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Critical Behavior of Random Resistor Networks Near the Percolation Threshold

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Critical Behavior of Random Resistor Networks Near the Percolation Threshold

Abstract
We use low-density series expansions to calculate critical exponents for the behavior of random resistor networks near the percolation threshold as a function of the spatial dimension $d$. By using scaling relations, we obtain values of the conductivity exponent $\mu$. For $d=2$ we find $\mu=1.43\pm0.02$, and for $d=3$, $\mu=1.95\pm0.03$, in excellent agreement with the experimental result of Abeles et al. Our results for high dimensionality agree well with the results of $\varepsilon$-expansion calculations.

Disciplines
Physics
Critical behavior of random resistor networks near the percolation threshold

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We use low-density series expansions to calculate critical exponents for the behavior of random resistor networks near the percolation threshold as a function of the spatial dimension \(d\). By using scaling relations, we obtain values of the conductivity exponent \(\mu\). For \(d = 2\) we find \(\mu = 1.43 \pm 0.02\), and for \(d = 3\), \(\mu = 1.95 \pm 0.03\), in excellent agreement with the experimental result of Abeles et al. Our results for high dimensionality agree well with the results of \(\epsilon\)-expansion calculations.

I. INTRODUCTION

In this paper, we discuss some new ideas concerning the properties of random resistor networks near the percolation threshold.\(^1\) We continue here the work of an earlier paper.\(^2\) The model we treat is that of an electrical network on a \(d\)-dimensional hypercubic lattice of \(N\) sites with real conductances \(\sigma_{ij}\) connecting nearest-neighbor pairs of lattice sites \(i\) and \(j\). Each \(\sigma_{ij}\) is an independent random variable assuming the values 0 and 1 with respective probabilities \(1 - p\) and \(p\). (Thus we will be considering the bond problem.) The macroscopic conductivity \(\Sigma\) is then defined to be the configurational average of \(\sigma N^{d-1}/V\), where \(\sigma = I/V\) and \(I\) is the dc current which flows when the potential difference \(V\) is applied between two opposite \((d - 1)\)-dimensional faces of the hypercube. We may define clusters as being groups of sites which are connected with respect to the unit (i.e., \(\sigma_{ij} = 1\)) conductances.

The statistics of cluster size and the associated pair connectedness correlation length \(\xi(p)\) were shown by Kasteleyn and Fortuin\(^3\) to be related to the thermodynamics of the \(s\)-state Potts model\(^4\) in the limit \(s = 1\), if the identification \(p = 1 - \exp(-J/kT)\) is made, where \(J\) is the nearest-neighbor interaction energy in the Potts model. The \(s\)-state Potts Hamiltonian can be written

\[ \mathcal{H} = -J N \sum_{ij} (\delta_{n_i n_j} - 1) - H s \sum_i (\delta_{n_i 1} - 1), \]

where each of the Potts variables \(n_i\) can take on any positive integer value less than or equal to \(s\). Note that if \(s = 2\), Eq. (1) becomes the Ising Hamiltonian. The free energy per site is given by the standard prescription

\[ F = -(kT/N) \ln[ \text{Tr}[e^{-\mathcal{H}/kT}] ] \]

If we make the identification for \(p\) given above, then it turns out that the fraction of sites in the infinite cluster \(P(p)\) is given by

\[ P(p) = \left. \frac{1}{s - 1} \frac{\partial F}{\partial H} \right|_{s = 1, H = 0}, \]

and the mean-square cluster size \(S(p)\) by

\[ S(p) = \left. \frac{1}{s - 1} \frac{\partial^2 F}{\partial H^2} \right|_{s = 1, H = 0}. \]

From Eqs. (3) and (4), we see that \(P(p)\) and \(S(p)\) are analogous to the magnetization and the magnetic susceptibility, respectively, of a ferromagnet.

This relation indicates that the usual scaling-exponent description for critical points can be applied to the percolation threshold \(p_c\), and that the various scaling relations and universality predictions can be expected to hold as well. Since the order parameter \(P(p)\) is a probability \(P(p) \geq 0\). Thus for \(d > d_c = 6\), the exponents for cluster statistics near \(p_c\) are those of the constrained mean-field theory (MFT) for a positive order parameter with a positive cubic term in the free energy\(^5\): \(\alpha = -1\), \(\beta = 1\), \(\gamma = 1\), and \(\nu = \frac{1}{2}\).

In view of scaling arguments which relate the diluted resistor network and percolation problems, de Gennes\(^6\) has suggested that \(d_c = 6\) for the resistor network also. Here we present numerical evidence which confirms that this suggestion is correct. We also discuss a new scaling relation.

II. SERIES FOR THE DILUTED RESISTOR NETWORK

It is possible to determine \(d_c\) for the ferromagnetic\(^7\) and spin-glass\(^8\) Ising models by analyzing the high-temperature series expansions for the order parameter susceptibilities, as a function of the dimensionality \(d\). This approach has been applied to the site percolation problem by Gaunt et al.\(^9\) These studies show that an
analysis of the series based on the assumption of a simple power law scaling form will produce an estimate for the exponent which is too large near \( d_x \), due to the confluent corrections.

We have carried out the same program for the diluted resistor network by analyzing low-concentration expansions for analogous susceptibilities. In order to do this, we must identify appropriate order parameters. For the undiluted \((p = 1)\) network, Kasteleyn and Fortuin\(^3\) have shown that the correlation function for the \( s = 0 \) state Potts model yields the resistance \( R_{ij} \) between lattice points \( i \) and \( j \). This correlation function can also be obtained by averaging over the Gaussian density matrix,

\[
\rho = \exp \left( -\frac{1}{2} \sum_{n} \sigma_{nn}(x_n - x_n^0)^2 \right).
\]

These observations suggest that \( R_{ij} \) plays the role of a correlation function for resistor networks.\(^2\) Therefore we define the resistive \((\rho)\), percolative \((p)\), and conductive \((\varphi)\) susceptibilities as

\[
\chi_\rho = \frac{1}{N^2} \sum_{i,j} \chi_{\rho ij},
\]

where

\[
\chi_{\rho ij} = [C_{ij}R_{ij}] - \frac{1}{N^2} \sum_{n} [C_{nn}R_{nn}],
\]

\[
\chi_p = [C_{ij}R_{ij}] - \frac{1}{N^2} \sum_{n} [C_{nn}R_{nn}],
\]

\[
= [C_{ij}] - p^2(p),
\]

and

\[
\chi_\varphi = [C_{ij}R_{ij}] - \frac{1}{N^2} \sum_{n} [C_{nn}R_{nn}],
\]

\[
= [R_{ij}] - \frac{1}{N^2} \sum_{n} [R_{nn}].
\]

The brackets \([\ ]\) denote a configurational average, and the cluster function \( C_{ij} \) is defined by

\[
C_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are in the same cluster,} \\ 0 & \text{otherwise.} \end{cases}
\]

For Eq. (5a) we take \( C_{ij} = 0 \) if \( C_{ij} = 0 \). Note that \( \chi = \chi(p) \), as defined by Eq. (4).

Our definition of \( \chi_\varphi \) is the same as that of Kasteleyn and Fortuin, and Harris \textit{et al.},\(^1\) but it differs from that of Sykes \textit{et al.}\(^2\) This difference will not affect the exponents, although it will, of course, lead to series expansions which are different from those of Sykes \textit{et al.}\(^1\)

The low-density series expansions for the various susceptibilities were calculated in a straightforward manner by the method of cumulants, as discussed extensively in Vol. III of the Domb and Green series.\(^1\) The computational procedure is considerably simplified by the fact that, due to the properties of the cumulant expansion, the weight factor for each diagram is proportional to \( p^n \), where \( n \) is the number of bonds in the diagram. Diagram weights are listed in Ref. 14. The series expansions for the susceptibilities \( \chi_\rho, \chi_\varphi \), and \( \chi_p \) are given in Tables I, II, and III, respectively.

### III. SCALING RELATIONS AND EXPONENTS

We define the conductivity exponent \( \mu \) by \( \Sigma \sim (p - p_c)^\mu \), for \( p > p_c \). The quantity \( L \) is defined to be the average resistance between two connected points which are separated by a percolation correlation length, \( \xi \):

\[
L = \frac{\chi_{\rho}(\xi)}{\chi_{\rho}(\xi)} = \frac{\chi_{\rho}(\xi)}{\chi_{\rho}(\xi)}, \quad \text{where } |x_i - x_j| = \xi.
\]

This definition of \( L \) makes sense both above and below \( p_c \). For \( p > p_c \) de Gennes\(^6\) has called \( L \) the "resistance between nodes," because on a Cayley tree, nodes are spaced a distance \( \xi \) apart, on

\begin{table}[h]
\centering
\begin{tabular}{cccccccccc}
\hline
\( n \) & \( m-1 \) & \( m-2 \) & \( m-3 \) & \( m-4 \) & \( m-5 \) & \( m-6 \) & \( m-7 \) & \( m-8 \) & \( m-9 \) & \( m-10 \) \\
\hline
1 & 1 & -2 & 3 & 13.5 & -75 & -292.1 & 2642.2 & 20298 & -167972.398 & -195630.113 \\
2 & 0 & 4 & -12 & 6.5 & 130 & 133.5 & -5362.3 & -27996.2 & 415426.816 & 3635613.528 \\
3 & 0 & 0 & 12 & -48 & 30 & 302.5 & 2272.4 & 2833.3 & -329864.541 & -1731450.299 \\
4 & 0 & 0 & 0 & 32 & -160 & 150 & 710.5 & 3414.95 & 74640.989 & -101778.107 \\
5 & 0 & 0 & 0 & 0 & 80 & -480 & 640 & 1592 & 3564.8 & 145332.757 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 192 & -1356 & 2564 & 3157.3 & -1651.2 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 448 & -3584 & 7968 & 4683.3 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1024 & -9216 & 24800 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2304 & -23040 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5120 \\
\hline
\end{tabular}
\caption{Coefficients \( a_{n,m} \) of the resistive susceptibility expansion defined by \( \chi_\rho = 2 \sum_{n,m \geq 1} a_{n,m} p^n p^m \).}
\end{table}
the average. However, the concept of a node seems somewhat ambiguous in networks which are not trees, and it will not be used here.

We define the exponent \( \xi \) by \( L \sim (p_c - p)^{\xi} \), for \( p < p_c \). We expect that \( L \) must grow at least as fast as \( \xi \), since clusters become increasingly ramified\(^{15(a)}\) as \( p \) approaches \( p_c \), so that in the critical region the contribution of independent parallel paths can be neglected.\(^{15(b)}\) Thus, the average resistance between two connected points will increase at least as fast as the distance between them. This means that \( \xi \approx \nu \), where \( \nu \) is defined, as usual, by \( \xi \sim (p_c - p)^{\nu} \). We also expect that \( L \) cannot be greater than the average length of a self-avoiding walk between two points separated by a distance \( \xi \). This gives us an upper bound \( \xi \leq \nu/\nu_s \), where \( \nu_s \) is the correlation length exponent for self-avoiding walks. Thus, \( \xi \) must satisfy the relation\(^{16}\)

\[
\nu \leq \xi \leq \nu/\nu_s .
\]

We also know that \( \nu = \xi \) for \( d = 1 \), and that \( \xi = \nu/\nu_s = 1 \) for \( d \geq 6 \), because the self-avoiding walk approximation (which is the appropriate MFT for this problem) gives the correct exponents for \( d \geq d_c = 6 \).

Following standard procedure, we define \( \xi' \) by \( L \sim (p_c - p)^{\xi'} \), for \( p > p_c \). Stailey\(^{17}\) has argued that a scaling picture of the usual sort is valid for this problem; therefore we expect that \( \xi' = \xi \).

Stinchcombe\(^{18}\) has verified that this relation is true for MFT, by showing that \( \xi' = 1 \) for a Cayley tree. De Gennes\(^{6}\) has shown that \( \xi \sim L^{-2} \xi^{2-d} \), which gives us the desired scaling relation,

\[
\mu = \xi + (d-2)\nu .
\]

### TABLE III. Coefficients \( c_{n,m} \) of the conductive susceptibility expansion defined by

\[
\chi_c = \sum_{n,m=1}^{\infty} c_{n,m} p^m .
\]

<table>
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<tr>
<th>( n )</th>
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<th>( m = 3 )</th>
<th>( m = 4 )</th>
<th>( m = 5 )</th>
<th>( m = 6 )</th>
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\(^{a}\) Bars over numbers indicate repeating decimal fractions.
Equation (9) has also been derived by other methods. From the definitions (5) and (7), we see that $\chi_\varepsilon = L \chi_p$ and $\chi_e = L^{-2} \chi_p$, so that we have

$$\gamma_r = \gamma_p + \xi,$$

(10a)

and

$$\gamma_e = \gamma_p - \xi,$$

(10b)

where the exponents $\gamma_\varepsilon$ are defined by $\chi_\varepsilon \sim (p - p_c)^{-\gamma_\varepsilon}$.

We have used our susceptibility series expansions to obtain $\gamma_r$, $\gamma_p$, and $\gamma_e$ directly, and thus check the validity of the scaling relations. The exponents were calculated by both Padé and ratio methods, and, in general, there was good agreement between the two techniques. Best values for these exponents are shown in Table IV. We also display values of the exponent $\gamma_e$ computed using the scaling relations Eqs. (10). The numbers in Table IV were computed by assuming the simple power law scaling form $\gamma \sim (p - p_c)^{\gamma}$; thus, the deviations of the exponents from their MFT values for $d \geq 6$ are believed to be spurious.

Our values for $\gamma_p$ are in excellent agreement with those of Gaunt et al. for $d \geq 3$, our confidence limits are significantly better than theirs. This is possible because the large value of $\gamma_\varepsilon$ allows a more precise determination of $\gamma_p$, thus substantially reducing the primary source of uncertainty. It is also probably significant that $\gamma_p$ is indeed given by the average of $\gamma_r$ and $\gamma_e$, as required by Eqs. (10a) and (10b).

Priest and Lubensky, and Amit have used the connection between these lattice statistics problems and the Potts model to compute $\varepsilon$ expansions to order $\varepsilon^2$ for the various exponents, where $\varepsilon = 6 - d$. They find $\gamma_p = 1.19$ for $d = 5$. On this basis, we feel that there is excellent agreement between the values of $\gamma_p$ derived from the series expansion and the $\varepsilon$ expansion, contrary to the opinion expressed by Gaunt et al. and Kirkpatrick. Recently, Dasgupta et al. have also computed $\xi = 1 + O(\varepsilon)$, and Wallace and Young have extended this result to show that $\xi = 1$ to all orders of perturbation theory near $d = 6$. Our calculations agree with this result, too. They are also consistent with the conjecture that $\xi = 1$ for $d \geq 4$.

Now we combine our series expansion results with the scaling relation, Eq. (9). In Table IV we list what we believe are best values of the correlation length exponent $\nu$, based on all information available to us. We also know that $\nu_p = 0.75$ for $d = 2$, $\nu_p = 0.588$ for $d = 3$, and $\nu_p = 0.5$ for $d = 4$; thus, our values for $\xi$ satisfy the inequalities, Eq. (4). Using Eq. (9) we obtain values for the conductivity exponent; these are listed in Table IV. We find that $\gamma = 1.43 \pm 0.02$ for $d = 2$, and $\gamma = 1.95 \pm 0.03$ for $d = 3$. Our result for $d = 3$ is in excellent agreement with the experiment of Abeles et al., who measured $\gamma = 1.9 \pm 0.2$ in amorphous W-Al$_2$O$_3$ cermet films.

By building models of resistor networks, Watson and Leath obtained $\gamma = 1.38 \pm 0.12$ for $d = 2$, and Adler et al. found $\gamma = 2$ for $d = 3$. However, extensive computer simulations by Kirkpatrick and by Straley give $\gamma = 1.1 \pm 0.1$ and $\gamma = 1.10 \pm 0.05$ for $d = 2$, and $\gamma = 1.6 \pm 0.1$ and $\gamma = 1.70 \pm 0.05$ for $d = 3$, respectively, in disagreement with our results. Their $d = 2$ results are difficult to reconcile with the scaling theory, since inserting $\gamma = 1.1$ into Eq. (9) gives $\xi = 1.1$ which does not satisfy $\xi \approx \nu$.

We believe that the discrepancy between our results and those of Kirkpatrick and Straley is associated with the fact that our values are obtained via a low-density series expansion, whereas theirs are obtained by using data for $p > p_c$. We remind the reader that Sykes et al. found that high-density series expansions for the percolation problem are poorly behaved. This would lead one to expect that corrections to the asymptotic scaling behavior should be unusually large for $p > p_c$, thus making the extraction of critical exponents from the data of Kirkpatrick and Straley extremely difficult. In view of the connection with the Potts model, we believe that it is unlikely that the high-density exponents actually differ from their low-density counterparts.

### IV. SUMMARY

In this paper we analyzed the properties of randomly diluted hypercubic resistor networks...
near the percolation threshold \( p_c \) as a function of the probability \( p \) that each resistor is present. The following results were obtained.

(i) A resistive susceptibility \( \chi_r \) and a conductive susceptibility, \( \chi_c \), were defined. Low-density series expansions for these functions and for the percolative susceptibility \( \chi_p \) were computed to order \( p^{10} \).

(ii) Ratio and Padé techniques were applied to the series expansions, and values of the susceptibility exponents \( \gamma_r, \gamma_c \), and \( \gamma_p \) were obtained as functions of \( d \). For \( d \) near six, there is good agreement between these results and the results of \( \epsilon \) expansion calculations.

(iii) The characteristic resistance \( L \) was defined as \( L = \chi_p(\xi)/\chi_p(\xi) \), where \( \xi \) is the percolation correlation length. It follows from this definition that the exponent \( \xi \), associated with the divergence of \( L \) at \( p_c \), is given by \( \xi = \gamma_r - \gamma_p = \gamma_p - \gamma_c \). The values of \( \xi \) obtained via this relation from the series expansions obey the inequalities \( \nu < \xi < \nu_\ast \), where \( \nu \) and \( \nu_\ast \) are the correlation length exponents for percolation and for self-avoiding walks, respectively. For \( d \) near 6, there is little or no dependence of \( \xi \) on \( d \).

(iv) By using the scaling relation \( \mu = \xi + (d-2)\nu \) values of the conductivity exponent \( \mu \) were obtained.

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