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Topics In Higgs Bundles And Modules Over Even Clifford Algebra

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Abstract

The first part of the thesis is a joint work with Sukjoo Lee. It was shown by Diaconescu, Donagi and Pantev that Hitchin systems of type ADE are isomorphic to certain Calabi-Yau integrable systems. In this paper, we prove an analogous result in the setting of meromorphic Hitchin systems of type A which are known to be Poisson integrable systems. We consider a symplectization of the meromorphic Hitchin integrable system, which is a semi-polarized integrable system in the sense ofKontsevich and Soibelman. On the Hitchin side, we show that the moduli space of unordered diagonally framed Higgs bundles forms an integrable system in this sense and recovers the meromorphic Hitchin system as the fiberwise compact quotient. Then we construct a new family of quasi-projective Calabi-Yau threefolds and show that its relative intermediate Jacobian fibration, as a semi-polarized integrable system, is isomorphic to the moduli space of unordered diagonally framed Higgs bundles.

The second part of the thesis studies the relation between the moduli spaces of modules over the sheaf of even Clifford algebra and the Prym variety, both associated to a conic bundle. In particular, we construct a rational map from the moduli space of modules over the sheaf of even Clifford algebra to the special subvarieties in Prym varieties, and check that the rational map is birational in some cases. As an application, we get an explicit correspondence between instanton bundles on cubic threefolds and twisted Higgs bundles.

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ABSTRACT

TOPICS IN HIGGS BUNDLES AND MODULES OVER EVEN CLIFFORD ALGEBRA

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The first part of the thesis is a joint work with Sukjoo Lee. It was shown by Diaconescu, Donagi and Pantev that Hitchin systems of type ADE are isomorphic to certain Calabi-Yau integrable systems. In this paper, we prove an analogous result in the setting of meromorphic Hitchin systems of type A which are known to be Poisson integrable systems. We consider a symplectization of the meromorphic Hitchin integrable system, which is a semi-polarized integrable system in the sense of Kontsevich and Soibelman. On the Hitchin side, we show that the moduli space of unordered diagonally framed Higgs bundles forms an integrable system in this sense and recovers the meromorphic Hitchin system as the fiberwise compact quotient. Then we construct a new family of quasi-projective Calabi-Yau threefolds and show that its relative intermediate Jacobian fibration, as a semi-polarized integrable system, is isomorphic to the moduli space of unordered diagonally framed Higgs bundles.

The second part of the thesis studies the relation between the moduli spaces of modules over the sheaf of even Clifford algebra and the Prym variety, both associated to a conic bundle. In particular, we construct a rational map from the moduli space of modules over the sheaf of even Clifford algebra to the special subvarieties in Prym varieties, and check that the rational map is birational in some cases. As an application, we get an explicit

correspondence between instanton bundles on cubic threefolds and twisted Higgs bundles.

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Chapter 1

Semi-polarized meromorphic Hitchin and Calabi-Yau integrable systems

1.1 Introduction

1.1.1 Introduction

Since the seminal work of Hitchin [Hit87b][Hit87a], Higgs bundles and their moduli spaces have been studied extensively. There have been numerous deep results on the moduli space of Higgs bundles related to other areas of mathematics such as the $P = W$ conjecture [CHM12][CMS19], the fundamental lemma in the Langlands program [Ngô06][Ngô10], the geometric Langlands conjecture [KW07] and mirror symmetry [HT03][DP12]. One of the striking properties of these moduli spaces is that they admit a holomorphic symplectic form and the structure of an integrable system, called the *Hitchin system*. In particular, the generic fiber of an integrable system is an abelian variety which turns out to be the Jacobian or (generalized) Prym variety of an associated spectral or cameral curve. This picture generalizes to the meromorphic situation where we allow the Higgs field to have poles along some divisors. While the meromorphic Hitchin system is no longer symplectic, it is still Poisson and integrable with respect to the Poisson structure.

On the other hand, Donagi-Markman and Donagi-Diaconescu-Pantev (DDP) introduced in [DM96a][DM96b][Dia+06][DDP07] integrable systems coming from some families of projective or quasi-projective Calabi-Yau threefolds, called *Calabi-Yau integrable systems*. A generic fiber is a complex torus or an abelian variety [Dia+06][DDP07], now obtained as the intermediate Jacobian of a Calabi-Yau threefold in the family.

It is shown in [DDP07] that for adjoint groups G of type ADE, there is an isomorphism between G-Hitchin systems and suitable Calabi-Yau integrable systems, which we call the *DDP correspondence*. An interesting aspect of the construction in [DDP07] is that although the relevant Calabi-Yau threefold is non-compact, the (a priori mixed) Hodge structure on its third cohomology happened to be pure of weight one up to Tate twist. Because of this, the corresponding intermediate Jacobian is a compact torus (in fact an abelian variety). Since the data of a weight 1 Hodge structure is equivalent to the data of an abelian variety, this isomorpshism can be rephrased as an isomorphism between variations of weight 1 Hodge structures equipped with the abstract Seiberg-Witten differential, see for example [DDP07] [Bec20].

It is worth mentioning that the origin of this story comes from physics, specifically, large N duality [Dia+06]. Recently, the correspondence has also found its place in the study of

T-branes in F-theory [AHK14][And+17].

The isomorphism between Hitchin and Calabi-Yau integrable systems has been generalized successfully to groups of type BCFG by the work of Beck et al. [Bec20][Bec19][BDW20] using the technique of foldings.

1.1.2 Main results

The goal of this paper is to extend the DDP correspondence to the setting of meromorphic $SL(n, \mathbb{C})$-Hitchin system $h : \mathcal{M}(n, D) \to B$ where D is a reduced divisor of the base curve. The best case scenario will be to construct a family of non-compact Calabi-Yau threefolds over the same base B and show that the associated Calabi-Yau integrable system is isomorphic to the meromorphic Hitchin system as Poisson integrable systems. However, since the deformation space of such non-compact Calabi-Yau's is strictly smaller than the base B, we do not expect to get a natural family which induces the Possion integrable system (see [KS14]).

Instead, we consider the notion of *semi-polarized integrable systems* introduced by Kontsevich-Soibelman [KS14]. These are non-compact versions of symplectic integrable systems whose fiber is a semi-abelian variety, an extension of an abelian variety by an affine torus. The main advantage is that they canonically induce the Poisson integrable systems as their compact quotients. In Section 2, we study this structure from the Hodge theoretic viewpoint. Since the data of a semi-polarized semi-abelian variety is equivalent to the data of a semi-polarized $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,-1),(-1,0),(0,-1)\}$ (see Appendix), the semi-polarized integrable system can be described as an admissible variation of $\mathbb{Z}$-mixed Hodge structures of such type with an abstract Seiberg-Witten differential as in the classical case.

The main objects on the Hitchin side are the moduli space of diagonally framed Higgs bundles (resp. unordered), introduced by Biswas-Logares-Peón-Nieto [BLP19][BLP20][1], and we denote these moduli space by $\overline{\mathcal{M}}^{\Delta}(n,D)$ (resp. $\mathcal{M}^{\Delta}(n,D)$). The moduli space $\overline{\mathcal{M}}^{\Delta}(n,D)$ is a subspace of the moduli space of framed Higgs bundles $\mathcal{M}_F(n,D)$ whose object is a triple (E,θ,δ) where (E,θ) is a $SL(n,\mathbb{C})$-Higgs bundle and δ is a framing of E at D. As the name suggests, an object in $\overline{\mathcal{M}}^{\Delta}(n,D)$ is a framed Higgs bundle such that the residue of its Higgs field is diagonal with respect to the framing δ. The unordered version $\mathcal{M}^{\Delta}(n,D)$ is obtained as the quotient of $\overline{\mathcal{M}}^{\Delta}(n,D)$ by $S_n^{|D|}$ where $S_n^{|D|}$ is the product of symmetric groups S_n acting on the space of the framings by permuting the order of components. The following diagram summarizes the relation among the moduli spaces:

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\Delta}(n,D) & \hookrightarrow & \mathcal{M}_F(n,D) \\ \downarrow q & & \downarrow f_1 \\ \mathcal{M}^{\Delta}(n,D) & \xrightarrow{f_2} & \mathcal{M}(n,D) \\ & \searrow^{h_\Delta} & \downarrow h \\ & & B \end{array} \tag{1.1.1}$$

where $q : \overline{\mathcal{M}}^{\Delta}(n,D) \to \mathcal{M}^{\Delta}(n,D)$ is the quotient map, f_1 and f_2 are the maps of forgetting the framings and $h_\Delta := h \circ f_2 : \mathcal{M}^{\Delta}(n,D) \to B$ is the Hitchin map on the moduli space of unordered diagonally framed Higgs bundles that we will study. In this paper, we will mainly work over the locus $B^{ur} \subset B$ of smooth cameral curves which are unramified over D and have simple ramifications. In particular, for a triple (E,θ,δ) over $b \in B^{ur}$, the residue of θ over D has distinct eigenvalues. We shall write the restrictions as $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur} := (h_\Delta \circ q)^{-1}(B^{ur})$ and $\mathcal{M}^{\Delta}(n,D)^{ur} := h_\Delta^{-1}(B^{ur})$.

[1] In [BLP20], what we call "diagonally framed" is referred to as "relatively framed" in [21].

We will show that $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ and $\mathcal{M}^{\Delta}(n,D)^{ur}$ are symplectic using deformation theoretic arguments. They also carry a smooth semi-polarized integrable system structure over the locus B^{ur}. The following is the first result of the paper.

Theorem 1.1.1. *(Proposition 1.3.25, Corollary 1.3.28) The moduli space of unordered diagonally framed Higgs bundle $\mathcal{M}^{\Delta}(n,D)$ is symplectic. The Hitchin fibration*

$$h_{\Delta}^{ur} : \mathcal{M}^{\Delta}(n,D)^{ur} \to B^{ur}$$

forms a smooth semi-polarized integrable system whose fiber is a semi-abelian variety.

In order to prove this, we study the fiber $(h_{\Delta}^{ur})^{-1}(b)$ over each $b \in B^{ur}$ via the spectral correspondence between unordered diagonally framed Higgs bundles on Σ and framed line bundles on the associated spectral cover $\overline{p}_b : \widetilde{\Sigma}_b \to \Sigma$. The framed line bundles on $\widetilde{\Sigma}_b$ are then parametrized by the Prym variety $Prym(\widetilde{\Sigma}_b^{\circ}, \Sigma^{\circ})$ associated to the restricted spectral cover $\overline{p}_b^{\circ} := \overline{p}_b|_{\widetilde{\Sigma}_b^{\circ}} : \widetilde{\Sigma}_b^{\circ} \to \Sigma^{\circ}$ where $\widetilde{\Sigma}_b^{\circ} := \widetilde{\Sigma}_b \setminus \overline{p}_b^{-1}(D)$ and $\Sigma^{\circ} := \Sigma \setminus D$. More precisely, $Prym(\widetilde{\Sigma}_b^{\circ}, \Sigma^{\circ})$ is a semi-abelian variety defined as the kernel of the punctured norm map $Nm^{\circ} : Jac(\widetilde{\Sigma}_b^{\circ}) \to Jac(\Sigma^{\circ})$.

Proposition 1.1.2. *(Proposition 1.3.12, Spectral correspondence) A generic fiber $h_{\Delta}^{-1}(b)$ is canonically isomorphic to the semi-abelian variety $Prym(\overline{\Sigma}_b^{\circ}, \Sigma^{\circ})$. In particular, the first homology $H_1(Prym(\overline{\Sigma}_b^{\circ}, \Sigma^{\circ}))$ admits a $\mathbb{Z}$-mixed Hodge structure of type* $\{(-1,-1),(-1,0),(0,-1)\}$.

On the Calabi-Yau side, we construct a family of Calabi-Yau threefolds $\pi : \mathcal{X} \to B$ by using the elementary modification technique in [Smi15]. To produce the relevant Calabi-Yau integrable systems, we should restrict the family $\pi : \mathcal{X} \to B$ to B^{ur}, denoted by $\pi^{ur} : \mathcal{X}^{ur} \to$

B^{ur}, whose fiber is smooth and its third homology admits a $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1),(-1,0),(0,-1)\}$ up to Tate twist. Now, by taking fiberwise intermediate Jacobians, we obtain a family of semi-abelian varieties $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to B^{ur}$. The local period map induces an integrable system structure of this family.

The main result of the paper is to establish an isomorphism between the two semi-polarized integrable systems:

Theorem 1.1.3. *(Theorem 1.5.1) There is an isomorphism of smooth semi-polarized integrable systems*

$$\begin{array}{ccc} \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \mathcal{M}^{\Delta}(n,D)^{ur} \\ & {\scriptstyle \pi^{ur}} \searrow \quad \swarrow {\scriptstyle h^{ur}_{\Delta}} & \\ & B^{ur} & \end{array} \tag{1.1.2}$$

The idea is to compare the admissible variations of $\mathbb{Z}$-mixed Hodge structures associated to the two semi-polarized integrable systems, by using the gluing techniques in [DDP07], [Bec20]. To complete the proof, we check that the comparison map intertwines the abstract Seiberg-Witten differentials on each side.

1.1.3 Related work

The ideas of the spectral correspondence for unordered diagonally framed Higgs bundles and the infinitesimal study of their moduli spaces are drawn from [BLP19]. We follow their approach closely in Section 1.3.3. However, we provide an improvement of their result in order to show that $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ and $\mathcal{M}^{\Delta}(n,D)^{ur}$ are symplectic which was not proved before. We also focus more on the Hodge structures of the relevant Hitchin fibers to prove Theorem 1.1.3.

A general construction of the moduli space of unordered diagonally framed Higgs bundles

$\mathcal{M}^{\Delta}(n, D)$ comes from symplectic implosion [GJS02] associated to the level group action on $\mathcal{M}_F(n, D)$, viewed as the cotangent bundle of the moduli of framed bundles [Mar94]. One can obtain the Hitchin fibration over the full base B, but it is a stratified space and very singular which makes it difficult to control. Indeed, as we only need the smooth part for our main result, we focus on Higgs fields that are diagonalizable over D throughout the paper.

Kontsevich-Soibelman proposed a different construction of the relevant Calabi-Yau integrable system as an affine conic bundle over a holomorphic symplectic surface containing a given spectral curve (see [KS14]). This can be done by blowing up intersections of spectral curves and the preimage of the divisor D in the total space of the twisted cotangent bundle $K_\Sigma(D)$. After removing the proper transform of the preimage, one gets the desired symplectic holomorphic surface. This model is birationally equivalent to the one we introduce in Section 4.

1.1.4 Plan

We first recollect the basics of integrable systems and introduce the notion of a semi-polarized integrable system in Section 2. In Section 3, we study the integrable system structure of the moduli space of unordered diagonally framed Higgs bundle. Also, we give both the spectral and cameral descriptions for completeness. In Section 4, we construct the semi-polarized Calabi-Yau integrable systems by using the technique of elementary modification. It is then followed by a Hodge theoretic computation. Finally, in Section 5, we give a proof of Theorem 1.1.3.

1.1.5 Notation

- Σ - a non-singular curve of genus g.
- D - an effective divisor of d reduced points.
- Σ° - the complement of the divisor D in Σ.
- $\mathcal{M}(n, D)$ - the moduli space of $K_\Sigma(D)$-twisted $SL(n, \mathbb{C})$-Higgs bundles.
- $\mathcal{M}_F(n, D)$ - the moduli space of framed Higgs bundles.
- $\overline{\mathcal{M}}^\Delta(n, D)$ - the moduli space of diagonally framed Higgs bundles.
- $\mathcal{M}^\Delta(n, D)$ - the moduli space of unordered diagonally framed Higgs bundles.
- $B = \oplus_{i=2}^n H^0(\Sigma, K_\Sigma(D)^{\otimes i})$ - the Hitchin base.
- $B^{ur} \subset B$ - the subset consists of smooth cameral curves which are unramified over D and have simple ramifications. Throughout the paper, we will always assume an element $b \in B$ is sitting in B^{ur}.
- $\overline{p}_b : \widetilde{\Sigma}_b \to \Sigma$ - the spectral cover for $b \in B$.
- $\widetilde{p}_b : \widetilde{\Sigma}_b \to \Sigma$ - the cameral cover for $b \in B$.

1.2 Semi-polarized integrable systems

In this section, we recall the notion of a *semi-polarized integrable system*, originally introduced in [KS14]. This is a non-compact generalization of the notion of algebraic integrable system [Hit87a] which provides a new way to view integrable systems in the Poisson setting. Similarly to the classical setting where algebraic integrable systems can be associated with variations of polarized weight one Hodge structures, we also have a Hodge-theoretic description of semi-polarized integrable systems. To make the paper self-contained, we shall begin reviewing basics of algebraic integrable systems by following [Bec20][Bec19].

1.2.1 Integrable systems and variations of Hodge structures

Definition 1.2.1. Let (M^{2n}, ω) be a holomorphic symplectic manifold of dimension $2n$ and B be a connected complex manifold of dimension n. A holomorphic map $\pi : M \to B$ is called an *algebraic integrable system* if it satisfies the following conditions.

1. π is proper and surjective;
2. there exists a Zariski open dense subset $B^\circ \subset B$ such that the restriction

$$\pi^\circ := \pi|_{M^\circ} : M^\circ \to B^\circ, \qquad M^\circ := \pi^{-1}(B^\circ)$$

has smooth connected Lagrangian fibers and admits a relative polarization of index 0.

In particular, if $B^\circ = B$, then (M, ω, π) is called a *smooth* algebraic integrable system.

The second condition that a generic fiber is Lagrangian puts rather restrictive constraints on the geometry of the fiber. To see this, first consider $\ker(d\pi^\circ)$, the sheaf of vector fields on M° which are tangent to the fibers of π°. Since the fibers of π° are Lagrangians, the holomorphic symplectic form ω induces an isomorphism $\ker(d\pi^\circ) \cong (\pi^\circ)^* T^\vee B^\circ$ via $v \mapsto \omega(v, -)$. By taking pushforward to B°, we have an isomorphism of coherent sheaves $\pi^\circ_* \ker(d\pi^\circ) \cong \pi^\circ_*(\pi^\circ)^* T^\vee B^\circ$. In fact, one can apply the projection formula and see $\pi^\circ_*(\pi^\circ)^* T^\vee B^\circ \cong T^\vee B^\circ$ because the fibers of π° are connected. Thus, the sheaf $\pi^\circ_* \ker(d\pi^\circ)$ is isomorphic to $T^\vee B^\circ$, hence locally free. We denote it by $\mathcal{V}$ and call it a vertical bundle of π°.

Next, choose a sufficiently small open subset $U \subset B^\circ$ and two local sections $u, v : U \to \mathcal{V}$ such that they are Hamiltonian vector fields $u = X_{(\pi^\circ)^* f}, v = X_{(\pi^\circ)^* g}$ for the functions $f, g : U \to \mathbb{C}$. As the fibers of π° are Lagrangians, we have $[u, v] = X_{\omega(u,v)} = 0$. It implies that the Lie algebra $(\mathcal{V}, [-,-])$ is abelian so that one can define a group action of $\mathcal{V}$ on M° via the fiberwise exponential map. In other words, the flows of the vector fields along the fibers of π° corresponding to the sections of $\mathcal{V}$ act on M° while preserving the fibers of π°.

The submanifold

$$\Gamma = \{v \in \mathcal{V} | \, \exists \, x \in M^\circ \text{such that} \quad v \cdot x = x\}$$

forms a full lattice in each fiber and induces a family of abelian varieties $\mathcal{A}(\pi^\circ) := \mathcal{V}/\Gamma \to B^\circ$ which acts simply transitively on $\pi^\circ : M^\circ \to B^\circ$. Therefore, a generic fiber of $\pi : M \to B$ is non-canonically isomorphic to an abelian variety.

From now on, we will focus on smooth integrable systems ($B^\circ = B$). From the viewpoint of Hodge theory, a family of polarized abelian varieties can be obtained from a variation of weight 1 polarized $\mathbb{Z}$-Hodge structures $\mathsf{V} = (V_\mathbb{Z}, F^\bullet V_\mathcal{O}, Q)$ over B where $V_\mathcal{O} := V_\mathbb{C} \otimes \mathcal{O}_B$ and $F^\bullet$ is the Hodge filtration. This is done by taking the relative Jacobian fibration so that we have the family

$$p : \mathcal{J}(\mathsf{V}) := \mathbf{Tot}(V_\mathcal{O}/(F^1 V_\mathcal{O} + V_\mathbb{Z})) \to B \tag{1.2.1}$$

whose vertical bundle is $\mathcal{V} := V_\mathcal{O}/F^1 V_\mathcal{O} \to B$.

A natural question is a condition for the family $p : \mathcal{J}(\mathsf{V}) \to B$ being an integrable system. In other words, we need a symplectic form on $\mathcal{J}(\mathsf{V})$ where fibers are connected Lagrangians. This can be achieved by the following theorem.

Theorem 1.2.2. *[Bec20] Let* $\mathsf{V} = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}}, Q)$ *be a variation of weight 1 polarized* $\mathbb{Z}$*-Hodge structures over* B *and* ∇^{GM} *be the Gauss-Manin connection on* $V_{\mathcal{O}}$*. Assume that there exists a global section* $\lambda \in H^0(B, V_{\mathcal{O}})$ *such that*

$$\begin{aligned} \phi_\lambda : & TB \to F^1 V_{\mathcal{O}} \\ & \mu \mapsto \nabla^{GM}_\mu \lambda \end{aligned}$$

is an isomorphism. Then the polarization Q *induces a canonical symplectic form* ω_λ *on* $\mathcal{J}(\mathsf{V})$ *such that the induced zero section becomes Lagrangian. Moreover, the symplectic form is independent of the polarization* Q *up to symplectomorphisms.*

Consider the dual variation of Hodge structure of V, $\mathsf{V}^\vee = \mathrm{Hom}_{\mathsf{VHS}}(\mathsf{V}, \mathbb{Z}_B)(-1)$ over B. The polarization Q identifies $\mathcal{V} = V_{\mathcal{O}}/F^1 V_{\mathcal{O}}$ with $F^1 V_{\mathcal{O}}^\vee$. Consider the compositions

$$\iota : \mathcal{V} \xrightarrow{\psi_Q} F^1 V_{\mathcal{O}}^\vee \xrightarrow{\phi_\lambda^\vee} T^\vee B. \tag{1.2.2}$$

where ψ_Q is the identification induced by the polarization Q and $\phi_\lambda^\vee$ is dual of ϕ_λ. Then the lattice $V_{\mathbb{Z}}$ in $\mathcal{V}$ embeds into $T^\vee B$ as a Lagrangian submanifold. Therefore, we obtain a symplectic structure from the canonical one on $T^\vee B$ by descending to $\mathcal{J}(V_{\mathcal{O}}) \cong T^\vee B/\iota(V_{\mathbb{Z}})$. We call such λ an *abstract Seiberg-Witten differential* [Bec20][Don97].

1.2.2 Semi-polarized integrable systems and variations of mixed Hodge structures

One can generalize the notion of an algebraic integrable system by allowing fibers to be non-proper. This is the main object of our study, first introduced in [KS14]. We recall the definition in a form convenient for our story.

Definition 1.2.3. Let (M^{2n+2k}, ω) be a holomorphic symplectic manifold of dimension $2n+2k$ and B be a connected complex manifold of dimension $n+k$. A holomorphic map $\pi : M \to B$ is called a *semi-polarized integrable system* if it satisfies the following conditions.

1. π is flat and surjective;

2. there exists a Zariski open dense subset $B^\circ \subset B$ such that the restriction

$$\pi^\circ := \pi|_{M^\circ} : M^\circ \to B^\circ, \qquad M^\circ := \pi^{-1}(B^\circ)$$

   has smooth connected Lagrangian fibers;

3. each fiber of π° is a semi-abelian variety which is an extension of a n-dimensional polarized abelian variety by a k-dimensional affine torus.

In particular, if $B^\circ = B$, then (M, ω, π) is called a *smooth* semi-polarized integrable system.

Similar to the classical case, the main example comes from an admissible variation of torsion-free $\mathbb{Z}$-mixed Hodge structures. Let $\mathsf{V} = (V_{\mathbb{Z}}, W_\bullet V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ be an admissible variation of $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1), (-1,0), (0,-1)\}$ over B where $V_{\mathcal{O}} := V_{\mathbb{C}} \otimes \mathcal{O}_B$ and $\mathrm{Gr}^W_{-1} V_{\mathbb{C}}$ is polarizable. In other words, we have

- $0 = W_{-3} \subset W_{-2} \subset W_{-1} = V_{\mathbb{Z}}$

- $0 = F^1 \subset F^0 \subset F^{-1} = V_{\mathcal{O}}$

and can choose a relative polarization on $\mathrm{Gr}^W_{-1} V_{\mathcal{O}}$. Throughout this paper, we choose a semi-polarization on $V_{\mathbb{Z}}$, a degenerate bilinear form $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}_B$ which yields the relative polarization on $\mathrm{Gr}^W_{-1} V_{\mathcal{O}}$. We call it a variation of semi-polarized $\mathbb{Z}$-mixed Hodge structures. Moreover, one can obtain a semi-abelian variety from a $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,-1),(-1,0),(0,-1)\}$ by taking the Jacobian (see Appendix). Therefore, we have a family of semi-abelian varieties by taking the relative Jacobian fibration

$$p:\mathcal{J}(\mathsf{V}):=\mathbf{Tot}(V_{\mathcal{O}}/(F^0V_{\mathcal{O}}+V_{\mathbb{Z}}))\to B \tag{1.2.3}$$

with its compact quotient $p_{\mathrm{cpt}}:\mathcal{J}_{\mathrm{cpt}}(\mathsf{V}):=\mathbf{Tot}(W_{-1}V_{\mathcal{O}}/(W_{-1}V_{\mathcal{O}}\cap F^0V_{\mathcal{O}}+V_{\mathbb{Z}}))\to B$.

To define an abstract Seiberg-Witten differential, we consider the dual variation of $\mathbb{Z}$-mixed Hodge structures $\mathsf{V}^\vee=(V_{\mathbb{Z}}^\vee,W_\bullet V_{\mathbb{Z}}^\vee,F^\bullet V_{\mathcal{O}}^\vee):=\mathrm{Hom}_{\mathrm{VMHS}}(\mathsf{V},\mathbb{Z}_B)$ of V. Note that we don't take a Tate twist so that it is of type $\{(0,1),(1,0),(1,1)\}$. Unlike the classical case, the Seiberg-Witten differential is defined as a global section of the dual vector bundle $V_{\mathcal{O}}^\vee$.

Definition 1.2.4. Let $\mathsf{V}=(V_{\mathbb{Z}},W_\bullet V_{\mathbb{Z}},F^\bullet V_{\mathcal{O}},Q)$ be an admissible variation of semi-polarized $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1),(-1,0),(0,-1)\}$ over B, and ∇^{GM} be the Gauss-Manin connection on $V_{\mathcal{O}}$. We define an abstract Seiberg-Witten differential as a global section of the dual bundle $V_{\mathcal{O}}^\vee$, $\lambda\in H^0(B,V_{\mathcal{O}}^\vee)$, such that the following morphism

$$\begin{aligned}\phi_\lambda:&TB\to F^1V_{\mathcal{O}}^\vee\\ &\mu\mapsto\nabla_\mu^{GM}\lambda\end{aligned} \tag{1.2.4}$$

is an isomorphism.

It is clear that the vertical bundle $\mathcal{V}$ of $\mathcal{J}(\mathsf{V})\to B$ can be identified with $(F^1V_{\mathcal{O}}^\vee)^\vee$ via the canonical non-degenerate pairing, $V_{\mathcal{O}}/F^0V_{\mathcal{O}}\otimes F^1V_{\mathcal{O}}^\vee\to\mathcal{O}_B$. Consider the composition

$$\iota:\mathcal{V}\to(F^1V_{\mathcal{O}}^\vee)^\vee\xrightarrow{\phi_\lambda^\vee}T^\vee B$$

under which the lattice $V_{\mathbb{Z}}\subset\mathcal{V}$ embeds into $T^\vee B$ as a Lagrangian submanifold. Similar to Theorem 1.2.2, we obtain a symplectic form from the canonical one on $T^\vee B$ with Lagrangian condition on a generic fiber. Moreover, the total space $\mathcal{J}(\mathsf{V})$ has a canonical Possion structure associated to the given symplectic form. As the action of the affine torus on $\mathcal{J}(\mathsf{V})$ is Hamiltonian, free and proper, the quotient space $\mathcal{J}_{\text{cpt}}(\mathsf{V})$ is a Poisson manifold. Thus, $\mathcal{J}_{\text{cpt}}(\mathsf{V})$ has a Poisson integrable system structure whose symplectic leaves are locally parametrized by $\phi_\lambda^{-1}(\mathrm{Gr}_2^W V_{\mathcal{O}}^\vee \cap F^1 V_{\mathcal{O}}^\vee)$ (see [KS14, Section 4.2] for more details). This proves the following proposition.

Proposition 1.2.5. *Let* $\mathsf{V} = (V_{\mathbb{Z}}, W_\bullet V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}}, Q)$ *be an admissible variation of semi-polarized* $\mathbb{Z}$*-mixed Hodge structures of type* $\{(-1,-1),(-1,0),(0,-1)\}$ *over* B *and* $\lambda \in H^0(B, V_{\mathcal{O}}^\vee)$ *be the Seiberg-Witten differential. Then, the relative Jacobian fibration*

$$p : \mathcal{J}(\mathsf{V}) := \boldsymbol{Tot}(V_{\mathcal{O}}/(F^0 V_{\mathcal{O}} + V_{\mathbb{Z}})) \to B \tag{1.2.5}$$

forms a semi-polarized integrable system. In particular, the compact quotient $\mathcal{J}_{cpt}(\mathsf{V}) \to B$ *admits a Poisson integrable system structure.*

Remark 1.2.6. The reason we take a global section of the dual vector bundle in the definition of Seiberg-Witten differential is that, unlike the classical case, the semi-polarization Q does not induce the canonical identification between V and $\mathsf{V}^\vee$. Moreover, this is also motivated by the geometric examples we will consider where $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}}^\vee$ are torsion-free integral homology and cohomology of a non-singular quasi-projective variety, respectively.

Remark 1.2.7. In [KS14], Kontsevich and Soibelman introduce the notion of a central charge $Z \in H^0(B, V_{\mathcal{O}}^\vee)$ which induces an local embedding of the base into $V_{\mathcal{O}}^\vee$. It is equivalent to the data of an abstract Seiberg-Witten differential which suits our story better.

1.3 Moduli space of diagonally framed Higgs bundles

In this section, we will study the moduli space of (unordered) diagonally framed Higgs bundles and the associated Hitchin map as introduced in [BLP19]. In particular, we will give the spectral and Hodge theoretic description of the generic Hitchin fiber. Then we prove that it is a semi-polarized integrable system in two different ways: using deformation theory and using abstract Seiberg-Witten differentials. As mentioned in Section 1, parts of this section will follow the approach of [BLP19]. For basic properties of Hitchin systems and spectral covers, we refer to [DM96b].

1.3.1 The moduli space of (unordered) diagonally framed Higgs bundles

We fix Σ to be a smooth curve of genus g, D a reduced divisor on Σ and $\Sigma^\circ := \Sigma \setminus D$.

Definition 1.3.1. A framed $SL(n, \mathbb{C})$-Higgs bundle on Σ is a triple (E, θ, δ), where E is a vector bundle of rank n with trivial determinant, $\delta : E_D \xrightarrow{\sim} \oplus_{i=1}^n \mathcal{O}_D$ is an isomorphism, i.e. a framing at D, and $\theta \in \Gamma(\Sigma, End_0(E) \otimes K_\Sigma(D))$ is a traceless Higgs field.

A morphism between framed Higgs bundles (E, θ, δ) and (E', θ', δ') is a map $f : E \to E'$ such that $\delta \circ f|_D = \delta'$ and $\theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta$.

Remark 1.3.2. A framed $GL(n, \mathbb{C})$-Higgs bundle and $PGL(n, \mathbb{C})$-Higgs bundle are defined in a similar way.

In order to discuss moduli spaces, we first define the stability conditions we will be using. We shall follow the definition of stability conditions in [BLP19]. Essentially, the stability condition for a framed Higgs bundle is just the stability condition for a $K_\Sigma(D)$-twisted Higgs bundle. More precisely, we say that a framed Higgs bundle (E, θ, δ) is stable (semistable respectively) if for every θ-invariant proper subbundle $F \subset E$, that is, $\theta(F) \subset F \otimes K(D)$, we have $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$ respectively). Here we write μ for the slope $\mu(E) = \deg(E)/\dim(E)$.

The following lemma and the next corollary can be found in [BLP19, Lemma 2.3]. We record them here for future reference. Let (E, θ) and (E, θ') be $K_\Sigma(D)$-valued semistable Higgs bundles on Σ with $\mu(E) = \mu(E')$.

Lemma 1.3.3. *Let $f : E \to E'$ be a $\mathcal{O}_\Sigma$-modules homomorphism such that*

1. $\theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta$,

2. *there is a point $x_0 \in \Sigma$ such that $f|_{x_0} = 0$,*

then f vanishes identically.

Corollary 1.3.4. *A semistable framed Higgs bundle admits no non-trivial automorphism.*

Proof. Indeed, suppose (E, θ, δ) admits an automorphism h, then the morphism $h - Id_E$ vanishes on D. By the Lemma 1.3.3 above, $h - Id_E$ vanishes identically or equivalently $h = Id_E$. □

We denote $\mathfrak{g} := \mathfrak{sl}_n$ ($\mathfrak{gl}_n$ respectively) and $\mathfrak{g}_E := End_0(E)$ ($End(E)$ respectively). For our discussion, we will only consider the case of $\mathfrak{sl}_n$. Let $\mathfrak{t}$ be the vector subspace of diagonal traceless $n \times n$ matrices and $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{t}$ with respect to the Killing form, i.e. the vector subspace of $n \times n$ matrices whose diagonal entries are all zero. We have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$. Given a framing δ of E, we can define the δ-restrictions to D as the compositions:

$$\mathfrak{g}_E \twoheadrightarrow \mathfrak{g}_E \otimes \mathcal{O}_D \xrightarrow{ad_\delta} \mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{q} \otimes \mathcal{O}_D$$

$$\mathfrak{g}_E \twoheadrightarrow \mathfrak{g}_E \otimes \mathcal{O}_D \xrightarrow{ad_\delta} \mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{t} \otimes \mathcal{O}_D$$

where the maps $\mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{q} \otimes \mathcal{O}_D$ and $\mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{t} \otimes \mathcal{O}_D$ are given by the projections for the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$.

Given a framed bundle (E, δ), we define subsheaves $\mathfrak{g}'_E, \mathfrak{g}''_E \subset \mathfrak{g}_E$ as the kernels

$$0 \to \mathfrak{g}'_E \to \mathfrak{g}_E \to \mathfrak{q}_D := i_* \mathfrak{q} \to 0$$

$$0 \to \mathfrak{g}''_E \to \mathfrak{g}_E \to \mathfrak{t}_D := i_* \mathfrak{t} \to 0$$

where $i : D \hookrightarrow \Sigma$ is the inclusion. In other words, a section of endomorphism in $\mathfrak{g}'_E$ ($\mathfrak{g}''_E$ respectively) restricted to $p \in D$ is diagonal (anti-diagonal respectively) with respect to δ.

Definition 1.3.5. We say that a framed Higgs bundle (E, θ, δ) is diagonally framed if $\theta \in H^0(\Sigma, \mathfrak{g}'_E \otimes K_\Sigma(D)) \subset H^0(\Sigma, \mathfrak{g}_E \otimes K_\Sigma(D))$.

By the results of [Sim94b][Sim94a] [BLP19, Section 2], it is shown that the moduli space of semistable framed $SL(n, \mathbb{C})$-Higgs bundles $\mathcal{M}_F(n, D)$ exists as a fine moduli space that is a smooth irreducible quasi-projective variety. The moduli space we are interested in is the moduli space of semistable diagonally framed $SL(n, \mathbb{C})$-Higgs bundle, denoted by $\overline{\mathcal{M}}^\Delta(n, D)$. It is clear that $\overline{\mathcal{M}}^\Delta(n, D)$ is a subvariety of $\mathcal{M}_F(n, D)$.

Remark 1.3.6. Unless mentioned otherwise, we will assume all diagonally framed Higgs bundles are semistable with structure group $SL(n, \mathbb{C})$ throughout the paper.

For each $p \in D$, there is a natural S_n-action on $\oplus_{i=1}^n \mathcal{O}_p$ by permuting the order of the components

$$\sigma : \oplus_{i=1}^n \mathcal{O}_p \xrightarrow{\sim} \oplus_{i=1}^n \mathcal{O}_p, \quad (s_1, ..., s_n) \mapsto (s_{\sigma(1)}, ..., s_{\sigma(n)}), \quad \text{where } \sigma \in S_n.$$

For each $p \in D$, this induces a S_n-action on the space of framings

$$\sigma \cdot \delta = \sigma \circ \delta : E|_p \to \oplus_{i=1}^n \mathcal{O}_p \xrightarrow{\sigma} \oplus_{i=1}^n \mathcal{O}_p.$$

Hence, the moduli spaces $\overline{\mathcal{M}}^{\Delta}(n, D)$ and $\mathcal{M}_F(n, D)$ admit a $S_n^{|D|}$-action: for $\underline{\sigma} \in S_n^{|D|}$,

$$\underline{\sigma} : (E, \theta, \delta) \mapsto (E, \theta, \underline{\sigma} \cdot \delta), \quad \text{where} \quad \underline{\sigma} \cdot \delta : E|_D \to \oplus_{i=1}^n \mathcal{O}_D \to \oplus_{i=1}^n \mathcal{O}_D.$$

Since the group is finite, we can consider the quotient $\mathcal{M}_F(n, D)/(S_n^{|D|})$. The effect of taking quotient is that, for a fixed Higgs bundle, framings that differ only in reordering of components will be identified. More precisely, a morphism between unordered framed Higgs bundles (E, θ, δ) and (E', θ', δ') is a map $f : E \to E'$ such that

$$\delta \circ f|_D = \underline{\sigma} \circ \delta' \text{ for some } \underline{\sigma} \in S_n^{|D|}, \quad \theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta.$$

In other words, $\mathcal{M}_F(n, D)/(S_n^{|D|})$ now parametrizes unordered framed Higgs bundles. However, this group action is not free. In order to get a free action by $S_n^{|D|}$, we will assume that the associated spectral curve is smooth and unramified over D, or equivalently, the residue of θ at D has distinct eigenvalues. More precisely, we define B^{ur} to be the locus of smooth cameral curves (see Section 1.3.4) which are unramified over D and have simple ramifications. Of course, the associated spectral curve for $b \in B^{ur}$ is automatically a smooth spectral curve that is unramified over D, and the necessity to work with smooth cameral curve with simple ramifications will be explained in Section 5. Moreover, we restrict to the subvariety $\overline{\mathcal{M}}^{\Delta}(n, D)^{ur} := \overline{h}_{\Delta}^{-1}(B^{ur})$ where $\overline{h}_{\Delta}$ denotes the composition $\overline{\mathcal{M}}^{\Delta}(n, D) \hookrightarrow \mathcal{M}_F(n, D) \xrightarrow{f_1} \mathcal{M}(n, D) \xrightarrow{h} B$ and f_1 denotes the forgetful map.

Lemma 1.3.7. *The $S_n^{|D|}$-action on $\overline{\mathcal{M}}^{\Delta}(n, D)^{ur}$ is free.*

Proof. Consider $(E, \theta, \delta) \in \overline{\mathcal{M}}^{\Delta}(n, D)^{ur}$ and suppose that there exists $\underline{\sigma} \in S_n^{|D|}$ and an isomorphism $\alpha : (E, \theta, \delta) \to (E, \theta, \underline{\sigma} \circ \delta)$. The compability condition $\delta \circ \alpha|_D = \underline{\sigma} \circ \delta$ implies that $\delta \circ \alpha|_D \circ \delta^{-1} = \underline{\sigma}$, while the compatibility condition $\theta \circ \alpha = (\alpha \otimes Id_{K_\Sigma(D)}) \circ \theta$ restricted to D is equivalent to $\theta_\delta \circ \underline{\sigma} = \underline{\sigma} \circ \theta_\delta$ where $\theta_\delta := \delta^{-1}\theta|_D\delta$. The last relation $\theta_\delta \circ \underline{\sigma} = \underline{\sigma} \circ \theta_\delta$ is clearly not possible as θ_δ is diagonal with distinct eigenvalues at each $p \in D$. $\square$

Since the $S_n^{|D|}$-action on $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$ is finite and free, we get a geometric quotient $\mathcal{M}^\Delta(n, D)^{ur} := \overline{\mathcal{M}}^\Delta(n, D)^{ur}/(S_n^{|D|})$. The variety $\mathcal{M}^\Delta(n, D)^{ur}$ parametrizes unordered diagonally framed Higgs bundles.

Clearly, there is a morphism $f_2 : \mathcal{M}^\Delta(n, D)^{ur} \to \mathcal{M}(n, D)^{ur} := h^{-1}(B^{ur})$ by forgetting the framings. For our purpose of proving Theorem 1.1.3, we will need to study the composition of the forgetful map f_2 and the Hitchin map h, denoted by $h^{ur}_\Delta : \mathcal{M}^\Delta(n, D)^{ur} \xrightarrow{f_2} \mathcal{M}(n, D)^{ur} \xrightarrow{h^{ur}} B^{ur}$. We summarize the relation among the moduli spaces over B^{ur}:

$$\begin{array}{ccc} \overline{\mathcal{M}}^\Delta(n, D)^{ur} & \hookrightarrow & \mathcal{M}_F(n, D)^{ur} \\ \downarrow q & & \downarrow f_1 \\ \mathcal{M}^\Delta(n, D)^{ur} & \xrightarrow{f_2} & \mathcal{M}(n, D)^{ur} \\ & \searrow^{h^{ur}_\Delta} & \downarrow h^{ur} \\ & & B^{ur} \end{array} \tag{1.3.1}$$

where $\mathcal{M}_F(n, D)^{ur} := (h \circ f_1)^{-1}(B^{ur})$.

1.3.2 Spectral correspondence

We explain the spectral correspondence for unordered diagonally framed Higgs bundles (see Proposition 1.3.12). After that, we describe the Hodge structures of a generic Hitchin fiber which will be used in the proof of the main theorem.

Definition 1.3.8. Let D be an effective reduced divisor on C. A D-framed line bundle on a curve C is a pair (L, β) where L is a line bundle and $\beta : L|_D \xrightarrow{\sim} \mathcal{O}_D$ is an isomorphism.

Remark 1.3.9. Unless mentioned otherwise, we will call (L, β) a framed line bundle whenever the divisor D is clear from the context.

Proposition 1.3.10. *Let C be a smooth curve and D a reduced divisor on C. Let $C^\circ = C \setminus D$, $j : C^\circ \to C$ and $i : D \to C$ be the natural inclusions. The isomorphism classes of degree 0 framed line bundles on C are parametrized by the generalized Jacobian*

$$Jac(C^\circ) := \frac{H^0(C, \Omega_C(\log D))^\vee}{H_1(C^\circ, \mathbb{Z})}. \tag{1.3.2}$$

Proof. By duality, we can identify

$$Jac(C^\circ) = \frac{H^0(C, \Omega_C(\log D))^\vee}{H_1(C^\circ, \mathbb{Z})} \cong \frac{H^1(C, \mathcal{O}(-D))}{H^1(C, D, \mathbb{Z})}$$

Consider the exponential sequence

$$0 \to j_!\mathbb{Z} \to \mathcal{O}_C(-D) \xrightarrow{\exp(2\pi i(-))} \mathcal{O}_C^*(-D) \to 0$$

where $\mathcal{O}_C^*(-D)$ is defined as the subsheaf of $\mathcal{O}_C^*$ consisting of functions with value 1 on D. It induces a long exact sequence

$$\begin{aligned} \cdots \to & H^1(C, j_!\mathbb{Z}) \cong H^1(C, D, \mathbb{Z}) \to H^1(C, \mathcal{O}_C(-D)) \to H^1(C, \mathcal{O}_C^*(-D)) \\ \xrightarrow{c_1} & H^2(C, j_!\mathbb{Z}) \cong H^2(C, D, \mathbb{Z}) \to H^2(C, \mathcal{O}_C(-D)) \to H^2(C, \mathcal{O}_C^*(-D)) \to \cdots \end{aligned}$$

where the map $c_1 : H^1(C, \mathcal{O}_C^*(-D)) \to H^2(C, j_!\mathbb{Z}) \cong H^2(C, D, \mathbb{Z}) \cong H^2(C, \mathbb{Z}) \cong \mathbb{Z}$ can be interpreted as the first Chern class map. The group $H^1(C, \mathcal{O}_C^*(-D))$ naturally parametrizes all framed line bundles. Indeed, the sheaf $\mathcal{O}_C^*(-D)$ sits in a short exact sequence

$$1 \to \mathcal{O}_C^*(-D) \to \mathcal{O}_C^* \to i_*\mathbb{C}^* \to 1$$

which induces a quasi-isomorphism $\mathcal{O}_C^*(-D) \to F^\bullet := [\mathcal{O}_C^* \to i_*\mathbb{C}^*]$ and hence an isomorphism $H^1(C, \mathcal{O}_C^*(-D)) \cong \mathbb{H}^1(C, F^\bullet)$. By choosing a Čech covering (U_α), a 1-cocyle in $Z^1(U_\alpha, F^\bullet)$ is a pair of $f_{\alpha\beta} \in H^0(U_{\alpha\beta}, \mathcal{O}_C^*)$ and $\eta_\alpha \in H^0(U_\alpha, i_*\mathbb{C}^*)$ such that $\eta_\alpha/\eta_\beta = f_{\alpha\beta}|_D$. The data $f_{\alpha\beta}$ represents a line bundle. By assumption, $f_{\alpha\beta}|_D = 1$ implies that

$\eta_\alpha|_D = \eta_\beta|_D \in \mathbb{C}^*$. Since a framing of a line bundle at a point is equivalent to a choice of a non-zero complex number, (η_α) defines a framing of the line bundle at D. In other words, the pair $(f_{\alpha\beta}, \eta_\alpha)$ represents a framed line bundle, and a class in $\mathbb{H}^1(C, F^\bullet)$ represents an isomorphism class of the framed line bundle.

In particular, we find that

$$Jac(C^\circ) \cong \frac{H^1(C, \mathcal{O}(-D))}{H^1(C, D, \mathbb{Z})} \cong \ker(\mathrm{c}_1 : H^1(C, \mathcal{O}_C^*(-D)) \to \mathbb{Z})$$

which paramatrizes degree 0 framed line bundles. □

We will apply the previous discussion to $C = \overline{\Sigma}_b$, a spectral curve of Σ corresponding to $b \in B^{ur}$.

Remark 1.3.11. Unless mentioned otherwise, we will omit the the subscript b in $\overline{\Sigma}_b$ and $\overline{\Sigma}_b^\circ$ in this section for convenience, as it is irrelevant to our discussion.

Since we are mainly interested in $SL(n, \mathbb{C})$-Higgs bundles, we will need to consider the Prym variety of the spectral cover $\overline{p} : \overline{\Sigma} \to \Sigma$. The norm map $Nm : Jac(\widetilde{\Sigma}) \to Jac(\Sigma)$ induces a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{C}^*)^{nd-1} & \longrightarrow & Jac(\widetilde{\Sigma}^\circ) & \longrightarrow & Jac(\widetilde{\Sigma}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle Nm^\circ} & & \downarrow{\scriptstyle Nm} & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{d-1} & \longrightarrow & Jac(\Sigma^\circ) & \longrightarrow & Jac(\Sigma) & \longrightarrow & 0 \end{array}$$

where $d = |D|$ and $Nm^\circ : Jac(\widetilde{\Sigma}^\circ) \to Jac(\Sigma^\circ)$ is defined by taking norms on line bundles and determinants on framings. Recall that $Nm(L) = \det(\overline{p}_* L) \otimes \det(\overline{p}_* \mathcal{O}_{\overline{\Sigma}})^\vee$ and for a framed line bundle $(L, \beta) \in Jac(\overline{\Sigma}^\circ)$, the natural framing

$$\overline{p}_* L|_x \xrightarrow{\sim} \bigoplus_{y \in \overline{p}^{-1}(x)} L_y \xrightarrow[\beta]{\sim} \bigoplus_{y \in \overline{p}^{-1}(x)} \mathcal{O}_y$$

induces a framing on $\det(\overline{p}_*L)|_x$ over each $x \in D$. Also, there is a natural framing on $\det(\overline{p}_*\mathcal{O}_{\overline{\Sigma}})^\vee|_x$ induced from the identity $Id : \mathcal{O}_{\overline{\Sigma}}|_{\overline{p}^{-1}(x)} \to \mathcal{O}_{\overline{\Sigma}}|_{\overline{p}^{-1}(x)}$. Both framings determine a framing on $Nm(L)$ and hence the map Nm°.

By taking the kernel of this morphism, we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{C}^*)^{(n-1)d} & \longrightarrow & Prym(\widetilde{\Sigma}^\circ/\Sigma^\circ) & \longrightarrow & Prym(\widetilde{\Sigma}/\Sigma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{nd-1} & \longrightarrow & Jac(\widetilde{\Sigma}^\circ) & \longrightarrow & Jac(\widetilde{\Sigma}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle Nm^\circ} & & \downarrow{\scriptstyle Nm} & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{d-1} & \longrightarrow & Jac(\Sigma^\circ) & \longrightarrow & Jac(\Sigma) & \longrightarrow & 0 \end{array} \tag{1.3.3}$$

where $Prym(\widetilde{\Sigma}^\circ/\Sigma^\circ) := \ker(Nm^\circ)$.

Proposition 1.3.12. *(Spectral correspondence [BLP19]).*

For a fixed $b \in B^{ur}$, there is a one-to-one correspondence between degree zero framed line bundles on $\widetilde{\Sigma}_b$ and unordered diagonally framed Higgs bundles on Σ. Moreover, the following results hold:

1. *The fiber $h^{-1}_{\Delta,GL(n)}(b)$ is isomorphic to $Jac(\widetilde{\Sigma}^\circ_b)$;*

2. *The fiber $h^{-1}_{\Delta,SL(n)}(b)$ is isomorphic to $Prym(\widetilde{\Sigma}^\circ_b/\Sigma^\circ)$.*

Proof. For simplicity, we assume $D = \{x\}, \overline{D} = \overline{p}^{-1}(x)$ in this proof. Let L be a line bundle on $\widetilde{\Sigma}_b$ and (E, θ) a Higgs bundle on Σ. Recall that there is a bijection between line bundles on $\widetilde{\Sigma}_b$ and Higgs bundles on Σ

$$\text{Line bundle } L \text{ on } \widetilde{\Sigma}_b \ \underset{coker(\overline{p}^*\theta - \lambda Id)}{\overset{\overline{p}_*}{\rightleftarrows}} \ \text{Higgs bundle } (E, \theta) \text{ on } \Sigma \tag{1.3.4}$$

where λ denotes the tautological section of $K_\Sigma(D)$. It remains to verify the bijection on framings.

Pushing forward a $\overline{D}$-framed line bundle (L,β) gives an unordered framed Higgs bundle $(\overline{p}_*L, \overline{p}_*\lambda, \delta)$ where

$$\delta : E|_x \xrightarrow{\sim} \bigoplus_{y\in p^{-1}(x)} L_y \xrightarrow[\beta]{\sim} \bigoplus_{y\in p^{-1}(x)} \mathcal{O}_y$$

is well-defined as an unordered framing. With respect to the unordered framing, the Higgs field $\overline{p}_*\lambda$ is diagonal as $\theta|_x := \overline{p}_*\lambda$ defines multiplication by λ_i on each eigenline L_i.

Conversely, given an unordered diagonally framed Higgs bundle (E,θ,δ), since we assume that $\theta|_x$ has distinct eigenvalues, for each $\lambda_i \in \overline{p}^{-1}(D)$, the natural composition

$$\ker(\overline{p}^*\theta - \lambda_i Id) \to E|_x \to \operatorname{coker}(\overline{p}^*\theta - \lambda_i Id)$$

is an isomorphism. The assumption that $\theta|_x$ is diagonal with respect to δ implies that there is a component $\mathcal{O}_x \xhookrightarrow{\alpha_i} \oplus_i^n \mathcal{O}_x$ such that

$$\begin{array}{ccc} \ker(\overline{p}^*\theta - \lambda_i Id) & \longrightarrow & E|_x \\ \cong\uparrow & & \cong\uparrow \\ \mathcal{O}_x & \xhookrightarrow{\alpha_i} & \oplus_i^n \mathcal{O}_x \end{array}$$

In particular, we get a framing $\mathcal{O}_x \xrightarrow{\sim} \operatorname{coker}(\overline{p}^*\theta - \lambda_i Id)$ for each λ_i.

Finally, claims (1), (2) follow from Proposition 1.3.10. □

Hodge structures

Recall that since $\widetilde{\Sigma}^\circ$ is non-compact, $H^1(\widetilde{\Sigma}^\circ, \mathbb{Z})$ carries the $\mathbb{Z}$-mixed Hodge structure whose Hodge filtration is given by

$$F^0 = H^1(\widetilde{\Sigma}^\circ, \mathbb{C}) \supset F^1 = H^0(\widetilde{\Sigma}, \Omega^1_{\widetilde{\Sigma}}(\log D)) \supset F^2 = 0. \tag{1.3.5}$$

This induces the mixed Hodge structure on $(H^1(\widetilde{\Sigma}^\circ, \mathbb{Z}))^\vee$ which is isomorphic to $H_1(\widetilde{\Sigma}^\circ, \mathbb{Z})/(\text{torsion}) \cong H_1(\widetilde{\Sigma}^\circ, \mathbb{Z})$ by the universal coefficient theorem. Note that

$\mathrm{Ext}(H_0(\widetilde{\Sigma}^\circ,\mathbb{Z}),\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0$, so there is no torsion in this case. The Hodge filtration of this dual mixed Hodge structure is given by

$$F^{-1} = H^1(\widetilde{\Sigma}^\circ,\mathbb{C})^\vee \supset F^0 = \left(\frac{H^1(\widetilde{\Sigma}^\circ,\mathbb{C})}{H^0(\widetilde{\Sigma},\Omega^1_{\widetilde{\Sigma}}(\log D))}\right)^\vee \supset F^1 = 0$$

Note that the weight filtration on $H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})$ is

$$W_{-3} = 0 \subset W_{-2} = \mathbb{Z}^{nd-1} \subset W_{-1} = H_1(\widetilde{\Sigma}^\circ,\mathbb{Z}).$$

Thus we can define as in [Car79] the Jacobian of this Hodge structure as

$$J(H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})) := \frac{H_1(\widetilde{\Sigma}^\circ,\mathbb{C})}{F^0 + H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})} \tag{1.3.6}$$

Lemma 1.3.13. *There is an isomorphism between*

$$J(H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})) \cong Jac(\widetilde{\Sigma}^\circ)$$

Proof. Indeed,

$$J(H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})) = \frac{H_1(\widetilde{\Sigma}^\circ,\mathbb{C})}{F^0 + H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})} = \frac{F^{-1}}{F^0 + H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})} \cong \frac{H^0(\widetilde{\Sigma},\Omega^1_{\widetilde{\Sigma}}(\log(D)))}{H_1(\widetilde{\Sigma}^\circ,\mathbb{Z})}.$$

□

Taking the first integral homology of every term in the diagram (1.3.3), we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z})^{(n-1)d} & \longrightarrow & H_{\Delta,SL(n)} & \longrightarrow & H_{SL(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathbb{Z})^{nd-1} & \longrightarrow & H_1(\widetilde{\Sigma}^\circ,\mathbb{Z}) & \longrightarrow & H_1(\widetilde{\Sigma},\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle Nm^\circ} & & \downarrow{\scriptstyle Nm} & & \\
0 & \longrightarrow & (\mathbb{Z})^{d-1} & \longrightarrow & H_1(\Sigma^\circ,\mathbb{Z}) & \longrightarrow & H_1(\Sigma,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where we define

$$H_{\Delta,SL(n)} := H_1(Prym(\widetilde{\Sigma}^\circ/\Sigma^\circ),\mathbb{Z}) \cong \ker(Nm^\circ : H_1(\widetilde{\Sigma}^\circ,\mathbb{Z}) \to H_1(\Sigma^\circ,\mathbb{Z})), \tag{1.3.7}$$

$$H_{SL(n)} := H_1(Prym(\widetilde{\Sigma}/\Sigma),\mathbb{Z}) \cong \ker(Nm : H_1(\widetilde{\Sigma},\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})). \tag{1.3.8}$$

Since the norm map is a morphism of mixed Hodge structures and taking the Jacobian is functorial, we immediately get the following result.

Corollary 1.3.14. *The Prym lattice $H_{\Delta,SL(n)}$ is torsion free and admits the $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,-1),(-1,0),(0,-1)\}$ induced by the map $H_1(Nm^\circ) : H_1(\widetilde{\Sigma}^\circ,\mathbb{Z}) \to H_1(\Sigma^\circ,\mathbb{Z})$. In particular, the Jacobian $J(H_{\Delta,SL(n)})$ is isomorphic to $Prym(\widetilde{\Sigma}^\circ/\Sigma^\circ)$.*

Remark 1.3.15. Note that the mixed Hodge structure of the above type on $H_{\Delta,SL(n)}$ is equivalent to the data of semi-abelian variety $J(H_{\Delta,SL(n)})$. A review is included in Appendix (1.6).

Remark 1.3.16. The Prym lattice $H_{\Delta,SL(n)}$ admits a sheaf-theoretic formulation which will be needed in later sections. Consider the short exact sequence

$$0 \to \mathcal{K} \to \overline{p}_*\underline{\mathbb{Z}} \xrightarrow{Tr} \underline{\mathbb{Z}} \to 0$$

The trace map $\overline{p}_*\underline{\mathbb{Z}} \xrightarrow{Tr} \underline{\mathbb{Z}}$ is defined by

$$\overline{p}_*\underline{\mathbb{Z}}(U) = \underline{\mathbb{Z}}(\overline{p}^{-1}(U)) \to \underline{\mathbb{Z}}(U), \quad (s_1,...,s_n) \mapsto \sum_{i=1}^n s_i$$

if U is away from the ramification divisor, where s_i is a section on each component of $\overline{p}^{-1}(U)$.

This short exact sequence induces a long exact sequence:

$$\begin{aligned} 0 &\to H^0_c(\Sigma,\mathcal{K}) \to H^0_c(\Sigma,\overline{p}_*\mathbb{Z}_{\widetilde{\Sigma}}) \cong H^0_c(\widetilde{\Sigma},\mathbb{Z}) \to H^0_c(\Sigma,\mathbb{Z}) \\ &\to H^1_c(\Sigma,\mathcal{K}) \to H^1_c(\Sigma,\overline{p}_*\mathbb{Z}) \cong H^1_c(\widetilde{\Sigma},\mathbb{Z}) \to H^1_c(\Sigma,\mathbb{Z}) \end{aligned}$$

Since the cokernel of the map $H^0_c(\widetilde{\Sigma},\mathbb{Z}) \to H^0_c(\Sigma,\mathbb{Z})$ is torsion and $H^1_c(\overline{\Sigma},\mathbb{Z})$ is torsion-free, it follows that the maximal torsion free quotient $H^1_c(\Sigma,\mathcal{K})_{\text{tf}} := H^1_c(\Sigma,\mathcal{K})/H^1_c(\Sigma,\mathcal{K})_{\text{tors}}$ can be identified as follows

$$H^1_c(\Sigma,\mathcal{K})_{\text{tf}} \cong \ker(H^1_c(\widetilde{\Sigma},\mathbb{Z}) \to H^1_c(\Sigma,\mathbb{Z})) \cong \ker(H_1(\widetilde{\Sigma},\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z}))$$

by Poincaré duality. Note that we could have used cohomology instead of compactly supported cohomology since the curve Σ is compact, but the above argument also works for the noncompact curve Σ°. In particular, the same argument implies that

$$H^1_c(\Sigma^\circ,\mathcal{K}|_{\Sigma^\circ})_{\text{tf}} \cong \ker(H_1(\widetilde{\Sigma}^\circ,\mathbb{Z}) \to H_1(\Sigma^\circ,\mathbb{Z})) \cong H_{\Delta,SL(n)}. \tag{1.3.9}$$

Note that $H^1_c(\Sigma^\circ,\mathcal{K}|_{\Sigma^\circ})_{\text{tf}}$ can also be written as $H^1(\Sigma,D,\mathcal{K})_{\text{tf}}$.

1.3.3 Deformation theory

In this section, we show that the moduli space of diagonally framed Higgs bundle $\overline{\mathcal{M}}^\Delta(n,D)$ is symplectic. For the following discussion in this section, we fix a diagonally framed Higgs bundle (E,θ,δ). Recall that we assume $b \in B^{ur}$ which means that the associated cameral curve is smooth, unramified over D, and has simple ramification. In particular, the residue of θ at D is diagonal with distinct eigenvalues with respect to the framing δ.

Denote by $\Sigma[\epsilon]$ the fiber product $\Sigma \times \text{Spec}(\mathbb{C}[\epsilon])$.

Definition 1.3.17. An infinitesimal deformation of diagonally framed Higgs bundle is a triple $(E_\epsilon,\theta_\epsilon,\delta_\epsilon)$ such that

- E_ϵ is a locally free sheaf on $\Sigma[\epsilon]$,
- $\theta_\epsilon \in H^0(\Sigma[\epsilon],\mathfrak{g}'_{E_\epsilon,D[\epsilon]} \otimes p^*_\Sigma K_\Sigma(D))$,

- $\delta_\epsilon : E|_{D[\epsilon]} \to \mathcal{O}_{D[\epsilon]}^{\oplus n}$ is an isomorphism,
- $(E_\epsilon, \theta_\epsilon, \delta_\epsilon)|_{D\times 0} \cong (E, \theta, \delta)$,

where as before $\mathfrak{g}'_{E_\epsilon, D[\epsilon]}$ is defined as the kernel of the map $\mathfrak{g}_{E_\epsilon} \twoheadrightarrow \mathfrak{q} \otimes \mathcal{O}_{D[\epsilon]}$ induced by δ_ϵ and $p_\Sigma : \Sigma[\epsilon] \to \Sigma$ denotes the natural projection.

Proposition 1.3.18. *The space of infinitesimal deformations of a diagonally framed Higgs bundle (E, θ, δ) is canonically isomorphic to $\mathbb{H}^1(C^\bullet)$ where*

$$C^\bullet : C^0 = \mathfrak{g}_E(-D) \xrightarrow{[\cdot,\theta]} C^1 = \mathfrak{g}'_E \otimes K_\Sigma(D) \tag{1.3.10}$$

Proof. Recall that [Mar94] the space of infinitesimal deformation of a framed Higgs bundles (E, θ, δ) is canonically isomorphic to $\mathbb{H}^1(C_F^\bullet)$ where

$$C_F^\bullet : C_F^0 = \mathfrak{g}_E(-D) \xrightarrow{[\cdot,\theta]} C_F^1 = \mathfrak{g}_E \otimes K_\Sigma(D). \tag{1.3.11}$$

Choose a Čech cover $U := (U_\alpha)$ of Σ which induces cover $U[\epsilon] := (U_\alpha[\epsilon])$ of $\Sigma[\epsilon]$. Imposing further the condition that the Higgs bundles are diagonally framed implies that $\theta \in \mathfrak{g}'_E \otimes K_\Sigma(D) \subset \mathfrak{g}_E \otimes K_\Sigma(D)$. Suppose that a 1-cocycle $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$ in $Z^1(U[\epsilon], C_F^\bullet)$ represents an infinitesimal deformation of (E, θ, δ) as framed Higgs bundles where $\dot{f}_{\alpha\beta} \in H^0(U_{\alpha\beta}[\epsilon], \mathfrak{g}_E(-D))$ and $\dot{\varphi}_\alpha \in H^0(U_\alpha[\epsilon], \mathfrak{g}_E \otimes p_\Sigma^* K_\Sigma(D))$. Then $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$ is an infinitesimal deformation of (E, θ, δ) as diagonally framed Higgs bundles if and only if $\dot{\varphi}_\alpha \in H^0(U_\alpha[\epsilon], \mathfrak{g}'_E \otimes p_\Sigma^* K_\Sigma(D))$. Hence, it follows that $\mathbb{H}^1(C^\bullet)$ parametrizes the infinitesimal deformations of diagonally framed Higgs bundles.

□

Recall that the Serre duality says that $\mathbb{H}^1(C^\bullet) \xrightarrow{\sim} (\mathbb{H}^1(\check{C}^\bullet))^\vee$ where

$$\check{C}^\bullet : (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D) \xrightarrow{[-,\theta]^t} \mathfrak{g}_E^\vee \otimes K_\Sigma(D) \tag{1.3.12}$$

is the Serre dual to $C^\bullet$. Combining the Serre duality isomorphism with the isomorphism in the next proposition, we get a non-degenerate skew-symmetric pairing on $\mathbb{H}^1(C^\bullet)$.

Proposition 1.3.19. *There is a canonical isomorphism*

$$\mathbb{H}^1(\check{C}^\bullet) \cong \mathbb{H}^1(C^\bullet). \tag{1.3.13}$$

Proof. We consider an auxiliary complex[2]

$$C_1^\bullet : \mathfrak{g}_E'' \to \mathfrak{g}_E \otimes K_\Sigma(D)$$

and show that this is isomorphic to both $C^\bullet$ and $\check{C}^\bullet$.

First, consider the morphism of complexes $t : C^\bullet \to C_1^\bullet$:

$$\begin{array}{ccccc} C^\bullet & \quad & \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) & \longrightarrow & \mathfrak{g}_E' \otimes K_\Sigma(D) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} \\ C_1^\bullet & & \mathfrak{g}_E'' & \longrightarrow & \mathfrak{g}_E \otimes K_\Sigma(D) \end{array}$$

Both t_0 and t_1 are injective. The diagram clearly commutes away from D, hence commutes everywhere. In particular, around D, choose an open subset U that trivializes all the bundles, we see that the maps become the natural maps

$$\begin{array}{ccc} \mathfrak{t}(-D) \oplus \mathfrak{q}(-D) & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{q}(-D)) \otimes K_\Sigma(D)|_U \\ \downarrow{\scriptstyle t_0|_U} & & \downarrow{\scriptstyle t_1|_U} \\ \mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{q}) \otimes K_\Sigma(D)|_U \end{array}$$

where we abuse notations by denoting $\mathfrak{t}$ and $\mathfrak{q}$ the trivial bundles with fibers $\mathfrak{t}$ and $\mathfrak{q}$, respectively. The cokernel of t is

$$\mathrm{coker}(t) : \mathfrak{q}_D \xrightarrow{[\cdot,\theta]|_D} \mathfrak{q}_D \otimes K_\Sigma(D)$$

Lemma 1.3.20.

$$\mathbb{H}^i(\mathrm{coker}(t)) = 0, \ \textit{for all } i.$$

[2]The complex $C_1^\bullet$ here coincides with the complex "$\mathcal{C}_\bullet^\Delta$" that is defined in [BLP19, Section 5].

Proof. Since the complex is supported at D, it reduces to a complex of $\mathbb{C}$-vector spaces. Assume D consists of a single point for simplicity. The complex reduces to

$$\mathfrak{q} \xrightarrow{[\cdot,\theta]|_D} \mathfrak{q}.$$

Recall our assumption that the associated spectral curve is unramified over D. The restriction $\theta|_D$ of the Higgs field to D is a diagonal matrix with distinct eigenvalues with respect to δ. In particular, $\theta|_D$ is regular and semisimple, so its centralizer $Z_{\mathfrak{g}}(\theta|_D) = \{x \in g|[x, \theta|_D] = 0\}$ is a Cartan subalgebra and coincides with $\mathfrak{t}$. Since $\ker([\cdot,\theta]|_D : \mathfrak{g} \to \mathfrak{g}) = Z_{\mathfrak{g}}(\theta|_D) = \mathfrak{t}$ which intersects $\mathfrak{q}$ trivially, it follows that the restricted map $([\cdot,\theta]|_D)|_{\mathfrak{q}} : \mathfrak{q} \to \mathfrak{q}$ is an isomorphism. Hence, all the cohomologies of the complex $\mathrm{coker}(t)$ must be zero.

□

The long exact sequence induced by $0 \to C^\bullet \to C_1^\bullet \to \mathrm{coker}(t) \to 0$ is:

$$\begin{aligned} 0 &\to \mathbb{H}^0(C^\bullet) \to \mathbb{H}^0(C_1^\bullet) \to \mathbb{H}^0(\mathrm{coker}(t)) = 0 \\ &\to \mathbb{H}^1(C^\bullet) \to \mathbb{H}^1(C_1^\bullet) \to \mathbb{H}^1(\mathrm{coker}(t)) = 0 \to ... \end{aligned}$$

and hence $\mathbb{H}^0(C^\bullet) \cong \mathbb{H}^0(C_1^\bullet)$ and $\mathbb{H}^1(C^\bullet) \cong \mathbb{H}^1(C_1^\bullet)$.

Finally, we claim that there is an isomorphism of complexes $C_1^\bullet \cong \check{C}^\bullet$

$$\begin{array}{ccccc} C_1^\bullet & & \mathfrak{g}_E'' & \longrightarrow & \mathfrak{g}_E \otimes K_\Sigma(D) \\ \downarrow \cong & & \downarrow r_0 & & \downarrow r_1 \\ \check{C}^\bullet & & (\mathfrak{g}_E')^\vee \otimes \mathcal{O}_\Sigma(-D) & \longrightarrow & (\mathfrak{g}_E)^\vee \otimes K_\Sigma(D) \end{array} \tag{1.3.14}$$

The map r_0 is defined as follows. Consider the composition of morphisms

$$\mathfrak{g}_E'' \hookrightarrow \mathfrak{g}_E \xrightarrow{\sim} \mathfrak{g}_E^\vee \hookrightarrow (\mathfrak{g}_E')^\vee \to (\mathfrak{g}_E')^\vee \otimes \mathcal{O}_D. \tag{1.3.15}$$

where the isomorphism $\mathfrak{g}_E \to \mathfrak{g}_E^\vee$ is given by the trace pairing. If we know that this composition is zero, then we will get a map

$$r_0 : \mathfrak{g}''_E \to \ker((\mathfrak{g}'_E)^\vee \to (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_D) = (\mathfrak{g}')^\vee_E \otimes \mathcal{O}_\Sigma(-D).$$

Away from D, the map (1.3.15) is clearly zero. Around D, we can find an open subset U such that each sheaf in the composition is trivial, then

$$\begin{array}{ccccccccc}
\mathfrak{g}''_E|_U & \longrightarrow & \mathfrak{g}_E|_U & \longrightarrow & \mathfrak{g}_E^\vee|_U & \longrightarrow & (\mathfrak{g}'_E)^\vee|_U & \longrightarrow & ((\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_D)|_U \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t} \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t}^\vee \oplus \mathfrak{q}^\vee & \longrightarrow & \mathfrak{t}^\vee \oplus \mathfrak{q}^\vee(D) & \longrightarrow & (\mathfrak{t}^\vee \otimes \mathcal{O}_D) \oplus (\mathfrak{q}^\vee \otimes \mathcal{O}_D(D))
\end{array}$$

Each component of the bottom row clearly composes to zero, hence the whole composition is zero. Locally over U, the map $r_0 : \mathfrak{g}''_E \to (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D)$ is induced by the trace pairing: $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee$ and $\mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^\vee$,

$$r_0|_U : \mathfrak{g}''_E|_U \cong \mathfrak{t}(-D) \oplus \mathfrak{q} \xrightarrow{\sim} \mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee \cong (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D)|_U$$

Since r_0 is clearly an isomorphism away from D, it follows that r_0 is an isomorphism.

The commutativity can be argued in the same way. Again, the diagram commutes away from D. Around D, the bundles trivialize and we get the diagram

$$\begin{array}{ccc}
\mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t} \oplus \mathfrak{q} \otimes K_\Sigma(D)|_U \\
\downarrow & & \downarrow \\
\mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee & \longrightarrow & \mathfrak{t}^\vee \oplus \mathfrak{q}^\vee \otimes K_\Sigma(D)|_U
\end{array}$$

which commutes on the nose.

All of this together gives

$$\mathbb{H}^1(C^\bullet) \cong \mathbb{H}^1(C_1^\bullet) \cong \mathbb{H}^1(\check{C}^\bullet). \tag{1.3.16}$$

as claimed. □

Let $\omega_\Delta : \mathbb{H}^1(C^\bullet) \times \mathbb{H}^1(C^\bullet) \to \mathbb{C}$ be the non-degenerate skew-symmetric pairing induced by Serre duality and the isomorphism in Proposition 1.3.19.

Proposition 1.3.21. *The nondegenerate 2-form ω_Δ is closed.*

Proof. Consider the following inclusion of complexes $C^\bullet \overset{u}{\hookrightarrow} C^\bullet_F$:

$$\begin{array}{ccccc} C^\bullet & & \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) & \longrightarrow & \mathfrak{g}'_E \otimes K_\Sigma(D) \\ \downarrow u & & \downarrow u_0 & & \downarrow u_1 \\ C^\bullet_F & & \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) & \longrightarrow & \mathfrak{g}_E \otimes K_\Sigma(D) \end{array}$$

where as before $C^\bullet_F$ is the complex whose first hypercohomology controls the deformations of the framed Higgs bundle (E, θ, δ). By the same argument as in Proposition 1.3.19, since u_0 is isomorphic and u_1 is injective whose cokernel has zero-dimensional support and concentrated in degree one, we have an injection

$$i : \mathbb{H}^1(C^\bullet) \hookrightarrow \mathbb{H}^1(C^\bullet_F).$$

Note that Serre duality induces a non-degenerate bilinear pairing on $\mathbb{H}^1(C^\bullet_F)$ which corresponds to the well-known symplectic form ω_F on $\mathcal{M}_F(n, D)$, see [BLP19]. We claim that the pairing ω_Δ is obtained by restricting ω_F to $\mathbb{H}^1(C^\bullet) \subset \mathbb{H}^1(C^\bullet_F)$. In other words, the corresponding 2-form on $\overline{\mathcal{M}}^\Delta(n, D)$ is obtained by pulling back the symplectic form ω_F on $\mathcal{M}_F(n, D)$. It then follows that ω_F is closed as well.

Our claim is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccccc} \mathbb{H}^1(C^\bullet_F) & \longrightarrow & \mathbb{H}^1(\check{C}^\bullet_F)^\vee & & \longrightarrow & & \mathbb{H}^1(C^\bullet_F)^\vee \\ i \uparrow & & \uparrow & & & & \downarrow i^\vee \\ \mathbb{H}^1(C^\bullet) & \longrightarrow & \mathbb{H}^1(\check{C}^\bullet)^\vee & \longrightarrow & \mathbb{H}^1(C^\bullet_1)^\vee & \longrightarrow & \mathbb{H}^1(C^\bullet)^\vee \end{array}$$

The left square diagram commutes by the functoriality of Serre duality. Then it remains to check the commutativity of the right square diagram. This follows from the commutativity of the diagram of complexes:

$$\begin{array}{ccccc}
\check{C}^\bullet_F & & \longleftarrow & & C^\bullet_F \\
\downarrow & & & & \uparrow \\
\check{C}^\bullet & \longleftarrow & C^\bullet_1 & \longleftarrow & C^\bullet
\end{array}$$

Away from D, the diagram clearly commutes. Around D, we again trivialize the bundles and the diagram looks like

$$\begin{array}{ccc}
\boxed{\mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee(-D) \to (\mathfrak{t}^\vee \oplus \mathfrak{q}^\vee)L} & \longleftarrow & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q}(-D) \to (\mathfrak{t} \oplus \mathfrak{q})L} \\
\downarrow & & \uparrow \\
\boxed{\mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee \to (\mathfrak{t}^\vee \oplus \mathfrak{q}^\vee)L} & & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q}(-D) \to (\mathfrak{t} \oplus \mathfrak{q}(-D))L} \\
& \nwarrow & \downarrow \\
& & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q} \to (\mathfrak{t} \oplus \mathfrak{q})L}
\end{array}$$

where we denote by L the operation "$\otimes K_\Sigma(D)$".

□

Proposition 1.3.22.

1. $\mathbb{H}^0(C^\bullet) = \mathbb{H}^2(C^\bullet) = 0$. *In particular, the deformations of a diagonally framed Higgs bundle* (E, θ, δ) *are unobstructed.*

2. *dim*$(\mathbb{H}^1(C^\bullet)) = (n^2 - 1)(2g - 2 + d) + (n - 1)d$.

Proof. (1) Since morphisms between diagonally framed Higgs bundles are in particular morphisms between framed Higgs bundles, automorphisms of diagonally framed Higgs bundles are the same as automorphisms as framed Higgs bundles. So Corollary 1.3.4 implies that the diagonally framed Higgs bundles are rigid. Hence, $\mathbb{H}^0(C^\bullet) = 0$.

On the other hand, again by Serre duality,

$$\mathbb{H}^2(C^\bullet) \cong (\mathbb{H}^0(\check{C}^\bullet))^\vee \cong (\mathbb{H}^0(C^\bullet_1))^\vee$$

where the second isomorphism comes from the isomorphism of the complex (1.3.14). Finally, recall that from the long exact sequence above, we have that $\mathbb{H}^0(C^\bullet) \cong \mathbb{H}^0(C_1^\bullet)$ which vanishes as we just proved, hence $\mathbb{H}^2(C^\bullet) = 0$.

(2) By the definition of $\mathfrak{g}'_E$, we have a short exact sequence

$$0 \to \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) \to \mathfrak{g}'_E \to i_*\mathfrak{t} \to 0$$

and thus

$$\begin{aligned}
\chi(\mathfrak{g}'_E \otimes K_\Sigma(D)) &= \chi(\mathfrak{g}'_E) + (n^2-1)\deg(K_\Sigma(D)) \\
&= \chi(\mathfrak{t} \otimes \mathcal{O}_D) + \chi(\mathfrak{g}_E(-D)) + (n^2-1)\deg(K_\Sigma(D)) \\
&= (n-1)d + \chi(\mathfrak{g}_E) + (n^2-1)\deg(\mathcal{O}_\Sigma(-D)) + (n^2-1)\deg(K_\Sigma(D)) \\
&= \chi(\mathfrak{g}_E) + (n-1)d + (n^2-1)(2g-2).
\end{aligned}$$

By (1), $\chi(C^\bullet) = \mathbb{H}^1(C^\bullet)$, so

$$\begin{aligned}
\mathbb{H}^1(C^\bullet) &= \chi(\mathfrak{g}'_E \otimes K_\Sigma(D)) - \chi(\mathfrak{g}_E(-D)) \\
&= \chi(\mathfrak{g}_E) + (n-1)d + (n^2-1)(2g-2) - \chi(\mathfrak{g}_E) + (n^2-1)d \\
&= (n^2-1)(2g-2+d) + (n-1)d.
\end{aligned}$$

□

Remark 1.3.23. In the case of $\mathfrak{g} = \mathfrak{gl}_n$, a similar computation shows that

$$\mathbb{H}^1(C^\bullet) = n^2(2g-2+d) + nd.$$

Remark 1.3.24. A direct computation by applying the Riemann-Roch theorem shows that

$$\begin{aligned}\dim(B) = \sum_{i=2}^{n} h^0(\Sigma, (K(D))^{\otimes i}) &= (2g-2+d)\left(\frac{n(n+1)}{2} - 1\right) + (n-1)(1-g) \\ &= \frac{1}{2}((n^2-1)(2g-2+d) + (n-1)d) \\ &= \frac{1}{2}\dim(\mathbb{H}^1(C^\bullet)).\end{aligned}$$

Proposition 1.3.25. *The open subset $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ of the moduli space $\overline{\mathcal{M}}^{\Delta}(n,D)$ is a smooth quasi-projective variety of dimension $(n^2-1)(2g-2+d)+(n-1)d$. The tangent space $T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ is canonically isomorphic to $\mathbb{H}^1(C^\bullet)$. Moreover, $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ admits a symplectic form ω_Δ which is the restriction of the symplectic form ω_F on $\mathcal{M}_F(n,D)$.*

Proof. All the claims follow immediately from Proposition 1.3.18, 1.3.21 and 1.3.22. The argument to show that ω_Δ is a restriction of ω_F is contained in the proof of Proposition 1.3.21. □

Proposition 1.3.26. *The fiber of the map $\overline{h}_\Delta : \overline{\mathcal{M}}^{\Delta}(n,D)^{ur} \to B^{ur}$ is Lagrangian with respect to ω_Δ.*

Proof. Denote by $(h_1, ..., h_l) := h \circ f_1 : \mathcal{M}_F(n,D) \to \mathcal{M}(n,D) \to \mathbb{C}^l = B$ the composition of the forgetful map and the Hitchin map. According to [BLP19, Theorem 5.1], the functions h_i Poisson-commute. Since the symplectic form ω_Δ on $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ is the restriction of the symplectic form ω_F on $\mathcal{M}_F(n,D)$, the functions h_i Poisson-commute as well when restricted to $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$.

Since the dimension of the fiber $h_\Delta^{-1}(b)$ for $b \in B^{ur}$ is exactly $\frac{1}{2}\dim(\overline{\mathcal{M}}^{\Delta}(n,D)^{ur})$ by Remark (1.3.24), it suffices to show that ω_Δ restricted to $\overline{h}_\Delta^{-1}(b)$ vanishes to prove our claim. This follows from Poisson-commutativity of $(h_i)|_{\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}}$.

□

Proposition 1.3.27. *The tangent space $T_{[(E,\theta,\delta)]}\mathcal{M}^{\Delta}(n,D)^{ur}$ is canonically isomorphic to $\mathbb{H}^1(C^\bullet)$. Moreover, the symplectic form ω_Δ on $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ is invariant under the $S_n^{|D|}$-action. In particular, ω_Δ descends to a symplectic form ω'_Δ on $\mathcal{M}^{\Delta}(n,D)^{ur}$.*

Proof. In the proof of Proposition 1.3.18, given an infinitesimal deformation $(E_\epsilon, \theta_\epsilon, \delta_\epsilon)$, the assignment of a 1-cocyle $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$ in $\mathbb{H}^1(C^\bullet)$ is independent of the reordering of components. That means we have the following commutative diagram

$$\begin{array}{ccc} T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^{\Delta}(n,D)^{ur} & \xrightarrow{\sim} & \mathbb{H}^1(C^\bullet) \\ \downarrow{\scriptstyle d\underline{\sigma}} & & \downarrow{\scriptstyle =} \\ T_{[(E,\theta,\underline{\sigma}\cdot\delta)]}\overline{\mathcal{M}}^{\Delta}(n,D)^{ur} & \xrightarrow{\sim} & \mathbb{H}^1(C^\bullet) \end{array}$$

for $\underline{\sigma} \in S_n^{|D|}$. The differential of the quotient map

$$dq : T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^{\Delta}(n,D)^{ur} \to T_{[(E,\theta,S_n^{|D|}\cdot\delta)]}\mathcal{M}^{\Delta}(n,D)^{ur}$$

is an isomorphism. Hence, the canonical identification $T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^{\Delta}(n,D)^{ur} \cong \mathbb{H}^1(C^\bullet)$ descends to the tangent space $T_{[(E,\theta,S_n^{|D|}\cdot\delta)]}\mathcal{M}^{\Delta}(n,D)^{ur}$ via dq and yields a canonical isomorphism $T_{[(E,\theta,S_n^{|D|}\cdot\delta)]}\mathcal{M}^{\Delta}(n,D)^{ur} \cong \mathbb{H}^1(C^\bullet)$. Since the group action of $S_n^{|D|}$ is trivial on $\mathbb{H}^1(C^\bullet)$, the symplectic form ω_Δ on $\overline{\mathcal{M}}^{\Delta}(n,D)^{ur}$ is invariant under $S_n^{|D|}$. □

Corollary 1.3.28. *The map $h_\Delta^{ur} : \mathcal{M}^{\Delta}(n,D)^{ur} \to B^{ur}$ forms a semi-polarized integrable system.*

Proof. By the spectral correspondence proved in Proposition 1.3.12, the fibers are semi-abelian varieties. Since ω'_Δ descends from the symplectic form ω_Δ, it follows immediately from Proposition 1.3.26 that the fiber of the map $h_\Delta^{ur} : \mathcal{M}^{\Delta}(n,D)^{ur} \to B^{ur}$ is Lagrangian with respect to ω'_Δ. □

Remark 1.3.29. For a fixed $b \in B^{ur}$, the fiber $(h^{ur}_{\Delta})^{-1}(b)$ is a semi-abelian variety $Prym(\widetilde{\Sigma}^{\circ}_{b}, \Sigma^{\circ})$ which admits a $(\mathbb{C}^*)^{(n-1)d}$-action. This group action can be seen by viewing $Prym(\widetilde{\Sigma}^{\circ}_{b}, \Sigma^{\circ})$ as parametrizing framed line bundles on $\widetilde{\Sigma}_b$ which correspond to unordered diagonally framed $SL(n, \mathbb{C})$-Higgs bundles under spectral correspondence. Then $(\mathbb{C}^*)^{(n-1)d}$ acts simply transitively on the space of framings over D for each fixed line bundle, and the quotient map is equivalent to the natural map $Prym(\widetilde{\Sigma}^{\circ}_{b}, \Sigma^{\circ}) \to Prym(\widetilde{\Sigma}_{b}, \Sigma)$ of forgetting the framings. Applying this fiberwise quotient by $(\mathbb{C}^*)^{(n-1)d}$ to the fibration $\mathcal{M}^{\Delta}(n, D)^{ur} \to B^{ur}$, we see that the quotient map is precisely the forgetful map $f_1 : \mathcal{M}^{\Delta}(n, D)^{ur} \to \mathcal{M}(n, D)^{ur}$. Thus, this provides a geometric interpretation of the fact that the Poisson integrable system $\mathcal{M}(n, D)^{ur} \to B^{ur}$ is realized as the fiberwise compact quotient of the semi-polarized integrable system $\mathcal{M}^{\Delta}(n, D) \to B^{ur}$ as discussed in Section 1.2.2.

1.3.4 Cameral description

Although the spectral curve description is more intuitive and straightforward, it only works for classical groups. To describe the general fiber of Hitchin system for any reductive group G as well as prove DDP-type results, it is more natural to use the cameral curve description and *generalized Prym variety*. In this section, we focus on the extension of classical results in our case (A-type). We refer to [DG02][DP12] for more basics and details about the cameral description.

In this section, we use general notation from algebraic group theory with an eye towards a generalization of the previous arguments to any reductive group G (see Remark 1.3.32).

As the Hitchin base B can be considered as the space of sections of $K_{\Sigma}(D) \otimes \mathfrak{t}/W$, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{\boldsymbol{\Sigma}} & \longrightarrow & \widetilde{\boldsymbol{U}} := \mathbf{Tot}(K_\Sigma(D)\otimes \mathfrak{t}) \\ \big\downarrow \widetilde{\boldsymbol{p}} & & \big\downarrow \phi \\ \Sigma \times B & \xrightarrow{ev} & \boldsymbol{U} := \mathbf{Tot}(K_\Sigma(D)\otimes \mathfrak{t}/W) \end{array} \tag{1.3.17}$$

where $\widetilde{\boldsymbol{\Sigma}}$ is the *universal cameral curve* of Σ. By projecting to B, we have a family of cameral curves $\widetilde{\boldsymbol{\Sigma}} \to B$ whose fiber is a W-Galois cover of the base curve Σ. An interesting observation is that in the meromorphic case, one can consider the universal cameral pair $(\widetilde{\boldsymbol{\Sigma}}, \widetilde{\boldsymbol{D}} := \widetilde{\boldsymbol{p}}^{-1}(D \times B))$, which allows us to extend the notion of generalized Prym variety [Don93]. Let's recall the definition of the generalized Prym variety. For a generic $b \in B$, we define a sheaf of abelian groups $\mathcal{T}_b$ by

$$\mathcal{T}_b(U) := \{t \in \widetilde{p}_{b*}(\Lambda_G \otimes \mathcal{O}^*_{\widetilde{\Sigma}})^W(U) |\, \alpha(t)|_{M^\alpha} = +1 \quad \forall \alpha \in R(G)\}$$

where $R(G)$ is a root system and Λ_G is the cocharater lattice and M^α is the ramification locus of $\widetilde{p}_b : \widetilde{\Sigma}_b \to \Sigma$ fixed by the reflection $S_2 \in W$ corresponding to α. We define the generalized Prym variety of $\widetilde{\Sigma}_b$ over Σ as the sheaf cohomology $H^1(\Sigma, \mathcal{T}_b)$.

Theorem 1.3.30 ([DG02][HHP10]). *For $b \in B^\circ$, the fiber $h^{-1}(b)$ in the meromorphic Hitchin system is isomorphic to the generalized Prym variety $H^1(\Sigma, \mathcal{T}_b)$:*

$$h^{-1}(b) \cong H^1(\Sigma, \mathcal{T}_b)$$

where B° is the locus of smooth cameral curves with simple ramifications.

Let $i_D : D \hookrightarrow \Sigma \hookleftarrow \Sigma \backslash D : j_D$ be inclusions. Associated to the cameral pair $(\widetilde{\Sigma}_b, \widetilde{D}_b)$, one can extend the generalized Prym variety to $H^1(\Sigma, j_{D!}j_D^*\mathcal{T}_b)$ which is isomorphic to $h_\Delta^{-1}(b)$.

Proposition 1.3.31. *For $b \in B^{ur}$, the unordered diagonally framed Hitchin fiber $(h_\Delta)^{-1}(b)$ is isomorphic to $H^1(\Sigma, j_{D!}j_D^*\mathcal{T}_b)$. In particular, it is a semi-abelian variety which corresponds to the $\mathbb{Z}$-mixed Hodge structure*

$$(H^1(\Sigma, D, (\widetilde{p}_{b*}\Lambda_{SL(n)})^W)_{tf}, H^1(\widetilde{\Sigma}_b, \widetilde{D}_b, \mathfrak{t})^W)$$

whose weight and Hodge filtration are induced from Hodge structure of $H^1(\widetilde{\Sigma}_b, \mathfrak{t})^W$ and $H^0(\widetilde{D}_b, \mathfrak{t})^W$.

Proof. For completeness, we use the spectral description of unordered diagonally framed Higgs bundles. The fiber $(h_\Delta)^{-1}(b)$ is isomorphic to the Jacobian of the relative $\mathbb{Z}$-mixed Hodge structure on $H^1(\Sigma, D, \mathcal{K}_b)$ where $\mathcal{K}_b := \ker(Tr : \bar{p}_{b*}\mathbb{Z} \to \mathbb{Z})$ (see Remark 1.3.16). To relate with the cameral description, we consider an isomorphism of sheaves,

$$(\widetilde{p}_{b*}\Lambda_{SL(n)})^W \cong \mathcal{K}_b \tag{1.3.18}$$

proved in Lemma 1.5.4. It induces the isomorphism of $\mathbb{Z}$-mixed Hodge structures on the relative sheaf cohomology:

$$H^1(\Sigma, D, (\widetilde{p}_{b*}\Lambda_{SL(n)})^W) \cong H^1(\Sigma, D, \mathcal{K}_b).$$

They agree on the torsion free part, hence we obtain the result by complexifying the lattice. □

Remark 1.3.32. In the forthcoming paper [LL], we develop the theory of diagonally framed Higgs bundle for arbitrary reductive group G and its abelianization by following [DG02]. In summary, note that an additional data of diagonal framing amounts to specifying W-equivariant section of T-bundle at D. This can be formulated as $H^0(D_b, \mathcal{T}) =$

$H^0(D, (\widetilde{p}_{b*}\Lambda_{SL(n)})^W \otimes \mathbb{C}^*)$ modulo the action of the center $Z(G)$. Moreover, the distinguished triangle in the constructible derived category of Σ, $D_c^b(\Sigma)$

$$j_{b!}j_b^* \to id \to i_{b*}i_b^*$$

induces the long exact sequence as follows

$$H^0(\Sigma, j_{b!}j_b^*\mathcal{T}_b) \to H^0(\Sigma, \mathcal{T}_b) \xrightarrow{i_D^*} H^0(D, \mathcal{T}_b) \to H^1(\Sigma, j_{b!}j_b^*\mathcal{T}_b) \to H^1(\Sigma, \mathcal{T}_b) \to 0. \quad (1.3.19)$$

Here, $H^0(\Sigma, \mathcal{T}_b)$ is the space of W-equivariant maps, $\mathrm{Hom}_W(\widetilde{\Sigma}_b, T)$, which takes values 1 on $M^\alpha_{\widetilde{\Sigma}_b}$ for every root α. Note that

$$Z(G) = \{t \in T^W | \alpha(t) = 1 \quad \text{for all} \quad \alpha \in R(G)\}.$$

Therefore, the cokernel of $i_D^* : H^0(\Sigma, \mathcal{T}_b) \to H^0(D, \mathcal{T}_b)$ can be identified with $T^{|D|}/Z(G)$, a level subgroup. Clearly this is a copy of $\mathbb{C}^*$'s, so we have the semi-abelian variety $H^1(\Sigma, j_{b!}j_b^*\mathcal{T}_b)$ as an extension of $H^1(\Sigma, \mathcal{T}_b)$ by $T^{|D|}/Z(G)$. In order to get the complete description of the general fiber, we should verify the precise torsor structure. For type A, this can be done easily with the help of spectral description.

1.3.5 Abstract Seiberg-Witten differential

Using the cameral description, one can define an abstract Seiberg-Witten differential. Note that in the classical case, the Seiberg-Witten differential is a holomorphic one-form which is obtained by the tautological section of the pullback of K_Σ under $\mathbf{Tot}(K_\Sigma) \to \Sigma$. Similarly, in the meromorphic case, the tautological section of the pullback of $K_\Sigma(D)$ under $\mathbf{Tot}(K_\Sigma(D)) \to \Sigma$ gives the logarithmic 1-form θ. For each $b \in B$, we define the Seiberg-Witten differential to be the restriction

$$\lambda_{\Delta,b} := \theta|_{\widetilde{\Sigma}_b} \in H^0(\widetilde{\Sigma}_b, \mathfrak{t} \otimes \Omega_{\widetilde{\Sigma}_b}(\log \widetilde{D}_b))^W = F^1H^1(\widetilde{\Sigma}_b \setminus \widetilde{D}_b, \mathfrak{t})^W$$

where $(H^1(\Sigma\backslash D, (\widetilde{p}_{b,*}\Lambda_{SL(n)})^W), H^1(\widetilde{\Sigma}_b\backslash\widetilde{D}_b, \mathfrak{t})^W)$ is the $\mathbb{Z}$-mixed Hodge structure associated to the cameral pair $(\widetilde{\Sigma}_b, \widetilde{p}_b^{-1}(D))$. This is the dual to the one we described earlier and is of type $\{(0,1),(1,0),(1,1)\}$. For simplicity, let's denote it by $V_b^\vee = H^1(\widetilde{\Sigma}_b \setminus \widetilde{D}_b, \mathfrak{t})^W$.

Note that having a variation of $\mathbb{Z}$-mixed Hodge structures over B corresponds to having the classifying map to mixed period domain; $\Phi : B \to \mathcal{D}/\Gamma$. It admits a holomorphic lift [MU87] $\widetilde{\Phi} : B \to \mathcal{D}$ which factors through relative Kodaira-Spencer map $\kappa : T_{B,b} \to H^1(\widetilde{\Sigma}_b, T_{\widetilde{\Sigma}_b}(-\log \widetilde{D}_b))$

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{\quad d\widetilde{\Phi}\quad} & T_{\mathcal{D},\widetilde{\Phi}(b)} \\ & {\scriptstyle\kappa}\searrow \qquad \nearrow{\scriptstyle m^\vee} & \\ & H^1(\widetilde{\Sigma}_b, T_{\widetilde{\Sigma}_b}(-\log \widetilde{D}_b)) & \end{array} \tag{1.3.20}$$

where $m^\vee : H^0(\widetilde{\Sigma}_b, \mathfrak{t} \otimes \Omega_{\widetilde{\Sigma}_b}(\log \widetilde{D}_b))^W \otimes H^1(\widetilde{\Sigma}_b, T_{\widetilde{\Sigma}_b}(-\log \widetilde{D}_b)) \to H^1(\widetilde{\Sigma}_b, \mathcal{O}_{\widetilde{D}_b})^W$ is the logarithmic contraction.

Proposition 1.3.33. *For each $b \in B^{ur}$, applying the Gauss-Manin connection to λ_Δ, one can obtain an isomorphism*

$$\nabla^{GM} : T_bB \xrightarrow{\cong} F^1V_b^\vee$$
$$\mu \mapsto \nabla_\mu^{GM}(\lambda_{\Delta,b})$$

Proof. The idea is to follow the local computation as in the original proof of the classical case [HHP10]. We can apply the same arguments because we restrict to cameral covers with no ramification over the divisors. First, given $\mu \in T_bB$, one can compute ∇_μ^{GM} by using the above diagram (1.3.20). Let's consider

$$C_\mu := pr \circ \nabla_\mu^{GM} : F^1 \to V_b^\vee \to V_b^\vee/F^1$$
$$C_\mu(\alpha) = \alpha \cup \kappa(\mu).$$

One can see that $\nabla_{\mu}^{GM}(\lambda_{\Delta}) \in F^1$ for all $\mu \in T_bB$ by noticing that $C_{\mu}(\lambda_{\Delta,b}) = 0$. This also follows from Griffiths' transversality of variation of mixed Hodge structures [PS08, Section 14.4]. On the other hand, using the isomorphism $T_bB \cong F^1V_b^{\vee} \cong H^0(\widetilde{\Sigma}_b, \mathfrak{t} \otimes K_{\widetilde{\Sigma}_b}(\widetilde{D}_b))^W$, we can assign a logarithmic one form α_{μ} to every $\mu \in T_bB$. From the definition of the Seiberg-Witten differential form, it now follows that

$$\nabla_{\mu}^{GM}(\lambda_{\Delta,b}) = \alpha_{\mu}$$

for all $\mu \in T_bB$. □

1.4 Calabi-Yau integrable systems

1.4.1 Construction

In this section, we shall generalize Smith's elementary modification idea [Smi15] to construct a (semi-polarized) Calabi-Yau integrable system.

First, we describe the construction of a family of Calabi-Yau threefolds. Let $V := \mathbf{Tot}(K_{\Sigma}(D) \oplus (K_{\Sigma}(D))^{n-1} \oplus K_{\Sigma}(D))$ and consider the short exact sequence

$$0 \to \mathcal{O}_{\Sigma}(-D) \xrightarrow{\alpha} \mathcal{O}_{\Sigma} \to i_{D*}\mathcal{O}_D \to 0.$$

Suppose u is a local frame of $\mathcal{O}_{\Sigma}(-D)$. In terms of a local coordinate z around a point of D where $z = 0$, $\alpha(u)$ is represented by $f \cdot u$ where f is a locally defined function that vanishes at $z = 0$. We define an elementary modification W of V along the first component:

$$W := \mathbf{Tot}(K_{\Sigma}(D - D) \oplus (K_{\Sigma}(D))^{n-1} \oplus K_{\Sigma}(D)) \to \mathbf{Tot}(K_{\Sigma}(D) \oplus (K_{\Sigma}(D))^{n-1} \oplus K_{\Sigma}(D))$$

and denote the projection map by $\pi_W : W \to \Sigma$.

For $b = (b_2(z), ..., b_n(z)) \in B = \oplus_{i=2}^n H^0(\Sigma, K_\Sigma(D))^{\otimes i}$, we define the threefold X_b as the zero locus of a section in $\Gamma(W, \pi_W^* K_\Sigma(D)^{\otimes n})$:

$$X_b := \{\alpha(x)y - s^n - \pi_W^* b_2(z)s^{n-2} - ... - \pi_W^* b_n(z) = 0\} \subset W \tag{1.4.1}$$

with the projection $\pi_b : X_b \to \Sigma$. Here we denote by x, y and s the tautological sections of $K_\Sigma, (K_\Sigma(D))^{n-1}$ and $K_\Sigma(D)$, respectively. Note that each term in the equation (1.4.1) is a section of $\pi_W^* K_\Sigma(D)^{\otimes n}$. More explicitly, we have

$$x \in \Gamma(W, \pi_W^* K_\Sigma), \quad \alpha(x) \in \Gamma(W, \pi_W^* K_\Sigma(D)), \quad y \in \Gamma(W, \pi_W^*(K_\Sigma(D))^{n-1})$$
$$s \in \Gamma(W, \pi_W^* K_\Sigma(D)), \quad \pi^* b_i \in \Gamma(W, \pi_W^*(K_\Sigma(D))^i)$$

This construction gives rise to a family of quasi-projective threefolds

$$pr_2 \circ \boldsymbol{\pi} : \mathcal{X} \to B.$$

Next, we show that the threefold X_b is indeed a non-singular Calabi-Yau threefold.

Proposition 1.4.1. *The threefold X_b has trivial canonical bundle.*

Proof. By the adjunction formula,

$$K_{X_b} = K_W \otimes \pi_W^*(K_\Sigma(D))^{\otimes n}|_{X_b}.$$

where $\pi_W : W \to \Sigma$. Note that

$$K_W = \pi_W^* \det(W^\vee) \otimes \pi_W^* K_\Sigma \cong \pi_W^*(K_\Sigma^{-n-1}(-nD)) \otimes \pi_W^* K_\Sigma \cong \pi_W^*(K_\Sigma^{-n}(-nD)).$$

So it follows that

$$K_{X_b} = \pi_W^*(K_\Sigma^{-n}(-nD)) \otimes \pi_W^*(K_\Sigma(D))^{\otimes n}|_{X_b} \cong \mathcal{O}_{X_b}.$$

□

Proposition 1.4.2. *For each $b \in B^{ur}$, the threefold X_b is non-singular.*

Proof. This is a local statement, so we can restrict to neighbourhoods in Σ. Around a point of D with local coordinate z, the local model of X_b is

$$\{f(z)xy - s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z) = 0\} \subset \mathbb{C}^3_{(x,y,s)} \times \mathbb{C}_z,$$

where $\widetilde{b}_i$ are now functions of z, and $f(z)$ is function with zero only at $z = 0$. We check smoothness by examining the Jacobian criterion. The equation

$$\frac{\partial}{\partial s}(s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z)) = 0$$

implies that, for each z, the equation $s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z) = 0$ must have repeated solutions, this happens only when z is at a critical value. The remaining equations in the Jacobian criterion are

$$f(z)y = 0, \quad f(z)x = 0, \quad f'(z)xy + \frac{\partial}{\partial z}(s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z)) = 0$$

When $x = y = 0$, the equation $\frac{\partial}{\partial z}(s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z)) = 0$ has no solution since we assume that the spectral curve associated to b is smooth. Hence, it must be the case $f(z) = 0$ or equivalently $z = 0$. However, since we assume $b \in B^{ur}$, this cannot happen and X_b is non-singular around D.

Away from D, a similar argument shows that the threefold is non-singular over the local neighbourhood. Hence, X_b is non-singular everywhere.

□

Again, by examining the defining equation (1.4.1), we can list the types of fibers of the map $\pi_b : X_b \to \Sigma$:

- For $p \in D$ with coordinate $z = 0$, the fiber is defined by the equation $s^n - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z) = 0$, i.e. disjoint union of n copies of $\mathbb{C}^2$.

- For a critical value p of π_b, the fiber is defined by $xy - \prod_{i=1}^{m}(s - s_i)^{k_i}$ where $\sum_{i=1}^{m} k_i = n$ $(m < n)$. Hence, the fiber is a singular surface with $A_{k_i - 1}$-singularity at s_i.

- For p away from D and the discriminant locus of π_b, the fiber is defined by $(xy - s^n) - \widetilde{b}_2(z)s^{n-2} - ... - \widetilde{b}_n(z) = 0$ and smooth, so it is isomorphic to a smooth fiber of the universal unfolding of A_{n-1}-singularity $\mathbb{C}^2/\mathbb{Z}_n$.

Next, we study the mixed Hodge structure of X_b. Let's denote the complement of $\pi_b^{-1}(D)$ by X_b°. The long exact sequence of compactly supported cohomologies associated to the pair $(X_b, \pi_b^{-1}(D))$ is

$$\cdots \to H_c^2(\pi_b^{-1}(D), \mathbb{Z}) \to H_c^3(X_b^\circ, \mathbb{Z}) \to H_c^3(X_b, \mathbb{Z}) \to H_c^3(\pi_b^{-1}(D), \mathbb{Z}) \to \cdots$$

As $H_c^2(\pi_b^{-1}(D), \mathbb{Z}) = H_c^3(\pi_b^{-1}(D), \mathbb{Z}) = 0$, we have an isomorphism of $\mathbb{Z}$-mixed Hodge structures

$$H_c^3(X_b, \mathbb{Z}) \cong H_c^3(X_b^\circ, \mathbb{Z}) \tag{1.4.2}$$

Moreover, the Leray spectral sequence for compactly supported cohomology associated to $\pi_b^\circ := \pi_b|_{X_b^\circ} : X_b^\circ \to \Sigma^\circ$ implies

$$H_c^3(X_b^\circ, \mathbb{Z}) \cong H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ\mathbb{Z}) \tag{1.4.3}$$

because the (compactly supported) cohomology of a fiber is non-trivial only for degree 0 and 2 [DDP07, Lemma 3.1]. As the Leray spectral sequence is compatible with mixed Hodge structures ([Ara05], [De 09, Corollary 2.10]), it is enough to compute the Hodge type of $H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ\mathbb{Z})$. For this, we need to deal with critical values of π_b° and the monodromy around D. First, note that the critical values do not determine Hodge type. This is because a local system $\mathcal{F}$ having finite monodromies M (only around the critical values) can be trivialized by pulling back to an order $|M|$ covering $\pi_M : \widetilde{\Sigma}_M \to \Sigma$. Then the sheaf cohomology $H^1(\Sigma, \mathcal{F})$ is the same as $H^1(\widetilde{\Sigma}_M, \pi_M^* \mathcal{F})^M$ whose Hodge type is determined by $H^1(\widetilde{\Sigma}_M, \pi_M^* \mathcal{F})$. Applying this to our case, we can ignore the critical values and it is enough to consider only the monodromy of $R^2\pi_{b!}^\circ\mathbb{Z}$ around D to compute the Hodge type. Since X_b is constructed via elementary modification from another threefold which has smooth fibers everywhere around D, we see that the monodromy of $R^2\pi_{b!}\mathbb{Z}$ around D is trivial. As $H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ\mathbb{Z}) \cong H^1(\Sigma, D, R^2\pi_{b!}\mathbb{Z})$, it admits the $\mathbb{Z}$-mixed Hodge structure of type $\{(-2,-2),(-2,-1),(-1,-2)\}$ due to the relative version of Zucker's theorem [Zuc79]. Therefore, we have the following result.

Proposition 1.4.3. *For $b \in B^{ur}$, the third homology group $H_3(X_b, \mathbb{Z})$ admits a $\mathbb{Z}$-mixed Hodge structure of type $\{(-2,-2),(-2,-1),(-1,-2)\}$. Moreover, the third cohomology group $H^3(X_b, \mathbb{Z})$ admits a $\mathbb{Z}$-mixed Hodge structure of type $\{(1,2),(2,1),(2,2)\}$.*

The homology version of the second intermediate Jacobian of X_b is defined to be Jacobian associated to the $\mathbb{Z}$-mixed Hodge structure of $H_3(X_b, \mathbb{Z})(1)$

$$J_2(X_b) := J(H_3(X_b, \mathbb{Z})(1)) = \frac{H_3(X_b, \mathbb{C})}{F^{-1}H_3(X_b, \mathbb{C}) + H_3(X_b, \mathbb{Z})} \tag{1.4.4}$$

Remark 1.4.4. The homology group $H_3(X_b, \mathbb{Z})(1)$ turns out to have torsion (see Theorem 1.5.2). To get the $\mathbb{Z}$-mixed Hodge structure on the lattice of the semi-abelian variety $J_2(X_b)$, we should consider the $\mathbb{Z}$-mixed Hodge structure on the torsion-free part $H_3(X_b, \mathbb{Z})_{\mathrm{tf}}(1)$.

Corollary 1.4.5. *For $b \in B^{ur}$, the homology version of the second intermediate Jacobian $J_2(X_b)$ is a semi-abelian variety.*

Remark 1.4.6. (Adjoint Type) Unlike the classical case, the cohomological intermediate Jacobian on $H^3(X_b, \mathbb{Z})$ is not a semi-abelian variety. This is one of the new features, so we need to consider different data to describe the case of $PGL(n, \mathbb{C})$, the adjoint group of type A. It turns out that the right object is a mixture of compactly supported cohomology and ordinary cohomology associated to $\pi_b : X_b \to \Sigma$:

$$H^1_c(\Sigma, R^2\pi_{b*}\mathbb{Z}) \cong H^1_c(\Sigma^\circ, R^2\pi^\circ_{b*}\mathbb{Z}).$$

1.4.2 Calabi-Yau integrable systems

Having constructed the family of Calabi-Yau threefolds $\mathcal{X}^{ur} \to B^{ur}$, we can consider the relative intermediate Jacobian fibration $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to B^{ur}$ whose fiber is $J_2(X_b) = H_3(X_b, \mathbb{C})/(F^{-1}H_3(X_b, \mathbb{C}) + H_3(X_b, \mathbb{Z}))$. One way to equip it with an integrable system structure is to find an abstract Seiberg-Witten differential (see Section 2). In the case of an intermediate Jacobian fibration, this can be achieved by finding a global nowhere-vanishing holomorphic volume form in each fiber. The resulting semi-polarized integrable system will again be called the Calabi-Yau integrable system.

Consider the subfamily of Calabi-Yau threefolds

$$(\mathcal{X}^\circ)^{ur} := \mathcal{X}^{ur} \setminus \boldsymbol{\pi}^{-1}(D \times B^{ur}) \subset \mathcal{X}^{ur} \to B^{ur}.$$

whose fiber is $X^\circ_b := X_b \setminus \pi_b^{-1}(D)$. From the relation (1.4.2), it is enough to find global holomorphic volume forms for the family $(\mathcal{X}^\circ)^{ur} \to B^{ur}$. The idea is that the family $(\mathcal{X}^\circ)^{ur} \to B^{ur}$ can be constructed alternatively by gluing Slodowy slices as in [DDP07][Bec20], which is the key ingredient used for the existence of global volume forms.

Claim 1.4.7. The family of quasi-projective Calabi-Yau threefolds $\pi^{ur} : (\mathcal{X}^\circ)^{ur} \to B^{ur}$ can be obtained by gluing Slodowy slices.

Recall that in the classical case [Slo80], the Slodowy slice $S \subset \mathfrak{g}$ provides a semi-universal $\mathbb{C}^*$-deformation $\sigma : S \to \mathfrak{t}/W$ of simple singularities via the adjoint map $\sigma : \mathfrak{g} \to \mathfrak{t}/W$. However, if we denote by d_j the standard ($\mathbb{C}^*$-action) weights of the generators of the coordinate ring $\mathbb{C}[\chi_1, ..., \chi_j]$ of $\mathfrak{t}/W$, then the weights on $\mathbb{C}[\chi_1, ..., \chi_j]$ must be chosen as $2d_j$ for σ to be $\mathbb{C}^*$-equivariant (see [BDW20, Remark 2.5.3], [Slo80]).

Now we choose a theta characteristic L on Σ, i.e. $L^2 \cong K_\Sigma$. Since $L^2|_{\Sigma^\circ} \cong K_\Sigma|_{\Sigma^\circ} \cong K_\Sigma(D)|_{\Sigma^\circ}$, we have an isomorphism of associated bundles over Σ°

$$L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W \cong K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W$$

where the weights of the $\mathbb{C}^*$-action on both sides are different: the left hand side has weights $2d_j$ and the right hand side has weights d_j. As the map $\sigma : S \to \mathfrak{t}/W$ is $\mathbb{C}^*$-equivariant, we can glue it along $\mathbf{Tot}(L)$ to obtain

$$\boldsymbol{\sigma} : \boldsymbol{S} := \mathbf{Tot}(L \times_{\mathbb{C}^*} S) \to \mathbf{Tot}(L \times_{\mathbb{C}^*} \mathfrak{t}/W)$$

and its restriction

$$\boldsymbol{\sigma}|_{\Sigma^\circ} : \boldsymbol{S}|_{\Sigma^\circ} := \mathbf{Tot}(L \times_{\mathbb{C}^*} S)|_{\Sigma^\circ} \to \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \cong \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) = \boldsymbol{U}|_{\Sigma^\circ}.$$

Pulling back under the evaluation map from $\Sigma \times B$, one gets a family of quasi-projective threefolds $(\mathcal{Y}^\circ)^{ur}$ as follows:

$$\begin{array}{ccc} (\mathcal{Y}^\circ)^{ur} & \longrightarrow & \boldsymbol{S}|_{\Sigma^\circ} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \boldsymbol{\sigma}|_{\Sigma^\circ}} \\ \Sigma^\circ \times B^{ur} & \xrightarrow{ev} & \boldsymbol{U}|_{\Sigma^\circ} \end{array} \tag{1.4.5}$$

Lemma 1.4.8. *We have an isomorphism of the families* $(\mathcal{Y}^\circ)^{ur} \cong (\mathcal{X}^\circ)^{ur}$ *over* B^{ur} *and, in particular,* $Y_b^\circ \cong X_b^\circ$ *where* Y_b° *is a member of the family* $\mathcal{Y}^\circ$.

Proof. For type A, we have a semi-universal $\mathbb{C}^*$-deformation of A_{n-1} singularities (see [KM92, Theorem 1]) as follows:

$$\begin{aligned} \sigma' : H := \{xy - s^n - b_2 s^{n-2} - ... - b_n = 0\} \subset \mathbb{C}^3 \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1} \cong \mathfrak{t}/W \\ (x, y, s, b_2, ..., b_n) \mapsto (b_2, ..., b_n) \end{aligned} \tag{1.4.6}$$

The map σ' is $\mathbb{C}^*$-equivariant if we endow the following $\mathbb{C}^*$-actions on $\mathbb{C}^3$ and $\mathbb{C}^{n-1}$:

$$(x, y, s) \mapsto (\lambda^2 x, \lambda^{2(n-1)} y, \lambda^2 s), \quad (b_2, ..., b_n) \mapsto (\lambda^4 b_2, ..., \lambda^{2n} b_n). \tag{1.4.7}$$

Since the semi-universal $\mathbb{C}^*$-deformation of a simple singularity is unique up to isomorphism, the two deformations σ and σ' are isomorphic. In other words, the Slodowy slice S contained in $\mathfrak{g}$ is isomorphic to the hypersurface H in $\mathbb{C}^3 \times \mathbb{C}^{n-1}$ as semi-universal $\mathbb{C}^*$-deformation. Note that it is important to choose the $\mathbb{C}^*$-action on $\mathbb{C}^{n-1} \cong \mathfrak{t}/W$ and $\mathbb{C}^3$ as above for S and H to be isomorphic as $\mathbb{C}^*$-deformation (see [BDW20, Remark 2.5.3]).

Next, let's turn to the global situation. We again have the isomorphism of associated bundles

$$L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^* \cong K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^*$$

with the weights of the $\mathbb{C}^*$-action on the left hand side being twice the weights on the right hand side. Hence, the associated bundle $L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^3$ is

$$L^2|_{\Sigma^\circ} \oplus L^{2(n-1)}|_{\Sigma^\circ} \oplus L^2|_{\Sigma^\circ} \cong (K_\Sigma(D) \oplus K_\Sigma(D)^{\otimes n-1} \oplus K_\Sigma(D))|_{\Sigma^\circ} \cong V|_{\Sigma^\circ}$$

Also, since the elementary modification is an isomorphism i.e. $V|_{\Sigma^\circ} \cong W|_{\Sigma^\circ}$ away from D, the previous construction (1.4.1) of the family $\pi^\circ : (\mathcal{X}^\circ)^{ur} \to B^{ur}$ as a family of hypersurfaces in the total space of $W|_{\Sigma^\circ}$ is equivalent to the construction as the pullback of the gluing of H and σ' over $K_\Sigma(D)|_{\Sigma^\circ}$:

$$\begin{array}{ccc} (\mathcal{X}^\circ)^{ur} & \longrightarrow & \boldsymbol{H}|_{\Sigma^\circ} \subset \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^3) \times \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \\ \downarrow \pi^\circ & & \downarrow (\boldsymbol{\sigma'})|_{\Sigma^\circ} \\ \Sigma^\circ \times B^{ur} & \xrightarrow{ev} & \boldsymbol{U}|_{\Sigma^\circ} = \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \end{array} \tag{1.4.8}$$

where we define $\boldsymbol{\sigma'} : \boldsymbol{H} = \mathbf{Tot}(K_\Sigma(D) \times_{\mathbb{C}^*} H) \to \boldsymbol{U}$ and all the $\mathbb{C}^*$-actions in the diagram are understood as having half the weights in (1.4.7). By the argument that S and H are isomorphic as $\mathbb{C}^*$-deformation, we have that $\boldsymbol{\sigma}|_{\Sigma^\circ} : \boldsymbol{S}|_{\Sigma^\circ} = \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} S) \to \boldsymbol{U}|_{\Sigma^\circ}$ and $\boldsymbol{\sigma'}|_{\Sigma^\circ} : \boldsymbol{H}|_{\Sigma^\circ} \to \boldsymbol{U}|_{\Sigma^\circ}$ are also isomorphic. By pulling back this isomorphism along the evaluation map to $\Sigma^\circ \times B^{ur}$, we get the isomorphism $(\mathcal{Y}^\circ)^{ur} \cong (\mathcal{X}^\circ)^{ur}$.

□

Proposition 1.4.9. *The relative intermediate Jacobian fibration $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to B^{ur}$ is a semi-polarized integrable system.*

Proof. By the relation (4.2), it is enough to show that there exists a Seiberg-Witten differential associated to the subfamily $(\mathcal{X}^\circ)^{ur} \to B^{ur}$. In other words, we need to construct a holomorphic volume form $\lambda_{CY}{}^\circ$ on $(\mathcal{X}^\circ)^{ur}$ which yields the nowhere vanishing holomorphic volume form $\lambda^\circ_{CY,b} \in H^0(X^\circ_b, K_{X^\circ_b})$ for each $b \in B^{ur}$ and satisfies the condition (1.2.4).

First, the holomorphic volume form $\lambda_{CY}{}^\circ$ is obtained from the holomorphic 3-form λ on $\boldsymbol{S}$. Note that the Kostant-Kirillov form on $\mathfrak{g}$ induces the nowhere vanishing section in $\nu \in H^0(S, K_\sigma)$. One can glue the sections over L by tensoring with local frames in the pullback of K_Σ, which turns out to be the holomorphic 3-form λ on $\boldsymbol{S}$ [DDP07][Bec20]. By restricting λ to Σ°, it becomes a global holomorphic 3-form whose pullback to $(\mathcal{X}^\circ)^{ur}$ is the desired volume form λ°_{CY}.

Next, the proof that λ°_{CY} becomes the Seiberg-Witten differential relies on our main result (Theorem 1.5.1). In particular, we identify the volume form λ°_{CY} with the Seiberg-Witten differential for the Hitchin system so that the form λ°_{CY} automatically satisfies the condition (1.2.4). Therefore, it follows from Proposition (1.2.5) that $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to B^{ur}$ is a semi-polarized integrable system. $\square$

1.5 Meromorphic DDP correspondence

1.5.1 Isomorphism of semi-polarized integrable systems

The goal of this section is to prove an isomorphism between the two semi-polarized integrable systems that have been studied so far: the moduli space of unordered diagonally framed Higgs bundles $\mathcal{M}^\Delta(n, D)^{ur} \to B^{ur}$ and the relative intermediate Jacobian fibration $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to B^{ur}$ of the family of Calabi-Yau threefolds $\mathcal{X}^{ur} \to B^{ur}$. The main result is stated as follows.

Theorem 1.5.1. *There is an isomorphism of semi-polarized integrable systems:*

$$\begin{array}{ccc} \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \mathcal{M}^\Delta(n,D)^{ur} \\ & {\scriptstyle \pi^{ur}} \searrow \quad \swarrow {\scriptstyle h^{ur}_\Delta} & \\ & B^{ur} & \end{array} \tag{1.5.1}$$

Recall that we have shown in Proposition 1.3.12 and Corollary 1.3.14 that $(h^{ur}_\Delta)^{-1}(b) \cong Prym(\overline{\Sigma}^\circ_b, \Sigma^\circ) \cong J(H_{\Delta,SL(n),b})$ where $H_{\Delta,SL(n),b} := H_1(Prym(\overline{\Sigma}^\circ_b, \Sigma^\circ), \mathbb{Z}) = H_1(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ})_{\text{tf}}$ and $\mathcal{K}_b := \ker(Tr : \overline{p}_{b*}\mathbb{Z} \to \mathbb{Z})$. By definition, the fiber $(\pi^{ur})^{-1}(b) = J_2(X_b) = J(H_3(X_b, \mathbb{Z})(1))$. The specialization of Theorem 1.5.1 to $b \in B^{ur}$ is equivalent to an isomorphism between the semi-abelian varieties $J_2(X_b)$ and $Prym(\overline{\Sigma}^\circ_b, \Sigma^\circ)$, or equivalently, between the $\mathbb{Z}$-mixed Hodge structures $H_3(X, \mathbb{Z})_{\text{tf}}(1)$ and $H_{\Delta,SL(n),b}$ of type $\{(-1,-1), (-1,0), (0,-1)\}$. We begin by proving the following result.

Theorem 1.5.2. *For $b \in B^{ur}$, there is an isomorphism of $\mathbb{Z}$-mixed Hodge structures:*

$$(H_3(X_b, \mathbb{Z})_{tf}(1), W_\bullet^{CY}, F_{CY}^\bullet) \cong (H_{\Delta,SL(n),b}, W_\bullet^{\Delta,b}, F_{\Delta,b}^\bullet). \tag{1.5.2}$$

Proof. We first fix some notations. Denote by $\Sigma^1 := \Sigma^\circ \setminus Br(\widetilde{p}_b^\circ), \widetilde{\Sigma}_b^1 := \widetilde{\Sigma}_b^\circ \setminus Ram(\widetilde{p}_b^\circ)$ the complement of the ramification and branch divisors in $\Sigma_b^\circ, \widetilde{\Sigma}_b^\circ$ respectively. Since the branch divisor of the spectral cover $\overline{p}_b : \overline{\Sigma}_b \to \Sigma$ is contained in the branch divisor of the cameral cover $\widetilde{p}_b^\circ : \widetilde{\Sigma}_b^\circ \to \Sigma^\circ$, we write $\overline{\Sigma}_b^1 := \overline{\Sigma}_b^\circ \setminus (\overline{p}_b^\circ)^{-1} Br(\widetilde{p}_b^\circ)$. The restricted maps of the spectral cover $\overline{p}_b^1 : \widetilde{\Sigma}_b^1 \to \Sigma^1$ and the cameral cover $\widetilde{p}_b^1 : \widetilde{\Sigma}_b^1 \to \Sigma^1$ are then unramified. Similarly, we write $X_b^1 \subset X_b^\circ$ the complement of $(\pi_b^\circ)^{-1}(D)$ in X_b° and the restricted map as $\pi_b^1 : X_b^1 \to \Sigma^1$.

Step 1. As argued in (1.4.2) and (1.4.3) of the previous section, we have the isomorphisms of $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1), (-1,0), (0,-1)\}$

$$H_3(X_b, \mathbb{Z})(1) \cong H_c^3(X_b, \mathbb{Z})(1) \cong H_c^3(X_b^\circ, \mathbb{Z})(1) \cong H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ \mathbb{Z})(1). \tag{1.5.3}$$

Step 2.

Lemma 1.5.3. *Over Σ°, we have an isomorphism of sheaves,*

$$R^2 \pi_{b\,!}^\circ \mathbb{Z} \cong (\widetilde{p}_{b*}^\circ \Lambda_{SL(n)})^W. \tag{1.5.4}$$

Proof. In the classical work of [Slo80], Slodowy provided a detailed study of the topology of the maps in the following diagram via its simultaneous resolution:

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S \\ \big\downarrow{\scriptstyle \widetilde{\sigma}} & & \big\downarrow{\scriptstyle \sigma} \\ \mathfrak{t} & \xrightarrow{\ \phi\ } & \mathfrak{t}/W \end{array} \tag{1.5.5}$$

It can be shown that there is an isomorphism of constructible sheaves

$$R^2 \sigma_*^1 \mathbb{Z} \cong (\phi_*^1 \Lambda_{SL(n)})^W \tag{1.5.6}$$

over an open subset $\mathfrak{t}^1/W \subset \mathfrak{t}/W$ defined as the image of another open subset $\mathfrak{t}^1 \subset \mathfrak{t}$ under ϕ. Here we denote $\phi^1 := \phi|_{\mathfrak{t}^1}$ and $\sigma^1 : \sigma^{-1}(\mathfrak{t}^1/W) \to \mathfrak{t}^1/W$. For details, see [Bec20, Lemma 5.1.3].

Next, we glue the maps σ and ϕ along $K_\Sigma(D)|_{\Sigma^\circ}$ as in (1.3.17) and (1.4.8)

$$\begin{array}{ccc} & & \boldsymbol{S}|_{\Sigma^\circ} = \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} S) \\ & & \downarrow{\scriptstyle \boldsymbol{\sigma}|_{\Sigma^\circ}} \\ \widetilde{\boldsymbol{U}}|_{\Sigma^\circ} := \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}) & \xrightarrow{\boldsymbol{\phi}|_{\Sigma^\circ}} & \boldsymbol{U}|_{\Sigma^\circ} = \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \cong \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \end{array} \tag{1.5.7}$$

Let us define $\boldsymbol{U^1} := \mathbf{Tot}(K_\Sigma(D) \times_{\mathbb{C}^*} \mathfrak{t}^1/W) \subset \boldsymbol{U}$. Since the varieties here are glued using the same cocyle of $L|_{\Sigma^\circ}$ (again, in taking the associated bundles here, $L|_{\Sigma^\circ}$ as a $\mathbb{C}^*$-bundle acts with twice the weights of the action by $K_\Sigma(D)|_{\Sigma^\circ}$), the isomorphism of constructible sheaves (1.5.6) over $\mathfrak{t}^1/W$ also glues together to another isomorphism of constructible sheaves over $\boldsymbol{U^1}|_{\Sigma^\circ}$:

$$R^2(\boldsymbol{\sigma})_!\mathbb{Z} \cong (\boldsymbol{\phi}_*\Lambda_{SL(n)})^W. \tag{1.5.8}$$

As argued in Claim (1.4.7), $\boldsymbol{\sigma}|_{\Sigma^\circ} : \boldsymbol{S}|_{\Sigma^\circ} \to \boldsymbol{U}|_{\Sigma^\circ}$ is equivalent to $(\boldsymbol{\sigma'})|_{\Sigma^\circ} : \boldsymbol{H}|_{\Sigma^\circ} \to \boldsymbol{U}|_{\Sigma^\circ}$, so we obtain

$$R^2(\boldsymbol{\sigma'})_!\mathbb{Z} \cong (\boldsymbol{\phi}_*\Lambda_{SL(n)})^W \tag{1.5.9}$$

over $\boldsymbol{U^1}|_{\Sigma^\circ}$. In both (1.5.8) and (1.5.9), we drop the notation of the restrictions of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'}$ and $\boldsymbol{\phi}$ to $\boldsymbol{U^1}|_{\Sigma^\circ}$ for convenience.

Recall from Claim (1.4.7) that $\pi_b^\circ : X_b^\circ \to \Sigma^\circ$ can be obtained by pulling back from $\boldsymbol{\sigma'}|_{\Sigma^\circ} : \boldsymbol{H}|_{\Sigma^\circ} \to \boldsymbol{U}|_{\Sigma^\circ}$ along the composition of the inclusion and the evaluation map $\Sigma^\circ \times \{b\} \hookrightarrow \Sigma^\circ \times B \to \boldsymbol{U}|_{\Sigma^\circ}$. For $b \in B^{ur}$, the section $b : \Sigma \to \boldsymbol{U}$ factorizes through $\boldsymbol{U^1}$ and then restricts to $b|_{\Sigma^\circ} : \Sigma^\circ \to \boldsymbol{U^1}|_{\Sigma^\circ}$, so the isomorphism (1.5.9) specializes to $R^2\pi_{b!}^\circ\mathbb{Z} \cong (\widetilde{p}_{b*}^\circ\Lambda_{SL(n)})^W$ by pulling back along $b|_{\Sigma^\circ}$.

□

Step 3.

Lemma 1.5.4. *Over Σ°, we have an isomorphism of sheaves,*

$$(\widetilde{p}^\circ_{b*}\Lambda_{SL(n)})^W \cong \mathcal{K}_b|_{\Sigma^\circ}. \tag{1.5.10}$$

Proof. To simplify the notation, we will write $\mathcal{K}_b^\circ := \mathcal{K}_b|_{\Sigma^\circ}$ in this proof. Recall that there is an isomorphism (see [Don93, (6.5)]) between the two sheaves away from the branch locus:

$$\overline{p}^1_{b*}\mathbb{Z} \cong (\widetilde{p}^1_{b*}R)^W \tag{1.5.11}$$

where $R := \mathbb{Z}[W/W_0]$ denote the free abelian group generated by the set of right (or left) cosets W/W_0. Then we see that

$$\mathcal{K}_b|_{\Sigma^1} = \ker(\overline{p}^1_{b*}\mathbb{Z} \to \mathbb{Z}) \cong \ker((\widetilde{p}^1_{b*}R)^W \to \mathbb{Z}) \cong (\widetilde{p}^1_{b*}\Lambda_{SL(n)})^W,$$

the last isomorphism holds because $\ker(R \to \mathbb{Z}) = \Lambda_{SL(n)}$.

Denote by $j : \Sigma^1 \to \Sigma^\circ$ the inclusion map. We first write $\mathcal{K}_b^\circ$ as $j_*j^*\mathcal{K}_b^\circ$. Indeed, as $\overline{p}^\circ_{b*}\mathbb{Z} = j_*\overline{p}^1_{b*}\mathbb{Z}$ and $\mathbb{Z} \cong j_*\mathbb{Z}$, applying the functor j_* to the short exact sequence $0 \to j^*\mathcal{K}_b^\circ \to \overline{p}^1_{b*}\mathbb{Z} \xrightarrow{Tr|_{\Sigma^1}} \mathbb{Z} \to 0$, we get

$$0 \to j_*j^*\mathcal{K}_b^\circ \to j_*\overline{p}^1_{b*}\mathbb{Z} = \overline{p}^1_{b*}\mathbb{Z} \xrightarrow{Tr} j_*\mathbb{Z} = \mathbb{Z} \to R^1j_*j^*\mathcal{K}_b^\circ \to ...$$

In particular, it follows that $j_*j^*\mathcal{K}_b^\circ \cong \ker(Tr) = \mathcal{K}_b^\circ$.

Hence, we get

$$(\widetilde{p}^\circ_{b*}\Lambda_{SL(n)})^W \cong j_*(\widetilde{p}^1_{b*}\Lambda_{SL(n)})^W \cong j_*j^*\mathcal{K}_b^\circ \cong \mathcal{K}_b^\circ$$

which means that the isomorphism (1.5.11) above extends from Σ^1 to Σ°. □

Step 4. Finally, since the isomorphic local systems $R^2\pi^\circ_{b!}\mathbb{Z} \cong (\widetilde{p}^\circ_{b*}\Lambda_{SL(n)})^W \cong \mathcal{K}_b|_{\Sigma^\circ}$ have trivial monodromy at D, one can argue as in [DDP07, Lemma 3.1] and the argument for Proposition 1.4.3 that it induces the $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,-1),(-1,0),(0,-1)\}$ on

$$H^1_c(\Sigma^\circ, R^2\pi^\circ_{b!}\mathbb{Z})(1) \cong H^1_c(\Sigma^\circ, (\widetilde{p}^\circ_{b*}\Lambda_{SL(n)})^W) \cong H^1_c(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ}).$$

Hence, taking the torsion free part, we achieve the isomorphism of $\mathbb{Z}$-mixed Hodge structures

$$H_3(X_b,\mathbb{Z})_{\mathrm{tf}}(1) \cong H^1_c(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ})_{\mathrm{tf}} \cong H_{\Delta,SL(n),b}.$$

□

By the equivalence between semi-abelian varieties and torsion free $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1),(-1,0),\ (0,-1)\}$, we immediately get the following result:

Corollary 1.5.5. *We have an isomorphism of semi-abelian varieties*

$$J_2(X_b) \cong h^{-1}_\Delta(b) \cong Prym(\widetilde{\Sigma}^\circ_b/\Sigma^\circ). \tag{1.5.12}$$

Now we return to the main theorem.

Proof of Theorem 1.5.1. Clearly, the argument in Theorem 1.5.2 works globally for the family of CY threefolds $pr_2 \circ \boldsymbol{\pi}^{ur} : \mathcal{X}^{ur} \to \Sigma \times B^{ur} \to B^{ur}$ and the family of punctured spectral curves $pr_2 \circ \overline{\boldsymbol{p}}^{ur} : \overline{\boldsymbol{\Sigma}}^\circ \to \Sigma^\circ \times B^{ur} \to B^{ur}$, so it yields an isomorphism of admissible variations of $\mathbb{Z}$-mixed Hodge structures:

$$R^3(pr_2 \circ \boldsymbol{\pi}^{ur})_!\mathbb{Z}(1) \cong R^1(pr_2)_!(\boldsymbol{\mathcal{K}}) \tag{1.5.13}$$

where $\boldsymbol{\mathcal{K}} := \ker(\boldsymbol{Tr} : \overline{\boldsymbol{p}}^{ur}_{*}\mathbb{Z} \to \mathbb{Z})$. By taking the relative Jacobian fibrations of both sides, we immediately get an isomorphism of varieties:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \boldsymbol{Prym}(\overline{\boldsymbol{\Sigma}}^{\circ}, \Sigma^{\circ}) \cong \mathcal{M}^{\Delta}(n, D)^{ur} \\ & {\scriptstyle \pi^{ur}} \searrow \quad \swarrow {\scriptstyle h^{ur}_{\Delta}} & \\ & B^{ur} & \end{array} \tag{1.5.14}$$

where $\boldsymbol{Prym}(\overline{\boldsymbol{\Sigma}}^{\circ}, \Sigma^{\circ})$ is the relative Prym fibration of the family of punctured spectral curves $\overline{\boldsymbol{\Sigma}}^{\circ} \to B^{ur}$. By the spectral correspondence proved in Proposition 1.3.12, we have $\boldsymbol{Prym}(\overline{\boldsymbol{\Sigma}}^{\circ}, \Sigma^{\circ}) \cong \mathcal{M}^{\Delta}(n, D)^{ur}$.

It remains to verify that the morphism $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \to \mathcal{M}^{\Delta}(n, D)^{ur}$ intertwines the abstract Seiberg-Witten differentials constructed on each side. This can be easily obtained by modifying the classical results in [DDP07] [Bec20] to our punctured case. Note that both the abstract Seiberg-Witten differentials come from the tautological section on $\widetilde{\boldsymbol{U}}$. In order to compare them, we again look at the simulteneous resolution of $S \to \mathfrak{t}/W$:

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S \\ \big\downarrow {\scriptstyle \widetilde{\sigma}} & & \big\downarrow {\scriptstyle \sigma} \\ \mathfrak{t} & \xrightarrow{\phi} & \mathfrak{t}/W \end{array} \tag{1.5.15}$$

and recall that $\widetilde{\sigma}$ is C^{∞}-trivial.

Taking a step further in (1.5.7), we can glue all the maps in the simultaneous resolution diagram to a commutative diagram

$$\begin{array}{ccc} \widetilde{\boldsymbol{S}}|_{\Sigma^{\circ}} & \xrightarrow{\boldsymbol{\Psi}} & \boldsymbol{S}|_{\Sigma^{\circ}} \\ \big\downarrow {\scriptstyle \widetilde{\boldsymbol{\sigma}}|_{\Sigma^{\circ}}} & & \big\downarrow {\scriptstyle \boldsymbol{\sigma}|_{\Sigma^{\circ}}} \\ \widetilde{\boldsymbol{U}}|_{\Sigma^{\circ}} & \xrightarrow{\phi|_{\Sigma^{\circ}}} & \boldsymbol{U}|_{\Sigma^{\circ}} \end{array} \tag{1.5.16}$$

where $\widetilde{\boldsymbol{S}}|_{\Sigma^{\circ}} := \mathbf{Tot}(L|_{\Sigma^{\circ}} \times_{\mathbb{C}^*} \widetilde{S})$.

The map $\boldsymbol{\Psi}$ induces an inclusion of cohomologies

$$\Psi^* : \mathcal{H}^3((\mathcal{X}^{\circ})^{ur}/B^{ur}, \mathbb{C}) \to \mathcal{H}^3((\widetilde{\mathcal{X}}^{\circ})^{ur}/B^{ur}, \mathbb{C}) \tag{1.5.17}$$

so that we can lift λ°_{CY} to $\widetilde{\mathcal{X}}^\circ$. As both are induced from the tautological section on $\widetilde{\boldsymbol{U}}$, under the following isomorphism

$$\mathcal{H}^3((\widetilde{\mathcal{X}}^\circ)^{ur}/B^{ur}, \mathbb{C}) \cong \mathcal{H}^1(\widetilde{\Sigma}^\circ, \mathfrak{t})$$

the two abstract Seiberg-Witten differentials λ°_{CY} and λ_Δ coincide [Bec20, Theorem 5.2.1].

□

Remark 1.5.6. (Adjoint type) The above argument is easily applied to the adjoint case, $PGL(n, \mathbb{C})$, so that there is an isomorphism between (unordered) diagonally framed $PGL(n, \mathbb{C})$-Hitchin system and Calabi-Yau integrable system. On the Hitchin side, we consider dual Prym sheaf $\mathcal{K}^\vee$. The key is to construct the relevant family of semi-abelian varieties on the Calabi-Yau side as mentioned in Remark 1.4.6.

1.6 Appendix: Summary of Deligne's theory of 1-motives

In [Del74], Deligne gave a motivic description of variations of (polarized) $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1), (-1,0), (0,-1), (0,0)\}$. We recall the arguments in [Del74] and study the special case which is of main interest in this paper.

Definition 1.6.1. An 1-motive M over $\mathbb{C}$ consists of

1. X free abelian group of finite rank, a complex abelian variety A, and a complex affine torus T.

2. A complex semi-abelian variety G which is an extension of A by T.

3. A homomorphism $u : X \to G$.

We will denote a 1-motive by (X, A, T, G, u) or $M = [X \xrightarrow{u} G]$.

Proposition 1.6.2. *The category of (polarizable) mixed Hodge structures of type* $\{(-1,-1),(-1,0),(0,-1),(0,0)\}$ *is equivalent to the category of 1-motives.*

Proof. Given a 1-motive M, Deligne constructed a mixed Hodge structure $(T(M)_{\mathbb{Z}}, W, F)$ of type $\{(-1,-1),(-1,0),(0,-1),(0,0)\}$ as follows. Define a lattice $T(M)_{\mathbb{Z}}$ as the fiber product

$$\begin{array}{ccc} T(M)_{\mathbb{Z}} & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle u} \\ Lie(G) & \xrightarrow{\exp} & G \end{array} \tag{1.6.1}$$

The weight filtration on $T(M)_{\mathbb{Z}}$ is given by setting $W_{-1}T(M)_{\mathbb{Z}} := H_1(G,\mathbb{Z}) = \ker(\beta)$ and $W_{-2}T(M)_{\mathbb{Z}} = H_1(T,\mathbb{Z})$. Also, by linearly extending $\alpha : T(M)_{\mathbb{Z}} \to Lie(G)$ to $\mathbb{C}$, we define $F^0(T(M)_{\mathbb{Z}} \otimes \mathbb{C}) := \ker(\alpha_{\mathbb{C}})$. By construction $\mathrm{Gr}^W_{-1}(T(M)_{\mathbb{Z}}) = H_1(A,\mathbb{Z})$ with the usual Hodge filtration and is therefore polarizable.

Conversely, if $H := (H_{\mathbb{Z}}, W, F)$ is a mixed Hodge structure of the given type with $\mathrm{Gr}^W_{-1}(H_{\mathbb{Z}})$ polarizable, then one can construct a 1-motive by taking

1. $A := \mathrm{Gr}^W_{-1}(H_{\mathbb{C}})/(F^0\mathrm{Gr}^W_{-1}(H_{\mathbb{C}}) + \mathrm{Gr}^W_{-1}(H_{\mathbb{Z}}))$
2. $\mathsf{T} := \mathrm{Gr}^W_{-2}(H_{\mathbb{C}})/\mathrm{Gr}^W_{-2}(H_{\mathbb{Z}})$
3. $G := H_{\mathbb{C}}/(F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$
4. $X := \mathrm{Gr}^W_0(H_{\mathbb{Z}})$

□

In particular, if X is trivial, the 1-motive M is equivalent to a semi-abelian variety G. By Proposition 1.6.2, we have an equivalence between the abelian category of semi-abelian varieties and the abelian category of (polarizable) $\mathbb{Z}$-mixed Hodge structures of type $\{(-1,-1),(-1,0),(0,-1)\}$.

Example 1.6.3. A typical example coming from geometry is the mixed Hodge structure on the first homology group of a punctured curve. Let C be a Riemann surface and $D \subset C$ be a reduced divisor. The first homology group $H_{\mathbb{Z}} = H_1(C \setminus D, \mathbb{Z})$ carries a $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,-1),(-1,0),(0,-1)\}$ where $\mathrm{Gr}^W_{-1}(H_{\mathbb{C}}) = H_1(C,\mathbb{Z}) \otimes \mathbb{C}$. Moreover, it admits a degenerate intersection pairing $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ whose kernel is $W_{-2}H_{\mathbb{C}} \cap H_{\mathbb{Z}}$. Note that it induces a polarization on $\mathrm{Gr}^W_{-1}(H_{\mathbb{C}})$ and so gives rise to the type of object in proposition 1.6.2. In other words, we get a semi-abelian variety G by taking the Jacobian of $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$ as follows

$$G := J(H) = H_{\mathbb{C}}/(F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

$$A := J_{\mathrm{cpt}}(H) = \mathrm{Gr}^W_{-1} H_{\mathbb{C}}/(Gr^W_{-1} F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

$$\mathsf{T} := W_{-2}H_{\mathbb{C}}/W_{-2}H_{\mathbb{Z}}$$

We call such integral mixed Hodge structure a *semi-polarized $\mathbb{Z}$-mixed Hodge structure*. Moreover, consider the dual mixed Hodge structure $H^\vee$ which is of type $\{(0,1),(1,0),(1,1)\}$. Geometrically it corresponds to the first cohomology $H^1(C \setminus D)$ of the punctured Riemann surface $C \setminus D$. The associated Jacobian $J(H^\vee) = H^\vee_{\mathbb{C}}/(F^1 H^\vee_{\mathbb{C}} + H_{\mathbb{Z}})$ is no longer a semi-abelian variety, but just a complex torus.

Chapter 2

Conic bundles, modules over even Clifford algebra, and special subvarieties of Prym varieties

2.1 Introduction

It is well-known that a smooth cubic threefold is irrational since the famous work of Clemens and Griffiths. They observed that if a threefold is rational, then its intermediate Jacobian must be isomorphic to a product of Jacobians of curves. The problem is then reduced to comparing the intermediate Jacobians of cubic threefolds with the Jacobians of curves as principally polarized abelian varieties by studying the singularity loci of their theta divisors.

Let us consider a cubic hypersurface $Y_n \subset \mathbb{P}^{n+1}$ of dimension n for a moment. The bounded derived category $\mathrm{D}^b(Y_n)$ of Y_n admits a semiorthogonal decomposition:

$$\mathrm{D}^b(Y_n) \cong \langle \mathcal{K}u(Y_n), \mathcal{O}_{Y_n}, \mathcal{O}_{Y_n}(1), ..., \mathcal{O}_{Y_n}(n-2) \rangle \tag{2.1.1}$$

where $\mathcal{K}u(Y_n) := \langle \mathcal{O}_{Y_n}, \mathcal{O}_{Y_n}(1), ..., \mathcal{O}_{Y_n}(n-2) \rangle^{\perp}$ is now known as the Kuznetsov component. In dimension four, it is conjectured by Kuznetsov that a smooth cubic fourfold is rational if and only if the Kuznetsov component $\mathcal{K}u(Y_4)$ is equivalent to the category of a K3 surface. While the conjecture has been checked to hold in some cases, the general conjecture remains unsolved.

Since $\mathcal{K}u(Y_n)$ is expected to capture the geometry of Y_n, an attempt to extract information out of the triangulated category $\mathcal{K}u(Y_n)$ is to construct Bridgeland stability conditions on $\mathcal{K}u(Y_n)$ and consider its moduli spaces of stable objects. In dimension three, it can be shown that $\mathcal{K}u(Y_3)$ reconstructs the Fano surface of lines of Y_3 as a moduli space of stable objects with suitable stability conditions. The reconstruction of Fano surface of lines then determines the intermediate Jacobian $J(Y_3)$ [Ber+12]. Alternatively, it is observed that instanton bundles are objects in $\mathcal{K}u(Y_3)$. Then by the work of Markushevich-Tikhomirov and others [MT98][IM00][Dru00][Bea02], it is shown that the moduli space of instanton bundles on a cubic threefold is birational to the intermediate Jacobian $J(Y_3)$. So $\mathcal{K}u(Y_n)$ can be thought of as the categorical counterpart of the intermediate Jacobian whose success in the rationality problem of cubic threefolds fits well into the philosophy of Kuznetsov's conjecture in $n = 4$.

A cubic hypersurface Y_n is defined by a homogeneous degree 3 polynomial in $n+2$ variables $x_0, ..., x_{n+1}$. Suppose Y_n contains the line defined by $x_2 = ... = x_{n+1} = 0$, then we can write the polynomial as

$$t_1x_0^2 + 2t_2x_0x_1 + t_3x_1^2 + 2q_1x_0 + 2q_2x_1 + f = 0$$

where t_i, q_j, f are polynomials in variables $x_2, ..., x_{n+1}$ of degree $1, 2, 3$ respectively. This means that we can think of it as a family of quadratic polynomials. More geometrically, let $l_0 \subset Y_n$ be a line that is not contained in a plane in Y_n. Then the blow-up $\widetilde{Y}_n := Bl_{l_0}(Y_n) \subset Bl_{l_0}(\mathbb{P}^{n+1})$ of Y_n along l_0 projects to a projective space $\mathbb{P}^{n-1}$, denoted by $p : \widetilde{Y}_n \to \mathbb{P}^{n-1}$. The map p is a conic bundle whose discriminant locus Δ_n is a degree 5 hypersurface. The idea of realizing a cubic hypersuface birationally as a conic bundle can be used to study its rationality. Following the idea of Mumford, it is shown that the intermediate Jacobian $J(Y_3) \cong J(\widetilde{Y}_3)$ is isomorphic to $\mathrm{Prym}(\widetilde{\Delta}_3, \Delta_3)$ where $\widetilde{\Delta}_3$ is the double cover parametrizing the irreducible components of the degenerate conics over Δ_3. By analyzing the difference between Prym varieties and the Jacobian of curves as principally polarized abelian varieties, it is again shown that a smooth cubic threefold is irrational.

The conic bundle structure of a cubic hypersurface also provides us information at the level of derived category. A quadratic form on a vector space defines the Clifford algebra which decomposes into the even and odd parts. We can apply the construction of Clifford algebra relatively for the conic bundles $\widetilde{Y}_n$ which is viewed as a family of conics over $\mathbb{P}^{n-1}$, and obtain a sheaf of even Clifford algebras $\mathcal{B}_0$ on $\mathbb{P}^{n-1}$. The bounded derived category $\mathrm{D}^b(\mathbb{P}^2, \mathcal{B}_0)$ of $\mathcal{B}_0$-modules appears as a component of the semiorthogonal decomposition of the conic bundle $\widetilde{Y}_n$:

$$\mathrm{D}^b(\widetilde{Y}_n) = \langle \mathrm{D}^b(\mathbb{P}^{n-1}, \mathcal{B}_0), p^*\mathrm{D}^b(\mathbb{P}^{n-1}) \rangle. \tag{2.1.2}$$

In the case $n = 3, 4$, by comparing the semiorthogonal decompositions in (2.1.1), (2.1.2) and the one for blowing up, one can show that there are embedding functors

$$\Xi_n : \mathcal{K}u(Y_n) \hookrightarrow \mathrm{D}^b(\mathbb{P}^{n-1}, \mathcal{B}_0) \tag{2.1.3}$$

for $n = 3, 4$ (see [Ber+12], [Bay+21]). The functors Ξ_n is useful in the study of $\mathcal{K}u(Y_n)$. For example, when $n = 3, 4$, the construction of Bridgeland stability conditions carried out in [Ber+12][Bay+21] uses the embedding functors Ξ_n as one of the key steps. Also, in the work of Lahoz-Macrì-Stellari [LMS15], the functor Ξ_3 is used to provide a birational map between the moduli space of instanton bundles and the moduli space of $\mathcal{B}_0$-modules.

Motivated by the relations found in the case of cubic threefold as described above: Prym/intermediate Jacobian, intermediate Jacobian/moduli space of instanton bundles, and instanton bundles/$\mathcal{B}_0$-modules, it is natural to search for a relation between $\mathcal{B}_0$-modules and the Prym varieties. In this paper, we will focus on three dimensional conic bundles (not necessarily obtained from a cubic threefold) and study the relation between the moduli spaces of $\mathcal{B}_0$-modules and the Prym varieties. Let $p : X \to \mathbb{P}^2$ be a three dimensional conic bundle over $\mathbb{P}^2$, Δ the discriminant curve on $\mathbb{P}^2$ and $\pi : \widetilde{\Delta} \to \Delta$ the double cover parametrizing the irreducible components of degenerate conics over Δ. We consider the moduli space $\mathfrak{M}_{d,e}$ of semistable $\mathcal{B}_0$-modules with fixed Chern character $(0, 2d, e)$, which means that the $\mathcal{B}_0$-modules are supported on plane curves. By taking the Fitting support, we get a morphism $\Upsilon : \mathfrak{M}_{d,e} \to |\mathcal{O}_{\mathbb{P}^2}(d)|$.

On the other hand, by the work of Welters [Wel81] and Beauville [Bea82], each linear system on Δ defines the special subvarieties in the Prym variety $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$ of the étale double cover $\pi : \widetilde{\Delta} \to \Delta$. We apply the construction to the linear system $|L_d| = |\mathcal{O}_{\mathbb{P}^2}(d)|_\Delta|$. For each k, there is an induced morphism $\pi^{(k)} : \widetilde{\Delta}^{(k)} \to \Delta^{(k)}$. As the linear system $|L_d|$ can be considered as a subvariety in $\Delta^{(k)}$ for $k = \deg(\Delta) \cdot d$, we define the variety of divisors lying over $|L_d|$ as $W_d = (\pi^{(k)})^{-1}(|L_d|)$. Then the image of W_d under the Abel-Jacobi map $\widetilde{\alpha} : \widetilde{\Delta}^{(k)} \to J^k\widetilde{\Delta}$ lies in the two components of (the translate of) Prym varieties, which are the special subvarieties. The variety W_d consists of two irreducible components $W_d = W_d^0 \cup W_d^1$, each of which maps to $|L_d|$. For $d < \deg(\Delta)$, we have $|\mathcal{O}_{\mathbb{P}^2}(d)| \cong |L_d|$, and we denote by $U_d \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ the open subset of smooth degree d curves intersecting Δ transversally. The main construction in this paper is to construct a morphism

$$\Phi : \mathfrak{M}_{d,e}|_{U_d} \to W_d^i|_{U_d} \tag{2.1.4}$$

for $d = 1, 2$, and show that it is an isomorphism. Moreover, we have the following:

Theorem 2.1.1. *For $d = 1, 2$, the moduli space $\mathfrak{M}_{d,e}$ is birational to one of the two components W_d^i of W_d. Moreover, if $\mathfrak{M}_{d,e}$ is birational to W_d^i, then $\mathfrak{M}_{d,e+1}$ is birational to W_d^{1-i}. In particular, the birational type of $\mathfrak{M}_{d,e}$ only depends on d and $(e \bmod 2)$.*

By composing with the Abel-Jacobi map $\widetilde{\alpha} : \widetilde{\Delta}^{(k)} \to J^k\widetilde{\Delta}$, we obtain a rational map

$$\widetilde{\alpha} \circ \Phi : \mathfrak{M}_{d,e} \dashrightarrow \mathrm{Prym}(\widetilde{\Delta}, \Delta) \tag{2.1.5}$$

whose image is an open subset of the special subvarieties.

Next, we apply the result above to the case of cubic threefolds. In [Kuz12] and [LMS15], it is observed that instanton bundles are objects in $\mathcal{K}u(Y_3)$. The authors use the functor $\Xi : \mathcal{K}u(Y_3) \hookrightarrow \mathrm{D}^b(\mathbb{P}^2, \mathcal{B}_0)$ to deduce a birational map between the moduli space $\mathfrak{M}_{Y_3}$ of instanton bundles on Y_3 and the moduli space $\mathfrak{M}_{2,-4}$ of $\mathcal{B}_0$-modules. In this case, the rational map $\mathfrak{M}_{2,-4} \dashrightarrow \mathrm{Prym}(\widetilde{\Delta}, \Delta)$ actually turns out to be birational. Hence, by composing the birational maps, we get

$$\mathfrak{M}_{Y_3} \dashrightarrow \mathrm{Prym}(\widetilde{\Delta}, \Delta) \tag{2.1.6}$$

As a point in $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$ can be interpreted as a ξ-twisted Higgs bundle on Δ by the spectral correspondence [BNR89], the birational map (2.1.6) gives an explicit correspondence between instanton bundles on Y_3 and ξ-twisted Higgs bundles on Δ.

Moreover, as mentioned above, the moduli space of instanton bundles is birational to the intermediate Jacobian $J(Y_3)$, so the birational map here gives a modular interpretation of the classical isomorphism $J(Y_3) \cong \mathrm{Prym}(\widetilde{\Delta}, \Delta)$ in terms of instanton bundles, $\mathcal{B}_0$-modules and Higgs bundles. From this viewpoint, we can think of the classical isomorphism $J(Y_3) \cong \mathrm{Prym}(\widetilde{\Delta}, \Delta)$ as a consequence of the embedding functor $\Xi_3 : \mathcal{K}u(Y_3) \hookrightarrow \mathrm{D}^b(\mathbb{P}^2, \mathcal{B}_0)$.

Philosophically, the result allows us to think of $D^b(\mathbb{P}^2, \mathcal{B}_0)$ as the categorical counterpart of $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$ associated to a conic bundle, just as $\mathcal{K}u(Y_3)$ is the categorical counterpart of $J(Y_3)$.

2.1.1 Convention

Throughout this paper we work over the complex numbers $\mathbb{C}$. All modules in this paper are assumed to be left modules. For a morphism $f : X \to Y$ of two spaces (schemes or stacks) and a subspace $Z \subset Y$, we will denote by $X|_Z := X \times_Y Z$ the fiber product and $f|_Z : X|_Z \to Z$.

2.2 Special subvarieties in Prym varieties

In this section, we recall the special subvariety construction of Prym varieties following the work of Welters [Wel81] and Beauville [Bea82]. Let $\pi : \widetilde{\Sigma} \to \Sigma$ be an étale double cover of two smooth curves. Then we denote by $Nm : J\widetilde{\Sigma} \to J\Sigma$ the norm map on the Jacobians of curves and also its translation $Nm : J^d\widetilde{\Sigma} \to J^d\Sigma$ by abusing notation.

Suppose g^r_d is a linear system of degree d and (projective) dimension r. Consider the Abel-Jacobi maps:

$$\widetilde{\alpha} : \widetilde{\Sigma}^{(d)} \to J^d\widetilde{\Sigma}, \quad \widetilde{x}_1 + ...\widetilde{x}_d \mapsto \mathcal{O}(\widetilde{x}_1 + ...\widetilde{x}_d)$$

$$\alpha : \Sigma^{(d)} \to J^d\Sigma, \quad x_1 + ... + x_d \mapsto \mathcal{O}(x_1 + ... + x_d)$$

they fit in the following commutative diagram

$$\begin{array}{ccc} \widetilde{\Sigma}^{(d)} & \xrightarrow{\widetilde{\alpha}} & J^d\widetilde{\Sigma} \\ \Big\downarrow{\scriptstyle \pi^{(d)}} & & \Big\downarrow{\scriptstyle Nm} \\ \Sigma^{(d)} & \xrightarrow{\alpha} & J^d\Sigma \end{array} \tag{2.2.1}$$

We assume that the linear system g^r_d contains a reduced divisor, so that g^r_d is not contained in the branch locus of $\pi^{(d)}$ by Bertini's theorem.

Now the linear system $g^r_d \cong \mathbb{P}^r$ is naturally a subvariety of $\Sigma^{(d)}$, we define $W = (\pi^{(d)})^{-1}(g^r_d)$ as the preimage of g^r_d in $\widetilde{\Sigma}^{(d)}$. The image of W under $\widetilde{\alpha}$ is denoted by $V = \widetilde{\alpha}(W)$. Recall that the kernel $\ker(Nm)$ consists of two disjoint components, each of which is isomorphic to the Prym variety, and we denote them by Pr^0 and Pr^1. By construction, we have that $V \subset \ker(Nm)$, so V also has two disjoint components $V^i \subset \mathrm{Pr}^i$ where $i = 0, 1$. Hence, W also breaks into a disjoint union of two subvarieties W^0 and W^1 such that $\widetilde{\alpha}(W^i) = V^i$. Welters [Wel81] called W the variety of divisors on $\widetilde{\Sigma}$ lying over g^r_d and the two irreducible components W^0 and W^1 the *halves* of the variety of divisors W. The subvarieties V^i are called the *special subvarieties* of Pr^i associated to the linear system g^r_d.

Remark 2.2.1. By [Mum71, Lemma 1], a line bundle $L \in \ker(Nm)$ can always be written as $L \cong M \otimes \sigma^*(M^\vee)$ such that if $\deg(M) \equiv 0$ (resp. 1) mod 2, then $L \in \mathrm{Pr}^0$ (resp. Pr^1). It follows that if $L \in \mathrm{Pr}^i$, then $L \otimes \mathcal{O}(x - \sigma(x)) \in \mathrm{Pr}^{1-i}$ where $x \in \Sigma$. This implies that if $x_1+...+x_d \in W^i$, then the divisor $\sigma(x_1)+...+x_d = (\sigma(x_1)-x_1)+(x_1+...+x_d)$ is contained in

W^{1-i}. In particular, we see that if we switch an even number of points x_i in $x_1 + ... + x_d \in W^i$, then the resulting divisor lies in the same component, i.e. $\sum_{i\in I} \sigma(x_i) + \sum_{j\notin I} x_j \in W^{1-i}$ if $x_1 + ... + x_d \in W^i$ and I has even cardinality.

Let $\overline{\Sigma} := \mathbb{Z}/2\mathbb{Z} \times \Sigma$ be the constant group schemes over Σ. The trivial double cover $p : \overline{\Sigma} \to \Sigma$ also induces a morphism on its d-th symmetric products $\overline{\Sigma}^{(d)} \to \Sigma^{(d)}$. Let $U \subset \Sigma^{(d)}$ be the open subset of reduced effective divisors.

Proposition 2.2.2. *The scheme $G' := \overline{\Sigma}^{(d)}|_U$ is a group scheme over U.*

Proof. Note that the map $G' \to U$ is étale. The multiplication map $m : \overline{\Sigma} \times_\Sigma \overline{\Sigma} \to \overline{\Sigma}$ induces the map $m^{(d)} : (\overline{\Sigma} \times_\Sigma \overline{\Sigma})^{(d)} \to \overline{\Sigma}^{(d)}$. On the other hand, the natural projections $pr_j : \overline{\Sigma} \times_\Sigma \overline{\Sigma} \to \overline{\Sigma}$ induces the maps $pr_j^{(d)} : (\overline{\Sigma} \times_\Sigma \overline{\Sigma})^{(d)} \to \overline{\Sigma}^{(d)}$ and so $r : (p^{(d)} \circ pr_1^{(d)})^{-1}(U) \to G' \times_U G'$ by universal property. It is easy to see that r is bijective on closed points. As G' and U are smooth, so $G' \times_U G'$ is also smooth and hence normal. Therefore, r is an isomorphism. Then we define the multiplication map on G' to be

$$G' \times_U G' \xrightarrow{r^{-1}} (p^{(d)} \circ pr_1^{(d)})^{-1}(U) \xrightarrow{m^{(d)}|_U} G'$$

The trivial double cover $\overline{\Sigma} \to \Sigma$ always has a section $\Sigma \to \overline{\Sigma}$ mapping to $q^{-1}(0)$ where $q : \overline{\Sigma} \to \mathbb{Z}/2\mathbb{Z}$ is the projection map. The identity map is defined as the restriction of $\Sigma^{(d)} \to \overline{\Sigma}^{(d)}$ to U, i.e. $e : U \to G'$.

The inverse map is simply the identity map $\iota : G' \to G'$. □

The map $q : \overline{\Sigma} \to \mathbb{Z}/2\mathbb{Z}$ induces $G' \to \overline{\Sigma}^{(d)} \xrightarrow{q^{(d)}} (\mathbb{Z}/2\mathbb{Z})^{(d)}$ and there is the summation map $\mathbb{Z}/2\mathbb{Z}^{(d)} \to \mathbb{Z}/2\mathbb{Z}$, and we denote by $s : G' \to \mathbb{Z}/2\mathbb{Z}$ the composition of the two maps. Then we define the preimage $G := s^{-1}(0)$.

Corollary 2.2.3. *The scheme G is a group scheme over U.*

We can denote a closed point of G as $\sum(\lambda_i, x_i)$ such that $\sum \lambda_i = 0$ in $\mathbb{Z}/2\mathbb{Z}$ where $\lambda_i \in \mathbb{Z}/2\mathbb{Z}$ and $x_i \in \Sigma$. In other words, G is the group U-scheme of even cardinality subsets of reduced divisors in Σ.

Proposition 2.2.4. *Let g^r_d be a linear system and consider the half $W^i \subset \widetilde{\Sigma}^{(d)}$ of the variety of divisors W lying over g^r_d. If we denote by $U_0 := U \cap g^r_d$ and $G_0 := G|_{U_0}$, then $W^i|_{U_0}$ is a pseudo G_0-torsor on U_0 i.e. the induced morphism $G_0 \times_{U_0} W^i|_{U_0} \to W^i|_{U_0} \times_{U_0} W^i|_{U_0}$ is an isomorphism.*

Proof. To simplify notation we denote $\widetilde{U} := \widetilde{\Sigma}^{(d)}|_U$. The construction is similar to the multiplication map defined in Proposition 2.2.2. We first define a group action $G' \times_U \widetilde{U} \to \widetilde{U}$. The involution action $\sigma : \overline{\Sigma} \times_\Sigma \widetilde{\Sigma} \to \widetilde{\Sigma}$ induces the map $\sigma^{(d)} : (\overline{\Sigma} \times_\Sigma \widetilde{\Sigma})^{(d)} \to \widetilde{\Sigma}^{(d)}$. The natural projections $pr_1 : \overline{\Sigma} \times_\Sigma \widetilde{\Sigma} \to \overline{\Sigma}$ and $pr_2 : \overline{\Sigma} \times_\Sigma \widetilde{\Sigma} \to \widetilde{\Sigma}$ induce the maps $pr_j^{(d)}$: $(\overline{\Sigma} \times_\Sigma \widetilde{\Sigma})^{(d)} \to \widetilde{\Sigma}^{(d)}$. Then we get by universal property $t : (p^{(d)} \circ pr_1^{(d)})^{-1}(U) \to G' \times_U \widetilde{U}$. Again, we can easily check that the map t is bijective on closed points and $G' \times_U \widetilde{U}$ is smooth and hence normal, the map t is an isomorphism. We define the group action as

$$G' \times_U \widetilde{U} \xrightarrow{t^{-1}} (p^{(d)} \circ pr_1^{(d)})^{-1}(U) \xrightarrow{\sigma^{(d)}|_U} \widetilde{U}$$

This defines another group action by restricting to $G \subset G'$. Finally, by Remark 2.2.1, we see that the restriction of the group action by G to G_0 defines a group action

$$G_0 \times_{U_0} W^i|_{U_0} \to W^i|_{U_0}$$

that is simply transitive on closed points, where $\widetilde{U}_0 := \widetilde{\Sigma}^{(d)}|_{U_0}$. Then it follows by the normality of $W^i|_{U_0} \times_{U_0} W^i|_{U_0}$ that the induced morphism $G_0 \times_{U_0} W^i|_{U_0} \to W^i|_{U_0} \times_{U_0} W^i|_{U_0}$ is an isomorphism. $\square$

Example 2.2.5. Consider the linear system $|K_\Sigma|$ i.e. the linear system of canonical divisors. In this case, we have $d = 2g-2$ and $r = g$. Observe that $\dim(W) = \dim(\mathrm{Pr})$ and the fiber of the morphisms $W^i \to \mathrm{Pr}^i$ at a point $[D] \in \mathrm{Pr}^i$ is $|D|$. It can be shown that [Mum71] [Mum74][Bea02]:

1. $\widetilde{\alpha}|_{W^1} : W^1 \to \mathrm{Pr}^1$ is birational.

2. $\widetilde{\alpha}|_{W^0} : W^0 \to \mathrm{Pr}^0$ maps onto a divisor $\Theta \subset \mathrm{Pr}^0$ and is generically a $\mathbb{P}^1$-bundle.

2.3 Modules over sheaf of even Clifford algebra

2.3.1 Sheaf of even Clifford algebra

Let $\pi : Q \to S$ be a conic bundle over a scheme S with simple degenerations, i.e. the fibers of degenerate conics have corank 1, which will be assumed throughout the paper. There is a rank 3 vector bundle F on S, together with an embedding of a line bundle $q : L \to S^2F^\vee$ which is also thought of as a section in $S^2F^\vee \otimes L^\vee$. Then Q is embedded in $\mathbb{P}(F) = \mathrm{Proj}(S^2F^\vee)$ as the zero locus of $q \in H^0(S, S^2F^\vee \otimes L^\vee) = H^0(\mathbb{P}(F), \mathcal{O}_{\mathbb{P}(F)/S}(2) \otimes (\pi')^*L^\vee)$ where we denote $\pi' : \mathbb{P}(F) \to S$.

We define the sheaf of even Clifford algebra by following the approach of [ABB14][1]. Consider the two ideals J_1 and J_2 of the tensor algebra which are generated by

$$v \otimes v \otimes f - \langle q(v,v), f\rangle, \quad u \otimes v \otimes f \otimes v \otimes w \otimes g - \langle q(v,v), f\rangle u \otimes w \otimes g \tag{2.3.1}$$

respectively, where the sections $u, v, w \in F$ and $f, g \in L$. Then the even Clifford algebra is

[1] Note that we write a line bundle-valued quadratic form as $\sigma : L \to S^2E^\vee$ where the authors in [ABB14] write it as $L^\vee \to S^2E^\vee$

defined as the quotient algebra

$$\mathcal{B}_0 := T^{\bullet}(F \otimes F \otimes L)/(J_1 + J_2). \tag{2.3.2}$$

The sheaf of algebra has naturally a filtration

$$\mathcal{O}_X = F_0 \subset F_1 = \mathcal{B}_0 \tag{2.3.3}$$

obtained as the image of the truncation of the tensor algebra $T^{\leq i}(F \otimes F \otimes L)$ in $\mathcal{B}_0$. Moreover, the associated graded piece $F_1/F_0 \cong \wedge^2 F \otimes L$. As an $\mathcal{O}_S$-module, we actually have $\mathcal{B}_0 \cong \mathcal{O}_S \oplus (\wedge^2 F \otimes L)$ which can be seen by defining the splitting $\wedge^2 F \otimes L \to F \otimes F \otimes L \to T^{\bullet}(F \otimes F \otimes L)/(J_1 + J_2)$ where $\wedge^2 F$ is thought of as a subbundle of antisymmetric 2-tensors of $F \otimes F$.

2.3.2 Root stacks

The main objects in this paper are $\mathcal{B}_0$-modules, i.e. modules over the sheaf of even Clifford algebra $\mathcal{B}_0$. In order to study the category of $\mathcal{B}_0$-modules, it is easier to work with a root stack cover of S. The advantage is that the category of $\mathcal{B}_0$-modules is equivalent to the category of modules over a sheaf of Azumaya algebras on the root stack. For details on root stacks, we refer the reader to [Cad07].

Let $\mathcal{L}$ be a line bundle on a scheme X and $s \in H^0(X, \mathcal{L})$ and r a positive integer. The pair $(\mathcal{L}, s)$ defines a morphism $X \to [\mathbb{A}^1/\mathbb{G}_m]$, and the r-th power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$ induces a morphism $\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$. Following $[Cad07]$, we define the r-th root stack $X_{\mathcal{L},s,r}$ as the fiber product

$$X \times_{[\mathbb{A}^1/\mathbb{G}_m],\theta_r} \left[\mathbb{A}^1/\mathbb{G}_m\right].$$

The r-th root stack $X_{\mathcal{L},s,r}$ is a Deligne-Mumford stack. Locally on X, when $\mathcal{L}$ is trivial, $X_{\mathcal{L},s,r}$ is just the quotient stack $[\operatorname{Spec}(\mathcal{O}_X[t]/(t^r - s))/\mu_r]$ where μ_r is the group of r-th roots of unity acting on t by scalar action. The root stack $X_{\mathcal{L},s,r}$ has X as its coarse moduli space. There is a tautological sheaf $\mathcal{T}$ on $X_{\mathcal{L},s,r}$ satisfying $\mathcal{T}^r \cong \psi^*\mathcal{L}$ where $\psi : X_{\mathcal{L},s,r} \to X$ is the projection. Then every line bundle on $X_{\mathcal{L},s,r}$ is isomorphic to $\psi^*G \otimes \mathcal{T}^k$ where k is unique and G is unique up to isomorphism. For our purposes, we will mainly consider the case $M = \mathcal{O}_X(D)$ for an effective Cartier divisor D and $s = s_D$ is the section vanishing at D. In this case, we will simply write $X_{\mathcal{O}_X(D),s_D,r} = X_{D,r}$ and the tautological sheaf $\mathcal{T}$ as $\mathcal{O}(\frac{D}{r})$.

Similarly, it is pointed out in [Cad07] that there is an equivalence of categories between the category of morphisms $X \to [\mathbb{A}^n/\mathbb{G}_m^n]$ and the category whose objects are n-tuples $(\mathcal{L}_i, s_i)_{i=1}^n$, where $\mathcal{L}_i$ is a line bundle on X and $s_i \in H^0(X, \mathcal{L}_i)$ and morphisms $(\mathcal{L}_i, s_i)_{i=1}^n \to (\mathcal{L}_i', s_i')_{i=1}^n$ are n-tuples $(\varphi_i)_{i=1}^n$ where $\varphi_i(s_i) = t_i$. If we let $\mathbb{D} := (D_1, ..., D_n)$ be an n-tuple of effective Cartier divisors and $\vec{r} = (r_1, ..., r_n)$, then the n-tuples $(\mathcal{O}_X(D_i), s_{D_i})_{i=1}^n$ will determine a morphism $X \to [\mathbb{A}^n/\mathbb{G}_m^n]$. Also, the morphisms on $\mathbb{A}^n$ and $\mathbb{G}_m^n$ sending $(x_1, ..., x_n) \mapsto (x_1^{r_1}, ..., x_n^{r_n})$ induces a morphism $\theta_{\vec{r}} : [\mathbb{A}^n/\mathbb{G}_m] \to [\mathbb{A}^n/\mathbb{G}_m]$. We define $X_{\mathbb{D},\vec{r}}$ as the fiber product

$$X \times_{[\mathbb{A}^n/\mathbb{G}_m^n],\theta_{\vec{r}}} [\mathbb{A}^n/\mathbb{G}_m^n] \,.$$

This can be interpreted as iterating the r-th root stack construction for $n = 1$. There are the tautological sheaves $\mathcal{O}\left(\frac{D_i}{r_i}\right)$ on $X_{\mathbb{D},\vec{r}}$ satisfying $\mathcal{O}\left(\frac{D_i}{r_i}\right)^{r_i} \cong \psi^*\mathcal{O}_X(D_i)$. Every line bundle F on $X_{\mathbb{D},\vec{r}}$ can be written as

$$F \cong \psi^*G \otimes \prod_{i=1}^{r} \mathcal{O}\left(\frac{D_i}{r_i}\right)^{k_i}$$

where $0 \le k_i \le r_i$ are unique and G is unique up to isomorphis and $\psi : X_{\mathbb{D},\vec{r}} \to X$ is the projection.

Lemma 2.3.1. *Let $D = D_1 + ... + D_n$ where D_i are pairwise disjoint effective Cartier divisors. If $r = r_1 = ... = r_n$, we have*

$$X_{\mathbb{D},\vec{r}} \xrightarrow{\sim} X_{D,r}.$$

Proof. An object of $X_{D,r}$ over scheme T consists of the quadruples (f, N, t, φ) where $f : T \to X$ is a morphism, N a line bundle, $t \in H^0(T, N)$ and $\varphi : N^r \xrightarrow{\sim} f^*\mathcal{O}(D)$ is an isomorphism.

On the other hand, an object of $X_{\mathbb{D},\vec{r}}$ consists of $(f, (N_i)_{i=1}^n, (s_i)_{i=1}^n, (\varphi_i)_{i=1}^n)$ where $f : T \to X$ a morphism, N_i is a line bundle, $s_i \in H^0(T, N_i)$ and $\varphi_i : N_i^{r_i} \xrightarrow{\sim} f_i^*\mathcal{O}(D_i)$ is an isomorphism. We see that there is a natural morphism $\alpha : X_{\mathbb{D},\vec{r}} \to X_{D,r}$ over X sending

$$(f, (N_i)_{i=1}^n, (s_i)_{i=1}^n, (\varphi_i)_{i=1}^n) \mapsto (f, \bigotimes_{i=1}^n N_i, \bigotimes_{i=1}^n s_i, \bigotimes_{i=1}^n \varphi_i). \tag{2.3.4}$$

To see that this is an isomorphism, we restrict to each open neighborhood U_i of D_i away from D_j $(j \neq i)$ such that $\mathcal{O}(D_j)|_{U_i} \xrightarrow{\sim} \mathcal{O}$ for $j \neq i$ and $\mathcal{O}(D_i)|_{U_i} \xrightarrow{\sim} \mathcal{O}(D_1 + .. + D_n)|_{U_i}$. Then it is clear that the functor (2.3.4) over U_i is essentially surjective i.e. the image of the quadruples $(f, (N_i)_{i=1}^n, (s_i)_{i=1}^n, (\varphi_i)_{i=1}^n)$ where $N_j \cong \mathcal{O}$ and $s_j = 1$ for $j \neq i$ is dense.

□

Example 2.3.2. Let $X = \mathrm{Spec}(R)$, $L = \mathcal{O}_X$ and s be a section of $\mathcal{O}_X$. Then $X_{L,s,r} \cong [\mathrm{Spec}R'/\mu_r]$, where $R' = R[t]/(t^r - s)$, and $\gamma \cdot t = \gamma^{-1}t$ and $\gamma \cdot a = a$ for $a \in R$ and $\gamma \in \mu_r$. A quasi-coherent sheaf on $[\mathrm{Spec}R'/\mu_r]$ is a R'-module M with a μ_r-action on M such that for $\gamma \in \mu_r, b \in R', m \in M$, we have

$$\gamma \cdot (b \cdot m) = (\gamma \cdot b) \cdot (\gamma \cdot m).$$

As μ_r is diagonalizable, there is a $\mathbb{Z}/r\mathbb{Z}$-grading $M \cong M_0 \oplus ... \oplus M_{r-1}$ where $\gamma \cdot m_i = \gamma^i m_i$ for $m_i \in M_i$. Note that the components are indexed by the group of characters of μ_r, which is $\mathbb{Z}/r\mathbb{Z}$. Similarly, $R' \cong R'_0 \oplus ... \oplus R'_{r-1}$ where $R'_0 = R$. In particular, we see that $\gamma : M \to M$ is an R-module homomorphism, and so each M_i is an R-module.

Example 2.3.3. When there exists a line bundle N such that $f : N^{\otimes r} \cong \mathcal{L}$, we can take the cyclic cover for section s, defined as

$$\phi : \widetilde{X} := \mathrm{Spec}\,(\mathcal{A}_X) \to X, \qquad \mathcal{A}_X := \mathcal{O}_X \oplus N^\vee \oplus ... \oplus (N^\vee)^{r-1}$$

where the algebra structure of $\mathcal{A}_X$ is given by the map $(N^\vee)^{\otimes r} \xrightarrow{f^\vee} (\mathcal{L})^\vee \xrightarrow{s^\vee} \mathcal{O}$.

By [Bor07, Théorème 3.4], we know that

$$\left[\widetilde{X}/\mu_r\right] \cong X_{\mathcal{L},s,r}.$$

Suppose X is a smooth curve and $D = p_1 + ... + p_k$ is a reduced divisor and $r = 2$. The cyclic cover $\phi : \widetilde{X} \to X$ is branched at p_i, we denote by w_i the ramification points such that $\phi(w_i) = p_i$. The points w_i are also the fixed points under the involution of $\widetilde{X}$.

Since the root stack $X_{D,2}$ is the quotient stack $\left[\widetilde{X}/\mu_2\right]$, a line bundle on $\left[\widetilde{X}/\mu_2\right]$ is the same as a μ_2-equivariant line bundle on $\widetilde{X}$. On $\mathcal{O}_{\widetilde{X}}(w_i)$, there is a group action on $\mathrm{Tot}(\mathcal{O}(w_i))$ which fixes the canonical section vanishing at w_i, we will denote by $L(w_i)$ the line bundle $\mathcal{O}(w_i)$ together with this μ_2-equivariant sheaf structure. In particular, the induced μ_2-action on the fiber of $\mathcal{O}(w_i)$ is $-Id$. The pull back of a line bundle F on X to $\widetilde{X}$ is equipped with a natural μ_2-equivariant sheaf structure, whose induced action on the fiber at w_i is Id and the μ_2-equivariant bundle is again denoted by ϕ^*F. Since $\phi^*N \cong \mathcal{O}\left(\sum_i w_i\right)$, we can write

$$\mathcal{O}(w_i) \cong \mathcal{O}\left(2w_i + \sum_{i \neq j} w_j\right) \otimes \phi^* N^\vee \cong \mathcal{O}\left(\sum_{i \neq j} w_j\right) \otimes \phi^*(N^\vee \otimes \mathcal{O}(p_i)).$$

So we see that $L\left(\sum_{i \neq j} w_j\right) \otimes \phi^*(N^\vee \otimes \mathcal{O}(p_i))$ has the same underlying line bundle as $L(w_i)$.

As discussed above, every line bundle on $X_{D,2}$ is of the form $\psi^*F \otimes \mathcal{O}\left(\sum_{i\in I} \frac{p_i}{2}\right)$. In terms of the language of μ_2-equivariant line bundles, we see that $\mathcal{O}\left(\frac{p_i}{2}\right)$ on $X_{D,2}$ corresponds to $L(w_i)$ on $\widetilde{X}$. Moreover, the pushforward $\psi_*\widehat{E}$ of a vector bundle $\widehat{E}$ on $X_{D,2}$ is the μ_2-invariant subbundle of the μ_2-equivariant bundle $\phi_*\widetilde{E}$, denoted by $(\phi_*\widetilde{E})^{\mu_2}$, where $\widetilde{E}$ is the μ_2-equivariant vector bundle corresponding to $\widehat{E}$.

Proposition 2.3.4. *([Bor07, Proposition 3.12]) Suppose that $div(s)$ is an effective Cartier divisor. Let $\mathcal{F}$ be a locally free sheaf on $X_{\mathcal{L},s,r}$. For each point $x \in X$, there exists a Zariski open neighborhood U of x such that $\mathcal{F}|_{\gamma^{-1}(U)}$ is a direct sum of invertible sheaves.*

In the case of a conic bundle $\pi : Q \to S$, recall that we assume the fibers of degenerate conic must have corank 1. We denote by $S_1 \subset S$ the discriminant locus of degenerate conics. We define the 2-nd root stack of S along S_1 as $\widehat{S} := S_{S_1,2}$ and $\psi : \widehat{S} \to S$ the projection. Then it is shown in [Kuz08, Section 3.6] that there is a sheaf of algebra $\widehat{\mathcal{B}}_0$ on $\widehat{S}$ such that $\psi_*\widehat{\mathcal{B}}_0 = \mathcal{B}_0$, so there is an equivalence of categories

$$\psi_* : \mathrm{Coh}(\widehat{S}, \widehat{\mathcal{B}}_0) \xrightarrow{\sim} \mathrm{Coh}(S, \mathcal{B}_0) \tag{2.3.5}$$

Moreover, the sheaf of algebra $\widehat{\mathcal{B}}_0$ is a sheaf of Azumaya algebra.

Suppose $C \subset S$ is a smooth curve, we restrict the conic bundle $Q \to S$ to a smooth curve $C \subset S$. We get the root stack $\widehat{C} := C_{S_1 \cap C,2} \cong \widehat{S}|_C$ and denote by $\widehat{B}_0$ the restriction

$\widehat{B}_0|_{\widehat{C}}$ by abuse of notation. The sheaf of algebra $\widehat{\mathcal{B}}_0$ on $\widehat{C}$ is a trivial Azumaya algebra. That means there exists a rank 2 vector bundle E_0 on $\widehat{C}$ (root stack construction is preserved under pull back) such that $\widehat{\mathcal{B}}_0 \cong \mathcal{E}nd(E_0)$ and it induces the equivalence of categories:

$$\begin{array}{ccccc} \mathrm{Coh}(C) & \xrightarrow{\sim} & \mathrm{Coh}(\widehat{C}, \widehat{\mathcal{B}}_0) & \xrightarrow{\sim} & \mathrm{Coh}(C, \mathcal{B}_0) \\ \mathcal{F} & \longmapsto & \mathcal{F} \otimes E_0 & \longmapsto & \psi_*(\mathcal{F} \otimes E_0) \end{array}$$

In particular, we have the following:

Corollary 2.3.5. *The rank of a $\mathcal{B}_0$-module $\psi_*(\mathcal{F} \otimes E_0)$ on C must be a multiple of 2.*

Let U be an open neighbourhood of $p \in S_1 \cap C$ where S_1 intersects C transversally. According to Proposition 2.3.4, $E_0|_{\psi^{-1}(U)} \cong L_1 \oplus L_2$ for some line bundle L_i on $\psi^{-1}(U)$. Each L_i defines a character $\chi_{i,p} : \mu_2 \to \mathbb{C}^*$ of μ_2 at the fiber of p.

Proposition 2.3.6. $\chi_{1,p}(-1) \cdot \chi_{2,p}(-1) = -1$.

Proof. Since we are interested in the fiber of E_0, we work in an affine neighbourhood $Z = \mathrm{Spec}(R)$ of p and the double cover $\widetilde{Z} = \mathrm{Spec}(R')$ where $R' := R[t]/(t^2 - s))$ and $div(s) = p$. So that the root stack restricted over Z is simply $\widehat{Z} = [\mathrm{Spec}(R[t]/(t^2 - s))/\mu_2]$. We can further reduce to the localization of R at p, we will again write the local ring as R and its unique maximal ideal $\mathfrak{m}$ which contains s.

As explained in Example 2.3.2, the rank 2 vector bundle E_0 on $\widehat{Z}$ is an R'-module M with a μ_2-action such that $M \cong M_0 \oplus M_1$ where M_i are R-modules. By Proposition 2.3.4, we can write $M \cong R'[l_1] \oplus R'[l_2]$ as $\mathbb{Z}/2\mathbb{Z}$-graded R'-modules where $l_i \in \{0, 1\}$, or equivalently, choose $e_1 \in M_{l_1}$ and $e_2 \in M_{l_2}$ such that $M \cong R'e_1 \oplus R'e_2$. Since $\chi_{i,p}(-1) = (-1)^{l_i}$ for $i = 1, 2$, it suffices to check $l_1 + l_2 = 1 \in \mathbb{Z}/2\mathbb{Z}$.

Suppose the contrary that $l_1 = l_2 = 0$ (or $l_1 = l_2 = 1$). Recall that E_0 satisfies $\psi_* \mathcal{E}nd(E_0) \cong \mathcal{B}_0$ as sheaf of algebras. Since the conic of Q over p is degenerate, its even Clifford algebra $\mathcal{B}_0|_p$ is not isomorphic to the endomorphism algebra of rank 2.

On the other hand, there is a natural morphism

$$\alpha : \psi_* \mathcal{E}nd(E_0) \to \mathcal{E}nd(\psi_* E_0).$$

Since E_0 corresponds to a $\mathbb{Z}/2\mathbb{Z}$-graded R'-module, $\mathcal{E}nd(E_0)$ also corresponds to a $\mathbb{Z}/2\mathbb{Z}$-graded R'-module and so $\psi_* \mathcal{E}nd(E_0)$ corresponds to the μ_2-invariant part i.e. $(\mathcal{E}nd(E_0))_0$ which is an R-module. In terms of the R'-basis $\{e_1, e_2\}$, $(\mathcal{E}nd(E_0))_0$ consists of the homogeneous R-module homomorphisms δ of degree 0:

$$e_1 \mapsto u_0 e_1 + u_0' e_2$$

$$e_2 \mapsto v_0 e_1 + v_0' e_2$$

where $u_0, u_0', v_0, v_0' \in R_0' = R$. Similarly, the module $\psi_* E_0$ is the μ_2-invariant part of $R'e_1 \oplus R'e_2$ which is freely generated by $\{f_1 = e_1, f_1 = e_2\}$ (or $\{f_1 = te_1, f_2 = te_2\}$ when $l_1 = l_2 = 1$) as R-modules. For $l_1 = l_2 = 0$, $\delta \in \psi_* \mathcal{E}nd(E_0)$ is mapped to an image in $\mathcal{E}nd(\psi_* E_0)$ of the form

$$f_1 = e_1 \mapsto u_0 e_1 + u_0' e_1 = u_0 f_1 + u_0' f_1$$

$$f_2 = e_2 \mapsto v_0 e_2 + v_0' e_2 = v_0 f_2 + v_0' f_2$$

Since u_0, v_0, u_0', v_0' are arbitrary elements in R, the image of α will be the endomorphism algebra over R i.e. α is an isomorphism of R-algebras, which is a contradiction. For $l_1 = l_2 = 1$, the image of α is also surjective for the same reason.

$\square$

2.3.3 Moduli space of $\mathcal{B}_0$-modules

Recall the definition of a sheaf of rings of differential operators from Simpson's paper [Sim94c]. Suppose S is a noetherian scheme over $\mathbb{C}$, and let $f : X \to S$ be a scheme of finite type over S. A sheaf of rings of differential operators on X over S is a sheaf of (not necessarily commutative) $\mathcal{O}_X$-algebras Λ over X, with a filtration $\Lambda_0 \subset \Lambda_1 \subset ...$ which satisfies the following properties:

1. $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$.

2. The image of the morphism $\mathcal{O}_X \to \Lambda$ is equal to Λ_0.

3. The image of $f^{-1}(\mathcal{O}_S)$ in $\mathcal{O}_X$ is contained in the center of Λ.

4. The left and right $\mathcal{O}_X$-module structures on $Gr_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ are equal.

5. The sheaves of $\mathcal{O}_X$-modules $Gr_i(\Lambda)$ are coherent.

6. The sheaf of graded $\mathcal{O}_X$-algebras $Gr(\Lambda) := \bigoplus_{i=0}^{\infty} Gr_i(\Lambda)$ is generated by $Gr_1(\Lambda)$ in the sense that the morphism of sheaves

$$Gr_1(\Lambda) \otimes_{\mathcal{O}_X} .... \otimes_{\mathcal{O}_X} Gr_1(\Lambda) \to Gr_1(\Lambda)$$

   is surjective.

Stability condition on Λ-modules are similar as coherent sheaves. Let $d = d(\mathcal{E})$ denote the dimension of the support of $\mathcal{E}$ and $p(\mathcal{E}, n)$ the Hilbert polynomial of $\mathcal{E}$. The leading coeffcient of $p(\mathcal{E}, n)$ is written as $r(\mathcal{E})/d!$ where $r(\mathcal{E})$ is the rank of $\mathcal{E}$. A Λ-module $\mathcal{E}$ is p-semistable (resp. p-stable) if it is of pure dimension, and if for any sub-Λ-module $\mathcal{F} \subset \mathcal{E}$

with $0 < r(\mathcal{F}) < r(\mathcal{E})$, there exists an N such that

$$\frac{p(\mathcal{F}, n)}{r(\mathcal{F})} \leq \frac{p(\mathcal{E}, n)}{r(\mathcal{E})} \tag{2.3.6}$$

(resp. $<$) for $n \geq N$.

Proposition 2.3.7. *The sheaf of $\mathcal{O}_{\mathbb{P}^2}$-algebra $\mathcal{B}_0$ is a sheaf of rings of differential operators.*

Proof. Recall that as an $\mathcal{O}_{\mathbb{P}^2}$-module, $\mathcal{B}_0 \cong \mathcal{O}_{\mathbb{P}^2} \oplus \wedge^2(E \otimes \mathcal{L})$ with the filtration $\Lambda_0 = \mathcal{O}_{\mathbb{P}^2}, \Lambda_i = \mathcal{B}_0$ for $i \geq 1$. Properties (1), (2), and (5) are clearly satisfied. The center of $\mathcal{B}_0$ is Λ_0, so (3) is also satisfied. The left and right $\mathcal{O}_{\mathbb{P}^2}$-module on $\mathcal{B}_0$ coincide by definition, so the induced left and right $\mathcal{O}_{\mathbb{P}^2}$-module structure also coincide on $Gr_i(\Lambda)$. Finally, since $Gr_i(\Lambda) = 0$ for $i > 1$, property (6) is satisfied trivially. □

Since $\mathcal{B}_0$ is a sheaf of rings of differential operators, [Sim94c, Theorem 4.7] guarantees the existence of a moduli space of semistable $\mathcal{B}_0$-modules with a fixed Hilbert polynomial whose closed points correspond to Jordan equivalence class of $\mathcal{B}_0$-modules. In this paper, we will be primarily interested in the moduli space of semistable $\mathcal{B}_0$-module with Chern character $(0, 2d, e)$, denoted by $\mathfrak{M}_{d,e}$.

By taking the Fitting support, we get the support morphism

$$\Upsilon : \mathfrak{M}_{d,e} \to |\mathcal{O}(2d)|.$$

However, as explained in Remark 2.3.5, the rank of a $\mathcal{B}_0$-module as a coherent sheaf on its support must be a multiple of 2. It is easy to show that if $i : C \to \mathbb{P}^2$ is the inclusion of a divisor C, then $c_1(i_*G) = \deg(C)\mathrm{rk}(G)$ for a coherent sheaf G on C. So we see that Υ factors through $|\mathcal{O}(d)|$:

$$\Upsilon : \mathfrak{M}_{d,e} \to |\mathcal{O}(d)| \hookrightarrow |\mathcal{O}(2d)|.$$

Theorem 2.3.8 ([LMS15]). *The moduli space $\mathfrak{M}_{d,e}$ is irreducible. Let $U_d \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ be the open subset of smooth degree d curves which intersect Δ transversally and $C \in U$. Then*

$$\Upsilon^{-1}(C) \cong \bigsqcup_I \operatorname{Pic}^{-|I|/2} C.$$

where I runs over the even cardinality subsets of $\{1, ..., dk\}$ and $k := deg(\Delta)$.

Proof. The proof of irreduciblity can be found in [LMS15, Theorem 2.11]. We will recall the description of the fiber $\Upsilon^{-1}(C)$ in [LMS15, Theorem 2.11] and provide more details as it will be important for our purposes in later sections. A $\mathcal{B}_0$-module $M \in \Upsilon^{-1}(C)$ is rank 2 vector bundle supported on C, so we can restrict our attention to $\mathcal{B}_0$-modules on C. Note that a $\mathcal{B}_0$-module M that is a rank 2 vector bundle on C is automatically p-stable. First, p-stability on C is reduced to slope stability on curve: let $\mu(M) = deg(M)/r(M)$, then M is slope (semi)stable if for all $\mathcal{B}_0$-submodule N, we have $\mu(N) < (\leq)\mu(M)$. Now, the rank of any $\mathcal{B}_0$-module must be a multiple of 2, so a $\mathcal{B}_0$-submodule N of M must have $r(N) = 2$. Then M/N is a sheaf of dimension zero with length l, which implies that $\mu(N) = \mu(M) - l/2 < \mu(M)$.

As explained in previous section, there is a rank 2 vector bundle E_0 on the 2nd-root stack $\widehat{C} := C_{C\cap\Delta,2}$ and an equivalence of category $\psi_* : \operatorname{Coh}(\widehat{C}, \mathcal{E}nd(E_0)) \xrightarrow{\sim} \operatorname{Coh}(C, \mathcal{B}_0)$ where $\psi : \widehat{C} \to C$ is the projection morphism. That means we are looking for line bundles $\widehat{L}$ on $\widehat{C}$ such that $\operatorname{ch}(i_*\psi_*(E_0 \otimes \widehat{L})) = (0, 2d, e)$ where $i : C \to \mathbb{P}^2$ the inclusion map. It is clear that $\operatorname{ch}_0(i_*\psi_*(E_0 \otimes \widehat{L})) = 0$ and $\operatorname{ch}_1(i_*\psi_*(E_0 \otimes F)) = 2d$. To compute ch_2, we use the fact that is easily computed by the Grothendieck-Riemann-Roch theorem:

$$\operatorname{ch}_2(i_*G) = \deg(G) - \frac{d^2}{2}\operatorname{rk}(G) \tag{2.3.7}$$

for a vector bundle G on C. So it is equivalent to finding all $\widehat{L}$ on $\widehat{C}$ such that $e = \deg(\psi_*(E_0 \otimes$

$\widehat{L}) - d^2$ or $\deg(\psi_*(E_0 \otimes \widehat{L})) = e + d^2$.

Case 1: When $k = \deg(\Delta)$ is even, in which case $\mathcal{O}_{\mathbb{P}_2}(k)|_C$ admits a square root $\mathcal{O}_{\mathbb{P}_2}(k/2)|_C$, so we can take the the cyclic cover $\phi : \widetilde{C} \to C$ of order 2 branched at $C \cap \Delta$ with an involution action. As explained in Example 2.3.3 the root stack $\widehat{C}$ is isomorphic to the quotient stack $\left[\widetilde{C}/\mu_2\right]$. Moreover, the morphism $\phi : \widetilde{C} \to C$ factors as $\widetilde{C} \xrightarrow{\eta} \widehat{C} \xrightarrow{\psi} C$.

Let $w_i \in \widetilde{C}, p_i = \phi(w_i) \in C \cap \Delta$ be the ramification and branch points respectively. Recall that a line bundle $\widehat{L}$ on $\widehat{C}$ can be written as $\psi^* F \otimes \mathcal{O}\left(\sum \lambda_i \frac{p_i}{2}\right)$ such that F is a line bundle on C, where $\lambda_i \in \{0, 1\}$. As a μ_2-equivariant line bundle $\widehat{L} = \phi^*(F) \otimes L\left(\sum \lambda_i w_i\right)$ on $\widetilde{C}$ (following the notation in Example 2.3.3),

$$\begin{aligned} \mathrm{c}_1\left(\phi_*\left(E_0 \otimes \phi^*(F) \otimes L\left(\sum \lambda_i w_i\right)\right)^{\mu_2}\right) &= \mathrm{c}_1\left(F \otimes \phi_*\left(E_0 \otimes L\left(\sum \lambda_i w_i\right)\right)^{\mu_2}\right) \\ &= 2\mathrm{c}_1(F) + \mathrm{c}_1\left(\phi_*\left(E_0 \otimes L\left(\sum \lambda_i w_i\right)\right)^{\mu_2}\right) \end{aligned} \tag{2.3.8}$$

Since we have the short exact sequence

$$0 \to \phi_*\left(E_0\right)^{\mu_2} \to \phi_*\left(E_0 \otimes L\left(\sum \lambda_i w_i\right)\right)^{\mu_2} \to \bigoplus \phi_*\left(E_0 \otimes L\left(\sum \lambda_i w_i\right) \otimes \mathcal{O}_{w_i}\right)^{\mu_2} \to 0$$

and $\mathrm{c}_1\left(\phi_*\left(E_0 \otimes \mathcal{O}_{w_i}\right)^{\mu_2}\right) = 1$ by Proposition 2.3.6, which implies

$$\mathrm{c}_1\left(\phi_*\left(E_0 \otimes \mathcal{O}_{w_i} \otimes L\left(\sum \lambda_i w_i\right)\right)^{\mu_2}\right) = 1,$$

the last expresssion of (2.3.8) becomes

$$2\mathrm{c}_1(F) + \mathrm{c}_1\left(\phi_*(E_0)^{\mu_2}\right) + |I|\,.$$

where I is the subset of $\{1, 2, ..., dk\}$ such that $\lambda_i = 1$ for $i \in I$ and $|I|$ is its cardinality. Since E_0 on $\widehat{C}$ is determined up to tensorization by a line bundle, this expression means that we can assume $\deg\left((\phi_* E_0)^{\mu_2}\right) = e + d^2$.

We also see that the condition $\deg(\psi_*(E_0 \otimes \widehat{L})) = e + d^2$ becomes

$$e + d^2 = 2\deg(F) + \deg(\psi_*(E_0)) + |I| \implies 0 = 2\deg(F) + |I|,$$

which is the same as saying that the degree of $\widehat{L}$ as a line bundle on $\widetilde{C}$ is 0. The condition $2\deg(F) + |I| = 0$ only makes sense if $|I|$ is even. We also see that for each fixed I, the set of line bundles satisfying the condition above is $\mathrm{Pic}^{-|I|/2}(C)$. Thus, $\Upsilon^{-1}(C) \cong \coprod_I \mathrm{Pic}^{-|I|/2}(C)$ where I runs over the set of even cardinality subsets of dk.

Case 2: When $k = \deg(\Delta)$ is odd, we will use the trick by choosing an auxiliary line $H \subset \mathbb{P}^2$ which intersects C transversally and $D_a := H \cap C$ is disjoint from $D := C \cap \Delta$. Then the line bundle $\mathcal{O}_C(D + D_a) \cong \mathcal{O}_{\mathbb{P}^2}(k+1)|_C$ has a natural square root $\mathcal{O}_{\mathbb{P}^2}((k+1)/2)|_C$, so we can again consider the cyclic cover $\widetilde{C}$ branched at $C \cap (\Delta + H)$. The root stack $\overline{C} := C_{D+D_a,2}$ is now isomorphic to the quotient stack $\left[\widetilde{C}/\mu_2\right]$. We again denote by $\widehat{C}$ the root stack $C_{C\cap\Delta,2}$. By Lemma 2.3.1, the stack $\overline{C}$ is isomorphic to $C_{D,D_a,(2,2)}$ which is constructed as a fiber product, so $C_{D,D_a,(2,2)}$ projects to C. We denote the composition by $f : \overline{C} \xrightarrow{\sim} C_{D,D_a,(2,2)} \to \widehat{C}$.

$$\begin{array}{ccccc} \widetilde{C} & & & & \\ \downarrow q & & \searrow^{\phi} & & \\ \overline{C} & \xrightarrow{f} & \widehat{C} & \xrightarrow{\psi} & C \end{array}$$

Let $\widehat{L}$ be a line bundle on $\widehat{C}$, we want to find all such line bundles such that $\mathrm{ch}(\psi_*(E_0 \otimes \widehat{L})) = (0, 2d, e)$. The same reasoning as in the previous case implies that this is equivalent to finding $\deg(\psi_*(E_0 \otimes \widehat{L})) = e + d^2$. By [Cad07, Theorem 3.1.1 (3)] (the proof there works for any vector bundle), we know that $\widehat{M} \cong f_* f^* \widehat{M}$ for any vector bundles on $\widehat{C}$, so

$$\psi_*(E_0 \otimes \widehat{L}) = \psi_* f_* \left(f^* \left(E_0 \otimes \widehat{L}\right)\right) = \phi_* \left(f^* \left(E_0 \otimes \widehat{L}\right)\right)^{\mu_2}$$

As $\overline{C} \cong \left[\widetilde{C}/\mu_2\right]$, $f^*\left(E_0 \otimes \widehat{L}\right)$ on $\overline{C}$ is a μ_2-equivariant vector bundle on $\widetilde{C}$ whose induced

μ_2-characters at the fixed points $w_i \in D_a$ is trivial. In other words, the problem now is to find all line bundles on $\widetilde{C}$ of the form $\phi^*(F) \otimes \mathcal{O}(\sum \lambda_i w_i)$ where $w_i \in \phi^{-1}(D)$ such that

$$\deg\left(\phi_*\left(E_0 \otimes \phi^*(F) \otimes \mathcal{O}\left(\sum \lambda_i w_i\right)\right)^{\mu_2}\right) = e + d^2.$$

The same argument as in Case 1 applies and implies that $2\deg(F) + |I| = 0$ where I is the subset of $\{1, ..., dk\}$ such that $\lambda_i = 1$ and $|I|$ is its cardinality. Hence, $\Upsilon^{-1}(C)$ is again isomorphic to $\bigsqcup_I \operatorname{Pic}^{-|I|/2}(C)$. Note that although we use the auxiliary line H and the divisor D_a in the argument, the result is independent of them.

□

Remark 2.3.9. Note that the isomorphism $\Upsilon^{-1}(C) \cong \bigsqcup_I \operatorname{Pic}^{-\tau_I/2}(C)$ here is not canonical, as E_0 is only determined up to tensorization by line bundles.

Proposition 2.3.10. *The moduli space $\mathfrak{M}_{d,e}|_{U_d}$ over U_d is smooth of dimension d^2+1.*

Proof. This is a consequence of [LMS15, Theorem 2.12] which states that the stable locus $\mathfrak{M}^s_{d,e}$ is smooth of dimension d^2+1. As argued in the proof of Theorem 2.3.8, a $\mathcal{B}_0$-module in $\Upsilon^{-1}(C)$ is automatically stable, so $\mathfrak{M}_{d,e}|_{U_d} \subset \mathfrak{M}^s_{d,e}$. □

Suppose $d = 1, 2$. For $d < \deg(\Delta)$, if we call the line bundle $L_d := O_{\mathbb{P}^2}(d)|_\Delta$ on Δ, it is easy to see that $|O_{\mathbb{P}^2}(d)| \cong |L_d|$. Recall the group scheme $G|_{U_d}$ over U_d defined in Section 2.

Corollary 2.3.11. *With the same notation as above, for $d = 1, 2$, $\Upsilon^{-1}(C)$ is a $G|_C$-torsor.*

Proof. For $d = 1, 2$, the Picard group $\operatorname{Pic}^a(C)$ is trivial for any a. Let $\sum p_i = C \cap \Delta$ be the divisor corresponding to C under $|\mathcal{O}_{\mathbb{P}^2}(2)| \cong |L_d|$. We can denote a closed point of $G|_C$ by

$\sum(\lambda_i, p_i)$ where $\lambda_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sum \lambda_i = 0$. Since we can write $M = \psi_* \widehat{M}$, the group $G|_C$ acts on $\Upsilon^{-1}(C)$ by

$$\left(\sum(\lambda_i, p_i)\right) \cdot M = \psi_* \left(\widehat{M} \otimes \mathcal{O}\left(\sum \lambda_i \frac{p_i}{2}\right) \otimes h_C^{-\frac{1}{2}\sum \lambda_i}\right)$$

where $h_C = \psi^* \mathcal{O}_C(1)$. To see that $G|_C$ acts simply transitively, fix E_0 such that for $M \in \Upsilon^{-1}(C)$, $M \cong \psi_*(E_0 \otimes \widehat{L})$, then the action becomes

$$\left(\sum(\lambda_i, p_i)\right) \cdot M = \psi_* \left(E_0 \otimes \widehat{L} \otimes \mathcal{O}\left(\sum \lambda_i \frac{p_i}{2}\right) \otimes h_C^{-\frac{1}{2}\sum \lambda_i}\right)$$

which is clearly simply transitive by the description of $\Upsilon^{-1}(C)$ in the proof of Theorem 2.3.8. □

2.4 Moduli spaces of $\mathcal{B}_0$-modules and special subvarieties of Prym varieties

In this section, we will construct the rational map from the moduli space $\mathfrak{M}_{d,e}$ to the Prym variety $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$. The key observation is that our $\mathcal{B}_0$-modules are supported on plane curves C which intersect the discriminant curve Δ in finitely many points. The $\mathcal{B}_0$-modules restrict to a representation of even Clifford algebra over each of these points, These representations then define a lift of the intersection $C \cap \Delta \subset \Delta$ to $\widetilde{\Delta}$, which will be a point in the variety of divisors lying over the linear system $|\mathcal{O}_{\mathbb{P}^2}(C)|_\Delta|$, and maps to $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$. So we begin by studying studying the representation theory for our purpose.

2.4.1 Representation theory of degenerate even Clifford algebra

In this subsection, we will restrict our attention to the fiber of the sheaf of even Clifford algebra $\mathcal{B}_0$ over a fixed $p \in C \cap \Delta$ which is a $\mathbb{C}$-algebra denoted by A. Note that all the fibers over the points in $C \cap \Delta$ are isomorphic as $\mathbb{C}$-algebra since the fiber $\mathcal{B}_0|_p$ over a point $p \in C \cap \Delta$ is defined by a degenerate quadratic form of corank 1 and all quadratic forms of corank 1 are isomorphic over $\mathbb{C}$. Let V be a vector space of dimension 3, and $q \in S^2V^*$ a quadratic form of rank 2. The even Clifford algebra is defined as a vector space $\mathbb{C} \oplus \wedge^2 V$ together with an algebra structure defined as follows. First, we can always find a basis $\{e_1, e_2, e_3\}$ of V such that q is represented as the matrix $\mathrm{diag}(1, 1, 0)$ and we denote by $\{1, x := ie_1 \wedge e_2, y := ie_2 \wedge e_3, z := e_1 \wedge e_3\}$ the basis of $\mathbb{C} \oplus \wedge^2 V$. The relations are given by

$$x^2 = 1, y^2 = z^2 = 0, xy = -z, xz = -y, xy = -yx, xz = -zx, yz = zy = 0. \tag{2.4.1}$$

Since A is an finite dimensional associative algebra, we can understand it via quivers and path algebras. We refer the reader to [ASS06] for the basics of quivers and path algebras.

Proposition 2.4.1. *The algebra A is isomorphic to the path algebra associated to the following quiver Q*

$$+ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} - \tag{2.4.2}$$

with relations $\alpha\beta = \beta\alpha = 0$.

Proof. We begin by finding the idempotents i.e. elements in A such that $x^2 = x$. This is achieved by setting up the equations

$$(a_0 + a_1x + a_2y + a_3z)^2 = (a_0 + a_1x + a_2y + a_3z)$$

and solving the equations in a_0, a_1, a_2, a_3. It is easy to check that the idempotents are:

$$0, 1, \frac{1}{2}(1 \pm x) + a_2y + a_3z$$

and that

$$e_+ := \frac{1}{2}(1 + x), \quad e_- := \frac{1}{2}(1 - x)$$

is a complete set of primitive orthogonal idempotents of A. From the description of idempotents, it is clear that the only central idempotents are $0, 1$, so A is connected.

We also need to compute the radical of A. Observe that the ideal $I = (y, z)$ is clearly nilpotent, i.e. $I^2 = 0$ and $A/I \cong \mathbb{C}[x]/(x^2 - 1) \cong \mathbb{C} \oplus \mathbb{C}$. By [ASS06, Corollary 1.4(c)], this implies that $\mathrm{rad}(A) = I = (y, z)$. It also follows that A is a basic algebra by [ASS06, Proposition 6.2(a)].

The arrows between $+ \to -$ of the associated quiver is described by

$$e_-(\mathrm{rad}(A)/\mathrm{rad}(A)^2)e_+ = \frac{1}{2}(1 - x)(y, z)\frac{1}{2}(1 + x) = \mathbb{C}(y + z).$$

Similarly, the arrows between $- \to +$ is described by

$$e_+(\mathrm{rad}(A)/(\mathrm{rad}(A)^2))e_- = \mathbb{C}(y - z)$$

and the arrows between $- \to -$ and $+ \to +$

$$e_-(\mathrm{rad}(A)/(\mathrm{rad}(A)^2))e_- = e_+(\mathrm{rad}(A)/(\mathrm{rad}(A)^2))e_+ = 0. \tag{2.4.3}$$

Hence, the associated quiver Q [ASS06, Definition 3.1] of A is given by

$$+ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} -$$

and we obtain a surjective map $\mathbb{C}Q \to A$ from the path algebra $\mathbb{C}Q$ associated to the quiver Q to A by sending the generators

$$e_+ \mapsto \frac{1}{2}(1 + x), \quad e_- \mapsto \frac{1}{2}(1 - x), \quad \alpha \mapsto \frac{1}{2}(y + z), \quad \beta \mapsto \frac{1}{2}(y - z). \tag{2.4.4}$$

It is easy to see that $\alpha\beta = \beta\alpha = 0$ and since any other paths of higher length must contain a factor of $\alpha\beta$ or $\beta\alpha$, we see that the kernel of $kQ \to A$ must be $J = (\alpha\beta, \beta\alpha)$. Therefore, we have an isomorphism $\mathbb{C}Q/J \cong A$.

□

Remark 2.4.2. We can prove the isomorphism in Proposition 2.4.1 directly by checking the map defined in (2.4.4) is indeed an isomorphism of $\mathbb{C}$-algebra. The detail with the idempotents and the radical ideal in the proof above is just to display a more systematic approach.

Since we are mainly interested in $\mathcal{B}_0$-modules that are locally free of rank 2, the fiber of such module over $p \in C \cap \Delta$ is a representation of A on $\mathbb{C}^2$. In light of the interpretation of A as a path algebra, we can easily classify all the isomorphism classes of representations on $\mathbb{C}^2$. The isomorphism classes of representations of $A \cong \mathbb{C}Q/J$ on $\mathbb{C}^2$ are listed as follows:

1. $$\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$$

2. $$\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$$

3. $$\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$$

4. $$\mathbb{C}^2 \underset{0}{\overset{0}{\rightleftarrows}} 0$$

5. $$0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}^2$$

2.4.2 Construction

Recall the geometric set-up: we have a rank 3 bundle F on $\mathbb{P}^2$ and an embedding of line bundle $q : L \hookrightarrow S^2F^\vee$. This defines a conic bundle $Q \subset \mathbb{P}(F)$ as the zero locus of $q \in H^0(\mathbb{P}^2, S^2F^\vee \otimes L^\vee) \cong H^0(\mathbb{P}(F), \mathcal{O}_{\mathbb{P}(F)/\mathbb{P}^2}(2) \otimes (\pi')^* L^\vee)$ in $\mathbb{P}(F)$ where we denote $\pi' : \mathbb{P}(F) \to \mathbb{P}^2$. The discriminant curve is assumed to be smooth and denoted by Δ. As an $\mathcal{O}_{\mathbb{P}^2}$-module, the sheaf of even Clifford algebra $\mathcal{B}_0$ on $\mathbb{P}^2$ is

$$\mathcal{B}_0 \cong \mathcal{O}_{\mathbb{P}^2} \oplus \wedge^2 F \otimes L.$$

We will restrict our attention to $\mathcal{B}_0$-modules supported on a degree d smooth curve $C \subset \mathbb{P}^2$ for $d = 1, 2$. Given such a $\mathcal{B}_0$-module M on C, for each $p \in C \cap \Delta$ we consider the vector subspace

$$K := \ker(\mathcal{B}_0|_p \to \mathcal{E}nd(M)|_p).$$

As we will see in Proposition 2.4.3 (1), K is a vector subspace of $\wedge^2 F|_p \otimes L|_p$. The natural isomorphisms $w : \wedge^2 F \xrightarrow{\sim} \det(F) \otimes F^\vee$ and $F \xrightarrow{\sim} (F^\vee)^\vee$ give rise to another vector space

$$K' := \ker(F|_p \xrightarrow{\sim} (F^\vee)^\vee|_p \xrightarrow{w_p^\vee \otimes \det(F)|_p \otimes L|_p} K^\vee \otimes (L \otimes \det(F))|_p)$$

where $w_p : K \hookrightarrow (\wedge^2 F \otimes L)|_p \to (\det(F) \otimes F^\vee \otimes L)|_p$ is the composition of the inclusion map and the isomorphism w restricted to p. Hence, $\mathbb{P}(K')$ is a linear subspace in $\mathbb{P}(F|_p)$. In the light of Proposition 2.4.3, K' is the two dimensional vector space in $F|_p$ that corresponds to the line K in $\wedge^2 F|_p$ (identified with $(\wedge^2 F \otimes L)|_p$).

Proposition 2.4.3.

1. $K \subset \wedge^2 F|_p \otimes L|_p \subset \mathcal{O}|_p \oplus \wedge^2 F|_p \otimes L|_p$;

2. $\dim(K) = 1$ and $\dim(K') = 2$;

3. The line $\mathbb{P}(K') \subset \mathbb{P}(F|_p)$ is one of the two irreducible components of the degenerate conic $Q|_p \subset \mathbb{P}(F|_p)$.

Proof. First of all, we can always choose a basis $\{e_1, e_2, e_3\}$ of $F|_p$ and a trivialization $i : L|_p \cong \mathbb{C}$ so that $q|_p$ is represented by $\mathrm{diag}(1,1,0)$. The trivialization i induces an isomorphism of $\mathbb{C}$-algebras $\mathcal{B}_0|_p \cong \mathcal{O}|_p \oplus \wedge^2 F|_p$ where the latter is generated by $1 \in \mathcal{O}|_p$ and $\{x := ie_1 \wedge e_2, y := ie_2 \wedge e_3, z := e_1 \wedge e_3\} \subset \wedge^2 F|_p$ with relation

$$x^2 = 1, y^2 = z^2 = 0, xy = -z, xz = -y, xy = -yx, xz = -zx, yz = 0.$$

The irreducible components of $Q|_p \subset \mathbb{P}(F|_p)$ are given by the projectivization of the isotropic planes in $F|_p$ with respect to q. If we write a vector $v \in F|_p$ as $\sum_{i=1}^{3} a_i e_i$, then the two isotropic planes are given by the two equations

$$a_1 + ia_2 = 0, \quad a_1 - ia_2 = 0 \tag{2.4.5}$$

which correspond to the lines in $\wedge^2 F|_p$

$$\mathbb{C}\langle ie_2 \wedge e_3 + e_1 \wedge e_3 \rangle = \mathbb{C}\langle y + z \rangle, \quad \mathbb{C}\langle ie_2 \wedge e_3 - e_1 \wedge e_3 \rangle = \mathbb{C}\langle y - z \rangle. \tag{2.4.6}$$

To prove all the claims, it suffices to show that $K \subset \mathcal{B}_0|_p$ corresponds to one of the these lines in the subspace $\wedge^2 F|_p \subset \mathcal{O}|_p \oplus \wedge^2 F|_p$. Indeed, then K' will correspond to one of the isotropic planes.

Recall that with the choice of basis $\{1, x, y, z\}$ of $\mathcal{O}|_p \oplus \wedge^2 F|_p$, we have an isomorphism $\mathbb{C}Q/J \xrightarrow{\sim} \mathcal{O}|_p \oplus \wedge^2 F|_p$:

$$e_+ \mapsto \frac{1}{2}(1 + x), \quad e_- \mapsto \frac{1}{2}(1 - x), \quad \alpha \mapsto \frac{1}{2}(y + z), \quad \beta \mapsto \frac{1}{2}(y - z).$$

Then the kernel of $\mathcal{B}_0|_p \to \mathcal{E}nd(M)|_p$ can be computed by the composition

$$\mathbb{C}Q/J \xrightarrow{\sim} \mathcal{O}|_p \oplus \wedge^2 F|_p \xrightarrow{\sim} \mathcal{B}_0|_p \to \mathcal{E}nd(M)|_p$$

which is a representation of the path algebra $\mathbb{C}Q/J$. By Proposition 2.4.5, the isomorphism classes of the representation of $\mathbb{C}Q/J$ on $\mathbb{C}^2$ in this case must be either type (1) and (2).

1. For type (1), the kernel of $\mathbb{C}Q/J \xrightarrow{\sim} \mathcal{B}_0|_p \to \mathcal{E}nd(M)|_p$ is $\mathbb{C}\langle\beta\rangle$ which corresponds to $K = \mathbb{C}\langle y - z\rangle \subset \mathcal{O}|_p \oplus \wedge^2 F|_p$.

2. For type (2), the kernel of $\mathbb{C}Q/J \xrightarrow{\sim} \mathcal{B}_0|_p \to \mathcal{E}nd(M)|_p$ is $\mathbb{C}\langle\alpha\rangle$ which corresponds to $K = \mathbb{C}\langle y + z\rangle \subset \mathcal{O}|_p \oplus \wedge^2 F|_p$.

All the claims follow immediately. □

Remark 2.4.4. The trivialization $i : L|_p \cong \mathbb{C}$ does not cause any ambiguity in the identifications as we are only interested in identification of vector subspaces, other trivializations will only differ in a scalar multiplication.

Proposition 2.4.5. *The representation of $\mathcal{B}_0|_p$ obtained from a $\mathcal{B}_0$-module M as the fiber $M|_p$ over $p \in C \cap \Delta$ must have isomorphism class of either type (1) or type (2).*

Proof. Fix $p \in C \cap \Delta$. Let $n = 3, 4, 5$ and M_n be a $\mathcal{B}_0$-modules such that its fiber over p is a $\mathcal{B}_0$-representation of type n isomorphism class. We can choose a local parameter $t \in \mathcal{O}_{C,p}$ as $\mathcal{O}_{C,p}$ is a discrete valuation ring.

Then M_n induces the homomorphisms over the local ring $\mathcal{O}_{C,p}$ and over the residue field $\kappa(p)$ (i.e. fiber)

$$\rho_n : \mathcal{B}_0 \otimes \mathcal{O}_{C,p} \to \mathcal{E}nd(M_n) \otimes \mathcal{O}_{C,p}, \quad \rho_n^0 : \mathcal{B}_0 \otimes \kappa(p) = \mathcal{B}_0|_p \to \mathcal{E}nd(M_n) \otimes \kappa(p) = \mathcal{E}nd(M_n)|_p.$$

Again, we can always choose a basis $\{e_1, e_2, e_3\}$ of $F|_p$ and a trivialization $i : L|_p \cong \mathbb{C}$ so that $q|_p$ is represented by $\mathrm{diag}(1, 1, 0)$. The trivialization i induces an isomorphism of $\mathbb{C}$-algebras $\mathcal{B}_0|_p \cong \mathcal{O}|_p \oplus \wedge^2 F|_p$ where the latter is generated by $1 \in \mathcal{O}|_p$ and $\{x := ie_1 \wedge e_2, y := ie_2 \wedge e_3, z := e_1 \wedge e_3\} \subset \wedge^2 F|_p$ with relation

$$x^2 = 1, y^2 = z^2 = 0, xy = -z, xz = -y, xy = -yx, xz = -zx, yz = 0.$$

Recall that the algebra $\mathcal{O}|_p \oplus \wedge^2 F|_p$ is isomorphic to $\mathbb{C}Q/J$ generated by e_+, e_-, α, β. If we call the isomorphism $j : \mathbb{C}Q/J \xrightarrow{\sim} \mathcal{O}|_p \oplus \wedge^2 F|_p \xrightarrow{\sim} \mathcal{B}_0|_p$, then clearly $\rho_n^0(j(\alpha)) = \rho_n^0(j(\beta)) = 0$. It follows that $\rho_n^0(y) = \rho_n^0(j(\alpha+\beta)) = 0$ and similarly $\rho_n^0(z) = 0$. So that means $\rho_n(y) = tP_y$ and $\rho_n(z) = tP_z$ for some $P_y, P_z \in \mathcal{E}nd(M_n) \otimes \mathcal{O}_{C,p}$.

As F is locally free of rank 3, by Nakayama lemma, we can lift the basis $\{e_1, e_2, e_3\}$ of $F|_p$ to a basis (also denoted as $\{e_1, e_2, e_3\}$ by abuse of notation) of $F \otimes \mathcal{O}_{C,p}$ over $\mathcal{O}_{C,p}$. The quadratic form q is represented by the matrix

$$(f_{ij}) = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \tag{2.4.7}$$

where f_{ij} are elements in $\mathcal{O}_{C,p}$. By the choice of basis $\{e_1, e_2, e_2\}$, we have $f_{11}|_p \neq 0$, $f_{22}|_p \neq 0$ and $f_{ij}|_p = 0$ for $(i,j) \neq (1,1), (2,2)$. It follows that $f_{ij} = tf'_{ij}$ for $(i,j) \neq (1,1), (2,2)$ and $f'_{ij} \in \mathcal{O}_{C,p}$. Then

$$\begin{aligned} yz &= (ie_2e_3)(e_1e_3) \\ &= if_{31}e_2e_3 - ie_2e_1e_3e_3 \\ &= if_{31}e_2e_3 - if_{33}e_2e_1 \\ &= f_{31}y - if_{33}f_{21} + f_{33}x \end{aligned}$$

(here we omit the " $\wedge$ " between the $(e_i)'s$) so it follows that

$$\rho_n(yz) = t^2 f'_{31} P_y - it^2 f'_{33} f'_{21} + tf'_{33}\rho_n(x).$$

Since

$$\rho_n(yz) = \rho_n(y)\rho_n(z) = t^2 P_y P_z$$

by equating the two expression, we get

$$t^2 P_y P_z = it^2 f'_{31} P_y - it^2 f'_{33} f'_{21} + t f'_{33} \rho_n(x) \implies t P_y P_z = it f'_{31} P_y - it f'_{33} f'_{21} + f'_{33} \rho_n(x).$$

Note that f'_{33} is invertible in $\mathcal{O}_{C,p}$ because otherwise $\det(f_{ij})$ will have zeros of order 2 with respect to t, which is not allowed since we assume that C intersects Δ transversally. Hence, we can write $\rho_n(x) = tP_x$ for some $P_x \in \mathcal{E}nd(M_n) \otimes \mathcal{O}_{C,p}$. In particular, we must have $\rho_n^0(x^2) = 0$.

On the other hand, we have

$$\begin{aligned} x^2 &= (ie_1e_2)(ie_1e_2) \\ &= -f_{21}e_1e_2 + e_1e_1e_2e_2 \\ &= if_{21}x + f_{11}f_{22} \end{aligned}$$

(again we omit the " $\wedge$ " between the $(e_i)'s$) and so $\rho_n^0(x^2) = (f_{11}f_{22})|_p \neq 0$ as $f_{21}|_p = 0$, $f_{11}|_p \neq 0$ and $f_{22}|_p \neq 0$. Hence, a contradiction.

□

For $d = 1, 2$, let $U_d \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ be the subset of smooth curves of degree d which intersect Δ transversally. For $d < \deg(\Delta)$, if we call the line bundle $L_d := O_{\mathbb{P}^2}(d)|_\Delta$ on Δ, it is easy to see that $|O_{\mathbb{P}^2}(d)| \cong |L_d|$. Hence, we can consider the variety of divisors W_d lying over $|L_d|$ and its two components as W_d^i for $i = 0, 1$.

For each $\mathcal{B}_0$-module $M \in \mathfrak{M}_{d,e}$ with support on $C \in U_d$, let $j : C \cap \Delta \hookrightarrow C$ be the inclusion, by Proposition 2.4.3, the assignment

$$M \mapsto \mathcal{K} := \ker\left(j^*\mathcal{B}_0 \to j^*\mathcal{E}nd(M)\right)$$

is argued to be contained in $j^*(\wedge^2 F)$ and it defines exactly a point in W_d.

This construction also works in family. Let T be a scheme and $\mathcal{M}_T$ be a flat family of $\mathcal{B}_0$-modules on $\mathbb{P}^2$ with Chern character $(0, 2d, e)$ i.e. $\mathcal{M}_T$ is a $p^*\mathcal{B}_0$-module on $\mathbb{P}^2 \times T$ where $p_1 : \mathbb{P}^2 \times T \to \mathbb{P}^2$ is the projection. Then we get a map $T \to \mathfrak{M}_{d,e}|_{U_d} \to |U_d| \subset \Delta^{(k)}$. We can restrict the family of $\mathcal{B}_0$-modules to $\Delta \times T \subset \mathbb{P}^2 \times T$. Consider the universal divisor

$$\begin{array}{ccc} & \mathcal{D} \subset \Delta \times \Delta^{(k)} & \\ \swarrow & & \searrow \\ \Delta & & \Delta^{(k)} \end{array}$$

By pulling back $\mathcal{D}$ along the map $\Delta \times T \to \Delta \times \Delta^{(k)}$, we get another divisor $\mathcal{D}_T \subset \Delta \times T$ and denote the inclusion by $i_T : \mathcal{D}_T \hookrightarrow \Delta \times T \hookrightarrow \mathbb{P}^2 \times T$.

We will write $F_T := i_T^* p_1^* F$ and $L_T := i_T^* p_1^* L$. The sheaf

$$\mathcal{K}_T := \ker\left(i_T^* p_1^* \mathcal{B}_0 \to i_T^* \mathcal{E}nd(\mathcal{M})\right)$$

has constant fiber dimension one and is contained in the rank 3 vector bundle $i_T^* p_1^* \left(\wedge^2 F_T \otimes L_T\right)$ on $\mathcal{D}_T$ by Proposition 2.4.3. Again, since there are the natural isomorphisms $w : \wedge^2 F_T \xrightarrow{\sim} \det(F_T) \otimes F_T^\vee$ and $F_T \xrightarrow{\sim} (F_T^\vee)^\vee$, we can define

$$\mathcal{K}'_T := \ker(F_T \xrightarrow{\sim} (F_T^\vee)^\vee \xrightarrow{w_p^\vee \otimes \det(F)|_p \otimes L|_p} \mathcal{K}_T^\vee \otimes L_T \otimes \det(F_T))$$

where $w_T : \mathcal{K}_T \hookrightarrow \wedge^2 F_T \otimes L_T \to \det(F_T) \otimes F_T^\vee \otimes L_T$ is the composition. As we checked in Proposition 2.4.3 that each fiber of the projectivization $\mathbb{P}(\mathcal{K}'_T) \subset \mathbb{P}(F_T)$ is a component of the fiber of a degenerate conic in the conic bundle $Q \to \mathbb{P}^2$, so we have $\mathbb{P}(\mathcal{K}'_T) \subset i_T^* p_1^* Q \subset \mathbb{P}(F_T)$. Since $\widetilde{\Delta} \to \Delta$ is the curve parametrizing the irreducible components of $Q|_\Delta \to \Delta$, it follows that $\mathbb{P}(\mathcal{K}'_T)$ over $\mathcal{D}_T$ defines a divisor $\widetilde{\mathcal{D}}_T \subset \widetilde{\Delta} \times T$ that maps to $\mathcal{D}_T$ via $\widetilde{\Delta} \times T \to \Delta \times T$. The divisor $\widetilde{\mathcal{D}}_T$ is a T-family of degree k divisors on $\widetilde{\Delta}$, so it defines a map $T \to \Delta^{(k)}$ which factors through $W_d|_{U_d}$ since $\mathcal{D}_T$ is induced from a map $T \to U_d$. It is easy to check that the assignment from $\mathcal{M}_T$ to $T \to W_d$ is functorial, hence we obtain a morphism over U_d:

$$\begin{array}{ccc} \mathfrak{M}_{d,e}|_{U_d} & \longrightarrow & W_d|_{U_d} \\ & \searrow \quad \swarrow & \\ & U_d & \end{array} \tag{2.4.8}$$

Since $\mathfrak{M}_{d,e}$ irreducible, the image of Φ is contained in one of the components W_d^i of W_d.

Proposition 2.4.6. *The morphism $\Phi|_C : \mathfrak{M}_{d,e}|_C = \Upsilon^{-1}(C) \to W_d^i|_C$ over $C \in U_d$ is a morphism of $G|_C$-torsors.*

Proof. Let $\sum(\lambda_i, p_i) \in G|_C$ and we write $M \in \mathfrak{M}_{d,e}|_C$ as $M = \psi_*(E_0 \otimes \widehat{L})$ by choosing a rank 2 bundle E_0 (recall that E_0 is determined up to a line bundle). We need to show that

$$\Phi\left(\sum(\lambda_i, p_i) \cdot M\right) = \Phi\left(\psi_*\left(E_0 \otimes \widehat{L} \otimes \mathcal{O}\left(\sum_i \frac{\lambda_i}{2} p_i\right) \otimes h_C^{-\frac{1}{2}\sum \lambda_i}\right)\right) = \left(\sum \lambda_i p_i\right) \cdot \Phi(M) \tag{2.4.9}$$

Since $\Phi(M)$ is determined at each point in $C \cap \Delta$, it suffices to check the equivariance property over p. As we checked that $\ker(\mathcal{B}_0|_p \to \mathcal{E}nd(\psi_* E_0 \otimes \widehat{L})|_p)$ always determines one of the two preimages of $p \in \Delta$ in the double cover $\widetilde{\Delta}$. To prove the proposition, it suffices to show that

$$\ker(\mathcal{B}_0|_p \to \mathcal{E}nd(\psi_* E_0 \otimes \widehat{L})|_p) \neq \ker(\mathcal{B}_0|_p \to \mathcal{E}nd(\psi_* E_0 \otimes \widehat{L} \otimes \mathcal{O}(p/2)|_p)$$

or equivalently,

$$\ker(\mathcal{B}_0|_p \to \mathcal{E}nd(\psi_* E_0 \otimes \widehat{L})|_p) \neq \ker(\mathcal{B}_0|_p \to \mathcal{E}nd(\psi_* E_0 \otimes \widehat{L} \otimes \mathcal{O}(-p/2)|_p). \tag{2.4.10}$$

In fact, we can simplify further by assuming $\widehat{L} = \mathcal{O}_{\widehat{C}}$.

The $\mathcal{B}_0$-module structure on $\psi_*(E_0 \otimes \widehat{L})$ can be described concretely by the composition of the isomorphism $\mathcal{B}_0 \cong \psi_* \mathcal{E}nd(E_0) \cong \psi_* \mathcal{E}nd(E_0 \otimes \widehat{L})$ and the natural morphism

$$\alpha : \psi_* \mathcal{E}nd(E_0 \otimes \widehat{L}) \to \mathcal{E}nd(\psi_*(E_0 \otimes \widehat{L})).$$

In particular, we can define

$$\alpha^0 : \psi_* \mathcal{E}nd(E_0) \xrightarrow{\sim} \psi_* \mathcal{E}nd\left(E_0 \otimes \mathcal{O}\left(-\frac{p}{2}\right)\right) \to \mathcal{E}nd\left(\psi_*\left(E_0 \otimes \mathcal{O}\left(-\frac{p}{2}\right)\right)\right)$$

$$\alpha^1 : \psi_* \mathcal{E}nd(E_0) \to \mathcal{E}nd\left(\psi_*(E_0)\right)$$

Hence, to check that (2.4.10) holds, it is equivalent to show that $\ker(\alpha^0|_p) \neq \ker(\alpha^1|_p)$.

To check these, we proceed as in Proposition 2.3.6 and work in an affine neighborhood $Z = \mathrm{Spec}(R)$ of p and the double cover $\widetilde{Z} = \mathrm{Spec}(R')$ where $R' := R[t]/(t^2 - s))$ and $div(s) = p$. So that the root stack restricted over Z is simply $\widehat{Z} = [\mathrm{Spec}(R[t]/(t^2 - s))/\mu_2]$. We can further reduce to the localization of R at p, we will again write the local ring as R and its unique maximal ideal $\mathfrak{m}$ which contains s.

As argued in Proposition 2.3.6, E_0 is a $\mathbb{Z}/2\mathbb{Z}$-graded R'-module $N = N_0 \oplus N_1$ and we can choose $e_1 \in N_0$ and $e_2 \in N_1$ such that $N \cong R'e_1 \oplus R'e_2$. In terms of the R'-basis $\{e_1, e_2\}$, $\psi_*(\mathcal{E}nd(E_0)) \cong (\mathcal{E}nd(E_0))_0$ consists of homogeneous R-module homomorphisms δ of degree 0:

$$\begin{aligned} e_1 &\mapsto u_0 e_1 + u_1 e_2 \\ e_2 &\mapsto v_1 e_1 + v_0 e_2 \end{aligned} \tag{2.4.11}$$

where $u_i, v_i \in (R')_i$. Then we can write $u_1 = t\widetilde{u}_1$ and $v_1 = t\widetilde{v}_1$ where $\widetilde{u}_1, \widetilde{v}_1 \in (R')_0 = R$.

As before, the module $\psi_* E_0$ is freely generated by $\{f_1 = e_1, f_2 = te_2\}$ as R-module. Suppose that $\delta \in \psi_* End(E_0)$ is of the form (2.4.11), then its image in $\mathcal{E}nd(\psi_* E_0)$ under α will be maps of the form

$$f_1 = e_1 \mapsto u_0 e_1 + \widetilde{u}_1(te_2) = u_0 f_1 + \widetilde{u}_1 f_2$$

$$f_2 = te_2 \mapsto \widetilde{v}_1 t(te_1) + v_0(te_2) = s\widetilde{v}_1 f_1 + v_0 f_2$$

If we choose the generators of $\psi_*\mathcal{E}nd(E_0)$ to be the following R-valued matrices (with respect to the basis $\{e_1, e_2\}$)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \tag{2.4.12}$$

their images in $\mathcal{E}nd(\psi_* E_0)$ are the corresponding R-matrices (with respect to the basis $\{f_1, f_2\}$):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & s \\ -1 & 0 \end{pmatrix} \tag{2.4.13}$$

As the homomorphism $\mathcal{O}\left(-\frac{p}{2}\right) \to \mathcal{O}_{\widehat{C}}$ is represented as the inclusion of R'-module homomorphism $R't \hookrightarrow R'$, the homomorphism $E_0 \otimes \mathcal{O}\left(-\frac{p}{2}\right) \to E_0$ corresponds to taking the R'-module homomorphism

$$R'(te_1) \oplus R'(te_2) \to R'(e_1) \oplus R'(e_2). \tag{2.4.14}$$

Note that $\psi_*\left(E_0 \otimes \mathcal{O}\left(-\frac{p}{2}\right)\right)$ corresponds to the R-module $R(sf_1) \oplus Rf_2$ as the μ_2-invariant submodule of $R'(te_1) \oplus R'(te_2)$. Pushing the homomorphism (2.4.14) forward $\psi_*\left(E_0 \otimes \mathcal{O}\left(-\frac{p}{2}\right)\right) \to \psi_* E_0$ corresponds to taking the μ_2-invariant part $R(sf_1) \oplus Rf_2 \to Rf_1 \oplus Rf_2$ which is represented by the R-valued matrix

$$\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the bases $\{sf_1, f_2\}$ of $\psi_* E_0 \otimes \mathcal{O}(-\frac{p}{2})$ and $\{f_1, f_2\}$ of $\psi_* E_0$.

For any $\delta \in \psi_*(\mathcal{E}nd(E_0))$, there are $\mathcal{O}_Z$-module homomorphisms $\alpha^0(\delta) : \psi_*(E_0 \otimes \mathcal{O}(-\frac{p}{2})) \to \psi_*(E_0 \otimes \mathcal{O}(-\frac{p}{2}))$ and $\alpha^1(\delta) : \psi_* E_0 \to \psi_* E_0$. They form a commutative diagram by the definition of a $\psi_*\mathcal{E}nd(E_0)$-module homomorphism

$$\begin{array}{ccc} \psi_*(E_0 \otimes \mathcal{O}(-\frac{p}{2})) & \xrightarrow{\alpha^0(\delta)} & \psi_*(E_0 \otimes \mathcal{O}(-\frac{p}{2})) \\ \downarrow & & \downarrow \\ \psi_* E_0 & \xrightarrow{\alpha^1(\delta)} & \psi_* E_0 \end{array}$$

In terms of the bases $\{f_1, f_2\}, \{sf_1, f_2\}$ of $\psi_* F$ and $\psi_*(F \otimes \mathcal{O}(-\frac{p}{2}))$, the morphism above can be written in

$$\begin{array}{ccc} R \oplus R & \xrightarrow{\alpha^0(\delta)} & R \oplus R \\ \Big\downarrow \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \\ R \oplus R & \xrightarrow{\alpha^1(\delta)} & R \oplus R \end{array}$$

When $\alpha^1(b) = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix}$, it is easy to check that $\alpha^0(b)$ must be $\begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}$. Similarly, we have the follwoing $\alpha^0(\delta)$ when $\alpha^1(\delta)$ is the other generator:

$$\alpha^1(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \alpha^0(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha^1(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \implies \alpha^0(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha^1(c) = \begin{pmatrix} 0 & s \\ -1 & 0 \end{pmatrix} \implies \alpha^0(c) = \begin{pmatrix} 0 & 1 \\ -s & 0 \end{pmatrix}$$

Finally, when $s = 0$ i.e. over p, we have

1. the kernel of $\alpha^1|_p$ is spanned by $(b + c)|_p$,

2. the kernel of $\alpha^0|_p$ is spanned by $(b - c)|_p$.

Hence, $\ker(\alpha^0|_p) \neq \ker(\alpha^1|_p)$ and we are done.

$\square$

Corollary 2.4.7. *For $d = 1, 2$, the moduli space $\mathfrak{M}_{d,e}$ is birational to one of the two components W_d^i of W_d. Moreover, if $\mathfrak{M}_{d,e}$ is birational to W_d^i, then $\mathfrak{M}_{d,e+1}$ is birational to W_d^{1-i}. In particular, the birational type of $\mathfrak{M}_{d,e}$ only depends on d and (e mod 2).*

Proof. Suppose $W_d^i|_{U_d}$ is the component corresponding to the image of Φ, so we have $\Phi : \mathfrak{M}_{d,e}|_{U_d} \to W_d^i|_{U_d}$. As $W_d^i|_{U_d}$ is smooth and hence normal, and by Proposition 2.4.6 the morphism Φ is bijective on closed points i.e. quasi-finite of degree 1, then the morphism Φ is an isomorphism.

Suppose $M = \psi_* \widehat{M}$ and $\Phi(M) = x_1 + ... + x_{dk} \in W_d^i$ where $x_i \in \widetilde{\Delta}$ and $k = \deg(\Delta)$. The computation in Theorem 2.3.8 shows that $\mathrm{ch}\left(\psi_*\left(\widehat{M} \otimes \mathcal{O}\left(\frac{p_i}{2}\right)\right)\right) = (0, 2d, e+1)$. By the proof of Proposition 2.4.6, we see that $\Phi\left(\psi_*\left(\widehat{M} \otimes \mathcal{O}\left(\frac{p_i}{2}\right)\right)\right) = x_1 + ... \sigma(x_i) + ... + x_{dk} \in W_d^{1-i}$. Hence, $\mathfrak{M}_{d,e+1}$ maps birationally to W_d^{1-i}. □

2.5 Cubic threefolds

We will apply the construction of the rational map $\Phi : \mathfrak{M}_{d,e} \dashrightarrow W_d$ for the conic bundles obtained by blowing up smooth cubic threefolds along a line. As a consequence, this yields an explicit correspondence between instanton bundles on cubic threefolds and twisted Higgs bundles on the discriminant curve.

Let $Y \subset \mathbb{P}^4$ be a cubic threefold and $l_0 \subset Y$ a general line. The blow-up $\sigma : \widetilde{Y} := Bl_{l_0} Y \to Y$ of Y along l_0 is known to be a conic bundle $\pi : \widetilde{Y} \to \mathbb{P}^2$. In this case, the rank 3 vector bundle is $F = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and the line bundle is $L = \mathcal{O}_{\mathbb{P}^2}(-1)$. The discriminant curve Δ of the conic bundle $\pi : \widetilde{Y} \to \mathbb{P}^2$ is a degree 5 curve and its étale double cover is denoted by $\widetilde{\Delta} \to \Delta$. Then we can consider the variety of divisors $W_2 \subset \widetilde{\Delta}^{(10)}$ lying over the linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|_\Delta|$, its two componets $W_2 = W_2^0 \cup W_2^1$ and the associated sheaf of even Clifford algebra $\mathcal{B}_0$, and the moduli space $\mathfrak{M}_{d,e}$ as considered in previous sections.

Proposition 2.5.1. *Let $e \in \mathbb{Z}$ be even. The image of $\Phi : \mathfrak{M}_{2,e} \dashrightarrow W_2$ is contained in the component W_2^1. In particular, $\mathfrak{M}_{2,e}$ is birational to the Prym variety.*

Proof. Note that in the case of Y, $\mathcal{O}_{\mathbb{P}^2}(2)|_\Delta \cong K_\Delta$, so can apply Example 2.2.5. Recall that in Example 2.2.5 the Abel-Jacobi map $\widetilde{\alpha} : \widetilde{\Delta}^{(10)} \to J^{10}\widetilde{\Delta}$ induces the morphism $\widetilde{\alpha}|_{W_2^1} : W_2^1 \to \mathrm{Pr}^1$ that maps birationally to the abelian variety Pr^1 and the morphism $\widetilde{\alpha}|_{W_2^0} : W_2^0 \to \mathrm{Pr}^0$ which is a generically $\mathbb{P}^1$-bundle over the theta divisor. By the work of [LMS15] (see Theorem 2.5.3 and Theorem 2.5.4), it is known that $\mathfrak{M}_{2,-4}$ is birational to another abelian variety, namely the intermediate Jacobian of the cubic threefold Y. In particular, the component of W_2 that is birational to $\mathfrak{M}_{2,-4}$ is birational to an abelian variety. But W_2^0 contains rational curves, which cannot happen for a variety birational to an abelian variety. Hence, the image of Φ must be contained in W_2^1. It follows immediately that the composition $\mathfrak{M}_{2,-4} \dashrightarrow W_2^1 \xrightarrow{\widetilde{\alpha}|_{W_2^1}} \mathrm{Pr}^1$ is a birational map. By Corollary 2.4.7, the same holds for $\mathfrak{M}_{2,e}$ when e is even.

□

Proposition 2.5.2. *The image of $\Phi : \mathfrak{M}_{2,e+1} \dashrightarrow W_2$ is contained in the component W_2^0 and its image in $\mathrm{Pr}^0 \cong Prym(\widetilde{\Delta}, \Delta)$ is an open subset of the theta divisor of the Prym variety.*

Proof. This follows immediately from Corollary 2.4.7, Proposition 2.5.1, and Example 2.2.5.

□

2.5.1 Instanton bundles on cubic threefolds and twisted Higgs bundles

A rank 2 vector bundle E on Y is called an instanton bundle if E is Gieseker semistable and $c_1(E) = 0$ and $c_2(E) = 2$.

Denote by $\mathfrak{M}_Y$ the moduli space of stable instanton bundles and $\overline{\mathfrak{M}}_Y$ its compactification by the moduli space of semistable instanton bundles. Now, the intermediate Jacobian $J(Y)$ of a cubic threefold Y has birationally a modular interpretation as the moduli space $\mathfrak{M}_Y$ of instanton bundles, via Serre's construction by the works of Markushevich, Tikhomirov, Iliev and Druel [MT98][IM00][Dru00][Bea02].

Theorem 2.5.3. *The compactification of $\mathfrak{M}_Y$ by the moduli space $\overline{\mathfrak{M}}_Y$ of rank 2 semistable sheaves with $c_1 = 0, c_2 = 2, c_3 = 0$ is isomorphic to the blow-up of $J(Y)$ along a translate of $-F(Y)$. Moreover, it induces an open immersion of $\mathfrak{M}_Y$ into $J(Y)$.*

We recall a theorem in [LMS15] relating instanton bundles and $\mathcal{B}_0$-modules. Recall that we can embed the Fano surface of lines $F(Y)$ in $J(Y)$ as $F(Y) \hookrightarrow Alb(F(Y)) \xrightarrow{\sim} J(Y)$ by picking l_0 as the base point. We denote by $\overline{F(Y)}$ the strict transform of $F(Y)$ under the blow-up in Theorem 2.5.3.

Theorem 2.5.4 ([LMS15])**.** *The moduli space $\mathfrak{M}_{2,-4}$ is isomorphic to the blow-up of $\overline{\mathfrak{M}}_Y$ along the strict transform $\overline{F(Y)}$ of $F(Y)$. In particular, $\mathfrak{M}_Y$ is birational to $\mathfrak{M}_{2,-4}$.*

For a stable instanton bundle $E \in \overline{\mathfrak{M}}_Y$, the image $\Xi_3(E) \in \mathfrak{M}_{2,-4}$ of E under the birational map in Theorem 2.5.4 is constructed explicitly as follows. First, we define the functor

$$\Psi : \mathrm{D}^b(\widetilde{Y}) \to \mathrm{D}^b(\mathbb{P}^2, \mathcal{B}_0), \quad E \mapsto \pi_*((E) \otimes_{\mathcal{O}_{\widetilde{Y}}} \mathcal{E} \otimes_{\mathcal{O}_{\widetilde{Y}}} \det F^\vee[1])$$

where $\mathcal{E}$ is a rank 2 vector bundle with a natural structure of flat left $\pi^*\mathcal{B}_0$-module. For details of the definition, we refer to [Kuz08]. Then $\Xi_3(E) = \Psi(\sigma^*(E))$. While $\Xi_3(E)$ is a priori a complex, it turns out that $\Xi_3(E)$ is concentrated in only one degree [LMS15, Lemma 3.9], so $\Xi_3(E)$ is indeed a $\mathcal{B}_0$-module.

On the other hand, recall that for an étale double cover $\widetilde{\Delta} \to \Delta$, there is an associated 2-torsion line bundle $\pi : \xi \to \Delta$ such that $\widetilde{\Delta}$ is recovered as the cyclic cover of ξ and the section $1 \in \xi$ i.e. $\widetilde{\Delta}$ is embedded in $\mathbf{Tot}(\xi)$ as the zero locus of $\pi^*s - 1$ where s is the tautological section of $\pi^*\xi$. Recall that a rank 2 traceless ξ-twisted *Higgs bundle* on a curve Σ is a pair (V, ϕ) consisting of a rank 2 vector bundle V and $\phi \in H^0(\Sigma, \mathcal{E}nd_0(V) \otimes \xi)$. Since we will only deal with this case, We simply call it a twisted Higgs bundle. The spectral correspondence [BNR89] says that pushing forward a line bundle N on $\widetilde{\Delta}$ gives a twisted Higgs bundle $(p_*N, p_*\lambda)$ on Δ, where λ is the tautological section of ξ. In fact, $\mathrm{Prym}(\widetilde{\Delta}, \Delta)$ parametrizes all twisted Higgs bundles on Δ with spectral curve the $\pi^*s - 1$. Since the Hitchin base $H^0(\Delta, \xi^{\otimes 2}) = H^0(\Delta, \mathcal{O}_\Delta) = \mathbb{C}$, all smooth spectral curves (defined away from $0 \in \mathbb{C}$) are isomorphic to each other.

Combining the functor Ξ_3 which induces a birational map $\mathfrak{M}_Y \dashrightarrow \mathfrak{M}_{2,-4}$, the birational map $\Phi : \mathfrak{M}_{2,-4} \dashrightarrow W^1_2$, the Abel-Jacobi map $\widetilde{\alpha} : \widetilde{\Delta} \to J^{10}\widetilde{\Delta}$ and the spectral correspondence, we obtain an explicit correspondence between instanton bundles on Y and ξ-twisted Higgs bundles on Δ:

$$
\begin{array}{c}
\boxed{\text{Instanton bundles on } Y} \\
\big\downarrow \Xi_3 \\
\boxed{\mathcal{B}_0\text{-modules on } \mathbb{P}^2} \\
\big\downarrow \widetilde{\alpha} \circ \Phi \\
\boxed{\text{Line bundles on } \widetilde{\Delta}} \\
\big\downarrow p_* \\
\boxed{\xi\text{-twisted Higgs bundles on } \Delta}
\end{array}
$$

The correspondence of different objects here holds as birational map between the corresponding moduli spaces.

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