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Differential Essential Dimension

Man Cheung Kevin Tsui
University of Pennsylvania

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Abstract

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In this thesis we define an analogue of essential dimension in differential algebra. As application, we show that the number of parameters in a general homogeneous linear differential equation over a field cannot be reduced via gauge transformations over the given field. We also bound the number of parameters needed to describe certain generic Picard-Vessiot extensions.

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Man Cheung Kevin Tsui

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Julia Hartmann, Professor of Mathematics

Dissertation Committee:
David Harbater, Christopher H. Browne Distinguished Professor in the School of Arts and Sciences
Robin Pemantle, Merriam Term Professor of Mathematics

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ABSTRACT

DIFFERENTIAL ESSENTIAL DIMENSION

Man Cheung Kevin Tsui

Julia Hartmann

Roughly speaking, the essential dimension of an algebraic object is the minimum number of parameters needed to specify the object. It was first introduced by J. Buhler and Z. Reichstein in [4] where it was used to bound the number of parameters that one may eliminate from a general polynomial by means of Tschirnhaus transformations.

In this thesis we define an analogue of essential dimension in differential algebra. As application, we show that the number of parameters in a general homogeneous linear differential equation over a field cannot be reduced via gauge transformations over the given field. We also bound the number of parameters needed to describe certain generic Picard-Vessiot extensions.

Contents

Chapter 1

Introduction

This thesis is about how much (or rather, how little) we can simplify a homogeneous linear differential equation with general coefficients by means of certain elementary operations called gauge transformations. To do this, we develop a differential analogue of the notion of essential dimension. Roughly speaking, the essential dimension of an algebraic object is the minimum number of parameters needed to specify the object. It was first introduced by J. Buhler and Z. Reichstein in [4] in the context of simplifying polynomials by means of Tschirnhaus transformations. To understand the present work, it is useful to consider the original context in which essential dimension was presented. In this thesis, all fields will be of characteristic zero.

1.1 Simplifying polynomials via Tschirnhaus transformations.

Simplifying polynomials has long been a goal of mathematics. Consider a quadratic polynomial $p(x) = x^2 + ax + b$ in two parameters a and b. The change of variables $x = y - a/2$ simplifies $p(x)$ to a polynomial of the form $q(y) = y^2 + c$ in the one parameter $c = b - a^2/4$. Similarly, a cubic polynomial may be simplified by a linear transformation to a polynomial of the form $y^3 + cy + c$, $y^3 + cy$, or $y^3 + c$ in one parameter c. However, linear changes of variables only simplify a quartic polynomial down to two parameters.

We can also try simplifying polynomials using more general transformations introduced by E. Tschirnhaus in 1683. Let F be a field. Given two polynomials $p(x)$ and $q(x)$ over F, we say that $q(x)$ is a *Tschirnhaus transformation* of $p(x)$ over F if there exists a polynomial over F that maps the set of roots of $p(x)$ bijectively onto the set of roots of $q(x)$ inside some common field containing these roots.

We now consider simplifying the *general polynomial*

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

whose coefficients $a_0, ..., a_{n-1}$ are algebraic indeterminates over F. We formalize the notion of the number of parameters to which one can reduce $p(x)$ to be the minimal number $d(n)$ of algebraically independent coefficients of $q(x)$, as $q(x)$ ranges over the Tschirnhaus transformations of $p(x)$ over the field $F(a_0, ..., a_{n-1})$. Above, we saw that $d(2) = d(3) = 1$ and $d(4) \leq 2$. Classical results by Klein and Hermite imply that $d(5) = 2$. By developing the theory of essential dimension, Buhler and Reichstein in [4] were able to prove $d(4) = 2$ and the bounds

$$\lfloor n/2 \rfloor \leq d(n) \leq n - 3$$

for $n \geq 5$. Later, A. Duncan [6] proved that $d(7) = 4$. However $d(n)$ remains unknown for $n \geq 8$, and the determination of $d(n)$ is a modern continuation of classical 19th century mathematics which resists a full resolution. For more on the history, see [4] and [5].

1.2 Simplifying differential equations via gauge transformations

We may ask a similar question about differential equations. Let F be a field of differentiable functions like $\mathbb{C}(x)$. Consider the *general homogeneous linear differential equation*

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

whose coefficients $a_0, ..., a_{n-1}$ are *differentially algebraically independent* over F, i.e., the coefficients and their formal derivatives

$$a_0, ..., a_{n-1}, a_0^{(1)}, ..., a_{n-1}^{(1)}, a_0^{(2)}, ..., a_{n-1}^{(2)}, ...$$

are algebraically independent over F. Then $p(y)$ has coefficients in the field

$$K = F\langle a_0, ..., a_{n-1}\rangle = F(a_0, ..., a_{n-1}, a_0^{(1)}, ..., a_{n-1}^{(1)}, ...)$$

generated by the indeterminates $a_0, ..., a_{n-1}$ and their formal derivatives over F.

We would like to simplify $p(y)$ by means of certain elementary transformations. Given two homogeneous linear differential equations $p(y) = 0$ and $q(y) = 0$ over F, we say that $q(y)$ is a *gauge transformation* of $p(y)$ if there exists a differential operator over F that maps the set of solutions of $p(y) = 0$ bijectively onto the set of solutions of $q(y) = 0$. We formalize the notion of the number of parameters to which one can reduce $p(y)$ to be the minimal number $e(n)$ of differentially algebraically independent coefficients of $q(y)$, as $q(y)$ ranges over the gauge transformations of $p(y)$ over K.

In Chapter 6, we will show that $e(n) = n$ for all $n \geq 1$ using similar arguments to those which establish the bounds for $d(n)$. This result roughly says that the general homogeneous linear differential equation cannot be simplified by gauge transformations alone, and that we must use auxiliary transformations like exponentiation and integration to help simplify the general equation.

Let us briefly explain how we will prove the result $e(n) = n$. Our overall approach is to follow that of [2] by developing the necessary analogous tools in differential algebra. For instance, in Chapter 4 we introduce a cohomology set which provides the appropriate substitute for Galois cohomology. This cohomology set allows us to translate from homogeneous linear differential equations to differential

GL_n-torsors (defined in Subsection 2.4.1). Under this correspondence, the general differential equation corresponds to a generic differential GL_n-torsor (defined in Chapter 5). The theory in Chapter 5 then lets us prove in Chapter 6 that the description of this generic differential GL_n-torsor requires at least n parameters.

1.3 Generic Picard-Vessiot extensions

We can also apply our work to the study of generic Picard-Vessiot extensions, similar to how the theory of essential dimension applies to the study of generic Galois extensions. One problem in Galois theory is to parametrize the family of all Galois extensions that have a given Galois group G. For instance, quadratic field extensions K/F always have the form $K = F(\sqrt{a})$ for some $a \in F$. Therefore we may view the quadratic field extension $F(\sqrt{a})/F(a)$ in one parameter a as being "generic", since any other quadratic extension is obtained by specifying a value of a. Many results about generic Galois extensions are documented in [10], and the theory of essential dimension is used to give lower bounds on the number of parameters a generic Galois extension requires.

We similarly have a Galois theory of differential equations for which there is a corresponding notion of generic extensions. This Galois theory was initiated by É. Picard and E. Vessiot and proceeds as follows. Starting with a homogeneous linear differential equation $p(y) = 0$ over a field F of differentiable functions, we construct a field extension K/F with good properties that contains a full set of solutions to $p(y) = 0$. Such an extension K/F is called a *Picard-Vessiot extension* of $p(y)$, and from it we construct the differential Galois group of $p(y)$. We expect Picard-Vessiot extensions with the same differential Galois group to have similar properties, and once again we seek to parametrize the Picard-Vessiot extensions with the same differential Galois group. We call such a parametrizing extension a "generic" Picard-Vessiot extensions (see Subsection 2.3.3 for a formal definition).

Many people have studied and constructed generic Picard-Vessiot extensions, including A.K. Bhandari, L. Goldman, L. Juan, A. Ledet, A. Magid, A. Pillay, and N. Sankaran (see for instance [9], [14], [12]). In Chapter 6, we will use an argument in [4] to give lower bounds on the number of parameters a generic Picard-Vessiot extension requires.

1.4 Outline of the thesis

This thesis proceeds as follows. In Chapter 2, we review differential algebra. In Chapter 3, we formalize certain classes of objects as functors and define the differential essential dimension of a functor. In Chapter 4, we introduce twisted forms and cohomology in the setting of Picard-Vessiot theory. In Chapter 5, we define and construct versal and generic differential torsors. Chapter 6 gives the main results of this thesis.

The general strategy we follow is that of [2]. The applications we give are the differential analogues of those in [4].

Chapter 2

Background

This chapter reviews background content and sets the notations and conventions used in this thesis. For content on differential algebra and the Picard-Vessiot theory, excellent sources include the article [29] and the textbook [30].

2.1 Algebra

The set $\{0, 1, 2, ...\}$ of natural numbers is denoted by $\mathbb{N}$. A *ring* means a (unital) commutative ring; a *noncommutative ring* means a possibly noncommutative ring. A *linear algebraic group* is a smooth affine group scheme of finite type over a field. By [22, Section 4(d)], any linear algebraic group is a closed subgroup of GL_n for some n. Over a field of characteristic zero, the smoothness condition is automatic (see [22, Theorem 3.23]). Given a linear algebraic group G over a field F and a field extension K/F, we let G_K denote the base change $G \times_{\mathrm{Spec}(F)} \mathrm{Spec}(K)$, and we let

$K[G] = K[G_K]$ denote the coordinate ring of G_K.

2.2 Differential algebra

2.2.1 Differential rings

A *derivation* on a ring R is an additive map $\partial : R \to R$ that satisfies the Leibniz rule $\partial(rs) = \partial(r)s + r\partial(s)$ for all $r, s \in R$. A *differential ring* is a ring equipped with a derivation. A *differential field* is a differential ring that is a field. The *constant ring* of R is the subring

$$C_R = \{r \in R \mid \partial(r) = 0\}$$

of R. It is easy to see that the constant ring of a differential field is a field. We call C_R the *constant field* of R if C_R is a field.

Example 2.2.1. The field $\mathbb{C}(x)$ of complex rational functions and the field $\mathbb{C}((x))$ of Laurent series in x are both differential fields with the derivation d/dx. Their constant fields equal $\mathbb{C}$.

When the context is clear, derivations are often denoted by the symbol ∂. We often refer to a differential ring (R, ∂) as R. We write $r^{(n)} = (\partial \circ \cdots \circ \partial)(r)$ for the n-fold application of the derivation ∂ on an element $r \in R$. We will not use the notation r' to mean $\partial(r)$.

Let (R, ∂_R) and (S, ∂_S) be differential rings. A *homomorphism* of differential rings from R to S is a ring homomorphism $\varphi : R \to S$ satisfying $\partial_S \circ \varphi = \varphi \circ \partial_R$.

Example 2.2.2. Let $(F, \partial) = (\mathbb{C}(x), d/dx)$. Let $S = F[y, 1/y]$ be a differential ring extension of F with derivation determined by $\partial(y) = y$. Since the element $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ in $\mathbb{C}((x))$ also satisfies $\partial(e^x) = e^x$, we can define an injective differential ring homomorphism $S \to \mathbb{C}((x))$ which takes y to e^x. By identifying S with its image in $\mathbb{C}((x))$, we may think of y as e^x.

Let R be a differential ring. An ideal (resp. prime ideal) I of R is a *differential ideal* (resp. *differential prime ideal*) if $\partial(I) \subseteq I$. Given a multiplicative subset S of R, the derivation on $S^{-1}R$ defined by

$$\partial(r/s) = (\partial(r)s - r\partial(s))/s^2$$

for all $r \in R$ and $s \in S$ is well-defined and makes the localization map

$$R \to S^{-1}R : r \mapsto r/1$$

a differential ring homomorphism. In particular, if R is an integral domain, the derivation ∂_R extends to a derivation on the field of fractions $\mathrm{Frac}(R)$.

One similarly defines notions such as *differential algebras*, *differential coalgebras*, and *differential Hopf algebras* to be algebras, coalgebras, and Hopf algebras with compatible derivations. We leave the definitions to the reader.

2.2.2 Differential polynomials and differential transcendence degree

Let I be a set and let $x_i^{(j)}$ be indeterminates for $i \in I$ and $j \in \mathbb{N}$. A *differential polynomial ring* over R in the indeterminates x_i for $i \in I$ is the differential R-algebra

$$R\{x_i\}_{i\in I} = R[x_i^{(j)}]_{i\in I, j\in\mathbb{N}}$$

equipped with a derivation ∂ determined by $\partial x_i^{(j)} = x_i^{(j+1)}$ for all $i \in I$ and $j \in \mathbb{N}$. Given a differential R-algebra S and a subset $\{s_i\}_{i\in I}$ of S, the image of the differential ring homomorphism

$$\varphi : R\{x_i\}_{i\in I} \to S : x_i \mapsto s_i$$

is denoted by $R\{s_i\}_{i\in I}$. If φ is injective, we say that $\{s_i\}_{i\in I}$ is *differentially algebraically independent* over R. Otherwise $\{s_i\}_{i\in I}$ is said to be *differentially algebraically dependent* over R. Finally, if F is a differential field, we let $F\langle s_i\rangle_{i\in I}$ denote the field of fractions of $F\{s_i\}_{i\in I}$.

Let K/F be a differential field extension. A *differential transcendence basis* of K/F is a differentially algebraically independent subset of K over F that is maximal with respect to inclusion. By [17, Chapter 2], any two differential transcendence bases of K/F have the same cardinality. Thus we define the *differential transcendence degree* of K/F, denoted by $\operatorname{trdeg}_F^\partial K$, to be the cardinality of any given differential transcendence basis of K/F.

Here is a standard proposition on the differential transcendence degree of a residue field.

Proposition 2.2.3. *Let R be a differential F-algebra that is an integral domain, and let $\mathfrak{p}$ be a differential prime ideal of R. Then*

$$\operatorname{trdeg}_F^{\partial} \kappa(\mathfrak{p}) \leq \operatorname{trdeg}_F^{\partial} \operatorname{Frac}(R)$$

where $\kappa(\mathfrak{p})$ is the residue field of R at $\mathfrak{p}$.

Proof. Any differential transcendence basis of $\kappa(\mathfrak{p})$ over F lifts to a differential transcendence basis of $\operatorname{Frac}(R)$ over F. $\square$

2.3 Picard-Vessiot theory

In this section, F is a differential field of characteristic zero, and its constant field C is algebraically closed and properly contained in F.

2.3.1 Differential modules

There are two other equivalent formulations to homogeneous linear differential equations. Let the derivation ∂ on F act on F^n by componentwise differentiation. A *matrix differential equation* of order n over F is an equation of the form $\partial(Y) = AY$ with $A \in M_n(F)$ and $Y = (Y_1, ..., Y_n)^t$ a column vector of indeterminates. If y_0 is a solution to a homogeneous linear differential equation

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0,$$

then $Y = (y_0, y_0^{(1)}, ..., y_0^{(n-1)})^t$ is a solution to the associated $n \times n$ matrix differential equation

$$\partial(Y) = A_p Y \qquad \text{where} \qquad A_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}. \tag{2.3.1}$$

Thus matrix differential equations provide one alternative formulation for homogeneous linear differential equations.

Here is the second formulation. Let R be a differential ring. We define $R[\partial]$ to be the free R-module on the basis $\{\partial^n\}_{n\in\mathbb{N}}$ where $\partial^0 = 1$. It becomes a noncommutative ring under the multiplication $\circ$ specified by $\partial^n \circ \partial^m = \partial^{n+m}$ and $\partial \circ r = r^{(1)} + r \circ \partial$ for $r \in R$. The elements of $R[\partial]$ are called *differential operators* over R. If

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

is a homogeneous linear differential equation of order n over R, then

$$p = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0$$

is the corresponding differential operator of order n in $R[\partial]$. We define a *differential R-module* to be a left $R[\partial]$-module. A *homomorphism* of differential R-modules over R is a homomorphism of left $R[\partial]$-modules. The *rank* of a differential R-module that is free as an R-module is just the rank of the underlying R-module.

If a differential R-module is generated by an element as a left $R[\partial]$-module, that element is called a *cyclic vector*.

Remark 2.3.1. Let $n \geq 0$. Over F, the connection between homogeneous linear differential equations (or differential operators), matrix differential equations, and differential modules are given below.

1. A differential equation $p(y) = 0$ of order n over F determines the differential F-module $F[\partial]/(F[\partial] \circ p)$ of rank n.

2. Conversely let M be a differential F-module of rank n. By [30, Proposition 2.9], M has a cyclic vector m. The homomorphism $F[\partial] \to M$ of differential modules taking p to $p(m)$ is then surjective, and its kernel is a left ideal of $F[\partial]$, which by [30, Corollary 2.3] is principal. This gives an isomorphism

$$F[\partial]/(F[\partial] \circ p) \cong M$$

for some differential operator $p \in F[\partial]$ with leading coefficient 1. We call p the *minimal differential operator* for m.

3. Moreover, $n \times n$ matrix differential equations over F correspond to differential F-modules of rank n. See [30, Section 1.2].

To better understand the above correspondence, we need to first make our definition of a gauge transformation more precise. We recall the following.

Proposition 2.3.2. *Let $p(y)$ be a homogeneous linear differential equation of order n over F. Then the set of solutions of $p(y) = 0$ in F is a C-vector space of dimension at most n.*

Proof. See [30, Lemma 1.10]. □

A homogeneous linear differential equation $p(y) = 0$ of order n over F is said to have a *full set of solutions* in F if the set of solutions to $p(y) = 0$ is a C-vector space of dimension n. In Subsection 2.3.2, we will see that given a homogeneous linear differential equation over F, there always exists a differential field extension K/F that contains a full set of solutions for the equation.

A differential F-module is *trivial* if it contains a F-vector space basis of elements with derivation zero. Any two trivial differential modules of the same vector space dimension are isomorphic. A homogeneous differential equation $p(y) = 0$ has a full set of solutions over F if and only if its corresponding differential module is trivial.

Let $p(y) = 0$ and $q(z) = 0$ be homogeneous linear differential equations of order n over F. We say that $q(z)$ is a *gauge transformation* of $p(y)$ if the following holds: for any differential field extension K/F containing full sets of solutions Y for $p(y) = 0$ and Z for $q(z) = 0$ in K, there exists a differential operator over F that maps Y bijectively onto Z.

Let $\mathbf{DiffEq}_n(F)$ be the set of homogeneous linear differential equations of order n over F up to gauge transformations and let $\mathbf{Diff}_n(F)$ be the set of isomorphism classes of differential F-modules of rank n.

Proposition 2.3.3. *Let $n \geq 1$. There is a bijection* $\mathbf{DiffEq}_n(F) \cong \mathbf{Diff}_n(F)$.

Proof. Let

$$\mathcal{F} : \mathbf{DiffEq}_n(F) \to \mathbf{Diff}_n(F) : p(y) \mapsto F[\partial]/(F[\partial] \circ p).$$

We check that $\mathcal{F}$ is well-defined. Let $q(z)$ be a gauge transformation of $p(y)$. Let K be a differential field over F that contains full sets of solutions Y for $p(y) = 0$ and Z for $q(z) = 0$, and let f be a differential operator that maps Y bijectively onto Z. Consider the maps

$$\varphi\colon F[\partial] \to F[\partial] : u \mapsto u \circ f \quad , \qquad \mathrm{ev}\colon F[\partial] \to \prod_Z K : v \mapsto (v(y))_{y \in Y}.$$

Any $v \in \ker(\mathrm{ev})$ is zero on the n-dimensional C-vector space Y and so has order $\geq n$ by Proposition 2.3.2. Since $p \in \ker(\mathrm{ev})$ has order n and since $\ker(\mathrm{ev})$ is a principal left ideal by [30, Corollary 2.3], we have $\ker(\mathrm{ev}) = F[\partial] \circ p$. A similar argument which additionally uses the surjectivity of $f : Y \to Z$ shows that $\ker(\mathrm{ev} \circ \varphi) = F[\partial] \circ q$. Therefore we have an injective homomorphism of differential modules

$$F[\partial]/(F[\partial] \circ q) \to F[\partial]/(F[\partial] \circ p).$$

By dimension count, this is an isomorphism.

Conversely let

$$\mathcal{G} : \mathbf{Diff}_n(F) \to \mathbf{DiffEq}_n(F)$$

take a differential module M to $p(y)$ where p is the minimal differential operator for a choice of cyclic vector of M.

We check that $\mathcal{G}$ is well-defined. By (2) of Remark 2.3.1, it suffices to show that given an isomorphism

$$\varphi : F[\partial]/(F[\partial] \circ q) \to F[\partial]/(F[\partial] \circ p)$$

and a differential field K/F containing full sets of solutions Y and Z to $p(y) = 0$ and $q(z) = 0$, there is a differential operator that maps Y bijectively onto Z. Choose differential operators f and g that represent $\varphi(1)$ and $\varphi^{-1}(1)$, respectively. Let $y_0 \in Y$ and consider the evaluation map

$$\mathrm{ev}_{y_0} : F[\partial]/(F[\partial] \circ p) \to K : v \mapsto v(y_0).$$

Under $\mathrm{ev}_{y_0} \circ \varphi$, the equality $0 = q$ becomes $0 = (q \circ f)(y_0)$ and so $f(y_0) \in Z$. Therefore f maps Y to Z and similarly g maps Z to Y. One easily sees both are inverse maps on Y and Z.

One easily checks that $\mathcal{F}$ and $\mathcal{G}$ are inverse maps. $\square$

Let M and N be differential modules over R. We define a derivation on the tensor product $M \otimes_R N$ by

$$\partial(m \otimes n) = \partial(m) \otimes n + m \otimes \partial(n)$$

for all $m \in M$ and $n \in N$, and extend linearly. One similarly defines the tensor product of two differential algebras.

Example 2.3.4. Let D be a C-algebra. We regard D as a differential ring with the trivial derivation $\partial = 0$. We view $D \otimes_C F$ as a differential ring. We will take this viewpoint for linear algebraic groups G defined over C, viewing $C[G]$ as a C-algebra with trivial derivation.

2.3.2 Main statements

In Galois theory, we start with a polynomial over a field and construct a splitting field and Galois group for the polynomial. Similarly in Picard-Vessiot theory, we start with a homogeneous linear differential equation over a differential field and construct its Picard-Vessiot ring extension and differential Galois group. We summarize this theory following the presentation in [7] and [30].

Recall that if R is a differential ring and $y_1, ..., y_n \in R$ are elements, the *Wronskian determinant* of $y_1, ..., y_n$ is the element

$$\mathrm{wr}(y_1, ..., y_n) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

in R. Let

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y \tag{2.3.2}$$

be a homogeneous linear differential equation over a differential field F. A differential algebra R/F is a *Picard-Vessiot (ring) extension* associated to (2.3.2) if there exist elements $y_1, ..., y_n \in R$ satisfying $p(y_1) = \cdots = p(y_n) = 0$ such that the following conditions hold.

1. The element $\mathrm{wr}(y_1, ..., y_n)$ is invertible in R, and R is generated as a differential ring by $y_1, ..., y_n$ and the multiplicative inverse of the Wronskian determinant $\mathrm{wr}(y_1, ..., y_n)$ over F, i.e.,

$$R = F\{y_1, ..., y_n, \mathrm{wr}(y_1, ..., y_n)^{-1}\}.$$

2. The ring of constants of R is C, i.e., $C_R = C$.

3. The ring R has no nontrivial proper differential ideal.

We call $\mathrm{Frac}(R)$ over F a *Picard-Vessiot (field) extension* associated to (2.3.2). When the context is clear, we often refer to Picard-Vessiot ring extensions and Picard-Vessiot field extensions as simply Picard-Vessiot extensions.

Remark 2.3.5.

1. Since C is an algebraically closed field, [30, Proposition 1.20] guarantees that a Picard-Vessiot extension exists for (2.3.2) and is unique up to (differential) isomorphism.

2. We can similarly define a Picard-Vessiot extension associated to a matrix differential equation. See [30, Definition 1.15].

Given a Picard-Vessiot ring extension R/F associated to (2.3.2), we define its *differential Galois group* $\mathrm{Gal}^\partial(R/F)$ to be the group of automorphisms of the differential ring R fixing F pointwise. Letting $K = \mathrm{Frac}(R)$, one similarly defines $\mathrm{Gal}^\partial(K/F)$ to be the group of automorphisms of the differential field K fixing F

pointwise. It can be shown that $\mathrm{Gal}^{\partial}(K/F) = \mathrm{Gal}^{\partial}(R/F)$. By [30, Theorem 1.27], this group is represented by the C-points of an closed subgroup of the general linear group GL_n over C. We also say that R/F (resp. K/F) is a *G-Picard-Vessiot extension* if G is an linear algebraic group over C such that $G(C)$ is isomorphic to $\mathrm{Gal}^{\partial}(R/F)$ as abstract groups.

There is a Galois correspondence for Picard-Vessiot field extensions.

Proposition 2.3.6. *Let K/F be a Picard-Vessiot field extension with differential Galois group G. Consider the set $\mathcal{S}$ of closed subgroups of G and the set $\mathcal{L}$ of differential subfields E of K containing F. Then there is a bijection $\mathcal{S} \to \mathcal{L}$ given by sending H in $\mathcal{S}$ to the fixed differential subfield $K^{H(C)}$. The inverse map is given by sending E in $\mathcal{L}$ to the closed subgroup H of G which satisfies $H(C) = \mathrm{Gal}^{\partial}(K/E)$.*

Proof. See [30, Proposition 1.34]. □

Finally there is an analogue of the separable closure of a field. We define the *Picard-Vessiot closure* F^{PV} of F to be the direct limit of Picard-Vessiot field extensions K over F, filtered by inclusion. Note that F^{PV} exists and is unique up to (differential) isomorphism; see [19, Section 3]. Unlike separable closures, there can be nontrivial Picard-Vessiot extensions over F^{PV}; see [30, Exercise 1.47(2)].

2.3.3 Generic Picard-Vessiot extensions

Let F be a differential field and let $K = F\langle y_1, ..., y_n\rangle$ where $y_1, ..., y_n$ are indeterminates over F. Let

$$\partial(Y) = A_{y_1,\dots,y_n}Y, \qquad A_{y_1,\dots,y_n} \in M_n(K) \tag{2.3.3}$$

be a matrix differential equation over K. Suppose that (2.3.3) determines a Picard-Vessiot extension L/K with differential Galois group G. Then L/K is said to be a *generic G-Picard-Vessiot extension* if for every G-Picard-Vessiot extension L'/K', there exist elements $a_1, ..., a_n \in K'$ such that $A_{a_1,\dots,a_n}$ is well-defined and L'/K' is the Picard-Vessiot extension for the differential equation $\partial(Y) = A_{a_1,\dots,a_n}Y$. We call such an L'/K' a *specialization* of L/K.

Corollary 6.3.2.1 will show that any generic G-Picard-Vessiot extension requires at least n parameters if G is either GL_n or $\mathbb{G}_m^n$.

Remark 2.3.7. There are several related definitions of generic Picard-Vessiot extensions. For instance, a *descent-generic* G-Picard-Vessiot extension is a generic Picard-Vessiot extension that additionally specialize to H-Picard-Vessiot extensions for any closed subgroup H of G. See [11, Section 4] for the precise definition of descent-generic extensions. Our definition is basically the one given in [11], and is less restrictive than the definition in [12]. In particular, the generic extensions constructed in [11] and [12, Section 6] are generic in our sense.

2.3.4 Differential Hopf-Galois extensions

We can characterize Picard-Vessiot extensions in another way, with definitions analogous to those in [24, Section 0].

Let S be a differential coalgebra over R with comultiplication Δ and counit ϵ. A *differential S-comodule* over R is a differential R-module M together with a differential R-linear map $\rho : M \to M \otimes_R S$ satisfying $(1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho$ and $(1 \otimes \epsilon) \circ \rho = 1$.

Let H be a differential Hopf algebra over R with comultiplication Δ_H, counit ϵ_H, and antipode σ_H. Suppose that S is a differential algebra over R equipped with a map $\Delta_S : S \to S \otimes_R H$ such that S is a differential H-comodule via the coaction map Δ_S. If M is both a differential S-module and a differential H-comodule with a differential R-linear map $\Delta_M : M \to M \otimes_R H$ satisfying $\Delta_M(ms) = \Delta_M(m)\Delta_S(s)$ for all $m \in M$ and all $s \in S$, we say that M is a *differential (H, S)-Hopf module* over R. The *H-coinvariants* of M is the differential R-submodule

$$M^{\mathrm{co}\, H} = \{m \in M \mid \Delta_M(m) = m \otimes 1\}$$

of M. If $S = R$ and $\Delta_S(s) = s \otimes 1$ for all $s \in R$, we simply call M a *differential H-Hopf module* over R.

A *differential H-Hopf-Galois extension* is a faithfully flat differential ring extension S/R such that S is a differential H-Hopf module over R, and such that the map

$$\begin{aligned} \mathrm{can}_S : S \otimes_R S &\to S \otimes_R H \\ x \otimes y &\mapsto (x \otimes 1)\Delta_S(y) \end{aligned} \tag{2.3.4}$$

is a (differential) isomorphism.

Remark 2.3.8.

1. Let R/F be a Picard-Vessiot ring extension with differential Galois group G. By [1, Proposition 1.12(a)], R/F is a differential $F[G]$-Hopf-Galois extension.

2. Let S/R be a differential H-Hopf-Galois extension. By [15, Chapter III, Proposition 1.1.1], the faithful flatness of S/R, gives an exact sequence:

$$0 \longrightarrow R \longrightarrow S \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} S \otimes_R S.$$

   Combined with (2.3.4), we get $R = S^{\mathrm{co}\, H}$.

2.4 Affine differential schemes

We can also formulate Picard-Vessiot extensions geometrically in the language of differential schemes. The advantage of this approach is that differential schemes have a topology called the Kolchin topology. This topology is used to define versal differential torsors in Chapter 5. For simplicity, we restrict ourselves to affine differential schemes. Our definition is similar to the one given in [18], but is not equivalent to it. In this section, F is a differential field of characteristic zero, its constant field C is algebraically closed and properly contained in F, and G is a linear algebraic group over C.

Let R be a differential ring. We define the *spectrum* of R to be the set

$$X = \mathrm{Diffspec}(R)$$

of differential prime ideals of R. For a subset S of X, we let $Z(S) = \{\mathfrak{p} \in X \mid S \subset \mathfrak{p}\}$. The $Z(S)$ define the closed sets of a topology on X, called the *Kolchin topology* on X. For $f \in R$, let $D(f) = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$. The collection $\{D(f)\}_{f \in R}$ defines a basis of open sets for the Kolchin topology on X. Such a pair (X, R) is called an *affine differential scheme*. We often write X to mean the pair (X, R). We call R the *coordinate ring* of X. If R is a differential algebra over over F, we denote the coordinate ring R by $F[X]$. If R is an integral domain, we let $F(X) = \mathrm{Frac}(F[X])$.

Remark 2.4.1. From the definition, it is clear that $\mathrm{Diffspec}(R)$ in the Kolchin topology has the subspace topology of $\mathrm{Spec}(R)$ in the Zariski topology. In particular, if the differential ring R has trivial derivation, then $\mathrm{Diffspec}(R)$ and $\mathrm{Spec}(R)$ coincide as topological spaces.

Let $X = \mathrm{Diffspec}\, R$ and $Y = \mathrm{Diffspec}\, S$. A differential ring homomorphism $\varphi : R \to S$ induces a map of differential schemes $\varphi^{\#} : Y \to X$ which takes a differential prime ideal $\mathfrak{p}$ of S to the differential prime ideal $\varphi^{-1}(\mathfrak{p})$ of R. This map $\varphi^{\#}$ is continuous (in the Kolchin topology). A *morphism* of affine differential schemes from (Y, S) to (X, R) is simply a pair $(\varphi^{\#}, \varphi)$. We often write $\varphi^{\#}$ to mean this pair. Together the affine differential schemes and their morphisms form a category that is equivalent to the category of differential rings.

Remark 2.4.2 (Open differential subscheme). Let $X = \text{Diffspec}(R)$ be a affine differential scheme and let $f \in R^{\times}$. As in algebraic geometry, the inclusion $i : R \to R\{1/f\}$ gives the morphism $i^{\#} : \text{Diffspec}(R\{1/f\}) \to X$ whose image is the open subset $D(f)$. We may therefore view $D(f)$ as the affine differential scheme $\text{Diffspec}(R\{1/f\})$.

A point of an affine differential scheme X is *generic* if its closure (in the Kolchin topology) is the entire topological space X. If R is an integral domain, then the point corresponding to the differential prime ideal (0) is the generic point of $\text{Diffspec}\, R$.

Fiber products exist in the category of affine differential schemes. Given $X = \text{Diffspec}\, R$, $Y = \text{Diffspec}\, S$, and $Z = \text{Diffspec}\, T$ and differential ring homomorphisms $T \to R$ and $T \to S$, the fiber product is given by $X \times_Z Y = \text{Diffspec}(R \otimes_T S)$.

Let $X = \text{Diffspec}\, R$ and $Y = \text{Diffspec}\, S$ be affine differential schemes. A *differential Y-point* (or *differential S-point*) of X is a morphism $Y \to X$. The set of differential Y-points of X is denoted by $X(Y)$ or $X(S)$. If K is a differential field, we may identify each differential K-point of X with its image in X and view $X(K)$ as a subset of X. This makes $X(K)$ a topological space under the subspace topology, and we again call this the *Kolchin topology* on $X(K)$.

Example 2.4.3 (Differential affine n-space). Given a differential ring R, the affine differential scheme $\text{Diffspec}\, R\{x_1, ..., x_n\}$ is called the *differential affine n-space* over R and denoted by $\mathbb{A}^n_R$. For a differential ring extension S/R, $\mathbb{A}^n_R(S)$ is a S-module, and there is an S-module isomorphism

$$\mathbb{A}_R^n(S) \cong S^n. \tag{2.4.1}$$

We can give S^n the *Kolchin topology* where the closed sets are now given by

$$Z(T) = \{x \in S^n \mid f(x) = 0 \text{ for all } f \in T\}$$

over the subsets $T \subseteq S\{x_1, ..., x_n\}$. This makes (2.4.1) a homeomorphism.

Here is a basic fact about the differential affine n-space that we will need in the proof of Proposition 5.0.2.

Proposition 2.4.4. *Let K/F be a differential field extension and let $f \in K\{x_1, ..., x_n\}$ satisfy $f(x) = 0$ for all $x \in F^n$. Then $f = 0$.*

Proof. See [27, page 35]. □

Corollary 2.4.4.1. *Let K/F be a differential field extension. Then F^n is a Kolchin dense subset of K^n.*

Proof. Let $Z(S)$ be the Kolchin closure of F^n in K^n. Any $f \in S$ satisfies $f(x) = 0$ for all $x \in F^n$, so by the previous proposition, $f = 0$. Thus $S = \{0\}$ and the closure of F^n is K^n. □

If $y \in X(Y)$, we may regard y as a affine differential scheme and so set $X_y = X \times_Y y$. We call $X_y \to y$ the *fiber* of $X \to Y$ at y. The fiber at a generic point is called the *generic fiber*.

A morphism of affine differential schemes is *dominant* if the map at the level of topological spaces has dense image. If R and S are differential rings which are integral domains, then $\mathrm{Diffspec}(S) \to \mathrm{Diffspec}(R)$ is a dominant morphism if and only if the differential ring homomorphism $R \to S$ is injective.

Let X and Y be affine differential schemes. Two morphisms $f : U \to Y$ and $g : V \to Y$ defined on dense open affine differential subschemes U and V of X are *equivalent* if $f|_W = g|_W$ on some dense open affine differential subscheme $W \subseteq U \cap V$ in X. A *rational morphism* $f : X \dashrightarrow Y$ is an equivalence class of such morphisms. A rational morphism is *dominant* if it has a representative that is a dominant morphism.

Example 2.4.5. Let $X = \mathrm{Diffspec}(R)$ and $Y = \mathrm{Diffspec}(S)$ be affine differential schemes and let $g \in S$. An injective differential ring homomorphism $\varphi : R \to S\{1/g\}$ determines a dominant morphism $\varphi^{\#} : \mathrm{Diffspec}(S\{1/g\}) \to X$. If R is an integral domain, $\mathrm{Diffspec}(S\{1/g\})$ is a dense open affine differential subscheme of X, and so $\varphi^{\#}$ defines a dominant rational morphism $\varphi^{\#} : Y \dashrightarrow X$.

Finally let G be a linear algebraic group over C. Recall that we view $C[G]$ as a differential ring with the trivial derivation. Then $G = \mathrm{Spec}(C[G]) = \mathrm{Diffspec}(C[G])$ as topological spaces, and we will let G denote $\mathrm{Diffspec}(C[G])$ as well. For a differential scheme Y over C, we let $G_Y = G \times_{\mathrm{Diffspec}(C)} Y$. For a differential field K/C, we let $K[G] = K[G_K]$.

2.4.1 Differential torsors

Let $X = \mathrm{Diffspec}(R)$ and $Y = \mathrm{Diffspec}(S)$ be affine differential schemes over F. Let $X \to Y$ be a morphism of affine differential schemes and let $X \times_Y G_Y \to X$ be a G_Y-action on X in the category of affine differential schemes over Y. We say that $X \to Y$ is a *differential G_Y-torsor* if the map

$$X \times_Y G_Y \to X \times_Y X : (x, g) \mapsto (x, xg) \tag{2.4.2}$$

is an isomorphism in the category of affine differential schemes. We will also call a differential G_Y-torsor a *differential G-torsor* over Y.

Given a differential G_Y-torsor $X \to Y$ and $y \in Y(K)$ for some differential field K/F, the fiber $X_y \to y$ is a differential G_y-torsor.

Remark 2.4.6. By [1, Proposition 1.12(a)], a G-Picard Vessiot ring extension R/F yields a differential G_F-torsor $\mathrm{Diffspec}(R) \to \mathrm{Diffspec}(F)$. Conversely, a differential torsor over a differential field is induced from a Picard-Vessiot extension in the following sense. Let $X \to \mathrm{Diffspec}(F)$ be a differential G_F-torsor. By [1, Proposition 1.12(b), Proposition 1.15], there exists a closed subgroup H of G over C and a differential H_F-torsor Y such that $F[Y]/F$ is a H-Picard-Vessiot ring extension, and such that X is isomorphic to the differential G_F-torsor $\mathrm{Ind}_{H_F}^{G_F}(Y)$. Here, $\mathrm{Ind}_{H_F}^{G_F}(Y)$ is the differential torsor whose coordinate ring is $(F[Y] \otimes_C C[G])^H$. (See [1, Remark 1.11].) In particular, Y has a differential $F(Y)$-rational point and so X has a $F(Y)$-rational point too.

Remark 2.4.7.

1. A differential G-torsor Diffspec(R) over F corresponds to a differential $F[G]$-Hopf-Galois extension R. A differential torsor over a differential ring need not correspond to a differential Hopf-Galois extension simply because we did not include a faithful flatness condition when defining differential torsors.

2. Note that [1] is written in the language of affine schemes whose coordinate rings are equipped with derivations. Since the part of their work that we cite can be written entirely using differential rings and differential Hopf-Galois extensions, we may apply their results to ours.

3. Our definition of differential torsor lacks a local trivialization property with respect to some Grothendieck topology. See [2, page 302] for a discussion on why [2, Definition 4.5] requires torsors to be defined with respect to the flat topology.

   In our setting, [1] already shows that differential torsors are induced from Picard-Vessiot extensions, which guarantees that our differential torsors over F will contain a differential K-point where K/F is a Picard-Vessiot field extension.

The following two examples give the structure of differential G-torsors over a differential field K when G is either $\mathbb{G}_m^n$ or $\mathbb{G}_a^n$.

Example 2.4.8. Let $G = \mathbb{G}_m^n$ and let $X = \text{Diffspec}(R)$ be a differential G-torsor over a differential field K. The multiplicative form of Hilbert's Theorem 90 implies that the underlying G-torsor structure of X is trivial. Thus

$$R = K[G] = K[y_1, ..., y_n, 1/(y_1 \cdots y_n)]$$

in the indeterminates y_i. The $C[G]$-coaction on R given by

$$\rho : R \to R \otimes_C C[G] : y_i \mapsto y_i \otimes y_i.$$

Since the derivation ∂ on R commutes with ρ and since the derivation on $C[G]$ is trivial, we have $\partial(y_i) = \partial(y_i) \otimes y_i$. Therefore the element $a_i = \partial(y_i)/y_i$ lies in the coinvariant ring $R^{\mathrm{co}\, C[G]} = K$.

Thus the differential G-torsor $\mathrm{Diffspec}(R)$ over K can be given explicitly as $R = F[y_1, ..., y_n, 1/(y_1 \cdots y_n)]$ with $\partial(y_i)/y_i \in K$.

Example 2.4.9. Let $G = \mathbb{G}_a^n$ and let $X = \mathrm{Diffspec}(R)$ be a differential G-torsor over a differential field K. The additive form of Hilbert's Theorem 90 implies that the underlying G-torsor structure of X is trivial. Thus

$$R = K[G] = K[y_1, ..., y_n]$$

in the indeterminates y_i. The $C[G]$-coaction on R given by

$$\rho : R \to R \otimes_C C[G] : y_i \mapsto y_i \otimes 1 + 1 \otimes y_i$$

As before, we find that the elements $\partial(y_i)$ lies in the co-invariant ring $R^{\mathrm{co}\, C[G]} = K$.

Thus the differential G-torsor $\mathrm{Diffspec}(R)$ over K can be given explicitly as $R = F[y_1, ..., y_n]$ with $\partial(y_i) \in K$.

2.5 Notations and conventions

For the rest of the thesis and unless explicitly stated otherwise, F is a differential field of characteristic zero, and its constant field C is algebraically closed and properly contained in F. Here are a few more notations and conventions that will be in force:

- $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of nonnegative integers.
- C_r is the cyclic group of order r.
- $\mathrm{trdeg}_F^{\partial} K$ is the differential transcendence degree of K/F (see Section 2.2).
- Fields_F is the category of fields over F.
- $\mathrm{Fields}_F^{\partial}$ is the category consisting of differential fields K/F.
- $\mathrm{Fields}_{F,C}^{\partial}$ is the category consisting of differential fields K/F such that the constant field of K coincides with C.
- F^{PV} is the PV closure of F (see Subsection 2.3.2).
- $\mathbf{DiffEq}_n(F)$ is the set of homogeneous linear differential equations of order n over F up to gauge transformations (see Subsection 2.3.1).
- $\mathbf{Diff}_n(F)$ is the set of isomorphism classes of differential F-modules of rank n (see Subsection 2.3.1).
- $\mathrm{Diffspec}(R)$ is the (differential) spectrum of a differential ring R (see Section 2.4).

- $X(S)$ is the set of differential S-points of an affine differential scheme X (see Section 2.4).

- $\mathbb{A}_R^n$ is the differential affine n-space over a differential ring R (see Section 2.4).

- $G\text{-}\mathbf{tors}^{\partial}(F)$ is the set of isomorphism classes of differential G-torsors over F (see Section 3.1).

- $\mathrm{ed}_F^{\partial}(a)$ and $\mathrm{ed}_F^{\partial}(\mathcal{F})$ are the differential essential dimension of an object a and the differential essential dimension of a functor $\mathcal{F}$ over F, respectively (see Section 3.2).

- $\mathbf{\Phi}\text{-Struc}_R$ is the category of $\mathbf{\Phi}$-structures over R (see Section 4.1).

- $\mathbf{\Phi}\text{-Struc}_S^{H'}$ is the category of H'-equivariant $\mathbf{\Phi}$-structures over S (see Section 4.1).

- $\mathrm{TF}(S/R, M)$ is the set of (differential) isomorphism classes of (S/R)-twisted forms of M (see Section 4.2).

- $H_{\partial}^1(\Gamma, G)$ is a cohomology set for the Picard-Vessiot theory (see Section 4.3).

Chapter 3

Differential Essential Dimension

In this chapter we formalize the notions of the classes of objects we consider and their differential essential dimensions. After this is done, we compute the differential essential dimension for classes of objects whose structure are known well.

3.1 Classes of objects as functors

We first organize the classes of objects we encountered by the differential field over which each object is defined. Formally, this entails writing down a functor from $\text{Fields}^{\partial}_{F,C}$ to Sets, as the following examples illustrate.

Example 3.1.1.

1. Let $n \geq 1$. Given a differential field K, recall that we defined $\mathbf{DiffEq}_n(K)$ to be the set of equivalence classes of homogeneous linear differential equations up to gauge transformations over K. Given a homomorphism of differential

fields $i : K \to L$, we define

$$\mathbf{DiffEq}_n(i) : \mathbf{DiffEq}_n(K) \to \mathbf{DiffEq}_n(L)$$

to be the map that takes the homogeneous linear differential equation $p(y) = 0$ to $i_*(p)(y) = 0$ where i is applied to the coefficients of p. This defines a functor $\mathbf{DiffEq}_n : \text{Fields}^\partial_{F,C} \to \text{Sets}$.

2. Let $n \geq 1$. Given a differential field K, recall that we defined $\mathbf{Diff}_n(K)$ to be the set of (differential) isomorphism classes of differential modules of dimension n over K. Given a homomorphism of differential fields $i : K \to L$, we define $\mathbf{Diff}_n(i)$ by extension of scalars, taking a differential module M over K to the differential module $M \otimes_K L$ over L. This defines a functor $\mathbf{Diff}_n : \text{Fields}^\partial_{F,C} \to \text{Sets}$.

3. Let G a linear algebraic group over C. Given a differential field K, we define $G\text{-}\mathbf{tors}^\partial(K)$ to be the set of (differential) isomorphism classes of differential G_K-torsors over K. Given an inclusion of differential fields $i : K \to L$, we define $G\text{-}\mathbf{tors}(i)$ by extension of scalars. This defines a functor $G\text{-}\mathbf{tors}^\partial : \text{Fields}^\partial_F \to \text{Sets}$.

4. Let R a differential algebra over F. Then $X = \text{Diffspec}(R)$ can be viewed as a functor

$$X : \text{Fields}^\partial_F \to \text{Sets}$$

taking a differential field K to $X(K)$. We often refer to this functor as the *functor of points* of X.

These classes of objects are related to each other. For instance, Proposition 2.3.3 gives a natural isomorphism $\mathbf{DiffEq}_n \cong \mathbf{Diff}_n$. This means that the notion of homogeneous linear differential equations is practically interchangeable with the notion of differential modules. In Corollary 4.5.2.1, we will prove the natural isomorphism $\mathbf{Diff}_n \cong \mathrm{GL}_n\textbf{-tors}^\partial$.

3.2 Differential essential dimension

To formalize the notion of counting parameters, consider the following motivating example. Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a matrix consisting of differential indeterminates a_{ij} over F and let $L = F\langle a_{ij}\rangle_{1\leq i,j\leq n}$. Consider the differential module M over L corresponding to the $n \times n$ matrix differential equation $Y' = AY$. This description of M requires the n^2 many parameters a_{ij}. By Remark 2, M is described by a homogeneous linear differential equation

$$p(y) = y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0y = 0$$

of order n over L. This description of M requires n parameters.

We can formalize this reduction of parameters by letting M' be the differential module corresponding to $p(y) = 0$ over

$$K = F\langle b_0, ..., b_{n-1}\rangle.$$

Then $M' \otimes_K L \cong M$. In other words, if $i : K \to L$ is the inclusion map, then

$$\mathbf{Diff}_n(i) : \mathbf{Diff}_n(K) \to \mathbf{Diff}_n(L)$$

takes M' to M. Because of this, we say that M "descends" to M' over K, or that M is "defined" over K. Thus the process of reducing from n^2 to n parameters amounts to realizing that while M is originally defined over L with $\operatorname{trdeg}_F^\partial L = n^2$, it is also defined over K with $\operatorname{trdeg}_F^\partial K = n$.

Definition 3.2.1. Let $\mathcal{F} : \mathcal{C} \to \text{Sets}$ be a functor. Let L be an object of $\mathcal{C}$ and let $a \in \mathcal{F}(L)$. We say that a is *defined* over K if there exists a homomorphism of differential fields $i : K \to L$ for which a lies in the image of $\mathcal{F}(i)$.

If a is defined over K, we also say that a *descends* to K, or that K is a *differential field of definition* of a. If $a = \mathcal{F}(i)(b)$ we say that a *descends* to b or that a is *defined* as b over K.

Definition 3.2.2. Let $\mathcal{F} : \mathcal{C} \to \text{Sets}$ be a functor. Let L be an object of $\mathcal{C}$ and let $a \in \mathcal{F}(L)$. We define the *differential essential dimension* of an element a to be the minimum

$$\operatorname{ed}_F^\partial(a) := \min \operatorname{trdeg}_F^\partial(K)$$

over all all differential field of definition K of a. We define the *differential essential dimension* of the functor $\mathcal{F}$ to be the number

$$\operatorname{ed}_F^\partial(\mathcal{F}) := \sup \operatorname{ed}_F^\partial(a)$$

over all $a \in \mathcal{F}(L)$ and L in $\mathcal{C}$.

Remark 3.2.3.

1. Let K be in $\mathrm{Fields}^{\partial}_{F,C}$ and let R/K be a G-Picard-Vessiot ring extension. If the differential G-torsor $\mathrm{Diffspec}(R) \to \mathrm{Diffspec}(K)$ is defined as $\mathrm{Diffspec}(R_0) \to \mathrm{Diffspec}(K_0)$, then R_0/K_0 is again a G-Picard-Vessiot extension by [1, Proposition 1.12(b)]. Similarly, if a G-Picard-Vessiot field extension L/K is defined over K_0, then the associated Picard-Vessiot ring extension is also defined over K_0. Because of this, we may sometimes write the differential essential dimension of a Picard-Vessiot ring extension or Picard-Vessiot field extension instead of the associated differential torsor.

2. Suppose that a G-Picard-Vessiot extension L/K descends to a G–Picard-Vessiot extension L_0/K_0 over a differential subfield K_0 of K. This means that there exists a G-equivariant (differential) isomorphism $L_0 \otimes_{K_0} K \cong L$ of differential K-algebras. Let N be a closed normal subgroup of G. Taking N-invariants of both sides of the isomorphism gives $L_0^N \otimes_{K_0} K \cong L^N$. Therefore the (G/N)-Picard-Vessiot extension L^N/K descends to L_0^N/K_0.

The differential essential dimension of a functor measures the size of the corresponding class of objects. If the functor is represented by a differential scheme, then the size should be the "dimension" of the differential scheme (see [2, Proposition 1.17]):

Proposition 3.2.4. *Let* $X = \mathrm{Diffspec}(R)$ *be an affine differential scheme over a* *differential field F. Viewing X as a functor* $\mathrm{Fields}_F^\partial \to \mathrm{Sets}$*, we have*

$$\mathrm{ed}_F^\partial(X) = \sup \mathrm{trdeg}_F^\partial \kappa(\mathfrak{p})$$

where the supremum is taken over all differential prime ideals $\mathfrak{p}$ of R. In particular, if R is an integral domain, then $\mathrm{ed}_F^\partial(X) = \mathrm{trdeg}_F^\partial F(X)$*.*

Proof. Let K be in $\mathrm{Fields}_F^\partial$ and let $x \in X(K)$. Then x corresponds to a differential homomorphism $x : R \to K$ which factors through $\kappa(\mathfrak{p})$ where $\mathfrak{p} = \ker(x)$. In other words, the smallest differential field of definition of x is its residue field. Therefore $\mathrm{ed}_F^\partial(x) = \mathrm{trdeg}_F^\partial \kappa(\mathfrak{p})$ and so $\mathrm{ed}_F^\partial(X) = \sup \mathrm{trdeg}_F^\partial \kappa(\mathfrak{p})$.

If R is an integral domain, by Proposition 2.2.3, we have $\kappa((0)) = F(X)$ has $\mathrm{trdeg}_F^\partial F(X) \geq \mathrm{trdeg}_F^\partial \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in X$. □

Basic properties of the usual essential dimension (see [2, Section 1]) adapt easily to the differential setting, with the exception of [2, Proposition 1.13] which requires the existence of composita in $\mathrm{Fields}_{F,C}^\partial$. One basic property is the following (see [2, Lemma 1.9]):

Proposition 3.2.5. *Let $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ be a natural transformation from $\mathcal{F} : \mathrm{Fields}_{F,C}^\partial \to \mathrm{Sets}$ and $\mathcal{G} : \mathrm{Fields}_{F,C}^\partial \to \mathrm{Sets}$. If η is surjective (on objects), then $\mathrm{ed}_F^\partial(\mathcal{F}) \geq \mathrm{ed}_F^\partial(\mathcal{G})$.*

Proof. Let $\mathrm{ed}_F^\partial(\mathcal{F}) = n$, possibly infinite. Let K in $\mathcal{C}$ and $b \in \mathcal{G}(K)$ be arbitrary. By surjectivity of η, there exists $a \in \mathcal{F}(K)$ such that $\eta_K(a) = b$. Since $\mathrm{ed}_F^\partial(\mathcal{F}) = n$, there exist E in $\mathcal{C}$ with $d(E) \leq n$ and a morphism $i : E \to K$ that induces a morphism $\mathcal{F}(i) : \mathcal{F}(E) \to \mathcal{F}(K)$ taking some object $a' \in \mathcal{F}(E)$ to a. By the commutativity of the diagram

$$\begin{tikzcd} \mathcal{F}(K) \arrow[r, "\eta_K"] & \mathcal{G}(K) \\ \mathcal{F}(E) \arrow[u, "\mathcal{F}(i)"] \arrow[r, "\eta_E"'] & \mathcal{G}(E), \arrow[u, "\mathcal{G}(i)"'] \end{tikzcd}$$

$\mathcal{G}(i)$ takes $b' := \eta_E(a')$ to b and thus $\mathrm{ed}_F^{\partial}(b) \le d(b') \le n$. Since $b \in \mathcal{G}(K)$ and K in $\mathcal{C}$ are arbitrary, we have $\mathrm{ed}_F^{\partial}(\mathcal{G}) \le n$, as desired. □

Remark 3.2.6. Here is a more general definition of essential dimension, given in two parts.

1. Consider a category $\mathcal{C}$ together with a function $d : \mathrm{Ob}(\mathcal{C}) \to \mathbb{N} \cup \{\infty\}$. The pair $(\mathcal{C}, d)$ is said to be a *field-like category* if for every morphism $K \to L$ in $\mathcal{C}$, we have $d(K) \le d(L)$. Given a field K, the pairs $(\mathrm{Fields}_K, \mathrm{trdeg}_K)$, $(\mathrm{Fields}_F^{\partial}, \mathrm{trdeg}_F^{\partial})$, and $(\mathrm{Fields}_{F,C}^{\partial}, \mathrm{trdeg}_F^{\partial})$ are field-like categories.

2. Given a field-like category $(\mathcal{C}, d)$ and a functor $\mathcal{F} : \mathcal{C} \to \mathrm{Sets}$, we define the *essential dimension* of an element $a \in \mathcal{F}(L)$ to be the number

$$\mathrm{ed}^{\mathcal{C}}(a) = \min d(K)$$

where the minimum ranges over all morphisms $i : K \to L$ in $\mathcal{C}$ such that a lies in the image of $\mathcal{F}(i) : \mathcal{F}(K) \to \mathcal{F}(L)$. The *essential dimension* of the functor $\mathcal{F}$ is defined to be the number

$$\mathrm{ed}^{\mathcal{C}}(\mathcal{F}) = \sup \mathrm{ed}^{\mathcal{C}}(a)$$

where the supremum ranges over all objects L of $\mathcal{C}$ and $a \in \mathcal{F}(L)$.

When $(\mathcal{C}, d) = (\mathrm{Fields}_K, \mathrm{trdeg}_K)$ and $(\mathrm{Fields}^{\partial}_{F,C}, \mathrm{trdeg}^{\partial}_F)$, this recovers the definition of essential dimension in [2] and the differential essential dimension. One may also consider a certain localization of the category $(\mathrm{Fields}^{\partial}_{F,C}, \mathrm{trdeg}^{\partial}_F)$ to obtain the definition of the essential p-dimension (see e.g. [21, Section 3b] for a definition).

Remark 3.2.7. As we briefly mentioned, essential dimension was first introduced by J. Buhler and Z. Reichstein for G-torsors when G is a finite constant group scheme in [4]. It was generalized to G-torsors for algebraic groups G by Z. Reichstein in [25]; to functors by A. Merkurjev in [2]; and to stacks by P. Brosnan, Z. Reichstein, and A. Vistoli in [3]. For results on essential dimension, one may consult for instance the survey articles [26] and [21]. There is also a related notion to essential dimension, called resolvent degree. See [8].

3.3 Differential essential dimension of $\mathbb{G}_m^n$ and $\mathbb{G}_a^n$

The differential essential dimension of G-$\mathbf{tors}^{\partial}$ can be determined if we understand the structure of the differential G-torsors well.

Proposition 3.3.1. *For $n \geq 0$, we have* $\mathrm{ed}^{\partial}_F(\mathbb{G}_m^n\text{-}\mathbf{tors}^{\partial}) = n$.

Proof. By Example 2.4.8, any differential $\mathbb{G}_m^n$-torsor X over a differential field K has coordinate ring $K[X]$ of the form $K[y_1, ..., y_n, 1/(y_1 \cdots y_n)]$ where the elements

$a_i = \partial y_i/y_i$ lie in K. Therefore X is defined over $F\langle a_1, ..., a_n\rangle$. This gives the upper bound $\mathrm{ed}_F^\partial(\mathbb{G}_m^n\textbf{-tors}^\partial) \le n$.

For the lower bound, let $n \ge 0$, $y_1, ..., y_n$ be differential indeterminates over F, $L = F\langle y_1, ..., y_n\rangle$, and $K = F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle$. To prove the assertion, it suffices to show that the $\mathbb{G}_m^n$-Picard-Vessiot extension L/K satisfies $\mathrm{ed}_F^\partial(L/K) = n$.

For the sake of contradiction, suppose this assertion is false. Let n be the smallest number for which it fails. Since the case of $n = 0$ trivially holds, we have $n \ge 1$. The extension L/K is then induced by some extension L_0/K_0 for some differential subfield K_0 of K satisfying $\mathrm{trdeg}_F^\partial K_0 < n$.

For $i = 1, ..., n$, the $\mathbb{G}_m$-subextension $K\langle y_i\rangle/K$ of L/K is induced by some $\mathbb{G}_m$-subextension of L_0/K_0, which is necessarily of the form $K_0\langle z_i\rangle/K_0$ for some $z_i \in L_0$ satisfying $\partial z_i/z_i \in K_0$ by Example 2.4.8. Thus we can write L_0 as $F\langle z_1, ..., z_n\rangle$ and K_0 as $F\langle \partial z_1/z_1, ..., \partial z_n/z_n\rangle$.

$$
\begin{tikzcd}
F\langle y_1, ..., y_n\rangle = & L \arrow[d, dash] \arrow[dr, dash] & & \\
F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle = & K \arrow[dr, dash] & L_0 \arrow[d, dash] & = F\langle z_1, ..., z_n\rangle \\
& & K_0 & = F\langle \partial z_1/z_1, ..., \partial z_n/z_n\rangle
\end{tikzcd}
$$

Let $L' = F\langle y_1, ..., y_{n-1}\rangle$, $K' = F\langle \partial y_1/y_1, ..., \partial y_{n-1}/y_{n-1}\rangle$, $L_0' = F\langle z_1, ..., z_{n-1}\rangle$, $K_0' = F\langle \partial z_1/z_1, ..., \partial z_{n-1}/z_{n-1}\rangle$, and $L'' = L'\langle z_n\rangle$. Note that the extension L_0'/K_0'

induces the $\mathbb{G}_m^{n-1}$-extension L'/K'.

$$\begin{tikzcd}[arrows=dash]
L'\langle z_n\rangle = & L'' \arrow[d] & & \\
F\langle y_1, ..., y_{n-1}\rangle = & L' \arrow[d] \arrow[dr] & & \\
F\langle \partial y_1/y_1, ..., \partial y_{n-1}/y_{n-1}\rangle = & K' \arrow[dr] & L'_0 \arrow[d] & = F\langle z_1, ..., z_{n-1}\rangle \\
 & & K'_0 & = F\langle \partial z_1/z_1, ..., \partial z_{n-1}/z_{n-1}\rangle
\end{tikzcd}$$

Since n is the minimal value for which the proposition fails, we have $\operatorname{trdeg}_F^{\partial} L'_0 = \operatorname{trdeg}_F^{\partial} K'_0 = n-1$. Noting that $L_0 = L'_0\langle z_n\rangle$, the inequalities

$$n-1 \geq \operatorname{trdeg}_F^{\partial} K_0 = \operatorname{trdeg}_F^{\partial} L_0 \geq \operatorname{trdeg}_F^{\partial} L'_0 = n-1$$

now force z_n to be differentially algebraic over L'_0 and hence over L'. Therefore $\operatorname{trdeg}_F^{\partial} L'' = \operatorname{trdeg}_F^{\partial} L' = n-1$.

To finish proving the proposition, it suffices to show that y_n is differentially algebraic over L'', since we would then get $n = \operatorname{trdeg}_F^{\partial} L = \operatorname{trdeg}_F^{\partial} L'' = n-1$, resulting in the desired contradiction. Recall that the extension $K\langle y_n\rangle/K$ is induced by the extension $K_0\langle z_n\rangle/K_0$. By the Kolchin-Ostrowski theorem [16, pages 1155-1156], there exist nonzero integers r, s and a nonzero element $d \in K$ such that $y_n^r z_n^s = d$ holds. Since $K = F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle = K'\langle \partial y_n/y_n\rangle$, we may view d as $f(\partial y_n/y_n)$ where f is a differential rational function in one variable T over K'. Therefore $y_n^r z_n^s = f(\partial y_n/y_n)$. Since d is nonzero, $f(T)$ is nonzero. Furthermore, the differential rational function $g(T) := f(\partial T/T) - T^r z_n^s$ over L'' is nonzero as it is not fixed by $T \mapsto 2T$. Therefore y_n satisfies the nonzero differential rational function $g(T)$ over L'' and so is differentially algebraic over L''. $\square$

A similar argument proves $\mathrm{ed}_F^\partial(\mathbb{G}_a^n\textbf{-tors}^\partial) = n$ and $\mathrm{ed}_F^\partial(C_r^n\textbf{-tors}^\partial) = n$ where C_r denotes the cyclic group of order r. The proof for $\mathbb{G}_a^n$ uses the other part of the Kolchin-Ostrowski theorem, while the proof for C_r^n uses Kummer theory. We give the proof for $\mathbb{G}_a^n$ below.

Proposition 3.3.2. *For $n \geq 0$, we have $\mathrm{ed}_F^\partial(\mathbb{G}_a^n\textbf{-tors}^\partial) = n$.*

Proof. By Example 2.4.8, any differential $\mathbb{G}_a^n$-torsor X over a differential field K has coordinate ring $K[X]$ of the form $K[y_1, ..., y_n]$ where the elements $a_i = \partial y_i$ lie in K. Therefore X is defined over $F\langle a_1, ..., a_n\rangle$. This gives the upper bound $\mathrm{ed}_F^\partial(\mathbb{G}_a^n\textbf{-tors}^\partial) \leq n$.

For the lower bound, let $y_1, ..., y_n$ be differential indeterminates over F, $L = F\langle y_1, ..., y_n\rangle$, and $K = F\langle \partial y_1, ..., \partial y_n\rangle$. To prove the assertion, it suffices to show that the $\mathbb{G}_a^n$-Picard-Vessiot extension L/K satisfies $\mathrm{ed}_F^\partial(L/K) = n$.

For the sake of contradiction, suppose the assertion is false and let n be the smallest number for which it fails. Since the case of $n = 0$ trivially holds, we have $n \geq 1$. The extension L/K is then induced by some extension L_0/K_0 for some differential subfield K_0 of K satisfying $\mathrm{trdeg}_F^\partial K_0 < n$.

For $i = 1, ..., n$, the $\mathbb{G}_a$-subextension $K\langle y_i\rangle/K$ of L/K is induced by some $\mathbb{G}_a$-subextension of L_0/K_0, which is necessarily of the form $K_0\langle z_i\rangle/K_0$ for some $z_i \in L_0$ satisfying $\partial z_i \in K_0$ by Example 2.4.9. Thus we can write L_0 as $F\langle z_1, ..., z_n\rangle$ and

K_0 as $F\langle \partial z_1, ..., \partial z_n \rangle$.

$$\begin{tikzcd}[arrows=dash]
F\langle y_1, ..., y_n \rangle = & L \arrow[d] \arrow[dr] & & \\
F\langle \partial y_1, ..., \partial y_n \rangle = & K \arrow[dr] & L_0 \arrow[d] & = F\langle z_1, ..., z_n \rangle \\
& & K_0 & = F\langle \partial z_1, ..., \partial z_n \rangle
\end{tikzcd}$$

Let $L' = F\langle y_1, ..., y_{n-1} \rangle$, $K' = F\langle \partial y_1, ..., \partial y_{n-1} \rangle$, $L_0' = F\langle z_1, ..., z_{n-1} \rangle$, $K_0' = F\langle \partial z_1, ..., \partial z_{n-1} \rangle$, and $L'' = L'\langle z_n \rangle$. Note that the extension L_0'/K_0' induces the $\mathbb{G}_a^{n-1}$-extension L'/K'.

$$\begin{tikzcd}[arrows=dash]
L'[z_n] = & L'' \arrow[d] & & \\
F\langle y_1, ..., y_{n-1} \rangle = & L' \arrow[d] \arrow[dr] & & \\
F\langle \partial y_1, ..., \partial y_{n-1} \rangle = & K' \arrow[dr] & L_0' \arrow[d] & = F\langle z_1, ..., z_{n-1} \rangle \\
& & K_0' & = F\langle \partial z_1, ..., \partial z_{n-1} \rangle
\end{tikzcd}$$

Since n is the minimal value for which the proposition fails, we have $\operatorname{trdeg}_F^\partial L_0' = \operatorname{trdeg}_F^\partial K_0' = n - 1$. Noting that $L_0 = L_0'\langle z_n \rangle$, the inequalities

$$n - 1 \geq \operatorname{trdeg}_F^\partial K_0 = \operatorname{trdeg}_F^\partial L_0 \geq \operatorname{trdeg}_F^\partial L_0' = n - 1$$

now force z_n to be differentially algebraic over L_0' hence over L'. Therefore $\operatorname{trdeg}_F^\partial L'' = \operatorname{trdeg}_F^\partial L' = n - 1$.

To finish proving the proposition, it suffices to show that y_n is differentially algebraic over L'', since we would then get $n = \operatorname{trdeg}_F^\partial L = \operatorname{trdeg}_F^\partial L'' = n - 1$, resulting in the desired contradiction. Recall that the extension $K\langle y_n \rangle / K$ is induced by the extension $K_0\langle z_n\rangle/K_0$. By the Kolchin-Ostrowski theorem [16, p1155-1156], there exist $r, s \in C^\times$ and $d \in K^\times$ such that $ry_n + sz_n = d$. Since $K = F\langle \partial y_1, ..., \partial y_n\rangle = K'\langle \partial y_n\rangle$, we may view d as $f(\partial y_n)$ where f is a differential rational function in one variable T over K'. Therefore $ry_n + sz_n = f(\partial y_n)$. Since d is nonzero, $f(T)$ is nonzero. Furthermore, the differential rational function $g(T) := f(\partial T) - rT - sz_n$ over L'' is nonzero as it is not fixed by $T \mapsto T - 1$. Therefore y_n satisfies the nonzero differential rational function $g(T)$ over L'' and so is differentially algebraic over L''. $\square$

Chapter 4

Twisted Forms and Cohomology

Certain classes of objects in differential algebra can be interpreted as twisted forms of a particular object. In this section, we will define the notions of such twisted forms as well as a cohomology set relevant to differential algebra. Finally we will show that the cohomology set is in bijection with the set of such twisted forms. This is analogous to the situation in algebra.

In this chapter, R denotes a differential F-algebra, and all unadorned tensor products are taken over R, i.e., $\otimes = \otimes_R$.

4.1 $\mathbf{\Phi}$-structures and descent along differential Hopf-Galois extensions

We loosely follow the formalism in [23, Section 1.3]. Let I be a set. We define a *tensor-type* to be a subset of $\mathbb{N}^4$ indexed by I. Let $\mathbf{\Phi} = \{(r_{1i}, r_{2i}, r_{3i}, r_{4i})\}_{i\in I}$ be a tensor-type, M a differential module over R, and H a differential Hopf algebra over R. We say that a tuple $(M, \{\Phi_i\}_{i\in I})$ is a $\mathbf{\Phi}$-*structure* over R if the Φ_i are differential R-module homomorphisms of the form

$$\Phi_i : M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \to M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}}.$$

Example 4.1.1.

1. A differential module M over R is a $\mathbf{\Phi}$-structure by taking $\mathbf{\Phi}$ to be the empty set.

2. A differential algebra A over R with multiplication map $m : A^{\otimes 2} \to A$ is a $\mathbf{\Phi}$-structure with $\mathbf{\Phi} = \{(2, 0, 1, 0)\}$.

3. A differential H-Hopf-Galois extension S/R consists of a multiplication map $m : S^{\otimes 2} \to S$ and a coaction map $\Delta_S : S \to S \otimes H$. Therefore S/R defines a $\mathbf{\Phi}$-structure with $\mathbf{\Phi} = \{(2, 0, 1, 0), (1, 0, 1, 1)\}$.

We will view the above classes of objects (differential modules, differential algebras, differential Hopf-Galois extensions) as $\mathbf{\Phi}$-structures with the $\mathbf{\Phi}$ given in the examples.

Remark 4.1.2. Note that the definition of a $\mathbf{\Phi}$-structure relies upon an implicit choice of differential Hopf algebra and indexing set, henceforth denoted by H and I, respectively.

Let $(M, \{\Phi_i\}_{i\in I})$ and $(N, \{\Psi_i\}_{i\in I})$ be $\mathbf{\Phi}$-structures over R. A *morphism of* $\mathbf{\Phi}$-*structures* over R is a differential R-module homomorphism $\varphi : M \to N$ such that $\varphi \circ \Phi_i = \Psi_i$ holds for all $i \in I$. Together the $\mathbf{\Phi}$-structures over R and the morphisms of $\mathbf{\Phi}$-structures over R form a category which we denote by $\mathbf{\Phi}\,\text{-Struc}_R$.

Let $\varphi : R \to S$ be a differential ring homomorphism. Then a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R induces the $\mathbf{\Phi}$-structure $(M \otimes S, \{\Phi_i \otimes 1_S\}_{i\in I})$ over S by extension of scalars. This map gives a functor

$$\mathbf{\Phi}\,\text{-Struc}_R \to \mathbf{\Phi}\,\text{-Struc}_S\,. \tag{4.1.1}$$

We may also write down an equivariant version of such structures. Let H' be a differential Hopf algebra over R. An *H'-equivariant $\mathbf{\Phi}$-structure* is a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R such that M is a differential H'-comodule with coaction map $\Delta_M : M \to M \otimes H'$, and such that Δ_M commutes with the Φ_i, i.e., the following diagram commutes for all $i \in I$:

$$\begin{tikzcd}
M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \arrow[r, "\Phi_i"] \arrow[d, "\Delta_M^{\otimes r_{1i}} \otimes 1"'] & M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}} \arrow[d, "\Delta_M^{\otimes r_{3i}} \otimes 1"] \\
M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \otimes H' \arrow[r, "\Phi_i \otimes 1_{H'}"'] & M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}} \otimes H'.
\end{tikzcd}$$

A *morphism of H'-equivariant $\mathbf{\Phi}$-structures* $(M, \{\Phi_i\}_{i\in I})$ and $(N, \{\Psi_i\}_{i\in I})$ is a morphism $\varphi : M \to N$ of $\mathbf{\Phi}$-structures that commutes with the H'-coactions on M and

N, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\Delta_M"'] & N \arrow[d, "\Delta_N"] \\ M \otimes H' \arrow[r, "\varphi\otimes 1_{H'}"'] & N \otimes H'. \end{tikzcd}$$

Together the H'-equivariant $\mathbf{\Phi}$-structures and their morphisms form a category which we denote by $\mathbf{\Phi}\text{-Struc}_R^{H'}$.

Let S/R be a differential H'-Hopf-Galois extension. By extension of scalars, a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R extends to $(M \otimes S, \{\Phi_i \otimes 1_S\}_{i\in I})$ over S. Since all the $\Phi_i \otimes 1_S$ commute with Δ_S, the new structure is H'-equivariant over S. Therefore (4.1.1) restricts to a functor

$$\mathbf{\Phi}\text{-Struc}_R \to \mathbf{\Phi}\text{-Struc}_S^{H'}. \tag{4.1.2}$$

Conversely, given a H'-equivariant $\mathbf{\Phi}$-structure $(N, \{\Phi_i\})$ over S, we may consider its coinvariant module $N^{\mathrm{co}\,H'} = \{n \in N \mid \Delta_N(n) = n \otimes 1\}$. Since each Φ_i commutes with Δ_N, the $\mathbf{\Phi}$-structure on N restricts to one on $N^{\mathrm{co}\,H'}$. This gives a functor

$$\mathbf{\Phi}\text{-Struc}_S^{H'} \to \mathbf{\Phi}\text{-Struc}_R.$$

We now show that the two functors just considered define an equivalence of categories. We follow the proof of [28, Theorem 7.3.1 (1) $\Rightarrow$ (2)].

Theorem 4.1.3 (Descent along differential Hopf-Galois extensions)**.** *Let H' be a differential Hopf algebra over R, S/R a differential H'-Hopf-Galois extension, and*

$\mathbf{\Phi}$ a tensor-type. Suppose that S/R is faithfully flat. Then extension of scalars defines an equivalence of categories

$$\mathbf{\Phi}\text{-Struc}_R \to \mathbf{\Phi}\text{-Struc}_S^{H'} .$$

The pseudo-inverse is given by taking an object N to the coinvariant module $N^{\mathrm{co}\,H'}$.

Proof. Naturality is clear. It suffices now to check that for all M in $\mathbf{\Phi}$-Struc$_R$ and N in $\mathbf{\Phi}$-Struc$_S^{H'}$, the two maps

$$\mu_N : N^{\mathrm{co}\,H'} \otimes_R S \to N : n \otimes s \mapsto ns$$

and

$$\iota_M : M \to (M \otimes_R S)^{\mathrm{co}\,H'} : m \mapsto m \otimes 1$$

are bijections.

Since S/R is a differential H'-Hopf-Galois extension, we have the (differential) isomorphism

$$\begin{aligned} \mathrm{can}_S : S \otimes_R S &\to S \otimes_R H' \\ x \otimes y &\mapsto (x \otimes 1)\Delta_S(y) \end{aligned}$$

Therefore for any P in $\mathrm{Mod}_S^{\partial}$, the map can_P given by

$$\begin{aligned} \mathrm{can}_P : P \otimes_R S &\to P \otimes_R H' \\ p \otimes s &\mapsto (p \otimes 1)\Delta_S(s) \end{aligned}$$

is also a (differential) isomorphism.

Consider the following two commutative diagrams.

$$\begin{tikzcd}
N^{\operatorname{co} H'} \otimes_R S \arrow[r] \arrow[d, "\mu_N"'] & N \otimes_R S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mathrm{can}_N"] & (N \otimes_S H') \otimes_R S \arrow[d, "\mathrm{can}_{N\otimes H'}"] \\
N \arrow[r, "\Delta_N"'] & N \otimes_S H' \arrow[r, shift left, "\Delta_{N\otimes H'}"] \arrow[r, shift right, "1_N \otimes \Delta_{H'}"'] & N \otimes_S H' \otimes_S H'
\end{tikzcd} \tag{4.1.3}$$

$$\begin{tikzcd}
(M \otimes_R S)^{\operatorname{co} H'} \arrow[r, "\iota"] & M \otimes_R S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & M \otimes_R S \otimes_R H' \\
M \arrow[r, "\iota"'] \arrow[u, "\iota_M"] & M \otimes_R S \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & M \otimes_R S \otimes_R S \arrow[u, "\mathrm{can}_{M\otimes S}"']
\end{tikzcd} \tag{4.1.4}$$

The top row of (4.1.4) is exact by the definition of $\operatorname{co} H'$. The top row of (4.1.3) is exact by the definition of $\operatorname{co} H'$ and the flatness of S/R. The bottom row of (4.1.3) is exact by coassociativity of Δ_N. Since S/R is faithfully flat, by [15, Chapter III, Proposition 1.1.1], the bottom row of (4.1.4) is exact. The vertical arrows can_N, $\mathrm{can}_{N\otimes H'}$, and $\mathrm{can}_{M\otimes S}$ are (differential) isomorphisms by our above discussion. Thus μ_N and ι_M are (differential) isomorphisms. $\square$

4.2 Twisted forms

Let $\mathbf{\Phi}$ be a tensor-type. Let M and N be $\mathbf{\Phi}$-structures over R, and let S/R be a differential ring extension. We say that M is a (S/R)-*twisted form* of N if there exists a (differential) isomorphism $\varphi : M \otimes_R S \cong N \otimes_R S$ of $\mathbf{\Phi}$-structures over S. We let $\mathrm{TF}(S/R, M)$ denote the set of (differential) isomorphism classes of (S/R)-twisted forms of M.

Example 4.2.1.

1. Any differential module over F is a (F^{PV}/F)-twisted form of the trivial differential module M of the same rank, giving a bijection between $\mathbf{Diff}_n(F)$ and $\mathrm{TF}(F^{\mathrm{PV}}/F, M)$.

2. Let H be a differential Hopf algebra over R. Let S/R be a differential H-Hopf-Galois extension. Then S is a (S/R)-twisted form of H via the (differential) isomorphism can_S.

Differential torsors are another important example of twisted forms. Given a differential ring extension S/R, and a linear algebraic group G over C, we say that a differential G_R-torsor X is a (S/R)-*twisted form* of a differential G_R-torsor Y if X_S is differentially isomorphic to Y_S as differential G_S-torsors; equivalently, $R[X]$ is a (S/R)-twisted form of $R[Y]$ as differential $R[G]$-Hopf-Galois extensions. For a given differential G_R-torsor X, we let $\mathrm{TF}(S/R, X)$ denote the set of (differential) isomorphism classes of (S/R)-twisted forms of X.

Proposition 4.2.2. *Let G be a linear algebraic group over C. Then any differential G_F-torsor is a (F^{PV}/F)-twisted form of the trivial differential G_F-torsor G_F. In particular, we have a bijection G-$\mathbf{tors}^{\partial}(F) \cong \mathrm{TF}(F^{\mathrm{PV}}/F, G_F)$.*

Proof. Let $H = F[G]$ and let X be a differential G_F-torsor over F. By [1, Proposition 1.15], there exists a closed subgroup G' of G and a simple differential G'_F-torsor Y with $C_{\mathrm{Frac}(F[Y])} = C$ such that $X \cong \mathrm{Ind}_{G'_F}^{G_F}(Y)$ as differential torsors. By [1, Proposition 1.12(b)], $F[Y]/F$ is a G'_F-Picard-Vessiot extension. Therefore $F[Y]/F$ is a (F^{PV}/F)-twisted form of H as a differential $F[G']$-Hopf-Galois extension. By [1, Remark 1.11], $F[X] \cong (F[Y] \otimes_C C[G])^{G'}$ and so

$$\begin{aligned} F[X] \otimes_F F^{\mathrm{PV}} &\cong (F[Y] \otimes_F F^{\mathrm{PV}} \otimes_C C[G])^{G'} \\ &\cong (F[G'] \otimes_F F^{\mathrm{PV}} \otimes_C C[G])^{G'} \\ &\cong F^{\mathrm{PV}} \otimes_C C[G] \\ &\cong F[G] \otimes_F F^{\mathrm{PV}}. \end{aligned}$$

Therefore $F[X]$ is a (F^{PV}/F)-twisted forms of H.

The final assertion of the proposition is now clear. □

Remark 4.2.3. Twisted forms in our sense have already been considered for the case of differential central simple algebras in [13], for differential Lie algebras in [20].

4.3 Cohomology

We now define a cohomology set that we will use to classify the twisted forms of the previous section. Recall that to specify a morphism of varieties over an algebraically closed field C, it suffices to do so on the C-points of the varieties.

Definition 4.3.1. Let Γ and G be linear algebraic groups over C. Suppose that G is a group object in the category of affine varieties with Γ-action, i.e., there is a morphism of varieties $\Gamma \times G \to G$ defining a Γ-action on G that is compatible with the group structure on G. A *1-cocycle* is a morphism of varieties $a : \Gamma(C) \to G(C)$

such that the following condition holds: for any $\sigma \in \Gamma(C)$ we let $a_\sigma := a(\sigma)$ and require that $a_{\sigma\tau} = a_\sigma \cdot \sigma(a_\tau)$ holds for all $\sigma, \tau \in \Gamma(C)$. Two 1-cocycles a and b are *equivalent* if there exists $c \in G(C)$ such that $a_\sigma = c \cdot b_\sigma \cdot c^{-1}$. We define $H^1_\partial(\Gamma, G)$ to be the set of 1-cocycles $\Gamma \to G$ modulo equivalence.

Note that the cocycles we have just defined are morphisms of varieties and not merely maps of sets, unlike the case of (finite) Galois cohomology. Note also that the cohomology set is functorial in both Γ and G: homomorphisms of linear algebraic groups $\Gamma' \to \Gamma$ and $G \to G'$ over C induce maps $H^1_\partial(\Gamma', G) \to H^1_\partial(\Gamma, G)$ and $H^1_\partial(\Gamma, G) \to H^1_\partial(\Gamma, G')$.

4.4 Cohomology classifies twisted forms

We now discuss how to classify the twisted forms by the cohomology set we introduced in the last section. Consider the following setup.

Let Γ be a linear algebraic group over C with Hopf algebra H_0' and let S/R be a differential $(H_0' \otimes_C R)$-Hopf-Galois extension. Let $(M, \{\Phi_i\}_{i \in I})$ be a $\mathbf{\Phi}$-structure over R and set $M_S := M \otimes_R S$. Furthermore assume that the automorphism group of M is represented by a linear algebraic group G over C, i.e., there exists an isomorphism

$$\mathrm{Aut}(M_S \otimes_C D) \cong G(D) \tag{4.4.1}$$

that holds for every C-algebra D and is functorial in D. We will always identify the two groups in (4.4.1).

Remark 4.4.1. Note that (4.4.1) gives a sequence of isomorphisms

$$\begin{aligned}\operatorname{Aut}(M_S \otimes_C H_0'^{\otimes n}) &\cong G(H_0'^{\otimes n}) \\ &\cong \operatorname{Hom}_{C-\mathrm{Algs}}(C[G], H_0'^{\otimes n}) \cong \operatorname{Mor}_{C-\mathrm{Sch}}(\Gamma^n, G)\end{aligned} \tag{4.4.2}$$

which takes an element $f \in \operatorname{Aut}(M_S \otimes_C H_0'^{\otimes n})$ to the morphism $\Gamma^n \to G$ given on C-points by

$$\begin{aligned}\Gamma^n(C) &\to G(C) \\ (\sigma_1, ..., \sigma_n) &\mapsto (1_{M_S} \otimes \sigma_1 \otimes \cdots \sigma_n)_* f.\end{aligned} \tag{4.4.3}$$

Here $(1_{M_S} \otimes \sigma_1 \otimes \cdots \sigma_n)_* f$ is the morphism obtained by extension of scalars (see 2.1), such that the following diagram commutes:

$$\begin{tikzcd} M_S \otimes_C H_0'^{\otimes n} \arrow[r, "f"] \arrow[d, "1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n"'] & M_S \otimes_C H_0'^{\otimes n} \arrow[d, "1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n"] \\ M_S \arrow[r, "(1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n)_* f"'] & M_S. \end{tikzcd} \tag{4.4.4}$$

Remark 4.4.2. We can define an action of Γ on G in the following way. First we define an action of $\Gamma(C)$ on S by letting an element $\sigma \in \Gamma(C)$ act on S via the automorphism $\boldsymbol{\sigma} = (1_S \otimes \sigma) \circ \Delta_S$:

$$\boldsymbol{\sigma} : S \xrightarrow{\Delta_S} S \otimes_C H_0' \xrightarrow{1_S \otimes \sigma} S. \tag{4.4.5}$$

This action extends to an action of $\Gamma(C)$ on $M_S = M \otimes_R S$. The action of $\Gamma(C)$ on M_S further gives an action of Γ on G on the C-points by conjugation: for any $\sigma \in \Gamma(C)$ and $\varphi \in G(C)$, we define the action to be $\sigma(\varphi) := \boldsymbol{\sigma} \circ \varphi \circ \boldsymbol{\sigma}^{-1}$. We will consider G with this Γ-action when discussing the cohomology set $H^1_\partial(\Gamma, G)$.

Remark 4.4.3. In the case $n = 1$ in Remark 4.4.1, an element $a \in \mathrm{Aut}(M_S \otimes_C H_0')$ corresponds to an element of $\mathrm{Mor}_{C-\mathrm{Sch}}(\Gamma, G)$ which we again denote by a. If for each $\sigma \in \Gamma(C)$ we let $a_\sigma := a(\sigma)$, we have the equality

$$(1_{M_S} \otimes \sigma)_* a = a_\sigma.$$

The commutativity of the diagram

$$\begin{tikzcd} M_S \arrow[r, "\Delta_{M_S}"] \arrow[dr, "\boldsymbol{\sigma}"'] & M_S \otimes_C H_0' \arrow[r, "a"] \arrow[d, "1\otimes\sigma"] & M_S \otimes_C H_0' \arrow[d, "1\otimes\sigma"] \\ & M_S \arrow[r, "a_\sigma"'] & M_S \end{tikzcd}$$

further gives the equality

$$(1_{M_S} \otimes \sigma) \circ (a \circ \Delta_{M_S}) = a_\sigma \circ \boldsymbol{\sigma} \tag{4.4.6}$$

which we will later use in Lemma 4.4.7.

Construction 4.4.4. We define a map

$$\mathcal{F} : \mathrm{TF}(S/R, M) \to H^1_\partial(\Gamma, G)$$

as follows. Let (N, φ) be a twisted form of M. We define $\mathcal{F}(N, \varphi)$ to be the cocycle $a : \Gamma(C) \to G(C)$ which sends an element σ in $\Gamma(C)$ to the element

$$a_\sigma := \varphi \circ \boldsymbol{\sigma}(\varphi) = \varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1} \text{ in } G(C).$$

We must verify that $\mathcal{F}$ is well-defined. That a is a cocycle follows from the standard computation in $G(C)$:

$$\begin{aligned} a_\sigma \cdot \sigma(a_\tau) &= a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1} \\ &= (\varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1}) \circ \boldsymbol{\sigma} \circ (\varphi \circ \boldsymbol{\tau} \circ \varphi^{-1} \circ \boldsymbol{\tau}^{-1}) \circ \boldsymbol{\sigma}^{-1} \\ &= \varphi \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau}) \circ \varphi^{-1} \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau})^{-1} \\ &= a_{\sigma\tau} \end{aligned}$$

which holds for all $\sigma, \tau \in \Gamma(C)$.

Next let (N', ψ) be differentially isomorphic to (N, φ) as twisted forms of M and let $b = \mathcal{F}(N', \psi)$. Setting $c = \psi \circ \varphi^{-1}$, we have

$$\begin{aligned} c^{-1} \circ b_\sigma \circ \sigma(c) &= c^{-1} \circ b_\sigma \circ \boldsymbol{\sigma} \circ c \circ \boldsymbol{\sigma}^{-1} \\ &= (\varphi \circ \psi^{-1}) \circ (\psi \circ \boldsymbol{\sigma} \circ \psi^{-1} \circ \boldsymbol{\sigma}^{-1}) \circ \boldsymbol{\sigma} \circ (\psi \circ \varphi^{-1}) \circ \boldsymbol{\sigma}^{-1} \\ &= \varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1} \\ &= a_\sigma \end{aligned}$$

for all $\sigma \in \Gamma(C)$. Thus $\mathcal{F}$ takes equivalent twisted forms to equivalent cocycles. We conclude that $\mathcal{F}$ is well-defined.

Construction 4.4.5. We define a map

$$\mathcal{G} : H^1_\partial(\Gamma, G) \to \mathrm{TF}(S/R, M) \tag{4.4.7}$$

as follows. Given a cocycle a representing an element of $H^1_\partial(\Gamma, G)$, we define $\mathcal{G}(a)$ to be the differential R-module

$$N := \{ m \in M_S \mid (a_\sigma \circ \boldsymbol{\sigma})(m) = m \text{ for all } \sigma \in \Gamma(C) \} . \tag{4.4.8}$$

We will soon check that $\mathcal{G}$ is a well-defined map (in Lemma 4.4.7). Our proofs of Lemma 4.4.7 and Theorem 4.4.8 below follow that of [24, Theorem 2.6] where a cohomology set was introduced to classify Hopf-Galois extensions for noncommutative rings. The following lemma allows us to convert from their "cochains" which are maps $M_S \to M_S \otimes_C H_0'$ to our cochains which are morphisms $\Gamma \to G$.

Lemma 4.4.6. *Let M be a C-vector space, X an algebraic variety over C, and $f, g \in M \otimes_C C[X]$. If $(1_M \otimes \sigma)f = (1_M \otimes \sigma)g$ for all $\sigma \in X(C)$ then $f = g$ in $M \otimes_C C[X]$.*

Proof. Since X is an algebraic variety over an algebraically closed field C, equality of functions on $X(C)$ implies equality of elements in $C[X]$. This gives the case $M = C$.

For a general M, let $\{m_i\}_{i \in I}$ be a basis of M over C and write $f = \sum m_i \otimes f_i$ and $g = \sum m_i \otimes g_i$ for some $f_i, g_i \in C[X]$. For all $\sigma \in X(C)$, we have $(1_M \otimes \sigma)(f) = (1_M \otimes \sigma)(g)$ hence $\sum m_i \otimes \sigma(f_i) = \sum m_i \otimes \sigma(g_i)$. The linear independence of $\{m_i\}_{i \in I}$ over C and the previous paragraph now give $f_i = g_i$ for all $i \in I$. $\square$

Lemma 4.4.7. *The map $\mathcal{G}$ in Construction 4.4.5 is well-defined.*

Proof. Step 1. We first check that given a cocycle a, $N = \mathcal{G}(a)$ is a twisted form of M. First note that a satisfies the following properties:

(a) $(a_\sigma \circ \boldsymbol{\sigma})(ms) = (a_\sigma \circ \boldsymbol{\sigma})(m)\boldsymbol{\sigma}(s)$ for all $\sigma \in \Gamma(C)$, $m \in M_S$, and $s \in S$;

(b) $a_1 = 1_{M_S}$;

(c) $a_{\sigma\tau} = a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1}$ for all $\sigma, \tau \in \Gamma(C)$.

Here (c) is the cocycle condition for a, (b) follows from (c) by letting $\sigma = \tau = 1$ in $\Gamma(C)$, and (a) follows from the S-linearity of a_σ.

We claim that the composite map Δ' given by $a \circ \Delta_{M_S} : M_S \to M_S \otimes_C H_0'$ defines a coaction on M_S making M_S a differential (H_0', S)-Hopf module. In other words, we must verify the following properties:

(A) $\Delta'(ms) = \Delta'(m)\Delta(s)$ for all $m \in M_S, s \in S$;

(B) $(1_{M_S} \otimes \epsilon_{H_0'}) \circ \Delta' = 1_{M_S}$;

(C) $(\Delta' \otimes 1_{H_0'}) \circ \Delta' = (1_{M_S} \otimes \Delta_{H_0'}) \circ \Delta'$.

Since $\epsilon_{H_0'} : H_0' \to C$ corresponds to $1 \in \Gamma(C)$, (B) follows from (b) by (4.4.6).

To show (A) and (C), by Lemma 4.4.6, it suffices to show that the equalities obtained by applying $(1_{M_S} \otimes \sigma)$ to (A) and $(1_{M_S} \otimes \sigma \otimes \tau)$ to (C) hold for all $\sigma, \tau \in \Gamma(C)$. Applying $(1_{M_S} \otimes \sigma)$ to (A) and simplifying by (4.4.6) gives (a). Thus (A) holds. Similarly, applying $1 \otimes (\sigma \circ \tau)$ to the right side of (C) gives

$$(1_{M_S} \otimes \sigma \otimes \tau) \circ (1_{M_S} \otimes \Delta_{H_0'}) \circ \Delta' = (1_{M_S} \otimes (\sigma \circ \tau)) \circ \Delta' = a_{\sigma\tau} \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau}). \quad (4.4.9)$$

Applying $1_{M_S} \otimes (\sigma \circ \tau)$ to the left side of (C) gives

$$a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1} \quad (4.4.10)$$

since the following diagram commutes:

$$
\begin{tikzcd}
M_S \arrow[d, "\Delta'"'] \arrow[drr, "a_\tau \circ \boldsymbol{\tau}"] & & \\
M_S \otimes_C H_0' \arrow[rr, "1\otimes\tau"] \arrow[d, "\Delta'\otimes 1_{H_0'}"'] \arrow[dr, "(a_\sigma\circ\boldsymbol{\sigma})\otimes 1_{H_0'}"] & (I) & M_S \arrow[d, "a_\sigma\circ\boldsymbol{\sigma}"] \\
M_S \otimes_C H_0' \otimes_C H_0' \arrow[r, "1\otimes\sigma\otimes 1"'] & M_S \otimes_C H_0' \arrow[r, "1\otimes\tau"'] & M_S.
\end{tikzcd}
$$

Here, the region (I) commutes by the computation

$$
\begin{aligned}
((a_\sigma \circ \boldsymbol{\sigma}) \circ (1 \otimes \tau))(m \otimes h) &= ((a_\sigma \circ \boldsymbol{\sigma})(m \cdot \tau(h)) \\
&= ((a_\sigma \circ \boldsymbol{\sigma})(m)) \cdot \tau(h) \\
&= ((1 \otimes \tau)\sigma(a_\sigma \circ \boldsymbol{\sigma}))(m \otimes h).
\end{aligned}
$$

Equating (4.4.9) with (4.4.10) gives (c). Thus (C) holds. This concludes checking that Δ' defines a coaction on M_S making M_S a differential (H_0', S)-Hopf module.

We are almost done with Step 1. Theorem 4.1.3 implies that the coinvariant module $(M_S)^{\mathrm{co}\,\Delta'}$ is a twisted form of M over R. Therefore it suffices to show that N equals $(M_S)^{\mathrm{co}\,\Delta'}$. Any $m \in (M_S)^{\mathrm{co}\,\Delta'}$ satisfies $\Delta'(m) = m \otimes 1$. For any $\sigma \in \Gamma(C)$, applying $(1 \otimes \sigma)$ to $\Delta'(m) = m \otimes 1$ and simplifying by (4.4.6) gives $(a_\sigma \circ \sigma)(m) = m$. Thus $(M_S)^{\mathrm{co}\,\Delta'} \subseteq N$. Invoking Lemma 4.4.6 gives the reverse inclusion and so $(M_S)^{\mathrm{co}\,\Delta'} = N$.

Step 2. The map $\mathcal{G}$ takes equivalent cocycles to isomorphic twisted forms. If b is a cocycle equivalent to a, there exists $c \in G(C) = \mathrm{Aut}(M_S)$ such that $b = c \circ a \circ c^{-1}$.

Let $N_b = \mathcal{G}(b)$. The automorphism $c : M_S \to M_S$ restricts to an isomorphism $N_a \cong N_b$, as desired.

□

We now prove the main theorem of this section.

Theorem 4.4.8. *Consider the above setup. The maps $\mathcal{F}$ and $\mathcal{G}$ are inverses. Hence there is a bijection between the two sets* $\mathrm{TF}(S/R, M)$ *and* $H^1_\partial(\Gamma, G)$.

Proof. We first check $\mathcal{G} \circ \mathcal{F} = 1$. Let (N, φ) be a twisted form of M_S with associated cocycle $a = \mathcal{F}(N, \varphi)$. Set $P := \mathcal{G}(a)$. We want to show $P \cong N$. Clearly the isomorphism $\varphi : N \otimes S \to M_S$ has image in P, so φ restricts to $\varphi|_N : N \to P$. Consider the commutative diagram

$$\begin{tikzcd} N \otimes S \arrow[r, "\varphi|_N \otimes 1_S"] \arrow[dr, "\varphi"', "\cong"] & P \otimes S \arrow[d, "\mathrm{mult.}\cong"] \\ & M_S. \end{tikzcd}$$

By definition of twisted form, the vertical map is an isomorphism. Thus $\varphi|_N \otimes 1_S$ is an isomorphism. By faithful flatness of S/R, $\varphi|_N$ is an isomorphism.

We next check $\mathcal{F} \circ \mathcal{G} = 1$. Let a be a cocycle, $(N, \varphi) := \mathcal{G}(a)$, and $b := \mathcal{F}(N, \varphi)$. Given $\sigma \in \Gamma(C)$, consider the diagram

$$\begin{tikzcd} M_S \arrow[rrr, "b_\sigma \circ \boldsymbol{\sigma}"] \arrow[dd, equal] & & & M_S \arrow[dd, equal] \\ & N \arrow[r, "\boldsymbol{\sigma}"] \arrow[ul, "\varphi"] \arrow[dl, "\varphi"] & N \arrow[ur, "\varphi"'] \arrow[dr, "\varphi"] & \\ M_S \arrow[rrr, "a_\sigma \circ \boldsymbol{\sigma}"'] & & & M_S \end{tikzcd}$$

where the triangles trivially commute and the upper trapezoid commutes by definition of b_σ. The bottom trapezoid commutes by the following two computations. For all $n \in N$ and $s \in S$, we have

$$\begin{aligned} \varphi(\boldsymbol{\sigma}(n \otimes s)) &= \varphi((n \otimes 1)(1 \otimes \sigma(s))) \\ &= n\sigma(s) \end{aligned}$$

where the last equality uses the isomorphism $N \otimes S \cong M_S$ given by scalar multiplication. Also

$$\begin{aligned} (a_\sigma \circ \boldsymbol{\sigma} \circ \varphi)(n \otimes s) &= (a_\sigma \circ \boldsymbol{\sigma})(ns) \\ &= (a_\sigma \circ \boldsymbol{\sigma})(n)\sigma(s) \\ &= n\sigma(s) \end{aligned}$$

where the second equality uses property (a) in the proof of Lemma 4.4.7 and the third equality uses the definition of N. Therefore the diagram above is commutative so $b_\sigma \circ \boldsymbol{\sigma} = a_\sigma \circ \boldsymbol{\sigma}$ for all $\sigma \in \Gamma(C)$. By (4.4.6) and Lemma 4.4.6, we have $b = a$. $\square$

4.5 Absolute cohomology

In this section we provide a variant to Theorem 4.4.8 which is more convenient to use in practice. While Theorem 4.4.8 is stated for twisted forms over differential Hopf-Galois extensions, which includes Picard-Vessiot ring extensions, sometimes it is easier to work with twisted forms over Picard-Vessiot field extensions.

Proposition 4.5.1. *Let R/F be a Picard-Vessiot ring extension with differential Galois group Γ, $K = \mathrm{Frac}(R)$, $\mathbf{\Phi}$ a tensor-type, and M a $\mathbf{\Phi}$-structure over F. Suppose that the automorphism group of $M \otimes_F R$ is representable by a linear algebraic group G over C. Then the inclusion map $\mathrm{TF}(R/F, M) \to TF(K/F, M)$ is a bijection. In particular, there is a bijection of the two sets $\mathrm{TF}(K/F, M)$ and $H^1_\partial(\Gamma, G)$.*

Proof. Given a (K/F)-twisted form of M, we obtain a cocycle $\mathrm{Gal}^\partial(K/F) \to G(C)$ as in Construction 4.4.4. This gives a map $\mathrm{TF}(K/F, M) \to H^1_\partial(\mathrm{Gal}^\partial(K/F), M)$ which is part of a commutative diagram

$$\begin{array}{ccc} \mathrm{TF}(R/F, M) & \longrightarrow & \mathrm{TF}(K/F, M) \\ \cong\downarrow & & \downarrow \\ H^1_\partial(\mathrm{Gal}^\partial(R/F), M) & \xrightarrow[\cong]{} & H^1_\partial(\mathrm{Gal}^\partial(K/F), M). \end{array}$$

In this diagram, the top arrow is an injection, the left vertical arrow is a bijection by Theorem 4.4.8, and the bottom arrow is a bijection since $\mathrm{Gal}^\partial(K/F) = \mathrm{Gal}^\partial(R/F)$. Therefore all arrows in the diagram are bijections. □

In Example 4.2.1 we considered (F^{PV}/F)-twisted forms. We can classify these by the following cohomology set. Any Picard-Vessiot field extension L/F with a Picard-Vessiot subextension K/F gives a map $\mathrm{Gal}^\partial(L/F) \to \mathrm{Gal}^\partial(K/F)$ which in turn induces a map

$$H^1_\partial(\mathrm{Gal}^\partial(K/F), G) \to H^1_\partial(\mathrm{Gal}^\partial(L/F), G). \qquad (4.5.1)$$

The maps of the form (4.5.1) forms a direct system. We call its direct limit

$$H^1_\partial(F, G) = \varinjlim H^1_\partial(\mathrm{Gal}^\partial(L/F), G)$$

the *absolute cohomology set over F with values in G.*

Proposition 4.5.2. *Let $\mathbf{\Phi}$ be a tensor-type and M a $\mathbf{\Phi}$-structure over F. Suppose that the automorphism group of $M \otimes_F F^{\mathrm{PV}}$ is representable by a linear algebraic group G over C. Suppose also that the map*

$$\varinjlim \mathrm{TF}(K/F, M) \to \mathrm{TF}(F^{\mathrm{PV}}/F, M) \tag{4.5.2}$$

induced by inclusion maps $\mathrm{TF}(K/F, M) \to \mathrm{TF}(F^{\mathrm{PV}}/F, M)$, where K/F are Picard-Vessiot extensions over F, is a bijection. Then there is a bijection of the two sets $\mathrm{TF}(F^{\mathrm{PV}}/F, M)$ and $H^1_\partial(F, G)$.

Proof. For any Picard-Vessiot field extension K/F, Proposition 4.5.1 gives a bijection $\mathrm{TF}(K/F, M) \cong H^1_\partial(\mathrm{Gal}^\partial(K/F), G)$. Now take direct limits over the Picard-Vessiot field extensions K/F. □

Proposition 4.5.2 lets us classify the classes of objects we encountered in Example 4.2.1.

Corollary 4.5.2.1 (Differential modules)**.** *There is a bijection $\mathbf{Diff}_n(F) \cong H^1_\partial(F, \mathrm{GL}_n)$.*

Proof. By Example 4.2.1, it suffices to show $\mathrm{TF}(F^{\mathrm{PV}}/F, M) \cong H^1_\partial(F, \mathrm{GL}_n)$, where M is the trivial differential module over F of rank n. We first verify that the hypotheses of Proposition 4.5.2 hold. Let D be a C-algebra. The elements of $\mathrm{Aut}(M \otimes_C D)$ consist of differential $F \otimes_C D$-linear isomorphisms and so lie in $\mathrm{GL}(M \otimes_C D) \cong \mathrm{GL}_n(F \otimes_C D)$. Since these maps commute with the trivial derivation on $M \otimes_C D$, they lie in the subgroup $\mathrm{GL}_n(C \otimes_C D) = \mathrm{GL}_n(D)$. This proves $\mathrm{Aut}(M \otimes_C D) \cong \mathrm{GL}_n(D)$.

Next a (F^{PV}/F)-twisted form N of M gives a (differential) isomorphism $\varphi : N \otimes_F F^{\mathrm{PV}} \to M \otimes_F F^{\mathrm{PV}}$. Pick F-bases $\{n_i\}$ and $\{m_j\}$ for N and M. Then $\varphi(n_i) = \sum c_{ij} m_j$ for some $c_{ij} \in F^{\mathrm{PV}}$. Therefore φ restricts to a (differential) isomorphism $N \otimes_F K \cong M \otimes_F K$, where K is the smallest Picard-Vessiot extension in F^{PV} generated by the n^2 coefficients c_{ij}. Thus (4.5.2) is surjective.

Since all hypotheses are verified, we may apply Proposition 4.5.2 to get the desired bijection. $\square$

Corollary 4.5.2.2 (Differential torsors)**.** *Let G be a linear algebraic group over C and consider G_F as a trivial differential G_F-torsor. Then there is a bijection $G\text{-}\mathbf{tors}^{\partial}(F) \cong H^1_{\partial}(F, G)$.*

Proof. By Proposition 4.2.2, it suffices to show $\mathrm{TF}(F^{\mathrm{PV}}/F, G_F) \cong H^1_{\partial}(F, G)$.

Let D be a C-algebra. The automorphisms of $G_{F \otimes_C D}$ as a $G_{F \otimes_C D}$-torsor are elements of $G(F \otimes_C D)$; the *differential* automorphism of the trivial differential $G_{F \otimes_C D}$-torsor $G_{F \otimes_C D}$ lie in the subgroup $G(C \otimes_C D) = G(D)$. Thus $\mathrm{Aut}(G(F \otimes_C D)) \cong G(D)$.

Next if X is a (F^{PV}/F)-twisted form of G_F, we get an isomorphism of differential Hopf-Galois extensions $F[X] \otimes_F F^{\mathrm{PV}} \cong F[G] \otimes_F F^{\mathrm{PV}}$. Since $F[X]$ and $F[G]$ are finitely generated as F-algebras, by a similar argument to the proof of Corollary 4.5.2.1, this isomorphism restricts to one over a Picard-Vessiot field extension K/F.

Therefore X is a (K/F)-twisted form of G_F. We conclude that (4.5.2) is surjective, with $M = F[G]$.

Since all hypotheses are verified, we may apply Proposition 4.5.2 to get the desired bijection. $\square$

Remark 4.5.3. The bijection in Corollary 4.5.2.2 can be made explicit by working with differential Hopf-Galois extensions. Here is a more straightforward description in terms of differential torsors. A differential G-torsor X in $\mathrm{TF}(F^{\mathrm{PV}}/F, G_F)$ lies in $\mathrm{TF}(K/F, G_F)$ for some Picard-Vessiot field extension K/F. For any such K, $X_K \cong G_K$ and so has a differential K-rational point x. Any $\sigma \in \mathrm{Gal}^\partial(K/F)$ defines a (differential) automorphism on $X(K)$, and we set $a_\sigma \in G(K)$ to be the unique element such that $\sigma(x) = x \cdot a_\sigma$ in $X(K)$. We can check that $\sigma \mapsto a_\sigma$ defines a cocycle $a \in H^1_\partial(\mathrm{Gal}^\partial(K/F), G)$, and so we have a map

$$\mathrm{TF}(F^{\mathrm{PV}}/F, G_F) \to H^1_\partial(\mathrm{Gal}^\partial(K/F), G).$$

The cocycle constructed is compatible with the maps

$$H^1_\partial(\mathrm{Gal}^\partial(K/F), G) \to H^1_\partial(\mathrm{Gal}^\partial(L/F), G)$$

and so we get an induced map

$$\mathrm{TF}(F^{\mathrm{PV}}/F, G_F) \to H^1_\partial(F, G).$$

One checks that this map coincides with the other description.

Chapter 5

Versal Differential Torsors

Following [2, Sections 4 and 6], we define and prove properties about versal differential torsors and generic differential torsors.

Let G be a linear algebraic group over C. Roughly speaking, a "versal differential G-torsor" should be a differential G-torsor that can "specialize" to any other differential G-torsor defined over fields. Here "specializing" a differential torsor should mean pulling back the differential torsor to one defined over a point.

Formally, let $f : X \to Y$ be a differential G-torsor and let K be in $\mathrm{Fields}^{\partial}_{F,C}$. We define a map

$$Y(K) \to G\text{-}\mathbf{tors}^{\partial}(K) \tag{5.0.1}$$

which takes $y \in Y(K)$ to the fiber $X_y \to y$. By Corollary 4.5.2.2, this can be rewritten as

$$Y(K) \to H^1_{\partial}(K, G) \tag{5.0.2}$$

which takes y to a cocycle a that corresponds to X_y according to Remark 4.5.3. We will refer to both (5.0.1) and (5.0.2) as the *specialization map*.

Definition 5.0.1. Let G be a linear algebraic group over C. A differential G-torsor $f : X \to Y$ is *versal* for G if $F[Y]$ is an integral domain, and for any K in $\mathrm{Fields}^{\partial}_{F,C}$ and any $Z \in G\text{-}\mathbf{tors}^{\partial}(K)$, under the specialization map

$$s : Y(K) \to G\text{-}\mathbf{tors}^{\partial}(K) \tag{5.0.3}$$

the fiber $s^{-1}(Z)$ is Kolchin dense in Y. The generic fiber of a versal differential G-torsor is called a *generic differential G-torsor.*

Equivalently, a differential G-torsor $X \to Y$ is versal if for any nonempty open differential subscheme W of Y and any K in $\mathrm{Fields}^{\partial}_{F,C}$, the specialization map

$$W(K) \to G\text{-}\mathbf{tors}^{\partial}(K) \tag{5.0.4}$$

is surjective.

Versal differential torsors arise in the following way.

Proposition 5.0.2. *Let G be a linear algebraic group over C. Let G_F act linearly on $\mathbb{A}^n_F$. Suppose that U is a G-invariant open differential subscheme of $\mathbb{A}^n$. Suppose also that $\pi : U \to Y$ is a differential G-torsor. Then $\pi : U \to Y$ is a versal differential G-torsor.*

Proof. Let K be in $\mathrm{Fields}^{\partial}_{F,C}$ and W an open differential subscheme of Y. We must show the map

$$W(K) \to H^1_{\partial}(K, G) \tag{5.0.5}$$

is surjective. Let $a \in H^1_\partial(K,G)$. We may suppose that a lies in $H^1_\partial(H,G)$ for some Picard-Vessiot extension L/K with differential Galois group H. We define the twisted action of H on $V(L)$ given by $\boldsymbol{\sigma}(v) := \sigma(v) \cdot a_\sigma^{-1}$ for all $v \in V(L)$ and $\sigma \in H$. Viewing $V(K)$ as the trivial differential K-module, $V(L)^{\boldsymbol{H}}$ is then a twisted form of $V(K)$ and so by Corollary 4.5.2.1, $V(L)^{\boldsymbol{H}} \otimes_K L \cong V(L)$. Thus $V(L)^{\boldsymbol{H}}$ is a n-dimensional K-linear subspace of $V(L)$, hence Kolchin dense in $V(L)$ by Corollary 2.4.4.1.

Since $\pi^{-1}(W)$ is open in V, there exists a point $x \in V(L)^{\boldsymbol{H}} \cap \pi^{-1}(W)(L)$. For any $\sigma \in \Gamma_K$, we then have $x = \boldsymbol{\sigma}(x) = \sigma(x) \cdot a_\sigma^{-1}$, giving

$$\sigma(\pi(x)) = \pi(\sigma(x)) = \pi(x \cdot a_\sigma^{-1}) = \pi(x).$$

Therefore $\pi(x)$ lies in $W(L)^H = W(K)$ and $\pi(x)$ maps to a under (5.0.5). Since $a \in H^1_\partial(K,G)$ is arbitrary, the map (5.0.5) is surjective. $\square$

To construct versal differential torsors explicitly, we specialize our situation further:

Lemma 5.0.3. *Let* $R = F\{x_1, ..., x_n\}$ *and* $w = \mathrm{wr}(x_1, ..., x_n)$. *Let* GL_n *act by right matrix multiplication on* $\mathbb{A}^n = \mathrm{Diffspec}(R)$ *and let* G *be a closed subgroup of* GL_n. *If* $R\{1/w\}^G$ *is differentially finitely generated over* F *and has field of fractions* $\mathrm{Frac}(R)^G$, *then*

$$\mathrm{Diffspec}(R\{1/w\}) \to \mathrm{Diffspec}(R\{1/w\}^G)$$

is a versal differential G*-torsor.*

Proof. Let $B := R\{1/w\}$ and $A := R\{1/w\}^G$. Note that $\mathrm{Diffspec}(B)$ is a G-invariant open subset of $\mathbb{A}^n$. By Proposition 5.0.2, it suffices to show that $\mathrm{Diffspec}(B) \to \mathrm{Diffspec}(A)$ is a differential torsor.

Let $K = \mathrm{Frac}(R)$. By [30, Exercise 1.35(4)], K/K^{GL_n} is a GL_n-Picard-Vessiot field extension for the differential equation

$$p(y) = \mathrm{wr}(y, x_1, ..., x_n)/w.$$

By the differential Galois correspondence, K/K^G is a G-Picard-Vessiot field extension for $p(y)$ over K^G. The Picard-Vessiot ring T for the extension K/K^G is then generated by the solutions $x_1, ..., x_n$ for $p(y)$ and the element w^{-1} over K^G. In particular, T contains R. The fact that $\mathrm{Diffspec}(T) \to \mathrm{Diffspec}(K^G)$ is a differential G-torsor gives a (differential) isomorphism

$$T \otimes_{K^G} T \cong T \otimes_F F[G] \tag{5.0.6}$$

given explicitly as follows. Let $F[\mathrm{GL}_n] = F[z_{ij}, 1/\det(Z)]$ where $Z = (z_{ij})$ is a $n \times n$ matrix of indeterminates. The closed embedding $G \to \mathrm{GL}_n$ induces a surjective F-algebra homomorphism $F[\mathrm{GL}_n] \to F[G] : z_{ij} \mapsto \overline{z_{ij}}$. Then (5.0.6) is determined by

$$\begin{array}{rcll} h \otimes 1 & \mapsto & h \otimes 1 & \text{for all } h \in T, \\ 1 \otimes x_j & \mapsto & \sum_{i=1}^n x_i \otimes \overline{z_{ij}} & \text{for } 1 \leq i \leq n, \\ 1 \otimes w^{-1} & \mapsto & w^{-1} \otimes \det(\overline{z_{ij}})^{-1}. & \end{array}$$

This restricts to a (differential) isomorphism

$$B \otimes_A B \cong B \otimes_F F[G] \tag{5.0.7}$$

which shows that $\text{Diffspec}(B) \to \text{Diffspec}(A)$ is a differential G-torsor. $\square$

We now write down explicit examples of versal differential G-torsors.

Proposition 5.0.4. *Let $R = F\{x_1, ..., x_n\}$ and let GL_n act by right matrix multiplication on $\mathbb{A}^n = \text{Diffspec}(R)$. For some $w \in R \setminus \{0\}$,*

$$\text{Diffspec}(R\{1/w\}) \to \text{Diffspec}(R\{1/w\}^G)$$

is a versal differential G-torsor for $G = \mathrm{GL}_n$ and $G = \mathbb{G}_m^n$.

Proof. By Lemma 5.0.3, it suffices to show that $R\{1/w\}^G$ is differentially finitely generated over F. Let $K = \text{Frac}(R)$.

Case 1: $G = \mathrm{GL}_n$. This is given in [14, Proposition 1] which we now reproduce. Let $w = \text{wr}(x_1, ..., x_n)$. By [30, Exercise 1.35(4)], K/K^G is a G-Picard-Vessiot field extension for the differential equation

$$p(y) = \text{wr}(y, x_1, ..., x_n)/w = y^{(n)} + w_{n-1}y^{(n-1)} + \cdots + w_0 y.$$

Therefore the coefficients $w_0, ..., w_{n-1}$ are differentially algebraically independent over F. Since $p(x_i) = 0$ for $i = 1, ..., n$,

$$\begin{aligned} R\{1/w\} &= F\{w_0, ..., w_{n-1}, 1/w\}[x_i^{(j-1)}]_{1 \le i,j \le n} \qquad (5.0.8) \\ &\cong F\{w_0, ..., w_{n-1}\} \otimes_F F[x_i^{(j-1)}, 1/w]_{1 \le i,j \le n}. \end{aligned}$$

We identify $F[x_i^{(j-1)}, 1/w]_{1 \le i,j \le n}$ with $F[\mathrm{GL}_n]$ and $F\{w_0, ..., w_{n-1}\}$ with the coordinate ring of a scheme V over F defined as follows. For any algebra S over F, we set

$$V(S) = \oplus_{j=1}^{\infty} \oplus_{i=1}^{n} S \cdot x_i^{(j-1)}$$

to be the free S-module on basis elements $x_i^{(j-1)}$.

For each j, we define a (right) $G(S)$-action on the free S-module $\oplus_{i=1}^{n} S \cdot x_i^{(j-1)}$ of rank n by right matrix multiplication in the coordinates $x_1^{(j-1)}, ..., x_n^{(j-1)}$. We define V_{triv} to be V with trivial G-action.

Considering $V \times G$ and $V_{\mathrm{triv}} \times G$ with the diagonal G-action, we have a G-equivariant isomorphism

$$V \times G \to V_{\mathrm{triv}} \times G$$

given by $(v, g) \mapsto (v \cdot g^{-1}, g)$. This induces an isomorphism on coordinate rings

$$R\{1/w\} \cong F[V \times G] \xrightarrow{\sim} F[V_{\mathrm{triv}} \times G] \cong F[V_{\mathrm{triv}}] \otimes F[G] \tag{5.0.9}$$

which restricts to G-invariants

$$R\{1/w\}^G \xrightarrow{\sim} F[V_{\mathrm{triv}} \times G]^G = F[V_{\mathrm{triv}}]. \tag{5.0.10}$$

Under (5.0.9), $F\{w_0, ..., w_{n-1}\}$ maps to $F[V_{\mathrm{triv}}]$ while $F[x_i^{(j-1)}]_{1 \le i,j \le n}$ maps to $F[G]$, so by (5.0.8), $F\{w_0, ..., w_{n-1}\}$ is precisely the preimage of $F[V_{\mathrm{triv}}]$. Comparing with (5.0.10) we see that $R\{1/w\}^G = F\{w_0, ..., w_{n-1}\}$ is differentially finitely generated over F.

Case 2: $G = \mathbb{G}_m^n$. A similar proof works here. Let $w = 1/(x_1 \cdots x_n)$. We note that K is a $\mathbb{G}_m^n$-Picard-Vessiot extension over $F\langle w_1, ..., w_n\rangle$ where $w_i = \partial x_i / x_i$.

Since the x_i are solutions to the differential equation $\partial Y_i/Y_i = w_i$ for $i = 1, ..., n$, we have

$$\begin{aligned} R\{1/w\} &= F\{w_1, ..., w_n, 1/w\}[x_1, ..., x_n] \\ &\cong F\{w_1, ..., w_n\} \otimes_F F[x_1, ..., x_n, 1/w]. \end{aligned} \tag{5.0.11}$$

Again we identify $F[x_1, ..., x_n, 1/w]$ with $F[G]$ and $F\{w_1, ..., w_n\}$ with a scheme V defined similarly to that of the previous case. Arguing as before gives $R\{1/w\}^G = F\{w_1, ..., w_n\}$.

□

Now that we have exhibited versal differential torsors, our next goal is to understand the differential essential dimension of their generic fibers.

Definition 5.0.5. Given differential G-torsors $f : X \to Y$ and $f' : X' \to Y'$ where $F[Y]$ and $F[Y']$ are integral domains, we say that f' is a *compression* of f if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"] & Y' \end{tikzcd} \tag{5.0.12}$$

where g is a G-equivariant rational dominant morphism and h is a rational morphism.

Proposition 5.0.6. *The compression of a versal differential torsor is again versal.*

Proof. Let $f' : X' \to Y'$ be a compression of $f : X \to Y$ and let g and h be as in (5.0.12). Let K be in $\text{Fields}^{\partial}_{F,C}$ and let $a \in H^1_\partial(K, G)$. Let $Z = \{y \in Y(K) \mid y \mapsto a\}$

and $Z' = \{y' \in Y'(K) \mid y' \mapsto a\}$. The domain of definition of h contains an affine open subset U of Y. For any $y \in U$, the fiber of f at y is isomorphic to the fiber of f' at $h(y)$, so $Z \cap U \subset h^{-1}(Z')$. To show that f' is versal differential torsor, we must show that Z is dense in $Y'(K)$. Let V' be an open subset of $Y'(K)$. We must check that $V' \cap Z'$ is nonempty. Well $h^{-1}(V' \cap Z') = h^{-1}(V') \cap h^{-1}(Z') \supset h^{-1}(V') \cap (Z \cap U)$. Since f is versal, Z is dense in $Y(K)$, and so the intersection of Z with the open set $h^{-1}(V') \cap U$ is nonempty. □

Proposition 5.0.7. *Let $X \to Y$ be a versal differential G-torsor. Suppose its generic fiber $X_0 \to Y_0$, where $Y_0 = \text{Diffspec}(F(Y))$, is defined over a subfield as $X_0' \to Y_0'$. Then there exists a differential G-torsor $X' \to Y'$ together with a compression*

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"'] & Y' \end{tikzcd}$$

such that at the generic fibers we get

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "f"'] & X_0' \arrow[d, "f'"] \\ Y_0 \arrow[r, "h"'] & Y_0'. \end{tikzcd}$$

Proof. Let $X, Y, X_0, Y_0, X_0', Y_0'$ be the differential spectra of B, A, P, K, P', K', respectively. Proving the proposition reduces to finding:

(a) a differential subring A' of K' such that $\text{Frac}\, A' = K'$;

(b) a differential algebra B' over A' satisfying $P' \cong B' \otimes_{A'} K'$ and such that the map $\text{Diffspec}(B') \to \text{Diffspec}(A')$, induced from the structure map $A \to B$, is a differential G-torsor; and

(c) elements $u \in K$ and $p \in P$ and a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{g^\#} & B\{1/p\} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow[h^\#]{} & A\{1/u\}. \end{array} \tag{5.0.13}$$

Since $\operatorname{trdeg}^\partial_F K$ and $\operatorname{trdeg}^\partial_F K'$ are finite and since P/K and P'/K' are algebras of finite type, we have $K = F\langle \boldsymbol{x} \rangle$, $K' = F\langle \boldsymbol{x}' \rangle$, $P = K[\boldsymbol{y}]$, and $P' = K'[\boldsymbol{y}']$ for some finite subsets $\boldsymbol{x} \subset K$, $\boldsymbol{x}' \subset K'$, $\boldsymbol{y} \subset P$, and $\boldsymbol{y}' \subset P'$.

The isomorphism

$$P' \otimes_{K'} P' \cong P' \otimes_F F[G] \tag{5.0.14}$$

is determined by $f \otimes 1 \mapsto f \otimes 1$ for $f \in P'$ and $1 \otimes y_j \mapsto \sum f_{ij} y_i \otimes c_{ij}$ for some $c_{ij} \in F[G]$, $f_{ij} \in K'$. Since any element of K' can be written as a ratio of elements of $F\{\boldsymbol{x}'\}$, we let f be the product the denominators of the terms f_{ij} which are nonzero. Set $A' = F\{\boldsymbol{x}', 1/f\}$, and $B' = A'[\boldsymbol{y}']$. Then (5.0.14) restricts to an isomorphism

$$B' \otimes_{A'} B' \cong B' \otimes_F F[G].$$

This gives (a) and (b).

For (c), since A' is differentially of finite type over F, by the same reasoning as above, the composite map

$$A' \hookrightarrow K' \xrightarrow{h^\#} K$$

has image inside $F\{\boldsymbol{x}, 1/u\}$ for some $u \in K$. Likewise $A = F\{\boldsymbol{x}, 1/v\}$ for some

$v \in K$. Thus we have a sequence of maps

$$A' \xrightarrow{h^\#} F\{\boldsymbol{x}, 1/u\} \hookrightarrow F\{\boldsymbol{x}, 1/u, 1/v\} = A\{1/u\}.$$

Similarly we get a map $B' \to B\{1/p\}$ compatible with this map, for some $p \in P$. This gives the commutative diagram (5.0.13). $\square$

Combining Propositions 5.0.6 and 5.0.7 we get the following:

Corollary 5.0.7.1. *If a generic differential G-torsor X descends to a differential G-torsor X', then X' is again a generic differential G-torsor.*

Corollary 5.0.7.2. *Let X be a generic differential G-torsor. Then* $\mathrm{ed}_F^\partial(X) = \mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$.

Proof. Clearly $\mathrm{ed}_F^\partial(X) \leq \mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$. The differential torsor X descends to a differential torsor $X_0 \to \mathrm{Diffspec}(K)$ for which $\mathrm{trdeg}_F^\partial K = \mathrm{ed}_F^\partial(X)$. Propositions 5.0.6 and 5.0.7 guarantee that X_0 is the generic fiber of some versal differential torsor $X' \to Y'$. By the definition of versal differential torsor, the specialization map

$$Y' \to G\text{-}\mathbf{tors}^\partial$$

is a surjective natural transformation. Therefore by 3.2.5, we have

$$\mathrm{ed}_F^\partial(X) = \mathrm{trdeg}_F^\partial K = \dim Y' \geq \mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial).$$

$\square$

With this, we have finished the most technical part of this thesis. Our next task is to use this corollary to give bounds on the number $\mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$.

Chapter 6

Applications

6.1 Bounds on differential essential dimension

Proposition 6.1.1. *We have* $\mathrm{ed}_F^\partial(\mathrm{GL}_n) = \mathrm{ed}_F^\partial(\mathbb{G}_m^n) = n$.

Proof. Let $G_1 = \mathrm{GL}_n$, $G_2 = \mathbb{G}_m^n$, $R = F\{x_1, ..., x_n\}$, $K = \mathrm{Frac}(R)$, $w_1 = \mathrm{wr}(x_1, ..., x_n)$, and $w_2 = 1/(x_1 \cdots x_n)$. By Proposition 5.0.4,

$$\mathrm{Diffspec}\, R\{1/w_i\} \to \mathrm{Diffspec}\, R\{1/w_i\}^{G_i}$$

is a versal differential G_i-torsor for $i \in \{1, 2\}$. Their generic fibers correspond to differential $F[G_i]$-Hopf-Galois extensions S_i/K^{G_i} for some differential subrings S_i of R with $\mathrm{Frac}(S_i) = K$ for $i \in \{1, 2\}$. Examining the proof of Proposition 5.0.4 shows that S_i/K^{G_i} are G_i-Picard-Vessiot extensions. For convenience, we work with the associated G_i-Picard-Vessiot field extensions K/K^{G_i} instead. By Corollary 5.0.7.2,

$$\mathrm{ed}^\partial(K/K^{G_i}) = \mathrm{ed}^\partial(G_i\text{-}\mathbf{tors}^\partial)$$

for $i = 1, 2$. Thus K/K^{G_1} descends to a G_1-Picard-Vessiot extension $K_0/K_0^{G_1}$ satisfying $\operatorname{trdeg}_F^{\partial} K_0 = \operatorname{ed}^{\partial}(G\text{-}\mathbf{tors}^{\partial})$. Since G_2 is a subgroup of G_1, K/K^{G_2} also descends to $K_0/K_0^{G_2}$. This gives

$$\operatorname{ed}_F^{\partial}(G_2\text{-}\mathbf{tors}^{\partial}) = \operatorname{ed}_F^{\partial}(K/K^{G_2}) \leq \operatorname{trdeg}_F^{\partial}(K_0) = \operatorname{ed}_F^{\partial}(G_2\text{-}\mathbf{tors}^{\partial}). \tag{6.1.1}$$

By Proposition 3.3.1, $\operatorname{ed}_F^{\partial}(G_2\text{-}\mathbf{tors}^{\partial}) \geq n$. We also know that $\operatorname{trdeg}_F^{\partial}(K_0) \leq \operatorname{trdeg}^{\partial} K = n$ so the terms in (6.1.1) all equal n. □

6.2 General differential equations

Proposition 6.2.1. *Let $a_0, ..., a_{n-1}$ be differential indeterminates over F. Consider the general differential equation $p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0$ over $L = F\langle x_0, ..., x_{n-1}\rangle$ as an element of $\mathbf{DiffEq}_n(L)$. Then $\operatorname{ed}_F^{\partial}(p(y)) = n$.*

Proof. Let $K = F\langle x_1, ..., x_n\rangle$ and $w = \operatorname{wr}(x_1, ..., x_n)$. By [30, Exercise 1.35(4)], the coefficients of the differential equation

$$\operatorname{wr}(y, x_1, ..., x_n)/w = y^{(n)} + w_{n-1}y^{(n-1)} + \cdots + w_0$$

are differential algebraically independent over F and so we may regard $p(y)$ as this differential equation in the coefficients w_i. Let $K_0 = K^{\mathrm{GL}_n}$. By Proposition 2.3.3, $p(y)$ corresponds to the differential module $N = K_0[\partial]/K_0[\partial] \circ p$ under the isomorphism $\mathbf{DiffEq}_n \cong \mathbf{Diff}_n$. By 4.5.2.1, and 4.5.2.2, there is a sequence of bijections

$$\mathbf{Diff}_n(K) \cong H^1(K^{PV}/K, \mathrm{GL}_n) \cong \mathrm{GL}_n\text{-}\mathbf{tors}^{\partial}(K). \tag{6.2.1}$$

Since (6.2.1) defines an isomorphism of functors $\mathbf{Diff}_n \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial$, it suffices to show that the differential module N corresponds to the generic differential GL_n-torsor S/K_0 (as constructed above; S is the Picard-Vessiot ring of the extension K/K_0) under this correspondence, for then

$$\mathrm{ed}_F^\partial(p(y)) = \mathrm{ed}_F^\partial(N) = \mathrm{ed}_F^\partial(S/K_0) = n.$$

First view N as the differential submodule $\bigoplus_{i=0}^{n-1} K_0 x_1^{(i)}$ of S. Consider any (differential) isomorphism $\varphi : S \otimes_{K_0} K \to K[\mathrm{GL}_n]$. Since $N \otimes_{K_0} K$ is a trivial differential module, its image is also a trivial differential module M_1 over K. For any $\sigma \in \mathrm{Gal}^\partial(K/K_0)$, consider the following diagram:

$$\begin{tikzcd}
 & & S \arrow[r, "\varphi"] \arrow[d, dotted, "\sigma"] & K[\mathrm{GL}_n] \arrow[d, "\sigma"] \\
N_K \arrow[r, "\varphi"] \arrow[d, "\sigma"] \arrow[urr, hook] & M_1 \arrow[d, "\sigma"] \arrow[urr, hook] & S \arrow[r, dotted, "\varphi"] & K[\mathrm{GL}_n] \\
N_K \arrow[r, "\varphi"] \arrow[urr, hook, dotted] & M_1 \arrow[urr, hook] & &
\end{tikzcd}$$

All the faces of this rectangular prism are commutative except the front and the back. Therefore the cocycle corresponding to the back face restricts to the the cocycle which is front face. In turn, they define equivalent cocycles in $H^1(K, \mathrm{GL}_n)$ once we fix identifications $\mathrm{Aut}(M_1) \cong \mathrm{GL}_n(C)$ and $\mathrm{Aut}(K[\mathrm{GL}_n]) \cong \mathrm{GL}_n(C)$.

$\square$

6.3 Generic Picard-Vessiot extensions

In this section, we define and prove lower bounds on generic Picard-Vessiot extensions. We follow the method of [4, Section 7].

Definition 6.3.1. Let $K = F\langle y_1, ..., y_n\rangle$ with indeterminates $y_1, ..., y_n$ over F. Let $p_{y_1,...,y_n}(y)$ be a homogeneous linear differential equation over K. Suppose that $p_{y_1,...,y_n}(y)$ determines a Picard-Vessiot extension R/K with differential Galois group G. Then R/K is said to be a *generic G-Picard-Vessiot extension* if for every G-Picard-Vessiot extension R'/K', there exist elements $a_1, ..., a_n \in K'$ such that R'/K' is the Picard-Vessiot extension for the differential equation $p_{a_1,...,a_n}(y)$.

Proposition 6.3.2. *Let G be a linear algebraic group over C, and let R/K be a generic G-Picard-Vessiot extension. Suppose that at least one generic differential G-torsor exists that corresponds to a G-Picard-Vessiot extension. Then* $\mathrm{trdeg}_F^{\partial}(K) \geq \mathrm{ed}_F^{\partial}(G\text{-}\mathbf{tors}^{\partial})$.

Proof. Let R'/K' be a G-Picard-Vessiot extension corresponding to a generic differential G-torsor. By the definition of a generic G-Picard-Vessiot extension, R/K is the Picard-Vessiot extension for some matrix differential equation $\partial(Y) = A_{y_1,...,y_n}Y$ over K, and there exist $a_1, ..., a_n \in K'$ such that R'/K' is the Picard-Vessiot extension for $\partial(Y) = A_{a_1,...,a_n}Y$ over K'.

We claim that R'/K' is defined over the differential subfield $K'' := F\langle a_1, ..., a_n\rangle$. Indeed, $\partial(Y) = A_{a_1,...,a_n}Y$ is defined over K'' and has a full set of solutions within R', thus determining a G-Picard-Vessiot ring R'' (over K'') inside R'. Since $R'' \otimes_{K''} K'$ determines a G-Picard-Vessiot subextension of R' over K', we must have $R'' \otimes_{K''} K' = R'$. Therefore R'/K' descends to R''/K''.

Now $K'' = F\langle a_1, ..., a_n\rangle$ implies that $\operatorname{trdeg}_F^{\partial} K = n \geq \operatorname{trdeg}_F^{\partial} K''$. Moreover R'/K' corresponds to a generic differential G-torsor so by Corollary 5.0.7.1, R''/K'' also corresponds to a generic differential G-torsor. Thus $\operatorname{trdeg}_F^{\partial} K'' \geq \operatorname{ed}_F^{\partial}(R'') = \operatorname{ed}_F^{\partial}(G\text{-}\mathbf{tors}^{\partial})$. Combining the inequalities gives $\operatorname{trdeg}_F^{\partial} K \geq \operatorname{ed}_F^{\partial}(G\text{-}\mathbf{tors}^{\partial})$.

□

Corollary 6.3.2.1. *Let G be either GL_n or $\mathbb{G}_m^n$. Let R/K be a generic G-Picard Vessiot extension. Then $\operatorname{trdeg}_F^{\partial} K \geq n$.*

Proof. We have directly constructed versal differential torsors for these groups G, the proof of which already showed their generic fibers corresponded to G-Picard-Vessiot extensions. Thus Proposition 6.3.2 applies. □

One can similarly construct versal differential torsors for the groups $\mathbb{G}_a^n$ and SL_n, and perhaps other groups, on a case-by-case basis, and therefore compute their differential essential dimensions using similar arguments. Since we expect to have more general arguments to take care of these in the near future, we do not write down individual proofs of each of these cases here.

In the case of $G = \mathbb{G}_m^n$, the extension $F\langle x_1, ..., x_n\rangle / F\langle x_1, ..., x_n\rangle^G$ is a generic Picard-Vessiot extension, and so the lower bound given in Proposition 6.3.2 is sharp. For an arbitrary group G, however, we do not know if there exists a generic G-Picard-Vessiot extension R/K for which equality holds in $\operatorname{trdeg}_F^\partial K \geq \operatorname{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$.

Bibliography

[1] Annette Bachmayr, David Harbater, Julia Hartmann, and Michael Wibmer. Differential embedding problems over complex function fields. *Documenta Mathematica*, 23:241–291, 2018.

[2] Grégory Berhuy and Giordano Favi. Essential dimension: a functorial point of view (after A. Merkurjev). *Doc. Math*, 8(106):279–330, 2003.

[3] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension and algebraic stacks. *arXiv preprint math/0701903*, 2007.

[4] Joe Buhler and Zinovy Reichstein. On the essential dimension of a finite group. *Compositio Mathematica*, 106(2):159–179, 1997.

[5] Joe Buhler and Zinovy Reichstein. On Tschirnhaus transformations. In *Topics in Number theory*, pages 127–142. Springer, 1999.

[6] Alexander Duncan. Essential dimensions of A_7 and S_7. *Mathematical Research Letters*, 17(2):263–266, 2010.

[7] Tobias Dyckerhoff. The inverse problem of differential Galois theory over the field $\mathbb{R}(z)$. *arXiv preprint arXiv:0802.2897*, 2008.

[8] Benson Farb and Jesse Wolfson. Resolvent degree, hilbert's 13th problem and geometry. *L'Enseignement Mathématique*, 65(3):303–376, 2020.

[9] Lawrence Goldman. Specialization and Picard-Vessiot theory. *Transactions of the American Mathematical Society*, 85(2):327–356, 1957.

[10] Christian U Jensen, Arne Ledet, Noriko Yui, et al. *Generic polynomials: constructive aspects of the inverse Galois problem*, volume 45. Cambridge University Press, 2002.

[11] Lourdes Juan and Arne Ledet. On generic differential SO_n-extensions. *Proceedings of the American Mathematical Society*, 136(4):1145–1153, 2008.

[12] Lourdes Juan and Arne Ledet. Generic Picard–Vessiot Extensions for Nonconnected Groups. *Communications in Algebra*, 44(7):2989–3004, 2016.

[13] Lourdes Juan and Andy Magid. Differential central simple algebras and Picard–Vessiot representations. *Proceedings of the American Mathematical Society*, 136(6):1911–1918, 2008.

[14] Lourdes Juan and Andy R Magid. Generic rings for Picard–Vessiot extensions and generic differential equations. *Journal of Pure and Applied Algebra*, 209(3):793–800, 2007.

[15] Max-Albert Knus. *Quadratic and Hermitian forms over rings*, volume 294. Springer Science & Business Media, 2012.

[16] Ellis R Kolchin. Algebraic groups and algebraic dependence. *American Journal of Mathematics*, 90(4):1151–1164, 1968.

[17] Ellis Robert Kolchin. *Differential algebra & algebraic groups*, volume 54. Academic press, 1973.

[18] Jerald J Kovacic. Differential schemes. In *Differential algebra and related topics*, pages 71–94. World Scientific, 2002.

[19] Andy R Magid. The Picard–Vessiot closure in differential Galois theory. *Banach Center Publications*, 58:151–164, 2002.

[20] Akira Masuoka and Yuta Shimada. Twisted Forms of Differential Lie Algebras over $\mathbb{C}(t)$ Associated with Complex Simple Lie Algebras. *Arnold Mathematical Journal*, pages 1–28, 2020.

[21] Alexander Merkurjev. Essential dimension. *Bulletin of the American Mathematical Society*, 54(4):635–661, 2017.

[22] James S Milne. *Algebraic groups: the theory of group schemes of finite type over a field*, volume 170. Cambridge University Press, 2017.

[23] Denis Nardin. The essential dimension of finite group schemes. 2012.

[24] Philippe Nuss and Marc Wambst. Non-abelian Hopf cohomology. *Journal of Algebra*, 312(2):733–754, 2007.

[25] Zinovy Reichstein. On the notion of essential dimension for algebraic groups. *Transformation Groups*, 5(3):265–304, 2000.

[26] Zinovy Reichstein. Essential dimension. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 162–188. World Scientific, 2010.

[27] Joseph Fels Ritt. *Differential algebra*, volume 33. American Mathematical Soc., 1950.

[28] Hans-Jürgen Schneider. Principal homogeneous spaces for arbitrary Hopf algebras. *Israel Journal of Mathematics*, 72(1-2):167–195, 1990.

[29] Marius van der Put. Galois theory of differential equations, algebraic groups and Lie algebras. *Journal of symbolic computation*, 28(4-5):441–473, 1999.

[30] Marius Van der Put and Michael F Singer. *Galois theory of linear differential equations*, volume 328. Springer Science & Business Media, 2012.