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Effects of Delay on the Functionality of Large-scale Networks

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Abstract

Networked systems are common across engineering and the physical sciences. Examples include the Internet, coordinated motion of multi-agent systems, synchronization phenomena in nature etc. Their robust functionality is important to ensure smooth operation in the presence of uncertainty and unmodelled dynamics. Many such networked systems can be viewed under a unified optimization framework and several approaches to assess their nominal behaviour have been developed. In this paper, we consider what effect multiple, non-commensurate (heterogeneous) communication delays can have on the functionality of large-scale networked systems with nonlinear dynamics. We show that for some networked systems, the structure of the delayed dynamics allows functionality to be retained for arbitrary communication delays, even for switching topologies under certain connectivity conditions; whereas in other cases the loop gains have to be compensated for by the delay size, in order to render functionality delay-independent for arbitrary network sizes. Consensus reaching in multi-agent systems and stability of network congestion control for the Internet are used as examples. The differences and similarities of the two cases are explained in detail, and the application of the methodology to other technological and physical networks is discussed.

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I. INTRODUCTION

In the past few years, there has been an increasing interest in understanding the collective behavior of systems that are formed by *arbitrary* (both in size and in structure) interconnections of smaller subsystems, also known as large-scale, networked systems [1]. In general, such systems behave differently than when the individual subunits are allowed to evolve on their own and research efforts concentrate in understanding how the subsystem interaction and the network interconnection affects the overall system behaviour. Towards this goal, there have been several approaches to develop unified frameworks for understanding and analyzing the functionality of such networks.

Apart from answering analysis questions, the task of *designing* control laws for the subsystems so that the networked system meets certain design objectives is also under intense research. Here the challenge is that the interconnection topology and its size may not be known *a priori* (apart from possible structural constraints), or may even change with time – therefore the designed system has to have *scalable*

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functional properties. These could be robust stability, performance etc., but also some times permanence and invariance.

There are several examples of systems that occur in nature which possess these features. One particular example is *oscillator synchronization* [2], i.e., the way oscillating objects behave differently when they are isolated but synchronize their frequencies or even lock their phases when coupled together to form a network. The synchronization framework has been used to explain many physical phenomena in which subsystems have the tendency to synergize and 'agree' to perform a common task when coupled together: the way pacemaker cells generate and pace the heartbeat, how fireflies flash in sync, crickets chirp in unison, crowds clap in synchrony etc. Beyond these physical examples are technological systems that have been designed to reach 'agreement' or 'consensus' for arbitrary system sizes. One such example is coordinated motion of multivehicle systems for arbitrary interconnection topologies which, under certain conditions, can also be allowed to change [3]–[8]. The design procedure results in closed loop systems that have dynamics closely related to the models used to describe synchronization in oscillator networks, which is not surprising. The same holds for the related issue of self-ordered particle motion [9], [10], asynchronous distributed computation [11] etc. Other examples that are closely related are agreement [12], and others [13]–[17] etc.

Another technological system whose design objective is scalable functionality (equilibrium optimality and stability in this case) for arbitrarily sized interconnections is Internet congestion control [18]. Here, the desired network properties can be formulated as a centralized convex optimization problem, drawing ideas from Economics and Utility maximization [19], [20]. A distributed solution can then be obtained using duality arguments, and a subgradient algorithm can be used to design TCP/AQM algorithms to steer the system to the optimal solution which in turn is rendered a globally asymptotically stable equilibrium for the overall system [21].

In this paper we first consider an optimization framework under which the behaviour of networked systems can be understood. In this framework, the designed distributed dynamics follow simple (sub)gradient rules on the Lagrangian in a dual decomposition procedure. In particular, we illustrate how dynamical models used in cooperative consensus, the well-known Kuramoto model for synchronization and the solution to general network flow problems can be viewed under this framework.

We then investigate the effect of communication delays [22]–[24] in the interactions of agents/vertices in large-scale networks described by nominal, nonlinear models. Time-delays are an indispensable feature of networked systems which is many times neglected to facilitate analysis. Delays can, e.g., be used to model the effect of propagation of state information between interacting agents or to capture the time required for information to be sent and acknowledgements received in an Internet network. Although many times delays are small, it is well known that they can deteriorate the system's performance or even destabilize it and this paper looks at the effect of delays on the functionality of two large-scale systems.

The first is coordination of multi-agent systems, for which the interactions between the agents are delayed by heterogeneous (multiple and non-commensurate) time-delays. We assume that the nominal

dynamics are nonlinear, continuous and locally passive, and that the graph capturing the interaction topology is directed but contains a spanning tree. Previous results have shown that coordination can be achieved for the un-delayed system [25], [26] and for the discrete-time delayed system with discrete linear dynamics [27] and switching topologies [28]. In [29], [30] the authors used a frequency domain analysis for a linear, continuous time system to show stability independent of delays, while the authors in [12] used nonlinear undelayed dynamics with a linear control law and a contraction theorem to show consensus is independent of delay, which was also identified in [31]. Here, we will build on our previous work [32], [33] on synchronization in oscillator networks to show that under the above assumptions, the consensus set is asymptotically attracting for nonlinear, continuous time dynamics and heterogeneous time-delays. Another problem that we will be investigating is whether coordination can be ensured for *switching* topologies within an admissible set even if delays are present in the system. This question is addressed using ideas from the stability analysis of systems with arbitrary switching with no chattering and with a finite dwell time, for which a sufficient but many times conservative condition is the existence of a common Lyapunov function for all the possible system instances (topologies) [34]. As we will see in the sequel, under (1) a dwell time condition and (2) that there exist contiguous intervals over which the *union graph* has a spanning tree, it can be established that the consensus set is asymptotically attracting.

The second class of systems that this paper is concerned with is nonlinear congestion control schemes for the Internet for arbitrary interconnection topologies with heterogeneous time-delays. To obtain stability conditions for the linearization of such systems, the multivariable Nyquist criterion was used in [35]. Here, we obtain conditions for the nonlinear system descriptions using appropriately structured Lyapunov functions, obtaining conditions similar to the ones obtained by linearization, again for arbitrary topologies under the assumption that the routing matrix of the network is full rank. Related work established stability for primal congestion control algorithms for arbitrary networks and heterogeneous time-delays [36], [37].

Consensus in multiagent systems and stability of network congestion control schemes are affected in different ways as the delay size is increased. In the case of coordination of multivehicle systems, the size of the (finite) delay does not affect attractivity to the consensus set (a *delay-independent* property), whereas in the case of the network congestion control schemes we will be investigating, the loop gains have to be compensated by the delay size to render a stability condition that is *delay-dependent*, non-dimensional in the delay, and hence delay-independent. Many other systems have these features, and we will be commenting on their properties in this paper, as well as the tools that can be used to analyze them. In particular, eventhough the tools we will be using will be Lyapunov-based, a so-called Lyapunov-Razumikhin function will be used in the case of consensus for multiagent systems, while a Lyapunov-Krasovskii approach will be used to ensure stability of a class of Internet congestion control schemes.

The paper is organized as follows. In Section II we consider a unified optimization framework for understanding the behaviour and designing dynamics for networked systems. In Section III we present our results on multi-agent system coordination and in Section IV the results on the stability of network congestion control schemes. The tools we will be using are standard and can be found in, e.g., [22]; a

short review can be found in the Appendix. In section V we discuss the importance of the results and provide an outlook for other system descriptions, concluding the paper in section VI.

A. Notation

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $|\cdot|$ (the 2-norm unless otherwise stated). Let $C = C([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The norm on C is defined as $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Moreover, let $\rho \geq 0$ and $x \in C([-\tau, \rho], \mathbb{R}^n)$; then for any $t \in [0, \rho]$, we define a segment $x_t \in C$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices $\mathcal{V} = \{v_i\}$, $i \in \mathcal{I} = \{1, \ldots, N\}$, and a set of edges $\mathcal{E} \subseteq \{(v_i, v_j) \mid v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$. If $v_i, v_j \in \mathcal{V}$ and $(v_i, v_j) \in \mathcal{E}$, then there is a directed edge from v_i to v_j and we say that v_i is the parent of v_j. A graph is said to be undirected if $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. No graph in this paper has a self-loop, i.e., $(v_i, v_i) \notin \mathcal{E}$. The adjacency matrix $A = [a_{ij}]$ of a graph $\mathcal{G}$ is an $N \times N$ real matrix such that $a_{ij} = 1 \Leftrightarrow (v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. If the graph is undirected, then $A = A^T$.

A directed path from vertex v_i to vertex v_j is a sequence of edges starting from v_i and ending at v_j so that consecutive edges belong to $\mathcal{E}$. A graph $\mathcal{G}$ is said to be strongly connected if there is a directed path between any two vertices in it. A directed tree is a directed graph in which for every vertex v_j there is exactly one v_i so that $(v_i, v_j) \in \mathcal{E}$ (i.e., v_j has exactly one parent) except the root of the tree. A spanning tree of a directed graph is a directed tree with the same vertex set but perhaps different edge set than the directed graph. In that case we say that a graph contains a spanning tree.

II. NETWORKED SYSTEMS FROM AN OPTIMIZATION VIEWPOINT

In this section, we will present a series of examples of networked systems that can be understood as solving large-scale, centralized optimization problems in a decentralized way [11], [21]. This standpoint can help us understand the properties of the dynamics of the agents and can shed some light on the structures of candidate scalable Lyapunov functions that can assess the network's nominal stability; but also its robustness to communication delays and time-varying topologies, as we will see in later sections.

Consider N nodes/agents, interacting over a network whose topology is given by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. These agents are collectively trying to solve a centralized, large-scale optimization problem in a decentralized way, i.e., agent j is allowed to send information to agent i only if $(v_j, v_i) \in \mathcal{E}$. In what follows, we assume that each node/agent holds a state $x_i \in \mathbb{R}$. Two important examples of collective tasks are:

- *Agreement Problems:* Such problems include formation control and flocking, cooperative transportation, synchronization, load balancing etc. In this case, the collective task is to minimize some norm of the difference of the states of all the agents, sometimes subject to state constraints. The aim is to achieve agreement, even if not all agents communicate with one another, and the question is under which conditions is this achievable.

- *Network Flow Problems:* Such problems include transportation, shortest path problems, assignment etc. In this case the collective task is to maximize some benefit function, which usually is an aggregate sum of individual benefit functions of the users, while certain state constraints are satisfied.

The solution methodology that is generally followed in order to decompose these problems and decentralize the computation, is to use ideas from convex optimization and Lagrange duality in conjunction with appropriate gradient/subgradient algorithms. This approach, and depending on the original problem structure, may lead to the construction of decentralized dynamics that the agents can use to steer the networked system towards or very close to the optimal value of the original, centralized optimization problem.

In particular, given a centralized *convex* optimization problem of the form:

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{such that} \quad & g_i(x) = 0, \quad i = 1, \ldots, N \\ \text{and} \quad & \sum_{i=1}^{N} R_{ji} h_i(x) \le c_j, \quad j = 1, \ldots, L \end{aligned} \tag{1}$$

where R is some routing matrix describing the graph structure

$$R_{ji} = \begin{cases} 1 & \text{if node } i \text{ is on edge } j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and the differentiable functions f, g_i, h_i and the constants c_j are given, the following procedure can be followed:

- *Dual Decomposition:* In this step, a dual program is formulated. First, the Lagrangian function is defined:

$$L(x, \nu, p) = f(x) + \sum_{i=1}^{N} \nu_i g_i(x) + \sum_{j=1}^{L} p_j \left(\sum_{i=1}^{N} A_{ji} h_i(x) - c_j \right)$$

where ν_i and $p_j \ge 0$ are (dual) Lagrange multipliers for the equality and inequality constraints (the x are called primal variables). Thereafter, the following function is computed:

$$g(\nu, p) = \inf_x L(x, \nu, p) \tag{3}$$

and the dual optimization problem is formulated:

$$\begin{aligned} \text{Maximize} \quad & g(\nu, p) \\ \text{such that} \quad & p_j \ge 0, \quad j = 1, \ldots, L \end{aligned}$$

Under the condition that the original problem is strictly feasible, then there is no duality gap (the original and the dual problems have the same optimum) and hence the dual problem, if easier to solve, can be solved instead of the original problem.

- *Gradient/Subgradient Algorithm:* One way to ensure that the dual problem is solved is to follow a gradient or subgradient descent (or ascent) towards the optimal point. In particular, for the dual optimization problem shown above, one can propose

$$\dot{p}_j = \alpha_j \left[\frac{\partial g}{\partial p_j} \right]_{p_j}^+, \quad j = 1, \dots, L$$

$$\dot{\nu}_i = \beta_i \frac{\partial g}{\partial \nu_i}$$

  where α_j and β_i are positive constants and the notation

$$[g(y)]_y^+ = \begin{cases} g(y) & \text{if } y > 0, \\ \max(0, g(y)) & \text{otherwise.} \end{cases}$$

  has been used; such a projection ensures that the p_j's stay positive.

- *Nominal Stability:* Once these dynamics have been constructed, then we can seek a Lyapunov function in order to establish the nominal stability properties of the equilibrium (which is also the optimal point of the original problem). This can be done by taking advantage of the fact that the dynamics were constructed following a gradient descent; in particular, a reasonable candidate for a Lyapunov function would be the energy-like function $-g(\nu, p)$ plus a term that imposes a barrier-like function on the primal variables x, that are constrained to infimize the Lagrangian through (3).

We now present representative examples of the above and review briefly previous and develop some new results on 'reverse-engineering' the optimization problem behind each. We also comment on the structure and existence of Lyapunov functions that show scalable stability of these systems to arbitrary sizes, setting up the stage for robust stability analysis with respect to communication delays between the agents later on in this paper.

Example 2.1: Consensus in Multi-agent Systems. Perhaps the easiest problem to be cast in an optimization framework, as already discussed in [6] and other articles, is the one in which nodes of the graph are agents which try to achieve consensus through bidirectional communication, with no constraints. Suppose the collective objective takes the form

$$\min V(x) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N A_{ji} \int_0^{x_j - x_i} f_{ji}(y) dy$$

where $f_{ji} = f_{ij}$ are continuous functions with $f_{ji}(0) = 0$ for all $i, j = 1, \dots, N$ which are globally passive, i.e., $y f_{ij}(y) > 0$ for all $y \neq 0$ and $f_{ij}(y) = 0$ if and only if $y = 0$. A simple gradient descent scheme reveals the following dynamics for agent i:

$$\dot{x}_i(t) = -k_i \frac{\partial V}{\partial x_i} = k_i \sum_{j=1}^N A_{ji} f_{ji} \left(x_j(t) - x_i(t) \right) \tag{4}$$

where the k_i are positive constants. The equilibrium (set) x^* of (4) satisfies:

$$\sum_{j=1}^N A_{ji} f_{ji} \left(x_j^*(t) - x_i^*(t) \right) = 0$$

and is $x^* = c\mathbf{1}$ where c is a constant and $\mathbf{1}$ is the vector of ones of dimension N. Denote this set by $\mathcal{X}$:

$$\mathcal{X} = \{x \in \mathbb{R}^n | x = c\mathbf{1}, \text{ for } c \in \mathbb{R}\} \tag{5}$$

Moreover, the function $V(x)$ is a Lyapunov function that can be used to conclude that the equilibrium set is asymptotically attracting, as in essence we have constructed a *gradient system*. In particular, we note that the time derivative of $V(x) > 0$ (with respect to the equilibrium set) takes the form:

$$\frac{dV(x)}{dt} = \sum_{i=1}^{N} \frac{\partial V}{\partial x_i} \dot{x}_i = -\frac{1}{k_i} \sum_{i=1}^{N} \dot{x}_i^2 \leq 0.$$

A simple LaSalle argument shows that indeed, the equilibrium set is asymptotically attracting, as the largest invariant set in $\dot{V} = 0$ is the equilibrium set, $x^* = c\mathbf{1}$.

The exact value of c depends on the initial condition; if the functions f_{ij} are such that $f_{ij}(y) = -f_{ij}(-y)$, then consensus is on the mean of the initial conditions. The case of a directed graph or the case of f_{ij} that do not satisfy the above conditions are much more complicated – see [38] for more details.

Also, the convergence rate of the algorithms will depend on the structure of the functions f_{ij}; from the linearization, we know that the value of f'_{ij} plays a role in the convergence; another important factor is the algebraic connectivity of the graph [5], [38].

In section III we will be considering multi-agent systems with the above dynamics, but we will concentrate on directed graphs, with *locally passive*, non-symmetric f_{ij}'s:

Definition 2.2: A C^1 function $f : \mathbb{R} \to \mathbb{R}$ is *locally passive*, if $yf(y) > 0$ for all $y \in [-\sigma^-, \sigma^+] \subset \mathbb{R}$ apart from $y = 0$, where $\sigma^- > 0$ and $\sigma^+ > 0$.

Assumption 2.3: The functions $f_{ij} : \mathbb{R} \to \mathbb{R}$ are *locally passive* on $[-\sigma_{ij}^-, \sigma_{ij}^+]$ for some $\sigma_{ij}^- > 0$ and $\sigma_{ij}^+ > 0$, for all $i, j = 1, \ldots, N$.
We will also define γ as:

$$\gamma = \min_{i,j=1,\ldots,N} \left(\sigma_{ij}^-, \sigma_{ij}^+\right) \tag{6}$$

In this case, it may be possible that there are equilibria other than the consensus set depending on the structure of the topology, the allowable initial conditions etc. For example, see [39], where the ring topology with six vertices and $f_{ij}(y) = \sin(y)$ yields a second equilibrium point. However, one can restrict the set of initial conditions or the type of functions f_{ij} (e.g., globally passive) to ensure that there is only one equilibrium set, which is the agreement set.

Moreover, the use of directed graphs makes the estimation of convergence rates and consensus values much more complex, and we will not be considering these questions in the current work.

Example 2.4: Synchronization in Oscillator Networks. A related problem is *synchronization*, in which oscillators attempt to entrain in frequency, and lock in phase. Consider N such oscillators with phases $\theta_i \in [0, 2\pi)$ and natural frequencies ω_i, which are coupled together over a network whose topology is given by an undirected graph $\mathcal{G}$.

Suppose the task of the oscillators is to minimize the misalignment of the phases, $\theta_{ij} = \theta_i - \theta_j$, but at the same time satisfy certain equality constraints:

$$\text{Minimize} \quad V(\theta) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} (1 - \cos \theta_{ij})$$

$$\text{such that} \quad \frac{\omega_i}{K} = \frac{1}{N} \sum_{j=1}^{N} A_{ij} \sin \theta_{ij}, \quad \text{for all } i = 1, \ldots, N.$$

Denoting by Θ the vector with elements θ_{ij}, the Lagrangian function is then

$$L(\Theta, \nu) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} [A_{ij} (1 - \cos \theta_{ij}) - A_{ij} \nu_i \sin \theta_{ij}] + \frac{1}{2} \sum_{i=1}^{N} \nu_i \frac{N \omega_i}{K}$$

where ν_i are constant Lagrange multipliers. The dual function takes the form

$$g(\nu) = \inf_{\Theta} L(\Theta, \nu) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \left[1 - \frac{1}{\sqrt{1 + (\nu_i - \nu_j)^2}} \right] + \frac{1}{2} \sum_{i=1}^{N} \nu_i \frac{N \omega_i}{K}$$

and therefore gradient dynamics on ν that will converge to the social optimum take the form:

$$\frac{d\nu_i}{dt} = k_i \frac{\partial g}{\partial \nu_i} = k_i \frac{N}{K} \omega_i - k_i \sum_{j=1}^{N} A_{ij} \frac{\nu_i - \nu_j}{\sqrt{1 + (\nu_i - \nu_j)^2}}$$

where $k_i > 0$. The latter dynamics are distributed and are very much related to the standard Kuramoto model [2], [64], which reads:

$$\frac{d\theta_i}{dt} = \omega_i - \frac{K}{N} \sum_{j=1}^{N} A_{ij} \sin(\theta_i - \theta_j)$$

if we use the first order condition

$$\nu_i - \nu_j = \tan(\theta_i - \theta_j)$$

and tune appropriately the k_i's. Asymptotic attractivity of the agreement set can be shown using an appropriate Lyapunov function whose structure, in the case of identical oscillators, looks very similar to $V(\theta)$ [40].

Example 2.5: Network Flow Problems. Problems in this class have as an objective to maximize a function of the state in a network with many nodes, while certain (usually inequality) constraints are satisfied. The problem formulation is very similar to (1) and takes the form:

$$\text{Maximize} \quad \sum_{i=1}^{N} f_i(x_i) \tag{7}$$

$$\text{such that} \quad \sum_{i=1}^{N} R_{ji} g_i(x_i) \le c_j, \quad j = 1, \ldots, L \tag{8}$$

where R is a routing matrix. Implicit in the above is the assumption that the objective function is the aggregate sum of 'local' objective functions, while the constraints couple the various nodes together, hence making the problem fully centralized. We can now formulate the Lagrangian:

$$L(x, p) = \sum_{i=1}^{N} f_i(x_i) - \sum_{i=1}^{N} \sum_{j=1}^{L} R_{ji} p_j g_i(x_i) + \sum_{j=1}^{L} p_j c_j$$
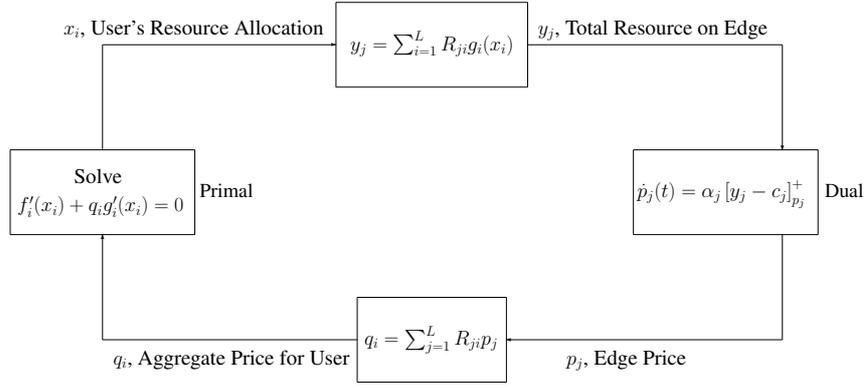
Fig. 1. Solving a centralized problem using dual decomposition.

The dual function can be computed to be

$$g(p) = \sup_x L(x, p) = \sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} \sum_{j=1}^{L} R_{ji} p_j g_i(x_i^*) + \sum_{j=1}^{L} p_j c_j$$

where $p_j \geq 0$, $j = 1, \ldots, L$ and x_i^* solve:

$$f_i'(x_i) + g_i'(x_i^*) q_i = 0, \quad i = 1, \ldots, N \tag{9}$$

where $q_i = \sum_{j=1}^{L} R_{ji} p_j$. If strong duality holds, i.e., when the original problem is a convex optimization problem with a strict feasible solution, then the above problem can be solved distributively using a subgradient algorithm on the dual variables,

$$\dot{p}_j(t) = \alpha_j \left[y_j - c_j \right]_{p_j}^+ \tag{10}$$

where $y_j = \sum_{i=1}^{N} R_{ji} g_i(x_i)$ and $\alpha_j > 0$. This is implemented in Figure 1.

In this case, too, a Lyapunov function can be constructed, which ensures that the dynamical system devised has an equilibrium that is asymptotically stable taking advantage of the gradient structure of the dynamics, as we will see in the next example.

Example 2.6: Network Utility Maximization in Network Congestion Control. One of the most important examples of resource allocation is the design of congestion control schemes for the Internet [18]. The aim is to allocate available bandwidth to competing users so as to avoid congestion collapse by ensuring that link capacities are not exceeded – but to do so in a distributed manner. This problem can be theoretically formulated as the fully centralized resource allocation (optimization) program shown above [19].

What is interesting to note is that the dual variables and the decomposition itself can be identified in the network structure and the protocols that have already been implemented in practice, and the scalability that the algorithm enjoys may be understood under this framework. Indeed, the dual variables p_j play the role of congestion signals which are generated by Active Queue Management (AQM) implemented at the links; in practice, the congestion measure is usually based on either delay or packet loss. On the other hand, the source rates x_i are adapted at the Transmission Control Protocol (TCP) part of the algorithm, according to the size of the aggregate price signals [20].

In addition to these, the sources can be assumed to possess a certain *utility* $U_i(x_i)$ (happiness) if allowed a certain transmission rate (these are the functions $f_i(x_i)$ in (7)), which is a monotonically increasing strictly concave function. Different protocols can be thought of as corresponding to different Utility functions U_i. The constraint functions in (8) are simply $g_i(x_i) = x_i$, which have the meaning that the aggregate sum of the rates on each link is less than the capacity c_l on the link. In total, relation (9) reduces to

$$x_i^* = U_i'^{-1}(q_i)$$

and the gradient algorithm on the dual algorithms takes the form (10) where the constants $\alpha_j > 0$ can be tuned to $\alpha_j = \frac{1}{c_j}$, thus giving the variables p_j the unit of time and the interpretation of processing delay at the links.

In total, the feedback structure is very similar to Figure 1. The closed loop system takes the form:

$$\dot{p}_l(t) = \left[\sum_{i=1}^{N} \frac{R_{li}}{c_l} U_i'^{-1} \left(\sum_{m=1}^{L} R_{mi} p_m(t) \right) - 1 \right]_{p_l}^{+} = [g_l(p)]_{p_l}^{+} \tag{11}$$

The following result is known [19]:

Theorem 2.7: For fixed full rank R, the (unique) equilibrium of (11) is asymptotically stable for all non-negative initial conditions.

The proof can be found in [18] and uses the following function as a Lyapunov function:

$$V(p) = \sum_{l=1}^{L} (c_l - y_l^*) p_l + \sum_{i=1}^{N} \int_{q_i^*}^{q_i} (x_i^* - U_i'^{-1}(Q)) dQ, \tag{12}$$

This function is positive definite for $p \geq 0$, with a minimum at the equilibrium. Differentiating $V(p)$ with respect to time, we get

$$\dot{V}(p) = -\sum_{m=1}^{L} c_m g_m(p) [g_m(p)]_{p_m}^{+} \leq 0.$$

Now $\dot{V}(p) = 0$ only when $g_m(p) = 0$ or $g_m(p) < 0$ and $p_m = 0$. This can only happen at the equilibrium of interest. Therefore, asymptotic stability is concluded using LaSalle's Theorem [41]. Moreover, V is radially unbounded, hence the equilibrium is globally (i.e., for $p_l \geq 0$) asymptotically stable.

In all the above examples, we have outlined how we can establish that the dynamics designed using the procedure described in the start of the section will converge to the optimal point of the original optimization problem. In particular, information on the structure of a scalable Lyapunov function can be obtained from the structure of the decomposition.

In the next two sections, we will investigate the effects of changing topologies and communication time-delays on the functionality of networked systems that are designed using the above methods. In particular, for consensus reaching (Example 2.1) we will consider the effect of heterogeneous communication time-delays and changes in the network topology on the attractivity of the consensus set; and for Internet congestion control, (Example 2.6) we will investigate the robustness of such schemes to heterogeneous propagation time-delays.

III. Coordination of Multi-agent systems

In this section we consider the robustness properties of the consensus algorithms for multi-agent systems given in Example 2.1 to communication time-delays and network topology changes. The undelayed problem has been studied by many authors [3], [17], [25], [26], and so has the delayed one. In particular, the work in [28], [42], shows stability with linear discrete dynamics, switching topologies and arbitrary heterogeneous delays. In a similar way, the work in [43] extends the work in [25] to include time-delays, considering a discrete-time system. The work in [30] considers linear systems with constant and time-varying delays. In [6], [44] a delay-dependent condition is obtained for the case of continuous-time dynamics, see Section V for a discussion. [31] and [27] consider a discrete-time linear system with commensurate delays and identify that consensus can be reached independent of delay. The paper [12] shows stability of continuous time nonlinear system models with linear heterogeneously delayed coupling, using a contraction principle. Lastly, the work in [45] shows consensus reaching in multi-agent packet-switched networks, while [46] discusses output synchronization of nonlinear systems with communication time-delays.

Here we are concerned with delayed versions of nonlinear continuous-time models of these systems, for which we aim to prove that consensus is retained irrespective of the size of the heterogeneous delays. We will use the formulation as it was detailed in Example 2.1, i.e., the network dynamics are described by Equation (4) where the functions f_{ij} satisfy Assumption 2.3. We first show that if the interaction graph has a spanning tree, the consensus set is asymptotically attracting using a different Lyapunov function argument, that will pave the way for the delayed case in the sequel. See also [14].

Theorem 3.1: Consider the system given by (4) where the f_{ji}'s are locally passive. Let the initial condition be chosen in the set D defined by

$$D = \left\{ x \in \mathbb{R}^n \mid |x_i| \leq \frac{\gamma}{2} \right\}$$

where γ is given by Equation (6). If the interaction graph has a spanning tree then the consensus set $\mathcal{X}$ is asymptotically attracting.

Proof: It is not difficult to verify that region D is positively invariant, something which will be established for the delayed case later on. Therefore all solutions are bounded.

Consider the following function

$$V(x) = \max_i x_i \tag{13}$$

which we will use as a candidate Lyapunov function - note that this function is not differentiable, but it can still be used to conclude the attractivity properties of equilibria [14], [47], [48]. In particular, the right hand side Dini derivative of $V(x)$ along the solutions of the system is defined as

$$\dot{V}(x(t)) = \lim_{\tau \to 0^+} \sup \frac{1}{\tau} \left[V(x(t+\tau)) - V(x(t)) \right]$$

Suppose I is the agent for which the maximum is achieved. If there are many such agents, we choose the one that has maximum $|\dot{x}_i|$ and if there are still many such agents, we choose any one of those,

but commit to that until a new agent holds the maximum value. Calculating the Dini derivative of the candidate Lyapunov function along (4) we get:

$$\dot{V}(x) = \dot{x}_I = k_I \sum_{j=1}^{N} A_{jI} f_{jI} \left(x_j(t) - x_I(t) \right)$$

We now argue that $\dot{V}(x) \leq 0$. Since $x_I = \max_{i=1,\ldots,N} x_i$ and $k_I > 0$, we immediately see that $\dot{x}_I \leq 0$ as the vector field is a positive summation of passive functions with non-positive arguments for $x \in D$. In conclusion, $\dot{V}(x) \leq 0$.

The x_i for which $\dot{V} = 0$ is the set for which $\dot{x}_I = 0$, where $x_I = \max_{i=1,\ldots,N} x_i = \alpha$. This holds if and only if all the parents of node I also hold the maximum value and so do their respective parents, etc. Since we have assumed that the graph contains a spanning tree, the root of the spanning tree and all the nodes in a directed path to the node v_I - denote all these nodes by $\mathcal{R}$ - also satisfy $x_i = \alpha$, $i \in \mathcal{R}$. Therefore the set for which $\dot{V} = 0$ is the set for which all vertices v_i, $i \in \mathcal{R}$ hold the value α, apart from nodes in $\mathcal{I} \setminus \mathcal{R}$ which may hold a value less than or equal to α.

A similar argument can be used to show that the function $W = -\min_i x_i$ is also a Lyapunov function, and in this case the set for which $\dot{W} = 0$ is the set for which $\dot{x}_K = 0$, where $x_K = \min_{i=1,\ldots,N} x_i = \beta$. Following the same route of thought as before, the root of the (same) spanning tree and all the nodes in the path from this root to v_K (again, denote these nodes by $\mathcal{Q}$) have to hold a value β, and nodes in $\mathcal{I} \setminus \mathcal{Q}$ must hold a value more than or equal to β.

Since the root of the spanning tree holds both the minimum and the maximum value, it is now easy to see that $\alpha = \beta$. An invariant set in the set $\{\dot{V} = 0\} \cap \{\dot{W} = 0\}$ is the consensus set $\mathcal{X}$ (5), which is asymptotically attracting by the invariance principle in [47]. ∎

We now return to the main objective of this section of the paper, which is to assess the consensus properties of the system with heterogeneously delayed dynamics

$$\dot{x}_i(t) = k_i \sum_{j=1}^{N} A_{ji} f_{ji} \left(x_j(t - \tau_{ji}) - x_i(t) \right). \tag{14}$$

Here $\tau_{ji} \geq 0$ are time-delays that model the propagation of state information from node v_j to v_i, for all $(v_j, v_i) \in \mathcal{E}$. The state-space is now infinite-dimensional, with the state $x_t \in C([-\tau, 0], \mathbb{R}^N)$ where $\tau = \max_{i,j=1,\ldots,N} \tau_{ji}$. The consensus set in this case is defined by:

$$\mathcal{X}_\tau = \{x_t \in C \mid x(t + \theta) = c\mathbf{1}, c \in \mathbb{R} \text{ for all } \theta \in [-\tau, 0], t \geq 0\} \tag{15}$$

In this case, the exact value of c will depend on the initial conditions in a complicated way.

We consider (14) and we will derive consensus conditions based on the properties of the functions f_{ji} and the structure of the interconnection topology. We first prove the following lemma:

Lemma 3.2: Consider (14) where the f_{ji}'s satisfy Assumption 2.2 and $\tau = \max_{i,j=1,\ldots,N} \tau_{ji}$. Define γ as in (6), and consider initial conditions ψ that satisfy

$$|\psi_i(\theta)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad \theta \in [-\tau, 0]. \tag{16}$$

Then

$$-\frac{\gamma}{2} \le x_i(t) \le \frac{\gamma}{2}$$

for all $t \ge -\tau$.

Proof: From the bounds on the initial condition, we have:

$$-\frac{\gamma}{2} \le \psi_i(\theta) \le \frac{\gamma}{2}$$

for $\theta \in [-\tau, 0]$. Denote the time at which this condition is violated by t^*. When this happens, the following hold:

1) $-\frac{\gamma}{2} \le x_i(t) \le \frac{\gamma}{2}$ for $t \in [-\tau, t^*)$ for all $i = 1, \ldots, N$;

2) At t^* there is an $i \in \{1, \ldots, N\}$ such that we have either:

$$\left\{ x_i(t^*) = \frac{\gamma}{2} \text{ and } \dot{x}_i(t^*) > 0 \right\} \text{ or } \left\{ x_i(t^*) = -\frac{\gamma}{2} \text{ and } \dot{x}_i(t^*) < 0 \right\}$$

Suppose the first case holds. Recall the structure of the dynamics:

$$\dot{x}_i = k_i \sum_{j=1}^{N} A_{ji} f_{ji}(x_j(t - \tau_{ij}) - x_i(t))$$

Now since $x_i(t^*) = \frac{\gamma}{2} \ge x_j(t^* - \tau_{ij})$ from the observations above, we can see that each term on the right hand side (the vector field) is non-positive as f_{ji} are locally passive functions; and so

$$\dot{x}_i(t^*) \le 0,$$

which leads to a contradiction. The same is true for the second case. In conclusion we have that

$$-\frac{\gamma}{2} \le x_i(t) \le \frac{\gamma}{2}$$

for all $t \ge -\tau$. ∎

The above Lemma has identified an invariant set. We will now proceed to show that the consensus set $\mathcal{X}_\tau$ is asymptotically attracting using Theorem A-7 in the Appendix, which is taken from [49]. The Lyapunov certificate is a functional of the form:

$$\overline{V}(\phi) = \max_{\theta \in [-\tau, 0]} V(\phi(\theta)).$$

where $V(\phi)$ is known as a 'Lyapunov-Razumikhin' function.

Theorem 3.3: Consider (14) with f_{ij} satisfying Assumption 2.2, where the digraph $\mathcal{G}$ contains a spanning tree. Define γ as in (6), let $\tau = \max_{i,j=1,\ldots,N} \tau_{ij}$ and consider initial conditions ψ in the set

$$\Omega = \left\{ \psi \in C([-\tau, 0], \mathbb{R}^N) \mid |\psi_i(\theta)| \le \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad \theta \in [-\tau, 0] \right\}.$$

Then the consensus set $\mathcal{X}_\tau$ is asymptotically attracting.

Proof: From Lemma 3.2, the set Ω is positively invariant, and hence solutions are bounded.

Consider now the function

$$V(x(t)) = \max_i x_i(t) \tag{17}$$

as a candidate Lyapunov-Razumikhin function. Let I be the index for which the maximum at time t is achieved. If there are many such indices, pick the one which satisfies $\max_i |\dot{x}_i|$, and if there is still a choice, pick any one of them, but commit to that index until another index achieves the maximum x_i.

In order to satisfy the first condition in Theorem A-7, we are interested in

$$V(\phi(0)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta))$$

This condition translates to

$$\phi_I(0) \geq \phi_j(\theta), \quad \theta \in [-\tau, 0], \quad j = 1, \ldots, N. \tag{18}$$

Moreover,

$$\dot{V}(\phi) = \dot{\phi}_I(0).$$

We want to ensure that while condition (18) holds, $\dot{V} \leq 0$. From Equation (18), we have that

$$\phi_j(\theta) - \phi_I(0) \leq 0$$

for all $j = 1, \ldots, N$, $\theta \in [-\tau, 0]$. Since f_{jI} are locally passive, we have $f_{jI}(\phi_j(-\tau_{jI}) - \phi_I(0)) \leq 0$ for all j for which $(v_j, v_I) \in \mathcal{E}$ and $\phi \in \Omega$. Therefore $\dot{\phi}_I(0) \leq 0$. In conclusion, while (18) holds, we have $\dot{V} \leq 0$.

We now proceed to compute the sets E and L defined by Equations (47-48). First of all, since $\phi = 0$ is in L, this set is non-empty. Suppose $\phi \in E$, i.e., let $\phi \in \Omega$ be such that

$$\max_i \max_{-\tau \leq \theta \leq 0} x_i(\phi)(t + \theta) = \max_i \max_{-\tau \leq \theta \leq 0} \phi_i(\theta) \tag{19}$$

for all $t \geq 0$ and $\theta \in [-\tau, 0]$. For $\phi \in E$ satisfying (18), there exists a t^* for which we have $\dot{V}(x_{t^*}(\phi)) = 0$, as V attains a relative maximum for such t^* (see Appendix). For such a t^*, we have:

$$\dot{V} = k_I \sum_{j=1}^{N} A_{jI} f_{jI} \left(x_j(t^* - \tau_{jI}) - x_I(t^*) \right) = 0$$

From (18) we have that $x_j(t^* + \theta) - x_I(t^*) \leq 0$ for all $\theta \in [-\tau, 0]$, and if the above condition is to hold, this means that $x_I(t^*) = x_j(t^* - \tau_{Ij})$ for all v_j that are parents of v_I. Continuing up a spanning tree we see that all nodes in the path from the root of the tree to v_I have to attain this value at some point in the past - call this vertex set $\mathcal{R}$, and define this value α. All other nodes can achieve a value less than or equal to α.

A similar argument can be used to show that the function $W = -\min x_i$ is non-increasing, and that the value held by v_K for which $x_K = \min_i x_i$ and nodes from the root of the same spanning tree to node v_K at some points in the past is β, while all the rest of the nodes can achieve a value more than or equal to β.

An invariant set in $\{\dot{V} = 0\} \cap \{\dot{W} = 0\}$ is indeed the consensus set $\mathcal{X}_\tau$, for which $\alpha = \beta$. Therefore the consensus set $\mathcal{X}_\tau$ is asymptotically attracting for digraph topologies that contain a spanning tree.

∎

We would like to make a few remarks about this system, and the technique that we have used to prove that consensus is reached. First, we have used a Lyapunov-Razumikhin function which is a direct extension of the Lyapunov function used for the stability of the undelayed system to establish the stability of the delayed case. This is due to the fact that many times Lyapunov-Razumikhin functions are easier to work with when dealing with nonlinear systems, especially if a condition that is independent of delay is being sought. Later on in this paper, we will see how to use a Lyapunov-Krasovskii functional for a large scale system to prove delay-dependent stability.

We also stress that stability of the equilibrium set is retained *independent of the size of the delay*; we only require that f_{ji} be locally passive and that the digraph contains a spanning tree. To see why this is so, consider the linear system

$$\dot{x} = -ax(t) + bx(t - \tau). \tag{20}$$

This system is asymptotically stable for all delays if $a > 0$ and $|b| < a$, and stable if $|b| = a$, for all delays – this latter condition is what we have when we linearize the system for the simplest network instance.

Also, the functions f_{ji} can be different for each link, and moreover f_{ij} can be different from f_{ji} in the case of bidirectional communication between agents i and j – this however makes it hard to say anything about convergence speed or the consensus value that is reached. Most importantly, our results are independent of the network topology. We want to investigate whether the same properties hold for the system when the topology changes, which is a more complicated issue. This is the subject of the following subsection.

A. Coordination of Multiagent Systems Under Switching Topologies

The issue of coordination under changing topologies has been investigated in [3], [4], [6], [28], [43], for the case in which the system does not have any delays that make the state-space infinite-dimensional. But even if time-delays are ignored, the problem of ensuring stability in this case is difficult. Switching arbitrarily among a set of possible topologies of size N can be regarded as a problem of establishing stability for a switching system with an unknown switching rule.

For the analysis of systems with linear subsystems under arbitrary switching, the (conservative) condition of *quadratic stability* has been used to ensure that a system comprised of M subsystems of the form $\dot{x} = A_p x$, $p = 1, \ldots, M$ is stable under arbitrary switching; the conditions in this case require the existence of a *common* Lyapunov function $V = x^T P x$, $P > 0$ so that $A_p^T P + P A_p < 0$ for all $p = 1, \ldots, M$. This argument is many times inconclusive, as this criterion is conservative. This conservativeness was observed in [3], where a different approach had to be taken to conclude coordination. Another problem of using such an argument in the class of systems we are interested is that an invariance principle would have to be invoked to conclude consensus, and the switching signal itself should satisfy certain conditions, as described in [50], [51]. One such condition is that the switching times $t_0 = 0 < t_1 < t_2 < \ldots$ are positively divergent and $\inf_k (t_{k+1} - t_k) \geq h$ where $h > 0$ is called the *dwell time* [52].

Let us consider a network of N agents each with state x_i, $i = 1, \ldots, N$ with an interaction topology chosen from a collection of graphs indexed by $p \in \mathcal{P} = \{1, \ldots, M\}$. Each graph $\mathcal{G}^{(p)} = (\mathcal{V}, \mathcal{E}^{(p)})$ has an adjacency matrix $A^{(p)}$ where $p = 1, \ldots, M$. Consider a piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ that is continuous from the right, which is non-chattering and with a dwell time $h > 0$.

Definition 3.4: Given a time interval $\Delta T_i = [T_i, T_{i+1}]$, denote by $\mathcal{P}(\Delta T_i) = \{p \in \mathcal{P} \mid \sigma(\theta) = p, \theta \in [T_i, T_{i+1}]\}$. Then the *union graph across* ΔT_i is defined by $\overline{\mathcal{G}}(\Delta T_i) = (\mathcal{V}, \bigcup_{p \in \mathcal{P}(\Delta T_i)} \mathcal{E}^{(p)})$.

When the topology is $p \in \mathcal{P}$, vertex v_i has the following dynamics:

$$\dot{x}_i = k_i \sum_{j=1}^{N} A_{ij}^{(p)} f_{ij} \left(x_j(t - \tau_{ij}) - x_i(t) \right). \tag{21}$$

Assume throughout that the functions f_{ji} are locally passive, as defined in Definition 2.2. The positive constant γ is defined as in Equation (6). It is easy to construct an invariant region even if the topologies change in a similar way as it was done for the fixed topology case, as long as there is no chattering. Then, one can show that the Lyapunov-Razumikhin function in Theorem 3.3 is a common Lyapunov-Razumikhin function and use this to construct conditions on the switching signal σ so that consensus is reached for any finite delays.

Theorem 3.5: Consider a piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ with a dwell time $h > 0$ where $\mathcal{P} = \{1, \ldots, M\}$. For interval $\Delta T_i = [T_i, T_{i+1}]$ let the collection of graphs $\mathcal{P}(\Delta T_i) \subseteq \mathcal{P}$ in N vertices be such that the union graph $\overline{\mathcal{G}}(\Delta T_i)$ contains a spanning tree. At time t when $\sigma(t) = p$ assign to vertex v_i the dynamics given by Equation (21). Define γ as in (6), and suppose the initial conditions ϕ belong to the set

$$\Omega = \left\{ \phi \in C([-\tau, 0], \mathbb{R}^N) \mid |\phi_i(s)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad s \in [-\tau, 0]. \right\} \tag{22}$$

where $\tau = \max_{i,j=1,\ldots,N} \tau_{ij}$. Then

$$-\frac{\gamma}{2} \leq x_i(t) \leq \frac{\gamma}{2}$$

for all $t \geq -\tau$. Moreover, the consensus equilibrium set is attracting even if the topology changes, as long as there exists a series of contiguous, nonempty, bounded time intervals ΔT_i so that $\overline{\mathcal{G}}(\Delta T_i)$ contains a spanning tree.

Proof: For a fixed topology, the first part has already been shown in the proof of Lemma 3.2. When the topologies switch, a similar argument can be used to show that indeed the region is invariant as long as chattering is avoided, as during switching, the next subsystem will have an initial condition that satisfies Equation (22); therefore this region cannot be escaped.

Having identified an invariant region, we will now show that the Lyapunov-Razumikhin function we used in the proof of Theorem 3.3 is a common Lyapunov-Razumikhin function for all the systems indexed by $p \in P$. Consider

$$V = \max_i x_i$$

Suppose that at time t, subsystem $p \in \mathcal{P}$ is active and vertex v_I achieves the above maximum. Then we have:

$$\dot{V}(x) = k_I \sum_{j=1}^{N} A_{jI}^{(p)} f_{jI} \left(\phi_j(-\tau_{jI}) - \phi_I(0) \right)$$

We are interested, furthermore, in the set for which

$$\phi_I(0) \geq \phi_j(s), \quad s \in [-\tau, 0]. \tag{23}$$

as explained in the proof of Theorem 3.3. Irrespective of p, for $\phi \in \Omega$ we have $\dot{V} \leq 0$ as the functions f_{jI} are locally passive and have non-positive arguments from condition (23). Therefore V is non-increasing for all subsystems while (23) holds.

We now proceed to show that even if topologies change, the consensus set $\mathcal{X}_\tau$ is asymptotically attracting. Consider a time interval $\Delta T_i = [T_i, T_{i+1}]$ within which the topology changes L_i times, and denote the times this happens by t_{i_l}, $l = 1, 2, \ldots, L_i$, assuming that $t_{i_0} = T_i$ is also a switching time. Note that since the switching signal has a dwell time $h > 0$, we have $t_{i_l} - t_{i_{l-1}} > h$, during which time the topology index takes a value $p_{i_l} \in \mathcal{P}$. For $t \in [t_{i_l}, t_{i_{l+1}}]$, the Dini derivative of V is given by:

$$\dot{V}(x(t)) = k_I \sum_{j=1}^{N} A_{jI}^{(p_{i_l})} f_{jI} \left(x_j(t - \tau_{jI}) - x_I(t) \right), \quad t \in [t_{i_l}, t_{i_{l+1}}]$$

where $x_I(t) = \max_{j \in \mathcal{I}} x_j(t)$. It has been argued earlier, owing to the fact that the f_{jI}'s are passive, that the function $V(x(t))$ is non-increasing when $V(x(t)) = \max_{-\tau \leq \theta \leq 0} V(x(t + \theta))$. Alternatively, the function $\overline{V}(x_t) = \max_{\theta \in [-\tau, 0]} V(x(t + \theta))$ is non-increasing for all time. Therefore, denoting $\theta_0(t)$ the value of θ for which the maximum is achieved at time t:

$$\overline{V}(x_{T_{i+1}}) - \overline{V}(x_{T_i})$$
$$= \sum_{l=0}^{L_i} \int_{t_{i_l}}^{t_{i_{l+1}}} k_I \sum_{j=1}^{N} A_{jI}^{(p_{i_l})} f_{jI} \left(x_j(t + \theta_0(t) - \tau_{jI}) - x_I(t + \theta_0(t)) \right) dt$$
$$= \sum_{l=0}^{L_i} \psi_l \leq 0,$$

where we have defined

$$\psi_l = \int_{t_{i_l}}^{t_{i_{l+1}}} k_I \sum_{j=1}^{N} A_{jI}^{(p_{i_l})} f_{jI} \left(x_j(t + \theta_0(t) - \tau_{jI}) - x_I(t + \theta_0(t)) \right) dt. \tag{24}$$

Consider now a sequence of contiguous intervals ΔT_i, $i = H, H + 1, \ldots, H + H_f$ with H and H_f positive integers. Then

$$\overline{V}(x_{T_{H+H_f}}) - \overline{V}(x_{T_H}) = \sum_{i=H}^{H+H_f} \sum_{l=0}^{L_i} \psi_l.$$

We have already established that the solution is bounded and therefore $\overline{V}$ is bounded from below. Since $\overline{V}$ is non-increasing, the left hand side of this equation goes to a constant as $H_f \to \infty$. In turn this means

that the series on the right hand side of this equation has to converge. Since $\psi_l \leq 0$, the general term has to go to zero, i.e.,:

$$\sum_{l=0}^{L_{H+H_f}} \psi_l \to 0 \text{ as } H_f \to \infty.$$

The integrand on the right of Equation (24) is not uniformly continuous, so a Barbalat argument cannot be immediately applied to conclude consensus. However, we can use the fact that switching happens with a dwell time and a Barbalat-like argument [51] to conclude that the integrand also goes to zero. But since I is the index of the vertex which holds $\max_i x_i$, and the union graph has a spanning tree for every interval, that means that the root of this union tree and all edges from node I to this root achieve this maximum asymptotically at some points in the past, which we call α.

A similar argument can show, using $\overline{W} = \min_{\theta \in [-\tau, 0]} \min_i x_i(t+\theta)$, that there is a time interval ΔT_{H+H_f} during which the consensus set between vertex v_K, for which $x_K = \min_i x_i$ and the vertex of the same spanning tree in the graph $\overline{\mathcal{G}}(\Delta T_{H+H_f})$ is asymptotically attracting to a value β.

The argument that $\alpha = \beta$ follows the same lines as in the proof of Theorem 2.7. In conclusion, the consensus set $\mathcal{X}_\tau$ is asymptotically attracting even if topologies change, as long as there exists a series of contiguous, nonempty, bounded time intervals ΔT_i so that $\overline{\mathcal{G}}(\Delta T_i)$ contains a spanning tree. ∎

We now present our results on another large-scale networked system, Internet networks employing congestion control.

IV. CONGESTION CONTROL FOR THE INTERNET

In Example 2.6, we have described the structure and architecture of a class of congestion control algorithms for the Internet. The Lyapunov function (12) showing the stability of the undelayed so-called dual algorithm is the basic building block that ensures scalability of the stability property irrespective of the size of the topology and other parameters, such as capacities of the links etc. However, when heterogeneous delays are taken into account, the problem becomes more difficult as the theory available for analysis of the nonlinear, delayed system description is involved. Inevitably, the first scalable analysis and design procedures centered on the investigation of the linearizations of the nonlinear equations, using for example, the generalized Nyquist criterion developed in [35] and other robust control tools [53]. See also [54].

The presence of heterogeneous time-delays in the communication between sources and links can be modelled as follows. The transmission rates x_i are in practice added with some forward time delay $\tau_{i,l}^f$. Hence, the aggregation at the links happens as follows:

$$y_l(t) = \sum_{i=1}^{N} R_{li} x_i(t - \tau_{i,l}^f) \triangleq r_f(x_i, \tau_{i,l}^f) \tag{25}$$

At the same time, the prices (dual variables) p_l of all the links that source i uses are aggregated to form q_i, the *aggregate price* for source i, through a delay $\tau_{i,l}^b$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau_{i,l}^b) \triangleq r_b(p_l, \tau_{i,l}^b) \tag{26}$$
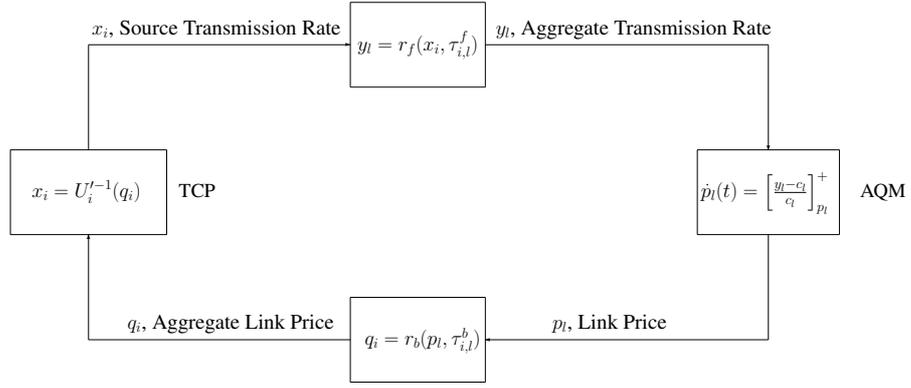
Fig. 2. The Internet as an interconnection of sources and links through delays.

The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau_{i,l}^f + \tau_{i,l}^b, \quad \forall\, l = 1, \ldots, L \tag{27}$$

Figure 2 shows the interconnection of sources and links that employ TCP and AQM schemes respectively, through forward and backward paths that are delayed (see Equations (25) and (26)). The interconnection topology is captured through the Routing matrix R (see Equation (2)), which is assumed to have full row rank. The dynamics at the sources and links shown in this Figure follow from the original problem description.

Lyapunov-based constructions have been used in the analysis of nonlinear congestion control schemes with delays [55]–[58], but only on simple network instances (single bottleneck). New tools have also been introduced, such as passivity theory formulations [59]. The analysis of such systems for arbitrary network sizes have only recently appeared in [36], [37], [60]. The first two references concern primal congestion control algorithms, for which delay-independent conditions are developed.

Here, we will describe a methodology based on Lyapunov-Krasovskii functionals (see Theorem A-2) to prove stability of the scheme shown in Figure 2. We will set the Lyapunov-Krasovskii functionals equal to the Lyapunov functions used in the undelayed systems, plus some simple integral terms.

In the delayed case, the closed loop dynamics take the form:

$$\dot{p}_l(t) = \left[\sum_{i=1}^N \frac{R_{li}}{c_l} U_i'^{-1} \left(\sum_{m=1}^L R_{mi} p_m(t - \tau_{i,l}^f - \tau_{i,m}^b) \right) - 1 \right]_{p_l}^+ \triangleq [g_l(p)]_{p_l}^+ \tag{28}$$

We also denote $h = \max_{\{i,l,m: R_{mi} = R_{li} = 1\}} \{\tau_{i,l}^f + \tau_{i,m}^b\}$, and let $C = C([-h, 0], \mathbb{R}^L)$ – see also the notation introduced earlier in Section I-A. We assume that the initial condition is non-negative, i.e.,

$$p(\theta) = \phi(\theta) \geq 0, \quad \theta \in [-h, 0], \quad \phi \in C.$$

This guarantees that the solutions to (28) satisfy $p_l(t) \geq 0$ for all time as a result of the projection nonlinearity in (28), which is the same as the one introduced in Example 2.6.

A particular congestion control scheme that we will be referring to is FAST [53], in which the sources are assumed to have the following Utility function:

$$U_i(x_i) = \frac{\tau_i M_i}{\alpha_i} x_i \left(1 - \log \frac{x_i}{\overline{x}_i} \right), \tag{29}$$

where $M_i = \sum_{l=1}^{L} R_{li}$, α_i are (positive) source gains and $\overline{x}_i$ are source constants. Since $x_i = U_i'^{-1}(q_i)$, we have:

$$x_i = \overline{x}_i e^{-\frac{\alpha_i q_i}{M_i \tau_i}}$$

which implies that $x_i \leq \overline{x}_i$, since $q_i \geq 0$.

The linearization of (28) was investigated in [53]. Assume that R refers to bottleneck links only, and for non-bottleneck links $p_l = 0$. This gives the system

$$\dot{p}_l(t) = \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{U_i''(x_i^*) \tilde{c}_l} p_m(t - \tau_{i,l}^f - \tau_{i,m}^b), \tag{30}$$

where $\tilde{c}_l$ is a target capacity set just below the true capacity c_l of link l and * denotes equilibrium quantities. For this system, the following result is known:

Theorem 4.1: [53] Let the matrix R that denotes the routing matrix in relation to the bottleneck links be full row rank. Then the system given by (30) is asymptotically stable if

$$\frac{1}{\tilde{c}_l} \sum_{i=1}^{N} \frac{R_{li} M_i \tau_i}{|U_i''(x_i^*)|} < \frac{\pi}{2}. \tag{31}$$

In particular, for the Utility function given by (29), the above condition reduces to $\alpha_i \leq \pi/2$, which is a decentralized condition on the sources' gains.

We now turn to the nonlinear system, given by Equation (28). Here R denotes the full routing matrix with non-bottleneck links (rows) included but is still full row rank so that equilibrium prices are uniquely determined. The existence and uniqueness of solutions of (28) is assumed. Recall that the Utility function is a continuously differentiable, non-decreasing, strictly concave function. Therefore, $U_i''(x_i) < 0$ everywhere. Let γ_i be the lower bound for $|U_i''(x_i)|$, so that

$$|U_i''(x_i)| \geq \gamma_i > 0, \quad \forall i. \tag{32}$$

The parameter $\gamma_i > 0$ serves as a global (i.e., for $x_i \geq 0$) Lipschitz constant for $U_i'^{-1}$ as:

$$\left| \left(U_i'^{-1}(q) \right)' \right| = \left| \frac{1}{U_i''(x)} \right| \leq \frac{1}{\gamma_i} \tag{33}$$

This means that:

$$\left| U_i'^{-1}(q_2) - U_i'^{-1}(q_1) \right| \leq \frac{1}{\gamma_i} |q_2 - q_1| \tag{34}$$

We then have the following result:

Theorem 4.2: The equilibrium of the system described by (28) is globally (for $p_l \geq 0$) asymptotically stable for arbitrary delays, provided that

$$\frac{1}{c_l} \sum_{i=1}^{N} \frac{R_{li} M_i \tau_i}{\gamma_i} < 1 \tag{35}$$

and the matrix R is an arbitrary full rank, fixed routing matrix.

Proof: Consider $V_1 = V$ given by (12). From the argument after Theorem 2.7, $V_1 > 0$ apart at the equilibrium, and is radially unbounded. Now

$$\dot{V}_1(p) = -\sum_{l=1}^{L} c_l g_{l,u}[g_l]_{p_l}^+ = -\sum_{l=1}^{L} c_l g_l [g_l]_{p_l}^+ - \sum_{l=1}^{L} c_l [g_l]_{p_l}^+ (g_{l,u} - g_l),$$

where $g_{l,u}(p)$ corresponds to the undelayed version of unprojected (28), the unprojected Equation (11). The second term is equal to:

$$
-\sum_{l=1}^{L} c_l [g_l]_{p_l}^+ (g_{l,u} - g_l)
$$

$$
\leq \sum_{l=1}^{L} \sum_{i=1}^{N} R_{li} |[g_l]_{p_l}^+| \left| U_i'^{-1}\left(\sum_{m=1}^{L} R_{mi} p_m(t) \right) - U_i'^{-1}\left(\sum_{m=1}^{L} R_{mi} p_m(t - \tau_{i,l}^f - \tau_{i,m}^b) \right) \right|
$$

$$
\leq \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{mi} R_{li}}{\gamma_i} |[g_l]_{p_l}^+| \left| p_m(t) - p_m(t - \tau_{i,l}^f - \tau_{i,m}^b) \right|
$$

$$
\leq \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{mi} R_{li}}{\gamma_i} \int_{-\tau_{i,l}^f - \tau_{i,m}^b}^{0} |[g_l]_{p_l}^+| |\dot{p}_m(t + \theta)| d\theta
$$

$$
\leq \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{mi} R_{li}(\tau_{i,l}^f + \tau_{i,m}^b)}{2\gamma_i} \left([g_l]_{p_l}^+\right)^2 + \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{mi} R_{li}}{2\gamma_i} \int_{-\tau_{i,l}^f - \tau_{i,m}^b}^{0} \dot{p}_m^2(t + \theta) d\theta
$$

where global Lipschitz continuity and the Leibniz rule were used (note that $p \geq 0$). Consider now the following function:

$$
V_2(p) = \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{2\gamma_i} \int_{-\tau_{i,l}^f - \tau_{i,m}^b}^{0} \int_{t+\theta}^{t} \dot{p}_m^2(\zeta) d\theta d\zeta.
$$

This differentiates to:

$$
\dot{V}_2(p) = \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}(\tau_{i,l}^f + \tau_{i,m}^b)}{2\gamma_i} \dot{p}_m^2(t) - \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{2\gamma_i} \int_{-\tau_{i,l}^f - \tau_{i,m}^b}^{0} \dot{p}_m^2(t + \theta) d\theta.
$$

Let the Lyapunov candidate be

$$
V = V_1 + V_2
$$
$$
= \sum_{l=1}^{L} (c_l - y_l^*) p_l + \sum_{i=1}^{N} \int_{q_i^*}^{q_i} (x_i^* - U_i'^{-1}(Q)) dQ + \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{2\gamma_i} \int_{-\tau_{i,l}^f - \tau_{i,m}^b}^{0} \int_{t+\theta}^{t} \dot{p}_m^2(\zeta) d\theta d\zeta \quad (36)
$$

Then we have:

$$
\dot{V} \leq -\sum_{l=1}^{L} c_l g_l [g_l]_{p_l}^+ + \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{mi} R_{li}(\tau_{i,l}^f + \tau_{i,m}^b)}{2\gamma_i} \left([g_l]_{p_l}^+\right)^2
$$
$$
+ \sum_{l=1}^{L} \sum_{i=1}^{N} \sum_{m=1}^{L} \frac{R_{li} R_{mi}(\tau_{i,l}^f + \tau_{i,m}^b)}{2\gamma_i} \dot{p}_m^2(t).
$$

Manipulation of the last two terms gives:

$$\dot{V} = \sum_{l=1}^{L} \left[-c_l g_l [g_l]_{p_l}^+ + \sum_{i=1}^{N} \frac{R_{li} M_i \tau_i}{\gamma_i} \left([g_l]_{p_l}^+ \right)^2 \right]$$

Now $\dot{V} \leq 0$ under the condition:

$$-c_l + \sum_{i=1}^{N} \frac{R_{li} M_i \tau_i}{\gamma_i} < 0$$

This implies that $\dot{V} \leq 0$, and stability of the equilibrium follows for $p_l \geq 0 \ \forall \ l = 1, \ldots, L$. The set $S = \{\phi \in C : \dot{V} = 0\}$ is the set for which $[g_l]_{p_l}^+ = 0$, i.e., for $l = 1, \ldots, L$ we have either

$$\sum_{i=1}^{N} \frac{R_{li}}{c_l} U_i'^{-1} \left(\sum_{m=1}^{L} R_{mi} \phi_m(-\tau_{i,l}^f - \tau_{i,m}^b) \right) = 1$$

or

$$\begin{cases} \sum_{i=1}^{N} \frac{R_{li}}{c_l} U_i'^{-1} \left(\sum_{m=1}^{L} R_{mi} \phi_m(-\tau_{i,l}^f - \tau_{i,m}^b) \right) < 1, \\ \phi_l(-\tau_{i,l}^f - \tau_{i,m}^b) = 0 \end{cases}$$

for $l = 1, \ldots, L$. The largest invariant set in S satisfies $\dot{p}_l = 0$, i.e. $p_l = K$, a constant. The only constant solution in S is the (unique) equilibrium (either $p_l^* = 0$ and $y_l^* < c_l$ or $p_l = p_l^*$). Therefore, the asymptotic stability of the equilibrium can be proven using LaSalle's theorem for time-delay systems (see Theorem 5.3.1 in [22] and the Appendix). Since V is radially unbounded, the equilibrium of the system is globally (i.e., for $p_l \geq 0$) asymptotically stable for arbitrary networks and delays, provided that condition (35) is satisfied and R is full rank. ∎

For the special case of FAST, we have the following corollary:

Corollary 4.3: The equilibrium of FAST is globally (for $p_l \geq 0$) asymptotically stable for arbitrary delays and network topologies, provided that

$$\alpha_i < \frac{x_i}{\overline{x}_i} \tag{37}$$

and the matrix R is an arbitrary full rank, fixed routing matrix.

Proof: For FAST, the Utility function is given by (29); this gives the following value for γ_i:

$$\gamma_i = \frac{\tau_i M_i}{\alpha_i \overline{x}_i}$$

since $x_i \leq \overline{x}_i$. Therefore, condition (35) of Theorem 4.2 becomes:

$$\frac{1}{c_l} \sum_{i=1}^{N} R_{li} \alpha_i \overline{x}_i < 1$$

Therefore, a sufficient condition for stability is

$$\alpha_i < \frac{x_i^*}{\overline{x}_i}.$$

for R full rank. ∎

We would like to emphasize that the result, even though conservative with respect to the condition in Theorem 4.1, holds for networks of arbitrary size, with heterogeneous delays and nonlinear dynamics. The tools that we have used to obtain these conditions are Lyapunov-Krasovskii functionals, which can treat delay-dependent conditions on high dimensional systems more effectively than Lyapunov-Razumikhin functions. These were simple extensions of the Lyapunov function used in the scalable stability of the undelayed system, see Theorem 2.7. Note that unlike the case of multi-agent systems, the stability condition for this network congestion control scheme requires the gains to be scaled down by the delay size. This essentially renders a delay-dependent condition non-dimensional in the delay size, and hence delay-independent.

V. CONNECTIONS AND OUTLOOK

In this paper we have presented a framework which can be used to formulate and understand the objectives of large-scale networked systems based on optimization and then investigated the robust functionality of two representative large-scale networked systems to heterogeneous delays and time-varying topologies. It was seen that:

- For the dynamics chosen for consensus reaching of multiagent systems, described by (14), the functionality is retained irrespective of the delay size, as long as the directed topologies contain a spanning tree. The topology can be allowed to change, as long as switches happen with a dwell time and the set of allowed digraphs contains a spanning tree over time.
- For dual Internet congestion control schemes described by (28), the gains have to be compensated for by the delay size for stability to be retained irrespective of the size of the network and arbitrary, heterogeneous delays.

Before we proceed, we should like to stress that even for independent, incommensurate delays $\tau_i, i = 1, \ldots, M$ the problem of deciding whether the system

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{M} A_k x(t - \tau_k)$$

is stable independent of delay, is NP hard [61]. Here $x \in \mathbb{R}^n$ and $A_i \in \mathbb{R}^{n \times n}$ are real matrices. The delay-independent condition for multi-agent consensus problem for a nonlinear model description for arbitrary delays is not conservative, and it is interesting that robust functionality can be ensured for this problem instance for arbitrary sizes and heterogeneous delays.

Similarly, in the delay-dependent case, the problem of deciding whether the system

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{M} A_k x(t - \tau_k)$$

is stable for all $\tau_k \in [0, T_k]$ with $T_k > 0$, is also NP-hard. In the case of network congestion control for the Internet we obtain conservative conditions with respect to the ones obtained by linearization. However,

these conditions are for nonlinear system descriptions of arbitrary size with many incommensurate delays, which do not 'degrade' as the system size increases. Therefore even though the general problem is NP-hard, the instances that we investigated in this paper, due to their structure and construction as explained in Section II, allow conclusions to be drawn for arbitrary network sizes with arbitrary, heterogeneous delays.

A question that comes to mind is whether there are cases of multi-agent systems for which the consensus reaching algorithm conditions are delay-dependent. Or, whether we can design congestion control schemes for the Internet and avoid compensating the gains by the delay size.

Consider a multi-agent coordination scenario in which the time delay τ_{ij} is introduced also in the state x_i for the dynamics of vertex v_i, i.e., which takes the form:

$$\dot{x}_i = k_i \sum_{j=1}^{N} A_{ij} f_{ij} \left(x_j(t - \tau_{ij}) - x_i(t - \tau_{ij}) \right).$$

In this case, which was investigated in [6], the condition on stability is delay-dependent. The simplest explanation for this, is that all terms in the vector field are delayed, in contrast to the dynamics in (14) where the x_i dynamics for vertex v_i are undelayed.

The exact opposite is true for *primal* network congestion control schemes, that take the form

$$\text{TCP scheme}: \quad \dot{x}_i = \kappa_i x_i(t - \tau_i) \left(\frac{a_i}{x_i^{n_i}(t)} - b_i x_i^{m_i}(t) q_i(t) \right)$$

$$\text{AQM scheme}: \quad p_l(t) = \left(\frac{y_l(t)}{c_l} \right)^{h_l}$$

where a_i, b_i and h_l are positive real numbers and m_i and n_i are real numbers satisfying $m_i + n_i > 0$. The rest of the variables have the same meaning as in Section IV. In this case, it was shown in [36] that if $n_i + m_i > \max_{l \in L} h_l > 0$ then the equilibrium is globally asymptotically stable. Therefore this structure of equations allows stability independent of delay for network congestion control.

It is important to observe that the Lyapunov functions used to analyze the delayed version of the systems under study are simple extensions of the Lyapunov functions used to analyze the undelayed systems. In particular, for the case of consensus reaching in multi-agent systems, the Lyapunov-Razumikhin function used for the delayed case (17) is the same as the Lyapunov function (13) used for the undelayed system. Similarly, the Lyapunov-Krasovskii functional used for the stability analysis of dual Internet congestion control schemes, Equation (36), is a simple extension of the Lyapunov function (12) used for the stability analysis of the undelayed system.

We used Lyapunov-Razumikhin functions to conclude delay-independent stability whereas we used Lyapunov-Krasovskii functionals for delay-dependent stability. There is no other reason for this choice, but the fact that Lyapunov-Razumikhin functions are easier to work with for nonlinear systems; however, they tend to give more conservative delay-dependent conditions, which is why we chose to use Lyapunov-Krasovskii functionals for the delay-dependent analysis of Internet congestion control schemes.

We would also like to mention that the dual network congestion control scheme defined by (28) in the single-link single-source case resembles closely a well-known system from population dynamics. Indeed

in this case, with U_i given by Equation (29), we have

$$\dot{p}(t) = \left[\frac{\overline{x}}{c}e^{-\frac{\alpha}{\tau}p(t-\tau)} - 1\right]_p^+ .$$

A simple change of variables $z(t) = \frac{\overline{x}}{c}e^{-\frac{\alpha}{\tau}p(t)} - 1$, ignoring the projection and rescaling time gives:

$$\dot{z}(t) = -\alpha[z(t)+1]z(t-1) \tag{38}$$

Equation (38) is *Hutchinson's Equation*, a well-known population dynamics model that models a single species striving for a common food. The delay represents the maturation of the population. The linearization of this system about the zero equilibrium is stable for $\alpha < \pi/2 \simeq 1.57$. In [62], E. M. Wright managed to prove global stability of the equilibrium of the nonlinear system for $\alpha < 37/24 = 1.54$ if the initial condition satisfies $z(t+\theta) \geq -1$, $\theta \in [-1,0)$ with $z(0) > -1$ (which corresponds to a non-negative p) by looking at the properties of the solution of (38). However, this result is difficult to scale for arbitrary population interactions – a similar problem to the arbitrary network topology case.

We would also want to point out an interesting connection between the results shown in Section III and synchronization in oscillator networks [2], [63], [64], see Example 2.4.

A. Synchronization in Oscillator Networks

We consider N coupled oscillators with phases $\theta_i \in [0, 2\pi)$ and natural frequencies ω_i. The phase of each oscillator θ_i (as well as its natural frequency ω_i) is associated to a vertex $v_i \in \mathcal{V}$ of an underlying *undirected* graph $\mathcal{G}$ with no loops and adjacency matrix A. The properties of the system

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin\left(\theta_j - \theta_i\right) \tag{39}$$

where K is the coupling strength was the subject of an earlier paper [40], for $M_i = N$, $i = 1, \ldots, N$. Here M_i is a scaling factor, which could be the number of neighbors of vertex v_i, i.e., the degree of vertex i, or N, the total population. Considering a network of identical oscillators, i.e., $\omega_i = \omega$, and switching to a rotating frame $\theta_i = \phi_i + \omega t$ it was shown in [40] that the phase-locked equilibrium set $\phi_i = c = $ constant is asymptotically attracting for arbitrary connected topologies. See also [65].

A time-delayed version for the above system can be analyzed using the tools developed in Section III. The dynamics of the i-th oscillator are:

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin\left(\theta_j(t-\tau_{ij}) - \theta_i(t)\right) \tag{40}$$

Most available results on the above system concern the linearization about the equilibrium set in the rotating frame given by $\phi_i(t) = c$, $i = 1, \ldots, N$, i.e., the system

$$\dot{\phi}_i(t) = \frac{K}{M_i}\sum_{j=1}^{N} G_{ij}(\phi_j(t-\tau_{ij}) - \phi_i(t)) \tag{41}$$

where $G_{ij} = A_{ij}\cos(\Omega\tau_{ij})$. In [66] and [67] the case of regular connected graphs (i.e. $M_i = d$, the (identical) degree of the vertices in the graph) with $\tau_{ij} = \tau$ was investigated, and synchronization criteria were established that required $G > 0$. In [68] the general connected graph case was considered with $M_i = d_i$ again for the case in which $\tau_{ij} = \tau$ yielding the condition $G > 0$. In both these cases the parameter Ω solves the 'self-consistency' relation

$$\Omega = \omega - K\sin(\Omega\tau). \tag{42}$$

In [32] the synchronization of oscillator networks for inhomogeneous delays τ_{ij} was investigated for scalings $M_i = N$ and $M_i = d_i$ for the linearization of system (40). In this case, (40) achieves uniform rotations $\theta(t) = \Omega t + \phi(t)$ when self-consistency relations of the following form are satisfied:

$$\Omega = \omega_i - \frac{K}{M_i}\sum_j A_{ij}\sin(\Omega\tau_{ij}) \tag{43}$$

for all i. Under the condition $G_{ij} > 0$, it was shown that the phase-locked equilibrium set is asymptotically attracting (even for non-identical oscillators).

For the nonlinear system given by (40) we can derive conditions for phase-locking to be achieved based on the results of Section III, assuming that given the oscillator frequencies ω_i and the coupling strength K, one chooses the delays τ_{ij} and Ω judiciously so that there exists a compatible solution to Equations (43). Note that if $\cos(\Omega\tau_{ij}) > 0$ for all $i,j = 1,\ldots,N$, then the functions

$$f_{ij}(y) = \sin(-\Omega\tau_{ij} + y) - \sin(\Omega\tau_{ij}) \tag{44}$$

have a positive derivative at $x = 0$, i.e. they are *locally passive*. Then the following corollary can be proven:

Corollary 5.1: Consider (40) and assume $\cos(\Omega\tau_{ij}) > 0$ for all $i,j = 1,\ldots,n$, where the graph $\mathcal{G}$ is connected. Define γ by

$$\gamma = \min\left(\left|\frac{2k\pi + \pi - 2\Omega\tau_{ij}}{2}\right|\right), \quad \forall\ k \text{ and } i,j = 1,\ldots,N,$$

and consider initial conditions ψ that satisfy

$$|\psi_i(\theta)| \le \frac{\gamma}{2}, \quad \forall i = 1,\ldots,N.$$

Then the phase-locked equilibrium is asymptotically attracting.

VI. CONCLUSION

In this paper we have presented a general framework for formulating and understanding the nominal properties of large-scale networked systems and then studied the robustness of two large-scale networked systems to communication time delays and time-varying topologies. In section V we referred to several other examples where the theory developed in this paper can be applied.

It is important to emphasize that what underscores the scalability of the functionality properties in these networked systems is the presence of an underlying optimization problem that the network as a

whole tries to solve through the interaction of its subunits. In the case of network congestion control, this takes the form of a network utility maximization objective subject to capacity constraints as it was mentioned in section IV. Such an underlying optimization framework also exists in the case of multi-agent system coordination; in this case, a disagreement function is being minimized subject to information flow constraints between the agents, if they exist. In both cases, the resulting dynamics are gradient-based, and hence a Lyapunov function for the undelayed system can be easily constructed based on these optimization frameworks. This indicates a possible design approach for large-scale networked systems, in which scalable functionality of the nominal system descriptions is guaranteed 'by construction'.

APPENDIX

In this Appendix we will present some background material on Functional Differential Equations that is used in this paper. For more details, see the book by [22].

Recall the notation introduced in Section I-A. Assume Ω is a subset of C, $f : \Omega \to \mathbb{R}^n$ is a given function, and '·' represents the right-hand derivative. Then we call

$$\dot{x}(t) = f(x_t) \tag{45}$$

a Retarded Functional Differential Equation (RFDE) on Ω. Given $\phi \in C$ and $\rho > 0$, a function $x(\phi)$ is said to be a solution to Equation (45) on $[-\tau, \rho)$ with initial condition ϕ, if $x \in C([-\tau, \rho), \mathbb{R}^n)$, $x_t \in \Omega$, $x(t)$ satisfies (45) for $t \in [0, \rho)$ and $x(\phi)(0) = \phi$. Such a solution exists and is unique under certain conditions; see [22] for more details.

An $x^* \in C$ is called a steady-state (equilibrium) of (45) if $x(x^*)(t) = x^*$ for all $t \geq 0$. Without loss of generality we assume that 0 is a steady-state for the system. Definitions of stability and asymptotic stability can be found in [22].

Just as in the case of nonlinear systems described by Ordinary Differential Equations (ODEs), a Lyapunov argument can be formulated for the stability analysis of RFDEs.

Definition A-1: A functional $V : \Omega \to \mathbb{R}$ is said to be a *Lyapunov functional* for (45) if it is continuous and $\dot{V}(\phi) \leq 0$ for all $\phi \in \Omega$ where

$$\dot{V}(\phi) \triangleq \lim \sup_{h \to 0^+} \frac{1}{h} \left(V(x_h(\phi)) - V(\phi) \right).$$

The condition $\dot{V}(\phi) \leq 0$ assures that V is *nonincreasing along solutions of (45)* that remain in Ω. For autonomous systems we have the following theorem (Lyapunov-Krasovskii) [22]:

Theorem A-2: Assume $f : \Omega \to \mathbb{R}^n$ is completely continuous[1] and the solutions of (45) depend continuously on initial data. Suppose that $V : \Omega \to \mathbb{R}$ is continuous and there exist nonnegative continuous functions $a(s)$ and $b(s)$ satisfying $a(0) = b(0) = 0$, $a(s)$ strictly increasing, such that:

$$V(\phi) \geq a(|\phi(0)|) \text{ on } \Omega$$

$$\dot{V}(\phi) \leq -b(|\phi(0)|) \text{ on } \Omega$$

---

[1]Recall that if Ω is a subset of a Banach space C and $A : \Omega \to C$, then A is *completely continuous* if A is continuous and for any bounded set $B \subseteq \Omega$, the closure of AB is compact.

Then the solution $x = 0$ of (45) is stable, and every solution is bounded. If in addition, $b(s) > 0$ for $s > 0$, then $x = 0$ is asymptotically stable.

For RFDEs, there is another Lyapunov-like theorem that can be used to prove stability: the so-called Lyapunov-Razumikhin theorem, which uses functions instead of functionals as certificates for stability.

Definition A-3: Let $D \subseteq \mathbb{R}^n$. By a *Lyapunov-Razumikhin Function $V = V(x)$* we mean a continuous function $V : D \to \mathbb{R}$; the upper right-hand derivative of V with respect to (45) is defined by:

$$\dot{V}(\phi) = \lim \sup_{h \to 0^+} \frac{1}{h}(V(\phi(0) + hf(\phi)) - V(\phi(0))).$$

We have the following Lyapunov-Razumikhin theorem:

Theorem A-4: Suppose $f : \Omega \to \mathbb{R}^n$ takes bounded subsets of Ω into bounded sets of $\mathbb{R}^n$ and consider (45). Suppose $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions, $a(s)$ positive for $s > 0$, $a(0) = 0$. Let $D \subseteq \mathbb{R}^n$. If there is a Lyapunov-Razumikhin Function $V : D \to \mathbb{R}$ such that:

1) $V(x) \geq a(|x|)$ for $x \in D$,

2) $\dot{V}(\phi(0)) \leq -b(\phi(0))$ if $V(\phi(\theta)) \leq V(\phi(0))$ for $\theta \in [-\tau, 0]$ (i.e., if $V(\phi(0)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta))$),

then the solution $x = 0$ of (45) is stable. If, furthermore there is a continuous non-decreasing function $p(s) > s$ for $s > 0$ such as the last condition is strengthened to

$$\dot{V}(\phi(0)) \leq -b(\phi(0)) \text{ if } V(\phi(\theta)) < p(V(\phi(0)))$$

for $\theta \in [-\tau, 0]$ then the solution $x = 0$ of (45) is asymptotically stable.

Note that the function V in Razumikhin's theorem may not be non-increasing along the system trajectories, but may indeed increase within a delay interval. The proof of Razumikhin's theorem is based on the fact that

$$\overline{V}(\phi) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) \tag{46}$$

is a Lyapunov-Krasovskii functional that is non-increasing along the system trajectories. This is an important observation; note that indeed $\overline{V}$ satisfies the first condition in Theorem A-2 if V satisfies the first condition in Theorem A-4. We now argue that the second condition of Theorem A-2 is also satisfied by $\overline{V}$ when the second condition in Theorem A-4 is satisfed by V. Let θ_0 is the value of θ for which the maximum in (46) is achieved, i.e.,

$$\overline{V}(\phi) = V(\phi(\theta_0)).$$

This means that $\dot{\overline{V}}(\phi) = 0$ if $\theta_0 < 0$ (as we have a local maximum at θ_0) and $\dot{\overline{V}}(\phi) \leq 0$ if $\theta_0 = 0$, from the second condition of Theorem A-4; therefore the second condition in Theorem A-2 is also satisfied, and so the two theorems are related as far as stability is concerned. For asymptotic stability relations, see [22].

This paper is also concerned with invariance. For this, we need to define ω-limit sets of solutions, and provide LaSalle-type theorems for functional differential equations. See [22] for more details.

Definition A-5: Let $\phi \in \Omega$. An element ψ of Ω is in $\omega(\phi)$, the ω-*limit set of ϕ*, if $x(\phi)(t)$ is defined on $[-\tau, \infty)$ and there is a sequence of non-negative real numbers $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\phi) - \psi\| \to$

0 as $n \to \infty$. A set $M \subset \Omega$ is said to be *positively invariant* for (45) if for any ϕ in M there is a solution $x(\phi)(t)$ of (45) that is defined on $[-\tau, \infty)$ such that $x_t \in M$ for all $t \geq 0$ and $x_0 = \phi$.

If $x(\phi)(t)$ is a solution of (45) that is defined and bounded on $[-\tau, \infty)$ then the orbit through ϕ, i.e., the set $\{x_t(\phi) : t \geq 0\}$ is precompact, $\omega(\phi)$ is non-empty, compact, connected and invariant, and $x_t(\phi) \to \omega(\phi)$ as $t \to \infty$.

We have already introduced two types of Lyapunov theorems for stability of time-delay systems, with Lyapunov-Krasovskii being the natural extension of Lyapunov's theorem for ODEs. It is therefore expected that the Lyapunov-Krasovskii theorem should have a LaSalle invariance principle extension. Indeed this is the case.

Let Ω be a subset of C. Consider a Lyapunov-Krasovskii functional $V = V(\phi)$ on Ω. We define

$$
\begin{aligned}
S &= \{\phi \in \overline{\Omega} : \dot{V}(\phi) = 0\}. \\
M &= \text{largest set in } S \text{ that is invariant with respect to Equation (45).}
\end{aligned}
$$

where $\overline{\Omega}$ denotes the closure of Ω. M here is the set of functions $\phi \in S$ which can serve as initial conditions for (45) so that the solution $x_t(\phi)$ satisfies $\dot{V}(\phi) = 0$ and $x_t(\phi)$ is in M.

Then we have the following LaSalle type invariance principle [22]:

Theorem A-6: Let V be a Lyapunov-Krasovskii functional of (45) on Ω, and for $\phi \in \Omega$ let $x_t(\phi)$ be a solution of (45) that is bounded on $[-\tau, \infty)$ such that x_t remains in Ω for all $t \geq 0$. Then $x_t \to M$ as $t \to \infty$.

An analogous invariance principle using Lyapunov-Razumikhin functions is more difficult to state. This is due to the fact that these functions are not non-decreasing along the trajectories of the system, as explained earlier. Such an invariance principle has, however, been developed in [49].

Let $V = V(x)$ be a Lyapunov-Razumikhin function. For a given set $\Omega \subset C$, define:

$$
E = \{\phi \in \Omega : \max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) \text{ for all } t \geq 0\} \tag{47}
$$

$$
L = \text{largest set in } E \text{ that is invariant with respect to Equation (45).} \tag{48}
$$

Again, L is the set of functions $\phi \in \Omega$ which can serve as initial conditions for (45) so that $x_t(\phi)$ satisfies

$$
\max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) = \max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta))
$$

for all $t \in (-\infty, \infty)$. Note that the above condition for $\overline{V}$ defined in (46) is indeed a condition that $\dot{\overline{V}}(\phi) = 0$, and so the set E is related to the set S defined earlier. In particular, for a Lyapunov-Razumikhin function V and for any $\phi \in E$, we have $\dot{V}(x_t(\phi)) = 0$ for any $t > 0$ such that $\max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta)) = V(x_t(\phi)(0))$.

In order to formulate an invariance-type theorem using a Lyapunov-Razumihkin function, one can build on its connection and state an invariance principle using Lyapunov-Razumikhin functions [49]:

Theorem A-7: Suppose there exists a Lyapunov-Razumikhin function $V = V(x)$ and a closed set Ω that is positively invariant with respect to (45) such that:

$$
\dot{V}(\phi) \leq 0 \text{ for all } \phi \in \Omega \text{ such that } V(\phi(0)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)).
$$

Then for any $\phi \in \Omega$ such that $x(\phi)(\cdot)$ is defined and bounded on $[-\tau, \infty)$, $\omega(\phi) \subseteq L \subseteq E$. Hence,

$$x_t(\phi) \to L \text{ as } t \to \infty.$$

It is important to note that since V is bounded from below along $x_t(\phi)$,

$$\lim_{t \to \infty} \left\{ \max_{-\tau \le \theta \le 0} V(x_t(\phi)(\theta)) \right\} = c \text{ exists.}$$

References

[1] R. M. Murray (Edited by), *Control in an Information Rich World: Report of the panel on Future Directions in Control, Dynamics and Systems*. SIAM, 2003.

[2] S. Strogatz, *SYNC: The emerging science of spontaneous order*. Hyperion Press, New York, 2003.

[3] A. Jadbabaie, J. Lin, and A. Stephen Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[4] L. Moreau, "Leaderless coordination via bidirectional and unidirectional time-dependent communication," in *Proceedings of the IEEE Conference on Decision and Control*, 2003.

[5] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents.," in *Proceedings of the American Control Conference*, 2003.

[6] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions On Automatic Control*, vol. 49, pp. 1520–1533, 2004.

[7] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 803–812, 1986.

[8] M. Arcak, "Passivity as a design tool for group coordination," in *Proceedings of the American Control Conference*, 2006.

[9] T. Vicsek, "A question of scale," *Nature*, vol. 411, p. 421, 2001.

[10] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[11] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Prentice Hall, 1989.

[12] W. Wang and J. J. E. Slotine, "Contraction analysis of time-delayed communications in group cooperation," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 712–717, 2006.

[13] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[14] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous-time coupled nonlinear systems," *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.

[15] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions On Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[16] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, "On syncrhonous robotic networks, parts i and ii," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2199–2228, 2007.

[17] Y. Liu and K. M. Passino, "Cohesive behaviours of multi-agent systems with information flow constraints," *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1734–1748, 2006.

[18] R. Srikant, *The Mathematics of Internet Congestion Control*. Birkhäuser, 2003.

[19] F. Kelly, A. Maulloo, and D. Tan, "Rate control in communication networks: shadow prices, proportional fairness and stability," *Journal of the Operational Research Society*, vol. 49, pp. 237–252, 1998.

[20] S. H. Low and D. E. Lapsley, "Optimization flow control, I: basic algorithm and convergence," *IEEE/ACM Transactions on Networking*, 1999.

[21] D. P. Palomar and M. Chiang, "Alternative distributed algorithms for network utility maximization: Framework and applications," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2254–2269, 2007.

[22] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. Applied Mathematical Sciences (99), Springer-Verlag, 1993.

[23] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*. Lecture Notes in Control and Information Sciences (269), Springer-Verlag, 2001.

[24] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of time-delay systems*. Birkhäuser, 2003.

[25] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[26] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[27] F. Xiao and L. Wang, "State consensus for multi-agent systems with switching topologies and time-varying delays," *International Journal of Control*, vol. 79, no. 10, pp. 1277–1284, 2006.

[28] J. N. Tsitsiklis, *Problems in Decentralized Decision Making and Computation*. PhD thesis, Laboratory for Information and Decision Systems, MIT, Boston, MA, 1984.

[29] D. Lee and M. W. Spong, "Agreement with non-uniform information delays," in *Proceedings of the American Control Conference*, 2006.

[30] P.-A. Bliman and G. Ferrari-Trecate, "Average consensus problems in networks of agents with delayed communications," 2008. Accepted for Publication in Automatica, preprint available from http://arxiv.org/pdf/math/0503009.pdf.

[31] H. G. Tanner and D. K. Christodoulakis, "State synchronization in local interaction networks is robust wth respect to time delays," in *Proceedings of 44th Conference on Decision and Control and European Contol Conference*, 2005.

[32] A. Papachristodoulou and A. Jadbabaie, "Synchronization in oscillator networks: Switching topologies and non-homogeneous delays," in *Proceedings of the IEEE Conference on Decision and Control*, 2005.

[33] A. Papachristodoulou and A. Jadbabaie, "Synchronization of oscillator networks with heterogeneous delays, switching topologies and nonlinear dynamics," in *Proceedings of the IEEE Conference on Decision and Control*, 2006.

[34] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.

[35] G. Vinnicombe, "On the stability of end-to-end congestion control for the Internet," tech. rep., Cambridge University, CUED/F-INFENG/TR.398, December 2000.

[36] L. Ying, G. E. Dullerud, and R. Srikant, "Global stability of internet congestion controllers with heterogeneous delays," *IEEE/ACM Transactions on Networking*, vol. 14, no. 3, pp. 579–591, 2006.

[37] P. Ranjan, R. J. La, and E. H. Abed, "Global stability of rate control with arbitrary communication delays," *IEEE/ACM Transactions on Networking*, vol. 14, no. 1, pp. 94–107, 2006.

[38] D. Bauso, L. Giarré, and R. Pesenti, "Nonlinear protocols for optimal distributed consensus in networks of dynamic agents," *Systems and Control Letters*, vol. 55, no. 11, pp. 918–928, 2006.

[39] P. Monzón and F. Paganini, "Global considerations on the kuramoto model of sinusoidally coupled oscillators," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005.

[40] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," in *Proceedings of the American Control Conference*, 2004.

[41] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, Inc., third ed., 2001.

[42] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus and flocking," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005.

[43] D. Angeli and P.-A. Bliman, "Stability of leaderless multi-agent systems. Extension of a result by Moreau," 2004. oai:arXiv.org:math/0411338.

[44] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Transactions On Automatic Control*, vol. 51, no. 3, pp. 401–420, 2006.

[45] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Multi-agent system consensus in packet switched networks," in *Proceedings of the European Control Conference*, 2007. Also under review for publication at the International Journal of Control.

[46] N. Chopra and M. W. Spong, "Output synchronization of nonlinear systems with time delay in communication," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006.

[47] N. P. Bhatia and G. P. Szegő, *Stability Theory of Dynamical Systems*. Springer, 2002.

[48] V. Chellaboina, A. Leonessa, and W. M. Haddad, "Generalized lyapunov and invariant set theorems for nonlinear dynamical systems," *Systems and Control Letters*, vol. 38, pp. 289–295, 1999.

[49] J. R. Haddock and J. Terjéki, "Liapunov-Razumikhin functions and an invariance principle for functional differential equations," *Journal of Differential Equations*, vol. 48, pp. 95–122, 1983.

[50] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 470–482, 2004.

[51] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag, "Nonlinear norm-observability notions and stability of switched systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 154–168, 2005.

[52] A. S. Morse, "Dwell-time switching," in *Proceedings of the 2nd European Control Conference*, pp. 176–181, 1993.

[53] F. Paganini, J. Doyle, and S. Low, "Scalable laws for stable network congestion control," in *Proceedings of the 40th IEEE Conference on Decision and Control*, 2001.

[54] R. Johari and D. Tan, "End-to-end congestion control for the Internet: Delays and stability," *IEEE/ACM Transactions on Networking*, vol. 9, no. 6, pp. 818–832, 2001.

[55] Z. Wang and F. Paganini, "Global stability with time-delay in network congestion control," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002.

[56] Z. Wang and F. Paganini, "Global stability with time-delay of a primal-dual congestion control," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, 2003.

[57] S. Deb and R. Srikant, "Global stability of congestion controllers for the Internet," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 1055–1060, 2003.

[58] F. Mazenc and S.-I. Niculescu, "Remarks on the stability of a class of TCP-like congestion control models," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, 2003.

[59] J. Wen and M. Arcak, "A unifying passivity framework for network flow control," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 162–174, 2004.

[60] A. Papachristodoulou, "Global stability analysis of a TCP/AQM protocol for arbitrary networks with delay," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004.

[61] O. Toker and H. Ozbay, "Complexity issues in robust stability of linear delay-differential systems," *Math., Control, Signals, Syst.*, vol. 9, pp. 386–400, 1996.

[62] E. M. Wright, "A non-linear difference-differential equation," *Journal für die Reine und Angewandte Mathematik*, vol. 194, pp. 66–87, 1955.

[63] S. H. Strogatz, "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D*, vol. 143, pp. 1–20, 2000.

[64] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*. Dover, 2003.

[65] N. Chopra and M. W. Spong, "On synchronization of kuramoto oscillators," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005.

[66] M. K. Stephen Yeung and S. H. Strogatz, "Time delay in the Kuramoto model of coupled oscillators," *Physical Review Letters*, vol. 82, no. 3, pp. 648–651, 1999.

[67] M. G. Earl and S. H. Strogatz, "Synchronization in oscillator networks with delayed coupling: A stability criterion," *Physical Review E*, vol. 67, 036204, no. 3, 2003.

[68] C. Li and G. Chen, "Synchronization in general complex dynamical networks with coupling delays," *Physica A*, vol. 343, pp. 263–278, 2004.