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Hyeongseok Ko
University of Pennsylvania

Norman I. Badler
University of Pennsylvania, badler@seas.upenn.edu

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Comments
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Hyeongseok Ko
Norman I. Badler

University of Pennsylvania
School of Engineering and Applied Science
Computer and Information Science Department
Philadelphia, PA 19104-6389

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Hyeongseok Ko, Norman I. Badler
Department of Computer and Information Science
School of Engineering and Applied Science
University of Pennsylvania
200 South 33rd Street
Philadelphia, PA 19104-6389
e-mail: ko@graphics.cis.upenn.edu, badler@central.cis.upenn.edu
Phone: (215) 898-1976
Fax: (215) 898-0587

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Abstract

The most prominent problems in utilizing the rotoscopy data for human walking animation can be summarized as preservation of the original motion characteristics in the generalization process and constraint satisfaction. Generalization is the process of producing the step of an arbitrary body and step length from the original measured step of one particular subject and step length. If we lose much of the original style in the generalization, it would be meaningless to use the measured data. We present a generalization technique that keeps the original spatial motion characteristics as much as possible. Two types of generalization are considered. One is the anthropometry generalization, which handles the non-uniform segment length ratio differences between the two bodies. The other is the step length generalization, which changes the steps to different step lengths of the same subject. These two generalizations are combined together to generate a step of arbitrary subject and step length. The constraint satisfaction is enforced within the generalization process.
1 Introduction

The dynamic model of human walking can be described by [3, 4, 14, 5]

\[ M\ddot{q} + D\dot{q} + Kq = f_{\text{internal}} + f_{\text{others}} \]

where \( q \) is the generalized coordinates, and \( M, D, \) and \( K \) are the matrices of inertia, damping, and stiffness, respectively. Generalized force is decomposed into two parts, the internal force (torque) \( f_{\text{internal}} \) and the other forces \( f_{\text{others}} \). The problem encountered in using the above equation in human walking animation is that there are unknowns in both sides of the equation, namely \( \dot{q} \) and \( f_{\text{internal}} \).

Bruderlin and Calvert built a keyframeless locomotion system [4, 3]. They generated every single frame based on both dynamics (for the movement of the underlying dynamic model of human body) and kinematics (for obtaining the detailed configuration). In the forward dynamics, he approximated \( f_{\text{internal}} \) according to the general biomechanical knowledge on human walking. Their system could generate a wide gamut of walking by changing the three primary parameters and other attributes. However, the dynamic model was not easy to control. For example, the walking motion is very sensitive to the hip height. 1cm difference in the height produce a big difference in the appearance of the resulting gait. Because their dynamic model was an approximation, it could generate the height error that made the motion unnatural. The dynamic correction phase required several iterations, due to its undirectness.

In the kinematic aspects, an enormous amount of measurement has been performed on human walking [11, 10], including Winter et. al’s work [15, 16, 12]. Therefore we tried to find a solution in a kinematic space, which by its nature promises realistic human-like motion.

There have been several attempts [10, 11, 6, 9] to obtain a general property of human walking, which is basically the average of the subjects considered. This generalization demonstrates the properties of human walking in general, but it does not provide exact data for one particular subject and step, which makes it difficult to be used in computer animation.

Boulic et al. tried a generalization of experimental data based on the normalized velocity of walking [2]. Their generalization could produce the parameters which might violate, in its direct application, some of the constraints imposed on walking. They overcame this problem through a correction phase based on inverse kinematics. To preserve the original characteristics of the walking data, they introduced the coach concept, which basically chooses the one among the multiple inverse kinematics solutions that is closest to the original motion. In our approach, the constraints are enforced within the generalization process, obviating the correction phase. Also the original locomotion style is maintained as much as possible, according to the definitions and discussions in the following sections.

Suppose a measured data set \( W(S_1, s_{l1}) \) of the subject \( S_1 \) and the step length \( s_{l1} \) is given. Our goal in this paper is to generate another data set \( W(S_2, s_{l2}) \) of arbitrary subject \( S_2 \) and step length \( s_{l2} \). In this way, from the data of one particular subject and step, we can produce steps of any subject and step length.

When another step data \( W(S_3, s_{l3}) \) is to be generated, most people may use the original measured one \( W(S_1, s_{l1}) \) rather than the generalized one \( W(S_2, s_{l2}) \) as the input of the generalization process, because some characteristics of the original motion may have been lost in producing \( W(S_2, s_{l2}) \). However, if both of \( W(S_1, s_{l1}) \) and \( W(S_2, s_{l2}) \) produce the same result, then we can
consider the original characteristics of \( W(S_1, s_{l1}) \) to be maintained in \( W(S_2, s_{l2}) \) during the generalization.

Furthermore, if the above is true for any \( S_3 \) and \( s_{l3} \) (transitive), then \( \tilde{W}(S_n, s_{l_n}) \) after the series of generalizations \( (W(S_1, s_{l1}), W(S_2, s_{l2}), \ldots, W(S_n, s_{l_n})) \) will be the same with \( \tilde{W}(S_n, s_{l_n}) \) after the direct generalization \( (W(S_1, s_{l1}), W(S_n, s_{l_n})) \). Under the transitivity, we can keep any one of the intermediate results instead of the original measured data for further generalization. The transitivity will be used as the measure of characteristic preservation, in the generalization of experimental data. We will show in the subsequent sections that our generalization scheme is indeed transitive.

One merit of our generalization method is that it can be extended incrementally. Because it basically imitates the original motion, we can simulate different locomotion styles by acquiring multiple sets of measurements. Thus, in one scene, several people can walk in their own walking pattern.

Any kinematic generalization just in the joint space has a drawback. For example, if the angle of the stance leg is given from the ball of the foot, ankle, knee, and hip, in that order, the error at the ball affects the position of the upper body more than the one at the hip. If we reverse the order, the same problem appears in locating the foot. To avoid this problem, we use Cartesian points for locating the hip and the ankle. Thus even if there is any error in the lower limb extreme angles, it is closed within the foot and the upper body is not affected.

A timed sequence \( Q \) is a set of 2-tuples

\[
Q = \{(t_i, v_i) \mid i = 1, \ldots, n\}
\]

where each \( t_i \) is a real number with \( t_i < t_{i+1} \) for \( i = 1, \ldots, n-1 \), and \( v_i \) can be any dimensional vector but should be the same dimensional for every \( i \). For any timed sequence \( Q \), we can define the function interpolation as:

\[
\text{interpolation}(Q, t) = v_i + \frac{t - t_i}{t_{i+1} - t_i} (v_{i+1} - v_i), \quad t_i \leq t \leq t_{i+1}.
\]

At a certain moment, if a leg is between its own heelstrike (beginning) and the other leg's heelstrike (ending), it is called the stance leg. If a leg is between the other leg's heelstrike (beginning) and its own heelstrike (ending), it is called the swing leg. For example, in Figure 1, left leg is the stance leg during the interval 1, and right leg is the stance leg during the interval 2. Thus at each moment we can refer to a specific leg by either stance or swing leg with no ambiguity. The joints and segments in a leg will be referred to using prefixes swing or stance. For example, swing ankle is the ankle in the swing leg.

Let \( HSM^- \) be the Heel Strike Moment just before the current step, \( HSM^+ \) be the Heel Strike Moment right after the current step, which is one step after \( HSM^- \), \( FGM \) be the moment when the stance foot is put flat on the ground (Flat Ground Moment), \( MOM \) be the Meta Off Moment when the toes begin to be off the floor and rotate around the tip of the toe, and \( TOM \) be the Toe Off Moment.

The anthropometry \( B(S) \) of a subject \( S \) is simply the m-tuple \( (l_1, \ldots, l_m) \), where \( l_i \) is the length of the \( i \)th segment and \( m \) is the total number of the segments. We say \( S_2 = \alpha S_1 \) iff \( B(S_2) = \alpha B(S_1) \). In this case \( S_2 \) is also denoted as \( \alpha S_1 \).

Among the body condition components, the one that mostly affects the lower body movement in walking would be the length of the leg. The leg length \( l_l(S) \) of the subject \( S \) is defined to be the
sum of the lengths of the thigh and the calf. We say \( S_2 \approx \alpha S_1 \) iff \( ll(S_2) = ll(\alpha S_1) \). In this case, \( S_2 \) is denoted as \( \alpha S_1 \), and we say

\[
\alpha = \frac{S_2}{S_1}
\]

Note that \( S_2 \approx \alpha S_1 \) and \( \alpha = \frac{S_2}{S_1} \) hold also when \( S_2 = \alpha S_1 \). The walk condition is simply the tuple \((S, s_l)\) of the subject and the step length.

The walk data \( W(S, s_l) \) of the subject \( S \) and the step length \( s_l \) is defined as the collection of the six timed sequences

\[
W(S, s_l) = \{H, F_i, F_3, M, A\}
\]

where each element represents hip trajectory, foot sole angle (simply foot angle later on) trajectories during \([HSM-, FGM]\), \([FGM, TOM]\), and \([TOM, HSM+]\), the meta angle trajectory, and the ankle trajectory, respectively. The meta angle is the angle between the floor and the toes during the meta off phase [4, 3].

Note that \( H \) and \( A \) are in the Cartesian space, and the other elements are in the joint space. \( H \) and \( F_i \), \((i = 1, 2, 3)\) govern the stance leg of the current step, from the start to the end of the step. \( M \) is for the current stance leg from the MOM until the TOM of the next step. \( A \) is for the swing leg.

All the time values used in defining the timed sequences are normalized according to the step duration. For example, \( HSM^- = 0 \) and \( HSM^+ = 1 \). Together with the function interpolation, \( W(S, s_l) \) provides enough information to generate the lower body movement during one step. If it is clear in the context, we denote them simply as \( W \) without the arguments. The difference between \( \hat{W} \) and \( W \) is that \( \hat{W} \) is the measured data whereas \( W \) is either measured or computed according to the generalization algorithm.

In the following sections, we will assume that we have the measured data of the subject \( S^* \) at the step length \( s_l^* \), \( \hat{W}(S^*, s_l^*) \), which will be called prototype walk data.

### 2 Generalization to Other Step Lengths

Suppose that we have the walk data \( W(S, s_l) \) of the subject \( S \) with the step length \( s_l \). Our goal in this section is to obtain the walk data \( W(S, ps_l) \) of the same figure with the step length \( ps_l \). Throughout this paper, we consider only sagittal plane movements. We have shown elsewhere that curved path motion may be computed form underlying sagittal plain walking data [7, 1].
2.1 The Position of the Hip at the Heel Strike Moment

The problem in this subsection is to find the position of the hip at the $HSM^+$ (at time $t_2$ in Figure 2) after stepping $sl$. We will use the step symmetry concept which was assumed in Bruderlin’s work [3, 4]. A closer look of the leg $L_2$ of Figure 2 is shown in Figure 3.

Length $a_1$ is the distance between HIP and $ANKLE_2$. It depends on the knee angle at the heel strike moment (KAHSM) and the foot angle at the heel strike moment (FAHSM). Inman showed the KAHSM depends on the subject and the step length [6]. But within one subject, the KAHSM is bigger for longer steps. We approximated the KAHSM by the function

$$\mu_1(S, sl) = -\alpha_1 \left(\frac{sl}{ll(S)} - sl^*\right) + \mu_1^*$$

(5)

where $\mu_1^*$ is the KAHSM of the prototype walk data. We can increase or decrease $\alpha_1$ within the range $[0, 0.3]$ without affecting the preservation property of our generalization. This specific interval is based on Inman’s work [6]. A similar function $\mu_2$ can be defined to approximate FAHSM by

$$\mu_2(S, sl) = \alpha_2 \left(\frac{sl}{ll(S)} - sl^*\right) + \mu_2^*$$

(6)

where $\mu_2^*$ is the FAHSM of the prototype walk data, and $\alpha_2$ is a positive constant we can control. As demonstrated by the formula, the FAHSM tends to be bigger for longer steps. This is also justified by Inman’s work [6].

Now we can compute $a_1$ of the figure $S$ by the formula

$$a_1^2 = thigh^2 + calf^2 - 2 \times thigh \times calf \times \cos \mu_1(S, sl)$$

(7)

for the step length $sl$.

Let the position $ANKLE_1 = (x_a, y_a)$. Then $ANKLE_2 = (x_a + sl, y_a)$. The heel position is given by

$$HEEL = (x_a + sl - a_2, 0).$$

(8)

The position of $A$ is obtained by rotating $ANKLE_2$ by $\mu_2(S, sl)$ around $HEEL$. Let the resulting position be

$$A = (x_a', y_a').$$

(9)
The $x$ coordinate of $HIP$ is $x_a + \frac{sl}{2}$. Therefore

$$a_3 = \sqrt{a_1^2 - (x'_a - x_a - \frac{sl}{2})^2}. \quad (10)$$

Therefore the height of the hip is $a_3 + y'_a$, and

$$HIP = (x_a + \frac{sl}{2}, a_3 + y'_a). \quad (11)$$

Note that the position of the hip relative to the previous ankle $ANKLE_1$ is completely determined by the given step length $sl$. That is, the hip position at the $HSM^+$ is determined independently of the previous steps.

### 2.2 Hip Trajectory in Cartesian Space

In this subsection, we will show how the hip trajectory $H^2$ of the step of the subject $S$ and step length $psl$ is generated, which is the first element of $W(S, psl)$. It will be obtained by modifying $H^1$ of the original walk data $W(S, sl)$. The other elements of $W(S, psl)$ will be discussed in the subsequent subsections.
Because the length of the leg is limited, the height of the hip is lower at the HSM+ of the bigger steps, as shown in Figure 4. Inman observed that the knee flexion during the stance phase is bigger, and therefore the maximum height of the hip is lower, for the longer steps [6]. The resulting hip trajectory will look like Figure 5.

Let the height of the hip just before the current step is \( y_1 \) which is given by the current posture, and the one right after the step be \( y_n \) which is given by the computation in the previous subsection. Let’s suppose the hip trajectory \( H^1 \) of \( W(S, sl) \) be

\[
H^1 = \{(t_i, x_i, y_i) | i = 1, \ldots, n\}
\]

where \( x_i \) is the value relative to the stance ankle at FGM of the current step, and \( y_i \) is relative to the height of the floor. For example, the \( x \) coordinate of the hip position just before the current step is negative, until the hip passes the current ankle. This convention will be used throughout this section.

The hip trajectory \( H^2 \) of \( W(S, psi) \) is defined as

\[
H^2 = \{(t_i, \rho' x_i, y_i') | i = 1, \ldots, n\}
\]

where

\[
y_i' = y_i + (1 - t_i)(\tilde{y}_1 - y_1) + t_i(\tilde{y}_n - y_n)
\]

and \( \rho' \) is to accommodate the step length difference. Note that it may be different from \( \rho \) because the step length is defined by the foot movement, not by the hip movement. It is given by

\[
\rho' = \frac{sl_{before} + \rho sl}{sl_{before} + sl}
\]

where \( sl_{before} \) is the step length of the previous step. These particular definitions will be justified in the later sections, by showing their characteristic preservation properties.

In the following subsections, we will consider the foot movement, in both angular and positional terms. The foot angle is enough to determine the stance foot configuration until MOM, because the toes stay flat on the floor. But from MOM until TOM, both the foot angle and the meta angle are needed to determine the configuration. Again, from the TOM until HSM+, because the swing foot is off the ground, we need not only the foot angle, but also at least one point (in our work, ankle) in the foot. That is why the \( F_1,F_2,F_3,M \), and \( A \) of \( W(S, sl) \) are defined in the specific intervals.

### 2.3 The Foot Angle Trajectory

Suppose \( F^1 \) is the profile of the foot angle of \( W(S, sl) \), which is the set union of the timed sequences \( F_1^1, F_2^1, \) and \( F_3^1 \). \( F^1 \) represents the foot angle during [HSM-,FGM], \( F_2^1 \) during [FGM,TOM], and \( F_3^1 \) during [TOM,HSM+].
and $F_3^1$ during $[TOM, HSM^+]$. I.e.,

\begin{align}
F^1 &= F_1^1 \cup F_2^1 \cup F_3^1 \\
F_1^1 &= \{(t_{1i}, f_{1i}) \mid i = 1, \ldots, n_1\} \\
F_2^1 &= \{(t_{2i}, f_{2i}) \mid i = 1, \ldots, n_2\} \\
F_3^1 &= \{(t_{3i}, f_{3i}) \mid i = 1, \ldots, n_3\}
\end{align}

with $t_{1i} = t_{2i}$ and $t_{2i} = t_{31}$.

The corresponding foot angle $F^2$ of $W(S, \rho sl)$ is defined by set union of the timed sequences $F_1^2$, $F_2^2$, and $F_3^2$. $F^2$ is defined as

\begin{equation}
F_1^2 = \{(t_{1i}, f_{1i} \mu_2(S, \rho sl \text{ before}) / f_{1i}) \mid i = 1, \ldots, n_1\}
\end{equation}

Inman [6] shows the tendency of bigger foot angles at TOM (FATOM) for the longer steps, which is approximated by the function $\mu_3$ as

$$\mu_3(S, sl) = \alpha_3 \left( \frac{sl}{l(S)} - sl^* \right) + \mu_3^*$$

where, $\mu_3^*$ is the FATOM of the prototype walk data, and $\alpha_3$ is a positive constant we can control.

$\mu_3$ can be used in defining $F_2^2$:

\begin{equation}
F_2^2 = \{(t_{2i}, f_{2i} \mu_3(S, \rho sl) / f_{2n2}) \mid i = 1, \ldots, n_2\}.
\end{equation}

Based on the monotonic property of the foot angle during $[TOM, HSM^+]$ [6], we define the last subset $F_3^2$ of $F^2$ as

\begin{equation}
F_3^2 = \{(t_{3i}, f_{3i}) \mid f_{3i} = f_{3i} + (1 - t_{3i})d_1 + t_{3i}d_2, i = 1, \ldots, n_3\}
\end{equation}

where $t_{3i} = HSM^+ / TOM$, $d_1 = \mu_3(S, \rho sl) - f_{3i}$, and $d_2 = \mu_2(S, \rho sl) - f_{3n2}$.

### 2.4 The Meta Angle Trajectory

After the FGM, meta angle is maintained close to zero until the MOM. From the MOM, it suddenly increases linearly to its maximum until the TOM [6]. The maximum value, i.e, the meta angle at TOM (MATOM), depends on the step length. It is approximated by the function $\mu_4$ as

$$\mu_4(S, sl) = \alpha_4 \left( \frac{sl}{l(S)} - sl^* \right) + \mu_4^*$$

where, $\mu_4^*$ is the MATOM of the prototype walk data, and $\alpha_4$ is a positive constant we can control.

Suppose $M^1$ is the timed sequence that represents the meta angle trajectory of $W(S, sl)$ during $[MOM, TOM]$:

$$M^1 = \{(t_i, m_i) \mid i = 1, \ldots, n_m\}.$$  

The corresponding meta angle trajectory $M^2$ of $W(S, \rho sl)$ is defined to be

$$M^2 = \{(t_i, m_i \mu_4(S, \rho sl) / m_{nm}) \mid i = 1, \ldots, n_m\}.$$
2.5 Ankle Trajectory During \([TOM, HSM^+]\)

The ankle positions at \(TOM\) and \(HSM^+\) can be computed using the functions \(\mu_3, \mu_4, \) and \(\mu_2\). As shown in Figure 6, the distance traveled along the \(x\) axis during \([TOM, HSM^+]\) is slightly shorter than the actual step length. Let the coordinates of the ankle just before the \(TOM\) and right after the \(HSM^+\) of the step length \(sl\) be \((b_{1x}, b_{1y})\) and \((b_{2x}, b_{2y})\), respectively; likewise, those of the step length \(\rho sl\) be \((b_{3x}, b_{3y})\) and \((b_{4x}, b_{4y})\), respectively.

If the ankle trajectory of \(W(S, sl)\) is given by

\[
A^1 = \{(t_i, x_i, y_i) \mid i = 1, \ldots, n_a\}
\]

then that of \(W(S, \rho sl)\) is defined as

\[
A^2 = \{(t_i', x_i', y_i') \mid i = 1, \ldots, n_a\}
\]

where

\[
x_i' = b_{3x} + \frac{b_{4x} - b_{3x}}{b_{2x} - b_{1x}}(x_i - b_{1x})
\]

and

\[
y_i' = y_i + (1 - t_i')d_1 + t_i'd_2
\]

with \(d_1 = b_{3y} - y_1, d_2 = b_{4y} - y_n, \) and \(t_i' = \frac{t_i - TOM}{HSM^+ - TOM}\).

3 Properties of Step Length Generalization

We will denote the whole step length generalization process in the previous section by \(\phi_\rho\). It can be interpreted as an operator whose input is walk data \(W(S, sl)\) and its output is another walk of the same subject with step length \(\rho\) times that of the original one. It can be compactly written as

\[
W(S, \rho sl) = \phi_\rho(W(S, sl)).
\]
Figure 7: The Constants defined in the Foot

Before introducing the lemmas and theorems about the property of the step length generalization, let’s define several lengths in the foot. \( ah, ab, hb, \) and \( bt \) are defined to be the lengths between the ankle and the heel, the ankle and the ball of the foot, the heel and the ball of the foot, the ball of the foot and the tip of toes, respectively. Let the point on the foot sole which is right below the ankle when the foot is flat on the ground be \( N \). Lengths \( aN, hN, \) and \( bN \) are defined to be the distances between the ankle and \( N \), the heel and \( N \), the ball of the foot and \( N \), respectively. The angles \( \theta_a, \theta_b, \theta_h \) are defined to be the inner angles of the hindfoot, the ankle, the ball of the foot, and the heel, respectively.

**Lemma 1** The coordinates of the ankle of the subject \( S \) just before the TOM and right after the \( HSM^+ \) of the step length \( sl \), \((b_{1x},b_{1y})\) and \((b_{2x},b_{2y})\), respectively are given by the following formulas.

\[
\begin{align*}
    b_{1x} &= bN + bt - bt \cos \mu_4(S,sl) - ab \cos \mu_3(S,sl) \\
    b_{1y} &= bt \sin \mu_4(S,sl) + ab \sin \mu_3(S,sl) \\
    b_{2x} &= sl - (ah \cos \theta_h - ah \cos(\theta_h + \mu_2(S,sl))) \\
    b_{2y} &= ah \sin(\theta_h + \mu_2(S,sl))
\end{align*}
\]  

**Lemma 2** In the step length generalization algorithm given in the previous section, the step length is generalized correctly. That is, the step length of \( \phi_p(W(S,sl)) \) is \( psl \). □

**Theorem 1** The step length generalization defined in the previous section, can be composed in the following way for any positive number \( p_1 \) and \( p_2 \).

\[
\phi_{p_1p_2} = \phi_{p_2} \circ \phi_{p_1}
\]  

□

**Corollary 1** For any two step length generalizations \( \phi_{p_1} \) and \( \phi_{p_2} \), their composition is commutative.

\[
\phi_{p_2} \circ \phi_{p_1} = \phi_{p_1} \circ \phi_{p_2}
\]  

□
Corollary 2 For any three step length generalizations $\phi_{p_1}$, $\phi_{p_2}$, and $\phi_{p_3}$, their composition is transitive.

$$\phi_{p_1} \circ (\phi_{p_2} \circ \phi_{p_3}) = (\phi_{p_1} \circ \phi_{p_2}) \circ \phi_{p_3} \quad (39)$$

The types of generalization considered in this paper are (1) $SL$ (step length), (2) $BC$ (body condition), and (3) $BOTH$. One of these will be denoted by the type variable $U$.

**Definition 1** Let $T$ be a walk data transformation on the type $U$. For instance, $\phi_{p}$ is a walk data transformation on $SL$. $T |_{(p_1,p_2)}$ is a walk data transformation from $p_1$ to $p_2$, for arbitrary parameters $p_1$ and $p_2$ within the type $U$. For example, if $T$ is a transformation on the step lengths, and $s_{l_1}$ and $s_{l_2}$ are step lengths, $T |_{(s_{l_1},s_{l_2})}$ is the transformation which tries to produce the walk data of step length $s_{l_2}$ out of the walk data of step length $s_{l_1}$. □

**Definition 2** Let $W$ be walk data. Let $U$ be one of the generalization types. $W |_{U}$ is the value of $U$ of $W$. For example, $W |_{SL}$ is the step length of the walk data $W$. □

**Definition 3** Let $T$ be a walk data transformation on $U$. If $T$ satisfies the following condition for an arbitrary walk data $W$ with $W |_{U} = p_1$, and arbitrary parameters $p_2$ and $p_3$ in $U$, it is called transitive.

$$T |_{(p_2,p_3)} \circ T |_{(p_1,p_2)} = T |_{(p_1,p_3)} \quad (40)$$

Suppose the original walk data $W_1$ is generalized into $W_2$, and the generalized one $W_2$ can again be used as an original walk data to produce another one $W_3$, and so on, until we get $W_n$. Specially if the result $W_n$ of the serial generalization $(W_1,W_2,W_3,\ldots,W_n)$ is identical with the result $W_n$ of the direct generalization $(W_1,W_n)$, for any $n$, the original property of the walk data seems to be preserved in each generalization. This intuition leads to the following definition.

**Definition 4** A walk data transformation is said to be characteristic preserving if the transformation is transitive. □

**Theorem 2** The walk data transformation $\phi$ on the step length defined in this section is characteristic preserving.

4 Generalization among Different Body Conditions

Let us imagine two human figures A and B. B's kinematic property is $\alpha$ times that of A's, in every aspect segmentwise. That is, $B = \alpha A$. In this situation, if B is walking at a step length that is $\alpha$ times as long as that of A's, what would be the joints angles of B compared to the corresponding joint angles of A? We assume that they are the same. Therefore if we have data for A at the step length $s_{l}$, we can use it for B at the step length $\alpha s_{l}$. This will be called the similarity assumption later on.

The above is justifiable by Murray et al.'s experiment [11]. They divided 60 subjects into 3 groups (20 subjects in each group) according to their height: tall, medium, and short. Each subject
was trained to walk freely as they usually do. A significant correlation between the height and the stride length was found.

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean Height(in)</th>
<th>Stride Length(in)</th>
<th>Ratio(Stride/Height)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tall</td>
<td>72.2</td>
<td>63.98</td>
<td>0.886</td>
</tr>
<tr>
<td>Medium</td>
<td>69.1</td>
<td>61.50</td>
<td>0.890</td>
</tr>
<tr>
<td>Short</td>
<td>66.0</td>
<td>59.37</td>
<td>0.899</td>
</tr>
</tbody>
</table>

The above table from Murray et. al shows the stride length is linearly related with the height. In this experiment they observed that there were no significant differences in the major joint (hip, knee, ankle) angles among the groups, which supports the above similarity assumption.

The flexion angle of the hip of the group Short was slightly bigger than the other two groups. That phenomenon can be explained by the slightly increasing ratio values in the above table. Because people live in a community, there tends to be a regression effect in walking. The shorter group's stride length relative to their height tends to be longer than that of the longer group. If the ratio value in the experiment was maintained constant, then the hip flexion of the Short might be closer to the other groups.

There have been many trials to find the relative size of the segments in human body. Since such kind of information depends on the individuals, the results depended on the sampled subjects from which the statistics were computed. The sampling may differ among the research groups. Therefore in using rotoscopy data for human walking animation in particular, the model used in the animation is more likely to be different from the subject on which the measurement was performed, not only in the total size but also in the individual ratios of the corresponding segments. So the similarity assumption alone can not cover the variety of locomotion phenomenon for general anthropometry.

The walk data $W(\alpha S, \alpha sl)$ is simply derivable from $W(S, sl)$, based on the similarity assumption. Generally speaking, the Cartesian quantity is scaled by $\alpha$ and the angular quantity remains the same. However, if the scale is not the same between the corresponding segments, the similarity assumption can not be applied directly. Let the walk data $W(S, sl)$ of the subject $S$ be given at the step length $sl$. We want to derive the walk data of an arbitrary subject $S'$ and step length $\alpha sl$, where $S' \approx \alpha S$. As defined earlier, the total leg lengths $ll(S')$ and $ll(\alpha S)$ are the same, therefore the hip trajectories of $S'$ and $\alpha S$ will be similar.

We assume that the trajectory of the hip of $S'$ will be the same as $\alpha S$ except for the ankle height difference. That is, if $\Delta aN$ is the difference between the ankle heights of $S'$ and $\alpha S$ (positive if $S'$ is higher), and if the hip trajectory of $S$ is $H^1$

$$H^1 = \{(t_i, x_i, y_i) \mid i = 1, \ldots, n\}$$

then that of $S'$ is defined by

$$H^2 = \{(t_i, x'_i, y'_i) \mid i = 1, \ldots, n\}$$

where

$$x'_i = \alpha x_i$$

$$y'_i = \alpha y_i + \Delta aN.$$
over anthropometry. That is, for any subject $S$, and for any step length $sl$,

$$\mu_i(\alpha S, \alpha sl) = \alpha_i(\frac{\alpha sl}{\mu(\alpha S)} - sl^*) + \mu^* = \mu_i(S, sl).$$  \hfill (45)

Once the foot angle and the meta angle trajectories are available, ankle positions of both $S$ and $S'$ at $TOM$ and $HSM^+$ can be computed. Let the coordinates of the ankle just before the $TOM$ and right after the $HSM^+$ of the subject $S$ with the step length $sl$ be $(b_{1x}, b_{1y})$ and $(b_{2x}, b_{2y})$, respectively; likewise, those of the subject $S'$ with step length $\alpha sl$ be $(b_{3x}, b_{3y})$, and $(b_{4x}, b_{4y})$, respectively. As in the step length generalization, $b_{4x}$ is determined so that the resulting step length is $\alpha sl$.

If the ankle trajectory of $W(S, sl)$ is given by

$$A^1 = \{(t_i, x_i, y_i) \mid i = 1, \ldots, n_a\}$$  \hfill (46)

then that of $W(S', \alpha sl)$ is defined as a slight modification of the ankle trajectory in $W(\alpha S, \alpha sl)$,

$$A^2 = \{(t_i, x'_i, y'_i) \mid i = 1, \ldots, n_a\}$$  \hfill (47)

where

$$x'_i = b_{3x} + \frac{b_{4x} - b_{3x}}{b_{2x} - b_{1x}}(x_i - b_{1x})$$  \hfill (48)

and

$$y'_i = \alpha y_i + (1 - t'_i)d_1 + t'_id_2$$  \hfill (49)

with $d_1 = b_{3y} - \alpha y_1$, $d_2 = b_{4y} - \alpha y_n$, and $t'_i = \frac{t_i - TOM}{HSM^+ - TOM}$. The model discrepancy between $\alpha S$ and $S'$ is absorbed at the knee angle, and somewhere in the foot.

The whole process above will be denoted by $\psi^S_{S_i}$, which maps the walk data $W(S_1, sl)$ to $W(S_2, \frac{\|S_2\|}{\|S_1\|}sl)$. I.e.,

$$W(S_2, \frac{\|S_2\|}{\|S_1\|}sl) = \psi^S_{S_i}(W(S_1, sl))$$  \hfill (50)

**Lemma 3** For any subjects $S_1$, $S_2$, and for any step length $sl$,

$$\psi^S_{S_i}(W(S_1, sl)) \mid_{sl = \frac{\|S_2\|}{\|S_1\|}sl}$$  \hfill (51)

I.e., the step length is generalized correctly over anthropometry. \hfill \Box

**Theorem 3** For any subjects $S, S_1, and S_2$,

$$\psi^S_{S_i} \circ \psi^S_{S_1} = \psi^S_{S_2}$$  \hfill (52)

Therefore the anthropometry generalization is characteristic preserving. \hfill \Box
5 Combining the Two Types of Generalization

To have a full generalization, the two kinds of generalization, namely, the step length generalization and the anthropometry generalization, should be combined together. Let’s suppose the original walk is \((S_1, s_1)\), and the desired walk is \((S_2, s_2)\). We can apply the anthropometry generalization first

\[ W(S_2, \frac{S_2}{S_1} s_1) = \psi_{S_2}^{S_1}(W(S_1, s_1)) \]

and then apply the step length generalization to get the final result

\[ W(S_2, s_2) = \phi_{S_2}^{S_1} \left( W(S_2, \frac{S_2}{S_1} s_1) \right). \]

Another way around is to apply step length generalization first

\[ W(S_1, \frac{S_1}{S_2} s_2) = \phi_{S_1}^{S_2} (W(S_1, s_1)) \]

and then apply the anthropometry generalization,

\[ W(S_2, s_2) = \psi_{S_2}^{S_1} (W(S_1, \frac{S_1}{S_2} s_2)). \]

One obvious question here is which way is correct or better. It seems desirable that the order of the applications of the generalization does not affect the result. In fact, our generalization algorithm does have that property.

**Theorem 4** In applying the generalizations \(\phi\) and \(\psi\) defined in the previous sections, the order of the application does not affect the final result. I.e., for any walk data \(W_1 = W(S_1, s_1)\), and for any walk \((S_2, s_2)\),

\[ \phi_{S_2}^{S_1} (\psi_{S_2}^{S_1} (W_1)) = \psi_{S_1}^{S_2} (\phi_{S_1}^{S_2} (W_1)) \]

or simply

\[ \phi_{S_2}^{S_1} \circ \psi_{S_2}^{S_1} = \psi_{S_1}^{S_2} \circ \phi_{S_1}^{S_2} \]

\(\Box\)

We have shown (theorems 2 that and 3) both types of generalization, i.e., the step length generalization \(\phi\) and the anthropometry generalization \(\psi\) are characteristic preserving when they are applied *homogeneously*. One obvious question here is whether it is characteristic preserving even when they are mixed up together. In fact, our generalization scheme does have that property.

**Theorem 5** Let \(WC_1 = (S_1, s_1)\), \(WC_2 = (S_2, s_2)\), and \(WC_3 = (S_3, s_3)\) be three arbitrary walk conditions. Let

\[ \tau_{12} = \psi_{S_2}^{S_1} \circ \phi_{S_2}^{S_1} \]

\[ \tau_{23} = \psi_{S_2}^{S_3} \circ \phi_{S_2}^{S_3} \]

\[ \tau_{13} = \psi_{S_2}^{S_3} \circ \phi_{S_2}^{S_3} \]

be the combined generalizations that try to transform \(WC_1\) to \(WC_2\), \(WC_2\) to \(WC_3\), and \(WC_1\) to \(WC_3\), respectively. Then

\[ \tau_{23} \circ \tau_{12} = \tau_{13} \]

Therefore the combined generalization is characteristic preserving.
6 Conclusion

The generalization algorithm was implemented in Jack™ [13]. In the implementation, the arm swing was done by a kinematic function that depends on the leg movement. The accompanying animation was based on the measured data from [15]. The following table shows the comparison between the subject measured and the figure animated, in their step lengths and anthropometry. For all the differences between the subject and the figure, our animation was quite successful.

<table>
<thead>
<tr>
<th></th>
<th>Subject (Measured)</th>
<th>Figure (Animated)</th>
<th>Ratio (Figure/Subject)</th>
</tr>
</thead>
<tbody>
<tr>
<td>step length</td>
<td>70.69</td>
<td>variable</td>
<td>variable</td>
</tr>
<tr>
<td>ankle to heel</td>
<td>7.629</td>
<td>13.49</td>
<td>1.77</td>
</tr>
<tr>
<td>ankle to ball</td>
<td>12.20</td>
<td>17.55</td>
<td>1.44</td>
</tr>
<tr>
<td>heel to ball</td>
<td>15.89</td>
<td>15.19</td>
<td>0.96</td>
</tr>
<tr>
<td>ball to toe</td>
<td>7.095</td>
<td>7.200</td>
<td>1.01</td>
</tr>
<tr>
<td>ball angle</td>
<td>0.4842</td>
<td>0.8378</td>
<td>1.73</td>
</tr>
<tr>
<td>heel angle</td>
<td>0.8400</td>
<td>1.312</td>
<td>1.56</td>
</tr>
<tr>
<td>ankle angle</td>
<td>1.818</td>
<td>0.9916</td>
<td>0.55</td>
</tr>
<tr>
<td>heel to N</td>
<td>5.093</td>
<td>3.450</td>
<td>0.67</td>
</tr>
<tr>
<td>ankle to N</td>
<td>5.680</td>
<td>13.04</td>
<td>2.30</td>
</tr>
<tr>
<td>ball to N</td>
<td>10.80</td>
<td>11.74</td>
<td>1.09</td>
</tr>
<tr>
<td>shin</td>
<td>38.05</td>
<td>34.53</td>
<td>0.91</td>
</tr>
<tr>
<td>thigh</td>
<td>35.85</td>
<td>40.72</td>
<td>1.14</td>
</tr>
<tr>
<td>leg length</td>
<td>73.90</td>
<td>75.25</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Even though the definition of the preservation was aimed for the animation, we should note the difference of its meaning, in the mathematics space and in the animation space. If the desired step is too different from the originally measured one, even though the characteristic is well preserved in mathematical way, it has less meaning in generating that step based on the normal one. For example, if the subject 2s tries to imitate the walk of (S, sl) in stepping 0.1sl, the goals of imitation and achieving the given step length will be in total conflict. In this case, to get a better animation we need a measurement of S at a smaller step.

In this work, we showed a new approach in generalizing rotoscopy data for human walking animation, which promises good results by the nature of the method. From the measured data of one step of a particular subject, we can generate the steps of any step length and anthropometry, that resemble the original step. We can extend our system to simulate multiple walking styles by acquiring other measurements. Also the characteristic preservation was suggested as a new criteria for determining the quality of generalization.

7 Acknowledgments

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References


A Appendix: Proofs of Theorems and Lemmas

Proof of Lemma 1  Note that the origin of the coordinates is put on the point N (defined above) of the current stance foot. The x axis is toward the walking direction, and y axis is upwards. The derivation of the four formulas is trivial, using the fact that the foot angle at TOM, the meta angle at TOM, and the foot angle at HSM+ are given by $\mu_3(S, sl), \mu_4(S, sl),$ and $\mu_2(S, sl)),$ respectively. □

Proof of Lemma 2  In defining $\phi$, the ankle is placed at

$$b_4x = psl - (ah \cos \theta_h - ah \cos(\theta_h + \mu_2(S, psl)))$$  (63)

in x direction at the end of the step. Because the foot rotates around the heel until the FGM, the heel is fixed during that period. Therefore the x coordinate $a_x$ of the ankle during $[HSM+, FGM_{next}]$ can be given

$$a_x = psl - (ah \cos \theta_h - ah \cos(\theta_h + \theta_{foot}))$$  (64)

where $FGM_{next}$ is the FGM of the next step, and $\theta_{foot}$ is the foot angle during that interval. Because $\theta_{foot}$ becomes zero at the $FGM_{next}$, the equation 64 is reduced to $psl$. □

In the proofs following, whenever a comparison is done between two walk data sets, we will have six paragraphs (i) through (vi), each of them showing the comparison between the corresponding $H, F_1, F_2, F_3, M,$ and $A$'s of the sets.

Proof of Theorem 1  (i)  The multiplier on x component of LHS is $\frac{sl_{before} + P_1P_2sl}{sl_{before} + P_1sl}$, by the definition. Meanwhile, in the RHS, the multiplier on x component in $\phi_{p_1}$ is $\frac{sl_{before} + P_2P_1sl}{sl_{before} + psl}$. By lemma 1, the resulting step length of $\phi(W(S, sl))$ is actually $\rho_1sl$. Therefore the multiplier on x component in $\phi_{p_2}$ out of $\phi_{p_1}$ is $\frac{sl_{before} + P_2P_1sl}{sl_{before} + P_1sl}$. Therefore, the multiplier of the composite transform $\phi_{p_2} \circ \phi_{p_1}$ is

$$\frac{sl_{before} + P_2P_1sl}{sl_{before} + P_1sl} \frac{sl_{before} + P_1sl}{sl_{before} + sl} = \frac{sl_{before} + P_1P_2sl}{sl_{before} + sl}$$  (65)

Therefore LHS = RHS.

Let the height of the hip just before the current step be $h_{before}$. Let the computed height of the hip after the step $\rho_1sl$ and $\rho_1\rho_2sl$ be $h_{rho1sl}$ and $h_{rho1rho2sl}$. The y component $y_{iLHS}$ of the LHS is

$$y_{iLHS} = y_i + (1 - t_i)(h_{before} - y_1) + t_i(h_{rho1rho2sl} - y_n)$$  (66)

Meanwhile, by $\phi_{p_1} y_i$ is transformed to

$$y_{iTEMP} = y_i + (1 - t_i)(h_{before} - y_1) + t_i(h_{rho1sl} - y_n)$$  (67)

Note that $y_{iTEMP} = h_{before}$ and $y_{nTEMP} = h_{rho1sl}$. Again, by $\phi_{p_2}$, $y_{iTEMP}$ is transformed to

$$y_{iRHS} = y_{iTEMP} + (1 - t_i)(h_{before} - y_{iTEMP}) + t_i(h_{rho2rho1sl} - y_{nTEMP})$$  (68)
which can be rewritten as

\[
y^\text{RHS} = y_i + (1 - t_i)(h_{\text{before}} - y_i) + t_i(h_{p1,sl} - y_n) \\
+ (1 - t_i)(h_{\text{before}} - y_1^{\text{TEMP}}) + t_i(h_{p2,sl} - y_n^{\text{TEMP}})
\]

\[
y_i + (1 - t_i)(h_{\text{before}} - y_i) + t_i(h_{p1,sl} - y_n + h_{p2,sl} - y_n^{\text{TEMP}})
\]

\[
y_i + (1 - t_i)(h_{\text{before}} - y_i) + t_i(h_{p2,sl} - y_n) = y^\text{LHS}
\]

(ii) The foot angle component of \( F_1 \) in LHS is

\[
f^{\text{LHS}}_{1i} = f_{1i} \frac{\mu_2(S, s_{\text{before}})}{f_{11}}
\]  

By \( \phi_{p1} \) in RHS,

\[
f^{\text{TEMP}}_{1i} = f_{1i} \frac{\mu_2(S, s_{\text{before}})}{f_{11}}
\]

Then by \( \phi_{p2} \),

\[
f^{\text{RHS}}_{1i} = f^{\text{TEMP}}_{1i} \frac{\mu_2(S, s_{\text{before}})}{f^{\text{TEMP}}_{11}}
\]

\[
= f_{1i} \frac{\mu_2(S, s_{\text{before}})}{f_{11}} \frac{\mu_2(S, s_{\text{before}})}{f^{\text{TEMP}}_{11}}
\]

By noting that

\[
f^{\text{TEMP}}_{11} = \mu_2(S, s_{\text{before}})
\]

\( f^{\text{RHS}}_{1i} \) is reduced to

\[
f^{\text{RHS}}_{1i} = f_{1i} \frac{\mu_2(S, s_{\text{before}})}{f_{11}}
\]

(iii) The foot angle component of \( F_2 \) in LHS is

\[
f^{\text{LHS}}_{2i} = f_{2i} \frac{\mu_3(S, \rho_{p2,sl})}{f_{2n2}}
\]

By \( \phi_{p1} \) in RHS,

\[
f^{\text{TEMP}}_{2i} = f_{2i} \frac{\mu_3(S, \rho_{p1,sl})}{f_{2n2}}
\]

Then by \( \phi_{p2} \),

\[
f^{\text{RHS}}_{2i} = f^{\text{TEMP}}_{2i} \frac{\mu_3(S, \rho_{p2,sl})}{f^{\text{TEMP}}_{2n2}}
\]

\[
= f_{2i} \frac{\mu_3(S, \rho_{p1,sl}) \mu_3(S, \rho_{p2,sl})}{f_{2n2}}
\]

Noting that

\[
f^{\text{TEMP}}_{2n2} = \mu_3(S, \rho_{p1,sl})
\]
$f^\text{RHS}_{2i}$ is reduced to

$$f^\text{RHS}_{2i} = f_{2i} \frac{\mu_3(S, \rho_{2p1sl})}{f_{2n_2}}$$

(iv)

$$f^\text{LHS}_{3i} = f_{3i} + (1 - t'_{3i})d^\text{LHS}_{1} + t'_{3i}d^\text{LHS}_{2}$$

where

$$d^\text{LHS}_{1} = \mu_3(S, \rho_{1p2sl}) - f_{31}$$

$$d^\text{LHS}_{2} = \mu_2(S, \rho_{1p2sl}) - f_{3n_3}$$

Meanwhile, in the right hand side,

$$f^\text{TEMP}_{3i} = f_{3i} + (1 - t'_{3i})d^\text{TEMP}_{1} + t'_{3i}d^\text{TEMP}_{2}$$

where

$$d^\text{TEMP}_{1} = \mu_3(S, \rho_{1psl}) - f_{31}$$

$$d^\text{TEMP}_{2} = \mu_2(S, \rho_{1psl}) - f_{3n_3}$$

and finally,

$$f^\text{RHS}_{3i} = f^\text{TEMP}_{3i} + (1 - t'_{3i})d^\text{RHS}_{1} + t'_{3i}d^\text{RHS}_{2}$$

where

$$d^\text{RHS}_{1} = \mu_3(S, \rho_{2p1sl}) - f^\text{TEMP}_{31}$$

$$d^\text{RHS}_{2} = \mu_2(S, \rho_{2p1sl}) - f^\text{TEMP}_{3n_3}$$

Therefore

$$f^\text{RHS}_{3i} = f_{3i} + (1 - t'_{3i})d^\text{TEMP}_{1} + t'_{3i}d^\text{TEMP}_{2} + (1 - t'_{3i})d^\text{RHS}_{1} + t'_{3i}d^\text{RHS}_{2}$$

$$= f_{3i} + (1 - t'_{3i})(d^\text{TEMP}_{1} + d^\text{RHS}_{1}) + t'_{3i}(d^\text{TEMP}_{2} + d^\text{RHS}_{2})$$

Noting the fact that

$$f^\text{TEMP}_{3i} = f_{3i} + d^\text{TEMP}_{1}$$

we have

$$d^\text{TEMP}_{1} + d^\text{RHS}_{1} = d^\text{LHS}_{1}$$

$$d^\text{TEMP}_{1} = d^\text{TEMP}_{1} + \mu_3(S, \rho_{2p1sl}) - f^\text{TEMP}_{31}$$

$$= d^\text{TEMP}_{1} + \mu_3(S, \rho_{2p1sl}) - (f_{31} + d^\text{TEMP}_{1})$$

$$= \mu_3(S, \rho_{2p1sl}) - f_{31}$$

$$= d^\text{LHS}_{1}$$

Similarly, we have

$$d^\text{TEMP}_{2} + d^\text{RHS}_{2} = d^\text{LHS}_{2}$$

Therefore $f^\text{LHS}_{3i} = f^\text{RHS}_{3i}$ holds.

(v) Proving this part is very similar to the part (iii).
(vi) By the definition
\[ x_i^{LHS} = b_3^{LHS} + \frac{b_4^{LHS} - b_5^{LHS}}{b_2 - b_1}(x_i - b_1) \]  
(103)
where
\[ b_3^{LHS} = bN + bt - bt \cos \mu_4(S, \rho_1 p_2 s_1) - ab \cos \mu_3(S, \rho_1 p_2 s_1) \]  
(104)
\[ b_4^{LHS} = \rho_1 p_2 s_1 - (ah \cos \theta_h - ah \cos(\theta_h + \mu_2(S, \rho_1 p_2 s_1))) \]  
(105)

Meanwhile in the RHS,
\[ x_i^{TEMP} = b_3^{TEMP} + \frac{b_4^{TEMP} - b_5^{TEMP}}{b_2 - b_1}(x_i - b_1) \]  
(106)
where
\[ b_3^{TEMP} = bN + bt - bt \cos \mu_4(S, \rho_1 s_1) - ab \cos \mu_3(S, \rho_1 s_1) \]  
(107)
\[ b_4^{TEMP} = \rho_1 s_1 - (ah \cos \theta_h - ah \cos(\theta_h + \mu_2(S, \rho_1 s_1))) \]  
(108)
and finally,
\[ x_i^{RHS} = b_3^{RHS} + \frac{b_4^{RHS} - b_5^{RHS}}{b_2^{TEMP} - b_1^{TEMP}}(x_i^{TEMP} - b_1^{TEMP}) \]  
(109)
where
\[ b_3^{RHS} = bN + bt - bt \cos \mu_4(S, \rho_2 p_1 s_1) - ab \cos \mu_3(S, \rho_2 p_1 s_1) \]  
(110)
\[ b_4^{RHS} = \rho_2 p_1 s_1 - (ah \cos \theta_h - ah \cos(\theta_h + \mu_2(S, \rho_2 p_1 s_1))) \]  
(111)

By substituting the equation 106 for \( x_i^{TEMP} \) in the equation 109
\[ x_i^{RHS} = b_3^{RHS} + \frac{b_4^{RHS} - b_5^{RHS}}{b_2^{TEMP} - b_1^{TEMP}}(b_3^{TEMP} + \frac{b_4^{TEMP} - b_5^{TEMP}}{b_2 - b_1}(x_i - b_1)) \]  
(112)
\[ -b_3^{TEMP} \]  
(113)
\[ = b_3^{RHS} + \frac{b_4^{RHS} - b_5^{RHS}}{b_2 - b_1}(x_i - b_1) \]  
(114)

As demonstrated above in the equations 104, 105, 110, and 111,
\[ b_3^{RHS} = b_3^{LHS} \]  
(115)
\[ b_4^{RHS} = b_4^{LHS} \]  
(116)

and therefore finally we have
\[ x_i^{LHS} = x_i^{RHS} \]  
(118)

The y component of the LHS is defined as
\[ y_i^{LHS} = y_i + (1 - t_i) d_1^{LHS} + t_i d_2^{LHS} \]  
(119)
Meanwhile, in the RHS,

\[ y_i^{\text{TEMP}} = y_i + (1 - t_i')d_1^{\text{TEMP}} + t_i'd_2^{\text{TEMP}} \]  

with

\[ d_1^{\text{TEMP}} = b_3^{\text{TEMP}} - y_1 \]  
\[ d_2^{\text{TEMP}} = b_4^{\text{TEMP}} - y_n \]  

and finally,

\[ y_i^{\text{RHS}} = y_i^{\text{TEMP}} + (1 - t_i')d_1^{\text{RHS}} + t_i'd_2^{\text{RHS}} \]  

with

\[ d_1^{\text{RHS}} = b_3^{\text{RHS}} - y_1^{\text{TEMP}} \]  
\[ d_2^{\text{RHS}} = b_4^{\text{RHS}} - y_n^{\text{TEMP}} \]  

By substituting the equation 122 for \( y_i^{\text{TEMP}} \) in the equation 125,

\[
y_i^{\text{RHS}} = y_i + (1 - t_i')d_1^{\text{TEMP}} + t_i'd_2^{\text{TEMP}} + (1 - t_i')d_1^{\text{RHS}} + t_i'd_2^{\text{RHS}}
\]

Here,

\[ d_1^{\text{TEMP}} + d_1^{\text{RHS}} = d_1^{\text{TEMP}} + b_3^{\text{RHS}} - y_1^{\text{TEMP}} \]
\[ = d_1^{\text{TEMP}} + b_3^{\text{RHS}} - (y_1 + d_1^{\text{TEMP}}) \]
\[ = b_3^{\text{RHS}} - y_1 \]

But

\[ b_3^{\text{RHS}} = bt \sin \mu_4(S, \rho_2 \rho_1 s) + ab \sin \mu_3(S, \rho_2 \rho_1 s) \]  
\[ = b_3^{\text{LHS}} \]  
\[ = b_3^{\text{LHS}} \]

Therefore

\[ d_1^{\text{TEMP}} + d_1^{\text{RHS}} = d_1^{\text{LHS}} \]  
\[ d_2^{\text{TEMP}} + d_2^{\text{RHS}} = d_2^{\text{LHS}} \]

In a similar way we can prove

\[ y_i^{\text{LHS}} = y_i^{\text{RHS}} \]
Proof of Theorem 2  Let $W_1$ be a walk data with step length $s_{l_1}$. Let $s_{l_2}$ and $s_{l_3}$ be two arbitrary step lengths. The transformation from $s_{l_1}$ to $s_{l_2}$ is $T_{|(s_{l_1},s_{l_2})} = \phi_{\frac{s_{l_2}}{s_{l_1}}}$ . The transformation from $s_{l_2}$ to $s_{l_3}$ is $T_{|(s_{l_2},s_{l_3})} = \phi_{\frac{s_{l_3}}{s_{l_2}}}$. The direct transformation from $s_{l_1}$ to $s_{l_3}$ is $T_{|(s_{l_1},s_{l_3})} = \phi_{\frac{s_{l_3}}{s_{l_1}}}$. By the theorem 1,

\[
T_{|(s_{l_2},s_{l_3})} \circ T_{|(s_{l_1},s_{l_2})} = \phi_{\frac{s_{l_3}}{s_{l_2}}} \circ \phi_{\frac{s_{l_2}}{s_{l_1}}} = \phi_{\frac{s_{l_3}}{s_{l_1}}} = T_{|(s_{l_1},s_{l_3})}
\]

\(\Box\)

Proof of Lemma 3  This is clear by the preceding definitions. \(\Box\)

Proof of Theorem 3  Let $\alpha_1 = S'_{S}$ and $\alpha_2 = S'_{S_1}$. Then $S'_{S} = \alpha_1 \alpha_2$.

(i) Showing the equality of the $x$ coordinates: By $\psi_{S_1}^{S}$, $x_i$ goes to $\alpha_1 x_i$, which in turn, by $\psi_{S_2}^{S}$, goes to $\alpha_2 \alpha_1 x_i$. But in the RHS, $x_i$ directly goes to $\alpha_1 \alpha_2 x_i$ by $\psi_{S}^{S}$. Showing the equality of the $y$ coordinates: Let the ankle heights of the subjects $S$, $S_1$, and $S_2$ be $aN$, $aN_1$, and $aN_2$, respectively. Let

\[
\Delta N_1 = aN_1 - \alpha_1 aN \\
\Delta N_2 = aN_2 - \alpha_2 aN_1
\]

By $\psi_{S_1}^{S}$, $y_i$ goes to

\[
y_i^{\text{TEMP}} = \alpha_1 y_i + \Delta N_1
\]

which again by $\psi_{S_1}^{S_2}$ goes to

\[
y_i^{\text{LHS}} = \alpha_2 y_i^{\text{TEMP}} + \Delta N_2 = \alpha_2 (\alpha_1 y_i + \Delta N_1) + \Delta N_2 = \alpha_2 \alpha_1 y_i + \Delta N_1 + \Delta N_2
\]

But in the RHS, $y_i$ directly goes to

\[
y_i^{\text{RHS}} = \alpha_1 \alpha_2 y_i + \Delta N_{12}
\]

by $\psi_{S}^{S_2}$, where

\[
\Delta N_{12} = aN_2 - \alpha_1 \alpha_2 aN
\]

But a simple manipulation of the equations 144 and 145 gives us

\[
\alpha_2 \Delta N_1 + \Delta N_2 = \Delta N_{12}
\]

and thus,

\[
y_i^{\text{LHS}} = y_i^{\text{RHS}}
\]
(ii)-(v) Because the foot angle and the meta angle trajectories remain the same during the body condition generalizations, the resulting angular trajectories in both sides will be identical to the original ones in $W(S, s_l)$.

(vi) Let the ankle positions of $S$ just before the TOM and right after the $HSM^+$ with the step length $s_l$ be $(b_{1x}, b_{1y})$ and $(b_{2x}, b_{2y})$, respectively. Likewise, let those of the subject $S_1$ after applying $\psi_{S_1}^S$ be $(b_{3x}, b_{3y})$, and $(b_{4x}, b_{4y})$, respectively. Let the ankle positions of the subject $S_2$ by applying $\psi_{S_1}^S$ to the result of $\psi_{S_1}^S$ be $(b_{5x}, b_{5y})$ and $(b_{6x}, b_{6y})$, respectively. Let the ankle positions of the subject $S_2$ by the direct application of $\psi_{S_2}^S$ be $(b_{7x}, b_{7y})$ be $(b_{8x}, b_{8y})$, and $(b_{9x}, b_{9y})$, respectively. Let the ankle positions of the subject $S_2$ by the direct application of $\psi_{S_2}^S$ be $(b_{10x}, b_{10y})$, and $(b_{11x}, b_{11y})$, respectively. Let the ankle positions of the subject $S_2$ by the direct application of $\psi_{S_2}^S$ be $(b_{12x}, b_{12y})$, and $(b_{13x}, b_{13y})$, respectively.

By the lemma 3, the final step lengths of both the LHS and RHS are identical. And the foot angles and the meta angles of the LHS and RHS at the TOM and the $HSM^+$ are identical. Moreover, eventually at the end of both the applications, the subject $S_2$ is considered. Therefore

$$
(b_{3x}, b_{3y}) = (b_{4x}, b_{4y})
$$

This can be also checked in the similar way as in the step length generalization. By $\psi_{S_1}^S$, $x_i$ goes to

$$
x_i^{\text{TEMP}} = b_{3x}^{\text{TEM}} + \frac{b_{4x}^{\text{TEMP}} - b_{3x}^{\text{TEM}}}{b_{2x} - b_{1x}} (x_i - b_{1x})
$$

which again by $\psi_{S_2}^S$ goes to

$$
x_i^{\text{LHS}} = b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{4x}^{\text{TEM}} - b_{3x}^{\text{TEM}}} (x_i^{\text{TEMP}} - b_{3x}^{\text{TEM}})
$$

$$
= b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{4x}^{\text{TEM}} - b_{3x}^{\text{TEM}}} (b_{3x}^{\text{TEM}} + \frac{b_{4x}^{\text{TEM}} - b_{3x}^{\text{TEM}}}{b_{2x} - b_{1x}} (x_i - b_{1x}))
$$

$$
= b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{2x} - b_{1x}} (x_i - b_{1x})
$$

In the RHS, $\psi_{S_2}^S$ sends $x_i$ directly to

$$
x_i^{\text{RHS}} = b_{3x}^{\text{RHS}} + \frac{b_{4x}^{\text{RHS}} - b_{3x}^{\text{RHS}}}{b_{2x} - b_{1x}} (x_i - b_{1x})
$$

By the results 154 and 155, we conclude that

$$
x_i^{\text{LHS}} = x_i^{\text{RHS}}
$$

Proof of Theorem 4 We use the superscript LTEMP to denote the result right after the application of $\psi_{S_1}^S$ in the left hand side, LHS after the application of the whole thing of the left hand side $\phi_{S_1} \circ \psi_{S_1}^S$, RTE M and RHS are used in the similar way.
(i) Let $s_{\text{before}}$ be the step length of the previous step of $S_1$. Then that of $S_2$ will be $\frac{S_2}{S_1} s_{\text{before}}$.

In the left hand side,

$$x_i^{\text{LTEMP}} = \frac{S_2}{S_1} x_i$$  \hspace{1cm} (163)  \\
$$x_i^{\text{LHS}} = \frac{\frac{S_2}{S_1} s_{\text{before}} + s_2}{\frac{S_2}{S_1} s_{\text{before}} + \frac{S_2}{S_1} s_1} s_{\text{before}} x_i$$ \hspace{1cm} (164)

In the right hand side,

$$x_i^{\text{RTEMP}} = \frac{s_{\text{before}} + \frac{S_2}{S_1} s_2}{s_{\text{before}} + S_1} x_i$$ \hspace{1cm} (165)  \\
$$x_i^{\text{RHS}} = \frac{S_2}{S_1} \frac{s_{\text{before}} + \frac{S_2}{S_1} s_2}{s_{\text{before}} + S_1} x_i$$ \hspace{1cm} (166)

But a simple manipulation of the equation 164 and 166 gives

$$x_i^{\text{LHS}} = x_i^{\text{RHS}}$$ \hspace{1cm} (167)

Now, let’s look at the $y$ component of the hip trajectory.

$$y_i^{\text{LTEMP}} = \frac{S_2}{S_1} y_i + \Delta a N$$ \hspace{1cm} (168) \\
$$y_i^{\text{LHS}} = y_i^{\text{LTEMP}} + (1 - t_i)(y_i^{\text{LHS}} - y_i^{\text{LTEMP}}) + t_i(y_i^{\text{LHS}} - y_n^{\text{LTEMP}})$$ \hspace{1cm} (169)  \\
$$= \frac{S_2}{S_1} y_i + \Delta a N + (1 - t_i)(y_i^{\text{LHS}} - y_i^{\text{LTEMP}}) + t_i(y_i^{\text{LHS}} - y_n^{\text{LTEMP}})$$ \hspace{1cm} (170)

$$y_i^{\text{RTEMP}} = y_i + (1 - t_i)(y_i^{\text{RTEMP}} - y_1) + t_i(y_i^{\text{RTEMP}} - y_n)$$ \hspace{1cm} (171)  \\
$$y_i^{\text{RHS}} = \frac{S_2}{S_1} y_i^{\text{RTEMP}} + \Delta a N$$ \hspace{1cm} (172)  \\
$$= \frac{S_2}{S_1} y_i + (1 - t_i)(\frac{S_2}{S_1} y_i^{\text{RTEMP}} - \frac{S_2}{S_1} y_1) + t_i(\frac{S_2}{S_1} y_n^{\text{RTEMP}} - \frac{S_2}{S_1} y_n)$$ \hspace{1cm} (173)  \\
$$+ \Delta a N$$ \hspace{1cm} (174)

We need to check the equalities of

$$y_i^{\text{LHS}} - y_i^{\text{LTEMP}} = \frac{S_2}{S_1} y_i^{\text{RTEMP}} - \frac{S_2}{S_1} y_i$$ \hspace{1cm} (175)  \\
$$y_n^{\text{LHS}} - y_n^{\text{LTEMP}} = \frac{S_2}{S_1} y_n^{\text{RTEMP}} - \frac{S_2}{S_1} y_n$$ \hspace{1cm} (176)

In the equation 168,

$$y_i^{\text{LTEMP}} = \frac{S_2}{S_1} y_1 + \Delta a N$$ \hspace{1cm} (177)

Note that $y_i^{\text{RTEMP}}$ is the height of the hip of $S_1$ just before the current step of step length $\frac{S_2}{S_1} s_2$, and $y_i^{\text{LHS}}$ is the height of the hip of $S_2$ just before the current step of step length $s_{\text{before}}$. Therefore they are related by the equation 44 as follows.

$$y_i^{\text{LHS}} = \frac{S_2}{S_1} y_i^{\text{RTEMP}} + \Delta a N$$ \hspace{1cm} (178)
Replacing the results 177 and 178 into the equation 175 proves the equality. In a similar way we can show the equation 176 holds.

(ii)

\[
\begin{align*}
\left[ f_{L_{\text{TEMP}}} \right]_{1i} &= f_{1i} \\
\left[ f_{L_{\text{H}}} \right]_{1i} &= f_{1i} \frac{\mu_2(S_2, S_2^{s_{\text{sl before}}})}{f_{11}} \\
&= f_{1i} \frac{\mu_2(S_1, S_1^{s_{\text{sl before}}})}{f_{11}}
\end{align*}
\]

because

\[
\begin{align*}
\mu_2(S_2, S_2^{s_{\text{sl before}}}) &= \alpha_2 \left( \frac{S_2^{s_{\text{sl before}}}}{ll(S_2)} - s_{l^*} \right) + \mu^*_2 \\
&= \alpha_2 \left( \frac{S_1^{s_{\text{sl before}}}}{ll(S_1)} - s_{l^*} \right) + \mu^*_2 \\
&= \alpha_2 \left( \frac{S_1^{s_{\text{sl before}}}}{ll(S_1)} - s_{l^*} \right) + \mu^*_2 \\
&= \mu_2(S_1, S_1^{s_{\text{sl before}}})
\end{align*}
\]

Meanwhile,

\[
\begin{align*}
\left[ f_{R_{\text{H}}} \right]_{1i} &= \left[ f_{R_{\text{TEMP}}} \right]_{1i} = \left[ f_{L_{\text{H}}} \right]_{1i} \\
&= \left[ f_{L_{\text{H}}} \right]_{1i}
\end{align*}
\]

(iii)-(v) Noting the fact that

\[
\mu_i(S_2, s_{l_2}) = \mu_i(S_1, S_1^{s_{\text{sl before}}}), \quad i = 1, 2, 3, 4
\]

we can show the equalities between the LHS and the RHS in a similar way as in (ii).

(vi) Let the ankle positions of $S_1$ just before the TOM and right after the $HSM^+$ with the step length $s_{l}$ be $(b_{1x}, b_{1y})$ and $(b_{2x}, b_{2y})$, respectively. Likewise, let those of the subject $S_2$ after applying $\psi_{S_1}^{S_2}$ be $(b_{3x}^{L_{\text{TEMP}}}, b_{3y}^{L_{\text{TEMP}}})$, and $(b_{4x}^{L_{\text{TEMP}}}, b_{4y}^{L_{\text{TEMP}}})$, respectively. Let the ankle positions of the subject $S_2$ by applying $\phi_{S_1}^{S_2}$ to the result of $\psi_{S_1}^{S_2}$ be $(b_{3x}^{R_{\text{TEMP}}}, b_{3y}^{R_{\text{TEMP}}})$ and $(b_{4x}^{R_{\text{TEMP}}}, b_{4y}^{R_{\text{TEMP}}})$, respectively. Let the ankle positions of the subject $S_2$ after the application of $\phi_{S_1}^{S_2}$ to the result of $\psi_{S_1}^{S_2}$ be $(b_{3x}^{R_{\text{H}}} , b_{3y}^{R_{\text{H}}})$ and $(b_{4x}^{R_{\text{H}}}, b_{4y}^{R_{\text{H}}})$, respectively.

By the lemmas 2 and 3, the final step lengths of both the LHS and RHS are the same. And the corresponding foot angles and the meta angles of the LHS and RHS are identical as shown in (ii)-(v) above. Moreover, eventually at the end of both side the applications, the subject $S_2$ is considered. Therefore

\[
\begin{align*}
(b_{3x}^{L_{\text{H}}} , b_{3y}^{L_{\text{H}}}) &= (b_{3x}^{R_{\text{H}}} , b_{3y}^{R_{\text{H}}}) \\
(b_{4x}^{L_{\text{H}}} , b_{4y}^{L_{\text{H}}}) &= (b_{4x}^{R_{\text{H}}} , b_{4y}^{R_{\text{H}}})
\end{align*}
\]
In the LHS,

\[
x_i^{\text{TEMP}} = b_{3x}^{\text{TEMP}} + \frac{b_{4x}^{\text{TEMP}} - b_{3x}^{\text{TEMP}}}{b_{2x} - b_{1x}}(x_i - b_{1x})
\]  

\[
x_i^{\text{LHS}} = b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{4x}^{\text{LTEMPP}} - b_{3x}^{\text{LTEMPP}}}(x_i^{\text{LTEMPP}} - b_{3x}^{\text{LTEMPP}})
\]  

\[
= b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{4x}^{\text{LTEMPP}} - b_{3x}^{\text{LTEMPP}}}(b_{3x}^{\text{LTEMPP}} + \frac{b_{4x}^{\text{LTEMPP}} - b_{3x}^{\text{LTEMPP}}}{b_{2x} - b_{1x}}(x_i - b_{1x}))
\]

\[
= b_{3x}^{\text{LHS}} + \frac{b_{4x}^{\text{LHS}} - b_{3x}^{\text{LHS}}}{b_{2x} - b_{1x}}(x_i - b_{1x})
\]

Similarly in the RHS,

\[
x_i^{\text{RHS}} = b_{3x}^{\text{RHS}} + \frac{b_{4x}^{\text{RHS}} - b_{3x}^{\text{RHS}}}{b_{2x} - b_{1x}}(x_i - b_{1x})
\]

Using the facts 189 and 190, we can deduce

\[
x_i^{\text{LHS}} = x_i^{\text{RHS}}
\]

As for the y component,

\[
y_i^{\text{LTEMP}} = \frac{S_2}{S_1}y_i + (1 - t'_i)d_1^{\text{TEMP}} + t'_i d_2^{\text{TEMP}}
\]  

\[
y_i^{\text{LHS}} = y_i^{\text{LTEMPP}} + (1 - t'_i)d_1^{\text{LHS}} + t'_i d_2^{\text{LHS}}
\]  

\[
= \frac{S_2}{S_1}y_i + (1 - t'_i)(a_1^{\text{LTEMPP}} + d_1^{\text{LHS}}) + t'_i(d_1^{\text{LTEMPP}} + d_2^{\text{LHS}})
\]

\[
y_i^{\text{RTEMP}} = y_i + (1 - t'_i)d_1^{\text{RTEMP}} + t'_i d_2^{\text{RTEMP}}
\]  

\[
y_i^{\text{RHS}} = \frac{S_2}{S_1}y_i^{\text{RTEMPP}} + (1 - t'_i)d_1^{\text{RHS}} + t'_i d_2^{\text{RHS}}
\]  

\[
= \frac{S_2}{S_1}y_i + (1 - t'_i)(\frac{S_2}{S_1}d_1^{\text{RTEMPP}} + a_1^{\text{RHS}}) + t'_i(\frac{S_2}{S_1}d_2^{\text{RTEMPP}} + d_2^{\text{RHS}})
\]

But

\[
d_1^{\text{LHS}} = b_{3y}^{\text{LHS}} - y_i^{\text{LTEMPP}}
\]  

\[
= b_{3y}^{\text{LHS}} - \frac{S_2}{S_1}y_i - d_1^{\text{LTEMPP}}
\]

Therefore

\[
d_1^{\text{LTEMPP}} + a_1^{\text{LHS}} = b_{3y}^{\text{LHS}} - \frac{S_2}{S_1}y_i
\]

Similarly

\[
d_1^{\text{RHS}} = b_{3y}^{\text{RHS}} - \frac{S_2}{S_1}y_i^{\text{RTEMPP}}
\]

\[
= b_{3y}^{\text{RHS}} - \frac{S_2}{S_1}(y_i + d_1^{\text{RTEMPP}})
\]
Therefore
\[
\frac{S_2}{S_1} \Delta_{\text{TEMP}} + d_1^{\text{RHS}} = b_{2y}^{\text{RHS}} - \frac{S_2}{S_1} y_1
\]  
(210)

Again by the fact 189, the right hand sides of the equations 207 and 210 are the same, and we have
\[
d_1^{\text{LTEMP}} + d_1^{\text{LHS}} = \frac{S_2}{S_1} d_1^{\text{RTEMP}} + d_1^{\text{RHS}}
\]  
(211)

Similarly we can derive the equality of
\[
d_2^{\text{LTEMP}} + d_2^{\text{LHS}} = \frac{S_2}{S_1} d_2^{\text{RTEMP}} + d_2^{\text{RHS}}
\]  
(212)

Putting these results in the equations 201 and comparing it with the equation 204, we can conclude that
\[
y_i^{\text{LHS}} = y_i^{\text{RHS}}
\]  
(213)

□

Proof of Theorem 5

\[
\tau_{23} \circ \tau_{12} = \psi_S \phi_1 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(214)

\[\text{thm4} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(215)

\[\text{thm1} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(216)

\[\text{thm4} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(217)

\[\text{thm3} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(218)

\[\text{thm3} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(219)

\[\text{thm3} \]
\[
= \psi_S \phi_2 \psi_{S_1} \phi_{S_1} \psi_{S_1} \phi_{S_1}
\]  
(220)

□