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Keywords
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QUOTIENTS OF FULLY NONLINEAR CONTROL SYSTEMS*

PAULO TABUADA† AND GEORGE J. PAPPAS‡

Abstract. In this paper, we introduce and study quotients of fully nonlinear control systems. Our definition is inspired by categorical definitions of quotients as well as recent work on abstractions of affine control systems. We show that quotients exist under mild regularity assumptions, and characterize the structure of the quotient state/input space. This allows to understand how states and inputs of the quotient system are related to states and inputs of the original system. We also introduce a notion of projectability which turns out to be equivalent to controlled invariance. This allows to regard previous work on symmetries, partial symmetries, and controlled invariance as leading to special types of quotients. We also show the existence of quotients that are not induced by symmetries or controlled invariance. Such decompositions have a potential use in a theory of hierarchical control based on quotients.

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1. Introduction. The analysis and synthesis problems for nonlinear control systems are often very difficult due to the the size and the complicated nature of the equations describing the processes to be controlled. It is therefore desirable to have a methodology that decomposes control systems into smaller subsystems while preserving the properties relevant for analysis or synthesis. From a theoretical point of view, the problem of decomposing control systems is also extremely interesting since it reveals system structure that must be understood and exploited.

In this paper we will focus on the study of quotient control systems since they can be seen as lower dimensional models that may still carry enough information about the original system. We will build on several accumulated results of different authors that in one way or another have made contributions to this problem. One of the first approaches was given in [17] where the analysis of the Lie algebra of a control system lead to a decomposition into smaller systems. At the same time in [35], quotients of control systems induced by observability equivalence relations where introduced in the more general context of realization theory. In [31], Lie algebraic conditions are formulated for the parallel and cascade decomposition of nonlinear control systems while the feedback version of the same problem was addressed in [24]. A different approach was based on reduction of mechanical systems by symmetries. In [39], symmetries were introduced for mechanical control systems, and further developed in [9] for general control systems. The existence of such symmetries was then used to decompose control systems as the interconnection of lower dimensionality subsystems. The notion of symmetry was further generalized in [26], where it was shown that the existence of symmetries implies that a certain distribution associated with the symmetries was controlled invariant. This related the notion of symmetry with the notion of controlled invariance for nonlinear systems. Controlled invariance [23, 11] was also used to decompose systems into smaller components. A different approach was taken

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in [22] were it was shown how to study controllability of systems evolving on principle fiber bundles through their projection on the base space. More recently, a modular approach to the modeling of mechanical systems has been proposed in [40], by studying how the interconnection of Hamiltonian control systems can still be regarded as a Hamiltonian control system. A different research direction was taken in [29], where instead of using structural properties of control systems, a constructive procedure was proposed to compute smaller control systems, called abstractions.

In several of the above approaches, some notion of quotienting is involved. When symmetries exist, one of the blocks of the decompositions introduced in [9] is simply the original control system factored by the action of a Lie group representing the symmetry. If a control system admits a controlled invariant distribution, it is shown in [23, 11] that it has a simpler local representation. This simpler representation can be obtained by factoring the original control system by the equivalence relation defined by considering the leaves of the foliation induced by the controlled invariant distribution, equivalence classes. The notion of abstraction introduced in [29] can also be seen as a quotient since the abstraction is a control system on a smaller dimensional state space defined by an equivalence relation on the state space of the original control system. These facts motivate fundamental questions such as existence and characterization of quotient systems. Existence questions have already been addressed in [35] but in a different setting. Only specific equivalence relations were considered (those induced by indistinguishability) and the input space remained unaltered by the factorization process. Furthermore, the quotients discussed in [35] are of a particular nature being characterized by the notion of projectability introduced in Section 6.

A thorough understanding of quotient systems has also important consequences for hierarchical control, since the construction of quotients proposed in [29] implicitly indicates that certain states of the original system may become inputs on the quotient control system. It is perhaps surprising that this methodology interchanges the role of state and input. However, this fact is the crucial factor that allows the development of a hierarchical control theory based on quotients. Since states of the original system may become inputs of the quotient system, a control design performed on a quotient system can serve as a design specification for the original system. A complete and thorough understanding of how the states and inputs propagate from control systems to their quotients will enable such a hierarchical design scheme. Preliminary work exploiting such hierarchical approach has been reported in [37].

In this paper, we take a new approach to the study of quotients by introducing the category of control systems as the natural setting for such problems in systems theory. The use of category theory for the study of problems in system theory also has a long history which can be traced back to the works of Arbib (see [2] for an introduction). More recently several authors have also adopted a categorical approach as in [19] where the category of affine control systems is investigated. We mention also [33], where a categorical approach has been used to provide a general theory of systems.

We define the category of control systems whose objects are fully (non-affine) nonlinear control systems, and morphisms map trajectories between objects. The morphisms in this category extend the notion of ϕ-related systems from [28]. In this categorical setting we formulate the notion of quotient control systems and show, in one of the main results, that:

under some regularity (constant rank) assumptions quotient control systems always exist.

This result implies that given a nonlinear projection map from the state space to some reduced state space, we can always construct a new control system on the reduced state space with the property that the nonlinear projection map carries trajectories of the original system into trajectories of the reduced system. This should be contrasted with several other approaches which rely on existence of symmetries or controlled invariance to assert the existence of quotients. We also introduce the notion of projectable control sections, which will be a fundamental ingredient to characterize the structure of quotients. This notion is in fact equivalent to controlled invariance, and this allows to regard quotients based on symmetries or controlled invariance as a special type of quotients. General quotients, however, are not necessarily induced by symmetries or controlled invariance and have the property that some of their inputs are related to states of the original model. This fact, implicit in [29], is explicitly characterized in this paper by understanding, how the state and input space of the quotient is related to the state and input space of the original control system. In particular, this paper main contribution states that:

in the absence of symmetries, states that are factored out in the quotient construction can be regarded as inputs of the quotient control system.

This result clearly distinguishes general quotients from previously studied quotients based on symmetries or partial symmetries in which inputs of the quotient system are the inputs (or a quotient) of the original system inputs. Since existence of symmetries can be regarded as rare phenomena[1], as shown in [32] for single-input systems, construction of quotients enables a widely applicable hierarchical approach to control design based on reconstruction of trajectories for the original system from quotient trajectories [37].

The outline of the paper is as follows. We start by introducing the relevant notions from differential geometry and control theory in Section 2. We then review the notion of ϕ-related control systems in Section 3 which was originally introduced in [28] and which will motivate the definition of the category of control systems presented in Section 4. In Section 5, we introduce the notion of quotient control systems and prove an existence and uniqueness result regarding quotients which roughly asserts that given a regular equivalence relation on the state space of a control system a quotient systems exists (under some regularity conditions) and is unique up to isomorphism. The characterization of quotients will be the goal of the remaining sections of the paper. We first introduce the notion of projectable control section at Section 6 and prove the main result of the paper characterizing the structure of the quotient state/input space at Section 7. We end with conclusions and some open questions for further research at Section 8.

2. Control Systems.

In this section we introduce all the relevant notions from differential geometry and control systems necessary for the remaining paper. The interested reader may whish to consult numerous books on these subjects, such as [1] for differential geometry and [14, 27] for control theory.

2.1. Differential Geometry.

We will consider that all the manifolds will be C^∞ and that all the maps will be smooth. Let M be a manifold and T_xM its tangent space at $x \in M$. The tangent bundle of M is denoted by $TM = \cup_{x\in M} T_xM$ and π_M is the canonical projection map $\pi_M : TM \longrightarrow M$ taking a tangent vector $X(x) \in T_xM \subset TM$ to the base point $x \in M$. Now let M and N be manifolds and $\phi : M \longrightarrow N$ a map, we denote by $T_x\phi : T_xM \longrightarrow T_{\phi(x)}N$ the induced tangent map which maps tangent vectors X at T_xM to tangent vectors $T_x\phi \cdot X$ at $T_{\phi(x)}N$. If ϕ is such

[1] We thank one of the anonymous reviewers for bringing this fact to the author's attention.

that $T_x\phi$ is surjective at $x \in M$ then we say that ϕ is a submersion at x. When ϕ is a submersion at every $x \in M$ we simply say that it is a submersion. Similarly, we say that ϕ has constant rank if the rank of the pointwise linear map $T_x\phi$ is constant for every $x \in M$. When ϕ has an inverse which is also smooth we call ϕ a diffeomorphism. We say that a manifold M is diffeomorphic to a manifold N, denoted by $M \cong N$, when there is a diffeomorphism between M and N. When this is the case we can use $\phi^{-1} : N \longrightarrow M$ to define a vector field on M from a vector field $Y \in TN$, denoted by $\phi^*Y = (\phi^{-1})_*Y$, and defined by $T_{\phi(x)}\phi^{-1} \cdot Y(\phi(x))$.

A fibered manifold is a manifold B equipped with a surjective submersion $\pi_B : B \longrightarrow M$. Manifolds B and M are called the total space and the base space, respectively. The surjection π_B defines a submanifold $\pi_B^{-1}(x) = \{b \in B : \pi_B(b) = x\} \subseteq B$ for every $x \in M$ called the fiber at $x \in M$. We will usually denote a fibered manifold simply by $\pi_B : B \longrightarrow M$. Since a surjective submersion is locally the canonical projection from $\mathbb{R}^i$ to $\mathbb{R}^j$, $i = \dim(B)$ and $j = \dim(M)$, we can always find local coordinates (x, y), where x are coordinates for the base space and y are coordinates for the fibers over the base space. We shall call these coordinates adapted coordinates.

A map $\varphi : B_1 \longrightarrow B_2$ between two fibered manifolds is fiber preserving iff there exists a map $\phi : M_1 \longrightarrow M_2$ between the base spaces such that the following diagram commutes:

$$\begin{tikzcd} B_1 \arrow[r, "\varphi"] \arrow[d, "\pi_{B_1}"'] & B_2 \arrow[d, "\pi_{B_2}"] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd} \tag{2.1}$$

that is to say, iff $\pi_{B_2} \circ \varphi = \phi \circ \pi_{B_1}$. In such a case we also refer to φ as a fiber preserving lift of ϕ. Given fibered manifolds B_1 and B_2, we will say that B_1 is a fibered submanifold of B_2 if the inclusion map $i : B_1 \hookrightarrow B_2$ is fiber preserving.

Given a map $h : M \longrightarrow N$ defined on the base space of a fibered manifold, its extension to the total space B is given by $\pi_B^* h = h \circ \pi_B$. We now consider the extension of a map $H : B \longrightarrow TM$ to a vector field in B. We will define local and global extensions of H. Globally, we define H^e as the set of all vector fields[2] $X : B \longrightarrow TB$ such that:

$$\begin{tikzcd} & TB \arrow[d, "T\pi_B"] \\ B \arrow[ur, "X"] \arrow[r, "H"'] & TM \end{tikzcd} \tag{2.2}$$

commutes, that is $T\pi_B(X) = H$. When working locally, one can be more specific and select a distinguished element of H^e, denoted by H^l, which satisfies in adapted local

[2]Global existence of such vector fields X follows from the existence of an horizontal space $\mathcal{H} \subseteq TB$, $\mathcal{H} \cong TM$ that allows to decompose TB as $TB = \mathcal{H} \oplus \ker(T\pi_B)$. A global extension of a map $H : B \longrightarrow TM$ to a vector field $X : B \longrightarrow TB$ is now uniquely defined as the vector field $X = H : B \longrightarrow TM \cong \mathcal{H} \subseteq TB$. Such horizontal space can be obtained, for example, as the orthogonal complement to $\ker(T\pi_B)$ given by a Riemannian metric on B.

coordinates (x, y), $H^l = H\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}$. A vector field $Y : M \longrightarrow TM$ on the base space M of a fibered manifold can also be extended to a vector field on the total space. It suffices to compose Y with the projection $\pi_B : B \longrightarrow M$ and recover the previous situation since $Y \circ \pi_B$ is a map from B to TM.

2.2. Control Systems. Since the early days of control theory it was clear that in order to give a global definition of control systems the notion of input could not be decoupled from the notion of state [4, 41]. Although the coupling between states and inputs is usually modeled through the use of fiber bundles, we shall consider more general spaces:

Definition 2.1 (Control System). *A control system $\Sigma_M = (U_M, F_M)$ consists of a fibered manifold $\pi_{U_M} : U_M \longrightarrow M$ called the control bundle and a map $F_M : U_M \longrightarrow TM$ making the following diagram commutative:*

$$\begin{tikzcd} U_M \arrow[r, "F_M"] \arrow[d, "\pi_{U_M}"'] & TM \arrow[dl, "\pi_M"] \\ M & \end{tikzcd} \tag{2.3}$$

that is, $\pi_M \circ F_M = \pi_{U_M}$, where $\pi_M : TM \longrightarrow M$ is the tangent bundle projection.

The input space U_M is modeled as a fibered manifold since in general the available control inputs may depend on the current state of the system. In adapted coordinates (x, v), Definition 2.1 reduces to the familiar expression $\dot{x} = f(x, v)$ with $v \in \pi_{U_M}^{-1}(x)$. The lack of local triviality assumptions on π_{U_M} is motivated by the need to model the construction of abstractions of control affine systems, as described in [29], in a fully nonlinear context. As the following example illustrates, even in simple situations the inputs of a control system resulting from an abstraction or quotient process can depend on the states in a way that cannot be modeled by a fiber bundle.

Consider control system $F_M : U_M \longrightarrow TM$ with $U_M = M \times U$, $M = \mathbb{R}^3$, $U =]0, 1[$ defined by:

$$F_M(x, y, z, u) = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})u$$

On the state space we define the following map $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ based on Reeb's foliation:

$$\phi(x, y, z) = (1 - r^2)e^z, \quad r = x^2 + y^2 \tag{2.4}$$

Computing the derivative of ϕ:

$$\mathrm{d}\phi = e^z(-4rx\,\mathrm{d}x - 4ry\,\mathrm{d}y + (1 - r^2)\,\mathrm{d}z)$$

we see that ϕ is a submersion since $1 - r^2 = 0$ for $r^2 = 1$ which implies that $x \neq 0$ or $y \neq 0$ and this in turn implies that $\mathrm{d}\phi \neq 0$. This shows that we can see $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ as a fibered manifold. If we now compute the projection of F_M on $\mathbb{R}$ by ϕ, we obtain:

$$\mathrm{d}\phi \cdot F_M = e^z(-4rx^2 - 4ry^2 + (1 - r^2)z)u$$

The set of vectors defined by the previous expression can be seen as a control system on $\mathbb{R}$ up to control parameterization, as it defines the possible directions of motion achievable by control. This is the principle underlying the notion of abstraction described in [29]. Such collection of vector fields admits the natural parameterization $\pi_{U_M}^{-1}(\phi^{-1}(w))$ for every $w \in \mathbb{R}$. However, such set of inputs cannot be given the structure of a fiber bundle. To see this, it suffices to note that the fibers $\phi^{-1}(w)$ are not homeomorphic for $w > 0$ and $w = 0$. For $w > 0$ we can solve $\phi(x, y, z) = w$ to obtain $z = \log \frac{w}{1-r^2}$ which defines $\phi^{-1}(w)$ as:

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ z = \log \frac{w}{1 - r^2} \quad \wedge \quad 0 \leq r^2 < 1\}$$

and which is homeomorphic to the open unit disk in $\mathbb{R}^2$. If $w = 0$, solving $\phi(x, y, z) = 0$ we obtain $r = 1$ which is diffeomorphic to a cylinder. We thus see that for any open set O in $\mathbb{R}$ containing 0, $\pi_{U_M}(\phi^{-1}(0))$ cannot be diffeomorphic to $O \times L$ for some manifold L describing the typical fibers of $\phi \circ \pi_{U_M}$ as they are not diffeomorphic for different points in O. It is precisely the need to capture and analyze situations like this, that forces one to consider models for the state/input space other than fiber bundles. The need to model these and other couplings between states and inputs has led to alternative approaches where the notion of control system and its properties are defined independently of states and inputs as in Willem's behavioral theory [30] and Fliess' differential algebraic approach [7].

We now return to our discussion of control systems by introducing the notion of control section[3] that is closely related with control systems and which will be fundamental in our study of quotients:

Definition 2.2 (Control Section). *Given a manifold M, a control section on M is a fibered submanifold $\pi_{\mathcal{S}_M} : \mathcal{S}_M \longrightarrow M$ of TM.*

We denote by $\mathcal{S}_M(x)$ the set of vectors $X \in T_xM$ such that $X \in \pi_{\mathcal{S}_M}^{-1}(x)$. We now see that under certain regularity assumptions, a control system (U_M, F_M) defines a control section by the pointwise assignment $\mathcal{S}_M(x) = F_M(\pi_{U_M}^{-1}(x))$. Conversely, a control section also defines a control system as we shall see in detail at Section 4. The notion of control section allows to refer in a concise way to the set of all tangent vectors that belong to the image of F_M by saying that $X \in T_xM$ belongs to $\mathcal{S}_M(x)$ iff there exists a $u \in U_M$ such that $\pi_M(u) = x$ and $F_M(u) = X$. When $\mathcal{S}_M(x)$ defines an affine distribution on TM, we call control system F_M control affine and fully nonlinear otherwise.

Having defined control systems the concept of trajectories or solutions of a control system is naturally expressed as:

Definition 2.3 (Trajectories of Control Systems). *A smooth curve $c : I \longrightarrow M$, $I \subseteq \mathbb{R}_0^+$ is called a trajectory of control system $\Sigma_M = (U_M, F_M)$, if there exists a (not necessarily smooth) curve $c^U : I \longrightarrow U_M$ making the following diagrams commutative:*

$$\begin{tikzcd} & U_M \arrow[d, "\pi_{U_M}"] \\ I \arrow[ur, "c^U"] \arrow[r, "c"'] & M \end{tikzcd} \qquad \begin{tikzcd} & U_M \arrow[d, "F_M"] \\ I \arrow[ur, "c^U"] \arrow[r, "Tc"'] & TM \end{tikzcd} \tag{2.5}$$

where we have identified I with TI.

[3]In some literature this notion is also know as field of admissible velocities.

The above commutative diagrams are equivalent to the following equalities:

$$\begin{aligned} \pi_{U_M} \circ c^U &= c \\ Tc &= F_M(c^U) \end{aligned}$$

which mean in adapted coordinates that $x(t)$ is a trajectory of a control system if there exists an input $v(t)$ such that $x(t)$ satisfies $\dot{x}(t) = f(x(t), v(t))$ and $v(t) \in \pi_{U_M}^{-1}(x(t))$ for all $t \in I$.

3. ϕ-related Control Systems. We start by reviewing the notion of ϕ-related control systems originally introduced in [28] and which motivates the construction of the category of control systems to be later presented.

DEFINITION 3.1 (ϕ-related Control Systems). *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. Given a map $\phi : M \longrightarrow N$ we say that Σ_N is ϕ-related to Σ_M iff for every $x \in M$:*

$$T_x\phi(\mathcal{S}_M(x)) \subseteq \mathcal{S}_N \circ \phi(x) \tag{3.1}$$

In [28] it is shown that this notion is equivalent to a more intuitive relation between Σ_M and Σ_N.

PROPOSITION 3.2 ([28]). *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively and let $\phi : M \longrightarrow N$ be a map. Control system Σ_N is ϕ-related to Σ_M iff for every trajectory $c(t)$ of Σ_M, $\phi(c(t))$ is a trajectory of Σ_N.*

Propagating trajectories from a system to another is clearly desirable. Since most control systems properties are properties of its trajectories, relating trajectories of different control systems allows also to relate the corresponding properties. If, in fact, system Σ_N is lower dimensional then system Σ_M, then we are clearly reducing the complexity of Σ_M. We can therefore regard Σ_N as an *abstraction* of Σ_M in the sense that some aspects of Σ_M have been collapsed or abstracted away, while others remain in Σ_N. This motivated the notion of abstraction based on trajectory propagation in [28], which defined an abstraction of a control system Σ_M as a ϕ-related control system Σ_N by a surjective submersion ϕ.

The idea of sending trajectories from one system to trajectories of another system has been used many times in control theory to study equivalence of control systems. We mention for example linearization by diffeomorphism [16] or feedback linearization [5, 10, 13]. In these examples the maps ϕ relating the control systems were in fact diffeomorphisms so that no aggregation or abstraction was involved. Related to the feedback linearization problem, is the partial feedback linearization problem where only partial linearization is thought. Such problem can be reduced to the feedback linearization problem by considering feedback linearization of a sub-system of the original control system [20]. The notion of sub-system can also be described by defining how sub-system trajectories relate to the original system trajectories. In this case, we require the existence of a map (satisfying certain injectivity assumptions) transforming sub-system trajectories into trajectories of the original system. The use of trajectory propagating maps can already be traced back to the works of Arbib (see [2] for an introduction) where by the use of category theoretic ideas it is shown that (discrete time) control systems and finite state automata are just different manifestations of the same phenomena.

4. The Category of Control Systems. Informally speaking, a category is a collection of *objects* and *morphisms* between the objects, that relate the structure of the objects. If one is interested in understanding vector spaces, it is natural to consider vector spaces as objects and linear maps as morphisms, since they preserve the vector space structure. This choice for objects and morphisms defines **Vect**, the category of vector spaces. Choosing manifolds for objects leads to the the natural choice of smooth maps for morphisms and defines **Man**, the category of smooth manifolds. In this section we introduce the category of control systems which we regard as the natural framework to study quotients of control systems. Besides providing an elegant language to describe the constructions to be presented, category theory also offers a conceptual methodology for the study of objects, control systems, in this case. We refer the reader to [18] and [3] for further details on the elementary notions of category theory used throughout the paper.

The category of control systems, denoted by **Con**, has as objects control systems as described in Definition 2.1. The morphisms in this category extend the concept of ϕ-related control systems described by Definition 3.1. Since the notion of ϕ-related control systems relates control sections and these can be parameterized by controls, the lifted notion should relate control sections as well as its parameterizations by inputs.

DEFINITION 4.1 (Morphisms of Control Systems). *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. A morphism f from Σ_M to Σ_N is a pair of maps $f = (\phi, \varphi)$, $\phi : M \longrightarrow N$ and $\varphi : U_M \longrightarrow U_N$ making the following diagrams commutative:*

$$
\begin{tikzcd}
U_M \arrow[r, "\varphi"] \arrow[d, "\pi_{U_M}"'] & U_N \arrow[d, "\pi_{U_N}"] \\
M \arrow[r, "\phi"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
U_M \arrow[r, "\varphi"] \arrow[d, "F_M"'] & U_N \arrow[d, "F_N"] \\
TM \arrow[r, "T\phi"'] & TN
\end{tikzcd}
\tag{4.1}
$$

It will be important for later use to also define isomorphisms:

DEFINITION 4.2 (Isomorphisms of Control Systems). *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. System Σ_M is isomorphic to system Σ_N iff there exist morphisms f_1 from Σ_M to Σ_N and f_2 from Σ_N to Σ_M such that $f_1 \circ f_2 = (id_N, id_{U_N})$ and $f_2 \circ f_1 = (id_M, id_{U_M})$.*

In this setting, feedback transformations[4] can be seen as special isomorphisms. Consider an isomorphism (ϕ, φ) with $\varphi : U_M \longrightarrow U_M$ such that $\phi = id_M$. In adapted coordinates (x, v), where x represents the base coordinates (the state) and v the coordinates on the fibers (the inputs), the isomorphism has a coordinate expression for φ of the form $\varphi = (x, \beta(x, v))$. The fiber term $\beta(x, v)$ representing the new control inputs is interpreted as a feedback transformation since it depends on the state at the current location as well as the former inputs v. We shall therefore refer to feedback transformations as isomorphisms over the identity since we have $\phi = id_M$. The control theoretic notion of feedback equivalence is captured in this framework by noting that two control systems are feedback equivalent iff there exists an isomorphism

[4]Some authors use the expression feedback transformation to denote any isomorphism in **Con**. We consider the more restrictive use where change of coordinates in the state space are disallowed as they cannot be realized by feedback.

(although not necessarily a feedback transformation) between the two systems. A related notion is that of system immersion. Although we cannot capture such notion in our framework, as we have not equipped control systems with observation maps, a restricted version of system immersion can still be defined within our framework. Recall that, according to [6], system Σ_M is said to be immersed in system Σ_N, if there exists an injective map $\phi : M \longrightarrow N$ such that the input-output behavior of Σ_M, when initialized at x, equals the input-output behavior of Σ_N, when initialized at $\phi(x)$. If we assume that $U_M = U \times M$ and $U_N = U \times N$ for some common input manifold U, that M is a submanifold of N, and that i is the canonical injection of M into N, then Σ_M is immersed into Σ_N, when (id_U, i) is a morphism from Σ_M to Σ_N. Note that existence of morphism (id_U, i) implies that $F_M(x,u) = Ti \cdot F_M(x,u) = F_N(i(x), id_U(u)) = F_N(x,u)$ for local coordinates $(x,u) \in U \times M \subseteq U \times N$ and this implies that Σ_M and Σ_N have the same input-output behavior when initialized at x and $i(x)$, respectively.

A control system can alternatively be defined by a control section $\mathcal{S}_M$ on M in the sense that at each point $x \in M$, $\mathcal{S}_M(x)$ defines all the possible directions along which we can flow or steer our system. However, there can be several control parameterizations for $\mathcal{S}_M$ and it matters to understand in what sense all those parameterizations represent the same control system. In order to obtain such equivalence we make the following assumptions about control systems that will be explicitly mentioned when needed:

AI: The fibers $\pi_{U_M}^{-1}(x)$ are connected for every $x \in M$.

AII: The map $F_M : U_M \longrightarrow T_M$ is an embedding.

Control systems satisfying assumption **AII** enjoy the following property:

Proposition 4.3. *Let (U_M, F_M) be a control system on manifold M satisfying* **AII** *and let (U'_M, F'_M) be any control system on manifold M such that $\mathcal{S}'_M(x) \subseteq \mathcal{S}_M(x)$ for every $x \in M$. Then, there exists a unique fiber preserving map $\overline{F_M}$ making the following diagram commutative:*

$$\begin{tikzcd} U_M \arrow[r, "F_M"] & TM \\ U'_M \arrow[u, "\overline{F_M}"] \arrow[ur, "F'_M"'] & \end{tikzcd} \tag{4.2}$$

The previous result is an immediate consequence of the fact that $F_M(U_M)$ is an embedded submanifold of TM. This is sufficient for the previous result to hold but not necessary. In fact, the existence of a unique map $\overline{F_M}$ is the property of interest and could be used as a definition, however it would be difficult to check in concrete examples if a given control system would satisfy such property. A different approach would relax the requirement that $F_M(U_M)$ is an embedded submanifold by the weaker assumption of initial submanifold, see [15] for the definition of initial submanifolds and its properties.

Since assumption **AII** implies the universal property [18] stated in Proposition 4.3, any two control systems satisfying **AII** and defining the same control section are isomorphic. It is in this sense that we do not need to distinguish between different parameterization of the same control section. They are the same control system, up to a change of control coordinates, that is, up to an isomorphism over the identity. This will be important when considering the effect of feedback since, as we have already seen, this change of control coordinates can be regarded as a feedback transformation.

The relation between the notions of ϕ-related control systems (3.1) and **Con** morphisms (4.1) is stated in the next proposition:

PROPOSITION 4.4. *Let Σ_M and Σ_N be two control systems defined on M and N, respectively. If $f = (\phi, \varphi)$ is a* **Con** *morphism from Σ_M to Σ_N then Σ_N is ϕ-related to Σ_M. Conversely, if Σ_N satisfies* **AII** *and Σ_N is ϕ-related to Σ_M by a smooth map $\phi : M \longrightarrow N$, then there exists a unique fiber preserving lift φ of ϕ such that $f = (\phi, \varphi)$ is a* **Con** *morphism from Σ_M to Σ_N.*

Proof. Definition 4.1 trivially implies Definition 3.1 so let us prove that Definition 3.1 implies Definition 4.1. If Σ_N is ϕ-related to Σ_M then by Definition 3.1, $T_x\phi(\mathcal{S}_M(x)) \subseteq \mathcal{S}_N \circ \phi(x)$. But $\mathcal{S}_M$ is parameterized by U_M, so that the map $T\phi \circ F_M : U_M \longrightarrow TN$ (see the diagram below) satisfies $T\phi \circ F_M(U_M) \subseteq \mathcal{S}_N$. Therefore, by Proposition 4.3, there is a unique fiber preserving map $\overline{F_N}$ such that:

$$\begin{tikzcd}
U_M \arrow[r, "\overline{F_N}"] \arrow[d, "F_M"'] & U_N \arrow[d, "F_N"] \\
TM \arrow[r, "T\phi"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\
M \arrow[r, "\phi"'] & N
\end{tikzcd}$$

commutes. By taking $\varphi = \overline{F_N}$, $\pi_{U_M} = \pi_M \circ F_M$ and $\pi_{U_N} = \pi_N \circ F_N$ one recovers Definition 4.1 and the equivalence is proved. ∎

The previous result shows that there is an equivalence between smooth maps ϕ, relating control systems and **Con** morphisms provided that we work on a suitable subcategory (where assumption **A.II** holds). This means that many properties of nonlinear control systems can be characterized by working with $\mathcal{S}_M$ instead of F_M. We also see that if there is a morphism f from Σ_M to Σ_N, then this morphism carries trajectories of Σ_M to trajectories of Σ_N in virtue of Proposition 3.2.

5. Quotients of Control Systems. Given a control system Σ_M and an equivalence relation on the manifold M we can regard the quotient control system as an abstraction since some modeling details propagate from Σ_M to the quotient while other modeling details disappear in the factorization process. This fact motivates the study of quotient control systems as they represent lower complexity (dimension) objects that can be used to verify properties of the original control system. Quotients are also important from a design perspective since a control law for the quotient object can be regarded as a specification for the desired behavior of the original control system. In this spirit we will address the following questions:

1. **Existence:** Given a control system Σ_M defined on a manifold M and an equivalence relation $\sim_M$ on M when does there exist a control system on $M/\sim_M$, the quotient manifold, and a fiber preserving lift p_U of the projection $p_M : M \longrightarrow M/\sim_M$ such that (p_M, p_U) is a **Con** morphism?
2. **Uniqueness:** Is the lift p_U of p_M, when it exists, unique?
3. **Structure of the quotient state/input space:** What is the structure of the fibers (input space) of the quotient control system?

To clarify our discussion we formalize the notion of quotient control systems:

DEFINITION 5.1 (Quotient Control System). *Let Σ_L, Σ_M, Σ_N be control systems defined on manifolds L, M and N, respectively and g, h two morphisms from Σ_L to Σ_M. The pair (f, Σ_N), where f is a morphism and Σ_N a control system, is a quotient control system of Σ_M if $f \circ g = f \circ h$ and for any other pair (f', Σ'_N) such that $f' \circ g = f' \circ h$ there exists one and only one morphism $\overline{f}$ from Σ_N to Σ'_N such that the following diagram commutes:*

$$\begin{tikzcd} \Sigma_L \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \Sigma_M \arrow[r, "f"] \arrow[dr, "f'"'] & \Sigma_N \arrow[d, "\overline{f}"] \\ & & \Sigma'_N \end{tikzcd} \tag{5.1}$$

that is, $f' = \overline{f} \circ f$.

Intuitively, we can read diagram (5.1) as follows. Assume that the set $\sim = \{(u, v) \in U_M \times U_M \ : \ (u, v) = (g(l), h(l)) \text{ for some } l \in U_L\}$ is a regular equivalence relation [1]. Then, the condition $f \circ g = f \circ h$ simply means that f respects the equivalence relation, that is, $u \sim v \quad \Rightarrow \quad f(u) = f(v)$. Furthermore, it asks that for any other map f' respecting relation $\sim$, there exists a unique map $\overline{f}$ such that $f' = \overline{f} \circ f$. This is a usual characterization of quotient manifolds [1] that we here use as a definition. The same chain of reasoning shows that if we replace control systems by the corresponding state space and the morphisms by the maps between the state spaces, then diagram (5.1) asks for N to be also quotient manifold obtained by factoring M by a regular equivalence relation $\sim_M$ on M defined by g and h. The same idea must, therefore, hold for control systems and this means that control system Σ_N must also satisfy a unique factorization property in order to be a quotient control system.

From the above discussion it is clear that a necessary condition for the existence of the quotient control system is the existence of the quotient manifold $M/\sim_M$. When $\sim_M$ is a regular equivalence relation the quotient space $M/\sim_M$ will be a manifold [1] and the equivalence relation can be equivalently described by a surjective submersion. We will, therefore, assume that the regular equivalence relation $\sim_M$ is given by a surjective submersion $\phi : M \longrightarrow N$. Similarly, the fiber preserving lift φ of ϕ will also have to be a surjective submersion. We now consider the following assumption which will be explicitly stated when required:

AIII: The map $T\phi \circ F_M : U_M \longrightarrow TN$ has constant rank and connected fibers.

The first two questions of the previous list are answered in the next theorem which asserts that quotients exist under moderate conditions:

THEOREM 5.2. *Let Σ_M be a control system on a manifold M, $\phi : M \longrightarrow N$ a surjective submersion and assume that* **AIII** *holds. Then there exists:*

1. *a control system Σ_N on N,*
2. *a* unique *fiber preserving lift $\varphi : U_M \longrightarrow U_N$ of ϕ such that the pair $((\phi, \varphi), \Sigma_N)$ is a quotient control system of Σ_M.*

Proof. By assumption **AIII** the map $T\phi \circ F_M$ has constant rank and we can define a regular and involutive distribution $\mathcal{D}$ on TU_M by $\mathcal{D} = \ker(TT\phi \circ TF_M)$. Furthermore, as $T\phi \circ F_M$ has connected fibers, also by assumption **AIII**, these are described by the integral manifolds of $\mathcal{D}$. We thus have a regular equivalence relation $\sim \subseteq U_M \times U_M$ obtained by declaring two points equivalent if the lie on the same integral manifold of $\mathcal{D}$. We now consider the manifold $U_M/\sim$ obtained as the quotient of U_M by $\sim$ and denote by $\pi : U_M \longrightarrow U_M/\sim$ the canonical projection. Since $T\phi \circ F_M(u) =$

$T\phi \circ F_M(u')$ iff $\pi(u) = \pi(u')$, it follows from the properties of quotient manifolds [1] that there exists a unique map $\alpha : U_M/\sim \longrightarrow TN$ such that $\alpha \circ \pi = T\phi \circ F_M$. We now define U_N as $U_M/\sim$, π_{U_N} as $\pi_N \circ \alpha$, F_N as α and φ as π. We note that φ is unique and claim that $((\phi, \varphi), U_M/\sim)$ is a quotient of Σ_M. The pair of maps (ϕ, φ) is a morphism from Σ_M to Σ_N since $T\phi \circ F_M = F_N \circ \varphi$ as required by the second diagram in (4.1) and composing $T\phi \circ F_M = F_N \circ \varphi$ with π_N we obtain:

$$\begin{aligned}
\pi_N \circ T\phi \circ F_M = &\ \pi_N \circ F_N \circ \varphi && \\
\Leftrightarrow \phi \circ \pi_M \circ F_M = &\ \pi_N \circ F_N \circ \varphi && \text{Since } \pi_N \circ T\phi = \phi \circ \pi_M \\
\Leftrightarrow \phi \circ \pi_{U_M} = &\ \pi_N \circ F_N \circ \varphi && \text{By commutativity of diagram (2.3)} \\
\Leftrightarrow \phi \circ \pi_{U_M} = &\ \pi_{U_N} \circ \varphi && \text{By definition of } \pi_{U_N}
\end{aligned}$$

which shows that the first diagram in (4.1) also commutes.

It remains to show that any other morphism $f' = (\phi', \varphi')$ such that ϕ' is compatible with the equivalence relation defined by ϕ factors uniquely through f. Since the equivalence relation defined by ϕ on M induces the equivalence relation $\sim$ on U_M, we see that $\varphi(u) = \varphi(u')$ implies $\varphi'(u) = \varphi'(u')$. It then follows from the universality of φ that φ' factors uniquely through φ, that is, there exists a unique map $\overline{\varphi}$ such that $\varphi' = \overline{\varphi} \circ \varphi$. Similarly, ϕ' factors uniquely trough ϕ via $\overline{\phi}$. It then remains to show that $(\overline{\phi}, \overline{\varphi})$ is a morphism from Σ_N to Σ'_N.

We first show that diagram (4.1) commutes. Let $u_n \in U_N$, as φ is a surjective map, there is a $u_m \in U_M$ such that $\varphi(u_m) = u_n$. We now have, by diagram chasing:

$$\begin{aligned}
F'_N \circ \overline{\varphi} \circ \varphi(u_m) &= F'_N \circ \varphi'(u_m) && \text{Since } \varphi' \text{ factors on } \varphi \\
&= T\phi' \circ F_M(u_m) && \text{By commutativity of the 2nd diagram in (4.1)} \\
&= T\overline{\phi} \circ T\phi \circ F_M(u_m) && \text{Since } \phi' \text{ factors on } \phi \\
&= T\overline{\phi} \circ F_N \circ \varphi(u_m) && \text{By commutativity of the 2nd diagram in (4.1)}
\end{aligned}$$

and replacing $\varphi(u_m)$ by u_n we see that $\overline{f}$ satisfies the second diagram in (4.1). Commutativity of the first diagram in (4.1) can be obtained similarly by diagram chasing. □

This result provides the first characterization of quotient objects in **Con**. It shows that given a regular equivalence relation on the base (state) space of a control system and a mild regularity condition, there exists a quotient control system on the quotient manifold. Furthermore it also shows that the regular equivalence relation on M or the map ϕ uniquely determines a fiber preserving lift φ which describes how the state/input pairs of the control system on M relate to the state/input pairs of the quotient control system. Furthermore, we also see that the map F_N is an injective immersion, a fact we will use several times in the remaining paper.

Existence of quotients under so weak conditions is perhaps surprising given the fact that in other contexts, quotients only exist in very specific situations: a quotient group can only be obtained by factoring a group by a normal subgroup and not by a general equivalence relation, a quotient linear spaces can only be obtained by factoring a linear space by a linear sub-subspace and not by a general equivalence relation, etc. This fact highlights the relevance of Theorem 5.2 at the theoretical level but also at the practical level since quotients can be constructively used to hierarchically design trajectories [37].

Having answered the first two questions from the previous list (existence and uniqueness), we concentrate on the characterization of the quotient control system input space. This problem requires a deeper understanding of how ϕ determines φ and will be the goal of the remaining paper. Since **Con** was defined over **Man**, that is morphisms in **Con** are smooth maps and control systems are defined on manifolds, the characterization of φ will require an interplay of tools from differential geometry and category theory.

6. Projectable Control Sections. We now extend the notion of projectable vector fields from [21] and of projectable families of vector fields from [22] to control sections. The notion of projectable control sections is weaker then projectable vector field or families of vector fields but nonetheless stronger than **Con** morphisms. The motivation for introducing this notion comes from the fact that projectability of control sections will be a fundamental ingredient in characterizing the structure of the quotient system input space . Furthermore, we will also see that projectability, as defined in this categorical setting, will correspond to the well known notion of controlled invariance.

Given a vector field X on M and a surjective submersion $\phi : M \longrightarrow N$ we say that X is projectable with respect to ϕ when $Y = T\phi \cdot X$, the projection of X, is a well defined vector field on N that satisfies $T\phi \cdot X = Y \circ \phi$ [21]. The vector field Y is also called ϕ-related to X [1]. This notion was extended to families of vector fields in [22] by requiring that the projection of each vector field in the family is a well defined vector field on N. However, when working with control sections, which can be regarded as *sets* of vectors at each base point $x \in M$, one should only require that the projection of these *sets* of vectors is the same *set* when the base points on M project on the same base point on N. Intuitively, we are asking for control sections that behave well under the projection defined by ϕ. This is formalized as follows:

Definition 6.1. *Let M be a manifold, $\mathcal{S}_M$ a control section on M and $\phi : M \longrightarrow N$ a surjective submersion. We say that $\mathcal{S}_M$ is projectable with respect to ϕ iff $\mathcal{S}_M$ induces a control section $\mathcal{S}_N$ on N such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{P}(TM) \arrow[r, "T\phi"] & \mathcal{P}(TN) \\ M \arrow[u, "\mathcal{S}_M"] \arrow[r, "\phi"'] & N \arrow[u, "\mathcal{S}_N"'] \end{tikzcd} \tag{6.1}$$

where $\mathcal{P}(TM)$ denotes the powerset of T_xM for every $x \in M$.

We see that if $\mathcal{S}_M$ is in fact a vector field we recover the notion of projectable vector fields. The notion of projectable control sections is stronger then the notion of **Con** morphism since for any $x_1, x_2 \in M$ such that $\phi(x_1) = \phi(x_2)$ we necessarily have $T\phi(\mathcal{S}_M(x_1)) = \mathcal{S}_N \circ \phi(x_1) = T\phi(\mathcal{S}_M(x_2))$ if $\mathcal{S}_M$ is projectable. On the other hand, if (ϕ, φ) is a **Con** morphism for a fiber preserving lift φ of ϕ, we only have the inclusions $T\phi(\mathcal{S}_M(x_1)) \subseteq \mathcal{S}_N \circ \phi(x_1)$ and $T\phi(\mathcal{S}_M(x_2)) \subseteq \mathcal{S}_N \circ \phi(x_1)$. Therefore projectability with respect to ϕ and assumption **A.II** implies that ϕ can be extended to a **Con** morphism but given a **Con** morphism $f = (\phi, \varphi)$ from Σ_M to Σ_N it is not true, in general, that $\mathcal{S}_M$ is projectable with respect to ϕ.

To determine the relevant conditions on $\mathcal{S}_M$ that ensure projectability we will need an auxiliary result:

Lemma 6.2. *Let $f : M \longrightarrow N$ be a map between manifolds and let X_t be the flow of a vector field $X \in TM$ such that $f \circ X_t = f$. Then the following equality holds for* *every $x \in M$:*

$$T_x f\, T_{X_t(x)} X_{-t} = T_{X_t(x)} f \tag{6.2}$$

Proof. The equality $f \circ X_t = f$ is equivalent to:

$$\begin{aligned} & f \circ X_t(x) = f(x) \\ \Leftrightarrow\ & f(X_t(x)) = f \circ (X_t)^{-1} \circ X_t(x) \\ \Leftrightarrow\ & f(X_t(x)) = f \circ X_{-t}(X_t(x)) \end{aligned} \tag{6.3}$$

and by differentiation of the previous expression we arrive at the desired equality:

$$T_{X_t(x)} f = T_x f\, T_{X_t(x)} X_{-t} \tag{6.4}$$

□

We can now give sufficient and necessary conditions for projectability of control sections.

PROPOSITION 6.3 (Projectable Control Sections). *Let M be a manifold, $\mathcal{S}_M$ a control section on M and $\phi : M \longrightarrow N$ a surjective submersion with connected fibers. Given any control system (U_M, F_M) satisfying* **AI** *and defining $\mathcal{S}_M$, and any $\widehat{F_M} \in F_M^e$, $\mathcal{S}_M$ is projectable with respect to ϕ iff:*

$$[\widehat{F_M}, \ker(T\pi^*_{U_M}\phi)] \subseteq \ker(T\pi^*_{U_M}\phi) + [\widehat{F_M}, \ker(T\pi_{U_M})] \tag{6.5}$$

Proof. We show necessity first. Assume that diagram (6.1) commutes. Then we have:

$$T_x\phi(\mathcal{S}_M(x)) = T_{x'}\phi(\mathcal{S}_M(x')) \tag{6.6}$$

for all $x, x' \in M$ such that $\phi(x) = \phi(x')$, that is, for any x and x' on the same leaf of the foliation induced by $\ker(T\phi)$. If we denote by K_t the flow of any vector field $K \in \ker(T\pi^*_{U_M}\phi)$, expression (6.6) implies that:

$$T_{\pi_{U_M} \circ K_t(u)}\phi(F_M \circ K_t(u)) \in T_x\phi(\mathcal{S}_M(x)) \tag{6.7}$$

for every $t \in \mathbb{R}$ such that K_t is defined and for every $u \in \pi^{-1}_{U_M}(x)$. Since the left hand side of (6.7) belongs to the right hand side and $\pi^{-1}_{U_M}(x)$ is connected by **AI**, we can always find a $Y \in \ker(T\pi_{U_M})$ such that its flow Y_t will parameterize the image of the left hand side, that is:

$$T_{\pi_{U_M} \circ K_t(u)}\phi(F_M \circ K_t(u)) = T_{\pi_{U_M} \circ Y_t(u)}\phi(F_M \circ Y_t(u)) \tag{6.8}$$

The previous equality implies that for any $\widehat{F_M} \in F_M^e$ we have:

$$T_{K_t(u)}\pi^*_{U_M}\phi(\widehat{F_M} \circ K_t(u)) = T_{Y_t(u)}\pi^*_{U_M}\phi(\widehat{F_M} \circ Y_t(u)) \tag{6.9}$$

however, the equalities $\pi^*_{U_M}\phi \circ K_t = K_t$, $\pi^*_{U_M}\phi \circ Y_t = Y_t$ and Lemma 6.2 allows to rewrite (6.9) as:

$$\begin{aligned} & T_u\pi^*_{U_M}\phi(T_{K_t(u)}K_{-t} \circ \widehat{F_M} \circ K_t(u)) = T_u\pi^*_{U_M}\phi(T_{Y_t(u)}Y_{-t} \circ \widehat{F_M} \circ Y_t(u)) \\ \Leftrightarrow \quad & T_u\pi^*_{U_M}\phi(K_t(u)^*\widehat{F_M}) = T_u\pi^*_{U_M}\phi(Y_t(u)^*\widehat{F_M}) \end{aligned} \tag{6.10}$$

Time differentiation at $t = 0$ now implies:

$$\begin{aligned} T_u\pi^*_{U_M}\phi([K(u), \widehat{F_M}(u)]) &= T_u\pi^*_{U_M}\phi([Y(u), \widehat{F_M}(u)]) \\ &\Rightarrow [K, \widehat{F_M}] \in [Y, \widehat{F_M}] + \ker(T\pi^*_{U_M}\phi) \end{aligned} \tag{6.11}$$

which trivially implies inclusion (6.5).

To show sufficiency we use a similar argument. Assume that (6.5) holds, then for any $K \in \ker(T\pi^*_{U_M}\phi)$ there exists a $Y \in \ker(T\pi_{U_M})$ such that:

$$\begin{aligned} &T_u\pi^*_{U_M}\phi([\widehat{F_M}(u), K(u)]) = T_u\pi^*_{U_M}\phi([\widehat{F_M}(u), Y(u)]) \\ \Leftrightarrow\ &T_u\pi^*_{U_M}\phi([\widehat{F_M}(u), K(u) - Y(u)]) = 0 \end{aligned} \tag{6.12}$$

Consider now the regular and involutive distribution $\ker(T\pi^*_{U_M}\phi)$. Involutivity and regularity imply that $Z^*_t W \in \ker(T\pi^*_{U_M}\phi)$ for any $W \in \ker(T\pi^*_{U_M}\phi)$ and the flow Z_t of any vector field $Z \in \ker(T\pi^*_{U_M}\phi)$ [34]. Since $K \in \ker(T\pi^*_{U_M}\phi)$ and $Y \in \ker(T\pi^*_{U_M}\phi)$ it follows that $K - Y \in \ker(T\pi^*_{U_M}\phi)$, but from (6.12), $[\widehat{F_M}, K - Y]$ also belongs to $\ker(T\pi^*_{U_M}\phi)$ so that we conclude:

$$T_u\pi^*_{U_M}\phi((K - Y)_t(u)^*[\widehat{F_M}, K - Y]) = 0 \tag{6.13}$$

where $(K - Y)_t$ denotes the flow of the vector field $K - Y$. However, the previous expression is equivalent to:

$$\begin{aligned} &T_u\pi^*_{U_M}\phi(\frac{d}{dt}(K - Y)_t(u)^*\widehat{F_M}) = 0 \\ \Leftrightarrow &\frac{d}{dt}T_u\pi^*_{U_M}\phi((K - Y)_t(u)^*\widehat{F_M}) = 0 \end{aligned} \tag{6.14}$$

where the last equality follows from the fact that $T\phi$ is a linear map. Since the time derivative is zero, we must have:

$$T_u\pi^*_{U_M}\phi((K - Y)_t(u)^*\widehat{F_M}) = T_u\pi^*_{U_M}\phi((K - Y)_0(u)^*\widehat{F_M}) = T_u\pi^*_{U_M}\phi(\widehat{F_M}(u)) \tag{6.15}$$

From the equality $\pi^*_{U_M}\phi = \pi^*_{U_M}\phi \circ (K - Y)_t$ we conclude that $T_u\pi^*_{U_M}\phi T_{(K-Y)_t(u)}(K - Y)_{-t} = T_{(K-Y)_t(u)}\pi^*_{U_M}\phi$ by Lemma 6.2 so that (6.15) can be written as:

$$T_{(K-Y)_t(u)}\pi^*_{U_M}\phi(\widehat{F_M} \circ (K - Y)_t(u)) = T_u\pi^*_{U_M}\phi(\widehat{F_M}(u)) \tag{6.16}$$

and projecting on TM we get:

$$T_{\pi_{U_M}(K'_t(u))}\phi(F_M \circ (K')_t(u)) = T_x\phi(F_M(u)) \tag{6.17}$$

with $K' = K - Y$. This equality shows that for any $X \in \mathcal{S}_M(x)$, $T_x\phi \cdot X \in T_{x'}\phi(\mathcal{S}_M(x'))$, therefore $T_x\phi(\mathcal{S}_M(x)) \subseteq T_{x'}\phi(\mathcal{S}_M(x'))$. However, replacing x by x' and K by $-K$ on (6.17) we get $T_{x'}\phi(\mathcal{S}_M(x')) \subseteq T_x\phi(\mathcal{S}_M(x))$ so that we conclude the equality:

$$T_x\phi(\mathcal{S}_M(x)) = T_{x'}\phi(\mathcal{S}_M(x')) \tag{6.18}$$

By connectedness of the fibers $\phi^{-1}(y)$ any point x'' satisfying $\phi(x'') = \phi(x)$ can be reached by a concatenation of flows induced by vector fields in $\ker(T\phi)$. Transitivity of equality between sets implies that (6.18) holds for any two points $x, x' \in M$ such that $\phi(x) = \phi(x')$ from which commutativity of diagram (6.1) readily follows. □

By now it is already clear that projectability and local controlled invariance are equivalent concepts. We recall the notion of locally controlled invariant distribution:

Definition 6.4 (Locally Controlled Invariant Distributions [27]). *Let $\Sigma_M = (U_M, F_M)$ be a control system on manifold M and let $\mathcal{D}$ be a distribution on M. Distribution $\mathcal{D}$ is locally controlled invariant for F_M if for every $x \in M$ there exist an open set $O \subseteq M$, containing x and a local (feedback) isomorphism over the identity α such that in adapted coordinates (x, v) the new control system $F_M \circ \alpha$ satisfies:*

$$[F_M \circ \alpha(x, v), \mathcal{D}(x)] \subseteq \mathcal{D}(x) \tag{6.19}$$

for every (x, v) in the domain of α.

If a control section is projectable then locally we can always choose $\widehat{F_M} = F_M^l$ and therefore recover the conditions for local controlled invariance from [8]:

Theorem 6.5 ([8]). *Let Σ_M be a control system on manifold M satisfying* **AI** *and $\phi : M \longrightarrow N$ a surjective submersion with connected fibers. The distribution* $\ker(T\phi)$ *is locally controlled invariant for F_M iff $\mathcal{S}_M$ is projectable with respect to ϕ.*

Event though controlled invariance and projectability are equivalent concepts, we shall use the notion of projectability to describe control sections that behave well under projection instead of controlled invariance which was introduced to described certain control enforced invariance properties of control systems [42].

From the study of symmetries of nonlinear control systems [9, 26] it was already known that the existence of symmetries or partial symmetries implies controlled invariance of a certain distribution associated with the symmetries. This shows that control systems that are projectable comprise quotients by symmetry and controlled invariance. Furthermore, quotients induced by indistinguishability, as discussed in [35], are also of this type. However, there are also quotients for which projectability does not hold as we describe in the next section. Furthermore, as the existence of symmetries can be considered a rare phenomena [32], it is especially important to understand the structure of general non projectable quotients.

7. The Structure of Quotient Control Systems. We have already seen that the notion of **Con** morphisms generalizes the notion of projectable control sections. This shows that it is possible to quotient control systems whose control sections are not projectable. In this situation the map φ and the input space of the quotient control system will be significantly different from the projectable case. To understand this difference we start characterizing the fiber preserving lift φ of ϕ. Recall that if $f = (\phi, \varphi)$ is a morphism from Σ_M to Σ_N we have the following commutative diagram:

$$\begin{tikzcd} U_M \arrow[r, "\varphi"] \arrow[d, "F_M"'] & U_N \arrow[d, "F_N"] \\ TM \arrow[r, "T\phi"'] & TN \end{tikzcd} \tag{7.1}$$

Since φ is a surjective submersion we know that U_N is diffeomorphic to $U_M/\sim$, where $\sim$ is the regular equivalence relation induced by φ. This means that to understand the structure of U_N it is enough to determine the regular and involutive distribution on U_M given by $\ker(T\varphi)$. However, the map φ is completely unknown, so we will resort to the elements that are available, namely F_M and ϕ to determine $\ker(T\varphi)$. Differentiating[5] diagram (7.1) we get:

$$\begin{tikzcd} TU_M \arrow[r, "T\varphi"] \arrow[d, "TF_M"'] & TU_N \arrow[d, "TF_N"] \\ TTM \arrow[r, "TT\phi"'] & TTN \end{tikzcd} \tag{7.2}$$

from which we conclude:

$$\ker(TT\phi \circ TF_M) = \ker(TF_N \circ T\varphi) = \ker(T\varphi) \tag{7.3}$$

where the last equality holds since F_N is an immersion, provided that assumption **AIII** holds. We can now attempt to understand what is factored away and what is propagated from U_M to U_N since $\ker(T\varphi)$ is expressible in terms of F_M and ϕ. The first step is to clarify the relation between $\ker(T\varphi)$ and $\ker(T\phi)$. Since φ is a fiber preserving lift of ϕ the following diagram commutes:

$$\begin{tikzcd} TU_M \arrow[r, "T\varphi"] \arrow[d, "T\pi_{U_M}"'] & TU_N \arrow[d, "T\pi_{U_N}"] \\ TM \arrow[r, "T\phi"'] & TN \end{tikzcd} \tag{7.4}$$

which implies that:

$$T\pi_{U_M}(\ker(T\varphi)) \subseteq \ker(T\phi) \tag{7.5}$$

However, this only tells us that the reduction on M due to ϕ cannot be "smaller" than the reduction on the base space of U_M due to φ. This leads to the interesting phenomena which occurs when, for *e.g.* :

$$T\pi_{U_M}(\ker(T\varphi)) = \{0\} \subseteq \ker(T\phi) \tag{7.6}$$

The above expression implies that the base space of U_M is not reduced by φ. However, U_N is a fibered manifold with base space N and therefore the points reduced by ϕ must necessarily move to the fibers of U_N. This means that points $u, u' \in U_M$ satisfying $\pi_{U_M}(u) \neq \pi_{U_M}(u')$ will be mapped by φ to points satisfying $\pi_{U_N} \circ \varphi(u) = \pi_{U_N} \circ \varphi(u')$ and $\varphi(u) \neq \varphi(u')$. This will not happen if we can ensure the existence of a distribution $\mathcal{D} \subseteq \ker(T\varphi)$ such that $T\pi_{U_M}(\mathcal{D}) = \ker(T\phi)$. The existence of such a distribution turns out to be related with projectability. To show such fact we need the following characterization of $\ker(T\varphi)$:

LEMMA 7.1. *Let $\Sigma_M = (U_M, F_M)$ be a control system on manifold M, $\phi : M \longrightarrow N$ a surjective submersion, $\varphi : U_M \longrightarrow U_N$ a fiber preserving lift of ϕ which is*

[5] The operator sending manifolds to their tangent manifolds and maps to their tangent maps is an endofunctor on **Man**, also called the tangent functor [15].

also a submersion and assume that **AIII** *holds. Under these assumptions, a regular distribution* $\mathcal{D} \subseteq TU_M$ *belongs to* $\ker(T\varphi)$ *iff:*

$$[\widehat{F_M}, \mathcal{D}] \subseteq \ker(T\pi^*_{U_M}) \tag{7.7}$$

where $\widehat{F_M}$ *is any vector field in* F^e_M*.*

Proof. Assume the existence of the distribution $\mathcal{D}$, then $\mathcal{D} \subseteq \ker(T\varphi)$ is equivalent to:

$$TT\phi \circ TF_M(\mathcal{D}) = \{0\} \tag{7.8}$$

since **AIII** holds. Let $Z \in \mathcal{D}$ and denote by Z_t the flow of Z. Expression (7.8) implies that:

$$\frac{d}{dt}\Big|_{t=0} T_{\pi_{U_M} \circ Z_t(u)}\phi(F_M \circ Z_t(u)) = 0 \quad \Rightarrow \quad \frac{d}{dt}\Big|_{t=0} T_{Z_t(u)}\pi^*_{U_M}\phi(\widehat{F_M} \circ Z_t(u)) = 0 \tag{7.9}$$

for any $\widehat{F_M} \in F^e_M$ and for all $t \in \mathbb{R}$ such that Z_t is defined.

Noticing that $Z \in \mathcal{D} \subseteq \ker(T\varphi)$ implies $\varphi = \varphi \circ Z_t$ (since φ is constant on the leaves of the foliation induced by $\ker(T\varphi)$) and $\pi_{U_N} \circ \varphi = \phi \circ \pi_{U_M}$ by commutativity of diagram (4.1), we conclude that $\pi^*_{U_M}\phi$ is also Z_t invariant:

$$\pi^*_{U_M}\phi \circ Z_t = \phi \circ \pi_{U_M} \circ Z_t = (\pi_{U_N} \circ \varphi) \circ Z_t = \pi_{U_N} \circ \varphi = \phi \circ \pi_{U_M} = \pi^*_{U_M}\phi \tag{7.10}$$

Lemma 6.2 now ensures that:

$$T_{Z_t(u)}\pi^*_{U_M}\phi = T_u\pi^*_{U_M}\phi \circ T_{Z_t(u)}Z_{-t} \tag{7.11}$$

and expression (7.11) allows to rewrite (7.9) as:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} T_{Z_t(u)}\pi^*_{U_M}\phi(\widehat{F_M} \circ Z_t(u)) = 0 &\Leftrightarrow \frac{d}{dt}\Big|_{t=0} T_u\pi^*_{U_M}\phi(T_{Z_t(u)}Z_{-t} \circ \widehat{F_M} \circ Z_t(u)) = 0 \\
&\Leftrightarrow \frac{d}{dt}\Big|_{t=0} T_u\pi^*_{U_M}\phi(Z_t(u)^*\widehat{F_M}) = 0 \\
&\Leftrightarrow T_u\pi^*_{U_M}\phi([Z(u), \widehat{F_M}(u)]) = 0 \qquad (7.12)
\end{aligned}$$

or equivalently $[Z, \widehat{F_M}] \in \ker(T\pi^*_{U_M}\phi)$. Since Z is any vector field in $\mathcal{D}$ it follows that $[\widehat{F_M}, \mathcal{D}] \subseteq \ker(T\pi^*_{U_M}\phi)$ as desired.

The converse is similarly proved. □

Using the Lemma 7.1, we can now characterize the existence of a distribution $\mathcal{D} \subseteq \ker(T\varphi)$ projecting on $\ker(T\phi)$.

Proposition 7.2. *Let* $\Sigma_M = (U_M, F_M)$ *be a control system on manifold* M *satisfying* **AI**, $\phi : M \longrightarrow N$ *a surjective submersion and* $\varphi : U_M \longrightarrow U_N$ *a fiber preserving lift of* ϕ *which is also a submersion. There exists a regular distribution* $\mathcal{D}$ *on* U_M *satisfying* $\mathcal{D} \subseteq \ker(T\varphi)$ *and* $T\pi_{U_M}(\mathcal{D}) = \ker(T\phi)$ *iff* $\mathcal{S}_M$ *is projectable with respect to* ϕ*.*

Proof. We start by showing that projectability implies the existence of $\mathcal{D}$. If $\mathcal{S}_M$ is projectable with respect to ϕ, then for every $x, x' \in M$ such that $\phi(x) = \phi(x')$ we have that $T_x\phi(\mathcal{S}_M(x)) = T_{x'}\phi(\mathcal{S}_M(x'))$. This means that for any $x \in M$, $u \in \pi^{-1}_{U_M}(x)$ and $X \in \ker(T\pi^*_{U_M}\phi)$ there exists a $Y \in \ker(T\pi_{U_M})$ (recall that $\pi^{-1}_{U_M}(x)$ is connected by **AI**) such that:

$$T_{\pi_{U_M}\circ X_t(u)}\phi(F_M \circ X_t(u)) = T_x\phi(F_M \circ Y_t(u)) \tag{7.13}$$

for all $t \in \mathbb{R}$ such that the flows X_t and Y_t of X and Y are defined. Considering now $T\phi$ as a map between the manifolds TM and TN, the time derivative of $T_{\alpha(t)}\phi(\beta(t))$ for $(\alpha, \beta) : \mathbb{R} \longrightarrow TM$ provides $T_{(\alpha(t),\beta(t))}T_{\alpha(t)}\phi(T\beta(t))$. The same considerations applied to (7.13) at $t = 0$ give:

$$T_{(x,F_M(u))}T_x\phi \circ T_uF_M(X(u)) = T_{(x,F_M(u))}T_x\phi \circ T_uF_M(Y(u)) \tag{7.14}$$

which we rewrite as:

$$T_{(x,F_M(u))}T_x\phi \circ T_uF_M(X(u) - Y(u)) = 0 \tag{7.15}$$

by linearity of the involved maps. Since (7.15) is true for any $X \in \ker(T\pi^*_{U_M}\phi)$ we can define the distribution:

$$\mathcal{D} = \bigcup_{K\in\ker(T\phi)} \{Z = X - Y \ : \ X \in K^e \ \wedge \ Y \in \ker(T\pi_{U_M}) \text{ is such that (7.15) holds}\} \tag{7.16}$$

This distribution clearly satisfies:

$$TT\phi \circ TF_M(\mathcal{D}) = \{0\} \quad \Leftrightarrow \quad \mathcal{D} \subseteq \ker(T\varphi) \tag{7.17}$$

is regular since $\dim(\mathcal{D}) = \dim(\ker(T\phi))$ by construction, satisfies $T\pi_{U_M}(\mathcal{D}) = \ker(T\phi)$ also by construction and is therefore the desired distribution.

The converse is proved as follows. Assume the existence of the distribution $\mathcal{D} \subseteq \ker(T\varphi)$. By Lemma 7.1 we have:

$$[\widehat{F_M}, \mathcal{D}] \subseteq \ker(T\pi^*_{U_M})$$

since $T\pi_{U_M}(\mathcal{D}) = \ker(T\phi)$ it follows that $\mathcal{D} + T\pi_{U_M} = T\pi^*_{U_M}\phi$ and:

$$\begin{aligned}
&[\widehat{F_M}, \mathcal{D}] \subseteq \ker(T\pi^*_{U_M})\\
\Rightarrow\ &[\widehat{F_M}, \mathcal{D} + \ker(T\pi_{U_M})] \subseteq \ker(T\pi^*_{U_M}) + [\widehat{F_M}, \ker(T\pi_{U_M})]\\
\Rightarrow\ &[\widehat{F_M}, \ker(T\pi^*_{U_M}\phi)] \subseteq \ker(T\pi^*_{U_M}) + [\widehat{F_M}, \ker(T\pi_{U_M})]
\end{aligned}$$

which combined with Proposition 6.3 shows that $\mathcal{S}_M$ is projectable with respect to ϕ as desired. □

Collecting the results given by Lemma 7.1 and Proposition 7.2 we can now characterize both φ and U_N. Intuitively, we will use projectability to determine if the fibers of the quotient control system will receive states from M and Lemma 7.1 to characterize the amount of reduction induced by φ on the fibers of π_{U_M}.

Theorem 7.3 (Structure of Control Systems Quotients). *Consider a control system $\Sigma_M = (U_M, F_M)$ over a manifold M satisfying* **AI**, *$(f, \Sigma_N) = ((\phi, \varphi), (U_N, F_N))$ a quotient of Σ_M where ϕ has connected fibers, and assume that* **AIII** *holds. Let $\mathcal{E}$ be the involutive distribution defined by $\mathcal{E} = \{X \in \ker(T\pi_{U_M}) : [\widehat{F_M}, X] \in \ker(T\pi^*_{U_M}\phi)\}$, which we assume to be regular, denote by $R_\mathcal{E}$ the regular equivalence relation induced by $\mathcal{E}$ and let $\widehat{F_M}$ be any vector field in F^e_M. Under these assumptions:*

1. **Reduction from states to states and no reduction on inputs**
Fibered manifold U_N has base space diffeomorphic to N, and fibers $\pi^{-1}_{U_N}(y)$ diffeomorphic to $\pi^{-1}_{U_M}(x)$, $\phi(x)=y$ iff:
(i) *$\mathcal{S}_M$ is projectable with respect to ϕ;*
(ii) *$\mathcal{E}=\{0\}$.*

2. **Reduction from states to states and from inputs to inputs**
Fibered manifold U_N has base space diffeomorphic to N, and fibers $\pi^{-1}_{U_N}(y)$ diffeomorphic to $\pi^{-1}_{U_M}(x)/R_{\mathcal{E}}$, $\phi(x)=y$ iff:
(i) *$\mathcal{S}_M$ is projectable with respect to ϕ;*
(ii) *$\mathcal{E}\neq\{0\}$.*

3. **Reduction from states to inputs and no reduction on inputs**
Fibered manifold U_N has base space diffeomorphic to N, and fibers $\pi^{-1}_{U_N}(y)$ diffeomorphic to $\pi^{-1}_{U_M}(\phi^{-1}(y))$, $\phi(x)=y$ iff:
(i) *$\forall K\in\ker(T\pi^*_{U_M}\phi)\quad [\widehat{F_M},K]\notin\ker(T\pi^*_{U_M}\phi)$;*
(ii) *$\mathcal{E}=\{0\}$.*

4. **Reduction from states to inputs and from inputs to inputs**
Fibered manifold U_N has base space diffeomorphic to N, and fibers $\pi^{-1}_{U_N}(y)$ diffeomorphic to $(\pi^{-1}_{U_M}/R_{\mathcal{E}})(\phi^{-1}(y))$, $\phi(x)=y$ iff:
(i) *$\forall K\in\ker(T\pi^*_{U_M}\phi)\quad [\widehat{F_M},K]\notin\ker(T\pi^*_{U_M}\phi)$;*
(ii) *$\mathcal{E}\neq\{0\}$.*

Proof. We note that in all four cases the base space of U_N is diffeomorphic to N, since U_N is equipped with a surjective submersion $\pi_{U_N}:U_N\longrightarrow N$. We will, therefore, only discuss the characterization of fibers of π_{U_N}. We follow the enumeration of the theorem.

1 and **2:** Since φ is fiber preserving, we denote by $\varphi_x:\pi^{-1}_{U_M}(x)\longrightarrow\pi^{-1}_{U_N}(\phi(x))$ the restriction of φ to the fibers $\pi^{-1}_{U_M}(x)$, $x\in M$. We now claim that projectability implies $\varphi_x(\pi^{-1}_{U_M}(x))=\varphi_{x'}(\pi^{-1}_{U_M}(x'))$ for every $x,x'\in M$ such that $\phi(x)=\phi(x')$. Recall that, by definition of projectability, we have $T_x\phi(\mathcal{S}_M(x))=T_{x'}\phi(\mathcal{S}_M(x'))$. However, $\mathcal{S}_M(x)=F_M(\pi^{-1}_{U_M}(x))$ so that we conclude $T_x\phi\circ F_M(\pi^{-1}_{U_M}(x))=T_{x'}\phi\circ F_M(\pi^{-1}_{U_M}(x'))$. From assumption **AIII** follows injectivity of F_N, which combined with commutativity of the second diagram in (4.1) leads to $\varphi_x(\pi^{-1}_M(x))=\varphi_{x'}(\pi^{-1}_M(x'))$, as desired. This equality also shows that φ_x is surjective since φ is. Furthermore, we conclude that to characterize $\pi^{-1}_{U_N}(y)$ it suffices to characterize the image of φ_x for some $x\in\phi^{-1}(y)$. We now consider $\ker(T\varphi(x))\cap\ker(T\pi^{-1}_{U_M}(x))$, which by Lemma 7.1 is equal to $\mathcal{E}$ and is regular by assumption. This shows that φ_x has constant rank and we now claim that it is a submersion. We first note that projectability implies via Proposition 7.2 and (7.5) that $T\pi_{U_M}(\ker(T\varphi))=\ker(T\phi)$. This shows that:

$$\dim(\ker(T\varphi))=\dim(\ker(T\phi))+\dim(\mathcal{E})\tag{7.18}$$

On the other hand:

$$\begin{aligned}
\dim(\pi^{-1}_{U_N}(y)) &= \dim(U_N)-\dim(N)\\
&= \dim(U_M)-\dim(\ker(T\varphi))-\dim(N)\\
&= \dim(U_M)-\dim(\ker(T\phi))-\dim(\mathcal{E})-\dim(N) && \text{by (7.18)}\\
&= \dim(U_M)-\dim(\ker(T\phi))-\dim(\mathcal{E})-\dim(M)+\dim(\ker(T\phi))\\
&= \dim(U_M)-\dim(\mathcal{E})-\dim(M)\\
&= \dim(\pi^{-1}_{U_M}(x))-\dim(\mathcal{E})=\operatorname{rank}(\varphi_x)
\end{aligned}\tag{7.19}$$

which shows that φ_x is a submersion. We thus see that $\pi_{U_N}^{-1}(y)$ can now be identified with $\pi_{U_M}^{-1}(x)/R_{\mathcal{E}}$ since every vector field $X \in \mathcal{E}$ satisfies $T\pi_{U_M}(X) = 0$ and therefore induces a vector field on $\pi_{U_M}^{-1}(x)$. If $\mathcal{E} = \{0\}$ it follows that $\pi_{U_N}^{-1}(y) \cong \pi_{U_M}^{-1}(x)/R_{\mathcal{E}} \cong \pi_{U_M}^{-1}(x)$ as required by case 1.

Conversely, since the base of U_N is diffeomorphic to the quotient of M by the regular equivalence relation induced by $\ker(T\phi)$ and the fibers of π_{U_N} diffeomorphic to $\pi_{U_M}/R_{\mathcal{E}}$ it follows that $\ker(T\varphi)$ can be locally described by $\mathcal{D} \oplus \mathcal{E}$ for $\mathcal{D} = \ker(T\phi)^l$ and $T\pi_{U_M}(\mathcal{E}) = \{0\}$. From the existence of $\mathcal{D}$ and Proposition 6.3 follows projectability of $\mathcal{S}_M(x)$. Furthermore, if the fibers of π_{U_M} are diffeomorphic to the fibers of π_{U_N} we have $\mathcal{E} = \{0\}$ (case 1) and otherwise, $\mathcal{E} \neq \{0\}$ (case 2).

3 and **4:** From assumption (i) and Lemma 7.1 we conclude that there exists no $X \neq 0$ belonging to $K(T\varphi)$ such that $T\pi_{U_M}(X) \in \ker(T\phi)$. Since $T\pi_{U_M}(\ker(T\varphi)) \subseteq \ker(T\phi)$ (see the discussion before (7.5)) it follows that $T\pi_{U_M}(\ker(T\varphi)) = \{0\}$. Consequently, every $X \in \ker(T\varphi)$ is tangent to $\pi_{U_M}^{-1}(x)$ and $\varphi(U_M)$ is diffeomorphic to a fibered manifold with base space M and fibers $\pi_{U_M}^{-1}(x)/R_{\mathcal{E}}$. Let us denote by $\pi : \varphi(U_M) \longrightarrow M$ the projection from total space to base space which clearly satisfies $\pi_{U_M} = \pi \circ \varphi$. We now use the fact $\pi_{U_N} \circ \varphi = \phi \circ \pi_{U_M}$ with $\pi_{U_M} = \pi \circ \varphi$ to get $\pi_{U_N} \circ \varphi = \phi \circ \pi \circ \varphi$ and by surjectivity of φ we finally conclude the equality $\pi_{U_N} = \phi \circ \pi$. It is now clear that $\pi_{U_N}^{-1}(y) \cong \pi^{-1}(\phi^{-1}(y)) \cong (\pi_{U_M}^{-1}/R_{\mathcal{E}})(\phi^{-1}(y))$ as required by case 4. Case 3 is obtained by setting $\mathcal{E} = \{0\}$ and obtaining $\pi_{U_N}^{-1}(y) \cong (\pi_{U_M}^{-1}/R_{\mathcal{E}})(\phi^{-1}(y)) \cong \pi_{U_M}^{-1}(\phi^{-1}(y))$.

The converse is proved as follows. Since the fibers of π_{U_N} are diffeomorphic to $(\pi_{U_M}^{-1}/R_{\mathcal{E}})(\phi^{-1}(y))$ we see that points $u, u' \in U_M$ satisfying $\pi_{U_M}(u) \neq \pi_{U_M}(u')$ and $\phi \circ \pi_{U_M}(u) = \phi \circ \pi_{U_M}(u')$ also satisfy $\varphi(u) \neq \varphi(u')$. This shows that no vector field $X \neq 0$ in $\ker(T\pi_{U_M}^* \phi)$ belongs to $\ker(T\varphi)$ since otherwise different points in a trajectory of X would violate the above remark. The nonexistence of such vectors X implies, via Lemma 7.1, condition (i) and also implies that $\ker(T\varphi) = \mathcal{E}$. It then follows that if $\pi_{U_N}^{-1}(y) \cong \pi_{U_M}^{-1}(\phi^{-1}(y))$, then $\mathcal{E} = 0$ (case 3) and $\mathcal{E} \neq 0$ otherwise (case 4). ∎

We see that the notion of projectability is fundamentally related to the structure of quotient control systems. If the controlled section $\mathcal{S}_M$ is projectable then the inputs of the quotient control system are the same or a quotient of the original inputs. Projectability can therefore be seen as a structural property of a control system in the sense that it admits special decompositions [12, 27]. However, for general systems not admitting this special structure, that is, for systems that are not projectable, it is still possible to construct quotients by moving the neglected state information to the fibers. The states of the original system that are factored out by ϕ are regarded as control inputs in the quotient control system. This shows that from a hierarchical synthesis point of view, control systems that are not projectable are much more appealing since one can design control laws for the abstracted system, that when pulled-down to the original one are regarded as specifications for the dynamics on the neglected states [37].

8. Conclusions. In this paper quotients of fully nonlinear control systems were investigated. We showed that under mild conditions quotients exist and we characterized the structure of the quotient state/input space. This was achieved by introducing the category of control systems which was the natural framework to discuss quotients of control systems. One of the important ingredients of the characterization of quotients was the notion of projectable control section, which being equivalent to controlled invariance allowed to understand the difference between general quotients and those induced by symmetries, partial symmetries or controlled invariance.

There are still innumerous directions to be explored. The correct relations of the results presented in this work with the notion of extended control system [25] are not yet understood. This seems to lead to a possible generalization of the constructive procedures presented in [29] to compute quotients of nonlinear control affine systems to fully nonlinear control systems. Other directions being currently investigated include similar results for mechanical control systems where the Hamiltonian structure is preserved by the factorization process [36] as well as hybrid control systems [38].

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