Rethinking Standards-Textbook Alignment: How Elementary Math Textbooks Are Interpreting And Enacting The Common Core State Standards

Rowan Eliot Machalow
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Rethinking Standards-Textbook Alignment: How Elementary Math Textbooks Are Interpreting And Enacting The Common Core State Standards

Abstract
The Common Core State Standards for Mathematics (CCSSM) set ambitious goals for conceptual understanding through the content standards and developing mathematical habits of mind through the performance standards. Textbooks often serve as a mediator between standards and classroom instruction, as they expand a few short statements into a year of lessons, tasks, and educative supports that teachers use as a primary resource for both making sense of the standards and bringing them to life with students. Because of this critical role, understanding how curriculum developers have interpreted standards contextually and enacted those interpretations through developing textbook structures and content. I explore the concept of alignment between standards and textbooks and argue that many approaches to evaluating alignment are not sufficiently nuanced. Instead, I advocate for an approach that attends to both the holistic intentions and the details of the standards. My intention is to shift alignment conversations from asking if or how much a textbook is aligned to asking how and in what ways it is aligned.

I analyzed how the CCSSM content and practice standards were interpreted and enacted in multiplication lessons across eight curriculum programs for grades 3-5. In each analysis, I addressed both structural features of the standards and structural features of the textbooks that seemed to support or inhibit full enactment of the standards. I identified several structural features of standards that seemed to impact both the depth and frequency at which they were addressed across the curriculum programs. Addressing textbooks, I found that the content standards have largely been successful in designating the topics covered and increasing conceptual understanding, which is an important achievement for mathematics education in the United States. However, I also found that only four of the eight programs meaningfully addressed the practice standards and more rigorous application of the content standards due to several structural features of lesson design. Based on these findings and additional research, I identify three instructional models that have emerged in textbooks in response to the CCSSM.

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RETHINKING STANDARDS-TEXTBOOK ALIGNMENT: HOW ELEMENTARY MATH TEXTBOOKS ARE INTERPRETING AND ENACTING THE COMMON CORE STATE STANDARDS

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A DISSERTATION in Education
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Rethinking standards-textbook alignment: How Elementary Math Textbooks Are Interpreting and Enacting the Common Core State Standards

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ABSTRACT

RETHINKING STANDARDS-TEXTBOOK ALIGNMENT: HOW ELEMENTARY MATHEMATICS TEXTBOOKS ARE INTERPRETING AND ENACTING THE COMMON CORE STATE STANDARDS

Rowan Machalow
Janine Remillard

The Common Core State Standards for Mathematics (CCSSM) set ambitious goals for conceptual understanding through the content standards and developing mathematical habits of mind through the performance standards. Textbooks often serve as a mediator between standards and classroom instruction, as they expand a few short statements into a year of lessons, tasks, and educative supports that teachers use as a primary resource for both making sense of the standards and bringing them to life with students. Because of this critical role, understanding how curriculum developers have interpreted standards contextually and enacted those interpretations through developing textbook structures and content. I explore the concept of alignment between standards and textbooks and argue that many approaches to evaluating alignment are not sufficiently nuanced. Instead, I advocate for an approach that attends to both the holistic intentions and the details of the standards. My intention is to shift alignment conversations from asking if or how much a textbook is aligned to asking how and in what ways it is aligned.
I analyzed how the CCSSM content and practice standards were interpreted and enacted in multiplication lessons across eight curriculum programs for grades 3-5. In each analysis, I addressed both structural features of the standards and structural features of the textbooks that seemed to support or inhibit full enactment of the standards. I identified several structural features of standards that seemed to impact both the depth and frequency at which they were addressed across the curriculum programs. Addressing textbooks, I found that the content standards have largely been successful in designating the topics covered and increasing conceptual understanding, which is an important achievement for mathematics education in the United States. However, I also found that only four of the eight programs meaningfully addressed the practice standards and more rigorous application of the content standards due to several structural features of lesson design. Based on these findings and additional research, I identify three instructional models that have emerged in textbooks in response to the CCSSM.
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CHAPTER 1: INTRODUCTION

The Common Core State Standards for Mathematics (CCSSM, 2010) set an ambitious agenda for reforming mathematics education in the United States, addressing both content—what students should know and be able to do by the end of a grade—and practices—the habits of mathematical thinking that students should acquire. However, a gap of roughly 180 days of teaching and learning lies between the first day of school and meeting these goals. To fill this gap, teachers rely heavily on textbooks to determine both what is taught and how it is addressed, especially in the United States where mathematics teachers tend to follow textbooks very closely (Ball & Cohen, 1996; Hiebert & Grouws, 2007; Pepin et al., 2013; Remillard, Harris, et al., 2014; Valverde et al., 2002).

Because of this critical role, textbooks act as mediators between standards and daily teaching and learning, often serving as the primary resource through which teachers both understand the standards and bring them to life through lessons and tasks (Ball & Cohen, 1996; Hiebert & Grouws, 2007; Remillard & Heck, 2014; Valverde et al., 2002). While history suggests that revising textbooks to meet ambitious goals is insufficient to change teacher practice without a level of support that is often lacking in the U.S. (Schoenfeld, 2004; Willoughby, 2000), studies also show that without rigorous textbooks, teachers are unlikely to increase rigor and conceptual understanding on their own (Stein et al., 2000, 2007). As such, there have been calls for research on textbooks as a critical measure for understanding the influence of the CCSSM on U.S. mathematics education (Heck et al., 2011; Polikoff, 2015).

Several studies that attempted to explore alignment between the CCSSM and textbooks a few years after the standards were released found that publishers were often
labeling their old materials with CCSSM standards as a sales technique without making substantive changes (Cogan et al., 2015; Remillard & Reinke, 2017; Reys & Reys, 2006). However, a decade after the release of the CCSSM, curriculum developers have had a chance to catch up, and the ways that mathematics learning is being presented in textbooks is undergoing some complex and important changes in response to the CCSSM and the messages that surround it. In this dissertation, I explore conceptual, practical, and methodological questions surrounding alignment between broad standards and how they are interpreted and enacted in daily lessons in elementary textbooks. My analysis investigates the structure and content of both elementary mathematics textbooks and the CCSSM standards themselves, addressing both how effectively the standards are able to communicate their intentions to curriculum developers and how the pedagogical philosophies and lesson structures of textbooks support or inhibit their enactment of the standards.

My dissertation is structured in three related but independent papers that explore how textbooks interpret and enact the CCSSM content standards (chapter 2) and the practice standards (chapter 3). Building on the findings from these two chapters and additional research, I explore how three instructional models, including one that is new to the United States, have been developed or heavily revised in textbooks to respond to the CCSSM (chapter 4). Each chapter uses different conceptual and analytical frameworks to addresses the unique questions and attributes of that analysis, as well as addressing different aspects of several common themes.
I introduce here three themes that span the research: textbooks as mediators of standards, reconceptualizing what it means for a textbook to be “aligned” to the CCSSM, and the challenges and opportunities of standards reform through textbooks.

**Textbooks as Mediators of Standards**

In standards-driven reform, textbooks present both an opportunity and a potential challenge. Textbooks play a pivotal role in translating a few abstract lines describing goals that students must reach into a year of lessons, tasks, and explanations are intended to help students meet these goals. From this perspective, curriculum developers have great power in interpreting and enacting the standards for teachers and students across the country (Remillard & Heck, 2014; Valverde et al., 2002). Because of this critical role, textbooks may be thought of as mediators for standards, as they are often the primary mode through which teachers access, come to understand, and teach the standards (Remillard & Heck, 2014; Valverde et al., 2002).

In this dissertation, I explore how curriculum developers have interpreted standards, a sense-making process that draws on prior knowledge, expectations, and linguistic interactions, and then enacted the standards through developing course structures, lesson structures, lesson content, tasks, and explanations in textbooks. This process involves a complex interchange that depends on both the ability of the standards developers to explain their intentions in ways that are accessible to readers, and on curriculum developers to make sense of the standards in a larger world of goals, pedagogical philosophies, expectations, and beliefs (Hill, 2001; Houang & Schmidt, 2008; Remillard, 2005; Spillane, 2004; Stein et al., 2007; Valverde et al., 2002).
I suggest that there are several forces that may shape this interchange that may not be well understood or sufficiently recognized. First, the language of the standards is open to interpretation based upon the readers prior knowledge, beliefs, and expectations, so that educators, including curriculum developers, may interpret or reinterpret them differently (Hill, 2001; Spillane, 2004). Second, curriculum developers may follow different pedagogical philosophies or hold different beliefs about what and how students should learn that heavily influence the decisions they make about how the CCSSM are interpreted and enacted through their textbooks (Munter et al., 2015; Remillard & Reinke, 2017). Third, curriculum developers may choose to produce textbooks that are driven by market forces that encourage conformity with other textbooks and less innovative or challenging approaches to teaching (Cogan et al., 2015; Remillard & Reinke, 2017; Reys & Reys, 2006). And fourth, curriculum developers’ perceptions of teachers’ beliefs and capacity to take on more challenging aspects of teaching and learning may shape their decisions about which aspects of the CCSSM to focus on, minimize, or ignore completely (Munter et al., 2015; Remillard & Reinke, 2017).

While this dissertation focuses on the CCSSM, I suggest that these issues are inherent to the process of translating any set of standards into curriculum resources, and many of the findings in this dissertation are widely applicable to other subject areas and national contexts. At the same time, there are some unique features of the CCSSM and the ways that mathematics education is conceptualized and enacted in the United States that heightened the way these forces are realized.
Reconceptualizing Alignment

Prior studies that explore alignment between standards and textbooks, seem to take one of two approaches. The first is a purely quantitative measure of coverage, either addressing the percentage of lessons in a textbook that addresses the standards or the percentage of standards that are addressed (Center for the Study of Curriculum, 2014). This provides a direct match between lessons and textbooks, but without attention to quality or depth and often suggests full coverage of a standard when only some components of it have been addressed. The second type either compares standards and lessons indirectly by matching them both to an outside lists of topics (Newton & Kasten, 2013; Polikoff, 2015; Valverde et al., 2002) or uses nebulous rubrics that rely on evaluators’ overall sense of alignment based on a range of qualities (EdReports, 2020a). This approach addresses rigor and depth but does not directly match up each standard to where it is present in the textbook. Both of these approaches leave considerable space for components, intentions, and nuances of the standards, if not entire standards, to be lost.

In this study, I reconceptualize what it means for a textbook to be “aligned” to standard and suggest an approach that puts the standards and the textbooks in close communication with each other. Rather than asking if or how much textbooks are aligned to the CCSSM, I instead ask how and in what ways textbooks are aligned to the CCSSM. My analysis presents a new approach by addressing both the spirit and the details of the CCSSM and looking for trends across multiple curriculum programs. Through this analysis, I address how both the structure and content of the standards and the structure and content of textbooks impacts trends in standards interpretation and enactment.
Key Shifts of the CCSSM for U.S. Mathematics Education

The CCSS initiative outlines three key shifts for education, which are valuable for understanding its intentions and potential impact: focus (addressing only the content listed rather than extraneous material), coherence (intertwining related content and following a learning progression when introducing new ideas), and rigor. Rigor consists of balanced attention to three smaller components, conceptual understanding (knowing why mathematical relationships and strategy work), procedural skills and fluency (developing automaticity through conceptual understanding), and application (utilizing conceptual understanding and fluency to solve rich and novel problems) (CCSS Authors, 2010; McCallum, 2012).

The CCSSM address these ambitious goals by outlining what students should know (through the content standards) and what mathematical habits of mind they should develop (through the practice standards). The content standards put a strong emphasis on conceptual understanding, expecting students to know both how to use strategies and why the strategies work. They specify a number of visual representations (borrowed from other countries, such as Singapore) and alternative algorithms that support students in developing mathematics concepts over time, as well as delaying memorization and standard algorithms (which obscure concepts) until conceptual understanding has been established through other strategies and models. While several prior mathematics education reform efforts have made similar suggestions, the CCSSM provide a clear grade-by-grade progression with a narrow range of topics to be covered more deeply in each grade. If the content standards are taken up by textbooks, it will represent a significant change for the United States, where the most popular textbooks have
historically focused on rote repetition of standard algorithms without understanding and often covered the same low rigor content year after year (Hiebert et al., 2005; Hill, Ball, et al., 2008; Ma, 2010; Remillard & Reinke, 2017; Schoenfeld, 2004).

The standards for mathematical practice (SMPs) are eight cross-grade standards that address the habits of mind possessed by mathematicians. These include addressing complex and messy situations by creating simpler models, searching for patterns and generalizing them, approaching novel problems by considering possible strategies and then assessing the fit of the strategy, and communicating mathematical ideas and critiquing the ideas of others. To develop these habits of mind, students need opportunities to wrestle with novel problems and complex situations, as well as having generative conversations with teachers and peers (Carpenter et al., 1989; Fennema et al., 1996; Stein et al., 2008). This style of learning, and the types of tasks that support it, although promoted in the 1990s, have also been infrequently realized in U.S. classrooms and school systems, which typically expect teachers to present strategies that students reproduce in highly similar tasks (Schoenfeld, 2004; Stigler & Hiebert, 1999; Willoughby, 2000).

**Opportunities and Challenges of the Standards-Based Reform through Textbooks**

The CCSSM are only the most recent wave of reform efforts that aim to make these shifts, and understanding this history is valuable to situate the current CCSS initiative. Several past reform efforts have positioned curriculum materials as change agents for developing the same types of knowledge, skills, and habits of mind expressed in the CCSSM, but were unable to substantially change teachers’ classroom practices (Ball & Cohen, 1996; Schoenfeld, 2004; Willoughby, 2000). Teachers who were
comfortable with didactic, rote models were expected to take on the challenging skills of facilitating group work and discussions, honing a sense of when to allow students to explore and when to guide them toward meaningful conclusions, and developing a deep conceptual grasp of new mathematics concepts and how to teach them to students (Sleep, 2012; Stein et al., 2008).

Textbooks were often expected to provide the necessary support for this complex set of knowledge and skills through educative features like sample discussions and sidebar explanations (Brown et al., 2009; Davis & Krajcik, 2005; Lloyd, 1999; Remillard & Bryans, 2004). Without additional support in the form of professional development and coaching, many teachers took up the superficial aspects of the reforms or were alienated, overwhelmed, or in disagreement over philosophical beliefs (Schoenfeld, 2004; Willoughby, 2000). In addition, many textbook publishers simply did not adapt their materials to address rigorous standards, and only a few smaller grant-funded organizations developed reform-oriented textbooks with limited market share (Schoenfeld, 2004; Willoughby, 2000).

While these reform efforts are not entirely new, the CCSSM differ from past attempts in several ways that increase the likelihood of them being incorporated meaningfully into both textbooks and teaching. Unlike the National Council for Teacher of Mathematics (NCTM, 1989, 2000) standards that suggested vague content goals across in three-year grade bands, the CCSSM content standards provide a concise list of what students should know and be able to do in each grade. Unlike the state standards movement that resulted in textbooks addressing a range of conflicting standards with curriculum that was “a mile wide and an inch deep,” the CCSSM have been adopted by
the majority of states allowing curriculum developers to focus on a single set of goals (McCallum, 2012; Remillard & Reinke, 2017; Schmidt et al., 1997). And unlike state standards that ranged dramatically in quality, consistency, and rigor, the CCSSM follow learning progressions based on other mathematically-successful countries that build up a small number of core concepts over several years with a focus on conceptual understanding (Carmichael et al., 2010; McCallum, 2012). And finally, there are significant financial and assessment incentives for schools to purchase textbooks that are aligned to the CCSS, which in turn motivate curriculum publishers to meet that need (No Child Left Behind Act of 2001, 2002).

**Methods and Sample Selection**

All of these complexities suggest that the process of mediating the standards through developing textbooks is inherently messy and requires curriculum developers to make many strategic decisions about interpreting and enacting the standards that go beyond their literal text. In this study, I first explored trends in how textbooks tended to interpret and enact the standards, and then identified factors—in the structure of the standards and the structure of textbooks—that correlated with those trends.

To approach this work, I selected one content area—multiplication—as a focus. Multiplication features heavily in the “major work” designated by the CCSS authors to cover a considerable portion upper elementary lessons (CCSS Authors, 2013; Porter et al., 2011b; Student Achievement Partners, 2010), it is addressed in multiple standards that cover several strands of the CCSS, and it includes strategies and models that were new to many U.S. educators with the arrival of the CCSS. For these reasons, I suggest
that this area of focus is representative of both a wider set of topics and grades in both textbooks and standards.

For my sample, I selected lessons from eight elementary mathematics curriculum programs that express a range of pedagogical philosophies and lesson structures. This allowed me to look for trends both within programs and across multiple programs. With this broad sample, I could look for trends in both the construction of textbooks and trends in how standards were being addressed across multiple textbooks.

**Overview of the Chapters**

I divided this work into three analyses, each of which is presented as a separate paper that can be read independently. The first paper addresses how the CCSSM content standards have been interpreted and enacted by elementary mathematics textbooks; the second paper presents a similar analysis for the CCSSM practice standards; and the third one builds on these two analyses and additional research to explore several new instructional models that have emerged in textbooks in response to the CCSSM. Each paper is briefly reviewed here.

*Rethinking Standards-Textbook Alignment: How Elementary Mathematics Textbooks Interpret the CCSSM Content Standards.*

In this chapter I first provide a conceptual framework that positions textbooks as mediators of standards and describes standards interpretation as a meaning-making and contextual process. In introduce three Key Shifts intended by the CCSS (CCSS Authors, 2010) as a lens for understanding how textbooks are addressing the overall goals of the CCSS initiative through their enactments. I then review two common approaches to evaluating standards in prior studies and introduce a new approach for exploring
alignment between standards and textbooks as an analytical model. I apply this analytical model to nine grade 3 multiplication standards from the CCSSM as they are interpreted and enacted in a total of 271 multiplication lessons from eight elementary textbooks.

In my findings, I first describe how the textbooks are meeting overall goals of the CCSS initiative in light of the Key Shifts. I then identify three structural factors of lesson design that seem to inhibit full enactment of the CCSSM standards, and three structural factors of standards that seem to support or inhibit their implementation across the eight textbooks. I explore each of these through two case studies of 3.OA.8 and 3.OA.5, to illustrate consistent and inconsistent enactment, as well as a short section on how examples are used across the standards. In the discussion I provide suggestions for standards developers, curriculum developers, and educators based upon these findings.

*Taking Matters into their Own Hands: How Elementary Mathematics Textbooks Interpret and Enact the CCSSM Standards for Mathematical Practice.*

In this chapter, I first share research supporting mathematical habits of mind, such as those described in the CCSSM Standards for Mathematical Practice (SMPs), as teachable skills. Through this lens, I describe some of the challenges and opportunities in developing mathematics textbooks that could support students in acquiring and practicing these skills, as well as the extremely limited past research in how the SMPs are being enacted in textbooks. I develop a theoretical framework addressing the process of interpreting and enacting SMPs. Arising from this framework and an iterative process with my analysis, I outline five tenets for analyzing how SMPs are enacted in textbooks. Using this framework, I closely analyze lessons from each of grades 3, 4, and 5 for each
of the eight textbooks (120 total lessons + 5 extras to address gaps) to understand how they interpret and enact the SMPs.

I present my findings in two sections. First, I address broad trends in how the SMPs are interpreted and enacted in textbooks at four levels: how SMPs are tagged (identified) in textbooks, how tasks are structured to position students as generators or receivers of knowledge, how SMPs are interpreted with attention to the holistic meanings or only the component parts, and how revisions to the standards titles to save space may be impacting ways they are enacted. I then illustrate these trends with three case studies: MP3 explores the nature of student interactions, MP4 addresses the challenges that arise when terms in the standards hold multiple meanings, and MP8 considers a situation where examples in the standards are used in place of generalized goal statements. In the discussion I provide suggestions for standards developers, curriculum developers, and educators based upon these findings.

*Continuity and Change: How Elementary Math Textbooks are Responding to the CCSSM.*

The two papers above revealed several trends in how the overall structures and pedagogical philosophies of textbooks relate to the ways developers interpret and enact the content and practice standards. In this chapter, I argue that historic distinctions between “traditional” and “reform” instructional models no longer describe the ways that textbooks are approaching instruction in response to the CCSSM. To do this, I first present a framework that grows from several other sources and considers how curriculum developers answer three questions through their textbooks:

- What is mathematics? (Skemp, 1976; Hiebert & Lefevre, 1986)
What is rigorous mathematics? (Hiebert & Grouws, 2007; Stein, Correnti, et al., 2016)

How should mathematics be taught and learned? (Munter, Stein, & Smith, 2015)

I use the responses to these questions, as demonstrated in all of the lessons from the prior two studies and additional analysis, to identify three new instructional models. Building on and expanding the work of Munter, Stein, and Smith (2015) I further describe the Direct and Dialogic models from their research, and then identify a third model which I label Guided Pathway. I describe how each of the models answers the questions above and discuss implications for teaching and learning through each of these models.

Across these three papers, I make the claim that the process of interpreting and enacting standards through textbooks is complex and nuanced, and that evaluations of alignment should be alert to these nuances. Based upon my findings with a wide set of sample curriculum programs, I suggest that the CCSSM content standards have been successful in introducing aspects of conceptual understanding to a broad range of U.S. textbooks (at least for multiplication in grade 3), but that deep enactment of the content standards, and any meaningful enactment of the practice standards, can only be found in some curriculum programs. I distinguish between three curriculum models, and suggest that the direct textbooks in this study largely address the content standards but are easier to teach, dialogic textbooks deeply address both the content and practice standards but may be difficult to teach, and guided pathway textbooks address the majority of the
content and practice standards in a modified way that may be easier for teachers while still providing many aspects of rigorous learning.
CHAPTER 2: RETHINKING STANDARDS-TEXTBOOK ALIGNMENT: HOW ELEMENTARY MATHEMATICS TEXTBOOKS INTERPRET THE CCSSM CONTENT STANDARDS

Abstract

Textbooks play a pivotal role in translating the Common Core State Standards for Mathematics (CCSSM) from a handful of brief statements into a full year of robust lessons, tasks, explanations, and educative supports for teachers. They are a primary resource for both teachers and students in making sense of and utilizing the standards. For this reason, a critical question to policy and education is “are textbooks aligned to the CCSSM?” This chapter responds by first questioning what alignment means and offering an approach to alignment that attends to both the spirit and the details of the standards, a technique that has not been used before in other major analyses. Using this approach, I analyze the alignment of eight grade 3 mathematics textbooks to the CCSSM for multiplication. Looking across the eight textbooks, I identify three factors of standards design that impact how successfully they are taken up by textbooks and three factors of textbook lesson design that impacts how successfully the standards are enacted. These findings have implications for standards developers, curriculum developers, and educators who purchase and utilize textbooks.

Introduction

Mathematics curriculum and standards in the United States have historically been accused of being “a mile wide and an inch deep,” with each grade repeating the same superficial content in a race to cover multiple topics without addressing deeper understanding for any of them (Hiebert et al., 2005; McCallum, 2012; Schmidt et al.,
1997; Stigler & Hiebert, 1999). Following the example of many successful countries, the Common Core State Standards (CCSSM, 2010) are designed to following a learning trajectory to deepen conceptual understanding of just a few topics within each grade using a range of algorithms and representations, and then build on those topics across grades (CCSS, 2012; Heritage, 2008; McCallum, 2012; Schmidt & Houang, 2012). These new standards represent an ambitious reform agenda for the United States that requires both significantly paring down content and introducing teachers and students to a range of new representations and algorithms that support conceptual understanding of core mathematical ideas and operations (McCallum, 2012; Remillard & Reinke, 2017).

This study explores conceptual, practical, and methodological questions surrounding alignment between broad standards and the day-to-day, unit-to-unit interpretation and enactment of these goals in curriculum materials. For this analysis, I examine the treatment of multiplication in eight grade 3 textbooks (a term which I use to also include teacher’s guides). While multiplication is used as an exemplar, my analytical methods address how language from standards is used, communicated, and interpreted by textbooks in a broader sense that can be applied to other grades, standards, and even content areas.

The CCSSM are comprised primarily of a set of mathematics content standards, which provide a concise list of what students should know and be able to do by end of each grade. (They also include a set of standards for mathematical practice, which I analyze separately in chapter 3.) The full range of concepts and skills that grade 3 students are expected to acquire in multiplication is encapsulated in nine standards of 2-5 sentences each. The work of expanding these nine brief statements into roughly 30-60
days of learning falls largely to curriculum developers, who interpret and enact the standards through developing textbooks with a year’s worth of pacing guides, lesson plans, tasks, activities, and educative supports for teachers (Ball & Cohen, 1996; Remillard & Heck, 2014; Remillard & Kim, 2020).

Because of this necessary expansion from brief, outcome-oriented statements to lessons and tasks that guide learning, textbooks serve as a mediator between standards and the actual teaching that happens in schools (Remillard & Heck, 2014; Valverde et al., 2002). They both reflect the ideas and intentions of the mathematics education community and transmit goals and philosophies to teachers who tend to follow textbooks closely for determining both what to teach and how to teach it, especially in the United States (Ball & Cohen, 1996; Houang & Schmidt, 2008; Polikoff, 2015; Remillard, 2005; Stein et al., 2007; Valverde et al., 2002). Because of this critical role in interpreting and enacting the standards, research on textbooks provides valuable insights for understanding the influence of the CCSSM on mathematics education in the United States (Heck et al., 2011; Polikoff, 2015).

After the Common Core State Standards (CCSS, 2010) were introduced, a wave of research followed to investigate alignment of mathematics textbooks to the new standards. These studies found that old curriculum programs were being disingenuously labeled with CCSSM standards to suggest alignment without any meaningful changes to content (Cogan et al., 2015; Polikoff, 2015). However, as curriculum developers have had more time to completely overhaul their existing programs or develop new ones, a number of mathematics programs have been found to be more closely aligned to the CCSSM in terms of addressing each of the standards at some level and not including
outside content (EdReports, 2020a). This finding represents a major success in the application of the CCSSM at the ground level, but also calls into question how alignment is being assessed.

While some reports show that standards-textbook alignment is on the rise, the answer to the question, “are recent elementary mathematics textbooks aligned to the CCSSM content standards?” is still “it depends.” It depends on how alignment is defined. It depends on the standard and how it is constructed linguistically and mathematically. It also depends on what textbooks are being explored and what pedagogical philosophies they support. Rather than answering this question with a simple yes or no, my aim is to understand alignment from a more nuanced perspective that explores how the CCSSM content standards for mathematics are being interpreted and enacted by textbooks from several perspectives.

I begin by asking “what does alignment mean?” I review several approaches to evaluating alignment between standards and textbooks and offer an approach that attends to both the spirit and the letter of the standards.

I then ask, “which standards are embraced, ignored, misinterpreted, or incompletely utilized?” and follow this by asking “what are the structural and mathematical characteristics of standards that appear to influence how they are interpreted and enacted?” That is, rather than evaluating the success of individual textbooks in aligning to some aspects of each of the standards, I explore how individual standards are taken up by textbooks. To do this, I identify patterns in standards implementation that span the eight elementary mathematics textbooks. From these
findings, I identify three factors in how standards are written that seem to support or inhibit full enactment by textbooks.

At the same time, I look for trends within curriculum programs to ask, “what lesson or task structures in textbooks support or inhibit full enactment of the standards?” By identifying places where curriculum developers state that they are enacting a standard and then analyzing what students are asked to do to meet that standard, I identify four features of textbooks that seem to inhibit full enactment of the CCSSM content standards.

While this study focuses on multiplication in grade 3, my findings have implications that extend beyond this multiplication and even beyond mathematics. For standards developers and policy makers, this research offers insights into ways to organize and phrase standards that makes them more likely to be taken up by textbooks. For curriculum developers, a more nuanced understanding of standards implementation may lead to more comprehensive coverage. For researchers and textbook evaluators, this methodology may provide an alternative approach to evaluating alignment that attends closely to both the letter and the spirit of the standards. And for educators and textbook purchasers, these findings may provide either resource for identifying types of standards that are less likely to be covered and should therefore be checked and supplemented, or a tool for identifying textbooks that cover the standards more fully when purchasing decisions are made.

*Research Questions*

This study explores conceptual, practical, and methodological considerations around alignment between content standards and the interpretation and enactment of
these standards through daily lessons in textbooks. This exploration is guided by the following research questions:

- **RQ1:** How are individual CCSSM content standards interpreted and enacted by curriculum developers across curriculum programs?
- **RQ2:** What characteristics of CCSSM content standards make them more or less likely to be interpreted or utilized in certain ways by curriculum developers?
- **RQ3:** What characteristics of textbooks support or impede holistic enactment of the CCSSM content standards?

**Conceptualizing Textbooks as Mediators of Standards**

Textbooks, especially in mathematics, have been characterized as a central mediator between policies and standards documents (the official curriculum) and what occurs in classrooms (the enacted curriculum) (Remillard, 2018b; Remillard & Heck, 2014; Stein et al., 2007; Valverde et al., 2002). In this analysis, I use the term *enactment* to describe how curriculum developers bring the CCSSM to life and communicate their interpretations through overall instructional approaches and lesson structures, the content of lessons and tasks, and educative features that describe their interpretations explicitly.

The majority of mathematics teachers, around the world and in the United States, rely heavily on textbooks as a teaching tool for determining topics, emphasis, lesson structures, tasks, and, to a lesser extent, philosophical approaches to learning (Houang & Schmidt, 2008; Polikoff, 2015; Remillard, 2005; Stein et al., 2007; Stigler & Hiebert, 1999; Valverde et al., 2002). Further, textbooks are one of the major resources through which teachers experience standards, as they both bring the standards to life and heavily influence how students and teachers will understand and enact them (Ball & Cohen,
In many practical ways, the ways that textbooks present standards may stand in for the standards themselves from teachers’ and students’ perspectives. This gives curriculum developers great power in determining or guiding how standards are translated into action in classrooms.

Because textbooks have such an influential position, they have often been envisioned and used as agents for educational reform. Since the 1960s, governmental and private grants have funded universities and small nonprofits in developing textbooks that support conceptual understanding, discovery, and discussion in mathematics classrooms; however, teachers struggled to use them due to insufficient support, lack of mathematical understanding or pedagogical skills, and/or incompatible beliefs (Fey & Graeber, 2003; Payne, 2003; Schoenfeld, 2004; Senk & Thompson, 2003; Stein et al., 2008; Willoughby, 2000). Though some of these reform-oriented programs have been available since the 1990s (including two in this study), they never gained substantial market share (Blazer et al., 2019; Opfer et al., 2018; Schoenfeld, 2004; Willoughby, 2000). Meanwhile, major textbook publishers (Pearson, McGraw Hill, and Houghton Mifflin Harcourt) have typically avoided reform-oriented approaches and developed textbooks with more marketable and easier-to-teach models of instruction that continue to dominate the market (Blazer et al., 2019; Opfer et al., 2018; Remillard & Reinke, 2017; Reys et al., 2004; Reys & Reys, 2006; Stigler & Hiebert, 1999; Willoughby, 2000). This article explores the role of CCSSM content standards in the development of four textbooks developed by major publishers and four from grant-funded organizations or universities.
Standards Interpretation as a Meaning-Making and Contextual Process

Standards may be thought of as messages that communicate a set of ideas from the developers to the people who use them. However, effective communication is possible only when the language is interpreted in the similar ways by the standards developers and those who read and use them (Hill, 2001; Otte, 1986; Vygotsky, 1978). Effective communication may disintegrate in several ways: First, the standards developers may not have articulated their ideas with sufficient clarity for readers to make sense of them due to linguistic choices, the developers’ schemata, and the developers’ assumptions about the readers’ schemata. This potential for unclear writing suggests that it is important to attend to what the author may have intended (and might articulate in other documents), but also to the static text that must be interpreted as it has been written and that takes on a life of its own once it leaves the writer’s hands (Otte, 1986). Second, readers’ personal backgrounds, beliefs, and prior knowledge may lead them to make sense of standards in ways that fit their expectations of what the standards should or might contain (Hill, 2001; Otte, 1986; Spillane, 2004; Vygotsky, 1978). Third, the CCSSM mathematics standards are written at the linguistic intersection of mathematical, pedagogical, and everyday language, where terms may have multiple connotations and denotations that are each defensible (Hill, 2001; Spillane, 2004). And fourth, those who utilize the standards to produce educational materials or experiences may have their own agendas, beliefs about mathematics education, or even marketing objectives that cause them to intentionally re-interpret the text of the standards to meet their own larger goals (Hill, 2001; Reys & Reys, 2006; Spillane, 2004). That is, interpreting standards is not a neutral act.
This process is illustrated in several studies. Hill (2001) found that when state standards used language with specific mathematical and pedagogical meanings, teachers and school/district administrators often reinterpreted them with locally-defined or conventional meanings for these terms, causing the intentions and details of the standards to become watered down. Similarly, Spillane (2004) found that district administrators often intentionally or unintentionally misinterpreted the messages of content standards based on their backgrounds and beliefs, and these misinterpretations were then passed along to teachers through a variety of local policies and materials. Like district administrators, curriculum developers yield a tremendous amount of power in interpreting standards.

While interpretation is an internal process, it also occurs at the community level, as individuals generate new linguistic and symbolic tools to wrestle with the ideas and demonstrate their understandings (Vygotsky, 1978). These symbolic tools are created with a range of levels of formality, audience and scope, participation, and perceived authority. For example, interpretations of standards might be collaboratively generated or communicated in staff meetings, informal conversations in the hall, YouTube videos, blogs, academic articles, articles and publications aimed at teachers, professional development sessions, and so on. Especially in a digital world, individuals can build their interpretations based on others’ interpretations, so that instead of starting each time with the original text of the standards individuals may read them in light of another interpretation, or even skip the original text of the standards completely.

Textbooks are likely to both absorb and transmit common interpretations of the standards within the mathematics education community. When curriculum developers
develop interpretations, they may borrow directly from each other, other resources on the internet, or organizational publications; take up interpretations that become widespread or ubiquitous across many resources; or modify and iterate on interpretations from any of these sources. Textbooks are then released into the educational community where the interpretations of the curriculum developers become authoritative and pervasive, as one of the primary tools that teachers utilize to understand the standards.

**Structure and Goals of the CCSSM Content Standards**

To understand the actual impacts of the CCSSM on textbooks, it is first valuable to understand their intended impacts. The CCSS Authors (2010) identify three ambitious shifts in how the CCSSM differ from previous U.S. mathematics teaching and standards. These shifts provide a useful anchor in understanding how textbooks have interpreted and enacted the content standards to attend holistically to the intentions of the CCSS authors.

I describe these shifts by drawing on the Key Shifts (CCSS Authors, 2010) document itself, as well as supporting documents like the Progressions of the CCSSM (The Common Core Standards Writing Team, 2011) resources, and the CCSSM themselves. I illustrate the shifts using the exemplar topic for this analysis, grade 3 multiplication. I chose this topic because it contains a number of strategies and representations that were not commonly used in the U.S. prior to the CCSSM, it draws on several content strands (numbers in base ten, operations and algebraic thinking, and measurement and data), it is central content for grade 3, and it is small enough to be focused but large enough to provide a meaningful representation of how standards are used. For these reasons, I believe that trends that may be seen in grade 3 multiplication are likely to be representative of wider trends.
**Shift 1: Focus**

The first shift addresses focus in two ways. First, educators are expected to address much less content in each grade, trading breadth for depth, by excluding any content that is not written explicitly in the content standards (CCSS Authors, 2010, 2013). Second, each grade has a list of the “major work” that is intended to be the focus of 65-85% of the year (CCSS Authors, 2013; Student Achievement Partners, 2010). In grade 3, there are seven major clusters, as well as four supporting or additional clusters that should be covered in the remaining 15-35% of the year. Of the major clusters, four of them address multiplication, making this one of the most important goals for the year.

One of the compelling reasons for choosing multiplication for this study is that it draws on several strands of the content standards and makes connections across these strands, providing a broad perspective on the CCSSM while maintaining focus. The majority of the grade 3 multiplication standards are in the Operations and Algebraic Thinking (OA) strand, which covers both the concepts that underly multiplication and several extensions such as solving two-step problems with variables in 3.OA.8 (Figure 2.1, left) and explaining arithmetic patterns with properties of operations in 3.OA.9 (Figure 2.1, right top). Multiplication also extends into the Numbers in Base Ten (NBT) strand as students prepare for multi-digit multiplication in fourth grade by using the associative property to multiply single digits by multiples of ten (e.g., $6 \times 80 = 6 \times (8 \times 10) = (6 \times 8) \times 10 = 48 \times 10 = 480$) in 3.NBT.3. In the Measurement and Data (MD) strand, students are expected to extend multiplication concepts and properties of operations from array models to finding areas of rectangles and combined rectangles in 3.MD.7c and 3.MD.7d (Figure 2.1, right bottom).
Figure 2.1

Clockwise from left: Example of a two-step problem with variables that is aligned to standard 3.OA.8. Example of using properties of operations to explain a pattern in 3.OA.9. Example of using multiples of 2 and 5 to find other products through the distributive property with area models in 3.MD.7c. Images from The Common Core Standards Writing Team (2011, pp. 26–28).

Shift 2: Coherence

The CCSSM content standards place a heavy emphasis on having students build an understanding of operations and ideas over multiple grades in a progression which may also be called a learning trajectory (CCSS Authors, 2013; Daro et al., 2011; McCallum, 2012; Remillard & Reinke, 2017). The learning trajectories in the CCSSM support the development of operations, like multiplication, with a range of representations and alternative algorithms that slowly build understanding of not just how to multiply but what multiplication means and why it works. Students’ strategies grow in sophistication both within each grade and across grades. For example, in multiplication of whole numbers, students work with equal groups in grade 1, arrays and skip counting in
grade 2, single-digit multiplication using visual representations and a range of strategies in grade 3, multiplication of one digit by up to four digits and two digits by two digits using place value and area models in grade 4, and finally learn the standard algorithm in grade 5. A few of the representations and algorithms in this progression are shown in Figure 2.2.

**Figure 2.2**

*Learning progression of some common multiplication representations and strategies from grades 1 to 5.*

<table>
<thead>
<tr>
<th>Equal Groups</th>
<th>Skip Counting</th>
<th>Arrays</th>
<th>Area Models</th>
<th>Distributive Property</th>
<th>Place Value</th>
<th>Standard Algorithm</th>
</tr>
</thead>
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<tr>
<td><strong>2, 4, 6</strong></td>
<td>2, 4, 6</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>12 × 3 = 36</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>12 × 3 = 36</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>12 × 3 = 36</td>
</tr>
</tbody>
</table>

**Shift 3: Rigor**

The CCSS Authors (2010) define rigor as an equal emphasis on conceptual understanding, procedural skills and fluency, and application. Conceptual understanding refers to knowing how strategies work and make connections between strategies using concepts like place value and properties of operations (CCSS, 2010). The CCSS publishers’ criteria (CCSS Authors, 2013, p. 10) explain that, “materials [should] amply feature high-quality conceptual problems and questions that can serve as fertile conversation starters in a classroom if students are unable to answer them” and go on to describe both shorter questions and prompts and extended tasks where students identify
correspondences between representations that are featured in the content standards. The representations and algorithms shown in figure 2.2 above demonstrate a range of ways of understanding the concepts that underly multiplication, when combined with explicit explanations and connections between the strategies. This makes a significant change from pre-CCSS approaches which often use rote memorization to teach multiplication facts in grade 3 and the standard algorithm in grade 4.

Procedural skills and fluency refer to the ability to calculate accurately and rapidly, including memorization of facts. By the end of grade 3, students are expected to have memorized all of single digit multiplication facts, however, the content standards are explicit that this fluency should be attained through first getting to know multiplicative relationships through a range of strategies, representations, and properties of operations, interweaving conceptual understanding with fluency.

Applications refer to using mathematical knowledge to solve problems in complex situations by building on both conceptual understanding and procedural skills and fluency. These situations may be both real world and purely mathematical, but they require students to applying known concepts in new ways through novel tasks as well as practicing with routine word problems. The CCSS publishers’ criteria (CCSS Authors, 2013, p. 11) explain that curriculum materials should “include an ample number of single-step and multistep contextual problems that… engage students in problem solving… in which students must make their own assumptions or simplifications in order to model a situation mathematically.” They further explain that “applications in the materials [should] draw only on content knowledge and skills specified in the content standards,” but also that “rich applications cannot always be shoehorned into the
mathematical topic of the day.” Details of how the application aspect of rigor should play out are predominantly covered by the CCSSM standards for mathematical practice (which are addressed in chapter 3), though it appears to some degree in content standards that address patterns and multi-steps problems.

As shown here, the shifts that the CCSS authors outline are substantial for both teachers and students. While many of today’s teachers may remember learning grade 3 multiplication through memorization and timed tests of decontextualized facts(Nanna, 2016), the CCSSM content standards require students to understand multiplication, represent it with multiple visual models, work flexibly with numbers and operations, apply multiplication in complex situations, and build up toward a meaningful understanding of multi-digit multiplication through incremental steps.

Approaches to Assessing Alignment: How Alignment is Understood and Evaluated

One of the major questions asked of a textbook in standards-based environment is “is it aligned to the CCSSM?” To answer that, it is first necessary to investigate what the term alignment means. The definition of this term may take on a range of meanings and practices depending upon the user’s needs and goals. Curriculum developers are likely to want to prove alignment to make their books more marketable. Independent evaluators and educators may wish to distinguish between textbooks that have higher and lower levels of alignment for making informed purchasing decisions. And researchers may wish to explore textbook alignment to gain insights into the impact of the CCSSM on curriculum, teaching, and learning in the United States.

To explore approaches to evaluating alignment, I build on the work of on Brown, Pitvorec, Ditto, and Kelso (2009) who identify four models of exploring alignment
between textbooks lessons and teacher enactment. After reviewing three other models, they recommend an “integrity” model that addresses both adherence to the content and instructional design of a lesson and enactment that supports the pedagogical orientation of the textbook.

In a similar way, I identify four approaches for studying alignment between the CCSSM and textbooks that have been used in past analyses and advocate for an integrity approach. I provide a description, benefits and drawbacks, and examples of these approaches with their findings, as shown in Table 2.1 and discussed below.
<table>
<thead>
<tr>
<th>Description</th>
<th>Checklist Approach</th>
<th>Outside List Approach</th>
<th>Intentions Approach</th>
<th>Integrity Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard is marked as present when at least part of it is present in a lesson</td>
<td>The textbook and the standards are compared to an outside list of mathematical topics, and then the overlap between the two is assessed.</td>
<td>Evaluators assess whether the textbook takes on the pedagogical characteristics of the standards, without making a systematic comparison between them.</td>
<td>Standards are interpreted as complex constructs with both literal requirements and pedagogical intentions, both of which must be met to support alignment.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Benefits</th>
<th>Checklist Approach</th>
<th>Outside List Approach</th>
<th>Intentions Approach</th>
<th>Integrity Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Alignment can be assessed relatively quickly and easily</td>
<td>• Valuable for broad cross-cultural textbook comparisons using a standard tool</td>
<td>• Encourages deeper and intentions-based interpretations of the standards</td>
<td>• Encourages deeper and intentions-based interpretations of the standards</td>
<td></td>
</tr>
<tr>
<td>• Relatively consistent and clear-cut</td>
<td></td>
<td></td>
<td>• Identifies when some aspects of a standard are not met</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Relatively consistent; issues of interpretation can be addressed directly</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Drawbacks</th>
<th>Checklist Approach</th>
<th>Outside List Approach</th>
<th>Intentions Approach</th>
<th>Integrity Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Does not recognize when some aspects of a standard are not met.</td>
<td>• Does not recognize when some aspects of a standard are not met.</td>
<td>• Does not recognize when some aspects of a standard are not met.</td>
<td>• Relatively slower and more intensive for reviewers</td>
<td></td>
</tr>
<tr>
<td>• Encourages superficial interpretations of the standards.</td>
<td>• Does not compare textbook content to the actual language of the CCSSM</td>
<td>• Lacks consistency in comparing textbook content to the actual language of the CCSSM</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Relatively slower</td>
<td>• Evaluations may be inconsistent and subjective</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Examples</th>
<th>Checklist Approach</th>
<th>Outside List Approach</th>
<th>Intentions Approach</th>
<th>Integrity Approach</th>
</tr>
</thead>
</table>
The Checklist Approach

The checklist approach identifies alignment as a binary: either the standard is met within a lesson or it is not. If there is evidence that at least a part of a standard is met in at least a part of a lesson, this is sufficient to mark it “true” in a true/false dichotomy. The checklist approach has often been used by textbook companies providing “cross-walk” documents to prove alignment to potential customers, and it often does not attend to details such as the quality or quantity of lessons that are classified as being aligned to each standard (CCSS Authors, 2013; Cogan et al., 2015; Remillard & Reinke, 2017; Reys & Reys, 2006). Thus, it may “miss the forest for the trees.” However, it is relatively fast, allowing researchers to cover a great deal of content very quickly, and it can provide a snapshot of analysis in a big-picture view.

The Textbook Navigator/Journal, a project led by William Schmidt at the University of Michigan, took a “checklist” approach toward studying alignment in 34 textbook series and 185 individual textbooks, including many that were developed prior to the release of the CCSS (Center for the Study of Curriculum, 2014; Cogan et al., 2015). Reviewers read the major focus of each lesson and then identified each CCSSM standard that was addressed by the lesson in some way using methodology developed for the original Third International Mathematics and Science Study in 1995 (TIMSS) (see Valverde et al. (2002) for more information) and adapted for the CCSSM. If only a portion of a standards was met by an individual lesson, this was still tagged as meeting the standard.

The Textbook Navigator/Journal analysis focused on the number of lessons that met each of the standards in some way, which were checked against the percentage of the
total lessons that should be dedicated to the major work of each grade, as designated by
the CCSSM. They found that the number of standards for each grade were covered by the
corresponding textbooks in the study ranged from 42% to 98%, with textbooks developed
after the CCSSM were released averaging higher alignment. They point out that when a
quarter of the lessons in a textbook are not aligned to any content standards of the
CCSSM, this equates to 8-13 weeks of learning spent on extraneous materials. As the
research was conducted in the first two or three years after the CCSSM was released
when many textbooks could not even meet a “checklist” level of alignment for many
standards, this big-picture view provides a measure of early alignment.

As suggested here, the checklist approach provides a solid and quick overview of
some superficial aspects of alignment. However, because the standards are evaluated as
all-or-nothing, this approach does not address quality, depth, or completeness of coverage
of a standard.

**The Outside List Approach**

The outside list approach begins with a list of possible mathematical topics, such
as the set of all possible mathematical topics covered across multiple textbooks in
multiple countries from grades K-12 (Newton & Kasten, 2013; Polikoff, 2015; Porter et
al., 2011a; Valverde et al., 2002). Reviewers then use this list to understand the topic
coverage of both standards and textbooks, without ever directly comparing the two
resources to each other.

Polikoff’s (2015) research uses the Survey of Enacted Curriculum (SEC)
methodology (Porter et al., 2011a), which was created as a stand-alone set of content and
rigor goals that predates the CCSS and can be used to make comparisons across
international textbooks and assessments. Polikoff used the SEC system of first mapping each CCSSM standard to a set of pre-existing short phrases (e.g., place value, number comparisons) paired with a level of cognitive demand (memorize, demonstrate understanding, etc.) to create codes such as “number theory (e.g., base-ten and non-base ten systems) X perform procedures.” Because this system was developed independently of the CCSSM, the evaluator statements do not address that, for example, non-based ten systems are not included anywhere in the CCSSM. Polikoff then assigned each whole-class or individual task across three CCSSM-aligned grade 4 textbooks with these paired topic and cognitive demand codes, without referring back to the text of the original standards. He also assumed that the CCSSM intended for each standard to have equal time assigned to it (p. 1201), an assumption that is in direct conflict with CCSS documents (CCSS Authors, 2013) indicating that the “major work” for each grade should take up the majority of the time. (Cobb and Jackson (2011) provide further critiques of the SEC approach.) While this definition of alignment moves so far away from comparing the actual text of the standards to textbook content that they are never examined simultaneously, the methodology was helpful in identifying that by 2014 or 2015, textbooks covered a high percentage of the topics of the standards (90-95%) but the alignment dropped significantly (to 76-84%) when cognitive demand was taken into account. Thus, although this “outside list” approach provides little input on the alignment of textbooks to the literal text of the standards, it indicates that if an important objective of the CCSSM initiative is to increase cognitive demand, that goal is not being met. Additionally, it does not provide much information on is how closely the textbook content is aligned to the text of the standards.
While the outside list approach provides a valuable tool for comparing topic coverage across cultures, it is at best a blunt instrument for aligning textbooks with standards. Using the SEC tool takes the reviewer far from the original text of the CCSSM, providing only general outlines of topics that may or may not be covered, without the level of meticulous detail that went into the authorship of the CCSSM.

**The Intentions Approach**

The intentions approach goes in the opposite direction and uses a nebulous sense of meeting the standards overall, without necessarily making a systematic checklist of the standards and matching them to places where they are covered in a textbook. This sense may be accentuated with examples or spot-checks, but does not feature any direct and consistent comparisons.

The “K–8 Publishers’ Criteria for the Common Core State Standards for Mathematics” (CCSS Authors, 2013) provides a set of guiding principles for evaluating textbook alignment to the CCSSM. The document advocates strongly against teaching isolated parts of standards out of context from the larger fabric of whole standards, clusters of standards, and the grade level progressions. It uses vivid metaphors like burning, tearing, breaking, and stripping trees of their branches to describe isolating a component of a standard, either for instruction or assessment, and warns that this risks a checklist mentality that removes opportunities for deep or extended learning. Instead, these guidelines focus on larger principles like the time spent on the designated “major work” of each grade and determining whether content is interwoven and structured to support the larger organization of the year’s work.
While this suggestion addresses the pedagogical importance of connecting mathematics concepts to each other, as an alignment approach it leaves space for many gaps. This approach may focus so much on the forest that it misses the trees. (Or, in the metaphors of the Publisher’s Criteria, attend so much to the health of the trees that the branches and twigs are overlooked.)

EdReports.org used the Publisher’s Criteria to develop a tool that is currently being used to evaluate each new or revised mathematics curriculum program that enters the market (EdReports, 2019b, 2019a). As of February 2019, there are over 30 reports available, though several of these are different editions of the same program. Using a rubric that mirrors the CCSS Publisher’s Criteria (CCSS Authors, 2013) in language, organization, and detail, evaluators are instructed to report on whether (1b) most of the year is spent on the major work of the grade, (1c) the supporting content enhances the major work, (1e) the content follows the grade-by-grade progression of the standards, and (1f) the content is shaped by the cluster headings. Alignment to the full range of the content standards is only addressed with the generalized statement “Note: ALL standards in CCSSM are accounted for in evidence gathering between indicators 1b, 1c, 1e, and 1f” (EdReports, 2019b, p. 4). The reports provide a few examples of individual tasks that the evaluators feel demonstrate alignment or lack of alignment to a handful of standards, but only to serve an evaluation of the broad indicator. No comprehensive list of alignment to individual standards is provided, nor is information provided on their website as to how this comparison is made or how evaluators are expected to assess this, meaning that different evaluators might interpret the “note” about the “full range of the content standards” differently. (Further critiques of the EdReports methodology can be found in
Overall, EdReports focuses on how the standards are used in conjunction with each other and paced over the year, a holistic approach that mostly attends to and provides information on the spirit of CCSSM alignment rather than the letter.

While the intentions approach provides some important insights about the quality of standards alignment, but it does not attend closely enough to the text of the original standards to provide clear and consistent information on content that may not be addressed adequately or at all. In the case of EdReports, reviewing coverage of all of the aspects of each standard is likely to be inconsistent because there is no tool for performing this type of evaluation.

**The Integrity Approach**

The integrity approach is an effort to fill gaps from the checklist, outside list, and intentions approaches by both noting when a statement is addressed in any form (quantitative) and evaluating the quality of the tasks and approaches used to teach it (qualitative). In this approach I propose that there is value in attending to standards at three levels: 1) sub-components of each standard that might not be noticed or met if they are not identified separately, 2) the goals of each standard as a whole, and 3) the quality, level of rigor, or consideration of specific actions within the standard. I also use this as the analytical framework in this study.

While the “Publishers’ Criteria” (CCSS Authors, 2013) warns strongly against fragmenting the standards in developing curriculum, I argue that it is not necessarily sound advice for how alignment should be measured because it allows sub-components of standards to be overlooked. With the knowledge that many standards have multiple
components that might not all be met in the same lesson or ever, there are compelling reasons to analyze standards and their interpretation with a small grain size.

Meyer (2015) utilized an “integrity” approach to analyze a single CCSSM practice standard, MP4 Model with Mathematics, across two high school textbooks. He began by closely reading the text of the standard to identify five actions described by it. He then identified characteristics of the action that must be present for it to be enacted in a curriculum program, and then used those to analyze each of the lessons that claimed to be aligned to the standard. For example, part of MP4 states that mathematically proficient students should be able to “identify important quantities in a practical situation and map their relationships” which he labeled as “identifying essential variables in a situation” and described as “deciding what information matters to a given task and also what does not matter.” He then identified seven tasks (out of 87 that claimed MP4 alignment) from the textbooks that required students to decide what information would be needed by intentionally withholding critical information. Overall, he found that the textbooks frequently provided opportunities to complete the two more computationally straightforward actions from MP4 (performing operations using models and interpreting the results of operations), but rarely gave them the chance to engage in the true work of modeling: identify variables, formulate models, or validate conclusions. By breaking apart the standard into smaller, discrete actions and thoroughly investigating each of them, Meyer suggests that while some components of the standard may be present in a curriculum program, it is necessary to address each of the actions of the standard with frequency and rigor to demonstrate integrity of alignment.
In developing the integrity approach, the other approaches and their findings each make contributions. Polikoff’s (2015) work suggests that textbooks might address the content of the standards, but not the rigor or quality. Meyer’s (2015) work suggests that textbooks may address only some sub-components of a standard rather than all of them, and further that the sub-components that are selected are likely to be the least challenging for students and teachers. The EdReports approach, which is based on the CCSS Publishers’ Criteria (CCSS Authors, 2013), suggest that full alignment to the standards involves looking at how they are met across a curriculum and in connection with each other, as well as assigning the rough percentage of total lessons that should be devoted to each topic.

One of the challenges of the integrity approach is that assessing the quality of alignment is different from assessing overall task quality or rigor. For example, Stein, Smith, and colleagues developed a tool for evaluating tasks at four different levels of rigor and looked for the presence of tasks at the two highest levels (doing mathematics and procedures with connections) (Margaret S. Smith & Stein, 1998; Stein et al., 2000; Stein & Smith, 1998). However, Polikoff’s approach, in which standards were assigned to five different types of rigor suggests that some standards set expectations for memorization while others set expectations for demonstrating understanding. By attending closely to the parts of the standard as well as the holistic meaning, the integrity approach would also encompass the different levels of task rigor inherent in the original standard.

Overall, the integrity approach suggests that to serve a greater vision of understanding whether standards are being implemented with the full complexity and
rigor at which they are written, it is important to assess their disparate parts, their holistic intentions, and the quality and rigor at which they are being implemented over the year. Although this method takes longer, it also offers a more precise and consistent approach to reviewing alignment, which I will use in this analysis.

Methods

In this study I use the major shifts of the CCSS and an integrity approach to analyze how the CCSSM multiplication standards for grade 3 are interpreted and enacted in textbooks. This section describes how I selected, prepared, and analyzed standards, how I identified and analyzed multiplication lessons in textbooks, and how I used trends across standards and across textbooks to identify factors that seemed to support and inhibit full enactment of the CCSSM content standards.

CCSSM Multiplication Standards Selection and Preparation

To prepare the standards, I first identified each of the standards that had multiplication concepts as their central focus. I included all grade three standards that addressed multiplication only, standards that used multiplication to understand area, and standards that applied multiplication in context (e.g., two-step problems). I did not include standards that were predominantly about division or about measurement in standard and metric units (which use multiplication only peripherally). This process resulted in 10 standards, one of which contained 4 smaller standards within it.

The integrity approach suggests that each part of a standard is important, and that the whole standard cannot be met if some of the parts are not met. To support this part of the analysis, I divided each standard whenever a phrase indicated that a different action, skill, representation, tool, etc. was to be used. I call each of these parts a statement. For
example, I decomposed standard 3.MD.7b into four statements as indicated by the numbers in parentheses: “(1) Multiply side lengths to find areas of rectangles with whole-number side lengths (2) in the context of solving real world (3) and mathematical problems, (4) and represent whole-number products as rectangular areas in mathematical reasoning.” I then referred to the statements by these numbers (e.g., 3.MD.7b-1, 3.MD.7b-2).

A full list of the standards, with their original text and separated statement, is available in Table 2.8 on p. 55.

**Textbook and Lesson Selection**

Eight elementary mathematics curriculum programs were included in this analysis. Programs were eligible for analysis if they were either fully developed after the CCSSM were released or underwent a significant revision to align with the CCSSM. One program, *Bridges in Mathematics*, provided only a limited number of units for review (providing 14 out of roughly 43 multiplication lessons), however I determined that these units were sufficient to provide meaningful information about standards alignment and therefore included them. For the remaining programs, the full text of the teachers’ guide and student materials were available. I was unable to include several other programs that would have met these criteria because I was unable to obtain review copies from the publishers.

In selecting lessons, my goal was to understand how the textbooks developed whole number multiplication concepts and skills over the course of the year. I used the table of contents, CCSSM alignment documents, lesson titles, and (in a few cases where it was not clear from the lesson titles) a brief look at lesson content to identify all lessons
where multiplication was the main focus of the lesson. I did not include chapter reviews, assessments, lessons which only included multiplication as part of a mixed review or warm-up, or lessons that peripherally utilized multiplication but focused primarily on other content. I also did not include lessons that focused primarily on division, measurement, or area/volume unless the name of the lesson or the way that it was described by the textbook authors indicated that the lesson was intended to deepen understanding of multiplication.

Of the eight programs, four of them were developed by commercial publishers and four were developed by other organizations using private or governmental funding (and then sometimes distributed by commercial publishers). While the focus of this analysis is on how CCSSM content standards are being interpreted across the full curriculum space, knowing the origin and format of the individual books was helpful in making sense of the results and often correlated with other important distinctions between the programs. Within my sample, all of the commercially developed programs were written following a single lesson format in which the teacher or text modeled a strategy that students repeated, while the grant-funded programs used a range of lesson formats that often involved students developing their own strategies. This is not to imply that all commercially made or grant-funded textbooks follow these outlines, only that these distinctions were visible in my sample and relevant for analysis; see chapter 4 for further discussion of lesson formats. To clarify the origins of each program throughout this chapter, I identify them with either a C for commercial or an F for funded. For example, Go Math! is abbreviated GO-C to indicate its commercial origins. The names, abbreviations, and number of eligible lessons for each program are listed in table 2.2.
Table 2.2

Programs sampled with abbreviations, developers, publishers, publication years, and number of multiplication lessons

<table>
<thead>
<tr>
<th>Program</th>
<th>Abbr.</th>
<th>Developer/Publisher</th>
<th>Year</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridges in Mathematics*</td>
<td>BRI-F</td>
<td>The Math Learning Center/Curriculum Associates</td>
<td>2015</td>
<td>14*</td>
</tr>
<tr>
<td>Eureka Math</td>
<td>EUR-F</td>
<td>Great Minds</td>
<td>2013</td>
<td>58</td>
</tr>
<tr>
<td>Everyday Mathematics 4</td>
<td>EVER-F</td>
<td>University of Chicago School Mathematics Project/McGraw Hill</td>
<td>2015</td>
<td>37</td>
</tr>
<tr>
<td>Investigations in Number, Data, and Space 3</td>
<td>INV-F</td>
<td>TERC/Pearson</td>
<td>2017</td>
<td>33</td>
</tr>
<tr>
<td>Go Math!</td>
<td>GO-C</td>
<td>Houghton Mifflin Harcourt</td>
<td>2015</td>
<td>27</td>
</tr>
<tr>
<td>HMH Into Math</td>
<td>INTO-C</td>
<td>Houghton Mifflin Harcourt</td>
<td>2020</td>
<td>28</td>
</tr>
<tr>
<td>My Math</td>
<td>MY-C</td>
<td>McGraw Hill</td>
<td>2018</td>
<td>30</td>
</tr>
</tbody>
</table>

* The Bridges in Mathematics analysis is based upon a limited number of units made available by the publisher in a sample. It does not represent the full program content, which contains roughly 43 multiplication-focused lessons.

Identifying the Major Work

While identifying multiplication lessons, I additionally used the unit and lesson titles in each Table of Contents to count the number of regular lessons (not review days, assessments, optional projects, etc.) that addressed a) multiplication and division including area in relation to multiplication, b) other major work: fractions (3.NF.A), measurement of time, liquid volume, and mass (3.MD.A), and c) lessons that addressed supporting or additional clusters.

From these, I was able to determine the percentage of total lessons over the year that addressed just multiplication and division (which make up five of the seven major
clusters) as well as the total percentage of time spent on major work, which is expected to be between 65% and 85%. This analysis supports a brief evaluation of the Focus shift.

**Analytical Methods**

This analysis explores how textbooks are interpreting the CCSSM grade 3 content standards for multiplication by asking a series of successive questions that support the integrity approach:

1) Where do textbooks claim that they are addressing the standards?
2) Which parts (statements) of the standards are textbooks addressing?
3) What is the level of quality and rigor in the way those statements are being addressed?
4) How is each standard addressed holistically across the year?

**Identifying where and how standards are applied through tagging**

All eight textbooks indicate the CCSSM content standards that they claim to cover in each lesson. I refer to these indicators as *tags* and view them as a form of communication between curriculum developers and teachers. For example, MY-C G3 1.1.5 lists the full text of the primary standard, 3.OA.3, and the numbers of supplementary standards at the top of the student textbook page (Figure 2.3, left) and at the top of the teacher’s guide to the lesson (Figure 2.3, right). EUR-F is the only program that does not tag all of the standards for each lesson, but instead tags standards for each cluster of 2-6 lessons that address the same topic, even though some lessons within each cluster may not address all of the standards.
I used the tags as an indicator of where the curriculum developers explicitly stated that they were addressing the CCSSM content standards to both identify and analyze data. In each lesson, I explored how the curriculum developers interpreted the tagged standards, as indicated by the explanations, tasks, and instructions to teachers and students. I looked for applications of the tagged standards in the main text of the teacher’s edition of each lesson, including both the student text and the instructions to the teacher about what should be discussed, modeled, or elicited. I did not include optional activities, differentiation options, homework, or suggestions.

As I was interested in how curriculum developers were interpreting the standards, I only analyzed the tagged standards in each lesson. For example, in Figure 2.3, the curriculum developers indicated that they were utilizing 3.OA.3, 3.OA.1, and 3.OA.5, so I analyzed the lesson in light of these standards. When I noticed that a lesson addressed standards that were not tagged, I typically made a note of this, but did not perform a full analysis of the other standards, on the grounds that curriculum developers did not intend to or acknowledge that they were addressing them.
Taking an integrity approach to standards alignment: Evaluating parts and wholes

For each lesson, I created a row in a Microsoft Excel spreadsheet with the following information: lesson name, lesson title, my own brief summary of the lesson (mostly for reference later), the tagged standards, and the standards that I saw evidence of the lesson addressing with numbered sub-categories. I also identified which representations students used and whether they were created by students or only used by the teacher or the text, and then noted the lesson quality using a set of emergent categories that are discussed in the following section. In some lessons, I added comments about interesting features in a Notes column. Table 2.3 shows the analysis of lesson ENV-C G3 1.3.

Table 2.3

Sample row from summary table used to analyze alignment between lessons and standards statements.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Title</th>
<th>Description</th>
<th>Standards: Tagged</th>
<th>Standards: Actual</th>
<th>Reps./ student generated?</th>
<th>Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>Arrays and properties</td>
<td>Find all the ways to arrange 12 into arrays; list in a table. Introduce arrays and the commutative property.</td>
<td>3.OA.1, 3.OA.3, 3.OA.5</td>
<td>3.OA.1-1, 3.OA.3-1, 3.OA.5-2</td>
<td>Arrays/yes</td>
<td>Sufficient</td>
</tr>
</tbody>
</table>

In doing this analysis, it is important to recognize that, like the textbook authors, I was also interpreting the standards. While I attempted to interpret them as literally as possible, with attention to the language of each statement and the whole standard as well as my own knowledge of language meanings within the mathematics education.
community, many of the standards were still ambiguous. I made note of these ambiguities, how they were interpreted by textbook authors, and how those choices compared with other possible or likely interpretations. Following the integrity approach, I looked for both stringent and loose matches to the written standards, but took the perspective that all text in the standards was included for a reason, and therefore should be enacted in textbooks over the course of the year.

When I finished analyzing the lessons of each curriculum program, I wrote a bulleted memo (usually around one page single-spaced) of interesting features relating to how the program interpreted the standards to capture qualitative observances that crossed multiple lessons. I also made notes about the sequence in which standards were addressed, how they were combined together, and how lessons were organized over time to understand how textbook authors were addressing focus, coherence, and rigor. From this foundation, I was able to identify factors of both the textbooks and the standards themselves that supported and inhibited full enactment.

**Quality and quantity of standards enactment**

The integrity approach suggests attending to the parts (statements) of the standard, its holistic meaning, the quality of enactment, and its placement within the course. While checking for the presence or absence of each statement in a standard is relatively straightforward, determining whether a textbook shows alignment to a statement with adequate quality, rigor, and attention to the “work” of the standard is more complex. There is no common metric for assessing these qualities, and any metric devised would be, at best, a defensible line in the sand. In addition, different statements have different levels of rigor and different expectations for student actions.
I began exploring the alignment between statements and lessons by looking for matches between the lesson content and the written instructions of each statement. However, as I took notes on ways in which some lessons failed to meet the tagged standards while others met them with great depth and clarity, several trends emerged. I codified these recognizable patterns into emergent codes with recognizable characteristics and the re-coded earlier lessons using the defined quality levels. Table 2.4 describes the four levels of alignment: robust, sufficient, insufficient, and absent, as well as three characteristic ways in which textbooks typically fell short of meeting the expectations of the standards: observing without doing, infrequent, and superficial. Each of these is expanded with examples in the Findings section.

I found that when any one characteristic of insufficient coverage was present, this served to prevent the entire standard from being enacted by the curriculum program. For example, a standard could be covered with deep attention to content, student ownership, and focused learning—but if it only happened in one lesson the impact of this standard was limited for the program. Similarly, a standard might be used frequently by students in a focused manner—but if only superficial aspects of the standard are ever addressed, this is still a minimal usage.
Table 2.4

Quality levels of lesson alignment with tagged standards, including three features that identify insufficient alignment. Colors are used in later analysis.

<table>
<thead>
<tr>
<th>Quality</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robust</td>
<td>Particularly strong and effective approach to the type of rigor (conceptual understanding, application, or fluency) that is the target of the standard, using the definitions in Shift 3: Rigor (see pp. 28-30).</td>
</tr>
<tr>
<td>Sufficient</td>
<td>Students complete the expectations of the standard actively (rather than by observing), conceptual understandings are made clear (either by the teacher, textbook, or students), and students have a way of demonstrating that they have gained the knowledge or skills described. Must appear in at least three lessons (or it will be coded for infrequency).</td>
</tr>
<tr>
<td>Insufficient</td>
<td>Standard is modeled by the teacher or textbook, or students fill in a few blanks of a process with heavy scaffolding, but students are not given the opportunity to perform the expectations of the standard themselves.</td>
</tr>
<tr>
<td>Infrequent</td>
<td>Standard is present only in 1-3 lessons, providing insufficient opportunities for understanding and practice. The number of lessons is indicated in each cell.</td>
</tr>
<tr>
<td>Superficial</td>
<td>Only superficial aspects of the mathematical concepts or skills that are described in a standard are enacted.</td>
</tr>
<tr>
<td>Absent</td>
<td>The standard is not present. (Note that the standard may be labeled as present, but no evidence is found of meeting the standard.)</td>
</tr>
</tbody>
</table>

Table 2.5 shows an example row from the master table that addresses the second statement from 3.OA.8. The full text of the statement is shown along with the color-coded cells indicating the quality of alignment. For example, BRI-F, EVER-F, and INV-F addressed this statement robustly, EUR-F addressed it sufficiently and MY-C did not address it at all. ENV-C, GO-C, and INTO-C each addressed it insufficiently for two reasons, as shown by the split cells: an infrequent number of lessons (3 lessons in ENV-C, 1 lesson in GO-C, and 2 lessons in INTO-C) and addressing only superficial aspects of the standard.
Table 2.5

Sample row of the master alignment table showing how a statement from 3.OA.8 is evaluated for quality in each of the eight curriculum programs.

<table>
<thead>
<tr>
<th>Standard</th>
<th>Standard Text</th>
<th>BRI-F</th>
<th>EUR-F</th>
<th>EVER-F</th>
<th>INV-F</th>
<th>ENV-C</th>
<th>GO-C</th>
<th>INTO-C</th>
<th>MV-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.OA.8-2</td>
<td>Represent these [two-step] problems using equations with a letter standing for the unknown quantity.</td>
<td>✅</td>
<td>✅</td>
<td>✅</td>
<td>3</td>
<td>✅</td>
<td>1</td>
<td>2</td>
<td>✅</td>
</tr>
</tbody>
</table>

Factors Influencing Standard Coverage in Curriculum Programs

After categorizing the quality of each standard statement across the master table, I was able to look at patterns that ran across the eight textbooks. Some statements were consistently addressed in sufficient and robust ways, some were often addressed insufficiently following different characteristics, some were often omitted, and others, like 3.OA.8-2 in Table 2.5, tended to elicit a mixed response.

I suggest that these patterns are not accidental, but rather that there is something about the standards themselves—either their content or the way that they are written—that influences how they are interpreted and enacted by textbooks. Looking across the statements, I identified three factors that seem to influence the uptake of standards by curriculum developers, which I discuss in the Findings section.

Findings

In this section I address each of the major shifts—focus, coherence, and rigor—as a lens for understanding how the CCSSM content standards for multiplication have been taken up by textbooks across grade 3. I address focus and coherence at the level of units...
and lessons, and then address rigor at the level of statements within the content standards following the integrity approach.

**Focus**

As the CCSSM were intended to narrow the U.S. math curriculum to cover only the content of the standards at any grade level, the first question of this analysis asks whether textbooks have followed this advice. Focus is defined as spending 65-85% of the time on the major work of grade and not including content beyond the standards (CCSS Authors, 2010, 2013; Student Achievement Partners, 2010). Across the board, textbooks wavered around the bottom edge of the 65% cut off in terms of the percentage of lessons that addressed major work, with some dipping slightly below, as shown in Table 2.6.

While there are no expectations about the amount of time dedicated to each topic in the major work, in grade 3 there are seven topics designated as major work and four of those address multiplication, so spending a bit more than half of the time for major work on multiplication might be a reasonable expectation.

**Table 2.6**

*Number and percent of lessons addressing multiplication and division and overall major work. Bold indicates values under the suggested minimum.*

<table>
<thead>
<tr>
<th>Multiplication and Division Units</th>
<th>Total Units</th>
<th>Percent of Lessons Addressing Multiplication and Division</th>
<th>Percent of Lessons Addressing Major Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>BRI-F</td>
<td>3</td>
<td>8</td>
<td><strong>28% (of modules)</strong></td>
</tr>
<tr>
<td>EUR-F</td>
<td>2</td>
<td>7</td>
<td><strong>28%</strong></td>
</tr>
<tr>
<td>EVER-F</td>
<td>7 (partial units)</td>
<td>9</td>
<td>43%</td>
</tr>
<tr>
<td>INV-F</td>
<td>3</td>
<td>8</td>
<td>39%</td>
</tr>
<tr>
<td>ENV-C</td>
<td>8</td>
<td>16</td>
<td>44%</td>
</tr>
<tr>
<td>GO-C</td>
<td>5</td>
<td>12</td>
<td>43%</td>
</tr>
<tr>
<td>INTO-C</td>
<td>6</td>
<td>20</td>
<td>44%</td>
</tr>
<tr>
<td>MY-C</td>
<td>6</td>
<td>14</td>
<td>43%</td>
</tr>
</tbody>
</table>
Two programs did not reach the 65% cut-off for lessons addressing major work, BRI-F and INV-F. Both of these programs spent a disproportionate amount of time on addition and subtraction, which is identified as an “additional cluster” because it is supposed to be covered thoroughly in grade 2 and only addressed lightly in grade 3 for fluency. In addition, BRI-F ends the year with a project-based, cross-topic unit on bridge-building which incorporates some multiplication but does not use it as a central feature, leaving less time available for multiplication. EUR-F also spent a disproportionately small amount of time on multiplication, largely because it focused extensively on the grade 3 major work of understanding fractions which was covered in less depth by other programs.

Aside from the extended focus on addition and subtraction from two programs, almost every program had a few lessons that went beyond or around the content standards. Sometimes, as in the case with a bridge-building unit in BRI-F, these lessons seemed designed to address the content standards through projects with high levels of rigor and application that build on content from the year. In other cases, such as a set of combinatorics lessons in My Math which were taught in a rote fashion and are better aligned to grade 7 content standards, it was unclear why they were present. These non-aligned lessons tended to make up a small minority of total textbook content, 2-9% depending on the program.

According to the checklist and intentions approaches, this extraneous content does not belong as it does not meet strict guidelines for teaching only the CCSSM content standards. However, it’s worth questioning whether alignment approaches should penalize small amounts of extended content. These additional topics tend to make up a
small proportion of the total content, provide opportunities for students to apply what they are learning in different ways, and typically extend learning within the grade rather than bringing in content from different grades. While the interpretation might be different if they made up a substantial portion of the book or took on substantive content from other grades, a small collection of outlier lessons might have little impact on the scope of the year and may be a starting point for interesting mathematical explorations.

Overall, the focus of the eight textbooks in the study was fairly strong, which demonstrates a tremendous shift from even a few years after the CCSSM were released (Center for the Study of Curriculum, 2014; Cogan et al., 2015).

**Coherence**

Coherence involves building up concepts using a developmental progression and using multiple content standards to support each other. Two interesting patterns arose in observing the development of multiplication topics over time.

Regarding the developmental progression, my summary tables indicated that all eight programs followed similar overall trajectories of learning multiplication through equal groups of objects, then skip counting and repeated addition, then arrays/area models, then the distributive property as a tool for finding unknown products, and then extended topics like multiplying by multiples of 10. However, after conceptual understanding was covered the programs progressed through the intermediate stages of gaining fluency differently. Some of the programs (BRI-F, EVER-F, and INV-F) focused on learning strategies that were based on the distributive property (e.g., using helper facts, doubling, add a group/subtract a group, etc.) and applied them to a range of factors in each lesson. In the other programs (ENV-C, EUR-F, GO-C, INTO-C, and MY-C),
there was a sequence of lessons that repeated the same cluster of strategies, but focusing on multiples of a different numbers (e.g., multiply by 8). While both progressions use the same strategies and the same factors, putting the emphasis on the strategy implicitly focuses students’ attention on flexible ways to solve problems, whereas focusing on the number being multiplied puts an implicit emphasis on getting correct answers and has a feeling of routinization.

Regarding the interweaving of standards to support each other, one of the major clusters for grade 3 is understanding the relationship between area and multiplication. In half of the programs (BRI-F, EVER-F, INTO-C, and INV-F) area models were introduced as a concept and then area models were incorporated into the tools that students had available to communicate ideas about multiplications, so that these two topics supported each other throughout the year. However, in other programs (ENV-C, EUR-F, GO-C, MY-C), area models were introduced as a tool for demonstrating the distributive property before laying the foundation of understanding area and then repeated later to address the measurement standards. This second introduction of the same topic both means that students may have been using area models for multiplication without understanding that they demonstrate area in two-dimensional space, and also that they were spending time repeating content that had already been taught.

Overall, the use of a learning trajectory with a range of representations and strategies was apparent across all of the textbooks and represents a substantial influence from the CCSSM. However, some programs attended to coherence more than others when sequencing topics across units.
**Rigor and the Integrity Approach**

In this section I dive deeply into the content of the standards themselves to understand how they are interpreted and enacted by textbooks with attention to both the shift of rigor (equal emphasis on conceptual understanding, fluency, and application) and an integrity approach (which equally emphasizes the importance of each part of the standard and the intentions of the whole).

The findings from this analysis are presented in three sections. First, I present a table that provides a visual overview of the quality and quantity with which each statement is addressed across the eight curriculum programs. Some standards were embraced fervently and interwoven thoroughly into the majority of the curriculum programs. Others showed trends of being minimally or incompletely implemented, not just in a single program or by happenstance, but in ways that were consistent across several of the programs. In further analysis of these trends and the language and content of the standards, I present three factors of how standards are designed that seem to impact how they are taken up by curriculum programs. I then explore two standards as cases to illustrate the relationship between features of standards design and features of lesson design across the eight textbooks.

**Alignment Across the Curriculum Programs**

In this section I present each of the standards as smaller statements along with the quality at which they are addressed by textbooks in Table 2.8. Breaking the standards apart into individual statements demonstrates that frequently one part of a statement may be addressed at a very different level of quality and quantity than other parts of the same standard, which is an important consideration when using an integrity approach to
explore alignment. While this table can be read vertically (by curriculum program) to identify stronger and weaker programs, the focus of this study is on reading it horizontally (by standards) to understand how the standards were covered across the curriculum programs. The color codes indicate the quality levels that are described in Table 2.4 in the Methods section and which are repeated in Table 2.7 for reference.

**Table 2.7**

*Color code used for quality levels in Table 2.8. For descriptions, see Table 2.4.*

<table>
<thead>
<tr>
<th>Robust</th>
<th>Sufficient</th>
<th>Insufficient</th>
<th>Absent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td><strong>Observe without Doing</strong></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td><strong>Infrequent</strong></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td><strong>Superficial</strong></td>
<td></td>
</tr>
</tbody>
</table>

When creating this table, it became apparent that there were three standards (3.OA.3, 3.OA.5, and 3.NBT.3) that textbooks often tagged for content that extended beyond the written text of the standards in consistent ways across the curriculum programs. That is, the curriculum developers indicated that they were using these standards to address content that fit the overall expectations of the standard but also extended beyond the limits of the literal text. As these extensions were part of how the curriculum developers were interpreting and enacting the standards, I added the extensions to the master table and continued the numbering system. For example, 3.OA.5-1 through 3.OA.5-6 address the original text of standard 3.OA.5, while 3.OA.5-7 is a logical extension of that standard and is indicated in blue text.
Table 2.8

Coverage of elementary multiplication standards across eight curriculum programs.
*Note: Only a sample set of lessons was available for BRI-F, which represent around one-third of the total multiplication lessons in the program. When the relevant lessons were not available to analyze, I indicated this with “could not assess.”

<table>
<thead>
<tr>
<th>Standard</th>
<th>Standard Text</th>
<th>BRI-F*</th>
<th>EUR-F</th>
<th>EVER-F</th>
<th>INV-F</th>
<th>ENV-C</th>
<th>GO-C</th>
<th>INT0-C</th>
<th>MY-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.OA.1-1</td>
<td>Interpret products of whole numbers, e.g., interpret $5 \times 7$ as the total number of objects in 5 groups of 7 objects each.</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
</tr>
<tr>
<td>3.OA.1-2</td>
<td>For example, describe a context in which a total number of objects can be expressed as $5 \times 7$.</td>
<td>Blue</td>
<td>Red</td>
<td>Green</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.OA.3</td>
<td>Use multiplication and division within 100</td>
<td>Too general: did not assess</td>
<td></td>
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<tr>
<td>3.OA.3-1</td>
<td>to solve word problems in situations involving</td>
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<tr>
<td>3.OA.3-2</td>
<td>equal groups,</td>
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<tr>
<td>3.OA.3-3</td>
<td>arrays,</td>
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<tr>
<td>3.OA.3-4</td>
<td>e.g., by using drawings</td>
<td>Meaning unclear: could not assess</td>
<td></td>
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<tr>
<td>3.OA.3-5</td>
<td>and equations</td>
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<tr>
<td>3.OA.3-6</td>
<td>with a symbol for the unknown number to represent the problem.</td>
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<td>1</td>
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<tr>
<td>3.OA.3-7</td>
<td>Beyond standards: e.g., number lines</td>
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</tr>
<tr>
<td>3.OA.3-8</td>
<td>Beyond standards: e.g., base-10 blocks and/or place value tables</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<tr>
<td>3.OA.3-9</td>
<td>Beyond standards: e.g., number bonds and/or lines that combine equations</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
<td>Green</td>
</tr>
<tr>
<td>3.OA.3-10</td>
<td>Beyond standards: e.g., tape/bar diagrams</td>
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<tr>
<td>3.OA.3-11</td>
<td>Beyond standards: e.g., counters and/or snap cubes</td>
<td></td>
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<td>1</td>
<td></td>
</tr>
<tr>
<td>Standard</td>
<td>Standard Text</td>
<td>BRI-F*</td>
<td>EUR-F</td>
<td>EVER-F</td>
<td>INV-F</td>
<td>ENW-C</td>
<td>GO-C</td>
<td>INTO-C</td>
<td>MY-C</td>
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<tr>
<td>3.OA.4</td>
<td>Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations ( 8 \times ? = 48 ), ( 5 = _ \div 3 ), ( 6 \times 6 = ? )</td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="blue" alt="" /></td>
<td><img src="red" alt="" /></td>
<td><img src="yellow" alt="" /></td>
<td><img src="green" alt="" /></td>
</tr>
<tr>
<td>3.OA.5-1</td>
<td>Apply properties of operations as strategies to multiply and divide. Examples:</td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
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<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
<td><img src="red" alt="" /></td>
</tr>
<tr>
<td>3.OA.5-2</td>
<td>If ( 6 \times 4 = 24 ) is known, then ( 4 \times 6 = 24 ) is also known. (Commutative property of multiplication.)</td>
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<td><img src="yellow" alt="" /></td>
</tr>
<tr>
<td>3.OA.5-3</td>
<td>( 3 \times 5 \times 2 ) can be found by ( 3 \times 5 = 15 ), then ( 15 \times 2 = 30 ), or by ( 5 \times 2 = 10 ), then ( 3 \times 10 = 30 ). (Associative property of multiplication.)</td>
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<td><img src="red" alt="" /></td>
</tr>
<tr>
<td>3.OA.5-4</td>
<td>Knowing that ( 8 \times 5 = 40 ) and ( 8 \times 2 = 16 ), one can find ( 8 \times 7 ) as ( (8 \times 5) + (8 \times 2) = 40 + 16 = 56 ). (Distributive property.)</td>
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</tr>
<tr>
<td>3.OA.5-5</td>
<td>Beyond standards: identity (( 5 \times 1 = 5 )) and zero property (( 5 \times 0 = 0 ))</td>
<td><img src="green" alt="" /></td>
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</tr>
<tr>
<td>3.OA.7-1</td>
<td>Fluently multiply and divide within 100</td>
<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
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</tr>
<tr>
<td>3.OA.7-2</td>
<td>Using strategies such as the relationship between multiplication and division (e.g., knowing that ( 8 \times 5 = 40 ), one knows ( 40 \div 5 = 8 ))</td>
<td>Did not assess because division lessons were not included.</td>
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</tr>
<tr>
<td>3.OA.7-3</td>
<td>or properties of operations</td>
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<tr>
<td>3.OA.7-4</td>
<td>By the end of Grade 3, know from memory all products of two one-digit numbers.</td>
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</tr>
<tr>
<td>3.OA.8-1</td>
<td>Solve two-step word problems using the four operations.</td>
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</tr>
<tr>
<td>3.OA.8-2</td>
<td></td>
<td><img src="green" alt="" /></td>
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<td><img src="green" alt="" /></td>
<td><img src="green" alt="" /></td>
</tr>
<tr>
<td>Standard</td>
<td>Standard Text</td>
<td>BRI-F*</td>
<td>EUR-F</td>
<td>EVER-F</td>
<td>INY-F</td>
<td>ENV-C</td>
<td>GO-C</td>
<td>INTO-C</td>
<td>MY-C</td>
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</tr>
<tr>
<td>3.OA.8-3</td>
<td>Represent these [two-step] problems using equations with a letter standing for the unknown quantity.</td>
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</tr>
<tr>
<td>3.OA.8-4</td>
<td>Assess the reasonableness of answers [to two-step problems with variables] using mental computation and estimation strategies including rounding.</td>
<td>Could not assess</td>
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</tr>
<tr>
<td>3.OA.8-5</td>
<td>Assess the reasonableness of answers [to any multiplication or two-step problems] using mental computation and estimation strategies including rounding.</td>
<td></td>
<td></td>
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<td></td>
<td>Tagged but not present</td>
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</tr>
<tr>
<td>3.OA.9-1</td>
<td>[Footnote to 3.OA.8] Students should know how to perform operations in conventional order when there are no parentheses to specify a particular order (Order of Operations).</td>
<td>Could not assess</td>
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<tr>
<td>3.OA.9-2</td>
<td>Identify arithmetic patterns (including patterns in the addition table or multiplication table),</td>
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</tr>
<tr>
<td>3.OA.9-3</td>
<td>and explain them [patterns in the addition and multiplication table] using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends. [Focus is on explanation.]</td>
<td>Could not assess</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>3.OA.9-4</td>
<td>and explain them [patterns in general] using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends. [Focus is on explanation.]</td>
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<td></td>
</tr>
<tr>
<td>Standard</td>
<td>Standard Text</td>
<td>BRI-F*</td>
<td>EUR-F</td>
<td>EVER-F</td>
<td>INV-F</td>
<td>ENV-C</td>
<td>GO-C</td>
<td>INTO-C</td>
<td>MY-C</td>
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</tr>
<tr>
<td>3.NBT.3-1</td>
<td>Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., $9 \times 80$, $5 \times 60$) using strategies based on place value.</td>
<td>Could not assess</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.NBT.3-2</td>
<td>and properties of operations.</td>
<td>C.N.A.</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 3.NBT.3-3| Beyond standards: Multiply one or more of the following:  
- one-digit by two-digit numbers (e.g., $5 \times 63$)  
- one-digit numbers by multiples of 100 (e.g., $9 \times 200$)  
- two-digit by two-digit numbers (e.g., $12 \times 12$) using properties of operations and/or visual models. | 1 (or more) | 2     | 1      | 1     |       |      |        |      |
| 3.MD.7.a | Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. |        |       |        |       |       |      |        |      |
| 3.MD.7.b-1| Multiply side lengths to find areas of rectangles with whole-number side lengths.                                       |        |       |        |       |       |      |        |      |
| 3.MD.7.b-2| in the context of solving real world and mathematical problems,                                                          |        |       |        |       |       |      |        |      |
| 3.MD.7.b-3| and represent whole-number products as rectangular areas in mathematical reasoning.                                     |        |       |        |       |       |      |        |      |
| 3.MD.7.c-1| Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths $a$ and $b + c$ is the sum of $a \times b$ and $a \times c$. |        |       |        |       |       |      |        |      |
| 3.MD.7.c-2| Use area models to represent the distributive property in mathematical reasoning.                                        |        |       |        |       |       |      |        |      |
Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts.

Looking across Table 2.8, one of the clearest observations is simply that statements tended to be addressed similarly across multiple programs. For example, 3.OA.5-1, -2, and -4 were nearly universally addressed at a sufficient level, and all eight programs extended the standard in the same way with 3.OA.5-5. However, 3.OA.5-3 was typically covered in only one or two lessons (infrequently). Meanwhile, 3.OA.8-3, -4, and -5 were nearly always omitted, while 3.OA.8-1 and -2 seemed to bring out the extremes, being addressed either robustly or insufficiently, depending upon the program.

Why was 3.OA.5 implemented by textbooks with such consistency while 3.OA.8 was not? I suggest that these patterns are not accidental, but rather that there is something about the standards themselves—either their content or the way that they are written—that influences how they are interpreted and enacted by textbooks. Looking across the statements, I identified three factors that seem to influence the uptake of standards by curriculum developers, which I present in Table 2.9.
Table 2.9

Three factors that seem to support and inhibit enactment of standards by textbooks with definitions.

<table>
<thead>
<tr>
<th>Supporting</th>
<th>Inhibiting</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linguistic complexity</strong></td>
<td></td>
</tr>
<tr>
<td>• one goal</td>
<td>• multiple goals</td>
</tr>
<tr>
<td>• clear/unambiguous meanings</td>
<td>• ambiguous phrases</td>
</tr>
<tr>
<td>• examples are used to <strong>clarify</strong></td>
<td>• examples that <strong>extend</strong> the meaning of a standard</td>
</tr>
<tr>
<td>the meaning of the standard</td>
<td></td>
</tr>
<tr>
<td><strong>Student struggle</strong></td>
<td></td>
</tr>
<tr>
<td>The standards:</td>
<td>The standards specify that students must:</td>
</tr>
<tr>
<td>• list concepts or skills without</td>
<td>• take an active role in solving</td>
</tr>
<tr>
<td>specifying how they are developed</td>
<td>problems or discovering concepts</td>
</tr>
<tr>
<td>• can be presented as a set of</td>
<td>• explain concepts in their own words</td>
</tr>
<tr>
<td>steps (including when underlying</td>
<td>• struggle with mathematically rigorous concepts</td>
</tr>
<tr>
<td>concepts are modeled by the</td>
<td></td>
</tr>
<tr>
<td>teacher/textbook)</td>
<td></td>
</tr>
<tr>
<td>• address simpler mathematical</td>
<td></td>
</tr>
<tr>
<td>concepts</td>
<td></td>
</tr>
<tr>
<td><strong>Repetition</strong></td>
<td></td>
</tr>
<tr>
<td>• concept appears in multiple</td>
<td>• concept appears in only one</td>
</tr>
<tr>
<td>standards</td>
<td>standard or only one part of a larger standard</td>
</tr>
</tbody>
</table>

I propose that there are features of lesson design related to insufficient standards alignment (see Table 2.4) and features of standards design (see Table 2.9) that each impact how standards are interpreted and enacted in textbooks. I explore them using two cases, one of incomplete and varied enactment with 3.OA.8 and one of consistently sufficient enactment with 3.OA.5. I then briefly discuss a special case of linguistic complexity relating to how the CCSSM use examples, as this is one of the most common causes for unclear communication.

**Incomplete and Varied Enactment: 3.OA.8**
Standard 3.OA.8 provides a valuable case of a standard that was often implemented incompletely and with highly varied quality. It demonstrates all three factors of standard design that inhibit enactment (high linguistic complexity, high student struggle, and no repetition), and also provided a good example of textbook enactment that ranges from robust in three of the grant-funded programs (BRI-F, EVER-F, and INV-F) to insufficient in three of the commercial programs (ENV-C, GO-C, and INTO-C), which made it ideal for illuminating the characteristics of these quality levels.

The full text of the standard is stated below with numbers in parentheses to show the separate statements, which I then unpack to consider possible meanings.

3.OA.8: Solve two-step word problems using the four operations (1). Represent these problems using equations with a letter standing for the unknown quantity (2). Assess the reasonableness of answers using mental computation and estimation strategies including rounding (3 & 4).

Footnote to 3.OA.8: Students should know how to perform operations in conventional order when there are no parentheses to specify a particular order (Order of Operations) (5).

The first two statements of 3.OA.8 are linguistically complex and ambiguous, resulting in several possible interpretations that have varying levels of conceptual rigor.

Statement 1, by itself, addresses solving two-step problems, which are somewhat familiar from textbooks that predate the CCSS. For example, “I had three bags with a dozen applies in each, and then I gave away six apples” is a two-step task with multiplication and subtraction. Statement 2 states that students should use variables to represent the unknowns. What is unclear is which of the following meanings of using variables for two-step problems is intended. Continuing the previous example with apples, three possible options are shown in Table 2.10.
Three possible interpretations of “two-step word problems” with “a letter standing for the unknown quality” in 3.OA.8.

<table>
<thead>
<tr>
<th>Option A</th>
<th>Option B</th>
<th>Option C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 12 = a$</td>
<td>$3 \times 12 - 6 = a$</td>
<td>$3 \times 12 = a$</td>
</tr>
<tr>
<td>$a - 6 = t$</td>
<td></td>
<td>$36 = a$</td>
</tr>
<tr>
<td>Alternate (giving away $a$ apples to keep 6):</td>
<td>$36 - 6 = t$</td>
<td>$30 = t$</td>
</tr>
<tr>
<td></td>
<td>$3 \times 12 - a = 6$</td>
<td></td>
</tr>
</tbody>
</table>

Option A is the most conceptually challenging as it requires algebraic thinking; students must recognize that while $a$ is substituted in for an unknown value, that its value is consistent between the two equations, and that when $a$ is determined in the first equation it can be substituted into the second equation. Option B is less challenging in terms of the use of variables, but it demonstrates an understanding that multiple operations can be combined in an expression and set equal to a variable. It can be made more rigorous if the variable is placed within the expression to the left of the equation, as shown in the alternate. The third option is the least rigorous and seems to bypass the purpose of using variables, as the 36 from the first equation is substituted into the second without requiring the variable at all. Options B and C additionally give the impression that variables should represent “the answer” on the right side of an equation, rather than a tool that can be used flexibly in any position. As 3.OA.8 does not distinguish between these options, it is at the discretion of the curriculum developers how to interpret it.

Statements 3 and 4 address a mathematical skill or habit of mind, checking answers using mental computation and estimation. This type of skill is more consistent with the standards of mathematical practice, such as MP1 which states that “Mathematically proficient students check their answers to problems using a different
method, and they continually ask themselves, ‘Does this make sense?’” While this is not conceptually challenging by itself (and checking answers with estimation has been a mainstay of rote mathematics learning), it is unclear whether this statement is supposed to be applied to statements 1 and 2 (representing two-step problems with variables) or not, though the linguistic structure implies that it should. In my analysis, I evaluate this statement first to ask if the textbooks have students use mental computation or estimation to check their answers at any point in the text, and then again to ask if they do so specifically for two-step problems with variables.

Finally, statement 5, which addresses the order of operations, can be found only in a footnote. On the CCSS website, it is written in miniscule letters at the bottom of the 3.OA page and is not present at all when viewing 3.OA.8 on its own page or when viewing all of the OA standards across grades K-5. The order of operations is not mentioned again in any following year, which suggests that if this content is not taught in grade 3 it might not be taught at all or could be taught anywhere. This footnote is additionally confusing because students are not expected to learn how to use parentheses to suggest order until grade 5 (in 5.OA.1), although they are used in many examples of the grade 3 standards.

Overall, 3.OA.8 is linguistically complex, with three separate statements that could each be applied in isolation from each other. It was also ambiguously written, making it unclear whether the two variables were supposed to appear in the same or different equations. It is also pedagogically complex, requiring a high level of student struggle. Representing two-step word problems in equations with two variables is much more conceptually challenging than the work that grade 3 students have traditionally been
expected to do in the U.S., and while the standard is presumably intended to prepare students for algebra (based upon its designation as an Operations and Algebraic Thinking standard), it’s unclear what students benefit from learning to write these equations as they are solving them in two separate steps that draw on more concrete representations.

In terms of frequency, all three components of this standard are mentioned only once across the grade 3 multiplication standards, which may tempt curriculum developers to focus on only the most prominent component, two-step problems. In addition, this standard does not rest easily in the cross-grade CCSSM learning trajectory: it involves a huge jump for third graders to move from using a single unknown to using a letter/variable and using two variables at once and putting them in a single equation (if it’s represented that way) and checking the resultant answer with mental math. The standard is then repeated in grade 4 with almost no alterations (4.OA.3) and then does not appear again in grade 5, which lacks any references to variables or unknowns. If variables were introduced in one of the other standards on unknowns (3.OA.3 or 3.OA.4), or if this level of complexity was built up more slowly and with more standards over grade 3 or across grades 3-5, it might be more accessible.

With all of this linguistic and conceptual confusion, how have curriculum developers chosen to interpret and enact 3.OA.8 in textbooks? The examples that follow from Go! Math (insufficient) and Everyday Mathematics (robust) have been carefully selected to showcase each of the characteristics of lesson quality and quantity that emerged from this analysis.
Insufficient Coverage

*Go! Math (GO-C)* addressed 3.OA.8 in lesson GO-C G3 7.10, from which two tasks are shown in Figure 2.4. This lesson was extremely similar to the lessons addressing this content in the other three commercial textbooks. From a checklist and perhaps an intentions perspective, the lesson might meet all the requirements of the standard: there were several two-step word problems, they utilized addition, subtraction, and division (but not multiplication), and some of the word problems used variables to represent unknowns. The lesson did not support checking work with mental computation or estimation at any point.

**Figure 2.4**

*Example from GO-C G3 7.10 that demonstrates insufficient coverage of 3.OA.8 because students observe the teacher without doing the work themselves, address the content in only one lesson (infrequently), and use a superficial interpretation.*
An integrity perspective, which considers the holistic intentions of the standard, whether each statement within the standard is addressed, and the quality at which it is addressed, would argue that this lesson does not meet the requirements of the standard. Each of the characteristics of insufficient coverage that impact this decision are introduced below.

**Observing without doing.**

This lesson demonstrated observing without doing, as students were never given the opportunity to write equations independently. Students wrote equations during the whole-group part of the lesson in two tasks with heavy scaffolding from the teacher and the textbook, and then practiced writing equations once using a fill-in-the-blank format. The fill-in-the-blank format removed a significant level of student struggle. For example, in task 1 of Figure 2.4, identifying that the problem could most effectively be solved by starting with the number 21 was a critical step that students were not allowed to take ownership of.

After completing the three tasks where students wrote equations with heavy scaffolding, all further tasks were in the format of task 2, where students provided a numeric answer with appropriate units, but did not write equations. The decision to have students meet the standard filling in blanks in a set of routine steps, rather than having them explore or derive the concepts, heavily impacted students’ ability to do the actual work of the standard. I.e., Could students be said to “represent the problems using equations” when the textbook provided most parts of the equations for them?

**Infrequency and omission.**
This lesson, GO-C G3 7.10, was the only lesson in the entire program that contained two-step problems or that used variables (letters) to represent unknowns. Whenever a standard was covered in three or fewer lessons, it received an Infrequent tag. The infrequency of this lesson is important because it is unlikely that students would be able to make sense of, incorporate, and apply this complex content after only a single exposure.

In addition, GO-C did not address using mental computation or estimation to check answers at any point in the year, including this lesson. In the following lesson, GO-C G3 7.11, students were introduced to the Order of Operations through direct instruction. The textbook models one problem with a story component and a single variable, but all tasks that students solved on their own involved only numbers. After these two lessons, students did not address any concepts from 3.OA.8 for the rest of the year. This suggests that, although the order of operations was covered, students would not see it enough times to understand and retain the information.

**Superficial interpretations.**

This lesson addressed only superficial aspects of the 3.OA.8 in several ways. As mentioned in the Observing without Doing section, students did not get to choose what variables to use or determine which equations they wrote. This led to a shallow conceptual implementation of the term *solve*, as students were not given the opportunity to solve any of these tasks without heavy scaffolding.

In addition, Go Math! chose the simplest interpretation of statements 1 and 2, option C, which treats each step of the equation as a separate math task. In every equation in this lesson, the variable was shown to the right of the equal sign (as “the answer”),
even though another standard 3.OA.4, suggests that students should be familiar with unknowns in multiple locations. This simplistic enactment of the standard suggests that variables serve little function and does not demonstrate that variables can represent unknown amounts in any position.

Students’ opportunity to engage in solving problems was also decreased by the scaffolding that students received throughout the textbook. Rather than engaging in productive struggle by approaching a complete task and making sense of it, word problems, such as the ones in this lesson, were taught by having students write lengthy descriptions of steps they intended to take before actually solving the problem, as shown in Figure 2.5. While thinking through a problem may be helpful, writing out sentences like “I will use the information to act out the problem” or rewriting almost the full text of the original question (“Chad bought 4 packs…”) with the whole class shifts attention away from the mathematical content to instead focus on rote steps. This scaffolding also decreased students’ opportunity to independently solve problems.

This approach to addressing 3.OA.8 was highly similar to the approach taken by ENV-C and INTO-C. It was also similar to the approach used in MY-C, though MY-C did not utilize any variables in its lessons, addressing statement 1 without statement 2.
While ENV-C did an excellent job of including a two-step task without variables or equations in every lesson (see Figure 2.6 from ENV-C GR3 10.2 as an example), in MY-C, GO-C, and INTO-C two-step problems were taught in only 1-3 lessons at the end of the academic year, rather than incorporated into ongoing learning, decreasing focus and coherence.

**Figure 2.6**

*Example from ENV-C G3 10.2 of a two-step problem without variables, frequent in ENV-C.*

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### Robust Coverage

As a counterpoint to the insufficient coverage situations, three grant-funded programs, BRI-F, EVER-F, and INV-F addressed aspects of 3.OA.8 in ways that were particularly nuanced, rich, and rigorous. The example below discusses how *Everyday Mathematics* (EVER-F) demonstrated robust coverage for the same standard, 3.OA.8. This example focuses on one lesson, EVER-F G3 6.11, though the description below puts the lesson into a larger context.

**Fluency through frequency.**

Procedural skills and fluency require multiple opportunities to both engage with the concepts behind an approach and practice it over multiple days and in multiple contexts and settings. EVER-F built up the components of 3.OA.8 over many lessons as well as dedicating several lessons just to that topic. The program began using unknowns
and variables early in the year, so that students might build familiarity over time. Two-step problems with variables, the focus of 3.OA.8, were covered in four days of lessons in which students first wrote their own story problems to match an existing equation (supporting 3.OA.1), shared and critiqued each other’s story problems (supporting mathematical practice standard 1), learned the order of operations within that context (the footnote to 3.OA.8), and then developed visual models to support writing and solving two-step problems in the target lesson, EVER-F G3 6.11. While there were only four lessons that covered statements 1 and 2 together, two-step word problems without variables and one-step problems with variables were used in both mixed review problems and multi-day explorations of more advanced concepts over the rest of the year.

**Conceptual understanding.**

*Everyday Mathematics* utilized a deep application and understanding of 3.OA.8. The program chose a more complex interpretation of the standard through Option B, where both steps of the problem were represented in the same equation with one variable, though the variable could be in any position. The EVER-F lesson demonstrated that variables could be placed anywhere in an equation, supporting the idea of variables standing in for numbers rather than being “the answer” and that the equal sign represents a balance of values on both sides of the equation. Figure 2.7 shows three representations of this concept that students co-constructed with their teachers to support deep understanding of both variables and operations in multi-step problems. While Option B is not as challenging as Option A, which places two variables in the same equation, this option is still ambitious for grade 3 students and has traditionally not been taught before late middle school or high school.
**Figure 2.7**

*Example from EVER-F G3 6.11 of interpreting 3.OA.8 using the more advanced form of Option B showing one equation with one variable that may be to the left of the equal sign.*

Below the change diagram, write a summary number model with 12 substituted for \( M \). Have children check that 12 makes the equation true and write the summary number model below the diagram on journal page 214.

![Change Diagram](image)

**Application.**

During EVER-F G3 6.11, students first worked with the teacher to represent a sample task by co-constructing the representations above, thereby addressing novel tasks as a class. After this introduction, students were expected to choose appropriate models, draw them independently, write equations, and discuss or share them with peers, providing them opportunities to make informed decisions about choosing strategies for new tasks rather than repeating rote steps. Figure 2.8 shows a sample task that students were expected to complete independently and then discuss.
In a separate two-day lesson (EVER-F G3 6.9 and 6.10), students were given equations such as \((10 - 2) \times 3 = G\) and asked to write their own story problems. While this is not an explicit part of 3.OA.8, this type of application supports deep conceptual understanding.

The approach that EVER-F took to addressing 3.OA.8 was similar to the approaches taken by the other grant-funded programs, BRI-F, INV-F, and some aspects of EUR-F (which addressed it sufficiently but not robustly). BRI-F and INV-F each interpreted this standard as requiring two variables in the same equation (Option A) and introduced the concept of variables and multi-step problems in multiple lessons throughout the year. EUR-F also chose Option B, the middle-level interpretation to meet this standard, but frequently asked students to solve two-step problems with two equations that each used their own variables (Option C) throughout the year.

Regarding 3.OA.8-3 and 3.OA.8-4, ENV-C and INV-F both often used mental computation and estimation, but not to assess the reasonableness of answers in two-step problems. EUR-F stood alone in fully addressing all three components of this standard.
(either together or separately), though this only occurred in one lesson. Figure 2.9 from EUR-F GR3 3.18, demonstrates both how EUR-F incorporated estimation into this lesson and the Option C style of interpreting 3.OA.8-1 and -2.

**Figure 2.9**

*The single example in the entire set of sample lessons where students check their work using estimation to meet 3.OA.8-3 and 3.OA.8.4 from EUR-F GR3 3.18.*

Looking across all eight programs, the ambiguous language and high student struggle in 3.OA.8 often led to it being only partial enacted and enacted in superficial or infrequent ways. Only EUR-F addressed checking answers with estimation (in the single example above) and order of operations was only addressed in three curriculum programs. Thus, 3.OA.8 was typically used only to address either two-step problems or variables, often only one at a time rather than holistically.

This standard was also often tagged when it was not warranted. For example, MY-C indicated this standard was covered whenever unknowns (without variables) were
used, often instead of the more appropriate 3.OA.4 (which addressed using unknowns in learning multiplication facts). Similarly, EUR-F often tagged 3.OA.8 for decontextualized problems with procedures that required two steps (such as using the distributive or associative properties), although this does not address any of the components of 3.OA.8. Both of these patterns occurred to some extent in several other programs.

As a case, the treatment of 3.OA.8 by textbooks was similar to how 3.OA.9 was interpreted and enacted. Both standards show similar profiles for high linguistic complexity, high student struggle, and low repetition, which may explain why they were either completely omitted or covered with insufficient quality in many textbooks.

**Sufficient and Consistent Enactment: 3.OA.5**

If there was one clear place that the CCSSM seemed to have a strong presence in these samples, it was using properties of operations to multiply numbers and understand multiplication conceptually. This theme is central to 3.OA.5, but it also comes up in four other standards (3.OA.7, 3.OA.9, 3.NBT.3, and 3.MD.7c), suggesting its importance to the CCSSM and perhaps leading to its frequent use, as shown by full rows of green in Table 2.8. This section will analyze 3.OA.5 as a case of a standard with consistently sufficient enactment across curriculum programs.

In this analysis, 3.OA.5 is divided into four statements, as shown by the numbers in parentheses. I have crossed out *and divide* from statement one, as this analysis focuses only on multiplication-focused lessons.

**3.OA.5:** Apply properties of operations as strategies to multiply **and divide** (1). *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication) (2). 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2
= 30, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. *(Associative property of multiplication)*

(3). Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find $8 \times 7$ as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. *(Distributive property)* (4).

**Footnote to 3.OA.5:** Students need not use formal terms for these properties.

This standard identifies the three basic properties of operations, commutativity, associativity, and distribution, and sets the ambitious expectation that students will not just memorize them as definitions, but use them as core strategies for the act of multiplication. Using operations in this way has been an established practice in programs like EVER-F and INV-F that were developed by universities and smaller organizations, but on a wider scale these approaches are new to the United States.

One of 3.OA.5’s strengths is that it is linguistically straightforward. It has a single, clear goal, it provides examples of three properties that are well known to mathematicians, and there is no extraneous or ambiguous information. In addition, the examples *clarify* the meaning of the overall goal. (Compare this to 3.OA.1 where the example adds a second goal that *extends* the meaning of the overall goal.) The linguistic structure leaves space for more than three properties of operations, however, there are only a handful of properties of operations in mathematics that could apply here.

In exploring this use of “for example,” it is interesting that the curriculum designers for all eight programs decided to include a lesson on the zero and identity properties (that $0 \times \text{any number} = 0$ and $1 \times \text{any number} = \text{that number}$). They reliably tagged these lessons as 3.OA.5, indicating that they interpreted the phrase “Examples:” to mean that the following text provided three necessary examples, but did not limit them from including other examples. It is relevant that mathematicians would likely consider the zero and identity properties to belong to the set of necessary properties for
understanding multiplication, so including them is not a far leap. Thus, an integrity approach to alignment would interpret this expansion as being within both the literal meaning and the intention of the standard.

In comparison to some of the other standards, 3.OA.5 has an undefined level of student struggle as it can be interpreted in either a more conceptual or a more procedural manner. On one hand, using the steps of statement 4 to solve $8 \times 7$ by finding the products of $8 \times 5$ and $8 \times 2$ requires more effort and a deeper conceptual understanding of multiplication and numeric relationships than simply memorizing $8 \times 7$. On the other hand, all of the statements in this standard can be taught through a set of memorized definitions and a series of proceduralized steps. Supporting this ambiguous level of student struggle, 3.OA.5 can be accessed through visual models which make the concepts and steps relatively accessible to students and teachers while also supporting conceptual understanding. Thus, 3.OA.5 has a higher level of student struggle than memorization strategies that might have been used prior to the CCSSM, but it may still be enacted through rote steps which decreases student struggle.

Given the somewhat increased levels of student struggle in 3.OA.5, it is likely that repetition across standards plays a large role in its sufficient quality across textbooks. Students are asked to reach other mathematical goals using “properties of operations” in three other multiplication standards (3.OA.7, 3.OA.9, 3.NBT.3), and one of the area standards, 3.MD.7c, which focuses specifically on the distributive property. In addition, the “properties of operations” language is echoed across grades in the CCSSM, and the distributive property in particular is invoked often in learning multi-digit and fraction multiplication in later grades.
What does a linguistically straightforward, cognitively intermediate, and frequently repeated standard look like in textbooks? Overall, nearly all of the textbooks met nearly all of the statements of 3.OA.5 at a sufficient level. The statement with the least coverage was the associative property (statement 3), which was often covered in only one or two lessons, perhaps because this property can be picked up by students with relative ease. Two programs, MY-C and INTO-C, tended to address the commutative property as a definition to memorize rather than a tool to multiply, but overall textbooks seemed to readily take up the call to utilize properties of operations as a primary approach to learning multiplication.

While each of the textbooks addressed 3.OA.5 at a sufficient level with respect to the specifications of the standard, this does not mean that they all met it with the same opportunities for students to explore and develop ideas. The ambiguous cognitive level of this standard allowed it to be interpreted in both more procedural and more conceptual ways by different programs, both of which are explored here.

To examine examples of 3.OA.5 in textbooks, I focus on the distributive property for three reasons: it was introduced to most textbook series by the CCSSM; it appears in multiple grade 3 standards; and the CCSSM use it as the foundation for understanding multi-digit and fraction multiplication in later grades. In textbooks, the distributive property often appeared in strategies which were given more student-accessible names such as add a group/subtract a group, doubling, using a known fact or breaking apart. It was also frequently presented with arrays or area models alongside equations, a format that could help students to make connections between multiple representations.
Importantly, knowing this strategy might encourage students to think flexibly with numbers, for example by breaking up an array into different known facts.

My Math (MY-C), like the other commercial programs, introduced the distributive property through a procedural approach where the student filled in the blanks in a heavily scaffolded demonstration. In MY-C G3 7.3 (Figure 2.10), the textbook provided an example of how the distributive property could be used to learn new multiplication facts through an array, an area model, and equations. Students then replicated these steps by drawing lines to break up array and area models that the textbook provided and to write their own equations using the procedure that was modeled. After this lesson, MY-C used the same three representations of the distributive property in multiple lessons focused on learning how to multiply by specific factors. After this introduction, students were not often encouraged to draw their own arrays or area models as a problem-solving strategy, though the repetition of these visual models in the guided practice portion of lessons could support them developing this as a mental math skill.
Figure 2.10

Example from MY-C G3 7.3 where students address 3.OA.5-4 with underlying conceptual understanding provided through heavy scaffolding: pink represents the expected student responses

While replicating a series of steps presented by the textbook lowers student struggle, students were still introduced to a conceptual understanding of the distributive property and asked to apply it to a range of story-based and purely numeric tasks. This low student struggle approach is also used in ENV-C, GO-C, and INTO-C. However, it represents a meaningful change from most pre-CCSS textbooks which did not use the distributive property or other strategies to support learning multiplication facts at all.

In comparison, the four grant-funded programs enacted this standard by having students first develop multiple visual models to explore the distributive property through novel tasks. For example, INV-F first introduces the distributive property in INV-F G3 1.2.4 (Figure 2.11) where students were asked to find the number of juice boxes in 5
packages of 6, and then in 8 packages of the same size. Students were encouraged to use bar models and then explain how they could use the first problem to help solve the second one, a question that leads students into “discovering” and making sense of the distributive property conceptually.

**Figure 2.11**

*Example from INV-F G3 1.2.4 of students discovering the distributive property through a contextual task and a bar model without generalizing the property.*

The distributive property was introduced again using arrays in INV-F G3 1.3.5 (Figure 2.12) where students were asked to both solve an immediate problem and then discuss whether the strategy of breaking the array into two smaller parts would “always work.” This question pushes students toward recognizing a generalizable strategy.
After each of these two lessons, students were frequently asked to solve open-ended problems with the distributive property as one possible approach to solving. They also played fluency games throughout the year that reinforced the distributive property as a strategy for finding products. INV-F’s lessons for this standard were written to expect a higher level of student struggle and the conceptual understanding aspect rigor than MY-C’s, though both represent a sufficient level of quality for meeting 3.OA.5-4 because they both fulfilled the written requirements of the standard.
Looking across the eight programs, 3.OA.5 tended to be tagged frequently and accurately, as the cluster of standards that overlap with 3.OA.5 were used to drive a significant portion of the grade 3 curriculum in most textbooks. Similar patterns of relatively low linguistic complexity, undefined student struggle, and repetition may also be the cause of consistent and mostly sufficient enactment of the overlapping standards 3.OA.1, 3.OA.3, 3.OA.4, 3.OA.7, and 3.MD.7. As in the examples here, these standards were often interpreted by the commercial textbooks through procedural sets of steps that students could follow, and often interpreted by the grant-funded textbooks as concepts that students should wrestle with and unpack. However, as these standards are written at an ambiguous level of student struggle that allowed them to be interpreted in either way, both sets of textbooks tended to meet these standards sufficiently.

Returning to the relationship between the shifts of focus and rigor, one interesting finding was that the curriculum programs that spent the least amount of time on multiplication (BRI-F and EUR-F) and on the major work in general (BRI-F and INV-F) (see Table 2.8) also tended to address multiplication at the highest level of conceptual understanding (along with EVER-F). After introducing the concepts, these programs tended to have more ongoing practice problems, games, and cross-standard lessons that used properties of operations without making them the central focus. This may also suggest that when properties of operations are used to teach multiplication conceptually that focusing larger portions of the year on these concepts is not as necessary.

**Interpreting Examples**

One aspect of linguistic complexity that is most potentially confusing in the CCSSM content standards is the use of examples. These occur frequently enough that I
address them separately here as a critique of the structural organization of the standards. The nine grade 3 multiplication standards selected for this analysis use the phrases *for example, examples, such as, or e.g.* eight times. A close reading led me to identify three binary categories that could be used to describe possible intended purposes of an example with respect to the primary goal of a standard: 1) clarification versus extension, 2) bounded versus unbounded, and 3) required versus suggested. I provide several examples of how these categories identify areas where examples can serve to make standards either clear or confusing, with an emphasis on the subjective nature of assigning these designations.

In my reading of 3.OA.5, which was discussed in the case above, assigning these categories felt relatively straightforward. The standard states a primary goal of having students “Apply properties of operations as strategies to multiply” and then describes how three properties of operations could be used to meet the primary goal. I personally interpret the examples in 3.OA.5 as clarifying (the examples demonstrate what the primary goal means), unbounded (educators could add the zero and identity properties while still meeting the primary goal), and required (there are only a handful of properties of operations, and textbooks should address all of the ones that are listed to fulfill this standard). Other readers could disagree about whether this standard is unbounded and required, though both of these interpretations are justifiable from a mathematics education perspective. This use of examples as clarifying, unbounded, and required seemed to enhance the quality and accessibility of the standard and support its consistent enactment in textbooks.
In 3.OA.7, the examples also seem to aid the quality and successful enactment of the standard. It contains two examples, one nested within each other, which are shown with the bold type:

3.OA7: Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations.

I describe the primary goal of this standard as using strategies to build multiplicative fluency. In my opinion, the first example, which begins with “such as,” seems to be clarifying (it provides examples of strategies), unbounded (other strategies could be used), and required (the strategies listed are repeated in other standards and important pedagogically). Textbooks tended to agree with these interpretations and covered the strategies listed (required) and added many others (unbounded). The second example, which starts with “e.g.,” seems to clarify the meaning of the “relationship between multiplication and division” in a bounded and required way that details the inverse relationship between multiplication and division facts. As both of these examples seemed to clarify the meaning of the primary goal of the standard, they seemed to enhance and support its consistent enactment in textbooks.

However, the meanings of examples become murkier in some other standards. For example, 3.OA.1 also contains two nested examples:

3.OA.1: Interpret products of whole numbers, e.g., interpret $5 \times 7$ as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as $5 \times 7$.

After reading this standard, it is extremely difficult to determine the boundaries and intentions of the primary goal, “interpret products of whole numbers.” The first example, which starts with “e.g.,” seems to be a clarification, but it is unclear whether it
is bounded or unbounded and required or optional. That is, is “interpreting $5 \times 7$ as the total number of objects in 5 groups of 7 objects each” the only way to “interpret products of whole numbers?” This example suggests drawing equally sized groups, but does this standard also apply when skip counting or using arrays because these strategies inherently rely on groups of objects? Taking the unbounded option one step further, might this standard also include interpreting products of whole numbers as scaling (e.g., representing $5 \times 7$ as 5 times as much as 7)?

The second example, which begins with “for example,” adds confusion. It might either be clarifying or extending the main goal of interpreting products, in either a bounded or unbounded fashion that could be required or suggested. If students are meant to meet the primary goal of 3.OA.1 only by giving examples of equal group situations, most textbooks addressed this standard in one or two lessons at the beginning of the year or omitted it entirely (though BRI-F and INV-F had robust lessons that focused on this specific skill). On the other hand, if 3.OA.1 can be applied whenever students are thinking about equal groups, which are an innate part of skip counting, repeated addition, and using arrays and area models, then 3.OA.1 could be applied in every lesson and the e.g., extension is a loose suggestion for one of many possible classroom activities. The design of this standard, which depends heavily on examples rather than a clearly worded primary goal, makes it extremely difficult to evaluate in textbooks.

Similarly, 3.OA.3 provides two lists of ways to represent multiplication, one of which is not listed as an example and one of which is.

3.OA.3: Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by
using drawings and equations with a symbol for the unknown number to represent the problem.

At first glance, the primary goal of this standard is solving word problems with multiplication and division that utilize several possible attributes. The “equal groups, arrays, and measurement quantities” are not an example and therefore seem to be required. They technically describe the problem types, though the lack of specificity about whether these components should be used in the question stem or as ways that students solve a task makes their role more ambiguous. Meanwhile, “drawings and equations with a symbol for the unknown number” are positioned as examples and therefore may be optional. Because of the way that the two lists are positioned relative to each other, the examples seem to extend (rather than clarify) and be unbounded and optional, though this interpretation is highly debatable, especially given that equations with unknowns are listed in two other standards and seem to also be required. Attempts to interpret this standard still raise many more questions, such as:

1) In comparing the two lists, who is supposed to provide the representations and equations? Does the structure of the standard suggest that students should solve problems where equal groups and arrays are provided for them (that is, rather than generating their own as a way to represent ideas), but generate their own drawings and equations?

2) The meaning of “drawings” is unclear—what type of drawings? Drawings created by the textbook or by students? Are the drawings of equal groups and arrays or something else? Does this refer to the act of creating drawings rather than viewing them?
3) How unbounded is this standard expected to be? If the overall goal of this standard is to use a range of visual and numeric representations, are other visual models and manipulatives such as number lines, base-10 blocks, number bonds, tape or bar diagrams, and counters included? What about equations that don’t include unknowns?

4) This standard specifies that all of these representations (and possible unbounded representations) are supposed to be used for the primary purpose of solving word problems. Can the standard still apply when the representations are being utilized for purely numeric tasks?

Given all of these questions about what visual, numeric, and physical models might or might not fall under this standard, textbooks tended to interpret it in two ways. First, 3.OA.3 was generally tagged whenever a visual or physical model was introduced for the first time or used to teach a new concepts (that is, in the majority of multiplication lessons) regardless of whether students were viewing the representations or generating them, and regardless of whether the representation was present in either of the lists or not. Second, this standard was tagged for both word problems and purely mathematical tasks, which seems to ignore a key requirement of the standard. Based upon the enactments of the standard by textbooks, a teacher might infer that the original text of the 3.OA.3 is something like “Use a range of visual, numeric, and physical models for multiplication,” a statement that overlaps only partially with the original standard. While this re-interpretation may arise largely from curriculum developers’ personal beliefs and expectations about what could or should be included in this 3.OA.3, some of the confusion is certainly due to the linguistic ambiguity around these two lists.
As a final thought on examples, 3.OA.8, which was described in the case above, is ambiguous because it *lacks* an example of what it means to represent two-step word problems with equations with variables. This standard would have been enhanced by a clarifying example to explain which of the three options for equation types was intended.

Looking across each of these standards, it appears that examples can serve a valuable role in effectively communicating the curriculum developers’ intentions, but they serve this function best when the primary goal is clearly stated rather than assumed through an example that extends the meaning. In the case of examples that extend the meaning, standards developers should also identify whether the extension is an optional suggestion of what the standard might look like in practice, or a required component for meeting the standards’ primary goal and whether the examples are bounded to those listed or unbounded and open to having other choices added.

**Discussion**

This article opened with a question that is being asked by a range of stakeholders for mathematics education across the United States: how well aligned are elementary mathematics textbooks to the CCSSM content standards? In this section, I answer this question in several ways. First, I address these findings in relation to the three Key Shifts (CCSS Authors, 2010) outlined for the CCSS and consider the implications for shifts that have been successful and those which remain only partially met. I then provide insights and suggestions for curriculum developers, standards developers, alignment evaluators, and educators based on these findings.

This study has a narrow focus, but potentially broad implications. While it covers only grade 3 multiplication standards and lessons, this is a major work for the grade,
covers a substantial proportion of lessons for the year, includes a range of standards from three strands of the CCSSM, and highlights an area where content has changed substantially from how multiplication was covered by most U.S. textbooks in the past. Thus, it is reasonable to expect that the ways that textbooks are addressing multiplication in grade 3 would be representative of how they are addressing other topics in other grades, especially given the high consistency in lesson format by each program. In addition, this study includes eight textbooks that are published by the three major U.S. publishing companies and some smaller organizations that collectively have substantial market share (Blazer et al., 2019; Opfer et al., 2018). While there were a few curriculum programs that I was unable to access for this study, these eight programs paint a broad picture of what resources are available and in use in the United States today. While this study would certainly be improved by adding more curriculum programs, the current sample provides sufficient representation to respond to the question of alignment between the CCSSM and elementary mathematics in a meaningful way.

Though these findings address only the grade 3 multiplication standards, they suggest a great deal of success in bringing the Key Shifts of the CCSSM to life through elementary mathematics textbooks. The level of agreement on content across these eight textbooks is unprecedented in the United States, which has several important implications. First, the “mile wide inch deep” programs of the past have been replaced with textbooks that attend well to focus: a high percentage of lessons address the major work of the grade and very little content goes beyond the standards. Assuming these trends continue in other topics and grades, this could help assure that students in different districts and states are all getting access to the same mathematical concepts each year,
which supports equity between students in different geographic regions and for students who switch schools.

Second, these findings suggest that textbooks have made great strides in addressing one aspect of coherence by following a learning progression within the grade. Rather than expecting students to memorize multiplication facts, these textbooks had students use a range of increasingly complex transparent algorithms to support them in understanding how and why multiplication works as a tool for reaching fluency. As the content in this analysis is closely tied to the grade-level standards, textbooks that follow the CCSSM equally closely in other grades would also follow a learning progression across years. Instead of repeating the same content over multiple years and then arriving in high school unprepared for the rigors of algebra, the adherence to the CCSSM content in these findings suggests that students will be exploring increasingly challenging, yet focused, content each year to prepare them for high school and beyond.

Third, conceptual understanding, one of the components of the key shift of rigor, showed strong representation across the curriculum programs. Rather than focusing on memorization and replicating standard algorithms by rote, the textbooks expected students to either observe or develop explanations for why multiplication works and using a range of strategies that make these concepts visible. This is a substantial and positive turn from many U.S. textbooks of the past. As conceptual understanding has increased, textbooks are still attending to fluency, another component of rigor, through practice with a range of task types as well as games in some programs.

Despite these successes, however, these findings suggest that there is still work to be done for addressing the key shift of rigor. This shift expects students to addresses the
content standards in novel and interesting contexts that go beyond replicating what has been demonstrated (application) and to wrestle with underlying mathematical ideas (conceptual understanding). The four commercially produced textbooks provided few or no opportunities for this type of learning, even when this was explicitly described in some of the standards like 3.OA.8 (solving two-step problems with variables) and 3.OA.9 (identifying and explaining patterns). This suggests that for many U.S. students, mathematics will remain a passive subject where they expect to be told what to do and how to think, rather than an active subject where they have the opportunity to create their own strategies, explanations, and solutions. Fortunately, there are now a range of curriculum programs available, including the four grant-funded programs in this study, that deeply address the CCSSM by inviting students to do the work of the content standards.

Using an integrity model to analyze standards-textbook alignment, as I did here, introduces a more nuanced look at what conceptual understanding means when it is used in conjunction with application, as the key shift of rigor requires. These findings suggest that when textbooks structure lessons so that students observe the teacher addressing a standard without enacting the standards themselves, only experience a standard in one or two lessons per year, address a standard in a mathematically superficial manner, or address only a part of a standard without addressing the whole that alignment to both the details and the intentions of the CCSSM content standards is far patchier than it initially appears. While on a large scale, there is great agreement on the major focus for grade 3 multiplication as suggested by enactments of 3.OA.5, students’ opportunity to engage with some of the other standards like 3.OA.8 varies dramatically by program.
This variable interpretation of the content standards seems to arise from both the structure of the standard and the structure of the textbook, and has implications for standards developers, curriculum developers, alignment assessors, and educators.

For standards developers, these findings suggest that the structure of standards plays an important role in the degree to which they are implemented by textbooks. Linguistic complexity often seemed to pose a barrier to enacting standards. Even within this small study of nine CCSSM content standards, I was unable to determine the intended meanings of several standards after spending substantial time analyzing the standards themselves, the progressions documents that unpack the standards (The Common Core Standards Writing Team, 2011), and the ways they had been interpreted by textbooks. The standards could be improved by having someone consider multiple possible interpretations of the written text and then improving the language to remove ambiguity. These findings also suggest that examples should only be used to clarify more general statements, not to serve in place of generalized statements. When examples are used, it should be made evident whether they are optional or required, and whether their range is bounded to the examples given or unbounded and open to allowing other possibilities as a way to meet the standard. Similarly, information in footnotes should be incorporated into the primary content of the standards where it is much more likely to be seen and utilized. Standards that have multiple goals, such as 3.OA.8, could also be improved by utilizing sub-standard lettering to ensure that each component is covered.

The level of student struggle in the standards has some important trade-offs that standards developers should consider closely. As the CCSSM content standards are currently written, the level of student struggle of many standards is ambiguous: the
content can be taught either as a series of memorized steps with the conceptual understandings identified by the textbook, or as a mathematical exploration of interrelated concepts that students discover and articulate as a class. When this type of ambiguity occurs, as in 3.OA.5, standards tended to be enacted sufficiently in most textbooks, but only some textbooks paired application with conceptual understanding, generally guided by their overall pedagogical philosophies. Thus, writing standards with ambiguous levels of student struggle tended to increase application across textbooks, but leave the actual level of student struggle to the curriculum developers.

Meanwhile, standards that required a higher level of student struggle through productive struggle, such as 3.OA.8, tended to be omitted or covered insufficiently by textbooks that positioned students to learn by replicating strategies modeled in the text. These omissions and superficial interpretations may stem from the pedagogical philosophies of curriculum developers who deem these standards less important, or may result from the practical limitations of incorporating student-led activities into textbooks with a fixed lesson structure based on modeling and repetition. These same standards were often covered with great richness and complexity by textbooks that positioned students as active generators of knowledge, where they served opportunities to extend learning in interesting ways. For standards developers, it may be valuable to know that writing standards where students must actively develop and wrestle with ideas increases the likelihood of the standard being addressed superficially, infrequently, without active student roles, or not at all.

Finally, standards developers may be interested in knowing that repetition has been largely successful in moving the most important new concepts about
multiplication—understanding how and why it works, as well as using a variety of representations and techniques to multiply—into the textbooks in this study. The overlapping content of 3.OA.1, 3.OA.3, 3.OA.4, 3.OA.5, and 3.MD.7 seems to have had a strong impact. However, the lack of repetition in 3.OA.8, 3.OA.9, and 3.NBT.3 is reflected in inconsistent enactment. While I do not suggest focusing on repetition as a pathway to ensuring textbook enactment, as this would make the standards document unnecessarily bulky and likely more confusing, breaking out some of the multi-part standards into smaller components with clarifying examples could support linguistic clarity and repetition simultaneously.

Given the complexity of the interactions between the structure of standards and the structure of textbooks, these findings suggest that several of the existing approaches to evaluating standards-textbook alignment are insufficient. A checklist approach, which gives credit to a textbook for meeting at least part of a standard regardless of quality, depth, or attention to each of the parts would have given an undeserved picture of success for this sample, as all eight textbooks addressed all nine standards in at least some way.

Using an intentions approach, it is unclear how each of these textbooks would fare. Because intentions approaches are either dependent upon tools that range far beyond the actual text of the standards or are highly susceptible to the evaluator’s beliefs about quality, instructional approaches, expectations of what the standards could or should contain, they do not provide the detailed information necessary to determine which actual CCSSM standards are being addressed and how. EdReports, a prominent textbook evaluator that uses an intentions approach, provides a good example of this inconsistency. For example, both BRI-F and INV-F spend disproportionate amounts of time in grade 3
on addition and subtraction and have very similar unit and lesson progressions (with INV-F following the CCSSM content standards more carefully), yet in the category of “focus and coherence” EdReports (2016a, 2017) gives BRI-F the highest possible score and INV-F a score of “partially meets expectations.” My analysis, which uses an integrity approach to closely track how often and in what order each component of each standard is met, suggests that both of these programs are highly similar in focus and coherence and should receive the same score. Similarly, in the category of rigor and application, ENV-C was given a high score despite having students only solve tasks that had been previously modeled for them, while EVER-F was given a low score despite having students frequently engage with novel tasks to develop and wrestle with ideas they generated (EdReports, 2016b, 2020b). As EdReports (EdReports, 2019b, 2019a) provides only a few hand-picked examples to back up their claims rather than a report on how individual standards are enacted in textbooks, it is difficult to understand where their claims originate, double-check their results, or identify concrete ways to improve the programs. To this end, I suggest that organizations that wish to evaluate standards move toward using an integrity approach where the standards themselves form the backbone of their alignment tools.

That said, cross-comparison tools such as the Survey of Enacted Curriculum (Polikoff, 2015) still serve a purpose in comparing standards and textbooks or assessments across multiple countries with multiple standards documents, but are not specific enough for assessing alignment between standards and textbooks when a single set of standards is in use. Overall, if textbook alignment evaluators begin using an integrity approach, they could provide valuable feedback to curriculum developers and
educators who are purchasing textbooks, as well as pushing the textbook industry toward greater alignment for all standards at a high level of rigor.

As a final thought for administrators and educators who are purchasing and implementing textbooks, these findings suggest that although all of the textbooks in the sample address all of the standards in at least some ways, there are substantial differences in how they address the standards. If educators believe that it is important to address all of the content standard thoroughly, rather than skipping over or superficially addressing the cognitively demanding ones, they are much more likely to find this in textbooks that use problem solving and discussion as the primary mode of learning. These programs not only addressed more parts of more standards, but also addressed the majority of the standards with greater quality, frequency, and opportunities for students to learn. The impact of these approaches to learning is further discussed in Chapter 3.
CHAPTER 3: TAKING MATTERS INTO THEIR OWN HANDS: HOW
ELEMENTARY MATHEMATICS TEXTBOOKS INTERPRET AND ENACT
THE CCSSM STANDARDS FOR MATHEMATICAL PRACTICE

Abstract

The Common Core State Standards (CCSS) Standards for Mathematical Practice (SMPs) outline eight practices, which may be thought of as habits of mind or approaches to problem solving, that mathematically proficient students should be using on a regular basis from grades K-12. This study explores how those practices were interpreted and enacted in eight U.S. textbooks, focusing on grades 3-5. Rather than evaluating if the textbooks met the SMPs, this article asks how textbooks addressed the SMPs. I conceptualize textbooks as mediators of standards, wherein curriculum developers interpret the SMPs based on their own backgrounds and pedagogical philosophies and then enact them through tasks, instructions, and teachers’ notes, which are then communicated to teachers through using the textbook.

This analysis found that textbooks’ approaches to addressing the SMPs varied based on whether they generally positioned students as generators or receivers of knowledge. The former tended to enact the standards in holistic and authentic ways, while the latter tagged them often but addressed them in name only. In addition, this analysis found trends in how SMPs were enacted, partially enacted, or largely misinterpreted across the eight textbooks, providing insights into factors that influence standards’ accessibility to readers. This chapter proposes a set of tenets for exploring textbook alignment to the SMPs that captures nuances in interpretation and enactment that relate to both the structure and content of the SMPs and the ways that textbooks
address learning. This approach to exploring alignment is illustrated through case studies of three standards: MP3 addresses the nature of student interactions in textbook lesson design, MP4 addresses the potential for misconceptions when terms in the SMPs have multiple meanings, and MP8 addresses challenges that can arise when SMPs are defined predominantly through examples. While this analysis is based on a small sample, the insights about both standards construction and textbooks’ views of learning have applicability for standards’ developers, curriculum developers, and educators across grades and subjects.

Introduction

Curriculum reform addresses the entire fabric of teaching and requires changes at a variety of levels including policy, teacher education, professional development, assessments, and family engagement, to name a few (Cohen et al., 2018; Marshall S Smith & O’Day, 1991). One critical tool is the textbook. This tool is central to teacher-student interactions, plays the role of translator between abstract policy and teacher practice, and often determines both what is taught and how it is addressed, especially in mathematics where U.S. teachers tend to follow textbooks very closely (Ball & Cohen, 1996; Hiebert & Grouws, 2007; Pepin et al., 2013; Remillard, Harris, et al., 2014; Valverde et al., 2002). While history suggests that reforming textbooks is insufficient for reform independently (Schoenfeld, 2004; Willoughby, 2000), studies also show that without rigorous textbooks, teachers are unlikely to increase rigor and conceptual understanding on their own (Stein et al., 2000, 2007). As such, there have been calls for research on textbooks (a term which I use to also include teacher’s guides) as an important aspect of understanding the influence of the most recent mathematics reform
tool, the Common Core State Standards for Mathematics (CCSSM, 2010), on the US educational system (Heck et al., 2011; Polikoff, 2015).

The CCSSM consist of grade-level content standards for K-8 and a set of eight cross-grade Standards for Mathematical Practice (SMPs) that are the focus of this article. The SMPs set out an ambitious agenda for mathematics learning. They expect students to acquire the skills and dispositions of mathematicians: making sense of and solving novel tasks, developing their own strategies, communicating their ideas with other mathematicians, contextualizing and decontextualizing, modeling messy data with simpler approximations, incorporating technology to deepen understanding, noticing underlying structure, and generalizing from specific examples to mathematical properties or general formulas.

Despite the ambitious nature of the SMPs, the CCSS authors have sent conflicting messages about their use in other documents that could impact how the SMPs are being interpreted and enacted. Unlike the standards documents from the National Council of Teachers of Mathematics (NCTM, 1989, 2000), which focused heavily on shifting instruction to be more problem-based, discussion-based, and supportive of students developing their own strategies, the CCSS claim to be agnostic to instructional approach (CCSS, 2012; McCallum, 2012; Munter et al., 2015). The authors explain that the “standards establish what students need to learn, but do not dictate how teachers should teach” (CCSS, 2012). At the same time, the SMPs are written in such a way that they can only be met when students have opportunities to struggle with novel problems and develop and share mathematical ideas, which has implications for the type of learning
that students experience in everyday lessons. This inconsistent messaging leaves space for a wide range of responses to the mathematical practices.

As textbooks both set and reflect larger beliefs and trends in the wider mathematics education community, they provide a valuable resource for investigating how the SMPs are being taken up in the United States. Past studies suggest that textbooks that claim to be CCSSM-aligned have often not been successful in meeting the SMPs, though there is often an immense range in levels of alignment between different programs (Cogan et al., 2015; DiNapoli, 2016; Meyer, 2015; Polikoff, 2015).

While most studies focus on the overall success or failure of each textbook in meeting the CCSSM, my goal in this analysis is to shift the conversation away from good/bad or yes/no approaches to determining if a textbook met the SMPs. Instead, I aim to understand how textbooks have interpreted and enacted the SMPs with questions like the following: When curriculum developers interpreted the standards for mathematical practice, what meanings did they take away and articulate, directly and indirectly, to teachers? When a textbook indicated that an SMP was being enacted, what were students doing and learning? How did the overall structural design of textbooks support and restrict opportunities for curriculum developers to enact the SMPs?

These questions have implications beyond the details of textbooks as they point to larger questions of policy and educational practices such as: How have the CCSSM, and messages surrounding them, impacted textbooks? Where has the CCSSM been unsuccessful in impacting textbooks, and what changes might make the SMPs more successful? Based on lesson plans from textbooks, what might mathematical learning
look like in the United States under the CCSSM? What information should educators have when making decisions about textbook purchases?

In this analysis, I explore how recent textbooks that claim CCSSM-alignment have interpreted and enacted the SMPs at several levels. First, I address the value of establishing mathematical habits of mind such as those described in the SMPs, as well as some of the challenges of entrusting such complex and ambitious practices to a standards document. From there, I develop a set of theoretical tenets for what authentic enactment of the SMPs would involve. I then use these tenets to explore the structural design features of eight elementary mathematics textbooks through their teachers’ guides. I address how textbook authors tagged and rephrased standards and the implications of those choices. I then use this analysis to unpack three SMPs that illustrated cases of important challenges and opportunities in interpreting and enacting the standards.

Research Questions

To this end, this analysis is structured around the following research questions:

RQ1: How do the language, structure, and expectations of students as active generators of knowledge in the SMPs relate to the ways that they are interpreted and enacted by textbooks?

RQ2: How do the structural features of curriculum programs, and the philosophies that underlie them, relate to the ways that SMPs are interpreted and enacted by textbooks?

Literature Review

The CCSSM Standards for Mathematical Practice lay out an ambitious reform agenda for students to gain a set of mathematical habits of mind as an approach to
learning content. However, reform efforts faces challenges from historical approaches to teaching mathematics in the United States, limitations of textbooks to promote change without additional professional learning, and linguistic complexities in interpreting the SMPs. I address each of these challenges and then use them to build a set of tenets for analyzing the use of SMPs in textbooks as my conceptual framework.

**Habits of Mind as Teachable Skills**

The SMPs describe what are often referred to as mathematical habits of mind—the actions, dispositions, and approaches to solving problems that are used by mathematicians. The assumption is that when students develop these habits of mind, they believe that mathematics involves creativity, sense-making, and problem solving, and they view themselves as being capable of tackling challenging tasks (Bass, 2005; Boaler, 2016; Cuoco et al., 1996; National Research Council, 2001; NCTM, 1989, 2000; Skemp, 1976).

Researchers argue that students develop the mindsets of mathematicians by doing the work of mathematicians. This work involves engaging in productive struggle, or expending effort to solve rich problems and wrestle with concepts that are within reach, but not immediately apparent or clearly formed (Hiebert & Grouws, 2007; National Research Council, 2001; NCTM, 2014; Stein, Correnti, et al., 2016). Productive struggle supports students in developing robust schemas (networks of concepts) that are less likely to deteriorate because memory is highly organized and meaningful, easier to access due to multiple linkages for recall, and supportive of transferring, developing, or reconstructing rules when new but related situations are presented (Boaler, 2016; Hiebert
This type of engagement is described in MP1, “Make sense of problems and persevere in solving them.”

When students discuss mathematical concepts, they expand the depth and complexity of their thinking through engaging in sustained reasoning, clarify their thinking by articulating ideas so that others can understand them, build on others’ ideas, make sense of unclear explanations, learn from mistakes, utilize representations and symbols for communicating ideas, and taking on the communication norms of the mathematics community (Boaler, 2016; Boerst et al., 2011; Vygotsky, 1978). This type of meaningful mathematical communication is described in MP3, “Construct viable arguments and critique the reasoning of others.”

While the SMPs are laid out as eight separate descriptions of what “mathematically proficient students” can do, mathematicians tend to use multiple habits of mind simultaneously, and can move dexterously between different approaches depending upon the task at hand (Boaler, 2016; Cuoco, 2018; Cuoco et al., 1996). Thus, most of the other six standards, which address solving problems by making sense of messy contexts (MP2), modeling (MP4), using tools strategically (MP5), attending to precision (MP6), identifying structures (MP7) and formalizing observations about structures (MP8), will almost always occur in conjunction with either MP1 or MP3 (or both). That is, the SMPs are clusters of overlapping practices that are typically used in tandem (Boaler, 2016; Cuoco, 2018).

While habits of mind are flexible and adaptable ways of thinking that experts can bring to bear on a task, they are built on skill sets that can be guided, supported, discussed, and most importantly, practiced (Boaler, 2016; Cuoco et al., 1996; Cuoco,
2018; Franke et al., 2007). For students to build these skills, they need access to tasks that allow them to grapple with new ideas, opportunities to develop and share their thinking, and support from teachers who can help students articulate their ideas and guide a conversation to a mathematical objective (Boerst et al., 2011; Cuoco et al., 1996; Hill, Blunk, et al., 2008; Stein et al., 2008; Stein & Smith, 1998). The requirement of regularly practicing these skills in order to learn them undergirds the instructions from the CCSS Authors (2013) to incorporate the SMPs into every lesson, and further, to use them in a holistic manner as a way to teach content (rather than as separate activities).

Supporting students as they engage in productive struggle and mathematically focused discussions, however, places heavy requirements on teachers. This is a particular challenge in the United States, where many elementary teachers did not learn math this way as students, and may not agree with the philosophy, have the necessary skills in facilitating productive discussions, or sufficiently understand the mathematical content and how to teach it effectively (Henningsen & Stein, 1997; Hiebert et al., 2005; Lortie, 1975; Ma, 2010; Willoughby, 2000). Even when U.S. teachers are using textbooks that support student learning through productive struggle, they tend to decrease rigor of the tasks by removing student opportunity to struggle or replacing mathematically-focused discussions with lecture or a focus on correct answers over strategies (Ball & Cohen, 1996; Grossman et al., 1999; Henningsen & Stein, 1997; Korthagen et al., 2006). Teachers also have a tendency to replace discussions that clarify students’ ideas and lead to a mathematical point with superficial characteristics of these types of lessons, such as having all of the students share their ideas without highlighting key points, making
connections between ideas, or encouraging efficient strategies (Boerst et al., 2011; Sleep, 2012; J. P. Smith, 1996; Stein et al., 2000, 2008).

Teaching these skills to teachers requires extensive professional learning about both mathematics content and facilitation of problem solving opportunities and discussions, but this level of professional learning is rarely available in the United States (Boerst et al., 2011; Ma, 2010; Willoughby, 2000). Instead, this responsibility usually falls on textbooks, which take on an educative role in explaining mathematics pedagogy, content, and facilitation skills (Davis & Krajcik, 2005; Drake et al., 2014; Remillard, 2009; Stein et al., 2007).

**Textbooks as Mediators of Standards**

Textbooks, especially in mathematics, have a considerable influence on what students learn and, to a lesser extent, how they learn it (Houang & Schmidt, 2008; Polikoff, 2015; Remillard, 2005; Stein et al., 2007; Valverde et al., 2002). The majority of mathematics teachers rely heavily on textbooks (including the teacher’s guides) as a teaching tool (Houang & Schmidt, 2008; Stein et al., 2007; Stigler & Hiebert, 1999). This influence is both direct, through interacting directly with the text on the page, and indirect, as teachers adapt, learn from, and make decisions about how the textbooks, suggestions in teachers’ guides, and supplemental resources are used (Ball & Cohen, 1996; Remillard, 2005; Valverde et al., 2002).

It is important to recognize that teachers are not automatons, and adapting textbooks to meet students’ needs, teachers’ beliefs, and a range of other factors is an important roles of teachers (Ball & Cohen, 1996; Remillard, 2005, 2018a; Son & Kim, 2015). However, research suggests, especially in mathematics, that content not covered in
a textbook is unlikely to be taught and that content that is written at low levels of student struggle is unlikely to be elevated to higher levels by teachers (Stein et al., 2007). In addition, mathematics textbooks have measurable impacts on the quality of classroom instruction and students’ test scores, even when the teachers using them have a range of backgrounds and pedagogical beliefs (Agodini & Harris, 2016; Hill & Charalambous, 2012; Remillard, Harris, et al., 2014).

Textbooks have been characterized as a central mediator between policies and standards documents (the official curriculum), what teachers intended to teach (the intended curriculum), what students experience (the enacted curriculum), and what students learn (Remillard, 2018b; Remillard & Heck, 2014; Stein et al., 2007; Valverde et al., 2002). That is, curriculum developers have significant power in interpreting the standards, and these interpretations become one of the major tools through which teachers experience the standards (Ball & Cohen, 1996; Stein et al., 2007; Valverde et al., 2002). And small but consistent decisions about the structure, language, roles, images, and tasks can index values, understandings, and dispositions that are communicated to teachers and students (Herbel-Eisenmann, 2007; Remillard, Van Steenbrugge, et al., 2014).

Textbooks have also been positioned as agents of reform, and since the 1960s, waves of textbooks have been developed with a goal of changing the ways that teachers practice their craft to support increased opportunities for problem solving, student autonomy, conceptual understanding, learning through discussion and student articulation of ideas, and student development of algorithms (Fey & Graeber, 2003; Payne, 2003; Schoenfeld, 2004; Senk & Thompson, 2003; Stein et al., 2008; Willoughby, 2000).
Textbooks published to meet these goals in the 1990s were often termed “standards-based” for their adherence to the Standards published by the National Council of Teachers of Mathematics (NCTM, 1989, 2000), which sought to impose changes in what how mathematics was taught (and learned) in K-12 schools (Senk & Thompson, 2003). However, laying the responsibility for reforming pedagogy on the backs of textbooks was often unsuccessful, as teachers felt overwhelmed with new ways of teaching; did not understand the purposed or content of the new materials; or did not philosophically agree, resulting in teachers reverting to traditional methods of modeling rote algorithms for students to repeat (Schoenfeld, 2004; Willoughby, 2000).

While teacher capacity and inadequate professional development pose a challenge to successfully implementing reform-oriented textbooks (Schoenfeld, 2004; Willoughby, 2000), there is another factor that often prevents these resources from reaching teachers in the first place. Textbook development in the United States is largely market driven, and curriculum publishers—who may spend millions developing a curriculum program—tend to emulate other best-selling programs with a result of remarkable uniformity and an avoidance of reform-oriented approaches (Remillard & Reinke, 2017; Reys et al., 2004; Reys & Reys, 2006; Stigler & Hiebert, 1999). Textbooks designed to embrace reform goals have mostly been developed by universities and small not-for-profits using funding from the National Science Foundation in the 1960s and 1980-90s and private foundations in the past decade, though the textbooks themselves are often printed and distributed by larger publishing houses (Remillard & Reinke, 2017; Schoenfeld, 2004; Willoughby, 2000). As a result, the largest three publishers, Pearson, McGraw Hill, and Houghton Mifflin Harcourt tend to offer two textbook series at each grade level, one that is more
traditional and one that is more reform-oriented, allowing them to maximize sales by reaching both markets (Remillard & Reinke, 2017).

This bifurcation of available textbooks in the U.S. market means that teachers and students in adjacent schools may be receiving materials with dramatically different philosophies about teaching, learning, and the nature of mathematics, depending upon the textbooks that their administrators choose.

**The Standards for Mathematical Practice in Textbooks**

Before digging into how the Standards for Mathematical Practice are used in textbooks, it is important to recognize that they are fundamentally different from the CCSSM content standards. The content standards lay out what students should know and be able to do by the end of a grade. While many of the content standards are written in a conceptually rigorous way, they can generally be taught in discrete chunks (although some of the chunks build on each other), they are easily testable on standardized exams, and focusing on them for a few lessons may be sufficient to cover them for the year as long as they are reviewed periodically. The SMPs, however, are intended to both describe desired habits of mind students should develop and guide educators in selecting tasks that will develop those habits of mind (CCSS Authors, 2013; McCallum, 2012; Munter et al., 2015).

The CCSSM SMPs build on two prior resources that each outline a reform agenda for mathematics education. The NCTM Standards (1989) outlined four process standards, which were increased to five in the updated Principles and Standards document (NCTM, 2000): problem solving, reasoning and proof, communication, connections (between mathematical ideas), and (mathematical) representation. Seeds of these standards (except
for proof) can be found in the CCSSM SMPs. The National Research Council’s (2001) review of research, *Adding it Up*, outlines five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition, which also show up in both the SMPs and the overall organization of the CCSSM, as expressed through the Key Shifts in Mathematics (CCSS Authors, 2010). Both documents set the expectation that practices that support developing these skills should be taken on by K-12 students as an overall vision for daily mathematics teaching and learning.

The eight SMPs cross all grades and are intended to be used throughout the year alongside all content being taught, rather than pulled out for special projects (CCSS Authors, 2013). So, while the CCSS authors claim that they are agnostic to how teachers teach, the SMPs, by listing habits of mind or skills possessed by “mathematically proficient students,” provide a set of expectations for the types of mathematics learning that students should be doing on a regular basis.

While most previous analyses of alignment between the CCSSM and textbooks have focused on the content standards, a few smaller studies have explored how individual SMPs are enacted in textbooks. Their findings suggest that meeting the SMPs is heavily dependent on the tasks that students are given, and also that mainstream textbooks often fail to provide adequate tasks (DiNapoli, 2016; Meyer, 2015). DiNapoli (2016) analyzed two algebra textbooks for MP1: make sense of problems and persevere in solving them. He used a framework of analyzed tasks on the degree to which they supported three aspects that supported students in persevering: a) low floor/high ceiling (both supporting initial engagement and offering high rigor), b) collaborative learning,
and c) perceptions of autonomy/student choice. He found that one textbook offered almost no opportunities for perseverance, while the other offered a great deal more through the design of its tasks.

Meyer analyzed all of the tasks from one algebra and one geometry textbook from the same publisher that claimed to meet MP5, model with mathematics. He found that the majority of the tasks involved having the textbook develop a model and leaving it to the students to perform operations based upon the model and interpret the results of those operations. Students were rarely asked to identify essential variables in a situation, use them to formulate models, or validate the fit of a model based upon the results—that is, the type of mathematics that is at the core of modeling. He recommends, both to meet MP5 and to engage students in mathematics, that teachers “help students see that the world will rarely fully validate the conclusions drawn from their models, that some uncertainty is to be expected, that mathematics is smooth and frictionless, whereas the world is rough and full of surprises” (p. 583).

Theoretical Framework: Interpreting and Enacting the Standard for Mathematical Practice

Defining Interpretation and Enactment

Interpretation is an act of making sense of a piece of text, typically with a goal of understanding the author’s intended meaning, although this may not always be possible, either theoretically or practically. Interpretation depends heavily on having shared understandings of both the denotations (possible literal definitions) and connotations (possible associations and emotional overtones) of terms, as well as what they mean in relation to each other (Hill, 2001; Otte, 1986). Interpreting even the simplest of the
standards or part of a standard requires going through a process of making meaning based on common, mathematical, and contextual definitions of each term, and will be guided by the reader’s personal background, beliefs, prior knowledge, and in some cases, personal goals or visions of what or how students should learn (Hill, 2001; Spillane, 2004). This meaning-making process is complicated when standards are written in vague, complex, and redundant language that can be difficult for teachers (and sometimes even researchers) to understand, as is often the case (Hill, 2001).

While interpretation is a subjective process, this does not mean that all standards interpretations are equally valid (Hill, 2001; Spillane, 2004). Instead, I argue that interpretation should be undertaken with a goal of understanding the intentions of the authors and reading each standard holistically to make sense of contextual clues and examples that clarify these intentions. That is, if standards are intended to unify student experiences around a common set of rigorous expectations, educators should strive to understand the CCSS authors’ intentions. (See chapter 2 for further development of this framework.)

For the purposes of this study, I define enactment of a standard by a textbook as a) developing lessons, tasks, explanations to teacher or students, or other content that addresses that standard and b) tagging that the standard is in use so that the curriculum developer and textbook user have a shared understanding of when the standard is being addressed. Enactment requires not only interpreting a standard, but also making intentional decisions about how to present that interpretation in a way that aligns with larger goals and beliefs (Cohen et al., 2018; Hill, 2001; Remillard, 1999; Remillard & Heck, 2014; Spillane, 2004).
In the field of curriculum research the term *enactment* is typically used to describe how a teacher makes sense of a textbook (the “intended curriculum”) and other goals, resources, and beliefs to construct a coherent lesson for students by following, modifying, omitting, or adding to what is given to them (Ball & Cohen, 1996; Remillard & Heck, 2014; Valverde et al., 2002). I use the term here to describe the similar process that curriculum developers go through in taking the standards, examples of tasks from a variety of resources, existing textbooks that might serve as models, as well as their beliefs and a range of other goals, to create coherent lessons in the form of the textbook and teacher’s guide (Remillard & Kim, 2020). Thus, in this article I use the term *enactment* to refer to the set of decisions that curriculum developers make to bring the standards to life for hypothetical teachers and students through the very real creation of tasks and lessons.

What does all of this look like in practice? Research suggests that the act of interpreting and enacting standards leads to a significant amount of both intentional and accidental modification from the intended meanings. Hill (2001) found that when state standards used language with specific mathematical and pedagogical meanings, teachers and school/district administrators often reinterpreted them with locally-defined or conventional meanings for these terms, causing the intentions and details of the standards to become watered down. Spillane (2004) found that district administrators, who hold roles similar to curriculum developers in establishing content for large groups of teachers and students, often intentionally or unintentionally misinterpreted the messages of content standards based on their backgrounds and beliefs, and these misinterpretations were then passed along to teachers through a variety of local policies and materials.
In addition, curriculum developers are motivated and constrained by market forces, physical space on the page, coverage of multiple standards within a year, making complex or confusing standards language accessible to teachers, and making their materials accessible to teachers who may not receive any professional development or support (Stein et al., 2007). Curriculum publishers have a history of producing new revisions of old textbooks that claim CCSSM alignment through tagging the standards when they happen to appear without making the substantial content or pacing changes necessary for alignment due to the complexity, cost, and time of overhauling a series while still needing to bring in revenue (Cogan et al., 2015; Polikoff, 2015; Remillard & Reinke, 2017).

**Conceptual Framework: Tenets of Analyzing SMPs**

As curriculum developers interpret and enact standards, they are also held responsible to internal and external authorities and users who want to know if their materials are *aligned to* or *meeting* standards (terms that I use interchangeably). The definitions of these terms in the context of standards-based reform can take on a range of meanings and practices depending upon the user’s needs and goals. In this section, I both offer some of the questions that go into defining these terms and propose my own conceptualization of standards-textbook alignment. The conceptualization that I offer is informed by a careful read of the CCSSM, supporting documents by the CCSS authors including their criteria for publishers, and what I see as a comprehensive and accurate enactment of the SMPs.

Before beginning, I want to mention that standards-textbook alignment may be evaluated for a range of purposes, some of which are better met by other approaches. For
example, comparing alignment across multiple sets of standards in multiple states or countries would not be served well by the analytical model that I propose. In addition, my primary purpose is to shift the conversation away from good/bad or yes/no methods of determining if a textbook met the SMPs, which also has its place. Instead, my goal is to understand how textbooks have interpreted and enacted the SMPs. Overall, I want to ask: When curriculum developers have interpreted standards through the process of designing curriculum artifacts, what meanings have they made? When a textbook indicated that an SMP has been enacted, what are students intended to do and learn?

In the K-8 Publisher’s Criteria for the CCSS, the CCSS Authors (2013) warn against separating the practice standards into their component parts, and stress that they need to be used holistically. They give as an example, “SMP.5 does not say, ‘Use tools.’ Or ‘Use appropriate tools.’ It says, ‘Use appropriate tools strategically.’ Thus, materials include problems that reward students’ strategic decisions about how to use tools, or about whether to use them at all.”

Even with this goal, determining the holistic application of and SMPs is not straightforward and raises a number of questions. What happens if an SMP describes two (or more) types of tasks or activities that are not easily or commonly combined? How much of an SMP has to be covered in a single task to “count?” What if a task addresses the organic “feel” of an SMP without including any of the specific types of activities described in it? What if only a small part of an SMP is being addressed in a particular task, but it is being enacted in a particularly effective and noticeable way?

In addition, some of those parts of SMPs are examples, and some of those examples are for different grade bands. Does this mean that students should perform
those example tasks if they are in the right grade band, and if so, does performing those
tasks mean the standard has been met? Should the examples be generalized and applied to
other grade bands when generalized statements are not offered in the SMP?

On the flip side, what if one part of an SMP is being summarized to the point
where it no longer meets the holistic goals of the SMP? Or what if one small part of an
SMP is isolated or taken out of context, such that the resulting task or explanation does
not bear a relationship to the SMP as a whole?

Are there measures for superficial interpretations or enactments of an SMP? And
if so, how can they be distinguished from rich or comprehensive applications?

The CCSS Publisher’s Criteria warn against partial interpretations of standards,
but how important is it to represent full titles in textbooks or teacher’s guides? For
example, while MP5’s whole title is “Use appropriate tools strategically,” is it harmful to
shorten this to “use tools” to avoid making a page too crowded and busy? What are the
implications for shortening or revising a standard to use language that’s more accessible
or clear for teachers and/or students? And do the answers to these questions change if the
content indicated by the label meets the intentions of the standard?

There is no single right answer to these questions, but I will lay out several tenets
that comprised an analytical framework, which I believe shows integrity by attending to
both the spirit and the details of the SMPs.

First, because each standard includes several parts, I propose that each of the
parts of an SMP is important, and all parts of a standard should be met within the
school year. This tenet assumes that each of the parts of the standards were included on
purpose, with intention that they should be applied. While a single task might not meet
every part of a standard, students should have opportunities within each grade of a curriculum program to meet each of the parts at least occasionally. For example, in MP6 (attend to precision), if students “use clear definitions” often but rarely or never “state the meanings of the symbols they choose,” then students have not fully met the SMP.

Second, I propose that the whole, a holistic understanding of the SMP, is important, so meeting an isolated part of a standard does not apply if it does not serve the overall goal of the standard. If a part of a standard is taken out of context in such a way that it doesn’t serve the overall purpose of a standard, then that part is not actually meeting the standard. For example, in MP8 (look for and express regularity in repeated reasoning), if students are expected to “continually evaluate the reasonableness of their intermediate results,” but do not do this for the larger purpose of generalizing from repeated situations to general formulas or equations, then they are not meeting the standard.

Third, because a number of standards use illustrative examples in place of more generalized ideas, I propose that examples included in the standards are intended to exemplify more generalized ideas, even if those ideas are not articulated in the standard. In MP7 (look for and make use of structure), the example states: “students will see $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property.” This example is relevant to grades 3-5 as it is written. However, it should also be interpreted as a generalized statement like “students will identify relationships between numbers that derive from underlying properties of operations,” which can apply to all grades and many topics. When an example addresses middle school or high school content, elementary school students should still be expected to meet
a generalized version of that example. At the same time, the tenet of attending to the whole still applies. If students utilize the distributive property without “looking closely to discern a pattern or structure,” an action that implies discovery and identification by students, then they are not meeting MP7.

The fourth tenet addresses who is doing the work. Because the verbs throughout the SMPs—making sense, persevering, reasoning, constructing arguments, modeling, using, looking—suggest that students should perform these actions, students must be expected to do the intellectual work indicated; teachers or textbooks cannot perform the work on the student’s behalf. If a standard states that students are supposed to discover, analyze, evaluate, justify, etc. a situation or concept, they are not able to do so when they are replicating an explanation or algorithm that was described by the teacher or textbook earlier in the lesson, because the opportunity to do the work of the standard has been removed. Examining the language of the standards indicates that when students are expected to replicate the exact procedures just demonstrated by the teacher or text, this violates the meaning of both the parts and the whole.

The fifth tenet addresses language in labeling the standards: textbooks may shorten standards titles for ease of use, but they are still responsible for addressing the parts and the holistic intentions of the standard. I recognize that space is at a premium on textbook pages, and unnecessary or age-inappropriate text can be overwhelming to students. Thus, if a publisher chooses to label the standards on student pages (which, notably, is not required or necessary), I support the decision to shorten the labels in a way that makes them both manageable and accessible. However, these shortened labels should not be confused with or replace the full text of the standards when curriculum developers
are interpreting or enacting the SMPs. That is, while a curriculum developer may apply a “use tools” tag as a visual marker, they should be held responsible for having students “use appropriate tools strategically” according to the full text of MP5 when they design tasks.

With these five tenets as an analytical framework, I explore questions of how textbooks interpret and enact the CCSSM Standards of Mathematics Practice. What does it look like when textbooks fully embrace both the spirit and the details of the SMPs? What are the trends in how textbooks might partially meet standards? That is, what are the consequences of violating one of the tenets above? How are textbooks interpreting, mis-interpreting, or re-interpreting the standards? And how are they failing to meet them altogether?

**Methods**

**Textbook and Lesson Selection**

This study aimed to offer a comprehensive understanding of how the CCSSM have been taken up in textbooks in the United States. Programs were eligible for analysis if they were either fully developed after the CCSSM were released or underwent a significant revision to align with the CCSSM. I was able to obtain access to eight curriculum programs, four developed by grant-funded organizations or universities and four from commercial publishers, as shown in Table 3.1. A handful of other curriculum programs would have also been eligible for this study but were not included because they publishers did not make them available.
Table 3.1

Curriculum programs used in the analysis with abbreviations, publishers, and publication years.

<table>
<thead>
<tr>
<th>Developer Type</th>
<th>Program</th>
<th>Abbr.</th>
<th>Developer/Publisher</th>
<th>Publication Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grant-Funded Organization or University</td>
<td>Bridges in Mathematics</td>
<td>BRI</td>
<td>The Math Learning Center/Curriculum Associates</td>
<td>2015</td>
</tr>
<tr>
<td>Grant-Funded Organization or University</td>
<td>Eureka Math</td>
<td>EUR</td>
<td>Great Minds</td>
<td>2013</td>
</tr>
<tr>
<td>Grant-Funded Organization or University</td>
<td>Everyday Mathematics 4</td>
<td>EVER</td>
<td>University of Chicago School Mathematics Project/McGraw Hill</td>
<td>2015</td>
</tr>
<tr>
<td>Grant-Funded Organization or University</td>
<td>Investigations in Number, Data, and Space 3</td>
<td>INV</td>
<td>TERC/Pearson</td>
<td>2017</td>
</tr>
<tr>
<td>Commercial Publisher</td>
<td>enVision Mathematics Common Core 2020</td>
<td>ENV</td>
<td>Pearson</td>
<td>2020</td>
</tr>
<tr>
<td>Commercial Publisher</td>
<td>Go! Math</td>
<td>GO</td>
<td>Houghton Mifflin Harcourt</td>
<td>2015</td>
</tr>
<tr>
<td>Commercial Publisher</td>
<td>Into Math</td>
<td>INTO</td>
<td>Houghton Mifflin Harcourt</td>
<td>2020</td>
</tr>
<tr>
<td>Commercial Publisher</td>
<td>My Math</td>
<td>MY</td>
<td>McGraw Hill</td>
<td>2018</td>
</tr>
</tbody>
</table>

I selected lessons for this analysis based on the needs of a larger study that also addressed CCSSM content standards and required restricting the study to a single content area. As a result, all lessons in this analysis address multiplication, which is central to older elementary grades, has a history of being taught using rote memorization, and has clear guidelines for being taught using new representations and conceptual approaches under the CCSSM content standards. As the CCSSM Practice Standards are supposed to be addressed through daily lessons, rather than concentrated to particular lesson types or topics, I assumed that SMP usage in multiplication lessons would be representative of SMP usage through the rest of the curriculum program. A brief visual confirmation of SMP tagging practices throughout the curriculum programs supported this assumption.
I used a random number generator to select five lessons per curriculum program in each of grades 3, 4, and 5 out of the multiplication lessons selected for the larger study, for a total of fifteen lessons per program and 120 lessons total. These lessons are titled in the format EVER-G G3 4.8, where EVER-G is the program abbreviation, G3 is grade 3, and 4.8 is unit 4, lesson 8.

Because some curriculum programs tagged around a dozen SMPs per lesson, while others tagged three or fewer per lesson (including zero in some lessons), the randomly selected lessons for a given curriculum program sometimes did not include sufficient examples of all eight SMPs. When two or fewer examples of an SMP were present in the sample, all additional lessons that tagged that standard in the larger study were identified, and then additional lessons were randomly selected from within that set. This increased the total number of lessons analyzed to 125.

**Analytical Methods: Standards**

This study involves two types of analysis: unpacking the SMPs and investigating how they’re interpreted by textbooks. To unpack the standards, I first divided each standard into shorter segments that I call *statements*. These might be sentences, phrases, or single words that encapsulate single ideas. My initial subdivisions were based upon sentence structure (usually with whole sentences as statements), but I revised them during analysis when it became apparent that textbooks were often addressing the SMPs at an even smaller grain size.

I labeled each statement with a letter for easy reference during the analysis, as shown in the first few lines of MP7 below. I initially labeled the first sentence as a single statement, but after realizing that it was important to distinguish between whether
textbooks were addressing patterns or structures, I separated the sentence into two statements, A and B. I gave each elementary example its own statement (as in C and D) because some textbooks developed tasks that were aligned to the SMP examples and often because the SMPs used examples in place of generalized statements, requiring the reader to generalize the examples as described in Tenet 3. When examples referred to middle and high school, I used strikethroughs to selectively remove grade-specific content with a goal of replacing it with more generalized language. For example, I removed the entire second sentence of E because it did not add anything meaningful to the example from the first sentence of E which both addresses a specific example that could appear in grades 3-5 textbooks and can be generalized as something like noticing patterns that arise from underlying mathematical properties. In F, I left the sentence structure intact while removing the high-school specific content to create a more generalized statement that reads something like “[Students] recognize the significance of [key structural features of a task] and can use the strategy [suggested by the key features] for solving problems.”

(A) Mathematically proficient students look closely to discern a pattern or structure. (C) Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may (D) sort a collection of shapes according to how many sides the shapes have.

(E) Later, students will see $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They (F) recognize the significance of an existing line in a
I then analyzed the standard’s content and structure by breaking down potential interpretations of both the separate statements and the holistic meaning, bringing in outside literature about the purpose and interpretations of the standard to support the analysis. When interesting trends appeared in the textbook analysis, I returned to the SMP text to better understand those relationships in an iterative process.

In unpacking the standards, one of my goals was to be transparent about my own acts of interpretation. I attempted to interpret the SMPs as literally as possible, with attention to the language of each statement and the whole standard, based on my own knowledge of language meanings within the mathematics education community. This exposed both the various ways that textbooks interpreted the standards, but also many ambiguities in the standards themselves and the multiple ways that they could be interpreted. Following the integrity approach, I looked for both stringent and loose matches to the written standards, but took the perspective that all text in the standards was included for a reason, and that each individual statement was intended to serve the holistic goal of the larger standard.

Analytical Methods: Interpretation by Textbooks

I analyzed lessons using Atlas.ti. Whenever a textbook indicated that an SMP was being used, which I call tagging, I analyzed the relevant lesson, task, student note, or teacher’s note associated with that SMP using two sets of codes. The first set included the statements that composed each of the SMPs, as described in the previous section. The second indicated the alignment style and quality of the task or note in meeting the
statements, using an emergent set of codes that are listed in Table 3.2. Several of these codes are discussed in greater detail in the findings section.

Table 3.2

Alignment styles and qualities with descriptions used in coding SMP enactment in textbooks.

<table>
<thead>
<tr>
<th>Alignment Style</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson alignment</td>
<td>The entire lesson was tagged (usually at the beginning of the teacher’s guide to the lesson) with a standard.</td>
</tr>
<tr>
<td>Task alignment</td>
<td>An individual task was tagged with a standard.</td>
</tr>
<tr>
<td>Definition/ explanation</td>
<td>The teacher’s guide contained a definition or explanation of a standard as an educative support to teachers. This was always co-coded with either Aligned, Isolated Statement, or Not Aligned to reflect the quality of the definition.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quality of alignment</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aligned</td>
<td>The task or note was aligned to the coded statements in a way that supports a holistic understanding of the standard and the individual parts.</td>
</tr>
<tr>
<td>Isolated statement</td>
<td>The task/note was aligned to one or two coded statements in a way that does not match a holistic understanding of the standard. (Violates tenet 2 and sometimes 3.)</td>
</tr>
<tr>
<td>Teacher/text demonstrated the SMP</td>
<td>The textbook or teacher was expected to demonstrate doing the work of the SMP while students were expected to observe, follow instructions, or confirm understanding. (Violates tenet 4.)</td>
</tr>
<tr>
<td>Students replicated steps</td>
<td>Students were expected to replicate a set of steps or an explanation that was given by the teacher or text earlier in the same lesson rather than doing the work themselves. (Violates tenet 4.)</td>
</tr>
<tr>
<td>Students described without doing</td>
<td>Students were expected to describe what they might do to address a statement, but not actually do it. (Violates tenet 4.)</td>
</tr>
<tr>
<td>Optional extension</td>
<td>The teacher’s guide tags the SMP in a question or task that is part of the core lesson but indicated as optional.</td>
</tr>
<tr>
<td>Not aligned</td>
<td>The task/note shows no visible alignment to either the individual statements or the holistic meaning of the standard (Violates tenets 1 and 2).</td>
</tr>
</tbody>
</table>

For example, the task below from EVER-G G3 4.8 (Figure 3.1) is tagged by the textbook as using MP7, “look for and make use of structure.” (The small yellow rectangle labeled GMP7.2 indicates that this standard is supposed to meet MP7, goal #2 in
Everyday Math’s SMP alignment system.) The instructions for the teacher and the relevant component of the student textbook were treated as a single quote, which received a Task Alignment code to show that the tagging was done at the task level. The task addressed statements 7B, 7F, 7G, and 7H which were each applied as separate codes. It also received a code of Aligned because the task addressed the individual statements coded in a way that supports a holistic interpretation of the MP7.

Figure 3.1

*Example task and instructions from EVER-G G3 4.8 that are aligned to statements from MP7.*

In this analysis I coded only the tasks where the curriculum developers indicated that they were meeting a standard with a tag. That is, my goal was not to determine which SMPs a lesson might potentially meet, but rather to understand how curriculum developers were interpreting the standards and communicating with teachers about meeting the standards. With that said, I did note when a task tagged for one standard was a particularly good example of a different standard that was not tagged, as consistent under-labeling and mislabeling also exposed important trends.
One curriculum program, Bridges in Mathematics (BRI-G), did not tag individual tasks and only tagged at the lesson level. In these lessons, I read the full text of the lesson and identified sections that met specific aspects of the tagged lesson standards.

As the total number of tagged standards could vary so widely from program to program (from around 20 in BRI-G and EUR-G even after adding supplementary lessons to cover all the standards to around 90 in GO-R and INTO-R), I did not feel that quantitative comparisons were either helpful or meaningful. Instead, I used Atlas.ti to sort, identify trends, and closely examine examples to reach a qualitative understanding of how the SMPs were being interpreted and enacted in textbooks.

Based on these codes, I then looked for trends in several areas: how SMPs were tagged by textbooks, how SMPs were enacted in textbooks that expressed different views of learning through their expected roles for students, how individual statements and holistic interpretations of SMPs were addressed by textbooks, and how rephrased standards titles seemed to relate to SMP enactment in tasks.

To reveal each of these aspects, I chose three SMPs to serve as case studies, one that focuses on the structure of textbooks (MP3) and two that focus on the language and structure of the standards (MP4 and MP8).

**Findings: Mathematical Practices in the Structural Design of Textbooks**

Interpreting and enacting SMPs happens at several levels in curriculum programs. Some decisions are embedded in the structural design of the entire curriculum program, such as determining what roles students and teachers will take on across the year and how SMPs will be tagged and explained to support teachers. With regards to addressing the
SMPs, there are several major questions that curriculum developers had to decide upon before writing lessons or tasks are written:

1. How and when were SMPs tagged to communicate to teachers that they were being used? (Educating and communicating with teachers)
2. How were students asked to solve problems and express ideas? Did these approaches give students the opportunity to do the work of the SMPs? (Tenet 4)
3. Were SMPs addressed holistically or only in parts? (Tenets 1-3)
4. Were SMPs addressed with full titles or were the titles shortened? If the titles were shortened, were the shortened titles treated as replacements for the holistic SMPs? (Tenet 5)

When I analyzed the eight textbooks in this study, I found that the ways that curriculum developers addressed each of these questions fell into two distinct categories based upon their overall philosophy of students’ roles in learning, as revealed through the structure of their lessons and tasks.

Four of the textbooks, BRI, EVER, EUR, and INV expressed a view of learning that positioned students as *generators of knowledge (GK)*. In these textbooks, lessons were generally structured to support students in taking active roles in solving novel tasks and generating mathematical ideas independently or as a class. While this view often resulted in meeting Tenet 4 (that students do the work of the SMPs), this view also corresponded with how the authors addressed and communicated about the SMPs more generally in the structural design of the lessons. For instance, the tended to tag SMPs for extended tasks, use the full language of the standards, and address the SMPs holistically in most lessons. For the remainder of this chapter, I will append a G to the abbreviation
of these textbooks to make them easy to identify. (E.g., BRI-G identifies *Bridges in Mathematics* as a GK textbook.)

ENV, GO, INTO, and MY expressed a view of learning that positioned students as *receivers of knowledge* (RK). These textbooks used a lesson structure in which the teacher or the textbook would demonstrate the steps of mathematical thinking through completed or partially completed examples with clear explanations, while students observed and sometimes described or explained these steps. When students were asked to solve open-ended tasks in these textbooks, they were always expected to replicate strategies that had already been taught rather than engage in productive struggle with novel tasks. In addition to preventing students from doing the work of the SMPs (tenet 4), these textbooks also used similar structural approaches to enacting the standards, by tagging SMPs for tasks that were too limited in time and complexity to meet the full SMPs, addressing the SMPs only in isolated parts without attending to holistic intentions, isolating SMPs in isolation from each other, and rephrasing SMPs in ways that seemed to negatively impact how they were interpreted. I use an R in the textbook abbreviations (e.g., ENV-R) to identify these textbooks.

While I based these categories entirely on the content and structure of the lessons in these textbooks and the ways that they addressed the SMPs, it is worth noting that all of the GK textbooks were developed by grant-funded small organizations or universities, while all of the RK textbooks were developed by major publishers. While this does not imply that these types of organizations always produce these types of textbooks, it does suggest that market forces may play some role in the willingness of curriculum developers to fully enact the SMPs. These emergent categories also suggest that
philosophical views about students’ roles in learning may impact how curriculum
developers interpret and enact the SMPs at a variety of structural levels that are detailed
below.

Lesson Structure: Tagging Standards

One interesting finding from this analysis is that textbooks that shared a
philosophical view about students’ roles in learning also used similar styles of tagging
SMPs within lessons. I define tagging as an act taken by curriculum developers to
communicate to teachers that an SMP is being used. That is, throughout this analysis I do
not explore overall lessons to identify potential applications of the SMPs, but I instead
analyze how SMPs are interpreted and enacted when a textbook explicitly announces,
through a tag, that the SMP is in use. This approach allows me to explore several things:
the alignment between the tasks themselves and the SMPs that they claim to meet, how
textbooks claim to address SMPs over multiple lessons, and the overall ramifications of
structural lesson designs as they relate to tagging.

I refer to the approach used by textbooks that position students as receivers of
knowledge as tagging individual tasks and the approach used by textbooks that position
students as generators of knowledge as highlighting lesson opportunities.

Tagging individual tasks

The tagging individual tasks approach was used by all four RK textbooks and
looked relatively similar across the four programs. In this approach, around one-third to
one-half of the tasks in a lesson were tagged with different standards using a distinctive
icon and text color. For example, Figure 3.2 from INTO-R G3 1.1.5 contained three tasks
that were each assigned to different SMPs along with explanatory text for the teacher (on
the right). In this tagging approach, the tasks could be related or unrelated to each other, and a single lesson could contain with up to a dozen different SMP tags. These tags were often accompanied by short explanatory text which described the goal of the task but did not necessarily connect it to the mathematical practice or support the teacher in deepening the practice. In this example, the explanatory text is unrelated to the SMPs that were tagged.

**Figure 3.2**

*Example of tagging individual tasks from INTO-R G3 1.1.5 that includes three short tasks that are each assigned to a different SMP (left) and a teacher’s note (right) that does not explain the role of the SMPs.*
The tagging individual tasks approach carries with it several implicit assumptions about time, space, and alignment. The first addresses time; this style asserts that extremely short tasks can provide students with ample opportunities to enact an SMP, which does not match the depth and complexity of the SMP expectations. The second addresses space with the assumption that small spaces, such as a single line where students report their answers, can capture the important information about a student’s response to the task. (By comparison, a larger space could encourage students to record their strategies and imply that pathways to reaching a solution are more important than the final answer.) Third, this style suggests that a given task can only meet one SMP, rather than considering how several standards might be used in combination with each other. And fourth, tagging so many unrelated SMPs in a short space suggest a “more is better” or “quantity over quality” attitude toward enacting SMPs that might support textbook sales but does not necessarily support teachers in deepening their practice.

**Highlighting lesson opportunities**

The second format, highlighting lesson opportunities, is a loose collection of approaches taken by the four GK textbooks. Each of these programs used a different format for recognizing practice standards, usually tagging zero to three SMPs per lesson that were of greater focus, even if other SMPs might have (and often did) apply to some or all of the lesson.

INV-G highlighted lesson opportunities through descriptive notes to the teacher. For example, INV-G G3 5.1.2 tagged only two SMPs in the lesson, each following a task that provided an especially appropriate opportunity to address that SMP. Although the majority of INV-G lessons could have been tagged with SMPs 1, 2, 3, 6, and either 7 or 8
because all lessons were organized around productive struggle and focused mathematical discussion, only one to three SMPs were tagged in each lesson. Each tag was accompanied by a description that often included the curriculum developers’ intentions, the goal of the activity, key components of the SMP, and suggested ways to reinforce the practice with students, as shown in Figure 3.3. This serves the purpose of both identifying the standard and educating the teacher about how it applies in context.

**Figure 3.3**

*Example of highlighting lesson opportunities from INV-G G3 5.1.2 that includes educative features about how MP2 applies in the task.*

In EVER-G, each lesson began with a list of the 2 or 3 SMPs that were addressed in that lesson. They were then tagged throughout the lesson text every time that they arose, sometimes as frequently as once per sentence. For example, **Figure 3.4** from
EVER-G G3 1.8 demonstrates four activities that supported MP2 and two that supported MP4. Notably, all of the tagged activities were part of the same larger exploratory discussion, rather than being separate and unrelated tasks, showing how the SMPs intertwine and support each other. This tagging approach supports teachers in developing teaching moves that address specific SMPs and can be transferred to other lessons. As in BRI-G, while most of the eight SMPs could have been tagged in this lesson, only two were selected to focus on.

Figure 3.4

Example of highlighting lesson opportunities from EVER-G G3 1.8 that identifies how goals within MP2 and MP4 are met by specific prompts and activities within a larger task-focused discussion. Note: GMP refers to EVER’s internal SMP labeling system.

Strategies may include counting the objects in the picture by 1s, counting by 6s, adding 6s, or doubling 6 and then adding 6 more. Make sure the class understands each representation and agrees that it matches the context of the problem. [GMP2.2, GMP4.1]

Ask: What do all these representations have in common? [GMP2.2]
Sample answer: They all show 3 groups of 6. Tell children that groups with the same number of objects are called equal groups. Stories that involve finding the total number of objects in a set of equal groups are called equal-groups number stories. Invite volunteers to explain how their representations show the equal groups from the story. [GMP4.1] Sample answers: My number model shows 6 added 3 times. I drew a square around each group of 6 stickers. The rows in my array are groups of 6.

Ask children to share number models that match the sketches in the margin. [GMP2.2] Children may suggest \(6 + 6 + 6 = 18\) and \(3 \times 6 = 18\). Record both, and ask: How do these number models match what is happening in our pictures? [GMP2.2] Sample answer: We are adding 6 three times because we have three equal groups of 6.

EUR-G typically tagged one SMP for one task in each lesson, with occasional lessons tagging zero or two tasks. The tags identified sections of a larger discussion that were particularly relevant to a given standard and identified them by standard number, as shown in Figure 3.5 from EUR-G G3 1.15. While MP7 could have been tagged in
multiple locations, this is the only section in the lesson that the curriculum developers chose to highlight. There is typically no explanation given, which provides less educative support to teachers than the other GK textbooks, though a few EUR-G lessons have educative notes describing the goals of the SMPs. While this is the only SMP tagged in this lesson, almost all EUR-G lessons could meet MP2, MP3, MP7, and/or MP8, and some meet other SMPs as well.

Figure 3.5

Example of highlighting lesson opportunities from EUR-G G3 1.15 that shows how a small section of the larger lesson is identified as meeting MP7.

BRI-G took the simplest approach and only listed SMPs at the beginning of the lesson, though a minority of BRI-G lesson also included a sidebar note to the teacher about the application of a selected standard. For example, BRI-G 2.3.5 (Figure 3.6) contained a task where students engaged in productive struggle to find a way to multiply $6 \times 26$ (MP1) using tools of their choice (MP5), and then had a mathematically focused discussion (MP3). The lesson is tagged with three SMPs, but it actually meets all eight SMPs. It is interesting that this example is one of the few places in the entire sample from eight textbooks where students were truly free to select their own tools to address MP5, yet the sidebar focused on MP3 which may have been considered more important or challenging for teachers to support.
Figure 3.6

Example from BRI-G 2.3.5 of a rare teacher’s note that explains how MP3 is utilized in a discussion. (Most BRI-G lessons only tag standards for the whole lesson.)

The decision to highlight only a few SMPs per lesson as an educative approach to teachers has both benefits and drawbacks. Because the GK textbooks all use approaches to learning that center around student discovery and discussion in various ways, it is rare to find a lesson in any of these programs that does not require students to make sense of problems and persevere in solving them (MP1), reason abstractly and quantitatively (MP2), construct viable arguments and critique the reasoning of others (MP3), look for and make use of structure (MP7), and look for and express regularity in repeated reasoning (MP8). Overall, if the GK textbooks tagged each of the SMPs each time the occurred, their lessons would be swimming in tags almost to the point of uselessness due to the redundancy and repetition. As likely intended by the curriculum developers, highlighting just one or two in each lesson draws attention to a few focal standards and can be educative to teachers.
However, this highlighting approach also results in a significant under-tagging of content that meets other SMPs, often many of them at once. This may send the message that not many standards are being met, which could detract from sales if reviewers are not aware of the full use of the SMPs in these textbooks. When taking an integrity approach to analyzing textbook-standards alignment that rests on identifying lessons where standards are tagged, this could also result in substantial under-recognition of SMPs that are actually being met (though typically the SMPs are met so robustly in other parts of these textbooks that it is less of an issue).

**Task Structure: Students Doing the Work**

With an understanding of where and how textbooks claim to address standards, I next pick up tenet #4: Students must do the work described by the standard; teachers or textbooks cannot perform the work on the student’s behalf. To address this tenet, I ask the questions: How were students asked to solve problems and express ideas? How did the structural features of tasks support or deny students the opportunities to fulfill the holistic intentions of the SMPs by doing the work described?

Language across the SMPs places an emphasis on student doing the work of planning solution pathways, considering appropriate representations, discerning trends, and so on. Reading the standards holistically, there is an assumption that students will be engaging with rich and complex tasks for which the solution strategy is not immediately obvious, requiring students to interpret, try, analyze, and justify and a number of other verbs that indicate a level of uncertainty and openness to the tasks that supports students in actively engaging in the intellectual work of mathematics. When tasks are not designed
to support students in actively engaging in this work, they may go through some of the superficial motions of aspects of the standards but cannot meet the full intentions.

I found four structural features of tasks that tended to reduce students’ opportunities to do the work of the SMPs. I used these as codes during analysis, which allowed me to see patterns in how both GK and RK textbooks either supported students in doing the work of the standards or reduced it. Table 3.3 provides a description of what a task looks like when it allows students to do the work of the SMP as a comparison, and then describes the four structural approaches that prevent students from engaging in this work.
### Table 3.3

*Tasks approaches to enacting SMPs with descriptions and examples.*

<table>
<thead>
<tr>
<th>Task Approach</th>
<th>Description</th>
</tr>
</thead>
</table>
| Students do the work (Aligned to SMPs) | Students do the intellectual work of the tagged SMP, with the teacher acting as a support or guide. *Example:* MP6 expects students to give clear explanations and definitions and MP8 expects students to notice repeated calculations and use them to generalize rules or equations. In EVER-G G5 6.1, students use the skills listed in MP6 and MP8 to complete an example table using understandings of place value, then write rules for the pattern in their own words. Then the teacher guides students in articulating clear rules in a class discussion. This task supports students in addressing the intentions of both SMPs.  
  > When most students have finished Problems 1–6, invite them to share the patterns they noticed, and ask volunteers to share their rules for multiplying and dividing decimals by powers of 10. *GMP6.1* Encourage students to use clear mathematical language as they share and discuss.  
  > *GMP6.3*  
  >  
  > **Sample rules:**  
  >  
  > • When multiplying by a power of 10, the decimal point moves to the right. The exponent in the power of 10 indicates the number of places the decimal point moves. Zeros are sometimes attached to the right of the starting digits to show how the digits have shifted. (See the Common Misconception note.)  
  > • When dividing by a power of 10, the decimal point moves to the left. The exponent in the power of 10 indicates the number of places the decimal point moves. Zeros are sometimes inserted to the left of the starting digits to show how the digits have shifted. |
| Teacher/text demonstrates SMP | Students observe the SMP being modeled by the teacher or textbook and are either asked to describe what occurs or fill in a few highly scaffolded blanks. Because mathematical ideas are provided for students, they do not have the opportunity to engage with the SMPs.  
*Example:* MP1 expects students to make sense of problems without immediately obvious solutions, test out solution paths, and evaluate the results. In MY-R G3 6.2, students observe a strategy being modeled and then the teacher guides them to fill in several blanks that summarized the results. Filling in blanks does not constitute the type of creative problem solving described in MP1. |
|---|---|
| Example 2 | Students are asked to replicate a strategy that has just been presented by the teacher or textbook. The textbook demonstrates how to go through the steps of a new strategy first, so that when students repeat it there is no longer an opportunity for productive struggle.  
*Example:* MP3 expects students to share ideas and strategies that they have generated and critique others' ideas, but this task in GO-R G5 1.5 expects students to articulate a rote procedure that has just been modeled by the teacher. |
| Students describe without doing | Students are asked to suggest ways that they could hypothetically meet an SMP (e.g., by suggesting alternative solution strategies or tools), but they do not actually complete this action. Typically this occurs when the textbook has just modeled a specific strategy, and students are asked in a sidebar of the teacher’s edition or an independent practice question to list other strategies they could have used instead. Listing alternatives is not the equivalent of the type of uncertainty and choice that supports productive struggle.  

*Example:* MP5 expects students to choose their own tools strategically while solving problems. In this task from INTO-R G3 2.3.1, students are instructed to complete a number line model on the textbook page and then brainstorm alternative tools that they could have used—but they are never given the opportunity to actually use these tools. |
|---|---|
| Optional extension | The SMP is addressed in an optional question that extends the core task, suggesting that teachers might not include it or that only subgroups of students would have the opportunity to meet the SMP. These extensions are not complete, stand-alone differentiation tasks (which were not coded), but rather options to extend the primary discussion or task.  

*Example:* MP2 expects students to represent contextual (real-life) problems with abstract symbols. In this task from INV-G G3 5.2.1, all students are instructed to practice fact fluency with a set of decontextualized array cards in a game format. As an optional extension, the teacher is advised to provide them with contextual situations, which has no direct bearing on the task and which the teacher might or might not do. Thus, the core task does not meet MP2 at all, but the teacher might optionally modify it to meet aspects of MP2. |
As a general rule, textbooks where students generate knowledge (GK) mostly have tasks that are designed to have student do the work, while textbooks where students receive knowledge (RK) mostly do not provide students with opportunities to do the intellectual work of the SMP because they use tasks where the teacher/text demonstrates the SMP, students replicate steps, and/or students describe without doing. However, there are some exceptions to both, which I was able to compare by coding each task individually for these approaches to doing (or not doing) the work of the SMP.

Optional extensions seemed to be used across the program types and appeared in five textbooks: BRI-G, GO-R, INTO-R, INV-G, and MY-R. Notably, optional extensions are predominantly a concern in RK programs where students might not have other opportunities to struggle, whereas in GK programs students have abundant opportunities to struggle and the optional extensions provide additional opportunities that might highlight a specific SMP.

*Enacting SMPs: Attending to Parts and Wholes*

The first three tenets of alignment address attending to the parts of each SMP (including the examples) as well as the whole. Whether curriculum authors chose to interpret and enact standards holistically or piecemeal had significant impacts on task development. Following tenets 1-3, textbooks may or may not have 1) attended to the overall goals of the SMP without covering some of the parts, 2) enacted some of the parts in ways that didn’t serve the overall goal, or 3) treated examples as parts in isolation from the whole or not expanded the examples to generalize them.
In my analysis, I was not able to address tenet 1 (covering all the parts) fully, as I used only a sample of multiplication lessons rather than analyzing all lessons in each grade. However, there were interesting trends in parts of standards that were addressed repeatedly across many or all of the programs, while other parts of the standards appeared rarely or not at all in the samples, which gives hints about meeting tenet 1.

I was able to assess tenets 2 and 3 in every task by identifying which parts of the standard were met, and whether they were met in ways that attended to the intentions of the whole. For example, MP6 expects students to “attend to precision” by giving clear definitions, stating the meanings of symbols, identifying the level of precision appropriate to a context (estimating), and calculating accurately and efficiently. A key component to “attending” to precision is choice—if there are no opportunities to be imprecise (or inaccurate, or inefficient), students cannot demonstrate showing attention.

This task in EVER-G G3 1.8 (Figure 3.7), provides an example of addressing only one part of MP6 while maintaining alignment to the overall intentions of the SMP. Students are given the opportunity to choose strategies and then discuss the relative efficiency of different strategies so that they can make more efficient choices in the future, which allows them to “attend” to how their decisions impact precision.

**Figure 3.7**

*Example of addressing one part of MP6 in a way that also attends to the goals of the whole standard from EVER-G G3 1.8.*
However, in ENV-R G5 8.5 (Figure 3.8) the instruction to calculate efficiently is taken out of the context of the larger SMP. In this task, this MP6 was interpreted to mean that students should follow a strategy that has already been modeled to calculate accurately and get a correct answer, without giving the student an opportunity to struggle with levels of precision, clear communication, or efficiency of strategies. Although students must calculate accurately to solve this problem (and every other problem in every textbook), they have not done so in a way that addresses the holistic intentions of MP6.

**Figure 3.8**

*Example of addressing part of MP6 without attending to the whole from ENV-R G5 8.5.*

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**Enacting SMPs: Rewriting the Standard Titles**

One of the challenges of the SMPs, as they are written, is that their length and complexity makes it difficult to reference the text within lessons in meaningful ways. Writing shorter versions of the SMPs can both make it easier to fit text into the limited space on a page and make the SMPs easier to remember and understand. However, summarizing complex ideas into simple statements comes with a loss of nuance, and may also result in intentional or unintentional changes in meaning. In the fifth tenet of alignment, I suggest that textbooks may shorten standards titles for ease of use, but they
are still responsible for addressing the parts and the holistic intentions of the standard, rather than only the goals suggested by the abbreviated titles.

The full text of the SMPs, which are formatted as long, dense paragraphs, is never printed in any of the textbooks. In comparison to the content standards, which are at most 3-4 sentences and often printed in full, teachers have far fewer opportunities to read the full text of the SMPs through their textbooks. At best, textbooks made use of the SMP titles, which range from around three to ten words, though even these are often shortened.

Five of the curriculum programs offered revised versions of the titles of the SMPs using three different approaches. When this happened, the curriculum authors seemed to base their lessons more on the revised statements than the original SMPs. This section explores a structural overview of these decisions and some of their implications.

**No Revisions: Bridges into Math, Investigations, and Eureka Math**

When BRI-G, INV-G, and EUR-G tagged or discussed the SMPs, they either used the full, original text of the SMP titles or used only the number in reference to it. As these textbooks, on the whole, tended to embrace the spirit of the SMPs in their lesson design, this tagging decision gave teachers more information about the skills that they were supposed to be developing. Working with only the titles still removes many nuances of the full text of the SMPs, but it holds true to their intention. Using numbers only may also encourage teachers to look up and read the entire SMPs themselves. In either case, these representations avoided misconceptions that could arise if revisions of the SMPs were mistaken for the full text.
Close Summaries: Everyday Math

Everyday Math (EVER-G), was the most intentional program in its revision of the standards into meaningful, student-friendly interpretations. This made it a good resource for exploring the implications of revising and summarizing the SMPs. The program provided a list of 23 Goals for Mathematical Practice (GMPs), with two to six GMPs used to summarize each SMP. Many GMPs improved the clarity of the SMPs by simplifying the language and the pedagogy while retaining key intentions. For example, when summarizing MP3 (construct viable arguments and critique the reasoning of others), the GMPs conveyed the primary goals of the standard in accessible language: “GMP3.1 Explain both what to do and why it works” and “GMP3.2 Work to make sense of others’ mathematical thinking.” However, some sets of GMPs substantially changed the meanings of the SMPs or left out essential components. For example, the GMPs that summarized MP5, Use appropriate tools strategically, did not mention technological tools although these compose the majority of the SMP.


In three of the RK programs, reducing the complex standards to short phrases often seemed to influence how the SMPs were interpreted and enacted. For example, MP5’s title is “use appropriate tools strategically,” which summarizes how the SMP expects students to “consider the available tools” and “make sound decisions about when each of these tools might be helpful,” and “identify relevant external mathematical resources.” ENV-R and MY-R rephrased MP5 to “use appropriate tools” and INTO-R rewrote it as only “use tools.” While these revisions seem to make the standard more concise by removing a potentially unimportant words, they actually served to change the
meaning of the whole standard. By removing the terms *appropriate* and *strategically* from this standard title, the curriculum developers also removed the expectation that students would do the work of choosing appropriate tools based upon considerations of their strengths and limitations.

**Redefining: Go! Math**

Any written text may have the occasional typo, or in this case, a mis-tagged standard. However, *Go Math!* (GO-R) mis-tagged SMPs so frequently that it called their correct tags into question. A typical GO-R lesson contained around eight individual task tags, and typically two of those eight showed no relationship between the SMP and the task. I hypothesize that this lack of relationship is related to the way that the standards were re-labeled in the student text. For example, MP6 (attend to precision), was variously labeled as *Explain, Explain to a Friend, Connect* (referring to explaining connections between representations), and *Explain a Method*. These phrases miss the holistic goal of being *precise* in making explanations and omit many important components of the SMP that address other forms of precision. This re-labelings suggest an overall lack of regard for both the spirit and the details of the SMPs.

**Findings: Interpretations by Textbooks**

This section moves from looking at the structural design of how textbooks addressed the SMPs to looking at the nuances of how individual SMP were interpreted and enacted. Although there was some variation between textbooks, the overall response to each of the SMPs bifurcated based upon the textbooks’ overall approaches to learning. Table 3.4 provides an overview of how each of the standards is enacted by textbooks that
positioned students as generators or receivers of knowledge, which I developed through my analysis of each standard across multiple textbooks using coding in Atlas.ti.
Table 3.4

Summaries of SMPs and their enactments by textbooks that positioned students as generators or receivers of knowledge (GK and RK, respectively).

<table>
<thead>
<tr>
<th>SMP</th>
<th>SMP Summary</th>
<th>GK Textbooks</th>
<th>RK Textbooks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: Make sense of problems and persevere in solving them</td>
<td>Students solve complex, novel problems by considering possible entry points, goals, and strategies, making conjectures, considering special cases.</td>
<td>Students solve novel problems with student-developed and student-chosen strategies as a way of learning concepts.</td>
<td>Students solve short, routine multi-step word problems using strategies demonstrated by the textbook earlier in the lesson.</td>
</tr>
<tr>
<td></td>
<td>Students self-monitor their progress while solving novel problems.</td>
<td>Students engage in self-monitoring because it is a necessary aspect of problem solving.</td>
<td>Students are taught decontextualized self-monitoring steps that they practice as separate exercises.</td>
</tr>
<tr>
<td></td>
<td>Students use multiple representations and strategies to find trends, check their answers, explain connections, and understand others’ strategies.</td>
<td>Students discuss multiple strategies generated by students and the connections between them.</td>
<td>Students read textbook explanations of relationships between strategies.</td>
</tr>
<tr>
<td>2: Reason abstractly and quantitatively.</td>
<td>Students flexibly solve messy, real-world tasks by decontextualizing (representing abstractly) and contextualizing (attending to the meaning of quantities).</td>
<td>Students solve simplified real-world tasks (supported by MP2) and purely mathematical tasks (not supported) by contextualizing and decontextualizing.</td>
<td>Students are taught two or more strategies which they use when indicated in the same lesson. Tasks use superficial contexts that are unimportant for solving.</td>
</tr>
<tr>
<td>3: Construct viable arguments and critique</td>
<td>Students make plausible arguments, using language and representations, based on previously established ideas, logical reasoning, cases and counterexamples, and context.</td>
<td>Students share their own strategies using their own language and representations in class discussions.</td>
<td>Students repeat explanations that are provided in the textbook or answer short extension questions in one or two sentences.</td>
</tr>
<tr>
<td>SMP</td>
<td>SMP Summary</td>
<td>GK Textbooks</td>
<td>RK Textbooks</td>
</tr>
<tr>
<td>-----</td>
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</tr>
<tr>
<td>the reasoning of others.</td>
<td>Students listen to or read the others’ arguments, compare their effectiveness, identify flaws, and ask questions to clarify or improve others’ arguments.</td>
<td>Students understand and critique each other’s written and oral strategies in class discussions.</td>
<td>Students explain why a misconception presented in the text is problematic.</td>
</tr>
<tr>
<td>4: Model with mathematics.</td>
<td>Students solve messy, real-world problems by developing simplified models, making assumptions and approximations, identifying and representing important quantities, interpreting their results in context, and improving their models.</td>
<td>Students grapple with real and messy data that they must approximate with reasonable models in a few lessons.</td>
<td>Students read superficial story problems with clean and simplified data, then write equations following assigned steps.</td>
</tr>
<tr>
<td>5: Use appropriate tools strategically.</td>
<td>Students make active choices about which physical tools (pencils, manipulatives, rulers, etc.) to use when solving a problem.</td>
<td>Students choose appropriate manipulatives (supported by MP5) and algorithms (not supported) to use.</td>
<td>Students are given step-by-step instructions for using concrete representations and algorithms.</td>
</tr>
<tr>
<td></td>
<td>Students make active choices about which technological tools (calculators, spreadsheets, mathematical software, etc.) to use when solving a problem.</td>
<td></td>
<td>This aspect of MP5 was never addressed.</td>
</tr>
<tr>
<td>6: Attend to precision.</td>
<td>Students communicate using precise language and clear definitions, including stating the meanings of symbols that they choose.</td>
<td>Students use precise language and labels to communicate their own thinking as a way of building shared knowledge.</td>
<td>Students are prompted to give clear definitions and explanations of concepts that have already been explained.</td>
</tr>
<tr>
<td></td>
<td>Students calculate efficiently, choose appropriate levels of precision, and label accurately.</td>
<td></td>
<td>These aspects of MP6 were rarely or never addressed.</td>
</tr>
<tr>
<td>SMP</td>
<td>SMP Summary</td>
<td>GK Textbooks</td>
<td>RK Textbooks</td>
</tr>
<tr>
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</tr>
<tr>
<td>7: Look for and make use of structure.</td>
<td>Students notice patterns that result from underlying mathematical structures, such as properties of operations or classes of problems, and can use these to solve problems (without formalizing their observations).</td>
<td>Students notice interesting structures that derive from underlying properties without formalizing their knowledge <em>(supported by MP7)</em> or by formalizing <em>(actually MP8)</em>.</td>
<td>Students solve pattern tasks <em>(not supported by MP7)</em>, derive formal properties <em>(actually MP8)</em> with heavy scaffolding, or read explanations of how patterns can be used to derive properties and then restate them in their own words.</td>
</tr>
<tr>
<td>8: Look for and express regularity in repeated reasoning.</td>
<td>Students notice repeated calculations within a problem or a class of problems and formalize their knowledge to develop general methods or short cuts.</td>
<td>Students identify shortcuts and general methods on their own or with the whole class.</td>
<td>Students are led to find shortcuts and general methods with heavy scaffolding or read explanations of shortcuts and general methods and then restate them in their own words or apply them.</td>
</tr>
</tbody>
</table>
This table provides a summary of the in-depth analysis that I made of each SMP. To illustrate these ideas, I share three cases that demonstrate trends in the relationships between standards and textbooks. The first one relates predominantly to the content and structure of textbooks and the other two relate more to the content and structure of the SMPs, though all three address both sides of the standards-textbook relationship.

- **The Nature of Student Interactions**: Two standards, MP1 (Make sense of problems and persevere in solving them) and MP3 (Construct viable arguments and critique the reasoning of others), describe the two major underlying pedagogical approaches that are used to guide lesson design in GK textbooks. When students are positioned as generators of knowledge, MP1 and MP3 describe the overall nature of students’ interactions with each other, the teacher, and mathematics, and provide a format through which all of the other standards are addressed. RK textbooks use a different lesson structure which results in addressing MP1 and MP3 as isolated activities. I use MP3 as a case to illustrate how these underlying pedagogical approaches are enacted by the two types of textbooks.

- **Key Terms Reinterpreted**: MP4 (Model with mathematics) and MP5 (Use appropriate tools strategically) have been largely interpreted differently from how they were written due to common use of the terms *model* and *tool* in the mathematics education community. I illustrate this trend with MP4.

- **Examples as Definitions**: MP7 (Look for and make use of structure) and MP8 (Look for and express regularity in repeated reasoning) are heavily defined by examples rather than generalized statements of broader skills. In addition, these
two standards seem to overlap with each other and the boundaries between them are vague. As a result, both the holistic intentions and the details of the standards are unclear, leading to confusing and inconsistent enactment in textbooks. This use MP8 as a case to illustrate the impact of example-based SMPs.

For each of the three cases, I provide in-depth analysis of how each of the standards is interpreted and enacted. Each analysis contains the following components:

Unpacking the standard

1. Summary
2. Full text of the standard, separated into statements with lettered labels that are used throughout the rest of the analysis
3. Analysis of the text of the standard focusing on structure, content, and possible interpretations

Interpretations by textbooks

4. Analysis of how the standard has been rephrased
5. Interpretation and enactment by textbooks that position students as generators of knowledge
6. Interpretation and enactment by textbooks that position students as receivers of knowledge

The Nature of Student Interactions: MP3 Construct viable arguments and critique the reasoning of others.

There are two standards, MP1 and MP3, that are of interest because they are broad enough that they can be used to structure lessons across a textbook. MP1 (Make sense of problems and persevere in solving them) addresses solving novel tasks where the solution strategy is not immediately obvious, and MP3 (Construct viable arguments and critique the reasoning of others) addresses the type of mathematically focused communication that can be used to guide learning through discussion. While each of
these SMPs addresses a wide swath of mathematical skills, they can also provide a learning framework for structuring lessons and accessing the other six SMPs. This case addresses the nature of students’ roles in the overall design of textbooks and illustrates how these two foundational standards can either be used to structure lessons (in GK textbooks) or be addressed as isolated skills (in RK textbooks). For this case, I use MP3 as an exemplar.

**Unpacking the standard.**

As suggested by its title, Construct viable arguments and critique the reasoning of others, MP3 has two major components that may or may not be used together. The first is constructing viable arguments through using prior assumptions and definitions, logical progressions, cases, counterexamples, induction (arguments based on contextual data), and concrete representations. The second is critiquing the reasoning of others through comparing effectiveness, identifying and explaining flaws, and asking questions to clarify or improve others’ arguments.

I’ve broken down MP3 into 10 statements for analysis, and removed one phrase (shown with a strikethrough) that refers to older students:

**MP3 Construct viable arguments and critique the reasoning of others.**

- **(A)** Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. 
- **(B)** They make conjectures and build a logical progression of statements to explore the truth of their conjectures. 
- **(C)** They are able to analyze situations by breaking them into cases, and **(D)** can recognize and use counterexamples. 
- **(E)** They justify their conclusions, communicate them to others, and respond to the arguments of others. 
- **(F)** They reason inductively about data, making plausible arguments that take into account the context from which the data arose. 
- **(G)** Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, **(H)** distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. 
- **(I)** Elementary students can construct arguments using concrete referents such as objects, drawings,
diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Mathematically focused communication offers significant opportunity for productive struggle as students realize that ideas that make sense in their minds might not be equally accessible to others, or vice versa, thereby forcing them to think adaptively, conceptualize ideas in different ways, or come to a deeper understanding of a topic as they find different ways to explain it. Notably, these skills can be used before or during problem solving (testing the truth of conjectures, breaking situations into cases, identifying counter examples), after solving a problem (justify conclusions and communicate them to others), or when making sense of patterns, strategies, or ideas presented by others, including teachers or textbooks (all aspects of the SMP).

At first glance, MP3 seems to be designed for the type of lessons that are common in many GK textbooks where students solve problems independently or in groups, then present the strategies that they’ve used and receive feedback from other members of the class while the teacher supports them in clarifying their arguments. This type of learning is based on an underlying assumption that students are solving open-ended tasks that have multiple possible solution strategies (as in MP1).

Hiebert and Grouws (2007) suggest that this type of mathematically focused communication may be able to occur in other lesson structures. This opens the door for a variety of opportunities for students to make arguments and critique the reasoning of others, and helps clarify distinctions between the mode of communication and the quality of student-generated ideas in communication. When unpacking MP3 for analysis, there
are some questions that arise: Are students still meeting MP3 if they write explanations that are never discussed with peers to receive feedback? What if students share their thinking with the class through the process of making sense of a task together, but don’t first attempt to solve it on their own?

Regarding doing the work of the standard, what if students make an argument, but it recapitulates a definition or strategy modeled earlier in the lesson? What if students critique an argument, but the argument was carefully selected and phrased by textbook authors to showcase a major misconception? What if students are invited to discuss their strategies, but the task restricts them to a single approach?

This section explores some of these options through the decisions made in the two types of textbooks.

**Interpretations by textbooks that position students as generators of knowledge**

In the GK programs (BRI-G, EVER-G, EUR-G, INV-G), where discussion is a primary mode of learning, MP3 was rarely tagged but frequently practiced. These lessons typically had students both share and critique strategies simultaneously. They also frequently offered sample conversations or discussion points to model what these types of interactions sound and feel like.

EVER-G (the only GK program that rephrased the standards instead of using their whole titles) separated MP3 into two goals that matched the two parts of the title: “Explain both what to do and why it works” and “Work to make sense of others’ mathematical thinking.” In almost every lesson in the GK textbooks where MP3 was tagged (and the majority of lessons where it was not), students used both parts together,
constructing and critiquing arguments at the same time, as well as building on others’ arguments.

BRI-G G3 5.3.2 (Figure 3.9) provides an example of how mathematically focused communication formed the backbone of an GK lesson. Students shared multiple strategies, built on others’ ideas, stated points of agreement, constructively disagreed, and had teacher support in pushing the discussion toward a mathematical point. (The green text represents notes that the teacher takes to record students’ ideas.) Throughout this task, students did the work of addressing MP3.
Figure 3.9

Example task from BRI-G G3 5.3.2 that positions students as generators of knowledge in a mathematically focused discussion to address MP3.

- Display the top portion of the Two Different Ways to Look at an Array Teacher Master, and give students a minute to quietly examine the two arrays.
- Then have them share observations, first in pairs and then as a whole class. What do they notice about these arrays, aside from the fact that they’re the same?

Students  They’re exactly the same!
They both have 4 across and 6 down.
They’re both 6-by-rectangles.
There’s 24 little squares in both of them. I knew because 4 and 4 is 8, then 8 more is 16, and 8 more is 24.
I agree with you, but I looked at the 6s. I did 6 and 6 is 12, then 12 and 12 is 24.
Teacher  I’m curious… did anyone use multiplication to find the total number of tiles in each array?
Marcus  I did. I knew it was 6 rows of 4, and 6 times 4 is 24.
George  I did it the other way. I saw 4 columns of 6, and I know that 4 × 6 is 24.

Across the sample GK textbooks, MP3 statements A, B, E, G, I, and J, which all refer to constructing, comparing, and critiquing arguments, were used frequently (and often used in lessons where MP3 was not tagged). However, there was only one example each of statements C (breaking into cases) and E (inductive reasoning), and no examples
of D (counterexamples). There was also only one explicit example of H, finding flaws in others’ arguments, though this style of sharing mathematical ideas as a daily classroom practice lends itself to this situation arising naturally.

**Interpretations by textbooks that position students as receivers of knowledge.**

In the RK programs (ENV-R, GO-R, INTO-R, and MY-R), MP3 was often summarized as two separate practices, constructing arguments and critiquing arguments, that were then tagged and used in isolation. For the first practice, ENV-R, INTO-R, and MY-R all had a tag called “Construct Arguments,” which GO-R slightly rephrased as “Make Arguments” and MY-R also added “Draw a Conclusion.” For the second practice, phrasing varied but the meaning was similar: ENV-R used “Critique Reasoning,” INTO-R used “Critique, Correct, and Clarify,” GO-R used “Verify the Reasoning of Others,” and MY-R used “Check for Reasonableness” and “Find the Error.” Notably, the phrases used by GO-R and MY-R reduced the complexity of the action from critiquing and questioning to verifying, checking, and finding an error that students had already been told was present. GO-R also summarized this standard with the nebulous and unrelated word “Apply,” resulting in tagged tasks that bore no relationship to MP3.

These textbooks typically addressed the first part, constructing arguments, in two ways: 1) asking students to explain short concepts to the class to supplement the explanation that is given by the textbook and 2) providing written explanations in one to two short write-on lines. The nature of the sample answers to these prompts, which usually showed one correct solution and no discussion, did not support students in the challenging work of making their ideas clear to others. This usually resulted from task
structures with teacher/texts demonstrating the SMP, students replicating steps that had already been modeled, or students describing without doing.

For example, in ENV-R G3 1.3 (Figure 3.10), students were shown a model of a 4 \times 5 array then shown how to find the total using repeated addition, skip counting, and multiplication. They were then asked to create a 5 \times 5 array with a nearly identical situation and “explain” it by listing the same steps. The sample response demonstrates a “replicate steps” task, and no justifications are present (that is, there is no argument being constructed). At the end of the notes in the teachers’ guide (on the right), students had the additional opportunity to consider whether an array with 6 \times 4 would have the same amount as an array of 4 \times 6. While this could be a rich discussion, it became a situation of “talk without doing” (students didn’t create the arrays) and “optional extension” (so that it might be overlooked by the teacher). Thus, students were practicing an extremely watered-down version of statement E, “justify their conclusions, communicate them to others,” without meeting the holistic intentions of the standard that address the productive struggle of generating a valid argument or making sense of others’ thinking. These types of tasks made up around half of the tasks tagged with MP3 in RK textbooks.
Figure 3.10

Example task from ENV-R G3 1.3 tagged with MP3 where students observe several strategies being modeled (to show $4 \times 5$) and then replicate the steps in a highly similar task (to show $5 \times 5$). This does not address the tagged SMP.

There were occasional instances in RK textbooks where questions in the teachers’ guide provided students a limited opportunity to generate ideas. For example, in GO-R G5 7.5 students were explicitly taught that multiplying a whole number by a fraction results in a product that is less than the whole number. In Figure 3.11, students were asked to flip the question to consider the size of the product relative to the fractional factor. This task demonstrated students doing the work although it was extremely brief and had an implicit correct answer, as it was not a direct replication of what had already been stated and allowed students some opportunity to make sense of mathematics independently. Around one quarter of the tasks tagged with MP3 in RK textbooks had this type of task with limited opportunities for students to do the work.
Figure 3.11

*Example of a limited opportunity for students to do the work from GO-R G5 7.5. Students observe a demonstration and then answer a question that slightly extends that learning, which meets limited goals of MP3.*

The other part of MP3, critiquing other’s arguments, was always addressed in RK textbooks by having students identify flaws in artificial student work clearly marked as demonstrating common errors or misconceptions. Students were usually expected to respond in one or two sentences that might not even address the underlying flaw in the reasoning. For example, Figure 3.12 shows a task from MY-R G3 9.4 in which students identify a known flaw and then explain it in a single sentence that is not shared with the class. While this addressed some aspects of statement H, it did not address the overall intentions of MP3.
Figure 3.12

Example of a task from MY-R G3 9.4 where students address a superficial aspect of MP3 by identifying known errors in a carefully designed task rather than participating in a conversation where they critique peers’ arguments.

By having students critique arguments that had be deliberately designed to be accessible, RK textbooks lost the essential component of having students engage in making sense of the sometimes confusing, novel, and exploratory approaches of their peers. Tasks where students identified errors in artificial student work make up around a quarter of MP3 tags in RK textbooks.

Overall, MP3 was addressed in RK textbooks at a largely superficial level that was based on replicating information that has been modeled by the textbook or occasionally generating brief responses to extension questions that had a single correct answer. Across the sample RK lessons, there were two examples of statement G, comparing the efficacy of different arguments (which were presented in the textbooks), and no examples of making conjectures (B), breaking situations into cases (C), using examples and counterexamples (D), or reasoning inductively (F).

**Key Terms Reinterpreted: MP4 Model with Mathematics**

MP4 (model with mathematics) and MP5 (use appropriate tools strategically) both address the linguistic complexities of standards. The interpretations of these SMPs may
vary widely depending upon how their key terms, *model* and *tools*, are defined. *Model* is often used in the mathematics education community as a noun, to refer to visual models (e.g., area models, arrays), while *tools* often refers to algorithms or problem-solving approaches. However, neither of these definitions are supported by the text of the SMPs. For example, MP5 discusses only physical tools (manipulatives, pencils, etc.) and technological tools (calculators, software). Yet the eight textbooks in this study interpreted the MP5 to almost exclusively address conceptual tools and algorithms (e.g., partial products multiplication) based upon common definitions of the key term, missing the written intentions of the SMP. This case explores the various meanings of the term *model*, and how it is described in MP4 and interpreted by textbooks.

**Unpacking the standard.**

The vision of modeling offered by the CCSSM in MP4 places an emphasis on data as messy and models as imperfect and adaptable tools that approximate the messy data in useful ways. Students are expected to use messy, real-life data, use approximation and assumptions to simplify the situation, construct a model that is a close enough approximation to be useful for testing hypotheses and drawing conclusions, and then assess the fit of the model during and after its construction to both improve the model and make sense of the simulation in context. There is a note that equations are a type of model.

I used 8 statements to analyze this standard; portions of two sentences are crossed out because they are intended for older grades, though some aspects may still be applied to elementary grades under tenet 5.
**MP4 Model with mathematics.** (A) Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. (B) In early grades, this might be as simple as writing an addition equation to describe a situation. (C) In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. (D) By high school, a student might use geometry [or] to solve a design problem or use a function to describe how one quantity of interest depends on another. (D) Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. (E) They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. (F) They can analyze those relationships mathematically to draw conclusions. (G) They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, (H) possibly improving the model if it has not served its purpose.

Before diving in to MP4, I briefly unpack the term “model” from a larger perspective. Models are conceptual systems that express relationships between key elements of larger systems. They may be constructed from objects, operations, equations, diagrams, rules, and so on, that are used with a goal of identifying, communicating, testing, or describing the relationships between quantities in a situation (English et al., 2005; Greer, 1997; Thompson, 1993; von Glasersfeld, 2003).

One of the potentially confusing linguistic challenges of this SMP is that the term *model* has two different but related meanings in mathematics education. Both of these types of models are mathematically important and meet the definitions above, but only one of these types is described in MP4. I briefly describe both types of models here.

In elementary education, the term *model* is most commonly used as a noun to describe concrete manipulatives like counters and linking cubes and visual representations like area models for multiplication. These models help students visualize abstract operations (like multiplication) or concepts (like the commutative property).
These representations serve as models in the important sense that counters or equations may stand in for any object, say birds or apples. As students develop increasingly abstract thinking, the lengths on the sides of an area model stand in for place value in multiplication, supporting an overall understanding of the relationships between quantities in that operation—regardless of what those quantities are in a specific multiplication problem. This is a valuable use of models, and one which is addressed in many of the CCSSM content standards.

The term *model* can also be used as a verb (*modeling*), as in the case with MP4. In this case, modeling is used as an action: taking real-life situations with messy, unpredictable, or incomplete data and developing a model that represents a simplified relationship or set of relationships between important elements of a more complex situation (Greer, 1997; Meyer, 2015; Thompson, 1993). This type of modeling (which overlaps with quantitative reasoning) also appears in MP2 (Reason quantitatively and abstractly), both of which focus on determining which information in a complex situation is important, what information must be assumed or estimated, and how that information can best be represented or manipulated to answer a question or solve a problem. MP4’s description of modeling would fit well with computer simulations of a weather forecast, where there are huge amounts of complex data and it is understood that the model offers, at best, a rough prediction.

Models of the first type (nouns) are robust tools that can be applied whenever the context suggest that the same relationship or operation (say, multiplication) is in use. Models of the second type (produced through modeling as a verb) are tools that are constructed for a specific situation and revised, discarded, replaced, and iterated on to
best meet the needs of the situation, the goals and preferences of the person doing the modeling, and the audience they are communicating with. Both types of models play an important role in developing mathematically proficient students, but the first is the subject of the CCSSM content standards and the second is the subject of the CCSSM practice standards. When models are only thought of in the first way, the uncertainty, messiness, approximation, assumptions, and revision that are the focus of MP4 (and MP2) are not addressed.

Similarly, there are two different definitions of what problems arising in everyday life may mean. In textbooks, this term is often interpreted to mean that there is a “story problem” or a superficial context (Greer, 1997; Meyer, 2015). In these superficial scenarios, the birds and apples are interchangeable, so swapping out the objects does not change the structure of the problem. These simple situations may also be addressed with many different representations that all show different aspects of the same underlying relationship. Students might model 2 bags with 3 apples each using a number line, by drawing two circles with three dots each, by skip-counting, or with an equation, because each of these models shows the relationship between quantities in multiplication. These superficial situations have a purpose, but also result in students ignoring the context because it is irrelevant (Greer, 1997). (Greer (1997) describes a researcher who posed problems like “There are 125 sheep and 5 dogs in a flock. How old is the shepherd?” to which students often responded with “125 ÷ 5 =25… he is 25 years old.”) When tasks are designed to address MP4, the original context is so integral to the strategies used and the solution that it cannot be ignored.
It is important to recognize that modeling mathematically “messy” situations with assumptions and approximations is well within the grasp of elementary students. For example, students might open three single-serve packages of apple slices, note the different numbers of slices in each one, and then use those to write equations that would allow them to make estimations about the number of apple slices in the whole box. They can also work on ill-defined but realistic problems, such as the middle-school examples of planning a school event or analyzing a problem in the community. If the fourth-grade class invites family members of all the students to a spring festival, how many gallons of punch will we need to buy? The key to this type of modeling is using tasks that are true to the complexity of the real world instead of the simplified proxies used so often by textbooks (Greer, 1997; Meyer, 2015).

**Interpretations by textbooks that position students as generators of knowledge**

GK textbooks tend to interpret MP4 in two ways: modeling simplified story problems and using visual models, both of which missed the holistic intentions of MP4. EVER-G’s goals summarized MP4 as: “apply mathematical ideas to real-world situations,” and “use mathematical models such as graphs, drawings, tables, symbols, numbers, and diagrams to solve problems.” Note that the second goal provided an almost completely different list of representations (the only overlap is *graphs*) than the one in statement E, though the EVER-G list was more aligned to the elementary content standards.

These two interpretations of MP4, modeling simplified story problems and using visual models, addressed superficial aspects of statement A (problems from everyday life, society, and the workplace), E (map relationships using tools like diagrams…), and G
(interpret results in the context of the situation). However, they did not represent D (making assumptions and approximations), G (improve the model if it hasn’t served its purpose), and the types of ill-defined scenarios exemplified in C (plan a school event or analyze a problem in the community). This suggests that to truly implement statements A, E, and G, students should be engaged with problems from everyday life that have some aspects of uncertainty, approximation, and assumption. Overall, GK textbooks did not address this holistic interpretation of MP4.

For example, BRI-G G4 1.1.4 (Figure 3.13) showed a typical GK task where students were engaged in modeling (creating models as a verb) but only for an extremely limited, concrete situation. Students were shown a picture of a box of crayons (with 4 rows of 8 crayons) and asked to figure out how many crayons were inside with an expectation of sharing their answers and explanations. The teacher’s guide suggested that students might use four different models to represent this situation (skip-counting with an open number line, repeated addition with a ratio table, multiplication with a tile array and multiplication with an area model), the first of which is shown below. Following the trend of GK texts, this task addressed a simplified interpretation of statements A, B, E, F, and G because there was a real-life context and students were choosing how they would like to model it. However, it did not provide an authentic problem that would support students in exploring assumptions, approximations, and improving models to better fit the data (C, D, and G). The Math Practice in Action sidebar note clarified this interpretation of MP4 as using visual models to represent thinking and solve (concrete, closed) problems. This is still type of modeling (as a verb), just not one that attends to both the parts and the whole of MP4.
Figure 3.13

Example of a task from BRI-G G4 1.1.4 where students model a known, closed situation that addresses parts of MP4 but not the holistic intentions.

<table>
<thead>
<tr>
<th>Student Strategies</th>
<th>Teacher Models</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Skip-Counting</strong></td>
<td><strong>Open Number Line</strong></td>
</tr>
<tr>
<td><em>Kendra</em> It's 24. I saw 3 rows of 8, so I just did skip-counting—8, 16, 24.</td>
<td><em>Teacher</em> Seems to me that the open number line is a good way to model your skip-counting strategy, Kendra. I'll start at 0, and take hops of 8, like this. What equation should I write to represent these hops?</td>
</tr>
<tr>
<td><em>Kendra</em> Three times 8 is 24!</td>
<td><em>Sasha</em> I think that's right because you took 3 hops of 8, and you landed on 24.</td>
</tr>
</tbody>
</table>

Often MP4 was applied in GK textbooks without a contextual situation and focused only on using visual models or manipulatives. For example, Figure 3.14 from EUR-G G4 3.15 discussed the relationships between three different multiplication models and why the final one was more efficient. This bears some superficial resemblance to MP4 statements (D) and (H), but even these statements refer to handling messy data, rather than increasing efficiency of general strategies. This task, and many others like it, were conceptually rich and highly aligned to the content standards, but did not meet the expectations of MP4.
Figure 3.14

*Example task from EUR-G G4 3.15 that interprets MP4 as referring using increasingly efficient visual models, not the act of developing increasingly efficient models to address messy situations.*

Display \(38 \div 4\).

T: In the Application Problem, you drew an array (pictured to the right) to solve. Represent the same problem using the area model on grid paper. (Allow two minutes to work.)

T: What do you notice about the array compared to the area model on graph paper?

S: The area model is faster to draw. Thirty-eight dots is a lot to draw. There are the same number of dots and squares when I used graph paper. Both get us the same answer of a quotient 9 with a remainder of 2.

T: Let’s represent \(38 \div 4\) even more efficiently without grid paper since it’s hard to come by grid paper every time you want to solve a problem.

T: (Give students one minute to draw.) Talk to your partner about how the array model and grid paper model supported you in drawing the rectangle with a given structure.

S: I knew the length was a little more than twice the width. I knew that the remainder was half a column. I knew that there was a remainder. It was really obvious with the array and grid paper.

In GK textbooks, modeling with simplified, closed problems was used in about two-thirds of tasks tagged with MP4 and the remaining third addressed visual models without a superficial story context. Both of these types of tasks engaged deeply with the use of visual models to represent core concepts of multiplication, and also expected students to actively choose and even develop model for these concepts. However, even when stories were present, the objects being used (e.g., crayons) could have been replaced with any other object without changing the meaning of the task, making them superficial.
The GK textbooks did not provide any examples of tasks where the initial conditions were uncertain or not fully known and would require students to approximate, estimate, or improve their models to better match messy data. This suggests that although students were addressing some aspects of modeling, many of the parts of MP4 were not met or not met in ways that supported the holistic intentions of the SMP, violating tenets 1 and 2.

**Interpretations by textbooks that position students as receivers of knowledge.**

While GK textbooks gave students opportunities to *model* (as a verb) with well-defined tasks, RK textbooks only gave students the opportunity to use pre-determined *models* (as a noun). This decision did not seem to stem from summaries of the title, as ENV-R, GO-R, INTO-R, and MY-R, all used the whole title (Model with Mathematics”) or some minor variation on it (e.g., “Model Math”). However the dual nature of the term *model* consistently led to enactments of MP4 in which students replicating steps of using visual models.

For example, ENV-R G3 2.6 (Figure 3.15) offered an explanatory box defining MP4 with five behaviors that bear little relationship to MP4 and misses its larger intentions. (The last statement, about estimation, would be appropriate if there were any questions in the textbook that required estimates as part of problem solving, but a digital search for the word “estimate” showed there were not.) The related task, where students were asked to complete a pre-drawn model under the teacher’s guidance addressed a superficial interpretation of A (everyday life), and E (representing with a model), as well as B (using equations). However, students did not create the model themselves, nor were
they modeling a complex, uncertain situation, so this did not meet any of the expectations of MP4.

**Figure 3.15**

*Examples from ENV-R G3 2.6 of explanatory text for MP4 that does not address any components of the standard (above) and a related task where students use a pre-drawn model of a known situation and do not meet the intentions of MP4 (below).*

Another frequent task type that was tagged with MP4 in the RK textbooks was using or understanding visual models or algorithms without an everyday context, as shown in Figure 3.16 from INTO-R G4 3.8.3. These tasks might be aligned to statement E if the words “practical situation” were ignored, but they otherwise showed no relationship to any of the MP4 text. In comparison to the GK textbooks, where students
developed the models, in RK textbooks these models were first taught to students and then replicated.

**Figure 3.16**

*Example from INTO-R G4 3.8.3 that interprets MP4 as referring to visual models but does not support any aspect of the standard.*

![Image](image_url)

**Problem 2 • Model with Mathematics** Students write equations with dimensions given in an area model.

In a smaller number of examples, isolated statements or even words were taken out of context from MP4, usually in reference to statement F (analyze relationships to draw conclusions) though this is supposed to be the second part of statement E (where students first create the models being interpreted). For example, Figure 3.17 from GO-R G4 3.4 asked students to read a pictograph, a task which did not meet any of the holistic intentions of MP4, but technically included an everyday life or workplace situation (A), a graph (from statement E, taken out of context), and interpretations of relationships (F).
Overall, RK textbooks not only avoided problem solving situations with messy data that would support approximating and improving models, but they also did not allow students to do the work of developing their own models for concrete, closed situations or purely mathematical problems. This misses alignment to both the whole and the parts of MP4.

**Examples as Definitions: MP8 Look for and express regularity in repeated reasoning.**

This case addresses the structural aspects of standards specifically relating to the use of examples. While examples may be used to clarify intentions of a standard, MP7 and MP8 rely heavily on examples to provide, rather than clarify, the overall goals. These standards led to the development of tenet 3: “When examples replace general statements, those example should be generalized to cover all grade levels and other topics. Addressing the content of an example only meets the standard if it serves the holistic purpose of the standard.” The structural complexity of MP7 and MP8 is compounded because they overlap with each other to a great degree, and could have been combined into a single, clearer standard. This case uses MP8 to surface questions and provide suggestions about how examples in standards should be interpreted.
Unpacking the standard.

Before digging into MP8, it is valuable to consider it in relation to MP7 (Look for and make use of structure). Both of these standards include the common elements of structures/repetition, making connections between structures/repetitions and generalizable properties or formulas, and attending simultaneously to both details and a big-picture view. By some readings of MP7, it might include within it all the components of MP8, making MP8 unnecessary.

In one possible interpretation, Goldenberg et al. (2017) suggest that “regularity in repeated reasoning (MP8) is a process that we use to generate a mathematical object, an equation, not (as in MP7) an analysis and use of the object.” That is, MP7 is about noticing and using patterns, while MP8 is about stating the general form of a pattern with an equation or logical statement. While other interpretation are possible, this is a solid explanation for why there are two separate SMPs covering this concept and helps focus MP8 on its explicit content rather than expanding its content to also include what is covered in MP7.

Read holistically, MP8 asks students to use repeated results to develop general methods, shortcuts, equations that model relationships, or general formulas. They are also expected to balance an awareness of oversight, details, and reasonableness of their results during the intermediate stages of solving a problem.

I divided this standard into six statements and removed those specific to middle school\(^1\) with strike throughs while retaining key terms that were more generalizable.

\(^1\) It is worth noting that the grade level references in MP8 seem to not reflect the those specified in the CCSSM content standards. Repeated decimals (statement C) appear in
MP8 Look for and express regularity in repeated reasoning. (A)
Mathematically proficient students notice if calculations are repeated, and (B) look both for general methods and for shortcuts. (C) Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. (D) By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation \((y - 2)/(x - 1) = 3\). (E) Noticing the regularity in the way terms cancel when expanding \((x - 1)(x + 1)\), \((x - 1)(x^2 + x + 1)\), and \((x - 1)(x^3 + x^2 + x + 1)\) might lead them to the general formula for the sum of a geometric series. (F) As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. (G) They continually evaluate the reasonableness of their intermediate results.

Like MP7, MP8 relies heavily on examples (statements C, D, and E) rather than general statements, which requires the reader to heavily abstract and interpret the examples. Assuming that each of these examples represents an approach to implementing MP8, I have generalized the three statements and then provided additional possible examples at the elementary level.

In the first (statement C), the same sub-calculations are repeated within a single larger calculation. Using the example from statement C, the long division steps to find 25 ÷ 11 would involve repeatedly performing the same two calculations: subtracting 22 from 30 alternating with subtracting 77 from 80. After students complete these same two calculations enough times, they should recognize that the decimals in 2.2727272727⋯ will repeat infinitely. Another example of this might be finding 9 × 9 through repeated

the CCSS content standards in seventh grade, not upper elementary. (Students might notice them in fifth grade while dividing with decimals, but they are not mentioned in the content standards.) The first “middle school” example (statement D) could perhaps be taught in an 8th grade study of linear equations although only \(y = mx + b\) format is expected. According to the CCSS content standards, the examples in statements D and E are only aligned to high school standards and would not be appropriate for a middle school context if curriculum developers follow instructions to not include content that goes beyond the listed grade.

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addition (9 + 9 +…) and noticing along the way that the intermediate sums follow the pattern of adding 10 and subtracting 1. Note that for this type of usage, students would have to recognize these patterns through repeated addition rather than another strategy.

In the second type of task, exemplified in statement D, students use a guess-and-check method with recorded intermediate steps to eventually abstract greater patterns or equations. An example for elementary grades might be the “postage stamp” problem: you only have two denominations of stamps, say 5 cents and 7 cents, and want to know which total amounts can be made from combinations of those stamps (e.g., 24 = 5 + 5 + 7 + 7). After carefully recording combinations up to a certain sum through guess and check, students will notice that there is a repeating pattern of calculations that can be used to prove that all future sums are possible to create. Goldenberg et al. (2017) see this type of task as central to MP8, though the other examples suggest that there are other task types that should also be included.

In the third type of task (statement E), students are asked to repeat a set of related calculations and use them to derive a more general formula. One elementary example might be finding the products of 9 × 1, 9 × 2, 9 × 3, etc. and then using them to discover a general formula for multiples of nine of adding ten and subtracting 1. (Note that this is the same general formula as in the example from type 1, but the method to reach it is different.)

In addition, Goldenberg et al. (2017)’s interpretation of the distinctions between MP7 and MP8 suggest a fourth type of task that might be included within statements B about using general methods and shortcuts. They suggest that MP7 addresses noticing key structural features without formalizing or generalizing (though this becomes clear
only be comparing it to MP8), and as a result that MP8 continues that work with a focus on creating general methods, shortcuts, equations, and formulas. Extending the examples from MP7, students might notice structural features that underlie the commutative or distributive properties in MP7, and then might take the next step of generalizing the properties in MP8. For example, while they notice that that an area model of $7 \times 8$ can be shown as two smaller rectangles that are each $7 \times 4$ in MP7, in MP8 they could explain with words or equations that $a \times (b + c) = (a \times b) + (a \times c)$ as a general rule. This problem type shows some overlap with statement E, and seems to be a reasonable extension of looking for “general methods and shortcuts” (statement B).

Notably, while the word “repeated” indicates patterns, traditional pattern problems (“what comes next?”) do not seem to be included in MP8, unless statement B is taken out of context from the rest of the standard.

Table 3.5 shows a summary of the types of repeated calculations that can lead to generalizations based on the example in MP8:
Table 3.5

Possible interpretations of MP8 with examples and references to statements from the standard.

<table>
<thead>
<tr>
<th>Approach to identifying repeated calculations</th>
<th>Examples of starting tasks used to make generalizations</th>
<th>Reference Statement</th>
</tr>
</thead>
</table>
| 1. Repeated sub-calculations                  | 25 ÷ 11 = 2.2727272…  
Repeated addition                              | C                   |
| 2. Guess and check                            | Sums that can be made with 5s and 7s  
(stamp problem)                                 | D                   |
| 3. Sets of repeated calculations              | 9 × 1, 9 × 2, 9 × 3…                                     | E                   |
| 4. Properties of operations or generalized formulas | Moving from 7 × 8 = 7 × 4 + 7 × 4 to a generalization of the distributive property | B, supported by E |
| 5. Traditional pattern problems (Not supported) | A, B, A, B, __, __  
Identify the rules:  
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

Statements F and G seem to describe how the discovery process around repeated calculations would work in the examples from statements A through E, as students must attend to their intermediate results within a larger calculation while maintaining oversight in order to recognize that a pattern is forming. If they are interpreted more broadly and in isolation from statements A through E, they offer a reminder to check your work while in the process of making calculations, a sentiment that is addressed more fully in MP1 and is not the focus for MP8. I suggest that holistic intentions of MP8 would only use statements F and G to serve the process of identifying and generalizing underlying structures.

Interpretations by textbooks that position students as generators of knowledge

MP8 was typically addressed by the GK textbooks (BRI-G, EVER-G, EUR-G, INV-G) with tasks of types 3 and 4. These are only mentioned in MP8 with “middle
school” examples, but GK textbooks seem to have generalized these examples to cover elementary grades.

EVER-G rephrased MP8 with three goals, one addressing the creation of rules and shortcuts, one about applying rules and shortcuts (which is possibly the domain of MP7, but bleeds into MP8), and one for reflecting before, during, and after solving a problem. EVER-G tended to use all of the goals together rather than separating them so that statements F and G could be applied to any of the task types. This interpretation was followed across the GK textbooks, where students both derived generalized formulas and applied them in the same task or lesson.

BRI-G G3 2.2.1 (Figure 3.18) provides an example of a type 3 (statement D) task in which students moved from repeated calculations (skip counting) to several more generalizable rules: all multiples of 6 are multiples of 3 and there will be fewer multiples of 9 than 6 below 90 because each “skip” is larger. These generalizations were further formalized in a subsequent lesson where students generalized the relationship between multiples of numbers that share a common factor. This task also addressed statements F and G with attention to intermediate results. Type 3 tasks like this were used in less than a quarter of MP8 tags in GK textbooks.
Figure 3.18

*Example of a task that meets MP8, type 3, in which students use repeated calculations to generalize rules.*

4 Then, have the students do a count-around for 6s up to 90, just as they did for 3s. Record the multiples of 6 under the multiples of 3. At the end of this count-around, ask students what they notice.

\[
3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90
\]

Have them explain and support their thinking by asking follow-up questions such as, “Why?” and, “What makes you think that?” When students notice that the multiples of 6 are also multiples of 3, be sure to discuss and emphasize the importance of this observation.

5 Tell students that they will do one more count-around with 9s. Ask them the following questions:

- Will there be more multiples or fewer multiples of 9? Why?
- Will everyone get to call out a number? Why or why not?
- Can you estimate how many people will get to call out a number? Tell us more about your estimate.
- What happens as the number we are counting by gets bigger?

*Students:* I think there will be more multiples of 9 because 9 is a bigger number.

I think there will be less because there were less multiples of 6 than there were with 3.

Task type 4 (aligned with statements B and E) made up around three-quarters of the MP8 tasks in the GK textbooks. It is demonstrated in INV-G G4 3.3.4 (Figure 3.19) where students were given a problem in which 120 apples were first packed in boxes that held 20 apples each and then re-packed into boxes that held 10 apples each. While students worked, they were encouraged to notice patterns which they first generalized into an awareness of doubling and halving relationships for the specific problem, and then extended during the next lesson to a generalizable formula. The discussion format focused on students’ strategies, and also addressed balancing general rules and details (statement F) and evaluating intermediate results (G).
Three of the GK textbooks (BRI-G, EVER-G, and INV-G) often included games that encouraged students to practice mathematical fluency and notice emergent rules that arose from numeric relationships. These games could address type 2, but these were rarely tagged with MP8 in the sample lessons. BRI-G G5 1.12 (Figure 3.20) contained one such game, called The Product Game, though the sidebar beside the task described how it relates to MP1, while MP8 was only listed as a general SMP for the whole lesson. In The Product Game, a student placed colored chips on two factors at the bottom, then
claimed the square with the product. The next student changed one of the factors, and then claimed the new product. The goal was to get four products in a row. After playing the game, the class discussed strategies and then completed a problem set which asked students what moves should be next and why in partial games. Through this game, students were expected to observe that numbers with more factors were easier to reach and notice numbers that shared common factors. These types of games were common across BRI-G, EVER-G, and INV-G, but with the exception of this example, were not tagged with MP8.

**Figure 3.20**

Example of The Product Game from BRI-G G5 1.12. This is one of the few games tagged with MP8, though most games in GK textbooks meet this SMP.

Elicit strategies such as choosing factors to yield products positioned in the middle of the board to yield more possible moves later; weighing the number of products possible at each turn with each of the two existing markers; and choosing factors the student is confident multiplying.

Teacher: I’d like you encourage all of you to take the time to make good game moves while you are playing and not to feel pressured to rush. Playing this game will improve your math ability if you challenge yourself to try combinations you need to think about, or explore several different possibilities rather than picking the first and easiest.
It is notable that types 3 and 4 match the strategies that are named in the CCSSM content standards, which may explain their prominence in GK textbooks. Task type 1 was found only in rare occasions (repeated sub-calculations are not a common occurrence generally in mathematics). Type 2 tasks, which use guess-and-check for more exploratory tasks that eventually lead to logical statements or proofs, were not found at all. (When guess-and-check was used, it was applied to individual situations, not generalizable rules, which does not meet MP8.) Goldenberg et al. (2017) would consider that an avoidance of type 2 tasks misses the intentions of MP8; at the minimum, one important part is being left out. Collectively, this analysis illustrates the importance of tenet 3 for generalizing from examples to overall statements when the general statements are missing.

**Interpretations by textbooks that position students as receivers of knowledge.**

The four RK programs tended to interpret MP8 in wide variety of ways that showed only a passing relationship to MP8. Some of them appeared to reference Type 3 (based on statement E) or Type 4 (based on statements B and E), but the textbook completed the actions of noticing repeated reasoning, generalizing, and applying generalizations on the students’ behalf. Students were excepted to explain or replicate practices as they were laid out in the textbook, which does not fulfil the requirements of MP8 due to tenet 4.

The full title of MP8 is “Look for and express regularity in repeated reasoning.” INTO-R shortens this to “Use repeated reasoning” which seems to focus on applications rather than discovering relationships and misses the purpose of MP8. ENV-R replaces this entire title with the term “Generalize,” which is a reasonably accurate approximation of MP8 in a single word. GO-R uses both of these phrases, though I was unable to
identify any differences in how the two terms were used when tagging tasks. MY-R adds the label “Look for a pattern,” which is used to tag traditional pattern tasks that are not aligned with MP8.

RK textbooks applied MP8 in four ways, three that address superficial aspects of MP8 and one that is outside the standard. The first approach, citing a definition, was used frequently in ENV-R and GO-R. In lesson ENV-R G4 7.4 (Figure 3.21), students cited definitions of terms or properties that had been given to them. While the text of the teacher’s edition suggested that students were generalizing, the textbook had already provided them with generalized definitions of prime and composite numbers and the application did not give students the opportunity to move from noticing a pattern to constructing a generalized statement of it. This seems to be a superficial version of task type 4.

Figure 3.21

*Example from ENV-R G4 7.4 that interprets MP8 as having students repeat a definition that was given to them, rather than generating their own generalizations of patterns.*
In the second RK approach, students were asked to apply a generalized rule in a specific, short task. For example, in GO-R G4 3.4 (Figure 3.22) students demonstrate knowledge of multiplying by multiples of 10 after seeing an identical example. Thus, students used a pattern that was laid out by the textbook but did not generalize a rule. The textbook then asked for an explanation of the steps and provided a misleading sample answer that did not attend to place value, two actions which also did not allow students to do the work of the standard in generalizing a rule. This approach was common across the RK textbooks and also appears to be a superficial version of type 4.

**Figure 3.22**

*Example from GO-R G4 3.4 in which MP8 is enacted by having students apply a generalized rule to a short task rather than generalizing the rule themselves.*

INTO-R G3 2.3.1 (Figure 3.23) provides another example of applying a generalized rule, this time in a superficial version of type 3, which relates to sets of repeated calculations. In this task, the teacher’s guide suggested that students show repeated reasoning by writing equations, but the task had students duplicate a shortcut that has been identified by the textbook (repetition) and fill in the blanks (teacher/text demonstrating the SMP). Students were not asked to generalize, so this task did not meet any of the goals of MP8.
Figure 3.23

Example from INTO-R G3 2.3.1 in which students repeat calculations but do not generalize a rule to meet MP8.

**Repeated Reasoning** There are 2 turtles. Each turtle has 4 legs. How many legs do the 2 turtles have?

- Add. \(4 + 4 = 8\)
- Multiply. \(2 \times 4 = 8\)
- The 2 turtles have 8 legs.

Problem 5 • Repeated Reasoning Students write an addition equation and a multiplication equation for a problem involving equal groups.

In the next approach, the textbook presented a pattern and then guided student through generalizing it, which appears to be a version of Type 3. For example, GO-R G5 1.5 (Figure 3.24) asked students to explain the pattern of exponential notation shown in the textbook, but due to a lessons structure where the teacher/text demonstrates the SMP and an explanation that featured procedures over concepts, students were doing minimal generalizing on their own.

Figure 3.24

Example from GO-R G5 1.5 where students generalize a pattern with heavy scaffolding from the teacher and textbook to address MP8.
Another type of task simply involved a set of related equations without drawing larger generalizations from them. For example, in INTO-R G5 3.8.1 (Figure 3.25), students repeated the calculations of multiplying 18 eggs by fractions, but they were not asked to notice patterns or use them to generalize rules. This seems to be a misinterpretation of type 3.

**Figure 3.25**

*Example from INTO-R G5 3.8.1 where students repeat calculations but do not notice or generalize patterns to address MP8.*

Finally, MP8 was often used to tag traditional pattern problems, which are not supported by the text of the standard. GO-R offered a teacher’s note at the beginning of lesson GO-R G5 9.5 (Figure 3.26, top) that defined MP8 as referring to these traditional pattern problems (type 5). In this example, students were asked to explain the steps in the pattern problem (Figure 3.26, bottom left) but the sample answer (Figure 3.26, bottom right) focused on procedural steps rather than an understanding of relationships between the numbers. In most of the type 5 tasks in RK textbooks, students were expected to only generate future steps without explaining them.
Figure 3.26

Example from GP G5 9.5 where MP8 is interpreted to refer to traditional pattern problems where students find the next value in a pattern in a teacher’s note (top), task (bottom left) and sample answer (bottom right).

There were also several tasks tagged as MP8 in INTO-R and GO-R that did not show any relationship to the standard. For example, this task in INTO-R G4 5.16.3 (Figure 3.27) is defined by the teacher’s guide as being aligned to MP8 because it has
multiple steps. There is no evidence of patterns or generalization here, so it is unclear why the task was given this tag.

**Figure 3.27**

*Example from INTO-R G4 5.16.3 of a task tagged with MP8 that bears no relationship to the standard.*

**Problem 17 • Use Repeated Reasoning**  Students solve a multistep problem.

Overall, RK textbooks did not address either the parts or the whole of MP8 because even when they included type 3 or 4 tasks the mathematical work was completed by the textbook. Instead, they seem to focus on the material from the CCSSM content standards, which heavily addresses multiple algorithms and understanding how operations and properties work. Across all four RK textbooks, students never engaged independently in the act of generalizing from a repeated situation to a rule or equation.

The number of tasks tagged with MP8 that do not bear any resemblance to the standard also suggest that this standard was written in a confusing way. This may stem from the heavy use of examples in the standard, which require an extra level of interpretation on the readers’ part.
Discussion

This chapter opened with a set of questions that reframed smaller questions about CCSSM interpretation and enactment in textbooks in a wider context: How have the CCSSM, and messages surrounding them, impacted textbooks? Where has the CCSSM been unsuccessful in impacting textbooks, and what changes might make the SMPs more successful? Based on lesson plans from textbooks, what might mathematical learning look like in the United States under the CCSSM? What information should educators have when making decisions about textbook purchases? Based upon these findings, I suggest some responses to each of these questions here.

Limitations and Opportunities for Textbooks

When the Common Core State Standards were released, the authors took a calculated risk in focusing on what would be taught (the content standards) while leaving decisions about how it would be taught to teachers, resulting in ambiguous expectations around the practice standards. These findings speak to the implications of this mixed messaging in both explicit and subtle ways.

At the widest level, the language of the standards for mathematical practice suggests that they can only be met when students have opportunities to engage in productive struggle with rigorous tasks (MP1) or through mathematically focused discussion (MP3). The other six SMPs may only be addressed through one or both of these approaches, as all require students to take an active role in constructing mathematical ideas and making informed mathematical decisions.

There are a number of both revised and new programs that position the student as generators of knowledge, such as BRI-G, EVER-G, EUR-G, and INV-G. These
programs, for the most part, use the SMPs as a pathway to learning content meaningfully and attend to the holistic intentions of the SMPs along with most of the parts. In addition to the older, revised programs (EVER-G and INV-G, as well as Math Trailblazers which was not available for this study), the creation of new programs that use these methods (BRI-G, EUR-G, and Illustrative Mathematics, also not available for analysis) is encouraging. As a result of the CCSS initiative and new public and private funding around it, elementary educators now have a choice of six (and growing) curriculum programs that largely meet the SMPs and are likely to be strongly aligned to the content standards (see chapter 2).

However, there are a number of programs, all developed by major publishing companies, that have interpreted the CCSS messages about pedagogical choice to support textbooks that position students as receivers of knowledge. These programs include revisions to ENV-R, GO-R, and MY-R, as well as new programs like INTO-R and Ready Mathematics (not available for analysis). These programs enact only superficial approximations of the SMPs, often missing both holistic intentions and most of the parts, due to their limited opportunities for students to do the work of the standards. In a round table conversation with some of the supporters and developers of programs that position students as receivers of knowledge, Munter, Stein, and Smith (2015) found that their definitions of what it means to learn mathematics results in devaluing the SMPs. For example, in response to MP3, they suggest that, “while a good student may have an internal dialogue concerning all of the other aspects of the third standard, communicating effectively with others is not a necessary capability” (p. 18). This is a fairly blatant re-definition of the importance of the SMPs and their role in full implementation of the
CCSSM, which does not match the Publisher’s Criteria (CCSS Authors, 2013). However, it may be more comfortable for teachers and provide greater sales to publishing companies (Remillard & Reinke, 2017; Reys & Reys, 2006). It may also match a belief by curriculum developers that students will learn mathematics more effectively if they do not address the intentions of the SMPs, a topic which is further discussed in chapter 4.

One of the challenges for educators is knowing how to distinguish between these types of textbooks in light of misleading claims from some publishers. Based on these findings, if educators value the type of mathematical learning espoused by the SMPs, they should be wary of any textbooks that tag multiple short tasks with different SMPs, as the SMPs cannot be authentically met in this way. Instead, these tags are likely to be superficial enactments or completely unrelated, though they give the impression that the text is highly aligned to the SMPs. Within this study, programs that authentically meet a lot of SMPs by interweaving them together in rich problem solving and discussion tended to label only a few SMPs per lesson. This decision to highlight lesson opportunities might leave GK textbooks at a disadvantage from a marketing perspective, as this more useful and authentic tagging system tends to under-tag SMPs, while RK programs tend to over-tag. To meet the SMPs, educators would be wise to purchase textbooks that tag only a few SMPs per lesson and show how they are utilized and intertwined within lengthier tasks.

Once textbooks have been purchased, tags serve as educative features (Davis & Krajcik, 2005; Stein et al., 2007) that interpret for teachers what the SMPs mean and look like in action. When these reinterpretations are inaccurate or incomplete, they do teachers and students a disservice. For example, all of the RK textbooks suggest that MP3 can be
met by “constructing mathematical arguments” and “critiquing the reasoning of others” as two separate acts rather than a holistic set of intertwined practices. Once these practices are isolated from each other, they can be enacted in short written responses instead of mathematically focused discussions, and this deconstructed and inaccurate interpretation is passed on to teachers. Even in EVER-G, an GK textbook, the ways that SMPs titles were revised impacted how they were implemented, both when they clarified the standards for greater accessibility (as in MP3 and MP8) and redefined them (as in MP4 with the definition of models).

These tagging choices seem to have also impacted one of the primary evaluators of textbooks in the United States, EdReports. Their reports do not seem to consider whether students are doing the work of the standards, especially when students are repeating strategies that have already been taught. Currently, EdReports evaluates ENV-R and INTO-R as “meets expectations” in the area of rigor and mathematical practices (which addresses both content and practice standards) because of the many instructions for teachers to launch discussions. However, my findings show that students are only asked to explain what has already been presented to them and teachers are not given any support to lead discussions beyond short, routine sample answers. At the same time, EdReports finds that EVER-G does not meet expectations, because the evaluators find the guided discussions to be prescriptive (which is also the case in other textbooks not evaluated in this way, such as EUR), the SMPs are revised into smaller, more concrete goals, and the program does not label all of the SMPs that are being used simultaneously in a lesson. This suggests that even with experienced evaluators, tagging decisions can significantly impact how SMPs are interpreted.
Limitations and Opportunities for the CCSSM

These findings also lead to several suggestions for the CCSS authors and standards authors in general. First, it is apparent that the SMPs are confusing and heavily open to interpretation. Considering that curriculum developers are expected to address these SMPs, it would be helpful for the authors to revise the SMPs to first state the intentions of the standards in clear language and then use examples to illustrate, rather than to replace generalized statements (as in MP7 and MP8). In some cases, it might be valuable to consider whether each of the standards is fully necessary: MP7 and MP8 could be made stronger and clearer if they were combined, and MP2 could be subsumed into MP4 and MP1. When the standards use language that commonly holds other meanings, it would be helpful for the SMPs to provide counterexamples. For example, to explain unequivocally whether models in MP4 and tools in MP5 may be used to refer to applying a choice of algorithms, these standards could state what is not included. The final few sentences of most of the SMPs, which explain how the rest of the standard should be enacted, might also be dropped to avoid situations where less relevant parts of the SMP are enacted without attention to the whole and to remove redundancy.

Second, while holistic enactments of the SMPs were largely guided by the instructional approach of the textbooks, enactments of the parts of the SMPs seemed to be guided by the CCSSM content standards. As a general rule, SMPs seemed to be utilized to serve the content standards, but when parts of the SMPs suggested the coverage of content and skills outside of the grade level, it was often not covered. For example, In MP3, the focus on constructing algorithms in the CCSSM meant that there were rarely opportunities for students to have conversations around tasks that involved
breaking situations into cases, inductive reasoning, and counterexamples. This carried through to MP4, where students tended to use models rather than engaging in the act of modeling authentic, real-life situations with estimation and approximation. Similarly, the technology requirements of MP5 may have been ignored because there are no technology components (e.g., calculator skills, using software) in the content standards.

With the understanding that CCSSM implementation is heavily guided by the content standards, the CCSS authors could consider moving some key areas that are being overlooked in the SMPs into the content standards. For example, if the grade 4 standards stated a set of goals for calculator and software use (MP5) or for solving authentic tasks with messy or ill-defined starting conditions (MP2, MP4, and support from MP1), there is a much better chance of these parts being addressed. The Publisher’s Criteria might also consider a more nuanced message than their stern dictates that textbooks should not contain any content that is not present in the content standards for that grade, as some of the parts of the SMPs cannot be met only through the content standards.

Overall, the SMPs, which provide recommendations for how content should be taught, have been treated as completely optional by RK textbooks except as a sales technique, while they have been addressed holistically but with many missing or misinterpreted parts by GK textbooks. Both of these decisions seem to follow the expectation that the content standards should both structure and limit the scope of a textbook and that the SMPs are more optional. This seems to reflect that politically savvy messaging that the CCSS do not dictate “how to teach.”
Now that the CCSS authors have had an initial victory in getting substantial alignment to the content standards, they might consider changing their messaging in a few years to emphasize the SMPs more heavily. Third-party evaluators like EdReports could also revise their evaluation tools to more carefully consider who is doing the work of the SMPs and whether both the holistic intentions and the parts are being addressed when evaluating SMP alignment in textbooks.
CHAPTER 4: CONTINUITY AND CHANGE: HOW ELEMENTARY MATHEMATICS TEXTBOOKS ARE RESPONDING TO THE CCSS

Abstract

Mathematics educational philosophies in the United States have historically been divided into two camps: a traditional approach in which teachers demonstrate procedures that are repeated by students and a reform approach in which students solve rich tasks using their own strategies, use them to discover underlying concepts, and build conceptual understanding through discussion. The Common Core State Standards for Mathematics (CCSSM) have attempted to navigate this dichotomy by setting clear expectations for what should be taught while leaving a greater range of freedom in how it should be taught. These messages have resulted in the development of elementary mathematics textbooks with several sets of features that are largely new to the United States and which decouple assumptions that rigorous mathematics learning can only proceed through a single instructional model. This paper explores how textbook publishers have interpreted the CCSSM using a framework that distinguishes three aspects of rigorous teaching and learning: Explicit Attention to Concepts (EAC), Student Opportunities to Struggle (SOS), and Discussion-Based Learning (DBL) (Hiebert & Grouws, 2007; Hill et al., 2018; Stein, Correnti, et al., 2016; Stein, Kelly, et al., 2016). Using data collected from eight elementary mathematics textbook series, I explore the theoretical and practical implications of textbooks that provide 1) only EAC, 2) EAC, SOS, and DBL, and 3) a newer model that I call guided pathway that involves EAC and DBL with a limited form of SOS. Exploring these instructional models can provide insight into research questions regarding the current state of mathematics curriculum in
the United States, policy questions about the impact of the CCSS as it moves from a static document to enacted tools, and instructional questions about benefits and drawbacks of different textbooks styles.

**Introduction**

Historically, mathematics reform efforts in the United States have been beleaguered by the “math wars,” in which traditionalists and reformers were often poised against each other. These entrenched battles have often addressed both what is taught and how it is taught, with reformers advocating for novel tasks, student-derived strategies, and making sense of mathematics through discussion, while traditionalists relied on repetition of teacher-taught strategies, memorization, and rote application (Munter et al., 2015; Schoenfeld, 2004; Willoughby, 2000). These disagreements are both philosophical and practical, and have played out in policy decisions, curriculum development, and daily decisions in teachers’ classrooms, as educators aimed to meet reform-oriented standards, but often defaulted to traditional models when they found reform models overwhelming, incomprehensible, or incompatible with their beliefs, knowledge, and skills (Munter et al., 2015; Schoenfeld, 2004; Willoughby, 2000).

The Common Core State Standards for Mathematics (CCSS, 2010) introduced a new wave of reform with a bold premise: the new standards would focus only on what to teach and remain intentionally agnostic regarding how it should be taught (CCSS, 2012; McCallum, 2012). The CCSS authors provide an ambitious set of content standards that list what students should know and be able to do by the end of each grade with clear messaging that this exact list of content—with minimal or no modifications—should comprise the work of each grade (CCSS, 2012; CCSS Authors, 2013; McCallum, 2012).
They provide an equally ambitious set of cross-grade mathematical practices that describe mathematicians’ habits of mind—wrestling with problems they do not immediately know how to solve, communicating ideas with others and critiquing others’ ideas, modeling messy tasks, deriving formulas and generalized approaches, and so on. The messaging about using these approaches as the basis for all mathematics learning (CCSS Authors, 2013) can be seen as in direct conflict with other messaging that the CCSS “do not dictate how teachers should teach” (CCSS, 2012, p. 2).

As educators have interpreted the CCSS and its messages, the ways that mathematics is being taught in the United States is undergoing some complex and important changes made visible in textbooks that have been revised or written to align to the Common Core. Textbooks play an important role as mediators, both reflecting and transmitting the intentions and ideas of the mathematics education community: their development is guided by pedagogical philosophies and specific approaches informed by larger trends, and in turn, they become one of the primary resource that teachers use in determining what and how to teach (Houang & Schmidt, 2008; Polikoff, 2015; Remillard, 2005; Stein et al., 2007; Valverde et al., 2002). In this way, they serve as a mediator between standards and the actual teaching that happens in schools by interpreting the CCSS and enacting them as lessons and tasks (Remillard & Heck, 2014; Valverde et al., 2002).

In this analysis, I examine the implicit philosophies of newly developed or revised textbooks and contrast them to the commonly accepted dichotomy of traditional and reform textbooks. Based on my prior research (see chapters 2 and 3) and new research in this chapter, I claim that textbook development that responds to the CCSS has moved the
United States into a new landscape of what mathematics teaching and learning might look like. This emergent landscape provides more opportunities for rigorous mathematics learning when it relates to what is taught—an important achievement for the nation. The question of how math is taught has become more complex; it has evolved in some ways but shows some roots to past approaches. In this chapter, I propose a new typology of instructional models that builds on Munter et al. (2015) to both deepen understanding of two updated models that they propose (direct and dialogic) and add a third model (guided pathway). This third model, though new to the United States, is common in other countries such as Germany and China and offers some potential advantages (Hiebert et al., 2005; Ma, 2010; Stigler & Hiebert, 1999).

As pedagogical philosophies are not always made explicit, I use an analytical framework that poses three essential questions that underly decisions about mathematics teaching and learning. My intention is not to find the best answer to these questions, but rather to these questions as a tool to surface authors’ pedagogical and instructional models. That is, I aim to understand how curriculum developers answer these three questions through their decisions about interpreting and enacting the CCSS in their textbooks. These three questions, along with the theoretical models that undergird them, are as follows:

1. What is mathematics? (Skemp, 1976; Hiebert & Lefevre, 1986)

2. What is rigorous mathematics? (Hiebert & Grouws, 2007; Stein, Correnti, et al., 2016)

3. How should mathematics be taught and learned? (Munter, Stein, & Smith, 2015)
I then use this framework set up by these questions to create a new typology for mathematics instructional models in the United States based on empirical textbook research. This research responds to the complex messaging from the CCSS authors that is clear about rigorous content standards, but ambiguous about the implementation of the equally rigorous performance standards that set expectations of *how* mathematics should be taught and learned. Through my analysis, I address the following research questions:

RQ1: What instructional models are employed in elementary school textbooks that have responded to the CCSS? What are the common features of these models and how do they differ?

RQ2: How are these models supported by differing views about the nature and purposes of teaching and learning mathematics?

RQ3: What are the implications of these differing models for understanding the alignment between the CCSSM and textbooks?

Exploring how curriculum developers have interpreted and enacted the CCSSM in their textbooks can shed light on several areas that extend beyond the textbooks themselves. First, as I discuss in chapters 2 and 3, textbooks have shifted in response to the CCSSM (Munter et al., 2015), and these changes have only begun to be conceptualized and studied to understand their potential impacts on teaching and learning (Hill et al., 2018; Stein, Correnti, et al., 2016). Second, as the CCSSM moves from a new initiative to an established resource, textbook analysis can address policy and practice questions about the impact of the CCSS initiative. Finally, understanding the qualities and potential impacts of different textbook styles can support educators in making informed purchasing and implementation decisions.
The Potentials and Pitfalls of Reform through Standards

Standards-based educational reform at a national level holds three great promises: equity, alignment, and rigor. The equity promise positions standards as a way to address the great disparities between students’ opportunities and achievements based upon their race, class, or the district or state in which they live, as all students are working toward the same academic goals each year (Martin, 2003; Schoenfeld, 2004). The alignment promise suggests that if textbooks and assessments are aligned to the same set of expectations, then teachers will understand what they are expected to teach, have the resources to teach it, and will have accurate measures of their success in teaching it (Cohen et al., 2018; Cohen & Ball, 1990; Hiebert & Morris, 2012). And the rigor promise suggests that standards can be a transformative tool in moving mathematics education in the United States from a laundry list of memorized steps to a subject that students see as creative, challenging, and meaningful (Cohen, 1995; Hiebert & Grouws, 2007; Hiebert & Morris, 2012).

Each of these promises come with a corresponding set of challenges. Regarding equity, demanding that students and teachers without the adequate support and resources meet high level standards is at best unrealistic and at worst punitive (Martin, 2003; Schoenfeld, 2004). Alignment brings up a number of challenges regarding what alignment looks like and who gets to decide, especially when standards are complex and open to interpretation (Hill, 2001; Polikoff, 2015; Spillane, 2004). I discuss issues of alignment in more depth in chapters 2 and 3.

The promise of rigor introduces an dilemma that is inextricably linked to the other two promises: truly rigorous, high-quality standards hold a risk of not being implemented
due to educators’ discordant beliefs or lack of relevant knowledge, skill or resources available (Hiebert et al., 2005; Schoenfeld, 2004; Stigler & Hiebert, 1999). However, lower-rigor standards may increase accessibility and attainability, which may increase alignment and conformity, by sacrificing other goals for improving teaching and learning (Cohen, 1995).

The type of rigorous problem solving and thinking that is used by mathematicians and is common for K-12 instruction in many other countries is not just uncommon in the United States, but often actively rejected or inaccessible to teachers without training that far exceeds what is available (Hiebert et al., 2005; Ma, 2010; Schoenfeld, 2004; Stigler & Hiebert, 1999). Cohen (1995, p. 754) explains this dilemma eloquently:

Most American educators are quite unfamiliar with high standards, as are most Americans. Our ignorance on this matter is one crippling inheritance of a school system that has long refused to offer intellectually demanding work to most students - in good part because few Americans have wanted it. That inheritance is a good reason to adopt higher standards, but it is also a great barrier to achieving or even seriously comprehending them.

Understanding these dilemmas positions ambitious standards as both a guiding light and one very small piece of a much larger puzzle that includes policy, teacher education, professional development, assessments, and family engagement, and many other areas (Cohen et al., 2018; Marshall S Smith & O’Day, 1991). In addressing the rigor versus accessibility dilemma, the CCSS authors chose to follow an ambitious agenda for rigorous reform benchmarked on other successful countries (McCallum, 2012; Remillard & Reinke, 2017), though they have presented it with mixed messaging that plays out in a multitude of complex ways and can be observed in textbooks.
The Role of Textbooks in Mediating Standards

By design, standards documents such as the CCSS provide an outline of goals and objectives for learning and/or teaching, but cannot be enacted without additional specification and reformulation (Remillard & Heck, 2014). The CCSS for mathematics outlines what students should know and be able to do by the end of each grade, as well as the habits of mind that should be incorporated into daily lessons. However, it does not provide a pathway or calendar for achieving those goals, tasks for student discovery or practice, lesson plans that specify the content and format of daily lessons, or guidance for teaches in implementing those lesson plans. In most countries, including the United States, the majority of these decisions are made by curriculum developers and delivered in the format of textbooks and their accompanying teacher’s guides (Remillard & Heck, 2014; Valverde et al., 2002).

Textbooks, especially in mathematics, have a considerable influence on what students learn and often how they learn it (Houang & Schmidt, 2008; Polikoff, 2015; Remillard, 2005; Stein et al., 2007; Valverde et al., 2002). The majority of mathematics teachers rely heavily on textbooks as a teaching tool (Houang & Schmidt, 2008; Stein et al., 2007; Stigler & Hiebert, 1999). Thus, textbooks (the written curriculum) act as a mediator between policies and standards documents (the official curriculum) and what teachers teach (the enacted curriculum) (Remillard, 2018b; Remillard & Heck, 2014; Stein et al., 2007; Valverde et al., 2002).

The importance of textbooks in shaping teaching gives curriculum developers a substantial role in interpreting the standards, including determining what they mean, their relative importance, and in some cases, how they could be reinterpreted to fit with the
beliefs and intentions of the individuals who are making the interpretations. Based on these interpretations, curriculum developers decide how the standards should be enacted in the progression of lessons across the year, as well as within lessons through discussions, tasks, fluency exercises, modifications for different learners, etc. During this process, small but consistent decisions about the structure, language, roles, images, and tasks can index values, understandings, and dispositions that are communicated to teachers and students (Herbel-Eisenmann, 2007; Remillard, Van Steenbrugge, et al., 2014). When these interpretations and enactments are compiled into textbooks, they become one of the major tools through which teachers experience the standards (Ball & Cohen, 1996; Stein et al., 2007; Valverde et al., 2002).

This process of interpretation and enactment plays an important role in communicating the CCSSM to teachers: it makes the sometimes unclear or abstract standards tangible and accessible, it provides teachers with some of the substantial expertise and design efforts of the curriculum developers, and it saves teachers, who often have little time and insufficient background knowledge and skills, from having to individually translate the standards into a useable format for teaching. It also has the potential to ensure greater coherence across teachers and schools, though at a wider scale this depends upon common interpretations across curriculum programs.

There are, however, some limitations to textbooks as mediators of standards. In past reform efforts in the United States, textbooks that were designed to address the reform-oriented NCTM standards (NCTM, 1989, 2000) proved unsuccessful for transforming education. One explanation for this failure is that teachers felt overwhelmed or did not philosophically agree with or understand the purpose or content of the new
materials (Schoenfeld, 2004; Willoughby, 2000). One of the main lessons of these efforts is that textbooks, in isolation, are not sufficient to educate teachers about new mathematical concepts and transform their teaching approaches unless they are combined with significant professional learning opportunities and accountability measures (Hiebert et al., 2005; Schoenfeld, 2004; Willoughby, 2000).

Another potential challenge of relying on textbooks as the primary messenger of rigorous standards is that there is substantial room for varying interpretations when reading standards. Moreover, enactments made by curriculum developers may not address the full rigor of the standards, depending upon how they are interpreted in light of the curriculum developers’ beliefs and aims. Several studies of textbooks that claimed alignment to the CCSS in the first few years after release indicate that publishing companies were slapping the CCSS sticker on their materials without making substantive changes to the content (Cogan et al., 2015; DiNapoli, 2016; Meyer, 2015; Polikoff, 2015). However, as the CCSS have persisted for a longer time period, curriculum developers have had time to revise their textbooks or develop new ones, and publishing companies have been forced to demonstrate meaningful alignment to remain competitive. As more time has passed, re-examining textbooks is worthwhile for understanding recent trends in mathematics education under the CCSSM.

Questions and Beliefs about Mathematics in Relation to the CCSSM

When curriculum developers interpret and enact the CCSSM, they do so by considering the text of the standards, their own pedagogical philosophies and prior knowledge, and, in some cases, the marketability of the product (Hill, 2001; Remillard & Heck, 2014; Reys & Reys, 2006; Spillane, 2004). This research seeks to understand the
implicit and explicit pedagogical philosophies of textbooks to understand their relationship with the CCSSM. This relationship is bi-directional, as messages around the CCSSM influence pedagogical philosophies while pedagogical philosophies also influence how curriculum developers will interpret and enact the standards.

Prior to the CCSSM, pedagogical philosophies in the United States tended to be enacted in textbooks following one of two models which are outlined in Table 4.1.

Table 4.1

<table>
<thead>
<tr>
<th></th>
<th>Traditional</th>
<th>Reform</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>What is mathematics?</strong></td>
<td>Math is a set of procedures to memorize</td>
<td>Math involves creative problem solving and conceptual understanding</td>
</tr>
<tr>
<td></td>
<td>Teachers should model procedures that students replicate without understanding</td>
<td>Students should solve problems, engage in rich discussion to build on others’ ideas, and understand why strategies work</td>
</tr>
</tbody>
</table>

However, as the rigor of the CCSSM and its inconsistent messaging has entered the U.S. curriculum space, the questions, relationships, and models have become more complex. I frame the conversation on pedagogical philosophies using three central questions about what mathematics education could or should be: *What is mathematics? What is rigorous mathematics?* and *How should mathematics be taught and learned?*

These questions, along with a framework for understanding how curriculum developers have responded to these questions, are shown in Figure 4.1.

The first row asks *What I mathematics?* and considers that in the United States this school subject can mean two entirely different things, which in the past have correlated roughly to traditional and reform pedagogies. The second row asks, *What is
rigorous mathematics? using a framework that attempts to disaggregate three separate features that have often been grouped together in the United States. By disaggregating these features, their relationships to the CCSS content and practice standards can be discussed in isolation from each other to understand where and how they appear. The third row asks, how should mathematics be taught and learned? Instead of the two historical models, this row proposes three models, two of which have previously been discussed in U.S. literature and one which I define here.
Figure 4.1

Conceptual framework addressing three central questions about what mathematics education could or should be with potential responses from curriculum developers.

<table>
<thead>
<tr>
<th>Question</th>
<th>Model</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is mathematics?</td>
<td>Procedural Understanding</td>
<td>Rote procedures and efficient computation</td>
</tr>
<tr>
<td></td>
<td>Conceptual Understanding</td>
<td>Creative problem solving and understanding why strategies work</td>
</tr>
<tr>
<td>What is rigorous mathematics?</td>
<td>Explicit Attention to Concepts (EAC)</td>
<td>Articulating concepts that underly strategies</td>
</tr>
<tr>
<td></td>
<td>Student Opportunity to Struggle (SOS)</td>
<td>Wrestling with and generating new ideas by solving novel tasks</td>
</tr>
<tr>
<td></td>
<td>Discussion-Based Learning (DBL)</td>
<td>Generating and clarifying ideas through discussion (Distinct from non-rigorous discussion)</td>
</tr>
<tr>
<td>How should mathematics be taught and learned?</td>
<td>Direct Model</td>
<td>Teacher presents procedures and students replicate</td>
</tr>
<tr>
<td></td>
<td>Guided Pathway Model</td>
<td>Teacher guides a discussion where students generate concepts and strategies through pre-determined steps</td>
</tr>
<tr>
<td></td>
<td>Dialogic Model</td>
<td>Students generate task solutions and consolidate ideas through discussion</td>
</tr>
</tbody>
</table>

In the following sections, I set up a framework of possible responses to each of these questions, and then use them to analyze how curriculum developers have used them to guide their adoption of the three models. This analytical framework provides tracible paths to philosophies that appear to both influenced and are reflected in curriculum design in my analysis of eight elementary textbooks.
**What is Mathematics?**

The costs and impacts of any decision about mathematics education reform ultimately come down to a question of “What does it mean to learn and do mathematics?” (Cohen et al., 2003; Hiebert & Grouws, 2007). This is not a straightforward question, and the way that it is answered has tremendous implications for the goals and methods of teaching mathematics in the United States, including how rigorous standards like the CCSSM are interpreted and applied.

Skemp (1976) suggests that in the United States, there are two different subjects being taught which have been confusingly given the same name, “mathematics.” (He suggests that one of them might be better labeled as “computation.”) Both subjects have some advantages, and each comes from a coherent set of beliefs about what the purpose of “mathematics” is in both school and later life.

One perspective, often termed *procedural*, views “mathematics” as the ability to perform routine operations quickly and accurately (Greeno & Johnson, 1985; Hiebert & Lefevre, 1986; Skemp, 1976). This perspectives offers several benefits: rote operations are relatively easy to learn and offer immediate rewards through quick and easy correct answers (Skemp, 1976). This mentality is compatible with the culture and exam focus of many schools, and is often preferred by teachers because there is a sense that deeper understanding takes too long to teach or assess and isn’t necessary for tests or life (Munter et al., 2015; Skemp, 1976).

A second perspective, often termed *conceptual*, views “mathematics” as using mathematical tools to creatively solve novel problems (Greeno & Johnson, 1985; Hiebert & Lefevre, 1986; Skemp, 1976). This type of thinking takes longer to learn, but the
understandings that are gained are more flexible and adaptable to new tasks. Algorithms become easier to remember because the learner can rely on underlying meaning rather than a meaningless jumble of steps and symbols, and can often re-derive algorithms if the exact steps are not well remembered (Greeno & Johnson, 1985; Hiebert & Lefevre, 1986; Skemp, 1976).

Skemp (1976) offers a compelling metaphor for these two views of mathematics. Procedural understanding is like asking for directions. You follow a set of steps which will typically take you to your destination quickly and efficiently. If you end up even a block away from your previous path you might be completely lost, but if you don’t expect to travel their often, this might be sufficient for your purposes. Conceptual understanding is like wandering around a neighborhood—getting to know its landmarks, building a mental map of where places are in relationship to each other, finding efficient or preferred routes between places, and sometimes even getting lost. It takes longer, but your mental map becomes stronger and self-reinforcing as you continue to explore, and you may start to appreciate discovering landmarks and paths for their own sake, enjoying the journey as well as the destination. Conceptual understanding also leads students to see themselves as competent problem-solvers, be motivated by the intrinsic rewards of persevering with a task, and view learning as self-continuing and self-rewarding (Skemp, 1976).

Both of these visions of “mathematics” offer some legitimacy; after all, many adults use only rote calculations in their daily lives and manage very well. However, if we believe in goals like preparing students for a technologically-based future, developing informed producers and consumers of complex information, remaining competitive in a
global data-based economy, equalizing educational opportunities for students from
diverse backgrounds, or giving all students a chance at success in mathematics-based
fields, there are compelling reasons to help them develop mindsets and skills for
traversing the land of “mathematics” with flexibility and ease (National Research
Council, 2001).

Conceptual understanding also supports students in developing mathematical self-
efficacy, a mindset where students see themselves as being capable problem solvers who
can approach novel problems with confidence and maintain that confidence even if early
attempts are not successful (Bandura, 1977; McGee, 2015; Skemp, 1976). In comparison,
procedural understanding is often correlated with math anxiety, a visceral experience of
fear when encountering mathematics (even among high scoring students), which leads to
avoidance of future mathematics courses or careers and pervasive feelings of
incompetence (Beilock et al., 2010; Grootenboer & Marshman, 2016; Machalow et al., in
press; McGlynn-Stewart, 2010).

The CCSSM offers a vision of mathematics that is based on conceptual
understanding in a nation where many parents, teachers, and policy makers hold a
procedural mindset. While there is an obvious misalignment between these two
perspectives, the implications of this are complex and multi-layered. To begin to tease
them apart, I next explore how conceptual understanding could be developed through
rigorous mathematics and how the CCSSM addresses these goals.
What is rigorous mathematics?

If we assume that conceptual understanding is the goal of mathematics reform through the CCSSM, the next set of questions that arises is “What teaching approaches result in conceptual understanding?” and “Where can they be seen in the CCSSM?”

To conceptualize the goals and structure of the CCSSM, I use a framework suggested by Hiebert and Grouws (2007) that has been further developed by Stein, Correnti, Moore, et al. (2016) and Hill, Litke, and Lynch (2018). Based on extensive research and reviews of the literature, this framework proposes that there are two essential features of teaching that promote conceptual understanding of mathematics: Explicit Attention to Concepts (EAC) and Student Opportunity to Struggle (SOS). Hiebert and Grouws (2007) also address a third feature, discussion, which they do not find to be necessary or sufficient for conceptual understanding as it is often conducted without EAC or SOS and therefore not a reliable indicator of rigorous mathematics. As a result, Stein et al. (2016) remove discussions from their analysis to simplify their model, while Hill et al. (2016) include several types of classroom interactions, as they still have many implications for student learning (Hiebert & Grouws, 2007). In this chapter, I suggest that discussion can be enacted in a rigorous way and label this as Discussion-Based Learning.

Explicit Attention to Concepts (EAC)

Explicit Attention to Concepts (EAC) involves making connections between mathematical ideas, exploring meanings underlying procedures, noting relationships between strategies, and connecting tasks to a greater mathematical point within a network of related concepts (Hiebert & Grouws, 2007; Hill et al., 2018; Stein, Correnti, et al.,
Important, the connections described above must be explicitly articulated—either by students or by teachers—rather than arranging tasks so that students might reach an implicit understanding that is never voiced or written.

This is the type of explicit attention to the concepts that underly mathematical relationships and procedures is espoused throughout the CCSSM content standards. For example, content standard 4.NBT.5 expects students to understand an operation (multi-digit multiplication) that has traditionally been taught by rote through the standard multiplication algorithm. However, 4.NBT.5 sets an expectation that the concepts underlying the procedure will be attended to explicitly through utilizing several wider concepts in mathematics (place value and properties of operations), by making connections to visual models, and through students’ own illustrations and explanations:

4.NBT.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models (CCSS, 2010).

Student Opportunity to Struggle (SOS)

Student Opportunity to Struggle (SOS) involves expending effort to wrestle with key mathematical ideas through solving problems when a strategy is not immediately apparent and when new understandings are within reach but not yet fully-formed (Hiebert & Grouws, 2007; Hill et al., 2018; Stein, Correnti, et al., 2016). This type of thinking is addressed in the CCSS standards for mathematical practice. MP1 defines SOS by describes the challenge of identifying or testing possibly strategies when they are not immediately obvious, making conjectures, trying multiple methods to solve a
problem, and making connections to other students’ approaches as a way to deepen understanding.

**MP1 Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt…. [They] continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

While SOS is the focal point of MP1, it is also an explicit component of each of the other SMPs, as demonstrated through heavy use of verbs and phrases such as *analyze, create, make conjectures, explore, justify, reason, make assumptions and approximations, interpret, reflect, make sense, consider,* and so on.

**Discussion and Discussion-Based Learning (DBL)**

As a pedagogical tool, discussion has often been associated with rigorous learning in the U.S. and has been the focus of mathematics reform efforts for decades. Compelling evidence, on the other hand, suggests that classroom discussions are neither necessary nor sufficient for rigorous mathematics learning or gains in student performance. Through an extensive literature review, Hiebert and Grouws (2007) find that while facilitated discussion can be an effective way to address SOS and EAC, these two features can also be found in lessons that do not involve discussions, and further that discussions are not effective in promoting conceptual understanding unless they also have EAC. Because discussion is not necessary for rigorous mathematical teaching, Stein and colleagues (2016) have chosen to remove this factor from their framework for assessing the rigor of tasks, though Hill (2018) considers a set of factors that shape the instructional format,
including discussion, based on additional research showing their impacts on student learning.

While I agree that discussion is neither necessary or sufficient for rigor on its own, it plays an important role in the United States in many pedagogical philosophies of mathematics education and in textbooks. When students engage in mathematically-focused discussion, they construct their own mathematical ideas, build on and wrestle with others’ ideas, learn the language and practice of engaging in academic discourse, view themselves as generators of knowledge (Boerst et al., 2011; Stein et al., 2008). Thus, rather than removing it from analysis, I have retained it and instead considered what might be involved in enacting it with high levels of rigor. I feel that understanding the role of discussion is critical for exploring how curriculum developers have chosen to interpret and enact the CCSSM in textbooks to support or undermine rigorous learning through SOS and EAC. I also suggest that rigorous discussion may encapsulate practices that extend beyond EAC and SOS. The definitions that follow arise from both the literature and my findings in this set of analyses.

In this chapter, I differentiate between discussion (having students converse with each other in the classroom, which may or may not be mathematically rigorous) and a mathematically and pedagogically rigorous set of practices that I term Discussion-Based Learning (DBL). I define DBL as using a discussion format to support students in generating and clarifying mathematical ideas in a way that promotes Explicit Attention to Concepts. In discussion-based learning, students share their ideas publicly, make clear arguments, work to understand and build on others’ arguments, and offer critiques as both a pedagogical approach for learning concepts and as an important mathematical skill
set in its own right. The key skills involved in DBL are described in MP3. (They are also referenced in the last sentence of MP1 (above) and MP6.)

**MP3 Construct viable arguments and critique the reasoning of others.** Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments…. They justify their conclusions, communicate them to others, and respond to the arguments of others…. [They] are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents… even though they are not generalized or made formal until later grades…. [They] can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

With that in mind, I distinguish DBL from other types of discussion in two ways. First, DBL positions students as generators of knowledge, unlike forms of discussion where students might repeat information that was previously introduced by a teacher. Second, DBL addresses EAC, and it is the teacher’s responsibility to guide students toward articulating clear mathematical ideas and then repeating or reinforcing the key mathematical concepts in a lesson (Boerst et al., 2011; Sleep, 2012; Stein et al., 2008). By comparison, when students share strategies as a form of participation and the teacher treats all strategies as being equally valid without highlighting mathematical points, this is discussion but not DBL. To elevate strategy sharing to DBL, the teacher would need to ensure that the mathematical ideas are comprehensible to other students, make connections between multiple strategies, discuss the relative efficiency of different strategies, and/or reinforce specific strategies that are the heart of a given lesson (Boerst et al., 2011; Sleep, 2012; Stein et al., 2008).

In the United States, DBL is often associated with a Launch-Explore-Discuss (LED) sequence that combines DBL with SOS. The teacher first launches (introduces) a
task in a way that stimulates thinking without giving away solving strategies, then
students explore the task with minimal teacher support (SOS), and then the whole class
discusses the strategies that can be used to solve the task as a way to generate and clarify
mathematical ideas (DBL) (Munter et al., 2015; Margaret S. Smith et al., 2008; Stein et
al., 2008). However, I suggest here that DBL may take on other forms so long as students
are generating their own mathematical ideas and teachers are supporting them in
clarifying those ideas.

As I am adding DBL to the potential list of rigorous lesson features defined by
Hiebert and Grouws (2015), I describe some of its background and context here.

As a skill set or habit of mind, there is inherent value in being able to
communicate clearly within the mathematical community and make sense of others’
thinking that might not be correct, fully developed, or fully articulated (National
Research Council, 2001; NCTM, 1989, 2000; Stein et al., 1996). As in real
communication among professional mathematicians, it may occur either orally or through
writing (that is, a discussion format is not necessary for developing arguments or making
critiques). And like other habits of mind, it requires time, practice, and guidance to
develop (Boaler, 2016; Cuoco et al., 1996; Franke et al., 2007; National Research
Council, 2001).

When mathematical communication is considered only as a skill or habit of mind
that mathematically proficient students should possess (as it is presented in MP3), it is
not necessarily more or less important than many others. One could argue that several
others skills from the SMPs like modeling (MP4) or identifying and making use of
patterns (MP7 and MP8), as well as other possibilities from beyond the SMPs like
developing proofs or having a productive disposition toward mathematics (National Research Council, 2001; NCTM, 2000) are of equal or greater value. However, I suggest that in the same way that MP1 (Make sense of problems and persevere in problem solving) functions as a pedagogical tool that supports the development of other mathematical concept and skills, that MP3 (Construct viable arguments and critique the reasoning of others) can be an underlying pedagogical approach through which other concepts and skills are addressed and through which the other six SMPs are met.

**Understanding the three features in CCSSM messaging**

This framework that separates out EAC, SOS, and DBL is of particular interest in understanding how the CCSSM has been interpreted and enacted because of mixed messaging regarding each of the different features. The CCSSM authors consistently message that the content standards, which address Explicit Attention to Concepts, should be followed as written with little deviation (CCSS, 2012; CCSS Authors, 2013; McCallum, 2012). However, the messaging around the how these concepts should be taught is inconsistent. On one hand, the CCSS authors proclaim that the practice standards must be integrated into every lesson across the curriculum and used in a holistic way that addresses their intentions (CCSS Authors, 2013). On the other hand, the CCSS authors explain that “The standards establish what students need to learn, but they do not dictate how teachers should teach” (CCSS, 2012, p. 2), a message which has been interpreted by many to mean that expectations set by the practice standards, including around SOS and DBL, are optional (Munter et al., 2015).
One important aspect of the Stein et al. (2016; 2016) framework is that instead of assuming that EAC, SOS, and DBL must be implemented together for any of them to be meaningful, which has been the message of mathematics reform in the U.S. for decades (NCTM, 1989, 2000; Schoenfeld, 2004), it questions the ramifications of considering each variable independently, as shown in Figure 4.2. While quadrant 1 (high/high) is associated with reform efforts, quadrant 4 (low/low) is associated with traditional teaching, and quadrant 3 is associated with reform approaches that are implemented poorly and superficially, this framework provides the space to investigate what happens when EAC is high while SOS is low (quadrant 2). (Stein et al., 1996; Stein, Kelly, et al., 2016).

Another interesting feature of this framework is that it removes discussion as a key factor, and instead views it as one possible approach through which SOS and EAC could be achieved. I have chosen to leave discussion in the framework because distinguishing it helps to tease out three different components which have historically been lumped together by reform movements within the United States, and also because it is addressed in MP3 of the CCSS standards for mathematics practice. However, I address concerns with discussion as a low-rigor practice by identifying DBL as a specific type of high-rigor practice.

Figure 4.2
Framework showing SOS and EAC as independent variables; taken from Stein, Kelly, et al. (2016).
Tying this back to the standards, messaging from the CCSS authors suggests that what to teach is mandatory with EAC spelled out in the content standards, but because the CCSSM do not dictate how to teach, that the practice standards involving SOS (MP1) and DBL (MP3) may be optional. The implications of these messages are explored in the next section.

**How Should Mathematics be Taught and Learned?**

Thus far, we have discussed that “mathematics” may be understood as having two different purposes, procedural understanding or conceptual understanding, and that the CCSSM has suggested that EAC is mandatory while SOS and DBL might be less important or more flexible. Beliefs, messages, and logistics come together here to ask the question: “how should mathematics be taught and learned?”

In a fascinating round-table discussion Munter, Stein, and Smith (2015) encouraged educators who held opposing views to share their visions, goals, methods, and beliefs about mathematics teaching and how they were utilizing and responding to the CCSSM. They aimed to neutralize past animosity between sides by looking for points of agreement as well as contrasts, and also provided new labels, *direct* and *dialogic*, to more accurately and respectfully label the instructional models proposed by the two groups. While they recognize that most teachers pull strategies from both models, sometimes within the same lesson, they argue that distinguishing them is valuable for making sense of two sets of behaviors that U.S. teachers often follow. I also offer international research on models from outside the United States and focus particularly on a model which I call *guided pathway* that is used in some other countries.
As a way of introducing all three of the models, and their global history, I find a quote about Japanese, German, and U.S. mathematics teaching from Stigler and Hiebert’s (1999, pp. 25–26) *The Teaching Gap* to provide an illuminating summary:

In Japanese lessons [similar to the dialogic model], there is mathematics on one hand, and students on the other. The students engage with the mathematics, and the teacher mediates the relationship between the two. In Germany [similar to the guided pathway model], there is the mathematics as well, but the teacher owns the mathematics and parcels it out to students as he sees fit, giving facts and explanations at just the right time. In U.S. lessons [the traditional model that preceded the direct model], there are students and there is the teacher. I have trouble finding the mathematics; I just see interactions between students and teachers.

I claim in this article that the dialogic model has an increased presence in the U.S. since *The Teaching Gap* was written, that the guided pathway model has been successfully introduced to the U.S. (and can also be seen in many textbooks that also have dialogic lessons), and that the direct model now contains significantly more mathematical concepts than its traditional predecessor. That is, mathematics can now be found in U.S. textbooks.

This section addresses prior research on each of these models, which I will build on in my own analysis.

**The dialogic model**

The dialogic model suggests that students learn mathematics effectively when they productively struggle with novel tasks (supporting SOS); participate in speculatively, supportive, and critical discourse (supporting DBL); develop their own increasingly efficient problem-solving strategies that eventually evolve into conventional procedures (supporting EAC); and also build fluency with some amount of routine practice. The teacher’s role is to monitor student’s work and provide suggestions that
avoid giving the students solutions that they can discover on their own. This model seeks to build conceptual understanding in which students have the skills and mindset to think with mathematical flexibility in new situations.

The dialogic model is supported by a range of theories and studies from multiple fields, all of which center around having students do the work of mathematicians to develop the mindsets of mathematicians. When students develop these habits of mind, they believe that mathematics involves creativity, sense-making, and problem solving, and they view themselves as being capable of tackling challenging tasks (Boaler, 2016; National Research Council, 2001; NCTM, 2000; Skemp, 1976). To develop these mindsets, they need opportunities to engage in productive struggle: expending effort to solve rich problems and wrestle with concepts that are within reach, but not immediately apparent or clearly formed (Hiebert & Grouws, 2007; National Research Council, 2001; NCTM, 2014; Stein, Correnti, et al., 2016). While both direct and dialogic groups believe that information is constructed as schemas (networks of organized information) (Munter et al., 2015), productive struggle supports students in developing more robust schemas which are less likely to deteriorate because memory is highly organized and meaningful, easier to access due to multiple linkages for recall, and supportive of transferring, developing, or reconstructing rules when new but related situations are presented (Boaler, 2016; Hiebert & Grouws, 2007; Hiebert & Lefevre, 1986; Skemp, 1976). In addition, dialogic approaches see language and communication as a critical tool for thinking and learning through engaging in sustained reasoning, articulating ideas so that others can understand them, making sense of others ideas that may be unclear or contain mistakes that offer additional learning opportunities, utilize representations and symbols for
communicating ideas, and taking on the communication norms of the mathematics community (Boaler, 2016; Boerst et al., 2011; Vygotsky, 1978). Thus, dialogic models interweave EAC, SOS, and DBL as a coherent approach to rigorous learning.

**The direct model**

The direct model suggests that students learn mathematics effectively when they observe clear demonstrations of how to solve a class of problems with attention to accurate and comprehensive explanations and definitions, practice similar problems with decreasing support and increasing complexity, and receive immediate feedback to correct mistakes and misconceptions. When students are solving problems, the teacher’s goal is to make things as simple as possible for students, and step in to support them quickly to avoid having them replicate and memorize incorrect approaches. This model implicitly (and in the words of some of the participants in the discussion) assumes that the goal of mathematics is procedural understanding—being able to replicate algorithms correctly to get correct answers under textbook conditions.

One finding of Munter, Stein, and Smith’s (2015) research, and a number of larger studies that they reference, was that proponents of both the direct and dialogic models agreed on the pacing, content, approaches, and quality of the CCSSM content standards and felt a commitment to following them. As issues of content were often at the center of the math wars, this level of agreement suggests a meaningful step toward meeting the goals of the CCSSM by increasing Explicit Attention to Concepts.

At the same time, one of the primary areas of disagreement among the direct and dialogic supporters related to the practice standards which focus heavily on SOS and DBL. While supporters of the direct model agreed that the practice standards were valid
theoretical statements of what it means to know and do mathematics, they felt that they could be ignored or minimized in daily lessons. For example, MP3 states that students should construct and critique arguments (DBL), but direct model supporters felt that it was sufficient for students to occasionally hold internal dialogues without communicating their ideas to others. Similarly, in MP1, MP2, and MP3, students are expected to propose and explore the truth of their own conjectures (part of SOS and DBL), but supports of the direct model restricted this to testing a few routine strategies in a structured setting. As a result, the direct model focuses heavily on EAC that is transmitted to students through direct instruction by the teacher or textbook. Students are then asked to offer their own written and verbal explanations of concepts that have already been modeled.

The direct model bears a strong resemblance to a specific type of task developed in the cognitive science community called *worked examples*, in which a strategy is modeled, and then students are given scaffolded questions to identify its key features before replicating it with decreasing levels of scaffolding. Research shows that when a student is learning new concepts and strategies for the first time with minimal background knowledge, worked examples are more effective than many other approaches (Booth et al., 2017; Chen et al., 2015; Kirschner, 2002; Paas et al., 2003; Sweller & Cooper, 1985). They are theorized to work because they decrease cognitive load, the amount of competing information that the brain has to process at once, while drawing attention to critical features and underlying concepts (Booth et al., 2017; Chen et al., 2015; Kalyuga et al., 2003; Sweller & Cooper, 1985). However, worked examples have their limits. While particularly effective for novices, their impact on even somewhat experienced students ranges from ineffective to harmful, in what is known as the
expertise-reversal effect (Kalyuga, 2007; Kalyuga et al., 2003; Moreno, 2006). Thus, the instructional model used in direct textbooks may be effective for introducing key concepts to novices, but to move students from being novices to experts, they require opportunities to productively struggle.

**International models**

While the direct and dialogic models can be traced to traditional and reform models (respectively) in the United States, understanding the instructional models used in other countries offers a helpful perspective for making sense of models now being used in the U.S. (Hiebert et al., 2005; Leung, 2005; Stigler & Hiebert, 1999; Valverde et al., 2002).

The direct model has some similarities to educational models in Singapore, China, and Hong Kong where teachers and textbooks use lecture and replication that focuses on a conceptual understanding of procedures (Ginsburg et al., 2005; Hiebert et al., 2005; Hoven & Garelick, 2007; Ma, 2010). These countries score very well on international assessments and the CCSSM content standards were designed to closely follow the national content standards in Singapore and other high-performing countries (Houang & Schmidt, 2008; McCallum, 2012; Remillard & Kim, 2017).

The dialogic model is similar to the model used in Japan, where students solve an open-ended task and then discuss it as a whole group with a focus on building on and making connections between multiple problem solving strategies (Hiebert et al., 2005; Stigler & Hiebert, 1999). However, it is important to note that this model requires expertise in mathematical content, pedagogical understanding of how to teach it effectively, and facilitation and behavior management skills (Ball et al., 2008; Shulman,
In Japan, this is supported by several hours of professional learning every week, which is unheard of in the United States, often leaving the burden of acquiring these skills to individual teachers and the educative features of the textbook (Davis & Krajcik, 2005; Stein et al., 2007; Stigler & Hiebert, 1999).

These are far from the only models. In the Czech Republic, students listen to lectures for several days to introduce all of the concepts in a unit. Then students spend several days “reviewing”—a rigorous activity in which students are called to the board to solve complex problems and justify their steps to the class while the teacher ask questions to highlight key features (Hiebert et al., 2005). In the Netherlands and Sweden, the primary role of teachers seems to be guiding students’ interactions with the textbook, which they spend most of their math lessons working on independently (Hiebert et al., 2005; Remillard et al., 2017). Students may each be working on separate lessons, even on material from different grades, and the short lessons that teachers present at the beginning may only apply to the content that a few of the students are presently learning.

In Germany, teachers and students walk through a series of problem-solving steps together, with the teacher asking students for immediate next steps or conceptual explanations of the current step (Stigler & Hiebert, 1999). While the teacher controls the progression and pacing of the task, the students are still responsible for developing their own next steps and explanations. This is similar to a model that is now appearing the United States which I call guided pathway.

The research in the rest of this paper explores how the direct, dialogic, and guided practice models have been enacted in eight U.S. elementary textbooks.
Methodology

This analysis builds on two related analyses that I conducted. These two projects asked *How are elementary mathematics textbooks interpreting and enacting the CCSSM content standards?* (chapter 2) and the equivalent question for the *CCSSM standards for mathematical practice (SMPs)* (chapter 3).

These analyses provided me with a close view of how the structure of lessons supported or discouraged different approaches to interpreting and enacting each type of standard. While conducting the other analyses, I simultaneously observed, recorded analytical notes, coded, and wrote memos on the relationship between the decisions made around the CCSSM and trends in instructional models.

This paper draws on the findings from the other two projects and the concurrent analysis of instructional models. Because the two foundational projects used different analytical methods, this project is strengthened by both a fine-grained and wide-spread analysis. For the standards of mathematics practice, I sampled five multiplication lessons from each of grades 3, 4, and 5 (for a total of 15 lessons per program). This in-depth analysis of a smaller number of lessons lent itself to observing how each of the components and tasks fit together within the structural design of a lesson to address the CCSSM SMPs and incorporate beliefs about the nature of mathematics and how it should be taught. I used this analysis to develop an understanding of the types of instructional models available and make detailed notes about their features.

I then conducted a larger study of the CCSSM content standards using all multiplication lessons in grade 3 for each curriculum program (ranging from 27-58 lessons per program) which allowed me to confirm or build on my initial observations. In
this study, I used the whole lesson as the unit of analysis and was able to label overall instructional models with a larger data set.

*Textbook and Lesson Selection*

Eight elementary mathematics curriculum programs were included in this analysis. Programs were eligible for analysis if they were either fully developed after the CCSSM were released or underwent a significant revision to align with the CCSSM. I was unable to include several other programs that would have met these criteria because I was unable to obtain program access from the publishers.

One program, Bridges in Mathematics, provided only a limited number of units for review (14 out of roughly 43 multiplication lessons for grade 3), so I expanded the sampled lessons for this study to also include all of the available multiplication-related lessons from grades 4 and 5 using a lesson-level approach based on the content standards analysis. For the remaining programs, the full text of the teachers’ guide and student materials were available and I used only the lessons included in the initial two analyses.

For convenience and clarity, I identify the programs, which are listed in Table 4.2, by the instructional models that arose from my findings. Descriptions and examples of these models appears in the following sections.
Table 4.2

Overview of textbook sampled for analysis with instructional models, publishers, years, and number of lessons per grade.

<table>
<thead>
<tr>
<th>Instr. Model(s)</th>
<th>Program</th>
<th>Abbr.</th>
<th>Developer/ Publisher</th>
<th>Year</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct</td>
<td>enVision Mathematics 2020</td>
<td>ENV</td>
<td>Pearson</td>
<td>2020</td>
<td>44</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Go! Math</td>
<td>GO</td>
<td>Houghton Mifflin Harcourt</td>
<td>2015</td>
<td>27</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Into Math</td>
<td>INTO</td>
<td>Houghton Mifflin Harcourt</td>
<td>2020</td>
<td>28</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>My Math</td>
<td>MY</td>
<td>McGraw Hill</td>
<td>2018</td>
<td>30</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Dialogic (with some Guided Pathway lessons)</td>
<td>Bridges in Mathematics*</td>
<td>BRI</td>
<td>The Math Learning Center/ Curriculum Associates</td>
<td>2015</td>
<td>14</td>
<td>11*</td>
<td>13*</td>
</tr>
<tr>
<td></td>
<td>Investigations in Number, Data, and Space 3</td>
<td>INV</td>
<td>TERC/Pearson</td>
<td>2017</td>
<td>33</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Guided Pathway and Dialogic</td>
<td>Everyday Mathematics 4</td>
<td>EVER</td>
<td>University of Chicago School Mathematics Project/ McGraw Hill</td>
<td>2015</td>
<td>37</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Guided Pathway</td>
<td>Eureka Math</td>
<td>EUR</td>
<td>Great Minds</td>
<td>2013</td>
<td>58</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

It is worth noting that the four programs that are labeled as direct were developed by traditional publishers (Pearson, Houghton Mifflin Harcourt, and McGraw Hill) while the other four that use dialogic or guided pathway models were developed by small organizations or universities using private or governmental funding. While I based my analysis of the programs entirely on their content and structure, and not their developers, understanding the origins of the textbooks is relevant to understanding the role of marketing pressures in curricular decisions, which will be discussed later.
Findings: Exploring How Textbooks Interpret and Enact the CCSSM

In my prior analyses of how eight elementary textbooks interpret and enact the CCSSM content and practice standards, I noticed several trends in the content and structure of the lessons that matched the descriptions of direct and dialogic instructional models described by Munter, Stein, and Smith (2015). In addition, I observed a third instructional model, guided pathway, that takes on some attributes of both the direct and dialogic models, but is more closely matched to some international models. My intention is to flesh out the work of Munter, et al. (2015) by investigating how the pedagogical philosophies of direct and dialogic curriculum developers were translated into textbook content, and extending their work by applying the same analysis to the guided pathway model.

In this section, I use the fundamental questions about curriculum design as a theoretical approach to exploring how the CCSSM content and practice standards are interpreted and enacted by the CCSSM in each of the instructional models. As an overview, Figure 4.3 expands the analytical framework shown in Figure 4.1 to include the placement of the CCSSM content and practice standards and specific textbooks that utilize each of the instructional models. Solid lines represent strong relationships, while dashed lines represent partial relationships.

For example, the guided pathway (middle of the third row), is aligned with conceptual understanding (top right) and has a strong relationship to EAC, but uses a modified version of SOS and DBL. It is the only model used in Eureka Math (EUR), makes up around half of the lessons in Everyday Math (EVER), and is used in a smaller
portion of the lessons in Bridges in Mathematics (BRI) and Investigations in Number, Data, and Space (INV).
Analytical framework addressing three central questions about what mathematics education with connections to the CCSSM and the eight textbooks in this study.

**What is mathematics?** (Skemp, 1976; Hiebert & Lefevre, 1986)

- **Procedural Understanding**
  - Rote procedures and efficient computation

- **Conceptual Understanding**
  - Creative problem solving and understanding why strategies work

**What is rigorous mathematics?** (Hiebert & Grouws, 2007; Stein, Correnti, et al., 2016)

- **Explicit Attention to Concepts (EAC)**
  - Articulating concepts that underly strategies

- **Student Opportunity to Struggle (SOS)**
  - Wrestling with and generating new ideas by solving novel tasks

**How should mathematics be taught and learned?** (Munter, Stein, & Smith, 2015)

- **Direct Model**
  - Teacher presents procedures and students replicate

- **Guided Pathway Model**
  - Teacher guides a discussion where students generate concepts and strategies through pre-determined steps

- **Dialogic Model**
  - Students generate task solutions and consolidate ideas through discussion

**CCSS influences:**

- **CCSS Content Standards**
- **CCSS Mathematical Practices (SMPs)**

**Sample textbooks:**

- enVision Mathematics (ENV)
- Go! Math (GO)
- Into Mathematics (INTO)
- Eureka! Math (EUR)
- Everyday Mathematics 4 (EVER)
- Bridges in Mathematics (BRI)
- Investigations in Number, Data, and Space 3 (INV)
This presents an overall picture of much greater complexity than before the CCSSM, but also tracible paths to philosophies that both influenced and are reflected in curriculum design. Two changes in mathematics curriculum programs are of particular interest. First, textbooks that use the Direct Model have picked up Explicit Attention to Concepts while still advocating Procedural Understanding. And second, a new Guided Pathway model has emerged that supports Conceptual Understanding through Explicit Attention to Concepts and modified versions of Student Opportunity to Struggle and Discussion-Based Learning. Each layer of this model, and its implications for the CCSSM and how they are interpreted and enacted in textbooks, is summarized in Table 4.3 and will be unpacked in the following sections.
### Table 4.3

**Summary of characteristics of the three instructional models with respect to EAC, SOS, DBL, and potential impacts.**

<table>
<thead>
<tr>
<th>Nature of Math</th>
<th>EAC</th>
<th>SOS</th>
<th>DBL</th>
<th>Potential Impacts</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Direct Model</strong> (ENV, INTO, GO, MY)</td>
<td>Procedural</td>
<td>EAC is presented to students by the textbook; substantial increase compared to pre-CCSS without requiring much teacher knowledge or skill.</td>
<td>Absent; SMPs are tagged for individual routine tasks, which sends misleading messages about the meaning and use of SMPs</td>
<td>Absent; two programs include DBL tasks that lack SOS</td>
</tr>
<tr>
<td><strong>Dialogic Model</strong> (BRI-most, EVER-half, INV-most)</td>
<td>Conceptual</td>
<td>Strong EAC in textbooks; heavy requirements for teacher knowledge and skills</td>
<td>Strong SOS; students have opportunities to develop strategies and wrestle with new concepts</td>
<td>Students learn through DBL; heavy requirements for teacher skills</td>
</tr>
<tr>
<td><strong>Guided Pathway Model</strong> (EUR-all, BRI-some, EVER-half, INV-some)</td>
<td>Limited Conceptual</td>
<td>Strong EAC focused carefully on the CCSSM content standards; clear instructions keep EAC central</td>
<td>Modified SOS; students solve problems and wrestle with concepts as a class along preset path; more SOS than direct, but less than dialogic</td>
<td>Structured dialogue where students suggest steps and provide explanations; easier for teachers</td>
</tr>
</tbody>
</table>
To make comparisons across programs easier, I showcase a single lesson from each model (one lesson from each of enVision, Bridges in Mathematics, and Eureka Math) that all address the topic of introducing the distributive property as a tool for finding unknown products of single digit numbers using known multiplication facts. I chose it because multiplication strategies based on the distributive property are the most frequently used strategies in grade 3, and because they were not used in traditional textbooks prior to the CCSSM (though they can be seen in earlier versions of EVER and INV). The distributive property is foundational for understanding factoring in algebra and a valuable tool for working flexibly with multiplication, so its heavy usage has substantial implications for learning.

This topic addresses content standards 3.OA.5 and 3.OA.7, which explain that the distributive property is not supposed to be taught as an abstract definition, but rather used as a tool to develop conceptual understanding and fluency with products of one-digit numbers.

3.OA.5 Apply properties of operations as strategies to multiply and divide. 
*Examples:... Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*

3.OA.7 Fluently multiply and divide within 100, using strategies such as… properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

In the following sections, I explore each of the three models, direct, dialogic, and guided pathway, by first introducing the model with an overview and an example, then showing how it addresses EAC, SOS, and DBL in light of the CCSSM. I end each section
with a discussion of impacts of the model on student learning and, in some cases, accessibility to teachers.

**The Dialogic Model in Textbooks**

The dialogic model appeared in my analysis as the primary instructional approach in Bridges in Mathematics and Investigations in Number, Data, and Space, version 3. It was also used in roughly half of the lessons in Everyday Mathematics 4.

The heart of the dialogic model is the Launch-Explore-Discuss (LED) cycle: launching an open-ended task in a way that draws on background knowledge without “giving away” strategies or solutions, allowing students to explore the task independently or with peers, and then using discussion to share, build on, and critique student’s strategies and ideas as a way of consolidating concepts (Boerst et al., 2011; Munter et al., 2015; Sleep, 2012; Margaret S. Smith et al., 2008). After the LED cycles, students often (but not always) have an opportunity to practice new skills and concepts, play fluency games, complete prior tasks.

While direct lessons used a consistent structure regardless of the content, dialogic lessons fluidly modified the placement, length, and timing of LED cycles to support different tasks and mathematical concepts. The most common lesson style consisted of 3-5 minutes to launch a task, another 3-10 minutes to explore, 10-25 minutes for discussion, and then 20-30 minutes for practice and fluency. However, this was far from the only option.

Another common lesson style utilized a rapid LED cycle (5-10 minutes total) and then had students spend the majority of the lesson on practice and other work. Some
lessons focused extensively on other parts of the LED cycle, giving twenty minutes to a whole class period to just the launch or the explore. Some lessons used up to six LED cycles back-to-back, with each LED task building on the concepts from the previous one. Some lessons lasted several days, in which a task would be launched on one day and discussed on the following day, with exploration on one or both days. And some lessons presented two or even three 15- to 25-minute LED cycles on related or unrelated topics, seeming to combine several distinct lessons into a single instructional period.

Unlike direct models, in which students worked predominantly with textbooks that provided completed or semi-completed strategies, many dialogic lessons were based on lengthy sample dialogues in the teachers’ guide. Students might begin with a blank sheet of paper or whiteboard, a set of manipulatives, or a single task presented on a slide or worksheet. As the class consolidated ideas through discussion, students’ strategies were recorded and displayed in hand-drawn posters, which position students, rather than the textbook, as the generators of ideas.

BRI GR3 2.2.3 provides an example of the flexibility of dialogic lessons with six LED tasks (this is higher than usual) that showcase several variations on the LED cycle. The lesson opens with a lengthy, story-based launch of a three-day exploration involving a window washer that is used to teach multiplication with arrays. Students determine the number of windowpanes in uncovered and partially covered arrays using three rapid LED cycles of increasing difficulty, of which the third is shown in Figure 4.4. The tasks are designed to elicit multiple strategies such as repeated addition, skip counting, the commutative property, and doubling. The window washer investigation continues for two
more days as students explore increasingly sophisticated strategies, but in this lesson they
next switch to a different set of tasks with different representations.

**Figure 4.4**

*Launch-Explore-Discuss task from BRI GR3 2.2.3: discovering doubling, an application
of the distributive property with arrays in a context of washing windows.*

| 4 × 4 = 16 | Students’ strategies will be inspired
| 16 + 16 = 32 | by how they see the picture. Many
|                  | will figure out how many panes they
|                  | see on the left and then double that
|                  | number because the second window
|                  | has the same number of panes as the
|                  | first. Some students, however, may
|                  | use the structure of rows and columns
to figure out how many there are alto-
|                  | gether. Others may use the problems
|                  | they have solved before. Encourage
|                  | students to solve the problem in
|                  | whatever way makes sense to them,
|                  | while pressing for efficiency.
| **Student** I counted by 4 to see how many were in one window. 4, 8, 12, 16 | **Big Idea**
| Then I doubled that to get the total. 16 + 16 = 32 | Thinking about equal groups can help
| **Student** I found all of the panes that I could see. Then I figured out the panes I couldn’t see. First I did 5 columns of 4; that’s 20. Then I saw a row of 3 on the top. That’s 23. Then, I knew there must be 3 more rows of 3 behind Wally. That’s 9, 23 and 9 make 32. It’s 32 windowpanes in all. | you find the total, especially when you can’t see every pane to count them.

In the next two LED cycles, students look at a cube train (stacked linking cubes
that alternate colors to show equal groups) and consider how they can determine the
placement and value of 8 × 6 if they already know 4 × 6 (Figure 4.5). Students share
several strategies and the teacher emphasizes the efficiency of doubling. They, they solve
several “number line puzzles” (see Figure 4.6) using the strategies that they developed
while working with arrays. At this point, the lesson instructs the teacher to confer with
individual students to either make their strategies more efficient or support them in
articulating conceptual connections, a small-group version of LED.
Figure 4.5

LED task from BRI GR3 2.2.3: discovering doubling, an application of the distributive property with number lines and “cube trains.”

![Number line with 4 × 6 and 8 × 6 shown on it.]

Emphasize the efficiency of using a doubling strategy. Many students will see that 8 × 6 is 4 × 6 doubled, so they will realize that they can double 24 to get 48.

Figure 4.6

LED task from BRI GR3 2.2.3: applying and discussing the distributive property with number line puzzles.

![Number line with 2 × 4, 4 × 4, 8 × 4, and 9 × 4 shown on it.]

**SUPPORT** For students who are skip-counting, help them to see a few different ways to make their work more efficient. Emphasize how laborious it is to skip-count up every time. Ask them how they could make it any easier. Ask them if they can combine any of their “jumps” to make it easier. For example, when skip-counting by 4s to find 8 × 4, help students to see that they could double their jumps to take jumps of 8.

**CHALLENGE** Challenge students to use and explain partial products. Both the number lines for 8s and 7s have been designed so that students see that they can use the first two problems to solve the third. For example, 2 × 7 + 3 × 7 = 5 × 7.

The lesson ends with the teacher introducing a fluency game called “Frog Jump Multiplication” in a final LED cycle. Students use dice to determine the number and size of hops they should make on a number line. The class discusses the first few turns with a focus on strategies that can be used to find the products (Figure 4.7).
Figure 4.7

LED task from BRI GR3 2.2.3: conversation about strategies used to find products to introduce the multiplication fluency game “Frog Jump Multiplication”

Teacher OK, you all rolled a 3 and a 6, so you’re going to take 3 jumps of 6. Where will you land? If you don’t know exactly, use what you do know to make a good estimate…. Who’d like to share?

Ahmad I don’t know exactly, but I did know if it was 3 jumps of 5, we would land on 15. So it’ll be a little more than 15.

Casey Hey! It would be 3 more than that. It would be 18.

Teacher Casey, when you said, ‘Hey!’ it sounded like you just thought of something new. What did you figure out?

Casey Well, we’re going to make 3 jumps. If each one was 5, like Ahmad said, it would be 15. But each jump is 6, that’s 1 more than 5. So if you have 3 jumps and they’re each 1 bigger, that’s like adding 3 to 15. It’s 18.

Kaitlyn I agree that it will be 18, but I thought, OK, 2 jumps of 6 is 12, because 6 plus 6 is 12. Then 6 more is 12 plus 6. I know 2 plus 6 is 8, so 12 plus 6 has to be 18.

Teacher Let’s take the jumps. Help me count while I draw them here on the number line.

Throughout the class, students take notes in their journals using their own words.

There is an optional student workbook page available for extra practice, if the teacher chooses to assign it.

This lesson will be used as an example for discussing how dialogic lessons address the three features that are often associated with rigor, DBL, SOS, and EAC, in the order of their prevalence in the lessons.

Discussion-Based Learning

One of the major strengths of dialogic lessons is that students are encouraged to communicate mathematical ideas as valued members of a mathematical community (Lave, 1991; Stein et al., 2008; Wenger, 2000). The philosophy behind the dialogic
model is based on two beliefs about mathematics teaching and learning: 1) communicating mathematical ideas is a necessary skill that requires practice, and 2) students should learn concepts by formulating ideas and discussing them with others under the guidance of a teacher who helps clarify or formalize ideas (Munter et al., 2015; Sleep, 2012; Stein et al., 2008).

The example lesson, BRI G3 2.2.3, provides several strong examples of one interpretation of what DBL can look like in practice through LED cycles. The window washer task (Figure 4.4) demonstrates how students can be guided toward reaching a specific mathematical goal (doubling as a multiplication strategy), by providing a task that is geared toward that understanding, examples of multiple student-generated strategies that include doubling along with alternatives, and notes to the teacher about emphasizing doubling while also supporting a range of different strategies. In this example, teachers are both given instruction on doubling as a strategy and about how to develop that understanding with their students.

The Frog Jump Multiplication game (Figure 4.7) provides a sample class dialogue that references multiplication strategies from the lesson, but predominantly demonstrates teaching techniques for DBL. One of the most important features of this example is that the teacher plays an encouraging role (“what did you figure out?”) but allows students to build on each other’s ideas, rather than jumping in prematurely. The quote from Ahmad (“It’ll be a little more than 15.”) provides a model for valuing incompletely developed ideas rather than expecting students to contribute only when they can offer complete solutions.
Three aspects of discussion-based learning are of particular importance in understanding the implications of this model: recording knowledge, teacher knowledge, and tagging SMPs in lessons.

*Recording knowledge generated by students.* In BRI G3 2.2.3, like most other dialogic lessons in BRI, EVER, and INV, the only records of learning are those created by the teacher or students. Student materials contain tasks, but do not typically contain explanations or models. In comparison to direct model lessons, which could be taught by handing students the textbook without further teacher guidance, dialogic lessons use textbooks as a source of rich tasks and the strategies are developed in conversations between students with guidance from the teacher. If a student does not understand a concept or forgets it, there are no summaries available in the textbook. Instead, students are given a set of tasks that allow them practice what they’ve learned, or if necessary, a second opportunity to derive the strategies. Students who still lack comprehension must fall back on the teacher or peers for support.

*Teacher knowledge and skills.* The format of dialogic lessons suggests that before students can learn, teachers must first develop knowledge of mathematics concepts and skills for supporting dialogue. Because strategies and concepts are found only in the teacher materials, not in the student materials, this places a heavy responsibility on the teacher to 1) understand the concepts, 2) be able to articulate the concepts in ways that students can understand, 3) be able to support students in articulating the concepts themselves, 4) manage the complex classroom dynamics of learning through discussion, and 5) guide open-ended discussions toward key mathematical points (Ball et al., 2008;
Hill, Ball, et al., 2008; Sleep, 2012; Stein et al., 2008; Willoughby, 2000; Windschitl, 2002). Thus, the instructions and sample discussions in dialogic lessons play a dual role of telling the teacher what to do and helping them develop knowledge and skills (Davis & Krajcik, 2005; Stein et al., 2007). If teachers are not able to develop these intertwined skills of supporting the development of mathematical concepts through discussion, they are likely to enact only superficial aspects of DBL, such as having students share out strategies without clarifying and formalizing concepts, addressing relationships between strategies, or guiding conversations toward mathematical goals, or they may transform the lessons into direct models without the comprehensive thought that is used in direct textbooks (Stein et al., 2008; Stein & Smith, 1998).

Tagging SMPs. As discussed on p. 213, DBL is the focus of CCSSM MP3 (Construct viable arguments and critique the reasoning of others) and mentioned as smaller components of MP1 (Make sense of problems and persevere in solving them) and MP6 (Attend to precision). As the sample lesson expects students to fulfill the conditions of all three of these standards, curriculum developers would be justified in tagging the lesson with all three of them. However, these three standards could be tagged in every lesson in the dialogic textbooks due to the dialogic nature of their lesson structure. While this ubiquitous tagging might be accurate, it would not be helpful for identifying unique features of different lessons,

Instead the dialogic curriculum developers took an alternative route of selecting up to three SMPs to tag when they were particularly prominent in each lesson. For example, in BRI G3 2.2.3, MP1 and MP4 were tagged as applying to the whole lesson,
but MP3 was not. The way that the SMPs were tagged as applying to the whole lesson indicates that they are used throughout the whole lesson and intertwined, suggesting that mathematical habits of mind are not used in isolation from each other, even when some are used more prominently at a given time. This decision to tag only a few highlighted standards may send the message to educators (and potential purchasers) of dialogic textbooks that they do not cover very many SMPs or do not cover them often.

**Student Opportunities to Struggle**

In almost all of the dialogic lessons in BRI, EVER, and INV, students were presented with novel tasks that required them to generate new strategies by using existing knowledge to wrestle with new ideas that were accessible but not immediately apparent. The time that students spent “struggling” varied from two minutes to two class periods. However, both the nature of the tasks themselves and the way that they were presented in the sequence of the lessons supported students in applying their existing knowledge in new ways or inventing new strategies, either individually, with peers, or with increasing clarity during the discussion.

BRI G3 2.2.3 provides good examples of the key features of task design to promote SOS: 1) novel strategies, 2) accessibility, and 3) focus on specific mathematical understandings.

To understand how tasks approach novel strategies, it is important to recognize that this is the first grade 3 lesson in which students work with arrays. Rather than first teaching students what an array is or demonstrating strategies for using them (as in direct lessons), the teacher tells a story in which the window washer “began wondering about
Students are given little instruction beyond trying to figure out the number of windows, which invites a range of solving strategies. In addition, the tasks are ordered so that each one requires students to make new discoveries and connections. These tasks are not novel because they are open-ended, but rather because they prompt students to develop strategies that have not been previously explored.

Regarding accessibility, these tasks were designed so that all students have an entry point. The teacher instructs students to think about the windows in a scaffolded order: an uncovered array, then in a partially covered array, and then in a pair of windows where one is partially covered, and one is not. Students approach the first array with no further instructions and may even count each individual windowpane. However, the strategies that work for one problem may not be sufficient for the next one, which forces students to move toward more sophisticated strategies if they are not already using them. Due to the strategy-sharing conversations in between, students can approach later problems by testing out strategies that others have used.

This scaffolding also focuses the lesson on specific mathematical understandings. Both the last window washer problem and the number line task are designed to have students “discover” doubling as an effective strategy. In the arrays, students may use a range of other strategies, but will still encounter doubling in the class conversation. The first number line task, where students explain how to find $8 \times 6$ when they have already labeled the location of $4 \times 6$, is specifically designed to have them formalize their new understanding of the doubling strategy. The number line task is both novel because
students are using a different visual model (number lines instead of arrays) and accessible because they have already articulated a doubling strategy with arrays.

Returning to how SMPs are tagged in dialogic lessons, BRI G3 2.2.3 tags MP1 (Make sense of problems and persevere in solving them; text on p. 213), MP1 outlines the core work of SOS as students consider problem situations, try a range of strategies, check their work during and after applying a strategy, and then discuss and connect their strategies with other strategies. As this type of work happens in nearly every dialogic lesson, the curriculum developers may have felt that it was particularly relevant here.

This lesson also tags MP4 which addresses taking complex, real world situations and creating models that approximate or represent the situation.

**MP4 Model with mathematics.** Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation…. [Students might] plan a school event or analyze a problem in the community… [or] solve a design problem… [They] are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later…. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. In this instance, the models that students are developing are equations that demonstrate the number of windowpanes in the complex situation of partially covered
windows. Note that this example only skims the periphery of MP4, which also addresses much more complex design and problem situations where students must make approximations and revise their models when dealing with messy data or tasks with incomplete information. In this task and every other task across the eight curriculum programs, SMPs like this were interpreted to heavily serve the content standards, rather than opening up mathematical modeling to more contextual and uncertain tasks. Thus, although there is significant SOS in this task, it is still bounded by a focus on covering required content standards rather than providing students with opportunities to engage with more realistic situations.

**Explicit Attention to Concepts**

EAC is both the greatest opportunity and the greatest challenge of dialogic lessons. Unlike SOS, in which students must create and apply their own strategies, in EAC students or teachers must articulate the concepts behind the strategies or explain the relationships between strategies. That is, EAC suggests that it is not sufficient to develop or use a strategy if there is no explanation for why it works and how it connects to other tasks or concepts. In a dialogic lesson, while students generate EAC in collaboration, the teacher still bears heavy responsibility: she must take students’ partially-formed explanations and make sure that the underlying concepts are a) made explicit and b) emphasized, rephrased, or repeated in ways that highlights the mathematical point of the lesson, and c) related to other possible strategies without making the lesson so diffuse that key points aren’t retained.
In BRI G3 2.2.3, both the instructions to the teacher and the sequence of tasks are carefully designed to push students toward discovering doubling, articulating where it can be used, and then applying it in more sophisticated ways (e.g., doubling $4 \times 6$ to get $8 \times 6$ and then add 6 more to get $8 \times 9$). The teacher is given multiple representations and examples of doubling (two examples in the window-washer task, one example with cube trains, and several examples with Frog Jump Multiplication), as well as instructions that doubling should be highlighted as a more efficient strategy in two places.

Like most dialogic lessons, BRI G3 2.2.3, is highly attentive to the concepts in the CCSSM content standard that are tagged: 3.OA.5 and 3.OA.7 (text on p. 232) and 3.OA.1:

**3.OA.1** Interpret products of whole numbers, e.g., interpret $5 \times 7$ as the total number of objects in 5 groups of 7 objects each….

In this lesson, 3.OA.1 can be seen when students generate strategies based on counting rows or columns in the window arrays, the way that the number trains alternate colors, and their use of strategies like repeated addition, skip-counting, and hops.

Doubling and extensions of doubling rely on the distributive property, which students use to flexibly solve multiplication problems (3.OA.5 and 3.OA.7). The Frog Jump task, in which students roll dice and then hop by equal groups is a fluency exercise that also incorporates properties of operations and a range of strategies as part of the process of learning and memorizing multiplication facts, addressing all three content standards.

While both examples and definitions of doubling, as well as its relationship to the distributive property and possible extensions, are present in the lesson, this does not
guarantee that they are accessible to teachers. The lesson consists of 6.5 pages of closely spaced text, covers three completely different task types, and uses manipulatives that take time to set up. It expects teachers to introduce a new story, a new visual model, a new game, the commutative property, the distributive property for advanced students, variables in equations, and doubling as an effective strategy—all in 110 minutes.

The sheer amount of text, the task and discussion format that assumes some students will not develop the intended strategies, and the conceptual rigor of the tasks all place a high demand on teachers to a) understand the primary goals of the lesson, b) give them sufficient time and attention in class, c) insure that students walk away from the lesson with the ability not only use doubling but understand the concepts that underly it, and d) meet the first three goals through guiding students’ discussions of strategies that they developed themselves. That is, although the teacher’s guide pays Explicit Attention to Concepts, this does not guarantee that a teacher, especially one with weak conceptual understanding of newer ways of teaching multiplication or who struggles with behavioral management, will be able to convey that same level of EAC to students.

**Impacts of the Dialogic Model**

The dialogic model, which is used in most of Bridges into Mathematics and Investigations in Number, Data, and Space, as well as about half of the lessons in Everyday Mathematics has a number of advantages, but is also challenging to implement. This section summarizes and discusses some of the key impacts of this model.

*Deep conceptual understanding and student generation of ideas.* Dialogic lessons tend to have high EAC, high SOS, and a meaningful use of DBL to guide learning and
develop mathematical communication skills. They position students as capable and creative mathematicians who contribute to a learning community. When they are implemented in ways that maintain the level of rigor and agency at which they are written, they have the potential to help students develop a deep conceptual understanding of mathematics, flexible mathematical thinking, and a sense of self-efficacy.

*Implementation challenges.* Unfortunately, the ways that the dialogic model supports high EAC, SOS, and DBL also make it challenging for teachers to implement. Teachers must possess strong knowledge of the concepts themselves, how they can best be taught to students, and how to guide students in discussions that maintain EAC and SOS. Research suggests that many elementary school teachers lack this knowledge and instead significantly decrease the rigor of their lessons, often by adapting them to use a direct model or by taking on only superficial features of DBL that remove SOS, EAC, or both (Chazan & Ball, 1999; Hill et al., 2011; Stein, Kelly, et al., 2016).

*Teaching the teacher.* The efficacy and rigor of dialogic lessons predominantly comes down to constraints of textbooks as learning resource for teachers. In Japan, where the dialogic model is used heavily, teachers spend hours every week observing each other’s mathematics lessons, planning and discussing ways to help students discover and articulate mathematical concepts, and improving existing lessons to more effectively move students toward understanding mathematical points through their own efforts (Stigler & Hiebert, 1999). In comparison, elementary teachers in the U.S. might, if they are lucky, receive a few hours a year of professional development or collaborative learning with a mathematics coach that focuses on effectively leading dialogic lessons.
This puts the onus of developing teachers’ content knowledge and discussion facilitation skills on the teacher’s guide. While BRI, EDM, and INV all go to extensive lengths to provide this support, there is an open question as to whether this work is beyond what a textbook can provide, and past U.S. mathematics reform efforts suggest that it is not (Schoenfeld, 2004; Stein et al., 2007; Willoughby, 2000).

**The Direct Model in Textbooks**

In my prior research (see chapters 2 and 3), I examined four textbooks that I identify as using a direct model: enVision Mathematics 2020, Go! Math, Into Mathematics, and My Math. All four followed a very similar structure in terms of unit organization, lesson organization, and tagging of standards within lessons. Their predominant mode of instruction involved providing students with completed problems or partially completed problems that explicitly taught a procedure for students to replicate. While the teacher introduced the new procedures or concepts, students were expected to listen and provide summaries of the information on the page to confirm understanding. After concepts or procedures were introduced by the teacher and textbooks, students replicated them in similar problems.

For example, in enVision grade 3 lesson 3.1 (ENV G3 3.1; shown in Figure 4.8), students are shown an example of how the distributive property can be used to find $4 \times 7$ with a contextual scenario, an array model, equations, and a clear explanation. The teacher prompts them with questions (to the right) that check their understanding of the information that has just been presented. After this, students complete four highly similar
tasks under the teacher’s guidance and 14 more fill-in-the-blank, multiple choice, and short-response task that use the same procedure.

**Figure 4.8**

*Introduction of the distributive property through teacher/textbook modeling with short prompts for students from ENV G3 3.1.*

This lesson will be used as an example for discussing how direct lessons address the three features that are often associated with rigor, EAC, SOS, and DBL. One short example from another textbook, Go Mathematics, is also briefly included.

**Explicit Attention to Concepts.**

ENV G3 3.1 presents a typical example of how direct lessons attend explicitly to concepts. As shown in Figure 4.8, students learn to apply the distributive property when the textbook demonstrates the strategy with arrays, explanations, and equations, shown
side-by-side to support making conceptual connections between them. This introduction connects mathematical ideas (e.g., using known facts and array models to find new products), addresses relationships between strategies (by presenting the distributive property in several ways with language to connect them), and identifies meanings underlying procedures (by leading students to notice that different ways of breaking up the array all result in the same product).

What is notable about this use of EAC is that prior to the CCSSM, the distributive property was often taught as a meaningless definition or ignored entirely in elementary textbooks. In contrast to older, traditional textbooks, all of the direct textbooks that I sampled feature the distributive property heavily as a tool for learning multiplication facts (grade 3) and understanding multi-digit multiplication (grades 4 and 5). As with ENV, they systematically present it through array and area models side-by-side with equations that make the relationships clear (as discussed in chapter 2).

This Explicit Attention to Concepts seems to stem from the precise language in 3.OA.5 and 3.OA.7 (text on p. 232), though it is mentioned in three other content standards as well. There are three important features of 3.OA.5 and 3.OA.7 that seem related to their frequent and in-depth use by direct textbooks. First, they are concrete: they are can be completed with simple visual models and a limited number of repeatable steps. Second, because of their concreteness and the way they are structured linguistically, they are compatible with a direct model of teaching. That is, according to the language of the standards, students can apply properties of operations in 3.OA.5 and use strategies to multiply in 3.OA.7 without either discovering these concepts on their
own (SOS) or discussing them with others (DBL). Third, important components of these standards is repeated in 3.OA.4, 3.NBT.3, and 3.MD.7d, so frequency may suggest intended importance (as discussed in chapter 2).

In comparison, several other content standards were not addressed with sufficient frequency or depth across the direct textbooks in my sample, typically when they required students to take on a more active role in generating knowledge or struggling with difficult content. (The concepts in these standards were also only addressed once each.) Thus, my analysis (see chapter 3) found that direct textbooks paid explicit attention to concepts that could be enacted procedurally, but often did not address or fully address those that required conceptual approaches to problem solving.

**Student Opportunity to Struggle.**

SOS is described in the CCSSM standards for mathematical practice, with an overview in MP1 that is echoed in the remaining standards. Based on the findings from Munter, Stein, and Smith (2015), one might assume that the SMPs are completely ignored or not included in direct textbooks. In fact, they are tagged far more often than in the other textbook types, with as many as six separate SMPs tagged per page in some lessons. However, as tasks in direct textbooks always ask students to replicate the procedure that was just modeled by the textbook, students do not have opportunities to struggle (SOS). Thus, the SMP tags typically addressed only superficial aspects or isolated sub-components of the practice standards.

For example, both parts of the task shown in Figure 4.8 are tagged with MP7 (look for and make use of structure), which addresses the active mental task of noticing
when a pattern or structural component of mathematics occurs and recognizing its significance for solving problems. In particular, MP7 focuses on identifying structures before they are formalized into properties or efficient strategies—a skill that is practiced when students wrestle to make sense of mathematical relationships through a novel task.

**MP7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure… [For example], students will see \(7 \times 8\) equals the well-remembered \(7 \times 5 + 7 \times 3\), in preparation for learning about the distributive property. They recognize the significance of an existing [structure] and can use the strategy… for solving problems…

In ENV G3 3.1, however, students are told what the existing structure is, provided with several examples and a definition, and then asked to replicate the strategy and repeat back the steps. At no point are students given the autonomy to productively struggle (engage in SOS), because the work has already been done for them by the textbook.

When students move on to their independent work, they continue to replicate the same strategy. For example, the final task of the lesson (Figure 4.9), which is designated as “higher order thinking” is actually a heavily-scaffolded, decontextualized, example of the demonstrated procedure. The limited space and limited time provided for tasks like these (students are asked to complete 14 independent tasks in around 20 minutes) may send the message that all mathematics tasks should be completed quickly and easily, and that open-ended tasks should be addressed as briefly as possible using

![Figure 4.9](image.png)

*Task from ENV G3 3.1 that expects students to replicate the procedure demonstrated in the textbook.*
previously taught procedures. Thus, these structural features both directly and indirectly support a procedural approach to learning.

In addition, these short tasks are often combined with individual standards for mathematics practice (as shown in Figure 4.8 previously and Figure 4.11 that follows). In the other two direct textbooks, GO and MY, several nearly identical tasks in a row may be tagged with different SMPs. For example, in MY G3 8.8 (Figure 4.10), the two tasks shown are highly similar examples of decontextualized, heavily scaffolded tasks where students apply the distributive property in a short area. However, the first is tagged with MP2 (Reason abstractly and quantitatively), which focuses on decontextualizing and contextualizing, and the second is tagged with MP5 (Use appropriate tools strategically), which focuses on appropriately choosing when to use physical or technological tools such as counters or calculators. These tasks do not meet the requirements of either standard. (See chapter 3 for further discussion and analysis.)
The approach of tagging individual tasks with SMPs, which is common throughout the direct textbooks, sends several messages about what it means to enact a mathematical practice. The way that each task receives a separate SMP (rather than assigning several SMPs to the entire lesson) implies that mathematical practices can be used in isolation from one another. In addition, the length of the task implies that SMPs can be met in short time periods with a single line of text. This becomes more than just a problem of alignment, although this is also an issue. From a wider perspective, when teachers and students are informed that they can “reason abstractly and quantitatively” (MP2) by implementing a routine procedure in less than two minutes, with a single line for showing work, and in isolation from other standards, this sends the message that this type of work is all that there is to mathematics.
This makes a false tagging of individual routine practice tasks with SMPs much more pernicious than it would be if they weren’t tagged at all. Simply ignoring the SMPs, as the direct model advocates claim to do (Munter et al., 2015) would be intellectually honest and perhaps signal to teaches that they should supplement their textbooks with tasks that encourage SOS (though it would doubtless cause sales to plummet and drive the program out of print). By labeling the SMPs when they do not apply, these direct textbooks are redefining the SMPs to imply that they can be met satisfactorily without engaging in SOS. This decision communicates to teachers, who often rely on textbooks to interpret the CCSSM, that they are meeting the SMPs when they are not. It also communicates a procedural understanding of mathematics; if teachers believe that tasks like those above meet the standards they are tagged with, it would be reasonable to conclude that mathematics consists of only applying routine procedures.

Discussion-Based Learning

Supporters of direct models in Munter, Stein, and Smith’s (2015) study stated that they did not see the value of learning through having students engage in discussion and sharing their own strategies. As a result, standard MP3 (Construct viable arguments and critique the reasoning of others), which describes the skills that students develop through discourse, was often subverted in direct textbooks.

For example, the same lesson, ENV G3 3.1, claims to address MP3, but only does so in a single, short task (Figure 4.11). This task was explicitly designed to highlight a common mistake using clear language and does not address the overall intentions of the standard, which focus on meaningful communication between student mathematicians.
Although students identify a flaw in a carefully contrived sample student’s work, which addresses a tiny part of MP3, this does not match the overall intentions of the standard.

**Figure 4.11**

*Task from ENV G3 3.1 that tags MP3 but subverts the meaning.*

To the extent that aspects of discussion are used in direct textbooks, it tends to appear during the beginning of the lesson when the teacher and textbook model a procedure, as shown in Figure 4.8. At this point, students are asked to respond to short prompts with brief answers, most of which can be found on the textbook page. Typically, once or twice per lesson students will be asked to explain a concept that builds one step beyond what has been presented, explain why a misconception might occur, and so on. While these extension questions require some innovative thought, they still use a format that insists on a single correct answer which a single student provides to the teacher before moving on to the next part of the model. Thus, this approach does not represent DBL.

While direct textbook models generally subvert MP3, two direct textbooks (enVision Mathematics 2020 and Into Mathematics) include a regular task type that uses
a Launch-Explore-Discuss format. Because LED is often assumed to be equivalent to DBL, these more superficial discussions may be confused with DBL. Instead, these tasks demonstrate how aspects of MP3 can be addressed in a limited way when discussion isn’t combined with student generation of knowledge, SOS, or EAC.

ENV and INTO begin each lesson with open-ended or somewhat open tasks that students may solve using multiple strategies that they then discuss as a class. For example, in ENV G3 3.2 (the lesson following the target lesson of this section), the introductory task asks students to find $3 \times 7$ using an array and provides two possible student responses as shown in Figure 4.12. Students also answer a question that leads them to name the distributive property and list the steps of using it to find an unknown product. At first glance, this LED format looks like DBL.
Figure 4.12

Task from ENV G3 3.2 which uses a launch-explore-discuss format to have students replicate the strategy they were taught the previous day.

This task has many superficial aspects of the LED cycle in a dialogic lesson—students are answering an open-ended problem and then discussing their answers—but it falls short of mathematical rigor for several reasons. First, and most importantly, the
teacher’s note claims that the purpose of the task is to “elicit productive struggle,” yet students are expected to replicate the procedure they were taught in the previous lesson. Thus, they are neither generating their own ideas (a necessary component of DBL) nor engaging in the critical components of SOS: solving a task where a strategy is not immediately apparent as a way to wrestle with emergent mathematical ideas. Throughout both ENV and INTO, almost all open-ended introductory tasks simply reviewed the prior day’s lesson. When they were occasionally used to introduce new material, they were much more heavily scaffolded.

Second, although students use visual models, choose their own strategies, and answer questions about the distributive property in Figure 4.12, this does not show EAC. EAC requires either the teacher, the students, or the textbook to articulate the concepts that underly the procedures or explicitly make connections between the strategies. Instead students demonstrate that they can use a previously taught strategy and describe the steps that they were taught without explaining why those steps work. Furthermore, the teachers are instructed to focus on the strategies in general rather than addressing the relative efficiencies of strategies or making connections between them, which may result in a common misconception that all strategies are equally appropriate and that students should choose their favorite rather than progressing toward more efficient strategies over time (Boerst et al., 2011; Sleep, 2012; Stein et al., 2008).

Finally, the structure of lesson ENV G3 3.2 overall suggests that these open-ended introductory tasks should not be given much time; after this task students observe the teacher model two more problems with multiple representations, complete eight tasks
with the teacher’s guidance, and then complete another 11 more tasks of varying length independently. Any teacher expecting to cover most or all of the material in a typical 45–60-minute period could probably spend, at best, 5 or 10 minutes on the opening task. This amount of time would be insufficient for learning new concepts through discussion. (By comparison, when dialogic lessons use LED cycles this short, it is usually because several LED cycles are used in sequence to build on each other.) This duration fits well, however, when students are merely demonstrating that they can apply previously used procedures.

When direct textbooks tag MP3 and use open-ended introductory tasks with an LED format, they signal to teachers (and potential customers) that they are meeting the SMPs and imply that they are using DBL. However, the heavy scaffolding, repetition of procedures that were directly taught to students, and short duration of tasks divorce these superficial discussions from rigorous learning through DBL, which requires student generation of ideas and EAC.

**Impacts of the Direct Model**

The direct model, which is used consistently in enVision Mathematics 2020, Go! Math, Into Mathematics, and My Math, shows some important advances for mathematics in the U.S., but also uses some approaches that are misleading to educators. This section summarizes the impacts of the decisions made by curriculum developers who use the direct model.

*Focused use of CCSSM content standards.* Recent CCSSM-aligned direct textbooks have come a long way from earlier traditional textbooks in paying explicit
attention to concepts. This shows an impressive and meaningful impact of the CCSSM on U.S. curriculum programs regarding the content standards. While not all of the content standards are addressed fully when they require higher-level thinking, for the most part the intended content of the CCSSM is being taught in the suggested distribution and with minimal outside material. Concerns of U.S. curriculum being “a mile wide and an inch deep” can be replaced with a sense that students are progressing along a learning trajectory that assigns specific content to each grade. This is an important shift in mathematics education for the United States and should not be overlooked.

* Misleading use of CCSSM practice standards and discussion. The CCSSM standards for mathematics practice expect students to develop a range of skills and habits of mind that require taking an active role in solving novel problems (SOS) and discovering and articulating concepts (DBL). This type of thinking is not compatible with the direct model, in which students are asked to solve simplistic tasks by replicating procedures and explanations that are provided by the textbook. Instead, direct textbooks in my analysis used a system of labeling short, routine tasks with SMPs which gave the impression that students were engaging with the SMPs when they were not. Similarly, while discussion is not a necessary component for rigorous mathematics teaching, it is often perceived that way in the U.S. When ENV and INTO utilized open-ended tasks to review the prior day’s lesson, they gave the appearance of using DBL, but in fact supported only low-rigor discussion without EAC, SOS, or student generation of knowledge.
This misleading labeling around the SMPs likely helps publishing companies to sell textbooks, but it is confusing to educators who may believe that they are fulfilling the SMPs when they assign routine tasks. This is perhaps most pernicious and difficult to identify in lessons that use LED tasks to imply that they are using DBL, as reviews may not notice that the content in each LED task is repeated from the previous day’s lesson or understand the ramifications of this choice. The direct textbooks that I studied, all of which were produced by major publishing companies, seem to be heavily focused on marketability by frequently tagging SMPs without actually enacting them.

*Trading rigor for ease of implementation.* The way that direct textbooks provide EAC through modeling by the teacher and textbook, but do not support SOS or DBL, makes them relatively accessible for teachers. The lessons are brief and colorful, and the textbooks are designed so that students could actually complete the lessons without the support of a teacher. By positioning students as receivers of knowledge, it is the teacher’s job to transmit the knowledge from the textbook, and outside ideas are neither necessary nor encouraged. This model is easier for teachers mathematically (they do not need to have a deep understanding of the mathematics in order to follow the instructions in the textbook) and pedagogically (students are expected to listen, reply to prompts with brief answers, and follow instructions).

This overall approach supports procedural understanding, resulting in easy and rapid knowledge acquisition that is just as easily forgotten because students are not necessarily making connections between topics in ways that support retention (Hiebert & Lefevre, 1986; Skemp, 1976). While students have the opportunity to observe how
concepts underly strategies, their passive role may result in decreased mathematics self-efficacy and applied skills, leaving them without either the conceptual tools or the mindsets necessary to solve novel math problems when they are asked to do mathematics outside of the narrow framework of their textbooks.

The Guided Pathway Model in Textbooks

The guided pathway model is used in all Eureka Math (EUR) lessons, for about half of the lessons in Everyday Mathematics (EVER), and in a smaller portion of lessons in Bridges in Mathematics (BRI) and Investigations (INV). While these lessons were present in prior versions of EVER and INV (both of which have been in print since the 1990s), the success of the newly developed EUR, which uses this model almost exclusively, suggests a shift in the U.S. mathematics education community in response to the messages of the CCSSM. Instead of seeing guided pathway lessons as a sub-type of the dialogic model (which is based on launch-explore-discuss cycles), the development of EUR suggests that this model should be considered on its own because it uses an entirely different lesson structure.

Like most direct and dialogic lessons, guided pathway lessons typically open with short fluency warm-ups and/or short open-ended tasks to review old information that may be especially relevant to new topics that will be addressed. Also like direct and dialogic lessons, they typically end with fluency-building practice based on the current and prior lessons.

In the core section of the lesson where the guided pathway model introduces new content, it shares some elements with the direct and dialogic models, and also has its own
set of unique identifying features. Like the direct model, the teacher controls the conversation, asking the class shorter questions and expecting relatively brief answers, explanations, and suggestions that have correct (and incorrect) answers. Like the dialogic model, students begin with a blank page (or task, set of manipulatives, etc.) rather than a completed model in a textbook, and are expected to solve novel tasks and construct concepts through dialogue. What differentiates the guided pathway model is that problem-solving and discussion are merged together and guided by the teacher, so that the class moves through solving a problem and explaining underlying concepts in intertwined steps as a whole group. Students come up with each calculation, idea, and explanation based upon their prior knowledge and creative approaches to the task, but only one step at a time, under the direction of a teacher.

For example, in EUR 1.16, students begin with two fluency exercises, skip counting by 4s and 5s, which is relevant to later parts of the lesson, and reading bar models, which is unrelated review. They then have an open-ended task that reviews content from the previous day (demonstrating how the commutative property can be shown with arrays in a contextual situation) that also utilizes a multiple of 4. Students are then guided through the steps of understanding both what the distributive property is and how it can be used to extend multiplication facts in a whole class problem solving discussion (Figure 4.13). Students draw their own models and provide explanations in their own words, but this all occurs in lockstep with the teacher and the rest of the class. After the group discussion, students have 10 minutes to complete a problem set of similar tasks and then a 10-minute open-ended discussion where they again explain how the
distributive property works and discuss how they could use it solve the open-ended task from the beginning of the lesson. (In many EUR lessons, the final open-ended discussion asks students to articulate new ideas that build on the lesson topic, but in EUR G3 1.16 they mostly reinforce students’ understanding.)
Two guided discussions from EUR G3 1.16 where students discover how to apply the distributive property to find unknown products.

**Problem 1:** Model the \(5 + n\) pattern as a strategy for multiplying using units of 4.

T: Shade the part of the array that shows \(5 \times 4\).
S: (Shade 5 rows of 4.)
T: Talk to your partner about how to box an array that shows \((5 \times 4) + (1 \times 4)\), and then box it.
S: The box should have one more row than what's shaded. (Box 6 \(\times 4\).)
T: What expression does the boxed array represent?
S: \(6 \times 4\).
T: Label the shaded and un-shaded arrays in your box with equations.
S: (Write \(5 \times 4 = 20\) and \(1 \times 4 = 4\).)
T: How can we combine our two multiplication equations to find the total number of dots?
S: \(6 \times 4 = 24\), or \(20 + 4 = 24\).

Repeat the process with the following suggested examples:
- \(5 \times 4\) and \(2 \times 4\) to model \(7 \times 4\)
- \(5 \times 4\) and \(4 \times 4\) to model \(9 \times 4\)

T: What expression did we use to help us solve all three problems?
S: \(5 \times 4\).
T: Talk to your partner. Why do you think I asked you to solve using \(5 \times 4\) each time?
S: You can just count by fives to solve it. \(\rightarrow\) It equals 20.
T: It’s easy to add other numbers to 20.
S: (Discuss. Identify the ease of skip-counting and that the products are multiples of 10.)
T: Now that you know how to use your fives, you have a way to solve 7 sixes as 5 sixes and 2 sixes or 7 eights as 5 eights and 2 eights.

**Problem 2:** Apply the \(5 + n\) pattern to decompose and solve larger facts.

Students work in pairs.

T: Fold the template so that only 8 of the 10 rows are showing. We’ll use the array that’s left. What multiplication expression are we finding?
S: (Fold two rows away.) \(8 \times 4\).
T: Use the strategy we practiced today to solve \(8 \times 4\).
S: (Demonstrate one possible solution.) Let’s shade and label \(5 \times 4\). \(\rightarrow\) Then, we can label the un-shaded part. \(\rightarrow\) That’s \(3 \times 4\). \(\rightarrow\) \(5 \times 4 = 20\) and \(3 \times 4 = 12\). \(\rightarrow\) \(20 + 12 = 32\). \(\rightarrow\) There are 32 in total.
T: (Write \(8 \times 4 = (5 \times 4) + (3 \times 4)\).) Talk with your partner about how you know this is true.
S: (Discuss.)
T: We can break a larger fact into two smaller facts to help us solve it. (Draw number bond shown to the right.) Here, we broke apart 8 fours into 5 fours and 3 fours to solve. So, we can write an equation, 8 fours = 5 fours + 3 fours. (Write equation on the board.)
T: \((5 + 3) \times 4\) is another way of writing \((5 \times 4) + (3 \times 4)\). Talk with your partner about why these expressions are the same.
S: (Discuss.)
T: True or false? In \(5 \times 4\) and \(3 \times 4\), the size of the groups is the same.
S: True!
T: Four represents the size of the groups. The expression \((5 \times 4) + (3 \times 4)\) shows how we distribute the groups of 4. Since the size of the groups is the same, we can add the 5 fours and 3 fours to make 8 fours.

\[
8 \text{ fours} = 5 \text{ fours} + 3 \text{ fours} \\
8 \times 4 = (5 \times 4) + (3 \times 4) \\
= (5 + 3) \times 4
\]
The guided pathway model heavily emphasizes EAC and uses modified or partial SOS and DBL, which are each discussed below. This results in both strengths and challenges for the model for supporting rigorous mathematics.

**Discussion-Based Learning**

The guided pathway model is of particular interest because it utilizes a different type of DBL than is typically seen in U.S. textbooks, though it involves both student generation of ideas and EAC. Following MP3 (text on p. 213), guided pathway lessons expect students to make claims, justify their conclusions, respond to others’ arguments, distinguish correct logic, and ask questions to improve arguments.

The heart of guided pathway lessons is a conversation between teachers and students. The teacher poses a step along the pathway and students are asked to complete the calculation, confirm the truth of the step, suggest a next step, explain the step, or discuss how the step could be applied. And although guided pathway lessons look like simple scripts because they emphasize only the key mathematical points that should emerge from discourse, the sample responses are often not where students begin, nor do they accurately portray the full range of interactions that occur during lessons. Instead, each sample response represents the mathematical point or place where students should arrive, and which the teacher should reinforce, making this a good example of DBL.

Teachers who use Eureka Mathematics (which contains only guided pathway lessons) report that they have lively class conversations that can go in unpredictable directions as students struggle to develop explanations, build on each other’s explanations, correct themselves and each other, and have “ah ha!” moments when they
finally make sense of concepts that they are building together. These “ah ha!” moments may come either through student’s own efforts to articulate their thinking, through other students’ explanations, through clarifications or summaries from the teacher, or as they progress further down the pathway and encounter additional examples and more opportunities to articulate the concepts. (Note that Eureka Mathematics positions the script-like parts of their lessons as “vignettes” to guide conversations rather than a script to be followed exactly. They encourage teachers to identify the “plot” and the “ladder” of each lesson and use these to move toward the lesson objectives using the steps as a framework.)

Like all EUR lessons, EUR G3 1.16 (Figure 4.13) is written in a series of interchanges between teachers and students, with the student components representing the goals that students should eventually meet through dialogue. For example, in the section that is tagged with MP7, the teacher says “Talk to your partner. Why do you think I asked you to solve using $5 \times 4$ each time?” Students are expected to have a conversation with their partners in which they come to the conclusions in the following line, that it is easier to count by 5s and add other numbers to 20, than it is to solve problems without the use of the distributive property or with less friendly factors, which gives the teacher to check their understanding. The teacher then asks them to compare their strategies for solving tasks with other multiples of 5 and leads a discussion that has two key points: 1) skip counting, especially with fives, is easier than other strategies, and 2) when 5 is multiplied by an even number, the product will be a multiple of 10. Both of these are nestled within a larger discussion that requires students to articulate how and why the
distributive property is an effective strategy and why adding on to groups of five is a particularly effective way to use it. Although the descriptions are terse, there is an underlying assumption that students will need to struggle to understand and articulate these concepts with some guidance from the teacher, which makes this a form of DBL.

**Explicit Attention to Concepts**

One of the strengths of the guided pathway model is a focus on EAC. In the lessons that I analyzed, students were frequently asked to explain concepts or make connections between strategies using their own words. Tasks and teacher prompts were carefully designed to make underlying connections visible and provide language for describing them. The more scripted nature of these lessons, in which students all work on the same strategy at the same time, was designed so that the whole class can focus on clarifying concepts.

For example, in EUR 1.16, students are asked at multiple points to explain how they can use $5 \times 4$ to find other multiples of 4 by creating a visual model, both write their own equations and explain equations given by the teacher, identify efficiencies within the strategy (e.g., adding on to easy numbers like 20, slip-counting with multiples of 10, etc.). Toward the end of the discussion (Figure 4.13, right), the teacher shows and explains the concept using another visual model (number bonds) that students are already familiar with. While the teacher determines the models that are used, at each point the students are responsible for making sense of the underlying models which the teacher then summarizes.
This lesson is aligned to 3.OA.5 and 3.OA.7 (text on p. 232), which both address applying properties of operations to multiply and divide as a path to gaining fluency by the end of grade 3. As in the sample lessons for the direct and dialogic models, this lesson attends closely to both teaching the distributive property and having students use it as a tool for solving multiplication problems flexibly.

In comparison to the direct model, where the teacher and textbook are responsible for modeling concepts to students for them to repeat back in their own words, in the guided pathway model students are generally expected to generate their own explanations. This requires cognitive effort and focused attention from students that is not present in direct lessons. Because students are asked for explanations at several points, if they struggle the first time, they can benefit of hearing peers’ explanations (in partnerships and with the whole group) as well as the teacher’s explanations, even though the questions changed slightly and require them to articulate the concepts in slightly different ways.

The guided pathway model has advantages over the direct model in that students have agency and must put forth cognitive effort to generate explanations, which supports understanding and memory. It also has some advantages over the dialogic model in having a tight focus that makes it more likely that teachers will support the students in reaching and understanding the key points, which could be lost or incompletely articulated in a broader discussion where many strategies are presented.
**Student Opportunities to Struggle**

SOS is defined as solving novel tasks where the strategy is not immediately apparent as a way of wrestling with new mathematical ideas. Guided practice tasks tend to either direct students’ steps or ask them to suggest an immediate next step. In fact, it often is not obvious from the beginning of a guided pathway discussion what the overall goal of the conversation will be or why it is important, as the bigger picture unfolds as the teacher and students travel down the path together.

So, does the guided pathway model provide students with opportunities to struggle? If the focus of SOS is solving a novel problem by considering multiple strategies and possibly testing a few or deriving new ones, then SOS is not present. However, if the focus of SOS is wrestling with new concepts that may not be fully developed through making sense of a task, then at least some form of productive struggle is embedded in guided pathway lessons.

This middle ground could be interpreted in several ways. MP1 (Make sense of problems and persevere in solving them) argues that mathematically proficient students should be able to take on novel tasks as a core mathematical practice, which is not supported by guided practice lessons. If students are not expected to solve novel tasks without the teacher’s guidance at least occasionally, they are not fulfilling MP1 and are missing the opportunity to develop what is perhaps the most important mathematical skill. In BRI, EVER, and INV, students have the opportunity to solve novel problems in the dialogic lessons that are interspersed with the guided pathway lessons, so that MP1 is addressed frequently throughout the year. However, EUR students only complete open-
ended tasks for which they have already learned the strategies and only approach novel
tasks through guided pathway discussions, so MP1 is never addressed.

However, students are still learning important mathematical habits of mind. For
example, the task in EUR G3 1.16 is tagged with MP7, in which students are expected to
take an active role in identifying generally applicable patterns.

**MP7 Look for and make use of structure.** Mathematically proficient students
look closely to discern a pattern or structure. Young students, for example, might
notice that three and seven more is the same amount as seven and three more,
or… see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for
learning about the distributive property… They also can step back for an
overview and shift perspective. They can see complicated things… as single
objects or as being composed of several objects.

Students demonstrate the ability to “discern patterns or structures” as they move
through the guided pathway, explaining the relevance of the structure at several points as
it emerges and becomes more generalized. At the end of the lesson, the teacher introduces
the term *distribute*, showing how students have progressed to building the concept (as
opposed to direct lessons where students begin with a definition of the property). As a
habit of mind, students are “making sense of complicated things” (arrays where they
don’t immediately know the product) as “single objects and as several smaller objects”
(smaller arrays for which they have memorized facts or can skip-count). By meeting this
SMP and articulating explanations for the structures, students are engaging in the
productive struggle, but not the type described in MP1 or pure SOS.

**SOS** is also considered valuable because it is theorized to support mathematical
self-efficacy, a mindset where students see themselves as being capable problem solvers
who can approach novel problems with confidence and maintain that confidence even if
early attempts are not successful (Bandura, 1977; McGee, 2015). Mathematical self-efficacy may or may not be decreased by the guided pathway model, depending upon how both the teacher and students perceive the value of students contributing explanations to a shared class understanding of a new concept. (Though students certainly take a more active role in guided pathway lessons than direct lessons.)

**Impacts of the Guided Pathway Model**

The guided pathway model is used in all Eureka lessons, about half of Everyday Mathematics, and for some lessons in Bridges into Mathematics and Investigations in Number, Data, and Space. As this model contains some aspects of both the dialogic and the direct model, it has some of the strengths and weaknesses of each them as well as its own unique characteristics.

*Rigor.* Of the three models, guided pathway is the most focused on EAC with the fewest distractions. With the whole class focusing on developing explanations of a single concept or strategy at a time, students have multiple opportunities to grab hold of key concepts and deepen their understanding which might not be made as clear in dialogic lessons where multiple strategies are being presented. In textbooks that make use of both dialogic and guided pathway lessons, using this model strategically to teach more challenging concepts could be a tremendous asset. However, in textbooks like Eureka, where there are no opportunities to solve novel tasks, the lack of SOS detracts from the overall rigor of the program.

*Students roles.* In the guided pathway model, teachers control the lessons, but students generate their own explanations. In comparison to the direct model, students are
encouraged to discover and articulate concepts (aspects of SOS and DBL), which may support them in developing deeper understanding and retaining ideas. However, in comparison to the dialogic model, students have decreased opportunities for developing and choosing their own strategies (SOS), which may result in decreased mathematics self-efficacy and insufficient preparation for solving novel math problems. Although the guided pathway model expects students to engage in productive struggle related to EAC, the level of teacher control and lack of full SOS may leave students more dependent on their teachers than the dialogic model.

*Ease of Use.* The flip side of strong teacher control is increased ease of implementation. One of the biggest challenges to the dialogic model is that most U.S. teachers lack the skills, knowledge, and ongoing professional learning opportunities to orchestrate classroom discussions that both draw on a variety of students’ strategies and clearly articulate the key mathematical ideas of the lesson. As guided practice model has the whole class focus on a single strategy and the teacher maintains control, this might be a more accessible tool for teachers to use while still focusing on EAC.

**Discussion**

This chapter opened by setting out three great promises of standards-based reform: equity, alignment, and rigor. Textbooks play a pivotal role in transforming the CCSSM from a succinct list of goals to a detailed set of daily lessons with substantial influence over teaching and learning for millions of children, and therefore provide a valuable lens for understanding how the CCSSM has made progress toward those promises. As the U.S. settles into a place where the CCSSM is no longer new and
textbook publishers have had an opportunity to develop or revise their programs, this analysis can provide one type of benchmark for understanding both the potentials and pitfalls for the current state of mathematics education under the CCSSM.

The U.S. is on the edge of some meaningful changes in how instructional models for textbooks (and teaching) are being conceptualized and developed. We are moving away from a simple traditional vs. reform system with simple binaries of rote/creative, bad/good, procedural/conceptual, and so on. While understanding the traditional and reform models continues to have historical importance—and certainly impacts answers to the questions of what mathematics is and how it should be taught—falling back on these categories will no longer be sufficient or accurate for studying curriculum, teaching, and learning.

Regarding alignment, the CCSS authors made it clear that the content standards were to be enacted as written, without adding, removing, or changing any material. This message seems to have been taken to heart, as textbooks show relatively high alignment to the content standards across the board. This is an immense win for the CCSS initiative, which has managed to wrangle most of the country onto a planned learning trajectory and away from the repetitive “mile wide, inch deep” textbooks of the past. Textbooks are now covering a narrower range of content in more depth, building on concepts from year to year instead of repeating it, and making the same content available to students regardless of where they live. In addition, these newer textbooks show an unprecedented level of Explicit Attention to Concepts, one of the factors that Hiebert and Grouws (2007) identify as necessary for rigorous mathematics teaching. The presence of visual models
and transparent algorithms that support conceptual understanding, clear explanations of concepts underlying strategies, and development of mathematical concepts along a developmental trajectory is indicative of a meaningful shift in how mathematics is being taught in the United States. From this perspective, the CCSSM has made huge progress on the promise of alignment.

However, alignment looks different when viewed through the lens of rigor. The CCSS authors have promoted an intentional agnosticism about how mathematics should be taught. This decision overlaps messily with the CCSS standards for mathematical practice, which require students to “make sense of problems and persevere in solving them” (MP1) and “construct mathematical arguments and critique the reasoning of others” (MP3).

The result has been three different interpretations on how necessary or relevant it is to follow the mathematical practices (as well as parts of the content standards that demand higher order thinking and student discovery). These interpretations are made concrete in the instructional models described in this article. Direct model textbooks essentially ignore the SMPs in lesson development, but then tag them inappropriately in the lessons, giving the incorrect impression that the SMPs can be met without allowing students to productively struggle, develop concepts on their own, or communicate mathematically with peers. Dialogic model lessons continue to incorporate the SMPs, especially MP1 and MP3, at the core of their instructional model. And guided pathway lessons provide opportunities for students to develop their own explanations and learn
through discussion, but not to productively struggle while solving novel tasks independently.

These three models suggest that the messaging around instructional agnosticism was effective: curriculum developers have felt free to make their own philosophically guided decisions around incorporating the SMPs, including, in the case of direct model programs, ignoring them and then tagging them incorrectly for marketing purposes. The result is an extremely unbalanced level of rigor across popular elementary mathematics textbooks.

And as long as curriculum developers are answering the question *What is mathematics?* with textbooks focused on procedural understanding, the CCSSM will not be able to meet the promise of equity. With the current set of available textbooks, two students at neighboring schools might be learning the same content, but one will be learning how to complete rote calculations while the other is learning the skills and habits of mind of a mathematician.

This raises the question: “Is it possible to be pedagogically agnostic while insisting on SOS and DBL in the text of the practice standards?” I propose that this is not possible, and that messaging about pedagogical agnosticism is in direct conflict with the text of every one of the practice standards. This does not mean that dialogic lessons are the only instructional model that can provide opportunities for rigor, as international models and the guided pathway model show.

This politically motivated concession on rigor shows great political savvy in cutting through the entrenched “math wars” to get, at a minimum, content standards
approved by states and into use. While the NCTM Standards (1989, 2000) encouraged SOS, EAC, and DBL equally and made relatively little impact on teaching (Schoenfeld, 2004; Stein, Kelly, et al., 2016; Willoughby, 2000), the CCSSM approach of focusing on the content standards is significantly more accessible for teachers. Even when new visual models and algorithms are introduced, they remain concrete, bounded, and compatible with instructional models that are comfortable to teachers. Put differently, by yielding on SOS and DBL in the practice standards, the CCSS authors may have giving the United States a real chance at achieving EAC on a wide scale.

With such a wide variation in interpretation of these messages, the decision around which textbook to purchase for a school or district can have a substantial impact on the rigor of the lessons that teachers and students receive. While direct model textbooks are now largely aligned to the CCSSM content standards, they present that content in a way that guarantees that students will be perpetual novices in the field of mathematics. Without SOS, they will never take on the adaptive expertise and competence that the SMPs suggest is necessary for mathematical proficiency. Further, cognitive research with worked examples (which are similar to the EAC-focused demonstrations in direct textbooks) finds that they become unhelpful or even harmful once students surpass a basic level of understanding (Kalyuga, 2007; Kalyuga et al., 2003; Sweller & Cooper, 1985). Based on this, I recommend that educators avoid purchasing direct textbooks or supplement them heavily with novel and rigorous tasks from other sources. Educators using textbooks that only use guided pathway lessons
could also benefit students by providing some SOS opportunities that are not teacher-guided.

Decision making around dialogic and guided pathway programs is more complex and may depend upon the resources that a school or district has for teachers’ professional learning. The dialogic model is both more rigorous but also more difficult for teachers to implement without decreasing that level of rigor. When implemented poorly, it may be stripped of the intended explicit attention to concepts and student opportunity to struggle in ways that are particularly confusing because there is no resource for students to fall back on. The guided pathway model may be easier for teachers to implement with sustained rigor and fulfills many of the SMPs, including having students productively struggle to develop explanations for concepts. The major drawback of the guided pathway model is that it does not include opportunities for students to productively struggle with novel tasks, though this could be supplemented in (for guided pathway only programs like Eureka Math) or addressed by choosing programs that use a balance of dialogic and guided pathway lessons (like Everyday Mathematics). Ultimately, making well-informed purchasing decisions between these two models requires having a strong sense of the benefits and drawbacks of each model, as well as an understanding of teachers’ knowledge, skills, and opportunities for further learning.

The nation is still divided on the nature of mathematics and how it should be taught. However, the CCSSM has made meaningful strides in incorporating one aspect of rigor, explicit attention to concepts, into curriculum programs that previously lacked this quality. In addition, it has opened the door to a newer instructional model, guided
pathway, that might balance agency in student discovery with a more teacher-centered approach that is easier for teachers to implement in situations where professional learning opportunities are typically limited. From a policy perspective, this is a huge success; from an instructional perspective, there is still work to be done in educating instructional leaders about the three models and their implications for teachers and students.
CHAPTER 5: CONCLUSION

Understanding how textbooks are interpreting and enacting the CCSSM may provide one clue into what to expect for the future of mathematics education in the United States as these textbooks move into circulation across the country. The previous chapters suggest that the CCSSM have been successful in narrowing the content focus of textbooks to address a few topics that develop along a learning progression using a number of new algorithms and representations that support conceptual understanding. At the same time, the lesson structures in textbooks, which arise from pedagogical philosophies of their developers, heavily influences the rigor and conceptual depth at which the content standards are addressed and whether the practice standards are enacted at all. Thus, while all textbooks now cover the same content, students and teachers may have radically difference experiences with what it means to learn and do mathematics with the CCSSM depending upon which textbooks they are using.

The complexities of standards-textbook alignment revealed in these findings suggest that approaching alignment as a simple question of percentages or a vague question of intentions, is not sufficient. To understand both the influence of the standards on mathematics education in the U.S. and how textbooks can be positioned to support their overall goals, I suggest that approaches to alignment should be considered much more carefully by policy makers, curriculum developers, alignment evaluators, textbook purchasers, and educators. In this dissertation I offer my analytical framework as an alternative approach and my findings as a proof of concept for unpacking some of the
In this section, I first address some limitations and next steps for research regarding standards-textbook alignment. I distinguish between findings that may be specific to this small sample and require further analysis and findings that are more structural and generalizable. I then consider implications for these findings and future research as textbooks move into usage by schools. I suggest that understanding how textbooks are structured to support or inhibit the more rigorous aspects of the CCSSM become not just an issue of alignment, but of access and equity.

Limitations and Next Steps

Perhaps the greatest limitation of this study is its relatively small size. While this study is broad in terms of the number of textbooks studied—many comparable studies use only one or two textbooks—it is still limited in terms of the subject area and grades covered. I was able to address only 30-60 grade 3 multiplication lessons per textbook when analyzing alignment with the CCSS content standards and only 15-18 grade 3-5 multiplication lessons per curriculum program when analyzing the SMPs. While my findings offer important evidence about how textbook authors enact standards, as well as a proof of concepts, for several emerging patterns, there is a further possibility that standards in other topics might be interpreted and enacted differently.

2 While I did not conduct rigorous analysis of other grades or topics, I performed spot-checks, digital searches for standards and key words indicating concepts, reviews of table of contents and standards alignment documents, and so on for all of the grade K-5 textbooks in the sampled curriculum programs to answer questions that arose during this research. While I cannot speak to specific standards at the level of detail in the findings, I can confidently state that the types of instructional models used in the multiplication lessons were used in consistent ways across the topics and grades in the sampled textbooks. Similarly, the modes of interpreting and enacting standards (with close adherence to CCSSM content standards in
With that said, much of my analysis of textbooks was not based on unique features of these particular samples, but on structural features across multiple textbooks. These structural features by design either support or inhibit the full implementation of the more cognitively demanding aspects of the CCSS. When students are positioned as receivers of knowledge, as they are in direct textbooks, there is no space for Student Opportunity to Struggle (SOS) or Discussion-Based Learning (DBL) because students cannot discover concepts and strategies have already been demonstrated for them. Even without a detailed analysis of standards-textbook alignment, the structure of these textbooks prohibits them from fulfilling any of the SMPs and the conceptual depth of many of the practice standards because they expect students to generate ideas.

A more significant limiting factor is that while I was able to deeply analyze all eight practice standards, I was able to study only nine content standards as a proof of concept. This makes the structural features of standards that I identified as impacting textbook enactment more tentative for the content standards. The factors that I suggest of linguistic complexity, student struggle, and repetition should be tested with other topic areas in other grades. Using examples instead of generalizable statements, however, seemed to arise as a problem equally in both content and practice standards, making the support for this factor more consistent.

As an immediate next step, further analysis of the content standards using the integrity model demonstrated in this dissertation, or similar integrity models that focus on general, but rigor and adherence to performance largely determined by the instructional model) were consistent across other grades and topics within these eight programs.
both the details and the holistic intentions of the standards, would be a valuable way to better understand how current U.S. curriculum programs are responding to the CCSSM.

**Textbooks in Context: Time, Access, and Equity**

As discussed in Chapter 4, Hiebert and Grouws (2007) proposed a distinction between explicit attention to concepts (EAC), student opportunity to struggle (SOS), and the structure of classroom lessons such as discussion-based learning (DBL). They noted that very few, if any, studies make these distinctions, so that typically programs which were rich in all three were compared to programs that were weak in all three, making it difficult to distinguish between them. (Stein, et al. (2016) and Hill, Litke, and Lynch (2018) are among the few who do, and they build on Hiebert and Grouws’ theoretical model.) Hiebert and Grouws note that EAC and SOS seem to be essential to deep mathematical understanding (and high scores on exams) but hypothesized that high EAC combined with focused and supported struggle within highly structured lessons might be able to meet the same goals, though they had not yet seen it in practice.

As the textbooks that were involved in this study move into use, future researchers in the United States may be able to gain some insights into the relative importance of these factors. Or perhaps more explicitly, is it enough to just have EAC without SOS and DBL? What if EAC is combined with bounded forms of SOS and DBL?

As curriculum developers have responded to the CCSSM with new instructional models, the United States is currently conducting a grand national experiment on exactly these questions. Direct textbooks address EAC in some ways, but they are not fulfilling
the intended meanings of some of the content standards or any of the practice standards because they do not SOS and DBL. Dialogic textbooks are structured to support EAC, SOS, and DBL, and within this sample, also attended deeply to almost all of the content and practice standards, though they are harder for teachers to implement. Guided pathway textbooks support EAC and modified SOS and DBL—the exact format Hiebert and Grouws hypothesized about, attend deeply to almost all of the content and practice standards, and are relatively easy for teachers to implement. These new instructional models have, in essence, removed the variable of EAC so that the effects of SOS and DBL in textbooks, in full and bounded forms, might be uncovered.

Research on textbook implementation that distinguishes between SOS, EAC, and DBL or other aspects of instructional format is just starting to occur in a few qualitative studies. Hill, et al. (2018) found that EAC was increasingly common in classrooms, though many teachers still lacked sufficient knowledge and skills to accurately articulate concepts. They also found that SOS and DBL were mostly present only when three factors occurred together: individual teachers’ knowledge and skill levels were relatively high, schools offered consistent training and support, and dialogic textbooks were in use. Stein, et al. (2016; 2016), who did not collect data on which textbooks their participants were using in a large-scale state-wide study, found that an impressive 30% of teachers were teaching with both high EAC and high SOS (using a dialogic model) though and even larger number (35%) were using SOS without the EAC to support it (often in a less mathematically clear version of the dialogic model). They found 18% of teachers to be using a high EAC/low SOS approach, which they expected to follow a direct format, but
instead was more similar to guided pathway for many participants, with teacher-guided discussion and bounded opportunities to struggle. They imply that teachers adapted their lessons to use a guided practice format and suggest that this format may be a more comfortable way for teachers to address the EAC requirements of the CCSSM. Most interestingly, they found that while students from the high EAC/high SOS classrooms had the highest scores on both skill-based and performance based tests, students in the high EAC/low SOS classrooms with bounded struggle (similar to guided pathway) did nearly as well.

At the moment, this type of research is just beginning to take place with carefully selected districts and schools. While textbooks with all three types of instructional models described in chapter 4 are currently available in the U.S. market, there are several limitations of timeline and access, a consideration which ultimately becomes an issue of equity.

Uniting the nation’s educational system under a single set of ambitious standards with sufficient incentives to impact textbooks, assessments, and teaching has often been viewed as a critical component for improving mathematics education in the United States (Hiebert et al., 2005; Schmidt & Houang, 2012; Senk & Thompson, 2003). However, deep changes on such a wide scale take time—multiple decades—to be realized. While the CCSS were released in 2010, the first newly-developed CCSS-aligned mathematics program (Eureka Math from Great Minds) did not emerge until 2013, and two major publishers only fully revised their programs (enVision Mathematics 2020 from Pearson and Into Math from Houghton Mifflin Harcourt) to address the CCSS in 2020. Consider
further that many school districts only adopt new curriculum every five to eight years, that teachers need several years to fully understand a new program and implement it well, and that students may struggle for several years when they are introduced to more rigorous mathematics or expected to switch from learning passively to generating ideas (Hudson et al., 2010; Kent & Spence, 2000; Reys & Reys, 2006; Stein et al., 2007). All of this is to say that in 2020, a decade after the release of the CCSS, many schools may still not be using textbooks designed for the CCSS or may just have started to use them. Many more may still be using older textbooks that claimed to be aligned to the CCSS but in fact did not make substantial changes, as was the case with earlier versions of ENV and other programs, (Cogan et al., 2015; Reys & Reys, 2006).)

Of perhaps greater concern is the question of which students and schools have access to which programs. In a study of the six CCSS states that track the curriculum programs their schools are using, the programs that use a dialogic model (BRI and EVER) were used in less than 7% of schools (Blazer et al., 2019) between 2014-15 and 2016-17. These schools had the lowest percentages of students eligible for free and reduced-price lunch (FRPL, a proxy for low socioeconomic status) and the highest percentage of parents who held a BA or higher. EUR, the only program that uses only a guided pathway model, was being used in around 15% of schools with some of the highest levels of FRPL students and lowest academic achievement by parents, possibly because it is available for free. The study conducted by Blazer et al. (2019) found no significant difference in state test scores based on curriculum programs across multiple states or years after accounting for other factors, which may not be surprising given the
remarkable conformity of direct textbooks and the very low prevalence and inequitable usage of dialogic textbooks.

This finding shifts the question of “what does textbook-standards alignment mean?” from an issue of compliance and pedagogical philosophies to an issue of equity. If Hiebert and Grouws (2007) correctly summarize from other research that both EAC and SOS are necessary for conceptual development, then direct textbooks are not just out of alignment with the CCSSM, they are also reinforcing social inequities and inhibiting any students who use them from developing a rich conceptual understanding of mathematics. Yet these textbooks collectively hold over three-quarters of the market, with disproportionately high use in schools with disadvantaged students.

This concern is exacerbated when organizations like EdReports use tools to evaluate standards alignment that seem to hold little internal consistency and give high scores to programs that do not allow students the opportunity to engage in practice standards and more rigorous aspects of the content standards. As an educational community, it would be valuable for stakeholders in a variety of positions, including policy makers, standards developers, curriculum developers, alignment evaluators, educational administrators, and teachers, to reconsider what alignment to the CCSS means and how they can support both its holistic intentions and its details. Distinguishing between superficial and deep alignment is not just a question of compliance, but in issue of supporting today’s students and future generations in opportunities to meaningfully learn mathematics.
BIBLIOGRAPHY


CCSS Authors. (2013). K–8 publishers’ criteria for the Common Core State Standards for Mathematics (pp. 1–22).


Pre-Service mathematics teachers’ narrative arcs and mathematical orientations over 20 Years. *Journal of Mathematics Teacher Education.*


Munter, C., Stein, M. K., & Smith, M. S. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record, 117*(11), 1–32.


Newton, J. A., & Kasten, S. E. (2013). Two models for evaluating alignment of state
standards and assessments: Competing or complementary perspectives? *Journal for Research in Mathematics Education, 44*(3), 550–580. https://doi.org/10.5951/jresmatheduc.44.3.0550


Remillard, J. T. (2018a). Examining teachers’ interactions with curriculum resource to
uncover pedagogical design capacity. In L. Fan, L. Trouche, C. Qi, S. Rezat, & S. Visnovska (Eds.), Recent Advances in Research on Mathematics Teachers’ Textbooks and Resources. Springer.


Student Achievement Partners. (2010). *CCSS Where To Focus Grade 3 Mathematics*. Achieve the Core. https://achievethecore.org/content/upload/SAP_Focus_Math_3.pdf


