December 2001

Abstractions of Hamiltonian Control Systems

Paulo Tabuada
Instituto Superior Técnico

George J. Pappas
University of Pennsylvania, pappasg@seas.upenn.edu

Abstractions of Hamiltonian Control Systems

Abstract
Given a control system and a desired property, an abstracted system is a reduced system that preserves the property of interest while ignoring modeling detail. In previous work, we considered abstractions of linear and nonlinear analytic control systems while preserving reachability properties. In this paper we consider the abstraction problem for Hamiltonian control systems, that is, we preserve the Hamiltonian structure during the abstraction process. We show how the mechanical structure of Hamiltonian control systems can be exploited to simplify the abstraction computations and we provide conditions under which the local accessibility properties of the abstracted Hamiltonian system are equivalent to the local accessibility properties of the original Hamiltonian control system.

Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Abstractions of Hamiltonian Control Systems[1]

Paulo Tabuada

Instituto de Sistemas e Robótica
Instituto Superior Técnico
1049-001 Lisboa - Portugal
tabuada@isr.ist.utl.pt

George J. Pappas

Department of EE
University of Pennsylvania
Philadelphia, PA 19104
pappasg@ee.upenn.edu

Abstract

Given a control system and a desired property, an abstracted system is a reduced system that preserves the property of interest while ignoring modeling detail. In previous work, we considered abstractions of linear and nonlinear analytic control systems while preserving reachability properties. In this paper we consider the abstraction problem for Hamiltonian control systems, that is, we preserve the Hamiltonian structure during the abstraction process. We show how the mechanical structure of Hamiltonian control systems can be exploited to simplify the abstraction computations and we provide conditions under which the local accessibility properties of the abstracted Hamiltonian system are equivalent to the local accessibility properties of the original Hamiltonian control system.

1 Introduction

Abstractions of control systems are important for reducing the complexity of their analysis or design. From an analysis perspective, given a large-scale control system and a property to be verified, one extracts a smaller *abstracted* system with equivalent properties. Checking the property on the abstraction is then equivalent to checking the property on the original system. From a design perspective, rather than designing a controller for the original large scale system, one designs a controller for the smaller abstracted system, and then refines the design to the original system while incorporating modeling detail.

A formal approach to a modeling framework of abstraction critically depends on whether we are able to *construct* hierarchies of abstractions as well as characterize conditions under which various properties propagate from the original to the abstracted system and vice versa. In [10], hierarchical abstractions of linear control systems were extracted using computationally efficient constructions. In the same spirit, abstractions of analytic control systems were considered in [11]. In this paper, we proceed in the spirit of [11] and consider abstractions of Hamiltonian control systems. Since Hamiltonian control systems are completely defined by controlled Hamiltonians we will simplify the computation of abstractions by performing them at the level of controlled Hamiltonians. On the other hand, to be able to relate the dynamics induced by the controlled Hamiltonians we need to restrict the class of abstracting maps to those that preserve the Hamiltonian structure. We also characterize abstracting maps for which the original and abstracted system are equivalent from a local accessibility point of view.

Reduction of mechanical control systems is a very rich and mature area [6, 7, 5, 8]. The approach presented in this paper is quite different from these established notions of reduction for mechanical systems. When performing an abstraction one is interested in ignoring irrelevant modeling details. In this spirit one factors the original model by group actions that do not necessarily represent symmetries. This extra freedom in performing reduction is balanced by the fact that information about the system is lost when performing an abstraction, whereas when reducing using symmetries no essential information is lost. However abstracting a control system and in particular an Hamiltonian one is always possible therefore leading to a more general notion of reduction.

2 Mathematical Preliminaries

In this section we review some basic facts from differential and Poisson geometry as well as control theory and Hamiltonian control systems, in order to establish consistent notation. The reader may whish to consult numerous books on these subjects such as [1, 2, 9, 4].

2.1 Differential Geometry

Let M be a differentiable manifold and T_xM its tangent space at $x \in M$. The tangent bundle of M is denoted by $TM = \cup_{x\in M} T_xM$ and π is the canonical projection map $\pi : TM \to M$ taking a tangent vector $X(x) \in T_xM \subset TM$ to the base point

[1]This work was performed while the first author was visiting the University of Pennsylvania. This research is partially supported by DARPA under grant F33615-00-C-1707, the University of Pennsylvania Research Foundation, and by Fundação para a Ciência e Tecnologia under grant PRAXIS XXI/BD/18149/98.

0-7803-7061-9/01/$10.00 © 2001 IEEE

$x \in M$. Dually we define the cotangent bundle as $T^*M = \cup_{x\in M} T_x^* M$, where T_x^*M is the cotangent space of M at x. Now let M and N be smooth manifolds and $\phi : M \to N$ a smooth map. Given a map $\phi : M \to N$, we denote by $T_x\phi : T_xM \to T_{\phi(x)}N$ the induced tangent which maps tangent vectors from T_xM to tangent vectors at $T_{\phi(x)}N$. A fiber bundle is a tuple $(B, M, \pi_B, U, \{O_i\}_{i\in I})$, where B, M and U are smooth manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map $\pi_B : B \to M$ is a surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of M such that for every $i \in I$ there exists a diffeomorphism $\Psi_i : \pi_B^{-1}(O_i) \to O_i \times U$ satisfying $\pi_o \circ \Psi_i = \pi_B$, where π_o is the projection from $O_i \times U$ to O_i. The submanifold $\pi^{-1}(x)$ is called the fiber at $x \in M$.

2.2 Poisson Geometry

For the purposes of this paper, it will be more natural to work with Poisson manifolds, rather than symplectic manifolds.[1] A Poisson structure on manifold M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$ called *Poisson bracket*, denoted by $\{f, g\}_M$ or simply $\{f, g\}$, satisfying the following identities:

$$\{f, g\} = -\{g, f\} \tag{2.1}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{2.2}$$

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{2.3}$$

A Poisson manifold $(M, \{,\}_M)$ is a smooth manifold M equipped with a Poisson structure $\{,\}_M$. Given a smooth function $h : M \to \mathbb{R}$, the Poisson bracket allows us to obtain an Hamiltonian vector field X_h with Hamiltonian h using:

$$\mathcal{L}_{X_h} f = \{f, h\} \qquad \forall f \in C^\infty(M) \tag{2.4}$$

where $\mathcal{L}_{X_h} f$ is the Lie derivative of f along X_h. Note that the vector field X_h is well defined since the Poisson bracket verifies the Leibnitz rule and therefore defines a derivation on $C^\infty(M)$ ([8]). Furthermore $C^\infty(M)$ equipped with $\{,\}$ is a Lie algebra, also called a Poisson algebra. Associated with the Poisson bracket there is a contravariant anti-symmetric two-tensor $B : T^*M \times T^*M \to \mathbb{R}$ such that:

$$B(x)(\mathrm{d}f, \mathrm{d}g) = \{f, g\}(x) \tag{2.5}$$

We say that the Poisson structure is non-degenerate if the map $B^\# : T^*M \to TM$ defined by:

$$\mathrm{d}g(B^\#(\mathrm{d}f)) = B(\mathrm{d}f, \mathrm{d}g) \tag{2.6}$$

is an isomorphism for every $x \in M$. Given a map $\phi : (M, \{,\}_M) \to (N, \{,\}_N)$ between Poisson manifolds, we say that ϕ preserves the Poisson structure or that ϕ is a Poisson map iff:

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi \tag{2.7}$$

for every $f, g \in C^\infty(N)$.

[1] More detailed expositions on symplectic and Poisson geometry can be found in [8, 3]

3 Hamiltonian Control Systems

Hamiltonian control systems are control systems endowed with additional structure. The extra structure comes from the fact that they model mechanical systems so they are essentially a collection of Hamiltonian vector fields parameterized by the control input. The following global and coordinate free description of Hamiltonian control systems is inspired from [12, 9].

Definition 3.1 (Control System) *An Hamiltonian control system $S_H = (U, H)$ consists of a control bundle $\pi_U : U \to M$ over a Poisson manifold $(M, \{,\})$ with non-degenerate Poisson bracket, and a smooth function $H : U \to \mathbb{R}$. With the Hamiltonian control system $S_H = (U, H)$ we associate the collection of Hamiltonians $\mathcal{H}$ as the collection of all smooth functions h of the form $h = H \circ \sigma$, where σ is a section[2] of $\pi_U : U \to M$. This family induces the set valued control section $\mathcal{D}_H$ of $\pi : TM \to M$ defined pointwise by $\mathcal{D}_H(x) = \cup_{h\in\mathcal{H}} X_h(x)$, where X_h satisfies $\mathcal{L}_{X_h} f = \{f, h\}$, for all $f \in C^\infty(M)$.*

The control space U is modeled as a fiber bundle since in general the control inputs available may depend on the current state of the system. The map H should be thought of as a controlled Hamiltonian since it (locally) assigns an Hamiltonian function to each control input. Note that the control bundle, and the controlled Hamiltonian completely specify the Hamiltonian control system. In particular, by fixing the control input, one obtains an Hamiltonian vector field. The concept of trajectories of is similar to ordinary control systems.

Definition 3.2 (Trajectories of Control systems) *A piece-wise smooth curve $c : I \to M$, $I \subseteq \mathbb{R}_0^+$ is called a* trajectory *of control system $S = (U, H)$, iff*

$$\frac{d}{dt}c(t) \quad \in \quad \mathcal{D}_H(c(t))$$

4 Hamiltonian Abstractions

Given an Hamiltonian control system[3] S_{H_M} defined on a Poisson manifold $(M, \{,\}_M)$ our goal is to construct a map $\phi : M \to N$, the *abstraction map* or *aggregation map* that will induce a new Hamiltonian control system S_{H_N} on the lower dimensional Poisson manifold $(N, \{,\}_N)$ having as trajectories $\phi(c^M)$, where c^M are S_{H_M} trajectories. The concept of abstraction map for continuous, not necessarily Hamiltonian, control systems is defined in [10] as:

Definition 4.1 (Abstracting Maps) *Let S_M and S_N be two control systems on manifolds M and N,*

[2] A section of $\pi_U : U \to M$ is a smooth map $\sigma : M \to U$ such that $\pi_U \circ \sigma$ =identity on M.

[3] From now on, $S_{H_M} = (U_M, H_M)$ or simply S_{H_M} denotes an Hamiltonian control system on Poisson manifold $(M, \{,\}_M)$.

respectively. A smooth surjective submersion $\phi : M \to N$ is called an abstraction or aggregation map iff for every trajectory c^M of S_M, $\phi(c^M)$ is a trajectory of S_N. Control system S_N is called a ϕ-abstraction of S_M.

From the above definition it is clear that an abstraction captures all the trajectories of the original system, but may also contain redundant trajectories. These redundant trajectories are not feasible by the original system and are therefore undesired. Clearly, it is difficult to determine whether a control system is an abstraction of another at the level of trajectories. One is then interested in a characterization of abstractions which is equivalent to Definition 4.1 but checkable. This leads to the notion of ϕ-related Hamiltonian control systems.

Definition 4.2 (ϕ-related Hamiltonian systems) *Let S_{H_M} and S_{H_N} be two Hamiltonian control systems defined on Poisson manifolds $(M, \{,\}_M)$ and $(N, \{,\}_N)$, respectively. Let $\phi : M \to N$ be a surjective Poisson submersion, and let ϕ_B be defined by $\phi_B = (B^\#_N)^{-1} \circ T\phi \circ B^\#_M$. Then Hamiltonian control system S_{H_M} is ϕ-related to S_{H_N} iff for all $x \in M$,*

$$\phi_B(\mathrm{d}\mathcal{H}_M(x)) \subseteq \mathrm{d}\mathcal{H}_N(\phi(x)) \tag{4.1}$$

Although the above definition is stated in terms of the exterior derivative of the family of Hamiltonians defining the control system, a canonical construction to be presented at Section 4.1 will allows us to compute $\mathcal{H}_N$ directly from from $\mathcal{H}_M$. The relationship between ϕ-abstractions and ϕ-related Hamiltonian control systems is now given.

Proposition 4.3 *Let S_{H_M} and S_{H_N} be Hamiltonian control systems on Poisson manifolds $(M, \{,\}_M)$ and $(N, \{,\}_N)$, respectively, and $\phi : M \to N$ a smooth Poisson map. Then S_{H_M} and S_{H_N} are ϕ-related if and only if S_{H_N} is a ϕ-abstraction of S_{H_M}.*

Before proving this result we recall the following Theorem from [10]:

Theorem 4.4 ([10]) *Let S_M and S_N be control systems on manifolds M and N, respectively, and $\phi : M \to N$ a smooth map. Then S_N is a ϕ-abstraction of S_M iff*

$$T_x\phi(\mathcal{D}_M(x)) \subseteq \mathcal{D}_N \circ \phi(x) \tag{4.2}$$

We now return to the proof of Proposition 4.3.

Proof: It is enough to show that if ϕ is a Poisson map then Definition 4.2 is equivalent to expression 4.2 for Hamiltonian control systems. The result then follows from Theorem 4.4. Definition 4.2 is equivalent to:

$$\begin{aligned} \phi_B(\mathrm{d}\mathcal{H}_M(x)) &\subseteq \mathrm{d}\mathcal{H}_N(\phi(x)) \Leftrightarrow \\ T_x\phi(B^\#_M(\mathrm{d}\mathcal{H}_M(x))) &\subseteq B^\#_N(\mathrm{d}\mathcal{H}_N(\phi(x))) \Leftrightarrow \\ T_x\phi(\mathcal{D}_{H_M}(x)) &\subseteq \mathcal{D}_{H_N}(\phi(x)) \end{aligned}$$

which is just (4.2). ∎

Proposition 4.3 tell us that the abstracting process can be characterized at the level of the controlled Hamiltonians. This result should be expected since the controlled Hamiltonians completely specify the dynamics of Hamiltonian control systems given a Poisson structure.

4.1 Canonical Construction

Given a Poisson map, Definition 4.2 provides us with a geometric definition for Hamiltonian abstractions which is useful conceptually but not computationally. We now present a canonical construction that will allow us to obtain an abstraction S_{H_N} from an Hamiltonian control system S_{H_M} and a Poisson map $\phi : M \to N$. Our construction is inspired from the canonical construction of [11], even though the construction presented here uses codistributions as opposed to distributions. This is natural for Hamiltonian systems since the differentials of the Hamiltonians capture all system information. We will assume from now on that all the Hamiltonian control systems are affine in controls, meaning that the associated set valued control sections will be affine distributions.

Definition 4.2 and, in particular, condition (4.1) require the union of all the values of $\phi_B(\mathrm{d}\mathcal{H}_M)$ evaluated at every $x \in \phi^{-1}(y)$. A way of constructing this union is to define another affine family of maps $\mathcal{F}$ such that $\mathrm{d}\mathcal{F}$ is *constant* on $\phi^{-1}(y)$ and furthermore satisfies $\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\mathcal{F}$. From this new family it suffices to compute $\mathrm{d}\mathcal{H}_N(y) = \mathrm{d}\mathcal{F}(x)$ for some $x \in \phi^{-1}(y)$ since $\mathrm{d}\mathcal{F}$ is the same for any $x \in \phi^{-1}(y)$. In other words, we would like to construct an affine space of maps $\mathcal{F}$ that we write as $\mathcal{F} = f_0 + F$, where f_0 is a smooth map and F is a linear space of smooth maps, such that:

1. $\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\mathcal{F}$
2. For all $x, x' \in M$ such that $\phi(x) = \phi(x')$, $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$.

Let $\mathcal{K}$ be the integrable distribution $Ker(T\phi)$. Then the leaves of the foliation $\mathcal{K}$ correspond to points on M that have the same image under ϕ. In this setting, we would like to design the family $\mathcal{F}$ so that the resulting codistribution $\mathrm{d}\mathcal{F}$ is invariant with respect to the vector fields in $\mathcal{K}$. This idea is captured in the following proposition:

Proposition 4.5 (Invariant Affine Codistributions) *An affine space $\mathcal{F} = f_0 + F$ of smooth functions satisfies $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ for all $x, x' \in M$ such that $\phi(x) = \phi(x')$ if and only if $\mathcal{L}_K \mathrm{d}f \in \mathrm{d}F$ for all $K \in \mathcal{K}$ and all maps $f \in \mathcal{F}$.*

Proof: We only provide a sketch due to space limitations. If $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ then $K_t^*\mathrm{d}f \in \mathcal{F}$ for the flow K_t of any vector field $K \in \mathcal{K}$ and any $f \in \mathcal{F}$. This means that $K_t^*\mathrm{d}f(x) = a_0(x,t)\mathrm{d}f_0(x) +$

$\sum_{i=1}^{n} a_i(x,t)\mathrm{d}f_i(x)$ for smooth scalar functions $a_i(x,t)$ and a basis $\{\mathrm{d}f_i\}_{i=1,2,\ldots,n}$ of $\mathrm{d}F$. Note that $a_0(x,t)$ is either 0 or 1 so that by continuity $\frac{d}{dt}a_0(x,t) = 0$ and we get by time differentiation at $t = 0$, $\mathcal{L}_K \mathrm{d}f = \dot{a}_i(x,t)\mathrm{d}f_i \in F$ as desired. Conversely we have that $\mathcal{L}_{\mathcal{K}}\mathrm{d}F \subseteq \mathcal{L}_{\mathcal{K}}\mathrm{d}\mathcal{F} \subseteq \mathrm{d}F$ and therefore the distribution $\mathrm{d}F$ is $\mathcal{K}$-invariant. Similarly one shows that the distribution $\mathbb{R}\mathrm{d}f_0 \oplus \mathrm{d}F$ is also $\mathcal{K}$-invariant and this means that $K_t^*\mathrm{d}f_0(x) = a_0(x,t)\mathrm{d}f_0 + \sum_{i=1}^{n} a_i(x,t)\mathrm{d}f_i(x)$. For $t = 0$ we see that $a_0(x,t) = 1$ and $a_i(x,t) = 0$. Since $\mathcal{L}_{\mathcal{K}}\mathrm{d}\mathcal{F} \subseteq \mathrm{d}F$ and $\mathcal{L}_K\mathrm{d}F \subseteq \mathrm{d}F$ imply that $\mathcal{L}_K\mathrm{d}f_0 \in \mathrm{d}F$ we get that $\dot{a}_0(x,t) = 0$ meaning that $a_0(x,t) = 1$ for all t and consequently $K_t^*\mathrm{d}f_0(x) \in \mathrm{d}\mathcal{F}(x)$. The assertion $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ for all x, x' such that $\phi(x) = \phi(x')$ now follows from $\mathcal{K}$-invariance. ∎

Proposition 4.5 motivates a canonical constructive procedure to obtain the abstracted Hamiltonian control system S_{H_N} given an Hamiltonian control system S_{H_M} and an abstracting Poisson map $\phi : M \to N$. If we denote the annihilating codistribution of $\mathcal{K}$ by:

$$\mathcal{K}^o = \{\beta \in T^*M \quad | \quad \beta(K) = 0 \quad \forall K \in \mathcal{K}\} \tag{4.3}$$

we can construct a collection of Hamiltonians $\mathcal{H}_N$ based on $\mathcal{H}_M$ as follows:

Definition 4.6 (Canonical construction) *Let $\phi : (M, \{,\}_M) \to (N, \{,\}_N)$ be a Poisson map between manifolds with non-degenerate Poisson brackets, and let $\mathcal{H}_M = h_0 + H$ be an affine space of Hamiltonians on M. Denote by $\overline{\mathcal{H}}_M$ the following affine family of smooth maps:*

$$\overline{\mathcal{H}}_M = h_0 \cup \overline{H} \cup \mathcal{L}_K\overline{H} \cup \mathcal{L}_K\mathcal{L}_K\overline{H} \cup \ldots \tag{4.4}$$

for $\overline{H} = H \cup \mathcal{L}_K h_0$ and all $K \in \mathcal{K}$. The collection of Hamiltonians $\mathcal{H}_N$ defined by:

$$\mathcal{H}_N = \overline{\mathcal{H}}_M \circ i \tag{4.5}$$

for any embedding $i : N \hookrightarrow M$ such that $B_M^{\#}(\mathcal{K}^o) \subseteq Ti(TN)$ is called canonically *ϕ-related to $\mathcal{H}_M$.*

It follows from the canonical construction 4.6 that the affine family of maps $\mathcal{H}_N$ canonically ϕ-related to $\mathcal{H}_M$ has the following properties

Proposition 4.7 (Minimal Abstraction) *The codistribution $\mathrm{d}\overline{\mathcal{H}}_M$ is the smallest codistribution satisfying:*

1. *$\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\overline{\mathcal{H}}_M$*
2. *For all $x_1, x_2 \in M$ such that $\phi(x_1) = \phi(x_2)$, $\mathrm{d}\overline{\mathcal{H}}_M(x_1) = \mathrm{d}\overline{\mathcal{H}}_M(x_2)$.*

and the Hamiltonian control system defined by $\mathcal{H}_N$ is the smallest Hamiltonian control system ϕ-related to $\mathcal{H}_M$.

As asserted by Proposition 4.7 the abstraction obtained by the canonical construction is the smallest Hamiltonian control system ϕ-related to S_{H_M}, therefore we are always able to compute the minimal ϕ-abstraction of any Hamiltonian control system given an abstracting Poisson map ϕ.

5 Local Accessibility Equivalence

In addition to propagating trajectories and Hamiltonians from the original Hamiltonian control system to the abstracted Hamiltonian system, we will investigate how accessibility properties can be preserved in the abstraction process. We first review several (local) accessibility properties for control systems [4, 9].

Definition 5.1 (Reachable sets [4]) *Let S_M be a control system on a smooth manifold M. For each $T > 0$ and each $x \in M$, the set of points reachable from x at time T, denoted by $Reach(x,T)$, is equal to the set of terminal points $c(T)$ of S_M trajectories that originate at x. The set of points reachable from x in T or fewer units of time, denoted by $Reach(x, \leq T)$ is given by $Reach(x, \leq T) = \cup_{t \leq T} Reach(x,T)$.*

Definition 5.2 *A control system S_M is said to be*

- *Locally accessible from x there is a neighborhood V of x such that $Reach(x,T)$ contains a non-empty open set of M for all $T > 0$ and $Reach(x,T) \subset V$.*
- *Locally accessible if it is locally accessible from all $x \in M$.*
- *Controllable if for all $x \in M$, $Reach(x,T) = M$ for some T.*

We now recall [9] that local accessibility properties of Hamiltonian control systems can be characterized by simple rank conditions of the Poisson algebra $\mathcal{P}(\mathcal{H}_M)$ associated with the affine collection of Hamiltonians $\mathcal{H}_M$ and defined as the smallest Poisson algebra containing $\mathcal{H}_M$ and satisfying $\{\mathcal{P}(\mathcal{H}_M), h_0\}_M \subseteq \mathcal{P}(\mathcal{H}_M)$.

Proposition 5.3 (Accessibility Rank Conditions) *Let S_{H_M} be an Hamiltonian control system on a Poisson manifold $(M, \{,\}_M)$ of dimension m and denote by $\mathcal{P}(\mathcal{H}_M)$ the Poisson algebra associated with the affine collection of Hamiltonians $\mathcal{H}_M$. Then*

- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$, then the control system S_{H_M} is locally accessible at $x \in M$.*
- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$ for all $x \in M$, then control system S_{H_M} is locally accessible.*
- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$ for all $x \in M$, $h_0 = 0$, $\mathcal{H}_M$ is symmetric, that is $h \in \mathcal{H}_M \Rightarrow -h \in \mathcal{H}_M$, and M is connected, then control system S_{H_M} is controllable.*

Theorem 4.4 immediately propagates local accessibility from the original Hamiltonian system to its abstraction.

Proposition 5.4 (Local Accessibility Propagation) *Let Hamiltonian control systems S_{H_M} and S_{H_N} be ϕ-related with respect to a Poisson map $\phi : M \longrightarrow N$. Then, if S_{H_M} is (symmetrically) locally accessible (at $x \in M$) then S_{H_N} is also (symmetrically) locally accessible (at $\phi(x) \in N$). Also, if S_{H_M} is controllable then S_{H_N} is controllable.*

We now determine under what conditions on the abstracting maps, local accessibility of the original system S_{H_M} is equivalent to local accessibility of its canonical abstraction S_{H_N}. In particular, we need to address the problem of propagating accessibility from the abstracted system S_{H_N} to the original system S_{H_M}. We start by exploring the relationship between the Poisson algebras associated with canonically ϕ-related Hamiltonian systems.

Lemma 5.5 *Let S_{H_N} be canonically ϕ-related to S_{H_M}, then for all $x \in M$ we have*

$$\phi_B\left(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))\right) = \mathrm{d}\mathcal{P}(\mathcal{H}_N)(\phi(x))$$

Using the above lemma whose proof we were forced to omit due to space limitations, accessibility equivalence between the two control systems can be now asserted.

Theorem 5.6 (Local Accessibility Equivalence) *Let S_{H_N} be canonically ϕ-related to S_{H_M}. If every vector field $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in \mathcal{P}(\mathcal{H}_M)$, then S_{H_M} is locally accessible if and only if S_{H_N} is locally accessible.*

Proof: We begin by showing how accessibility properties of S_{H_M} are propagated to S_{H_N}. Suppose that S_{H_M} is locally accessible, that is $\mathrm{d}\mathcal{P}(\mathcal{H}_M)(x) = T_x^*M$ for all $x \in M$, then by Lemma 5.5 $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(\phi(x)) = \phi_B(x)T_x^*M$. Since $\phi_B = (B_N^\#)^{-1} \circ T\phi \circ B_M^\#$ and both $B_N^\#$ and $B_M^\#$ are isomorphisms, and $T\phi$ is surjective, ϕ_B is also surjective. We conclude therefore that $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(y) = T_y^*N$, for all $y = \phi(x)$. But ϕ is surjective so S_{H_N} is locally accessible.

Let us now show how accessibility properties of S_{H_N} can be pulled back to S_{H_M}. We proceed by contradiction. Assume that every $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in \mathcal{P}(\mathcal{H}_M)$ and that S_{H_N} is locally accessible while S_{H_M} is not. Then $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(y) = T_y^*N$ and by Lemma 5.5 $\phi_B\mathrm{d}\mathcal{P}(\mathcal{H}_M)(x) = T_y^*N$ for all x such that $\phi(x) = y$. Since S_{H_M} is not locally accessible there exists some $g \in C^\infty(M)$ such that $\mathrm{d}g(x) \notin \mathrm{d}\mathcal{P}(\mathcal{H}_M)(x)$, but ϕ_B is surjective so $\mathrm{d}g(x)$ must belong to $Ker(\phi_B(x))$. Taking into consideration that $\mathrm{d}g(x) \in Ker(\phi_B(x)) \Leftrightarrow X_g(x) \in Ker(T_x\phi)$ we have a contradiction since we were assuming that all Hamiltonian functions of the vectors belonging to $Ker(T_x\phi)$ were also in $\mathcal{P}(\mathcal{H}_M)(x)$ and $g(x) \notin \mathcal{P}(\mathcal{H}_M)(x)$. This shows that S_{H_M} is in fact locally accessible from x. Since the argument does not depend on the particular point x, S_{H_M} is locally accessible. ∎

Corollary 5.7 *Let S_{H_N} be canonically ϕ-related to S_{H_M}. If every $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in \mathcal{P}(\mathcal{H}_M)$, $h_0 = 0$, both $\mathcal{H}_M$ and $\mathcal{H}_N$ are symmetric and furthermore both M and N are connected then S_{H_M} is controllable iff S_{H_N} is controllable.*

Theorem 5.6 provides moderate conditions to propagate accessibility properties in a hierarchy of abstractions. In fact, when dealing with affine Hamiltonian control systems we can always build a map ϕ satisfying the conditions of Theorem 5.6 by defining its kernel to be X_{h_i} for some i provided that the conjugate of h_i belongs to the Poisson algebra associated with the control system. An example of this construction is presented in the next section.

6 A spherical pendulum example

As an illustrative example, consider the spherical pendulum as a fully actuated mechanical control system. This system can be used to model, for example, the stabilization of the spinning axis of a satellite or a pan and tilt camera. Consider a massless rigid rod of length l fixed in one end by a spherical joint and having a bulb of mass m on the other end. The configuration space for this control system is S^2, however we will work locally with spherical coordinates described by $\theta \in]0,\pi[$ and $\alpha \in]0, 2\pi[$. The kinetic energy of the system is given by:

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\alpha}^2) \tag{6.1}$$

and the potential energy of the system is:

$$V = -mgl\cos\theta \tag{6.2}$$

Trough the Legendre transform of the Lagrangian $L = T - V$ one arrives at the Hamiltonian:

$$h_0 = \frac{1}{2ml^2}p_\theta^2 + \frac{1}{2ml^2\sin^2\theta}p_\alpha^2 - mgl\cos\theta \tag{6.3}$$

where p_θ is given by $p_\theta = ml^2\dot{\theta}$ and $p_\alpha = ml^2\sin^2\theta\,\dot{\alpha}$. Since the system is fully actuated the Hamiltonian control system S_{H_M} defined over $M = T^*S^2$ with the canonical Poisson bracket is given by:

$$H_M = h_0 + h_1u_1 + h_2u_2 \tag{6.4}$$

with $h_1 = \theta$ and $h_2 = \alpha$ and where u_1 and u_2 are the control inputs.

The drift vector field associated with h_0 is invariant under rotations around the vertical axis and could be reduced using this symmetry. However to emphasize the advantages to the abstraction method we will abstract away precisely the directions where there are no symmetries. Consider the local abstracting map:

$$\phi : T^*S^2 \quad \to \quad T^*S^1 \tag{6.5}$$

$$(\theta, \alpha, p_\theta, p_\alpha) \quad \mapsto \quad (\alpha, p_\alpha) \tag{6.6}$$

It is clear that $\theta \in \mathcal{P}(\mathcal{H}_M)$ and the conjugate variable to θ, p_θ, also belongs to $\mathcal{P}(\mathcal{H}_M)$ since $\{h_0, \theta\}_M = -\frac{1}{ml^2}p_\theta$, so the conditions of Theorem 5.6 are fulfilled. Following the steps of the canonical construction one computes:

$$\begin{aligned} \mathcal{L}_{K_1} h_0 &= \frac{\partial h_0}{\partial \theta} = \frac{1}{ml^2 \sin^2 \theta} \cot \theta \, p_\alpha \\ \mathcal{L}_{K_2} h_0 &= \frac{\partial h_0}{\partial p_\theta} = \frac{1}{ml^2} p_\theta \end{aligned} \tag{6.7}$$

where $K_1 = \frac{\partial}{\partial \theta}$ and $K_2 = \frac{\partial}{\partial p_\theta}$. The collection $\overline{H}$ is therefore given by $\overline{H} = H \cup \mathcal{L}_K h_0 = \{h_1, h_2, \mathcal{L}_K h_0\}$ and $\mathcal{H}_N = \overline{\mathcal{H}}_M \circ i$:

$$\begin{aligned} \mathcal{H}_N \;=\; \{ & \frac{1}{2ml^2}u_4^2 + \frac{1}{2ml^2 \sin^2 u_3} p_\alpha^2 - mgl \cos u_3 \, , \, u_3 u_1 \, , \\ & \alpha u_2 \, , \, \frac{1}{ml^2 \sin^2 u_3} \cot u_3 \, p_\alpha^2 \, , \, \frac{1}{ml^2} u_4 \} \end{aligned} \tag{6.8}$$

where we treated as formal new inputs u_3 and u_4 the abstracted variables θ and p_θ, respectively. This is possible since $\mathcal{H}_N$ does not depend on the embedding $i : N \hookrightarrow M$ and therefore the abstracted variables can take any value on $\phi^{-1}(y)$. In Equation 6.8 the first and third terms of the first function as well as the second and fifth functions are constants multiplied by inputs, this means that they are associated with the null vector field so we can discard them without altering the Hamiltonian control system defined by $\mathcal{H}_N$. We thus obtain:

$$\mathcal{H}_N = \{ \frac{1}{2ml^2 \sin^2 u_3} p_\alpha^2 \, , \, \alpha u_2 \, , \, \frac{1}{ml^2 \sin^2 u_3} \cot u_3 \, p_\alpha^2 \} \tag{6.9}$$

which can be further simplified by discarding the third function since its exterior derivative is linearly dependent on the exterior derivative of the first function. We finally get:

$$\mathcal{H}_N = \{ \frac{1}{2ml^2 \sin^2 u_3} p_\alpha^2 \, , \, \alpha u_2 \} \tag{6.10}$$

and after renaming the control inputs to $v_1 = u_3$ and $v_2 = u_2$ we obtain the following Hamiltonian control system defined by $\mathcal{H}_N$:

$$\dot{\alpha} \;=\; \frac{1}{ml^2 \sin^2 v_1} p_\alpha \tag{6.11}$$

$$\dot{p}_\alpha \;=\; v_2 \tag{6.12}$$

which is a controllable Hamiltonian control system on $N = T^*S^1$.

7 Conclusions

In this paper, we have presented a hierarchical abstraction methodology for Hamiltonian nonlinear control systems. The extra structure of mechanical systems was utilized to provide constructive methods for generating abstractions while maintaining the Hamiltonian structure. Furthermore we have characterized accessibility equivalence through easily checkable conditions.

These results are very encouraging for hierarchically controlling mechanical systems. Refining controller design from the abstracted to the original system is clearly an important issue to address. Other research topics under current research include the propagations of nonholonomic constraints among the different levels of the hierarchy, and better understanding the relationship between Hamiltonian abstractions and more established notions of reduction based on symmetries.

References

[1] R. Abraham, J. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis and Applications.* Applied Mathematical Sciences. Springer-Verlag, 1988.

[2] W. M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Academic Press, 1975.

[3] Ana Cannas da Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics.* Springer-Verlag, 2001.

[4] Velimir Jurdjevic. *Geometric Control Theory.* Number 51 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[5] W. S. Koon and J. E. Marsden. The poisson reduction of nonholonomic mechanical systems. *Reports on Mathematical Physics*, 42:101–134, 1998.

[6] J. E. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on Mathematical Physics*, 5:121–120, 1974.

[7] Jerold Marsden and Tudor Ratiu. Reduction of poisson manifolds. *Letters in Mathematical Physics*, 11:161–170, 1986.

[8] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry.* Texts in Applied Mathematics. Springer-Verlag, 1999.

[9] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer-Verlag, 1995.

[10] George J. Pappas, Gerardo Lafferriere, and Shankar Sastry. Hierarchically consistent control systems. *IEEE Transactions on Automatic Control*, 45(6):1144–1160, June 2000.

[11] George J. Pappas and Slobodan Simic. Consistent hierarchies of nonlinear abstractions. In *Proceedings of the 39th IEEE Conference in Decision and Control.* Sydney, Australia, December 2000.

[12] A. J. van der Schaft. Hamiltonian dynamics with external forces and observations. *Mathematical Systems Theory*, 15:145–168, 1982.