 1 and 2). We provide a polynomial-time algorithm that finds the optimal solution in the access allocation problem. This algorithm can be used to maximize the auctioneer's revenue or, in conjunction with the VCG mechanism, to maximize the social welfare of the bidders. In the auction model in Section V, we provide a bidding language that allows bidders to completely express their bids when they have no knowledge of the channel usage behavior (defined in Section IV) on a channel of the other bidders and approximately express their bids when they have this knowledge. Section VI provides a polynomial-time algorithm to optimally solve the access allocation problem for the model in Section V when the number of cardinalities of the set of secondary networks on a channel is upper-bounded. When bidders have no knowledge of the channel usage behavior of other bidders, this algorithm can be used to maximize the auctioneer's revenue or to maximize social welfare in conjunction with the VCG mechanism. When bidders have this knowledge, the algorithm can be used to maximize the auctioneer's revenue or, in conjunction with the extended second-chance mechanism, to maximize social welfare. Finally, in the auction model in Section VII, we provide a bidding language and a 2-approximation algorithm for the access allocation problem that is polynomial-time for an arbitrary number of cardinalities of the set of secondary networks on a channel. This algorithm can be used to approximate the maximum revenue of the auctioneer or in conjunction with the extended second-chance mechanism to approximate the maximum social welfare.

For notational convenience, throughout the paper, we assume that $z_i(.)$ are the bids expressed by bidder i and view the access allocation problem as the problem of maximizing the revenue of the auctioneer. However, our framework applies without change to the problem of maximizing social welfare.

IV. Auction With Dependent Bids

A primary or secondary network on a channel shares the channel with other networks, and hence its actual utility from the channel depends on the transmissions of those networks. A network may have some knowledge or beliefs about the typical transmission patterns of the other bidding networks. For example, the agency owning the network may conduct a survey on the typical transmission patterns of the other networks in its region, or if auctions are periodically conducted to allocate spectrum in the region, the agency may gain this knowledge about the networks with whom it shared channels previously. Thus, the conjectured utilities, and hence the bids of a network, would depend on which networks it will share different channels with.

A. Basic Model

In the basic model with dependent bids, we consider the model described in Section II with the following additional assumptions.

Assumption 1: There is only one secondary network on each channel.

Assumption 2: Each network can be either the primary or the secondary network on only one channel.

We explore the effect of relaxing either of these assumptions in Section IV-B. We assume that $N \geq 2M$, so that all M channels can be allocated.

A secondary network on a channel can use the channel whenever the primary network is not using it. Therefore, the throughput and delay of the secondary network on the channel depends on the *channel usage behavior* of the primary on the channel, i.e., on the rate of its transmissions on the channel and how these transmissions are spread over time. On the other hand, the primary network on a channel has prioritized access to the channel. That is, when the secondary network wants to transmit on the channel, it senses the channel and can transmit only if it finds that the primary network is not transmitting. However, due to the imperfect nature of sensing, the secondary network will sometimes transmit while the primary network is transmitting, resulting in a collision. Hence the primary network's utility depends on the channel usage behavior of the secondary network on the channel. Thus, the actual utility of a primary or secondary network depends on which network it shares a channel with. As explained, a network may in general have certain beliefs about the channel usage behavior of other networks and hence may wish to express bids dependent on the network with whom it shares the channel. To model this, let

$$z_i^p(j),\ j \in \{1,\ldots,N\}\backslash\{i\}$$

be the bid of network i for the case when it is the primary network on a channel and network j is the secondary network on the channel. Similarly, let

$$z_i^s(j),\ j \in \{1,\ldots,N\}\backslash\{i\}$$

be the bid of network i for the case when it is the secondary network on a channel and network i is the primary network.

Let

$$k = \{(i_1,j_1),\ldots,(i_M,j_M)\}$$

be an allocation of the M channels to a set of networks. k is a set of M orderered pairs (i_t, j_t) such that network i_t is the primary network on channel t and network j_t is the secondary network on channel t. Note that the revenue of the allocation k is

$$\sum_{t=1}^{M}(z_{i_t}^p(j_t) + z_{j_t}^s(i_t)).$$

We describe an algorithm for determining k^*, the allocation that maximizes the revenue, by reduction to a maximum weight matching problem in a graph. Let G be a weighted undirected graph with N nodes, one node corresponding to each network. G is a complete graph, i.e., between every pair of nodes, there is an edge. Let the weight of the edge joining nodes i and j be

$$w_{ij} = \max(z_i^p(j) + z_j^s(i), z_j^p(i) + z_i^s(j)). \tag{5}$$

Note that the weights are nonnegative real numbers. The interpretation of the weights w_{ij} is as follows. If network i (respectively, network j) is the primary network on a channel and network j (respectively, network i) is the secondary network, then the sum of the amounts paid by networks i and j is $z_i^p(j)+z_j^s(i)$ (respectively, $z_i^s(j)+z_j^p(i)$). Therefore, w_{ij}, the greater of these two quantities, is the maximum sum of payments of networks i and j if they are the two networks on the same channel.

A *matching* $\mathcal{M}$ in a graph is defined to be a subset of the edges such that no two edges in the subset share a common node. The weight of a matching is the sum of the weights of its edges. The following algorithm finds the channel allocation k^* that maximizes the revenue.

STEP1: In graph G, find a matching $\mathcal{M}_M^*$ of maximum weight among matchings with exactly M edges[1] (we say how later).

STEP2: Let $e_1,\ldots,e_M$ be the M edges in the matching $\mathcal{M}_M^*$. Let e_t^1 and e_t^2 be the two endpoints of edge e_t. The allocation k^* is chosen such that for $t = 1,\ldots,M$, networks e_t^1 and e_t^2 become the two networks (primary and secondary) on channel t. If

$$z_{e_t^1}^p(e_t^2) + z_{e_t^2}^s(e_t^1) \geq z_{e_t^2}^p(e_t^1) + z_{e_t^1}^s(e_t^2)$$

then network e_t^1 becomes the primary network on channel t and network e_t^2 becomes the secondary network; otherwise, network e_t^2 becomes the primary network on channel t and network e_t^1 becomes the secondary network.

Theorem 1: The allocation k^* found from the matching $\mathcal{M}_M^*$ in the above algorithm is the one that maximizes the revenue.

[1]Note that there exists a matching with exactly M edges since there are $N \geq 2M$ nodes and G is a complete graph.

Proof: There is a many-to-one correspondence between the set of channel allocations and the set of matchings with exactly M edges. (It is many-to-one since the allocations obtained from any allocation by swapping the roles of the primary and secondary networks on one or more channels correspond to the same matching.) From the interpretation of the weight of an edge given, it follows that the weight of a matching $\mathcal{M}_M$ has the maximum revenue among the revenues of the channel allocations that correspond to it. Therefore, the weight of the maximum weight matching $\mathcal{M}_M^*$ equals the maximum revenue among the revenues of all the channel allocations. Also, note that Step 2 of the algorithm ensures that we select the channel allocation k^*, whose revenue is the same as the weight of $\mathcal{M}_M^*$. It follows that the allocation k^* found from the matching $\mathcal{M}_M^*$ is the one that maximizes the revenue. ■

Now, it remains to show how to find the matching $\mathcal{M}_M^*$. Edmonds [14] gave a polynomial-time algorithm for finding the maximum weight matching (with any number of edges) in a graph. However, we are interested in a maximum weight matching among matchings with M edges, which cannot be directly obtained by Edmonds' algorithm. It can be obtained in $O(M^4 + M^2N^2)$ time[2] using White's modification [15], [16] to Edmonds' algorithm.

B. Intractability of Extensions

We now explore the effect of relaxing either one of Assumptions 1 and 2. Suppose Assumption 1 is relaxed and Assumption 2 is retained. That is, we assume that each network can be the primary or a secondary network on only one channel. However, there can be multiple secondary networks on a channel. We show that even if there are two secondary networks on a channel, the problem of finding a channel allocation that maximizes the revenue is NP-complete.

Let $z_i^p(j_1, \ldots, j_{v-1})$ be the bid of network i for the case in which it is primary on a channel and networks $j_1, \ldots, j_{v-1}$ are secondary. Let $z_{j_1}^s(i, j_2, \ldots, j_{v-1})$ be the bid of network j_1 for the case in which network i is the primary and networks $j_1, \ldots, j_{v-1}$ are the secondary networks. Also, let z_i^p be the bid of network i for the case in which it is primary on a channel with no secondary on the channel. We now define the r-Network Dependent Bid Access Allocation Problem (r-DBA).

Definition 1 (The r-DBA Problem): Suppose M channels are to be allocated to N bidders such that on each channel, one network is primary and at most $r-1$ networks are secondary, where r is a fixed positive integer. Each bidder can be a primary or secondary network on at most one channel, and the bids of networks are as given above. Find the allocation that maximizes the revenue.

We show that r-DBA is NP-Complete. To this end, we first show that a simpler version of r-DBA, which we call the Exactly r-Network Dependent Bid Access Allocation Problem (r-EDBA), is NP-complete. The r-EDBA problem is defined in the same way as r-DBA, except that on each channel, *exactly* $r-1$ networks are secondary instead of *at most* $r-1$ networks.

Note that if in an instance of r-EDBA, $N < rM$, then there is no channel allocation with r networks on each channel. In this case, we define the optimal revenue of the r-EDBA instance to be $-\infty$.

[2]Recall that a function $f(n)$ is said to be $O(g(n))$ if there exist positive constants c and n_0 such that $f(n) \le cg(n)$ for all $n \ge n_0$ [13].

The decision version of r-DBA or r-EDBA is as follows: Given a bound D, is there a channel allocation such that the revenue under the allocation is at least D? We next show that (the decision version of) 3-EDBA is NP-complete.

Lemma 1: 3-EDBA is NP-complete.

Proof: Given an allocation of the M channels, we can verify in polynomial time whether the revenue under the allocation is at least D. This shows that 3-EDBA is in the class NP.

Next, we show that the 3-Dimensional Matching problem (3DM), which is known to be NP-complete [19], is polynomial-time-reducible to 3-EDBA, i.e., 3DM $\le_p$ 3-EDBA. An instance of 3DM is as follows [19]: Given disjoint sets A, B, C of m elements each and a set T of ordered triples of the form (a, b, c), where $a \in A$, $b \in B$ and $c \in C$, does there exist a set of m triples in T so that each element of $A \cup B \cup C$ is contained in exactly one of these triples?

From this instance of 3DM, we construct an instance of 3-EDBA as follows. Let there be $M = m$ channels and $3m$ networks—one network corresponding to each element of $A \cup B \cup C$. We now design the bids, which will complete the construction. For every set $\{i, j, l\}$ of three networks such that (i, j, l) (or one of its permutations (j, l, i), (l, j, i), etc.) is a triple in T, define all of the following bids to be equal to $\frac{1}{3}$: $z_i^p(j, l)$, $z_j^p(i, l)$, $z_l^p(i, j)$, $z_i^s(j, l)$, $z_i^s(l, j)$, $z_j^s(i, l)$, $z_j^s(l, i)$, $z_l^s(i, j)$, $z_l^s(j, i)$. For every set $\{i, j, l\}$ of three networks such that no permutation of (i, j, l) is a triple in T, let all of the above bids be equal to $\frac{1}{6}$. In this 3-EDBA problem, we ask: Is there a channel allocation of the m channels with revenue of at least $D = m$? We claim that the answer is yes if and only if the answer in the original 3DM problem is yes.

To prove sufficiency, suppose there exists a subset $T' \subseteq T$ of m triples such that each element of $A \cup B \cup C$ is contained in exactly one of these triples. Let

$$T' = \{(a_t, b_t, c_t) : t = 1, \ldots, m\}.$$

Then, allocate the m channels such that network a_t is the primary network and networks b_t and c_t are the secondary networks on channel t, $t = 1, \ldots, m$. The revenue of this allocation is

$$\sum_{t=1}^{m} \{z_{a_t}^p(b_t, c_t) + z_{b_t}^s(a_t, c_t) + z_{c_t}^s(a_t, b_t)\} = \sum_{t=1}^{m} \left\{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right\} = m.$$

Hence, the answer in the 3-EDBA problem is yes.

Conversely, suppose there exists an allocation of the m channels with revenue of at least m. In this allocation, let a_t be the primary and b_t and c_t be the secondary networks on channel t, $t = 1, \ldots, m$. If (a_t, b_t, c_t) or its permutation is a triple in T, then the sum of payments of networks a_t, b_t and c_t is 1, else it is $\left\{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right\} = \frac{1}{2}$. Since there are m channels and the revenue of the allocation is at least m, it follows that the revenue is exactly m and that for each t, (a_t, b_t, c_t) or one of its permutations is a triple in T. Moreover, since each network can be the primary or a secondary network on only one channel, it follows that each of the $3m$ networks is a primary or secondary network on exactly one channel. Hence, the m triples in T corresponding to (a_t, b_t, c_t) or its permutation for $t = 1, \ldots, m$ are such that each element of $A \cup B \cup C$ is contained in exactly one of the triples. Thus, the answer to the 3DM problem is yes. This shows that 3DM $\leq_p$ 3-EDBA and hence that 3-EDBA is NP-complete. ■

By an analogous reduction from r-dimensional matching, it can be shown that r-EDBA is NP-complete for fixed $r > 3$. Note that for $r > 3$, r-dimensional matching is NP-complete, which follows from a trivial reduction from 3-dimensional matching.

Now, we show that for any fixed $r \geq 3$, (the decision version of) r-DBA is NP-complete by a reduction from r-EDBA.

Theorem 2: For $r \geq 3$, r-DBA is NP-complete.

Proof: Clearly, r-DBA is in the class NP.

Now, we show that r-EDBA $\leq_p$ r-DBA. From any instance of r-EDBA with given M, N, D, and bid functions $z_i^p(.)$ and $z_i^s(.)$, we construct an instance of r-DBA as follows. The number of channels, number of networks, and the bound on revenue are the same as in the original r-EDBA instance (M, N, and D, respectively). The bids of network i are given by

$$\tilde{z}_i^p(j_1,\ldots,j_{v-1}) = \begin{cases} z_i^p(j_1,\ldots,j_{v-1}), & \text{if } v = r \\ 0, & \text{if } 2 \leq v < r \end{cases}$$

$$\tilde{z}_i^s(j_1,\ldots,j_{v-1}) = \begin{cases} z_i^s(j_1,\ldots,j_{v-1}), & \text{if } v = r \\ 0, & \text{if } 2 \leq v < r \end{cases}$$

and

$$\tilde{z}_i^p = 0.$$

Recall that if $N < rM$, then there is no channel allocation in the r-EDBA instance with exactly r networks on each channel. Hence, the answer to the decision version is negative. Thus, let $N \geq rM$. We now show that there exists a channel allocation with revenue at least D in the r-EDBA instance if and only if there exists one such in the r-DBA instance. If there is a channel allocation with revenue at least D in the r-EDBA instance, then by construction of the bids in the r-DBA instance, the revenue of that channel allocation is the same in the r-DBA instance and hence at least D.

Conversely, suppose there is a channel allocation k with revenue at least D in the r-DBA problem. From this channel allocation, construct a channel allocation k' for the r-EDBA instance as follows: If there are $r-1$ secondary networks on a channel in k, let the primary and secondary networks on the channel be the same in k'. From the construction of bids in the r-DBA instance, it follows that the sum of payments of the networks on this channel in k' is the same as that in k. If there are $v-1$ secondary networks on a channel l in k, where $v < r$, then on channel l in k', let the same v networks be primary and secondary and, in addition, let $r-v$ more networks be secondaries, which were not primary or secondary on any channel in k. Such networks exist since $N \geq rM$. By the construction of the bids in the r-DBA instance, the sum of payments of the networks on channel l in k is 0, whereas that in k' is at least 0. Thus, the revenue of allocation k' is at least as much as the revenue of channel k and hence is at least D.

This shows that r-EDBA $\leq_p$ r-DBA. Since r-EDBA is NP-complete as shown, it follows that r-DBA is NP-complete. ■

Note that in the r-DBA problem, if r is unbounded, then each bidder i would have to submit an exponential number of bids $z_i^p(j_1,\ldots,j_{v-1})$ and $z_i^s(j_1,\ldots,j_{v-1})$.

Now, suppose we relax Assumption 2 and retain Assumption 1. Then, each network can become a primary or secondary network on up to M channels. As explained, the utility of a network from the primary or secondary rights on a given channel depends upon the channel usage behavior of the network it shares the channel with. However, the channel usage behavior of this network on the channel may in turn depend upon the number of channels on which it has primary and secondary rights and the channel usage behavior of the networks it shares those channels with and so on. Thus, in general, the utility of a network may depend upon which networks are the primary and secondary networks on each channel. The number of possible ways of choosing the primary and secondary networks on the M channels is clearly exponential. Thus, relaxing Assumption 2 would require each network to express an exponential number of bids in the auction with dependent bids, which is computationally intractable.

V. Auction With Independent Bids

In Section IV, we noted that when networks have some knowledge of the channel usage behavior of other networks, they would like to express bids dependent on which networks they will share channels with. However, it is quite possible in some scenarios that networks have no knowledge of the channel usage behavior of the other bidding networks. In this case, their conjectures about the utility that they will actually get from a channel allocation would be based only on the number of channels on which they will get primary and secondary rights and the *number of* other networks they will share these channels with in the allocation and would be independent of *which* other networks they will share channels with. Thus, they would submit bids, based on these conjectured utilities, that are independent of which networks share different channels with them.

Moreover, in Section IV-B, we showed that bids of exponential size are needed in the auction with dependent bids when Assumptions 1 and 2 are relaxed. This motivates the idea that even when networks have some knowledge of the channel usage behavior of the other networks, we can obtain a compact bidding language—that is, a means for networks to approximately convey their bids—by imposing the restriction that the bids of a network be independent of which other networks it shares different channels with. We study the auction resulting from imposing this restriction in this section.

A. Model

Consider the model in Section II with the following additions. On each channel, one network can be the primary network and $m_1, m_2, \ldots, m_{(n-1)}$, or m_n networks can be the secondary networks, where $1 \leq m_1 < m_2 < \cdots < m_n$. Note that n is the number of possible cardinalities of the set of secondary networks on a channel.

When the results of the auction are declared, let $n_{i,0}$ be the number of channels on which bidder i is the primary network. Let $n_{i,j}, j = 1,\ldots,n$ be the number of channels on which bidder i is a secondary network along with $m_j - 1$ other secondary networks.

Suppose there are m_j secondary networks on a channel. Recall from Section II that each of these m_j secondary networks have equal rights on the channel. The share of each of these networks in the secondary rights on the channel is called a *secondary part of type* j. Also, the channel is said to be divided into m_j secondary parts of type j. Similarly, since exactly one network becomes a primary network on a channel, if a network is the primary network on l channels, we say that it is allocated l *primary parts*. Also, we refer to the throughput received by a network as a secondary network as its *secondary throughput*.

In general, network i's utility may depend not only on the total expected secondary throughput that it gets, but also on the distribution of this secondary throughput over the M channels. For example, it may get the same expected secondary throughput: 1) if it is the secondary network on two channels with one other secondary network on each; and 2) if it is the sole secondary network on one channel. However, it may prefer one of these scenarios over the other. This is because a network has to sense different channels on which it has secondary rights for ongoing transmissions and also communicate on them. There may be costs due to delays for switching the antennas of the network's nodes between different channels. To take into account this possibility, in this section, we assume that the utility of network i depends not just on the expected secondary throughput (and the number of primary parts) it receives, but on the vector $(n_{i,0}, n_{i,1}, \ldots, n_{i,n})$. We allow bidder i to submit bids as a function of this vector.

Each bidder i submits the following bid vector to the auctioneer:

$$\{z_i(n_{i,0}, n_{i,1}, \ldots n_{i,n}) : 0 \le n_{i,0}, n_{i,1}, \ldots n_{i,n} \le M,\\ n_{i,1} + n_{i,2} + \cdots + n_{i,n} \le M; n_{i,j} \text{ integer, } j=0,1,\ldots n\}$$

where $z_i(n_{i,0}, n_{i,1}, \ldots n_{i,n})$ is network i's bid for becoming the primary network on $n_{i,0}$ channels and becoming a secondary network on $n_{i,j}$ channels along with $m_j - 1$ other secondary networks, for $j = 1, 2, \ldots n$.

B. Feasible Allocation

We say that an allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ is *feasible* if it is possible to assign to networks, the rights to be primary and secondary on each of the M channels such that network i, $i = 1, \ldots, N$, is allocated $n_{i,0}$ primary parts and $n_{i,j}$ secondary parts of type j for $j = 1, \ldots, n$. The following lemma describes necessary and sufficient conditions for an allocation to be feasible.

Lemma 2: An allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ is feasible if and only if $n_{i,0}, n_{i,1} \ldots n_{i,n}$, for $i = 1, \ldots, N$, are integers such that for some nonnegative integers $M_j, j = 1, \ldots n$, satisfying $M_1 + \cdots + M_n = M$

$$0 \le n_{i,0} \le M, \quad i = 1, \ldots, N \tag{6}$$

$$\sum_{i=1}^{N} n_{i,0} = M \tag{7}$$

$$0 \le n_{i,j} \le M_j, \quad i = 1, \ldots, N; \; j = 1, \ldots, n \tag{8}$$

$$\sum_{i=1}^{N} n_{i,j} = m_j M_j, \quad j = 1, \ldots, n. \tag{9}$$

Note that the integer M_j in the lemma corresponds to the number of channels that are divided into m_j secondary parts of type j. We assume that the number of bidders is at least m_1 so that a feasible allocation exists.

Proof: The necessity of all conditions is obvious. Now we show sufficiency. Suppose all the above conditions are satisfied. We construct a feasible allocation. Allocate $n_{i,0}$ primary parts to network i for $i = 1, \ldots, N$. Since $\sum_{i=1}^{N} n_{i,0} = M$, each primary part is allocated exactly once. Now, consider the M_j channels divided into m_j secondary parts. Label the m_j secondary parts of type j on each of these channels from 1 to m_j. Also, label the M_j channels from 1 to M_j. Now, consider the following order of the $m_j M_j$ secondary parts of type j: secondary part 1 of channel 1, part 1 of channel $2, \ldots,$ part 1 of channel M_j, part 2 of channel 1, part 2 of channel $2, \ldots,$ part 2 of channel $M_j, \ldots,$ part m_j of channel 1, part m_j of channel $2, \ldots,$ part m_j of channel M_j. Now, with secondary parts in the above order, first allocate $n_{1,j}$ secondary parts to network 1, then $n_{2,j}$ parts to network $2, \ldots$, then $n_{N,j}$ parts to network N. Since $\sum_{i=1}^{N} n_{i,j} = m_j M_j$, in this way it is possible to allocate each secondary part of type j exactly once. Also, since $n_{i,j} \le M_j \forall i$, it is clear that no network is assigned two or more secondary parts on the same channel. Hence, the allocation is feasible. ■

From a feasible allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$, it is easy to construct a consistent specification of the primary and secondary networks on each channel. Hence, the access allocation problem reduces to finding a feasible allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ that maximizes the auctioneer's revenue given the submitted bid vectors $z_i(.)$.

Let

$$k = \{n_{i,j} : i = 1, \ldots N; j = 0, \ldots, n\} \tag{10}$$

denote a feasible allocation. Let K be the set of all feasible allocations.

VI. Optimal Solution of Access Allocation Problem

In this section, we present an algorithm for optimally solving the access allocation problem for the auction described in Section V. The algorithm is polynomial-time when n, the number of possible cardinalities of the set of secondary networks on a channel, is fixed (and m_n is allowed to grow with the problem size). This special case can be useful in practice because even with small n, flexibility in channel allocation can be achieved by choosing $m_1, \ldots, m_n$ judiciously. For example, with $n = 3$, we can choose $m_1 = 1$, $m_2 = 4$, and $m_3 = 8$. In this case, large chunks of secondary throughput can be allocated to a network by having it the sole secondary network on several channels and small chunks can be allocated to networks by having four or eight networks share a channel.

A. Algorithm Description

A dynamic programming algorithm is given in [11] and [12] for the winner determination problem in a combinatorial auction with multiple units of a fixed number of different types of objects. We generalize the algorithm in [11] and [12] in two directions: 1) the objects in a combinatorial auction are indivisible, whereas we need to decide into how many secondary parts to divide each channel; and 2) in our auction, the allocation has to be feasible according to the conditions in Lemma 2.

We first summarize the algorithm. Given the bids $z_i(.)$, our goal is to find the allocation k^* that maximizes revenue. For each set of nonnegative integers $M_1, \ldots, M_n$ such that $M_1 + \cdots + M_n = M$, a dynamic programming algorithm is used to find out the maximum revenue and the maximizing channel allocation when M_j channels are divided into m_j secondary parts, $j = 1, \ldots, n$. Then, we maximize over all sets of $M_1, \ldots, M_n$ to find the optimal set $M_1^*, \ldots, M_n^*$.

We now give the details of the algorithm. Fix $M_1, \dots, M_n$ such that $M_1 + \cdots + M_n = M$. Let $T(j_0, j_1, \dots j_n, i)$ denote the maximum possible revenue from all participating networks when j_0 primary parts and j_t secondary parts of type t, $t = 1, \dots, n$, are to be allocated and networks $1, \dots, i$ are participating in the auction. More precisely, let $K(j_0, j_1, \dots, j_n, i)$ be the set of allocations $k_i = \{n_{v,t} : v = 1, \dots, i; t = 0, \dots, n\}$ satisfying the following conditions, which parallel the conditions in Lemma 2:

$$0 \le n_{v,0} \le M, \ \ v = 1, \dots, i \tag{11}$$

$$\sum_{v=1}^{i} n_{v,0} = j_0 \tag{12}$$

$$0 \le n_{v,t} \le M_t, \ \ v = 1, \dots, i; \ t = 1, \dots, n \tag{13}$$

$$\sum_{v=1}^{i} n_{v,t} = j_t, \ t = 1, \dots, n. \tag{14}$$

Then

$$T(j_0, j_1, \dots j_n, i) = \max\left\{\sum_{v=1}^{i} z_v(n_{v,0}, n_{v,1}, \dots, n_{v,n}) : k_i \in K(j_0, j_1, \dots, j_n, i)\right\}.$$

Thus, $T(M, m_1M_1, \dots m_nM_n, N)$ is the maximum revenue from networks $1, \dots, N$ when M_j channels are divided into m_j secondary parts of type j, for $j = 1, \dots, n$. We now give a dynamic programming algorithm to find $T(M, m_1M_1, \dots m_nM_n, N)$.

The following expression is used for finding the values of $T(j_0, j_1, \dots j_n, 1)$.

$$T(j_0, j_1, \dots j_n, 1) = \begin{cases} z_1(j_0, j_1, \dots, j_n), \\ \ \text{if } j_0 \le M, \ j_t \le M_t, \ t = 1, \dots, n \ . \\ -\infty, \quad \text{otherwise} \end{cases} \tag{15}$$

The reason the equation holds is as follows. Since there is only one network (network 1), the only possibility is to allocate all parts to network 1. However, if $j_0 > M$, then $n_{1,0} > M$, which violates condition (11). Similarly, if $j_t > M_t$, then $n_{1,t} > M_t$, which violates condition (13). Hence, if $j_0 > M$ or $j_t > M_t$, then $T(.)$ is set to $-\infty$.

The recursion (16) at the bottom of the page is used for finding the values of $T(j_0, j_1, \dots j_n, i)$ for $i \ge 2$. In the recursion, if l_0 primary parts and l_v secondary parts of type v, $v = 1, \dots, n$, are allocated to network i, then it is willing to pay $z_i(l_0, \dots, l_n)$, and the maximum revenue obtainable from networks $1, \dots, i-1$ for the remaining parts is by definition $T(j_0 - l_0, j_1 - l_1, \dots, j_n - l_n, i-1)$. Moreover, $l_v \le j_v$ since j_v secondary parts of type v are available and $l_v \le M_v$ by (13). Therefore, $l_v \le \min(j_v, M_v)$ for $v = 1, \dots, n$, and similarly $l_0 \le \min(j_0, M)$. Equation (16) follows by maximizing the revenue from networks $1, \dots, i-1$ over all possible values of $l_0, l_1, \dots, l_n$.

$$\begin{aligned} T(j_0, j_1, \dots, j_n, i) = \max(&T(j_0 - l_0, j_1 - l_1, \dots, j_n - l_n, i-1) + z_i(l_0, l_1, \dots, l_n) \\ &: l_0 \in \{0, 1, \dots, \min(j_0, M)\}, l_v \in \{0, 1, \dots, \min(j_v, M_v)\}, v = 1, \dots, n) \end{aligned} \tag{16}$$

A feasible channel allocation that achieves the maximum revenue $T(M, m_1M_1, \dots m_nM_n, N)$ for the fixed values $M_1, \dots, M_n$ considered can be found from the array $T(.)$ by repeatedly finding the $l_0, l_1, \dots, l_n$ that achieve the maximum in the right side of (16).

For all sets $M_1, \dots, M_n$ such that $M_1 + \cdots + M_n = M$, $T(M, m_1M_1, \dots m_nM_n, N)$ and the revenue maximizing allocation are found as explained above. Then, the optimal set $(M_1^*, \dots, M_n^*)$ is found as follows:

$$(M_1^*, \dots, M_n^*) = \underset{M_1 + \cdots + M_n = M}{\operatorname{argmax}} T(M, m_1M_1, \dots m_nM_n, N). \tag{17}$$

The revenue maximizing allocation with $M_1 = M_1^*, \dots, M_n = M_n^*$ is the one that maximizes revenue over all channel allocations.

B. Running Time

The maximum in (16) is taken over $O(MM_1M_2 \dots M_n)$ values. Moreover, $T(j_0, j_1, \dots, j_n, i)$ is calculated for i from 1 to N, j_0 from 0 to M, j_1 from 0 to $m_1M_1, \dots, j_n$ from 0 to m_nM_n, that is (since $m_j = O(m_n)$ for $j = 1, \dots, n$), for $O(MM_1M_2 \dots M_n m_n^n N)$ values. Finally, this process is carried out for all $M_1, \dots, M_n$ such that $M_1 + \cdots + M_n = M$. Hence, the time to compute k^* is

$$\begin{aligned} &\sum_{M_1 + \cdots M_n = M} O((MM_1 \dots M_n)^2 m_n^n N) \\ &\le \sum_{M_1=0}^{M} \cdots \sum_{M_n=0}^{M} O((MM_1 \dots M_n)^2 m_n^n N) \\ &= O(M^{3n+2} m_n^n N). \end{aligned}$$

Thus, the running time is $O(M^{3n+2} m_n^n N)$, which is polynomial for fixed n.

C. Space Complexity

Each network i submits its bid $z_i(n_{i,0}, n_{i,1}, \dots, n_{i,n})$ for $n_{i,0} \in \{0, 1, \dots, M\}$ and all sets $n_{i,1}, \dots, n_{i,n}$ satisfying $\sum_{j=1}^{n} n_{i,j} \le M$. There are $O(M^{n+1})$ such bids. Summing over the N networks, the storage requirement for bids is $O(M^{n+1}N)$.

To find the revenue maximizing allocation for a fixed set $M_1, \dots, M_n$, we need to store the array $T(j_0, j_1, \dots, j_n, i)$, $j_0 \in \{0, 1, \dots, M\}$, $j_t \in \{0, 1, \dots, m_tM_t\}$, $t = 1, \dots, n$, $i = 1, \dots, N$. This requires $O(M^{n+1} m_n^n N)$ amount of storage. Once the allocation has been found, only the allocation and the value of $T(M, m_1M_1, \dots m_nM_n, N)$ can be stored, which require $O(nN)$ and $O(1)$ storage, respectively, and the rest of the array $T(j_0, j_1, \dots, j_n, i)$ can be discarded.

We need to store the revenue maximizing allocation and the value $T(M, m_1M_1, \ldots m_nM_n, N)$ for all sets $(M_1, \ldots, M_n)$ such that $M_1 + \cdots + M_n = M$. The number of such sets is $O(M^n)$, so the storage required is $O(M^n Nn)$.

Thus, the maximum amount of storage required at any given time during the entire algorithm to compute k^* is $O(M^{n+1}m_n^n N)$.

VII. Greedy 2-Approximation Algorithm

The scheme described in Section VI is computationally tractable for fixed n, the number of possible cardinalities of the set of secondary networks on a channel. However, if n is allowed to grow, the set of bids of a network is exponential in size as shown in Section VI-C, and hence the scheme is computationally intractable. In this section, we first provide a compact bidding language for the case with large n. We conjecture that under this bidding language, the access allocation problem is NP-hard. We give a basis for this conjecture in Section IX. We provide a polynomial-time algorithm that approximates the maximum revenue of the auctioneer within a factor of 2.

We describe the bidding language in Section VII-A. In Section VII-B, we introduce residual bid functions, a concept used in the approximation algorithm. We describe the algorithm in Section VII-C and prove that it achieves an approximation ratio of 2 in Section VII-D. Finally, in Section VII-E, we describe an efficient implementation of the algorithm.

A. Bidding Language

Consider the model in Section V with the following changes. Let the bandwidth of each of the M channels be W b/s. We assume that the primary network on a channel uses the channel for an expected fraction of time α, where $0 < \alpha < 1$. When auctions are repeated periodically to assign spectrum, α can be estimated based on long-term measurements of the primary networks' channel usage. Alternatively, it can be estimated via simulations. Since secondary networks can use the channel whenever the primary is not using it, an expected bandwidth of $W(1-\alpha)$ is available on a channel for the secondary networks. So when m_j secondary networks share a channel, each one of them can get an expected secondary throughput of $\frac{W(1-\alpha)}{m_j}$ on the channel.[3]

In this section, we allow a network to express bids as a function of the number of channels $n_{i,0}$ on which it is primary and its total expected secondary throughput T_i^s on all M channels. Note that

$$T_i^s = \sum_{j=1}^{n} \frac{n_{i,j}W(1-\alpha)}{m_j}. \tag{18}$$

In the sequel, for brevity, we simply say secondary throughput instead of expected secondary throughput. Moreover, we assume that the utility, and hence the bid $z_i(n_{i,0}, T_i^s)$, of each network i when it is primary on $n_{i,0}$ channels and has T_i^s units of secondary throughput, is separable, i.e., of the form

$$z_i(n_{i,0}, T_i^s) = w_i(n_{i,0}) + y_i(T_i^s) \tag{19}$$

where $w_i(n_{i,0})$ is its bid for being primary on $n_{i,0}$ channels and $y_i(T_i^s)$ is its bid for T_i^s units of throughput as a secondary network. This assumption is a good approximation since networks transmit different kinds of traffic (delay-sensitive and elastic, respectively) as a primary and secondary network.

Under this assumption, the access allocation problem separates out into two independent problems: allocating the primary parts and allocating the secondary parts. The problem of allocating the primary parts can be optimally solved in $O(M^2N)$ time using the dynamic programming algorithm in Section VI with $n = 0$. In this section, we focus on giving a 2-approximation algorithm for the problem of allocating the secondary parts. In the rest of the section, "revenue" refers to the auctioneer's revenue from selling the secondary rights on the M channels.

Assume that $y_i(.)$ is a concave increasing function for each network i. We use piecewise linear concave functions to compactly represent the bid functions of the networks. They can be used to closely approximate arbitrary concave functions [17] and have been previously used in the context of spectrum auctions in [6]. Each network i specifies its bid for at most P different levels of secondary throughput, for a positive integer P. More precisely, let $P_i \le P$ be a positive integer, and let

$$0 = q_{i,1} < q_{i,2} < \cdots < q_{i,P_i}. \tag{20}$$

For $v = 1, \ldots, P_i$, network i specifies $y_i(q_{i,v})$, which is its bid for $q_{i,v}$ units of secondary throughput. Network i's bid for q units of secondary throughput, where $q_{i,v} < q < q_{i,v+1}$, is found by linear interpolation

$$y_i(q) = y_i(q_{i,v}) + \left(\frac{y_i(q_{i,v+1}) - y_i(q_{i,v})}{q_{i,v+1} - q_{i,v}}\right)(q - q_{i,v}). \tag{21}$$

Note that $q_{i,1}, \ldots, q_{i,P_i}$ are the breakpoints of the piecewise linear function $y_i(.)$.

We assume that for each network i, $q_{i,1} = 0$, that $y_i(q_{i,1}) = y_i(0) = 0$ and that

$$q_{i,P_i} \ge MW(1-\alpha). \tag{22}$$

Since $MW(1-\alpha)$ is the total secondary throughput available on the M channels, the second assumption means that network i's bid for any amount of secondary throughput on the M channels can be found by linear interpolation.

B. Residual Bid Functions

Our algorithm uses the following concept.

Definition 2: Let $\tilde{q} \ge 0$. The $\tilde{q}$-residual bid function of network i is the function $\tilde{y}_i(.)$ given by

$$\tilde{y}_i(q) = y_i(\tilde{q} + q) - y_i(\tilde{q}). \tag{23}$$

[3]Note that an expected bandwidth of at least $\frac{W(1-\alpha)}{m_j}$ is available to each of the m_j secondary networks. If some of them do not use this full bandwidth, then more than $\frac{W(1-\alpha)}{m_j}$ is available to the other networks.

We will sometimes say "the residual bid function after accounting for $\tilde{q}$" instead of the $\tilde{q}$-residual bid function. Informally, once network i has been allocated $\tilde{q}$ units of secondary throughput, $\tilde{y}_i(.)$ acts as its bid function for allocations of additional secondary throughput. The following lemma gives some simple properties about the $\tilde{q}$-residual bid function.

Lemma 3: Let $\tilde{y}_i(q)$ be the $\tilde{q}$-residual bid function of network i for some $\tilde{q} \geq 0$. Then:

1) $\tilde{y}_i(q) \leq y_i(q) \, \forall q \geq 0$;
2) $\tilde{y}_i(q)$ is a piecewise-linear concave increasing function of q.

Proof:

$$y_i(q+\tilde{q}) \leq y_i(q) + y_i(\tilde{q}), \ \ \forall q \geq 0$$

by concavity of $y_i(.)$. Hence

$$\tilde{y}_i(q) = y_i(q+\tilde{q}) - y_i(\tilde{q}) \leq y_i(q) \ \ \forall q \geq 0$$

which proves part 1.

Now, $y_i(q)$ is piecewise-linear, concave, and increasing by assumption. Thus, $y_i(q+\tilde{q})$ is a piecewise-linear, concave ,and increasing function of q as well. Part 2 follows by (23). ■

The significance of the $\tilde{q}$-residual bid function is given by the following lemma.

Lemma 4: Suppose the bid function of network i is $y_i(.)$, and it is successively allocated secondary throughputs of $q_1, q_2, \ldots, q_f$. Let $y_i^v(.)$ denote the $(q_1 + \cdots + q_v)$-residual bid function of network i, for $v = 1, \ldots, f$. Then

$$y_i(q_1 + \cdots + q_f) = y_i(q_1) + y_i^1(q_2) + \cdots + y_i^{f-1}(q_f). \quad (24)$$

Proof: By definition

$$y_i^1(q_2) = y_i(q_1+q_2) - y_i(q_1)$$

which implies that

$$y_i(q_1+q_2) = y_i(q_1) + y_i^1(q_2). \quad (25)$$

Similarly

$$\begin{aligned} y_i(q_1+q_2+q_3) &= y_i(q_1+q_2) + y_i^2(q_3) & (26)\\ &= y_i(q_1) + y_i^1(q_2) + y_i^2(q_3) & (27) \end{aligned}$$

where the second step follows from (25). Similarly proceeding for f steps, we get the desired result (24). ■

Thus, the significance of the residual bid function is that if a network i is successively allocated chunks $q_1, \ldots, q_f$ of secondary throughput (e.g., by successive steps of an algorithm), then we can keep track of its residual bid function after every allocation so that the extra money that network i is willing to pay for the vth allocation q_v is simply $y_i^{v-1}(q_v)$. Moreover, this tracking can be done using the update rule in part 1 of the following lemma to calculate $y_i^{v+1}(.)$ from $y_i^v(.)$.

Lemma 5: Let $\tilde{y}_i(.)$ and $y_i^+(.)$ be the $\tilde{q}$-residual bid function and $(\tilde{q}+\hat{q})$-residual bid function of network i, respectively. Then:

1) $y_i^+(q) = \tilde{y}_i(q+\hat{q}) - \tilde{y}_i(\hat{q}) \ \ \forall q \geq 0$;
2) $y_i^+(q) \leq \tilde{y}_i(q) \ \ \forall q \geq 0$.

Proof:

$$\begin{aligned} &\tilde{y}_i(q+\hat{q}) - \tilde{y}_i(\hat{q}) \\ &\quad = (y_i(q+\hat{q}+\tilde{q}) - y_i(\tilde{q})) - (y_i(\hat{q}+\tilde{q}) - y_i(\tilde{q})) \\ &\quad = y_i(q+\hat{q}+\tilde{q}) - y_i(\hat{q}+\tilde{q}) = y_i^+(q). \end{aligned}$$

Hence, $y_i^+(.)$ is the $\hat{q}$-residual bid function corresponding to the bid function $\tilde{y}_i(.)$. Thus, by Lemma 3, $y_i^+(q) \leq \tilde{y}_i(q) \, \forall q \geq 0$. ■

C. Algorithm Description

We now describe the greedy 2-approximation algorithm. The algorithm determines S_l^G, the set of secondary networks on channel l for $l = 1, \ldots, M$. Denote by $q_{i,l}^G$, the amount of secondary throughput allocated by the greedy algorithm to network i in the lth channel. Since each network in S_l^G equally shares the secondary throughput on channel l, we have

$$q_{i,l}^G = \begin{cases} \frac{W(1-\alpha)}{|S_l^G|}, & \text{if } i \in S_l^G \\ 0, & \text{else.} \end{cases} \quad (28)$$

Let $y_i^l(.)$ be the $(q_{i,1}^G + \cdots + q_{i,l}^G)$-residual bid function of network i, that is, its residual bid function after accounting for the secondary throughput allocated to it in channels 1 to l. Note that for each network i, $y_i^0(.)$ is the bid function $y_i(.)$.

The greedy algorithm successively determines S_l^G, for $l = 1, \ldots, M$, one channel at a time. Suppose the algorithm has determined $S_1^G, S_2^G, \ldots, S_{l-1}^G$, and for each network i, has found the residual bid function $y_i^{l-1}(.)$. It assigns channel l using the following steps:

STEP1: For $j = 1, \ldots, n$, find the maximum increase in revenue R_j^l obtainable from channel l by dividing the channel into m_j secondary parts using the following rule. Sort the set of numbers $y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right), i = 1, \ldots, N$ into decreasing order. Let $y_{(v)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$ denote the vth largest element. Then, R_j^l is given by

$$R_j^l = \sum_{v=1}^{m_j} y_{(v)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right).$$

STEP2: Find the maximum among $R_1^l, \ldots, R_n^l$. Suppose R_j^l is the maximum. Then, divide the lth channel into m_j secondary parts. On the lth channel, the m_j networks with the m_j largest values $y_{(1)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right), \ldots, y_{(m_j)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$, which were determined in STEP1, become secondary networks. This determines S_l^G.

STEP3: For each $i \in S_l^G$, find the function $y_i^l(.)$ from its bid function $y_i(.)$ and $q_{i,1}^G, \ldots, q_{i,l}^G$. Note that $q_{i,l}^G$ is given by (28).

Comments on Algorithm:

1) Once channels $1, \ldots, l-1$ have been allocated, steps 1 and 2 allocate channel l so as to obtain the maximum possible increase in revenue over the revenue from channels $1, \ldots, l-1$. This property will be crucial in proving the approximation ratio of 2.

2) In Step 3, for conceptual clarity, we have not presented the most efficient implementation. Specifically, the function $y_i^l(.)$ need not be computed from scratch. It can be found iteratively from $y_i^{l-1}(.)$ using the update rule in part 1 of Lemma 5. See Section VII-E for details.

D. Approximation Ratio

Theorem 3: Let R^* be the maximum possible revenue under any allocation of the rights to be secondary networks on the M channels, and let R^G be that achieved by the above greedy algorithm. Then, $R^G \geq \frac{R^*}{2}$.

Proof: Let R^l be the increase in revenue obtained by the greedy algorithm from allocating the lth channel. By part 2 of Lemma 5

$$y_i^l(q) \leq y_i^{l-1}(q) \ \ \forall q \geq 0. \tag{29}$$

From the discussion after Lemma 4, it follows that after channels $1,\ldots,l$ were allocated, the extra money network i was willing to pay for its share in channel $(l+1)$ is $y_i^l(q_{i,l+1}^G)$. Moreover, if the greedy algorithm were to allocate the lth channel to the same set of networks, S_{l+1}^G, to whom it actually allocated the $(l+1)$st channel, then we would have the following.

1) Each network in S_{l+1}^G would have received on the lth channel, a throughput of $\frac{W(1-\alpha)}{|S_{l+1}^G|}$, which equals $q_{i,l+1}^G$ by (28).
2) After channels $1,\ldots,l-1$ were allocated, the extra money network i would be willing to pay for its share in channel l would have been $y_i^{l-1}(q_{i,l+1}^G)$, and hence by (29),
3) the increase in revenue from the lth channel would have been at least R^{l+1}.

However, the actual increase in revenue from the lth channel, R^l, is, by definition of the greedy rule, the maximum possible from allocating the lth channel. Hence, $R^l \geq R^{l+1}$. Thus, we get

$$R^1 \geq R^2 \geq \cdots \geq R^M.$$

Since $R^G = R^1 + \cdots + R^M$, we get

$$R^M \leq \frac{R^G}{M}. \tag{30}$$

Now, let q_i^* be the total secondary throughput allocated by the optimal algorithm to network i and q_i^G be that allocated by the greedy algorithm. Also, let S_l^* be the set of secondary networks on the lth channel, $l = 1,\ldots,M$, in the optimal allocation. Next, we will upper-bound $R^* - R^G$, the excess revenue of the optimal allocation over the greedy allocation. To this end, for each network i, we account for its payment for $\max(q_i^* - q_i^G, 0)$, the excess secondary throughput if any, of the optimal allocation over the greedy algorithm's allocation, by accounting for its payments for the chunks $q_{i,l}^e, l = 1,\ldots,M$. Here, $q_{i,l}^e$ is the contribution of channel l to the excess $\max(q_i^* - q_i^G, 0)$, once the contributions of channels $1,\ldots,l-1$ have been accounted for, and is given by

$$q_{i,l}^e = \min\left(\frac{W(1-\alpha)}{|S_l^*|}, \max(q_i^* - q_i^G - q_{i,1}^e - \cdots - q_{i,l-1}^e, 0)\right), \quad i \in S_l^* \tag{31}$$

$$q_{i,l}^e = 0, \quad i \notin S_l^*. \tag{32}$$

We motivate the expressions above. The second term in the min in (31) is equal to the as-yet unaccounted for excess, if any, obtained by subtracting the contributions $q_{i,1}^e,\ldots,q_{i,l-1}^e$ of channels $1,\ldots,l-1$ from the total excess throughput $\max(q_i^* - q_i^G, 0)$. Also, since channel l is shared by $|S_l^*|$ networks, $q_{i,l}^e \leq \frac{W(1-\alpha)}{|S_l^*|}$. Hence, $q_{i,l}^e$ is the minimum of the two terms in (31).

From (31) and (32), it can be shown using a simple, yet tedious, case-by-case analysis that

$$q_i^* - q_i^G \leq \sum_{l=1}^{M} q_{i,l}^e, \ \ i = 1,\ldots,N. \tag{33}$$

Let $y_{i,l}^e(.)$ be the $(q_i^G + q_{i,1}^e + \cdots + q_{i,l}^e)$-residual bid function of network i.

Now

$$\begin{aligned} R^* &= \sum_{i=1}^{N} y_i(q_i^*) \leq \sum_{i=1}^{N} y_i\left(q_i^G + \sum_{l=1}^{M} q_{i,l}^e\right) \\ &\quad \text{(by (33) and since } y_i(.) \text{ is increasing)} \\ &= \sum_{i=1}^{N}\left(y_i(q_i^G) + \sum_{l=1}^{M} y_{i,l-1}^e(q_{i,l}^e)\right) \ \text{(by Lemma 4)} \\ &= R^G + \sum_{l=1}^{M}\sum_{i=1}^{N} y_{i,l-1}^e(q_{i,l}^e) = R^G + \sum_{l=1}^{M}\sum_{i\in S_l^*} y_{i,l-1}^e(q_{i,l}^e) \end{aligned} \tag{34}$$

where the last step follows since $q_{i,l}^e = 0$ if $i \notin S_l^*$ by (32) and since $y_{i,l-1}^e(0) = 0$.

Now, by the definitions of $y_i^{M-1}(.)$ and $y_{i,l-1}^e(.)$, part 2 of Lemma 5, and (34), we get

$$R^* - R^G \leq \sum_{l=1}^{M}\left(\sum_{i\in S_l^*} y_i^{M-1}(q_{i,l}^e)\right). \tag{35}$$

Now, $q_{i,l}^e \leq \frac{W(1-\alpha)}{|S_l^*|}$ by (31) and (32), and since $y_i^{M-1}(.)$ is increasing by part 2 of Lemma 3, we get the following inequality from (35):

$$R^* - R^G \leq \sum_{l=1}^{M}\left(\sum_{i\in S_l^*} y_i^{M-1}\left(\frac{W(1-\alpha)}{|S_l^*|}\right)\right). \tag{36}$$

Now, we have

$$\sum_{i\in S_l^*} y_i^{M-1}\left(\frac{W(1-\alpha)}{|S_l^*|}\right) \leq R^M \tag{37}$$

because when the greedy algorithm was about to allocate channel M, the increase in revenue it would have got from the channel if it allocated the channel to the $|S_l^*|$ networks in the set S_l^* is equal to the expression on the left-hand side of (37) (refer to Lemma 4 and the discussion immediately following it). This expression is at most R^M since the greedy algorithm allocates the Mth channel so as to maximize the increase in revenue from it.

By (36) and (37), we get

$$R^* - R^G \leq \sum_{l=1}^{M} R^M \leq R^G \text{ (from (30))}.$$

The result follows. ■

E. Efficient Implementation

We now describe an efficient implementation of the greedy algorithm.

We first discuss how to store the function $y_i^l(.)$ so that $y_i^l(q)$ can be found for any q in $O(\log P)$ time. Recall from part 2 of Lemma 3 that $y_i^l(.)$ is piecewise-linear. Similar to the representation of the bid function $y_i(.)$, $y_i^l(.)$ is stored by storing its value at P_i^l values $q_{i,1}^l, \ldots, q_{i,P_i^l}^l$, which are the breakpoints of the piecewise-linear function $y_i^l(.)$. Also, $y_i^l(q)$, where $q_{i,v}^l < q < q_{i,v+1}^l$ is found by linear interpolation similar to (21)

$$y_i^l(q) = y_i^l(q_{i,v}^l) + \left(\frac{y_i^l(q_{i,v+1}^l) - y_i^l(q_{i,v}^l)}{q_{i,v+1}^l - q_{i,v}^l} \right) (q - q_{i,v}^l). \quad (38)$$

Now, the numbers $q_{i,1}^l, \ldots, q_{i,P_i^l}^l$ and the numbers $y_i^l(q_{i,1}^l), \ldots, y_i^l(q_{i,P_i^l}^l)$ can be stored in two sorted arrays, so that for any v, $q_{i,v}^l$ and $y_i^l(q_{i,v}^l)$ can be accessed in constant time. Also, since $P_i^l \leq P_i \leq P$ (see the last step in the steps below), given any q, we can find v such that $q_{i,v}^l \leq q < q_{i,v+1}^l$ by binary search [13] in $O(\log P)$ time. Once this v is found, we can find $y_i^l(q)$ in constant time using (38).

Suppose the algorithm has allocated the first $l-1$ channels and hence has computed $y_i^{l-1}(.)$ and $q_{i,v}^{l-1}, v = 1, \ldots, P_i^{l-1}$. Also, suppose the lth channel has been divided into m_j secondary parts. While allocating channel l, in Step 3, $y_i^l(.)$ can be found as follows from $y_i^{l-1}(.)$ using the update rule in part 1 of Lemma 5. For network i, first find out v such that

$$q_{i,v}^{l-1} \leq \frac{W(1-\alpha)}{m_j} < q_{i,v+1}^{l-1}.$$

Then, find $y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$ using (38). Next, perform the following steps:

$q_{i,1}^l = 0$
$y_i^l(q_{i,1}^l) = 0$
for $t = 2, 3, \ldots, P_i^{l-1} - v + 1$ **do**
  $q_{i,t}^l = q_{i,t+v-1}^{l-1} - \frac{W(1-\alpha)}{m_j}$
  $y_i^l(q_{i,t}^l) = y_i^{l-1}(q_{i,t+v-1}^{l-1}) - y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$
end for
$P_i^l = P_i^{l-1} - v + 1.$

The second statement in the **for** loop implements the update rule in part 1 of Lemma 5. Also, it can be checked that the first statement in the **for** loop appropriately sets the breakpoints of the function $y_i^l(.)$.

It can be shown that the running time of the greedy algorithm is $O(nMN \log NP + MPm_n)$ when the presented implementation is used.

VIII. Simulations

In Section VII-D, we proved that the greedy approximation algorithm achieves an approximation ratio of 2. In this section, we show via simulations that in fact, for a variety of scenarios, the greedy algorithm achieves the optimal revenue.

In all our simulations, we used the values $n = 2$, $m_1 = 1$, $m_2 = 4$, and $W(1-\alpha) = 4$ units. First, we simulated the case in which the bid function of every network is different and is a piecewise-linear approximation of a quadratic function. Let $C_{\min}$, $C_{\max}$, and MAX be parameters such that $C_{\max} > C_{\min} > 0$ and $MAX > 0$. Consider the following quadratic function:

$$\hat{y}_i(q) = c_i \left(1 - \frac{(q - MAX)^2}{(MAX)^2} \right), \quad i = 1, \ldots, N. \quad (39)$$

The bid function $y_i(q)$ of network i is chosen to be a piecewise-linear approximation of the above function, where the parameters c_i, $i = 1, \ldots, N$ are uniformly spaced in the interval $[C_{\min}, C_{\max}]$

$$c_i = C_{\min} + (i-1) \frac{(C_{\max} - C_{\min})}{N-1}, \quad i = 1, \ldots, N. \quad (40)$$

With these bid functions, we found the revenue using the greedy approximation algorithm and the optimal revenue using the dynamic programming algorithm in Section VI. We used small values for n and M since the running time of the dynamic programming algorithm grows rapidly with these parameters (see Section VI-B). For different values of the parameters N, $C_{\min}$, $C_{\max}$, and MAX, we evaluated the revenues of the greedy algorithm and the optimal revenue for M varying from 5 to 60 and found that the greedy algorithm achieves the optimal revenue.

Next, we considered the case in which there are two classes of networks and the bid function of each network in the same class is the same. The bid functions $y_i(q)$ of networks $i = 1, \ldots, N_1$ and of networks $i = N_1 + 1, \ldots, N$ are piecewise-linear approximations of the following exponential functions, respectively

$$\hat{y}_i(q) = B_1(1 - \exp(-a_1 q)), \quad i = 1, \ldots, N_1 \quad (41)$$
$$\hat{y}_i(q) = B_2(1 - \exp(-a_2 q)), \quad i = N_1 + 1, \ldots, N \quad (42)$$

where a_1, a_2, B_1, B_2, and N_1 are parameters. For different values of these parameters, we evaluated the revenues of the greedy algorithm and the optimal revenue for M varying from 5 to 60 and found that the greedy algorithm achieves the optimal revenue.

Thus, although the worst-case approximation ratio of the greedy algorithm is 2, in a variety of scenarios, it achieves the optimal revenue.

Nevertheless, we could construct some pathological examples in which the greedy algorithm achieves a revenue equal to $\frac{5}{6}$ times the optimal revenue and is therefore strictly suboptimal. We now describe one such example. Let $M = 2, N = 3, n = 2$, $m_1 = 2, m_2 = 3$, and $W(1-\alpha) = 6$. The bid function of network i, $i \in \{1, 2, 3\}$, is given by

$$y_i(q) = \begin{cases} q\beta_i, & \text{if } q \leq 4 \\ 4\beta_i, & \text{if } q > 4 \end{cases} \tag{43}$$

where $\beta_1 = \beta_2 = 1$, $\beta_3 = 1 - \epsilon$, and ϵ is a small positive constant. It can be checked that the greedy algorithm assigns channel 1 to networks 1 and 2 and channel 2 to networks 1, 2, and 3 and achieves a revenue of $R^G = (10 - 2\epsilon)$. The optimal algorithm assigns each one of channels 1 and 2 to networks 1, 2, and 3 and achieves a revenue of $R^* = (12 - 4\epsilon)$. Note that $\frac{R^G}{R^*}$ equals $\frac{5}{6}$ in the limit as ϵ tends to 0.

In summary, the greedy algorithm is suboptimal only for pathological input instances and is optimal for a large variety of "well-behaved" inputs; thus, it performs well in practice.

IX. Future Work

We now describe some directions for future research. We conjecture that the access allocation problem described in Section VII-A is NP-hard. Our conjecture is motivated by the facts that: 1) the bid function of each network can be an *arbitrary real-valued* function satisfying the conditions in Section VII-A; 2) the number of secondary networks on each channel can be selected from a possibly large set $\{m_1, \ldots, m_n\}$; and 3) the set of secondary networks on each channel can be an arbitrary subset of the set of all N networks. The proof of the conjecture remains an open problem for future research.

Also, we considered the case when the M channels are identical. The extension to nonidentical channels remains an open problem.

When the auctioneer's objective is to maximize its revenue, note that the algorithms that we designed for the access allocation problem can be used to maximize the auctioneer's revenue *given the bids* $z_i(.)$ *of the bidders*. To compute its bid, a bidder i may use different strategies, which it thinks will maximize its net utility in (1). For example, when auctions are conducted periodically, a bidder may compute its bid based on its knowledge of the outcomes of previous auctions. An open problem is the design of allocation strategies for the auctioneer and bidding strategies for the bidders when each player chooses its strategies based on the outcomes of previous auctions in order to influence the other players to act to its own advantage.

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