Essays On Heterogeneity In Macroeconomics

Hanbaek Lee

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Abstract
This dissertation is composed of three chapters. In the first two chapters, I study how micro-level heterogeneity affects aggregate fluctuations in an economy. The third chapter develops a novel computational method that solves the nonlinear dynamic stochastic general equilibrium with heterogeneous agents.

In the first chapter, I study how heterogeneous firm-level lumpy investments affect the business cycle. I develop a heterogeneous-firm business cycle model where large firms’ lumpy investments closely follow the empirical patterns. In the model, synchronized large-scale investments of large firms significantly amplify productivity-driven aggregate fluctuations and lead to investment cycles even in the absence of aggregate shocks. In the second chapter, I study how the pass-through businesses of top income earners affect the aggregate fluctuations in the U.S. economy. Using a heterogeneous-household real business cycle model with endogenous labor supply and occupational choice, I argue that the business-income-driven top income inequality has made the following changes in the productivity-driven aggregate fluctuations: 1) lower volatility of aggregate output and 2) stronger negative correlation between labor hour and productivity. In the third chapter, I develop and test a novel algorithm that solves heterogeneous-agent models with aggregate uncertainty. This method computes the nonlinear dynamic stochastic general equilibrium with a high degree of accuracy. And the computational gain compared to existing methods is significant when a non-trivial market-clearing condition is present in the model.

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To my lovely wife, Minji and my entire family
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Introduction

The recent rise of micro-level data availability along with the increased computational power have led macroeconomists to revisit a traditional question of how well an economy can be represented by a representative-agent economic model. This dissertation seeks the answer to the question focusing on the role of micro-level heterogeneity on the business cycle. In the first chapter, I study the aggregate implications of firm-level heterogeneity in the interest-elasticity of lumpy investments. In the second chapter, household-level heterogeneity in the income source is considered to study its role over the business cycle. The third chapter develops and tests a novel methodology to solve a nonlinear business cycle model with heterogeneous agents.

In the first chapter, I argue that synchronized large-scale investments of large firms can significantly amplify productivity-driven aggregate fluctuations and lead to investment cycles even in the absence of aggregate shocks. Using U.S. Compustat data, I show that years preceding recessions display investment surges among large firms. Furthermore, after the investment surges, large firms become inelastic to interest rates and display persistent inaction duration. I then develop a heterogeneous-firm real business cycle model in which a firm needs to process multiple investment stages for large-scale investments and can accelerate it at a cost. In the model, following a TFP shock, the synchronized timings of lumpy investments are persistently synchronized. These synchronized investment timings result in endogenous nonlinear fluctuations in the aggregate investment, which I call an
echo effect. And TFP-induced recessions are especially severe after the surge of large firms’ lumpy investments. This is because a lowered interest rate during the recession does not motivate the large firms that have just finished large-scale investments to make another round of lumpy investments. In the calibrated model, a negative TFP shock has a 29% greater impact on aggregate investment after a surge of lumpy investments. In support of the model prediction, I present evidence for the investment cycle in the post-shock period in macro-level data on nonresidential fixed investment.

In the second chapter, I study how the observed trend of rising top income inequality driven by pass-through business income has affected the business cycle. I develop a heterogeneous-household real business cycle model with endogenous labor supply and occupational choice and calibrate the model to capture the observed top income inequality. Compared to the counterfactual economy with the factor-income-driven top income inequality, the economy in the baseline model features the aggregate fluctuations that outperform in explaining the recent changes in the business cycle: 1) lower volatility of aggregate output and 2) stronger negative correlation between labor hour and productivity. Heterogeneous labor demand sensitivities to TFP shocks between pass-through businesses and C-corporations build the core of the aggregate dynamics, and the aggregate employment dynamics display substantial nonlinearity due to this heterogeneity.

In the third chapter, I develop and test a novel algorithm that solves heterogeneous agent models with aggregate uncertainty. The first two chapters use this algorithm to solve the nonlinear dynamic stochastic general equilibrium. The algorithm iteratively updates agents’ expectations on the future path of aggregate states from the transition dynamics on a single path of simulated shocks until the expected path converges to the simulated path. The nonlinear dynamic stochastic general equilibrium could be computed with a high degree of accuracy by this method; the market clearing prices and the expected aggregate states are directly computed at each point on the path without relying on the parametric law of motions. Using the algorithm, I analyze a heterogeneous-firm business cycle model.
where firms are subject to an external financing cost and hoard cash as a buffer stock up to a target level. Based on the model, I discuss the business cycle implications of the corporate cash holdings.
Chapter 1

Striking While the Iron Is Cold: Fragility after a Surge of Lumpy Investments

1980, 1998, and 2007 were the three years with the largest surges in the fraction of large firms making large-scale investments since 1980. And these three years were followed by recessions within two years.¹ Is it merely a coincidence that investment surges of large firms precede recessions?

This paper studies a mechanism that makes an economy more fragile to a negative TFP shock after a surge in lumpy investments of large firms. I develop and analyze a business cycle model with heterogeneous firms that reflects empirical findings from micro-level data. Then using the model, I qualitatively and quantitatively analyze the amplification of productivity-driven aggregate fluctuations. Through lumpy investment decisions at firm

¹. Following Cooper and Haltiwanger (2006), I define an investment in a year beyond 20% of existing capital stock as a large-scale capital adjustment. Firms that hold capital stocks greater than the 90th percentile of the capital distribution in each industry based on two-digit NAICS code are defined as large firms.
level, the interaction between endogenous and exogenous sources of aggregate fluctuations builds the core of this analysis.

I document two empirical characteristics of firm-level lumpy investments that matter for aggregate fluctuations: the interest-inelasticity of large-scale investment timings and the highly persistent inaction durations. First, using the U.S. Compustat data, I show that large firms’ timings of lumpy investments are inelastic to interest rate changes. Therefore, if a negative aggregate TFP shock hits an economy after a previous surge of large firms’ lumpy investments, new aggregate investment drops substantially. These large firms are not willing to make another large capital adjustment on top of their recent investments despite a lowered interest rate.

Second, I document that inaction periods of a firm’s capital adjustment are highly persistent across periods. The observed persistence for all firms is substantially higher than the level implied by the stochastic investment \((S, s)\) cycle in models with fixed costs in the literature. This high persistence has an important aggregate implication: the mean-reversion of the synchronized investment timings across firms is sluggish. Thus, an aggregate TFP shock effect lasts longer when the high persistence in the length of inaction periods at firm level is higher.

Therefore, it is necessary to capture these two empirical findings in the model to study how firm-level lumpy investments affect business cycle. Based on existing evidence from the literature (Yang et al., 2020), I argue that large firms’ structured decision-making process such as capital budgeting accounts for the observed inelasticity and the persistence. In particular, capital budgeting is a universal tool for CFO’s to plan and evaluate an investment project. Almost 99.5% of CFO’s from Fortune 1,000 firms rely on capital budgeting. I provide a suggestive evidence that firms become inelastic to interest rate change and insensitive to investment opportunities during the capital budgeting process. From this, I claim implementation lags from structured decision-making process is a critical component to be modeled to capture the empirical findings.
Then I develop and analyze a model with lumpy investment in which a firm needs to process a required number of investment stages for a large-scale investment. Here the investment stages capture the bureaucratic steps in capital budgeting such as meetings of the board of directors for the large-scale investment decision or auditing procedures for large-scale investments (Malenko, 2019). Each firm decides the optimal number of stages to process each period. The processing cost convexly increases both in the number of stages to be processed and in the size of a firm’s capital stock. I name this cost as “acceleration cost.” The convexity of acceleration cost in the number of stages disturbs firms’ nimble capital adjustment. The convexity in the size of capital stock makes large firms face larger cost of agile capital adjustment. These features make large firms’ lumpy investments inelastic to interest rate changes.

When an aggregate TFP shock hits the economy in the model, the timing of lumpy investments is synchronized across firms. Then, aggregate investments nonlinearly respond to the aggregate TFP shock because synchronized lumpy investments are not mitigated by changes in the interest rate. Furthermore, high persistence in inaction duration persistently synchronizes future investment timings. This leads to long-run echo effects in the economy. In this model, the impulse response of the economy depends on the aggregate state of the economy. Specifically, the mass of large firms that are ready to adjust their own existing capital is the key conditioning state variable.

Using the calibrated model, I decompose the total response of aggregate investments to an aggregate TFP shock into an exogenous effect and an endogenous effect. The endogenous effect accounts for substantial portion of the aggregate investment response: it explains up to 15% of the total response. The endogenous effect is largest when a negative aggregate TFP shock hits the economy after a surge of lumpy investments: the same negative aggregate TFP shock has up to 29% greater impact on aggregate investment after a surge of lumpy investments than in other aggregate states.
In the model, if a group of firms is extremely inelastic to interest rate changes and has an extremely high persistence of inaction duration, the echo effect can permanently persist in the post-shock period. The synchronized investment timings are then permanently synchronized, leading to a stationary cycle. To characterize this stationary cycle formally, I first define a cyclical competitive equilibrium that conceptually extends stationary recursive competitive equilibrium. In this equilibrium, aggregate allocations fluctuate without relying on exogenous shocks. Different endogenous fluctuations can arise depending on the synchronized pattern in the initial distribution. I explore this theoretical possibility in Section 1.5.

I found the model prediction of echo effects is empirically supported by macro-level data. First, I analyze echo effects from historical events that were followed by large aggregate TFP shocks. According to Ohanian (2001), aggregate TFP dropped by around 18% in the Great Depression. Using a Fisher g-test, I show that there was significant deterministic periodicity in the manufacturing industries’ investment growth rate in non-residential structures after the Great Depression. Similarly, after the oil crisis in 1979, the oil industry’s investment growth rate in structures displays significant deterministic periodicity. Second, I provide an evidence of echo effects from the impulse response of non-residential structure investment from BEA data. These empirical results validate nonlinear dynamics implied by the acceleration cost model.

Related literatures This paper contributes to the literature that studies how firm-level lumpy investments affect business cycle. Within this literature, Abel and Eberly (2002) empirically showed that there are statistically and economically significant nonlinearities in firm-level investments. They point out that it is necessary to track the cross-sectional distribution of firm-level investments to account for aggregate investment. Cooper et al.

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2. Under the calibrated baseline model, the echo effects die out after around 25 years from the point of an aggregate TFP shock. A permanent echo is generated under a parameterization with higher acceleration cost where large firms become extremely inelastic to the operating environment.
and Gourio and Kashyap (2007) found aggregate investment is largely driven by establishment-level capital adjustment in extensive margin. Especially, Cooper et al. (1999) found synchronized lumpy investments can generate echo effect of aggregate shocks in partial equilibrium. Gourio and Kashyap (2007) pointed out that if a fixed cost is drawn from a highly concentrated non-uniform distribution, aggregated lumpy investments show different impulse response than frictionless models in partial equilibrium. In contrast, Khan and Thomas (2008) found that lumpiness in investment at the establishment level is washed out after aggregation, due to strong general equilibrium effect.

In this paper, I empirically show that there are firm-level lumpy investments that are inelastic to interest rate dynamics, thus not smoothed out by changes in the interest rate after aggregation. Therefore, irrelevance result does not hold even in general equilibrium if a model captures those interest-inelastic firms. I conclude that large firms’ lumpy investments contributes to nonlinear aggregate fluctuations under the interest-inelasticity.

Relatedly, Koby and Wolf (2019) shows that observed dampening effect of factor price is not as strong as the implied level in models with fixed cost. Specifically, the paper empirically analyzed price-inelasticity of firm-level investments to the bonus depreciation stimulus policies. Despite the empirical findings, modeling a micro-level investment inelastic to price change still remains as a difficult task. House (2014) pointed out that a conventional model with fixed cost cannot capture inelastic lumpy investments due to strong general equilibrium effect; model-implied lumpy investments are highly price-elastic. To overcome this limitation in the fixed cost model, Bachmann et al. (2013) introduce maintenance and replacement investments under the high fixed cost parameter. In their model, micro-level lumpiness does not wash away after aggregation, leading to state-dependent sensitivity of aggregate investment in general equilibrium. However, the model in Bachmann et al. (2013) does not fit well to the firm-level lumpy investments because the implied level of persistence in the inaction durations is substantially lower than
the empirical level.\textsuperscript{3} Winberry (2018) included habit formation in the household’s utility function so that aggregate TFP sensitivity of real interest rate becomes counter-cyclical. Combined with convex adjustment cost, counter-cyclically responsive real-interest rate does not dampen aggregated lumpy investments over the business cycle.

Differently from these approaches, I introduce a convex acceleration cost in the model. This allows the model to capture large firms’ interest-inelastic lumpy investment timings and high persistence of inaction durations across periods. The latter is relatively less highlighted feature in the literature despite its important role in the aggregation. Specifically, high persistence of inaction durations across periods contributes to the persistent synchronization of firm-level lumpy investments in the post shock periods. This generates nonlinearity in the impulse response of the aggregate investment to an aggregate TFP shock that mimics echoes. In support of this theoretical prediction, I present evidence for the nonlinear investment dynamics from the macro-level data.

Second, this paper contributes to nonlinear business cycle literature. A large body of researches has focused on the nonlinearity in aggregate fluctuations that arise when heterogeneous agents are subject to micro frictions. Bachmann et al. (2013) found firm-level lumpiness in investments leads to pro-cyclical sensitivity of aggregate investments to an aggregate shock. Similarly, Berger and Vavra (2015) concludes lumpiness in households’ durable adjustment result in pro-cyclical responsiveness of aggregate durable expenditures to an aggregate shock. Fernandez-Villaverde et al. (2020) found that financial frictions can generate endogenous aggregate risk under the heterogeneous household model. In this setup, the aggregate allocations display state-dependent responsiveness to an aggregate TFP shock. Volatility shock in real interest rate studied in Fernandez-Villaverde et al. (2011) and uncertainty shock in Bloom et al. (2018) also lead to nonlinear aggregate

\textsuperscript{3} The conventional lumpy investment models’ implied persistence in inaction periods is around 0.65 ~ 0.7. In contrast, the level observed from data is around 0.9, and the acceleration cost model can match this level. I make more detailed comparison across the models in Section 1.4.3.
fluctuations. To this literature, this paper contributes by modeling interest-inelastic firm’s lumpy investments as an additional source of nonlinearity in the business cycle.

Third, this paper contributes to endogenous business cycle literature by generating aggregate fluctuations in general equilibrium without increasing-returns-to-scale technologies (Benhabib and Farmer, 1994; Farmer, 2016). After an aggregate TFP shock hits the economy, the equilibrium allocations fluctuate, forming echo patterns. This is due to persistently synchronized investment timings among interest-inelastic firms. Depending on the level of insensitivity to idiosyncratic shock process, this echo can be a decaying echo or a permanent echo. Both types of echoes are possible sources of endogenous fluctuations in an economy. I show that these echoes are empirically supported by statistically significant deterministic periodicity in the post-crisis period from macro-level data.

**Roadmap**

Section 1.1 empirically analyzes characteristics of lumpy investments for large and small firms. Based on the empirical analysis, Section 2.2 develops a business cycle model with heterogeneous firms subject to acceleration cost. In Section 2.3, I explain calibration used for this model. Using the model under the calibrated parameters, Section 3.4 quantitatively analyze nonlinear effect of lumpy investments in business cycle. In Section 1.5, endogenous aggregate fluctuations arising from a permanent echo effects are studied as a theoretical possibility an acceleration cost model can lead to. Section 1.6 suggests empirical evidence for nonlinearity in macro-level data. Section 3.5 concludes. Proofs and other detailed figures and tables are included in appendices.

### 1.1 Firm-level empirical analysis

For the firm-level empirical analysis, I use U.S. Compustat data. While Compustat data covers only public firms, its coverage is relatively less an issue in this analysis because the focus is on firms with large capital stocks. Throughout the whole empirical analysis,
large firms are defined as firms that hold capital stocks greater than the 90th percentile of the capital distribution in each industry of two-digit NAICS code. Sample period covers from 1980 to 2016. Firms with negative asset and zero employment are excluded from the sample. All the firm-level variables except capital stock and investment are deflated by GDP deflator. Investment is deflated by nonresidential fixed investment deflator available from National Income and Product Accounts data (NIPA Table 1.1.9, line 9). Firm-level real capital stock is obtained from applying perpetual inventory method to net real investment. Industry is categorized by the first two-digit NAICS code.

<table>
<thead>
<tr>
<th></th>
<th>Large</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aggregate Sales ($t bil.)</td>
<td>9007.2</td>
<td>4641.8</td>
</tr>
<tr>
<td>Aggregate Employment (1 mil.)</td>
<td>30.3</td>
<td>22.3</td>
</tr>
<tr>
<td><strong>Firm-level</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Avg. Sales ($1 mil.)</td>
<td>8143.9</td>
<td>332.2</td>
</tr>
<tr>
<td>Avg. Employment (1K)</td>
<td>28</td>
<td>1.7</td>
</tr>
<tr>
<td>Avg. Age after IPO</td>
<td>20.1</td>
<td>7.6</td>
</tr>
<tr>
<td>Num. Obs.</td>
<td>1111</td>
<td>13985</td>
</tr>
<tr>
<td><strong>Financial constraint</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Liability / Total Asset (%)</td>
<td>61.7</td>
<td>98.4</td>
</tr>
</tbody>
</table>

Table 1: Large and small firms’ summary statistics

Table 1 reports summary statistics for large and small firms during the sample periods. Under the given definition of large firms, around 60% of aggregate sales and employments belong to large firms. On average, large firms are 25 times greater than small firms in sales and employment. Large firms are on average old firms, having been listed around 13 years longer than small firms. Large firms’ ratio of total liability out of total asset is around 61.7%, and is smaller than the small firms’ fraction 98.4%. Thus, large firms are less financially constrained on average.

---

4. If only SIC code is available for a firm, I imputed NAICS code following online appendix D.2 of Autor et al. (2020). If both of NAICS and SIC are missing, I filled in the next available industry code for the firm.
1.1.1 Motivating facts

I define an investment spike as a firm-specific event where a firm makes a large-scale investment greater than 20% of the firm’s existing capital stock.\(^5\) I refer to this investment spike as a lumpy investment or capital adjustment in extensive margin, interchangeably. Throughout the empirical analysis, the fraction of firms making lumpy investments is the key variable. I define the key variable, spike ratio as follows:

\[
\text{Spike ratio}_{j,t} := \frac{\sum_{i \in j} \text{Investment spike}_{i,t}}{\text{# of } j\text{-type firms at } t}, \quad j \in \{\text{small}, \text{large}\}
\]

The numerator is counting all the investment spikes of firm type \(j \in \{\text{small}, \text{large}\}\) at time \(t\), and it is normalized by the total number of \(j\) type firms.

Figure 1 plots the time series of spike ratio of large firms. On average, 15.3% of large firms adjust their existing capital stocks in extensive margin in a year. As can be seen from Figure 1, since 1980 there have been only three periods (1980, 1998, and 2007) where the fraction of large firms making spiky investments surged beyond 20%. All three events were followed by recessions within two years.

Conversely, there were four recessions in the U.S. over the same periods, and three out of four recessions were preceded by the surge of large firms’ lumpy investments. The exception was the recession in 1990, and it was the mildest recession among the four recessions.

Table 2 summarizes the deviation of large and small firms’ spike ratios from the mean level in each year before the recessions.\(^6\) Before the recessions in 1981, 2000 and 2008, the spike ratio of large firms were greater than the average level by more than 25%. In contrast,

---

5. 20% cutoff is from non-convex adjustment cost literature including Cooper and Haltiwanger (2006), Gourio and Kashyap (2007), and Khan and Thomas (2008). If a firm’s acquired capital stock is greater than 20% of existing capital stock in a certain year, I do not count the year as the firm’s lumpy investment period.

6. \(\Delta \text{Spike}_j(\%)\) is obtained from demeaning and normalizing spike ratio by the mean separately for large and small firms.
Figure 1: Three surges of large firms’ lumpy investments preceded recessions

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \text{Spike}_{\text{Large}}$ (%)</td>
<td>53.53</td>
<td>3.70</td>
<td>26.84</td>
<td>33.49</td>
</tr>
<tr>
<td>$\Delta \text{Spike}_{\text{Small}}$ (%)</td>
<td>11.78</td>
<td>-0.20</td>
<td>16.97</td>
<td>7.34</td>
</tr>
</tbody>
</table>

Table 2: Deviation of spike ratios from mean before recessions

the spike ratio of small firms did not increase dramatically before each recession as shown in the second line of the table.

Relatedly, in the following analysis, I show aggregate investment rate is conditionally heteroskedastic on the average lagged spike ratio of large firms. That is, residualized volatility of aggregate investment rate is high if a great portion of large firms have made lumpy investments in the recent years.

For this analysis, I use aggregate data on non-residential investment (NIPA Table 1.1.5, line 9) and aggregate capital (Fixed Asset Accounts Table 1.1, line 4) from BEA. The thick line in Figure 2 plots logged estimates of standard deviation of residuals from autoregression of aggregate investment rates as a function of the recent average of large firms’ spike ratio. The recent average is based on the average spike ratio of past two years. As can be seen from this figure, aggregate investment rates are heteroskedastic

---

7. This empirical analysis is motivated from the conditional heteroskedasticity analysis in Bachmann et al. (2013) (Figure 1).
conditional on the lagged average spike ratio. Table A.1.1 reports the regression coefficients for the fitted line. According to the regression result, one standard-deviation increase (3.18%) in the large firms’ spike ratio is associated with 35% increase in the standard deviation of the residualized aggregate investments. From this result, I conclude that aggregate investments respond more strongly to an aggregate shock after a surge of lumpy investments of large firms. Consistent with the patterns in Figure 1, the three recession years of interest are located at the top-right corner in Figure 2.

Figure 2: Conditional heteroskedasticity of aggregate investments

I claim this is not a mere coincidence that the surges of large firms’ lumpy investments precede the recessions. I suggest a novel mechanism where an economy responds more strongly to a negative aggregate productivity shock after a surge of large firms’ lumpy investments based on the empirical findings at the firm level. The key mechanism is in the interest-inelastic investment timings of large firms. I empirically investigate the characteristics of large firms’ lumpy investments in the following sections.
1.1.2 Interest-inelasticity of large firms’ lumpy investments

In this section, I run a Vector Autoregression (VAR) to analyze different investment behavior of large and small firms when the interest rate changes. Then, using high-frequency monetary policy shocks, I estimate the heterogeneous interest-elasticity of investments in the extensive margin for large and small firms.

In the VAR, aggregate TFP, federal funds rate, and the fraction of large/small firms that make large-scale investments ($SpikeRatio_{jt}$) are included in the stated order; and one-period-ahead CPI is included as an exogenous control variable:\(^8\)

$$X_{j,t+1} = \Phi_0 + \Phi_1 X_{j,t} + \Phi_2 C_t + \epsilon_t \quad j \in \{Small, Large\}$$

$$X_{j,t} = [TFP_t, FedFund_t, SpikeRatio_{jt}], \quad C_t = \mathbb{E}_t CPI_{t+1} \cong CPI_{t+1}$$

Figure 3 plots impulse responses of the fractions of large (solid line) and small (dot-dashed line) firms adjusting capital stocks in extensive margin ($SpikeRatio_{jt}$) to an interest rate shock. Dashed line is 95% confidence interval of the estimated response.

Upon impact, there are no significant contemporaneous responses from large and small firms’ lumpy investments. However, in the following years, small firms display significant drop in the fraction of adjusting firms. Two years from the shock period, the response drops by around 1.3%. In contrast, the large firms’ response does not show any significant deviation from zero for the whole post-shock period. From this evidence, I claim large firms do not significantly change their lumpy investment timings in response to interest rate changes.

---

8. All the macro variables are at annual frequency and are HP-filtered with smoothing parameter 6.25 following Ravn and Uhlig (2002). In calculation of the fraction of investment spikes, a firm’s consecutive two investment spikes are considered as one spike. The impulse response is obtained from the orthogonalized VAR. The impulse variable is the federal funds rate. The lag order $p = 1$ is chosen by AIC criterion.
This is consistent with the finding of Cloyne et al. (2019) that the investment of large firms paying dividends are inelastic to interest rate changes.\textsuperscript{9} Similarly, Crouzet and Mehrotra (2020) found large firms are less cyclically sensitive than small firms.\textsuperscript{10} Also, there exists survey evidence that supports interest-inelasticity of firm-level investments. According to the survey results in Sharpe and Suarez (2013), 68\% of the respondent firms do not change their investment plan despite the interest rate drops.\textsuperscript{11} And almost 80\% of the respondent firms do not change their investment plans unless the interest rate jumps up more than 3\%. Considering the survey respondents are large firms that hire CFO for their financial management, the reported inelastic investments to interest rate change are consistent with the result of VAR analysis in this paper.

However, the VAR analysis does not rule out the possibility that other exogenous variations than TFP can simultaneously affect the spike ratio and the interest rate. Therefore, the result obtained from the VAR is about a correlation rather than a response to the pure interest rate change. For the sharp identification of heterogeneous interest-elasticity in the extensive margin, I construct an exogenous monetary policy shock following Jeenas\textsuperscript{9}. The result of Cloyne et al. (2019) combines both intensive and extensive margin responses, while the result in this paper singles out the response in extensive margin.\textsuperscript{10} According to Crouzet and Mehrotra (2020), this discrepancy between small and large firms is not driven by financial distress.\textsuperscript{11} The survey was conducted by Duke University and CFO magazine, and around 1,000 companies responded to the survey. Table A.1.2 summarizes the key results of the survey.
(2018) and Ottonello and Winberry (2020). The monetary policy shock is obtained by
time aggregating high-frequency monetary policy shock identified from the unexpected
jump (drop) in the federal funds rate during a 30-minutes window around the FOMC
announcement.\footnote{The result is robust over the choice of a wider window (one-hour window).}
To capture the unexpected component in the federal funds rate, I use the
change in the rate implied by the current-month federal funds futures contract. All the
data on the timings of the FOMC announcement and the high-frequency surprise are from
Gurkaynak et al. (2005) and Gorodnichenko and Weber (2016). The sample period covers
from March 1990 until December 2009. I follow the convention that the positive monetary
policy shock is an unexpected increase in the federal funds futures rate, so it implies the
contractionary monetary policy.

To match the data frequency between the firm-level data and the monetary policy shock,
I time aggregate the monetary policy shocks. Specifically, I compute the one-year backward
weighted average monetary policy shock at each firm’s financial yearend. The weight of
each surprise is determined by the number of days between the corresponding FOMC
announcement and the next FOMC announcement.\footnote{A higher weight is assigned for a monetary policy shock when there was greater amount of time for a firm to respond to the shock (Ottonello and Winberry, 2020).} If the next FOMC announcement
was made after the financial yearend, the days are counted until the financial yearend. By
this data joining process, a firm’s balance sheet information and the monetary policy shock
is matched at the same financial year. The weighted moving average monetary policy
shock is plotted in Figure A.2.1.

To study the heterogeneous firm-level investment responses in the extensive margin to
the monetary policy shock, I estimate the following probit regression separately for large
firms and small firms.

\[ \Pr(\text{spike}_{i,t}) = \beta MP_t + \alpha_i + \alpha_{s,t} + \Omega \text{Control}_{i,t} + \eta_{i,t}, \quad \eta_{i,t} \sim iid \ N(0, \sigma) \]
where $MP_t$ is the monetary policy shock, $a_i$ is the firm $i$ fixed effect, and $a_{s,t}$ is the sector-year fixed effect. The control variables include the current account and current liability normalized by total asset, log of total asset (size), and log of sales. The standard errors are two-way clustered across sectors and years.

<table>
<thead>
<tr>
<th>Dependent variable: $P(spike_{i,t})$</th>
<th>Large</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MP_t$</td>
<td>-0.0022</td>
<td>-0.0124</td>
</tr>
<tr>
<td></td>
<td>(0.0306)</td>
<td>(0.0063)</td>
</tr>
<tr>
<td>Observations</td>
<td>7,635</td>
<td>84,300</td>
</tr>
<tr>
<td>Psuedo $R^2$</td>
<td>0.0501</td>
<td>0.0511</td>
</tr>
<tr>
<td>Firm Fixed Effect</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Sector-year Fixed Effect</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Firm-level Control</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Two-way Cluster</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 3: Persistence in inaction durations

Table 3 reports the coefficient of the monetary policy shock in the probit regression separately for large and small firms with the standard errors in the bracket. In the estimated result, a contractionary (expansionary) monetary policy shock significantly reduces (increases) the probability of making large-scale investment for small firms while large firms stay unaffected. From the marginal effect analysis on the estimated probit regression, I find one basis point increase in the monetary policy shock (from zero) is associated with around 2% drop in the probability of making lumpy investments for small firms. The same variation in the monetary policy shock is associated with only negligible variation in the large firms’ investments in the extensive margin.

The interest-inelasticity of large firms’ investments in the extensive margin has an important macroeconomic implication in the aggregation of micro-level investments. Under the presence of interest-inelasticity, micro-level lumpiness does wash out after aggregation as the timings of lumpy investments are not smoothed by the interest rate changes over the business cycle. Therefore, the micro-level lumpiness leads to macro-level lumpiness.
after aggregation. This macro-level lumpiness generates nonlinear aggregate fluctuations in the economy. In the quantitative analysis section, using the heterogeneous-firm business cycle model, I analyze the role of interest-inelastic lumpy investments of large firms on the business cycle.

1.1.3 Insensitivity of large firms’ lumpy investments to idiosyncratic TFP shock

In this section, I show investments of large and small firms have different sensitivity in extensive margin to their idiosyncratic TFP shocks. For this empirical analysis, I measure the firm-level TFP following Ackerberg et al. (2015). The detailed steps for the firm-level TFP estimation are described in Appendix A.3.

Specifically, I implement an event study separately for large and small firms to study the response of capital adjustment in extensive margin to the shock in the firm-level TFP. The probit regression for the event study is specified as follows:

\[
\Pr(\text{spike}_{i,t}) = \sum_{\tau=-4}^{3} \beta_{\tau} \mathbb{I}\{\tau = t\} + \alpha_{industry} + \alpha_{year} + \epsilon_{i,t}, \quad \epsilon_{i,t} \sim iid \ N(0, \sigma)
\]

where \(\tau = 0\) is the event time, and \(\text{spike}_{i,t}\) is a binary variable indicating whether a firm makes a large-scale investment in extensive margin. The event is defined as a firm-specific year when an innovation in the firm-level TFP deviates more than one standard deviation from zero. The innovation in TFP (\(\text{TFP}_{\text{innovation}}\)) is obtained from the residuals after fitting the TFP process into AR(1) process:

\[
\text{TFP}_{i,t} = \rho \text{TFP}_{i,t-1} + \text{TFP}_{\text{innovation}}_{i,t}
\]

The periods of interest are from four years prior to the event until three years after the event. Full observations of eight years around the event (including the event year) are required to be included in the sample.
Each panel of Figure 4 plots the estimated coefficients $\beta_t$ of large and small firms (solid line) and its 95% confidence interval (dashed line) around the event time $\tau = 0$, for both positive event (panel (a) and (c)) and negative event (panel (b) and (d)). The dotted line is the time series of average idiosyncratic TFP across firms around the event.

As can be seen from panel (a) and (b), large firms’ extensive-margin adjustment does not significantly respond to idiosyncratic productivity shocks. In contrast, small firms display strong responsiveness to both positive and negative idiosyncratic productivity shocks as shown in panel (c) and (d). For a positive innovation in the firm-level TFP, the small firms’ probability of making large-scale investment jumps up by 10%. For the negative innovation in the firm-level TFP, the small firms’ probability of making large-scale investment drops by 14%.

Figure 4: Event study: sensitivity to idiosyncratic TFP innovation

For the robustness check, I estimate the firm-level TFP in two other ways: one is from Solow residuals and the other is from Olley and Pakes (1996). The results stay unchanged.
for these alternative TFP measures. The results based on the other two TFP measured are reported in Appendix A.4.

To sum up the results, the extensive-margin investments of small firms strongly respond to idiosyncratic productivity shocks. In contrast, large firms’ extensive-margin adjustments do not strongly respond to idiosyncratic productivity shocks. This insensitivity possibly comes from difficulty of catching a sudden investment opportunity for large firms relying on capital budgeting for their internal resource allocations. The detailed discussion for why this insensitivity arises will be made in section 1.1.5.

Heterogeneous sensitivity to idiosyncratic TFP shock is important in aggregate investment dynamics because it determines an allocation’s speed of reversion to a steady-state level. If an aggregate TFP shock hits an economy, the distribution of micro-level allocations departs from the stationary distribution. Then, large firms with low sensitivity to an idiosyncratic shock converge slower to the stationary allocation than small firms do.

1.1.4 Firm-level persistence in inaction duration

In this section, I document large firms’ inaction duration of capital adjustment is highly persistent across periods.

Figure 5 plots the distributions of inaction periods between neighboring spiky investments for large and small firms.¹⁴ Large firms’ inaction durations are longer than small firms’ inaction durations on average by around a year and have a fatter tail in the distribution.

To study underlying regularity in the lumpy investment timings at the firm level, I compare each firm’s inaction duration with the lagged inaction duration. Specifically, I check how well aligned the inaction duration and the lagged duration are along the 45-degree line in a scatter plot. To this end, I fit the inaction duration into autoregressive

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¹⁴. Consecutive investment spikes are assumed to have no inaction periods.
process. Table 4 reports the AR(1) regression results for logged inaction periods ($t2Inv$) of large and small firms. Inaction duration ($t2Inv$) at $t$ is defined as a time interval (in years) between the spike at period $t$ and the most recent investment spike. The numbers in the bracket are the standard errors. Both types of firms have fairly high persistence in the inaction durations. The level of persistence is even higher than the measured persistence in the firm-level productivity shocks that is around 0.58 (Bachmann and Bayer, 2013).  

<table>
<thead>
<tr>
<th>Dependent variable: $log(t2Inv_{i,j})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
</tr>
<tr>
<td>$log(t2Inv_{i,j-1})$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Large - Small</td>
</tr>
<tr>
<td>($p$-value)</td>
</tr>
<tr>
<td>Observations</td>
</tr>
</tbody>
</table>

Table 4: Persistence in inaction durations

15. The implied level of average persistence in inaction duration in Khan and Thomas (2008) is around 0.7. Calibration used in Gourio and Kashyap (2007) gives slightly higher persistence around 0.75, but it does not achieve the observed high persistence level in the data.
To summarize the results, the inaction durations are highly persistent across periods for both large and small firms. The high persistence in the inaction duration has an important macroeconomic implication for aggregate investment dynamics: once firms’ investment timings are synchronized, the timings are persistently synchronized. Also, it takes long time for firms to revert back to the stationary distribution once they deviate from it.

1.1.5 Why some firms are insensitive? Capital budgeting

*It [i.e., capital budgeting] sucks the energy, time, fun, and big dreams out of an organization. It hides opportunity and stunts growth. It brings out the most unproductive behaviors in an organization, from sandbagging to settling for mediocrity.*

— Jack Welch, General Electric

In this section, I suggest a possible economic explanation for insensitivity of large firms’ lumpy investments to interest rate changes and investment opportunities. A chief financial officer (CFO) of a firm faces complicated inflow and outflow of capital during the firm’s operation. Thus, having the capital flow under complete control is one of the most important things to do for the position. For this, most of CFO’s rely on capital budgeting for their decision on capital allocation and investment plan for longer horizon than a year. According to a survey conducted by Ekholm and Wallin (2000) towards 650 Finnish companies with a turnover greater than 16.7 million euros, 86% of respondents answered they use annual capital budgeting. In the survey conducted by Ryan and Ryan (2002) towards Fortune 1000 companies, they found 99.5% of the respondents answered they use capital budgeting. Likewise, capital budgeting is a universal tool for CFO’s to plan and evaluate an investment project.

However, as pointed out in the quote from Jack Welch, a former CEO of General Electric, the budgeting process often involves inflexibility that possibly leads to decision lags. Yang et al. (2020) showed how structured decision making such as capital budgeting
can affect the decision lags. According to Yang et al. (2020), CEOs at large firms make a
decision based on significantly more structured style than CEOs at small firms. In their
estimates, a one standard deviation increase in the score of structured style is associated
with a 1.92-fold increase in firm size. Then, they found structured decision-making process
takes longer time than unstructured (intuition-driven) process. A one standard deviation
increase in the score of structured style is associated with 28% longer time required to
reach a decision. This indicates that large firms tend to display decision lags on average
due to their structured style of decision making. This does not imply the structured
decision making is inefficient. Rather, Yang et al. (2020) points out that the structured
style helps making a greater number of decisions than the unstructured style. It could be
understood as large firms adopt the structured style due to a great number of issues to
deal with, and this leads to decision lags. This gives an explanation on why large-scale
investment timings of large firms are insensitive to a change in the real interest rate.

In a survey from Duke University/CFO Magazine Business Global Outlook completed
by around 800 CFO’s of the U.S. companies reported by Sharpe and Suarez (2013), 89% of
respondents answered they would not change investment plan despite more than 3
percentage point decrease in the interest rate.\(^\text{16}\) The reason for interest-inelasticity is
summarized in Table 5. Among those who answered that their reasons are non-financing
related, nearly half of CFO’s answered that it is because their investment plans are set on
long-term basis.\(^\text{17}\) This answer shows inflexible lumpy investment timings due to long-run
planning horizon in capital budgeting process. The second largest group of CFO’s chose
lack of profitable investment opportunities as a reason for their interest-inelastic investment
plans. This can be also attributed to a limitation in capital budgeting practice that might
have masked good opportunities according to Jagannathan et al. (2016).

---

\(^{16}\) 80\% of respondents answered they would not change investment plan despite more than 3 percentage point increase in the interest rate.

\(^{17}\) For the financing related reasons, 32\% of all respondents answered they are interest-inelastic because their firms are financially unconstrained, and 27\% of all respondents answered that it is because their hurdle rate from capital budgeting is already higher than interest rate. Around 35\% of respondents chose non-financing related answers.
Q. Reasons for not changing investment plan despite the interest rate change (among respondents who chose non-financing related answers)

<table>
<thead>
<tr>
<th>Reasons</th>
<th>Despite price drop</th>
<th>Despite price jump</th>
</tr>
</thead>
<tbody>
<tr>
<td>Based on long-term plan, not current rates</td>
<td>49%</td>
<td>47%</td>
</tr>
<tr>
<td>Lack of profitable opportunities</td>
<td>29%</td>
<td>31%</td>
</tr>
<tr>
<td>High uncertainty</td>
<td>9%</td>
<td>3%</td>
</tr>
<tr>
<td>Firm is not capital intensive / Other</td>
<td>14%</td>
<td>19%</td>
</tr>
</tbody>
</table>

Table 5: CFO survey results (Sharpe and Suarez, 2013): inelasticity to interest rate changes

Consistently, Ekholm and Wallin (2000) found from their survey that “incapability of signaling changes in the competitive environment” is the most agreed problem among CFO’s in annual capital budgeting convention. This incapability makes firms insensitive not only to price fluctuations, but to investment opportunities. Then, it is natural that the firm sticks to their convention of investment routines, displaying high persistence in inaction periods.\(^{18}\)

The insensitivity to competitive environment including investment opportunity and interest rate change, is not only an issue to CFO’s in large firms. It matters also for the whole economy, as it leads to nonlinear dynamics of aggregate investments once aggregated. In the next section, I model this firm-level insensitivity by introducing a technological restriction that disturbs a nimble reaction to changes in operating environments including idiosyncratic productivity and interest rate.

### 1.2 Model

I develop and analyze a heterogeneous-firm real business cycle model that captures the empirical findings of this paper.

\(^{18}\) The surveys I included in this section did not explicitly distinguish large and small firms except for Yang et al. (2020). However, all the respondents are CFO’s of firms. Assuming firms that hire CFO are on average large firms, the evidence supports the claim of this paper.
In the model, time is discrete, and lasts forever. There is a continuum of measure one of firms that own capital, produce business outputs, and make investment. The business output can be reinvested as capital, after a firm pays adjustment costs.

1.2.1 Technology

A firm owns capital. It produces a unit of goods that can be converted to a unit of capital after an adjustment cost. The production technology is a Cobb-Douglas function with decreasing returns to scale:

\[ z_t A_t f(k_t, l_t) = z_t A_t k^\alpha_t l^{1-\alpha}_t \]

where \(k_t\) is capital input; \(l_t\) is labor input; \(z_t\) is idiosyncratic productivity; \(A_t\) is aggregate TFP, and \(\alpha + \gamma < 1\). Idiosyncratic productivity \(z_t\) and aggregate TFP \(A_t\) follow the stochastic processes as specified below:

\[
\ln(z_{t+1}) = \rho_z \ln(z_t) + \epsilon_{z,t+1}, \quad \epsilon_{z,t+1} \sim iid \ N(0, \sigma_z)
\]

\[
\ln(A_{t+1}) = \rho_A \ln(A_t) + \epsilon_{A,t+1}, \quad \epsilon_{A,t+1} \sim iid \ N(0, \sigma_A)
\]

where \(\rho_i\) and \(\sigma_i\) are persistence and standard deviation of \(i.i.d\) innovation in each process \(i \in \{z, A\}\), respectively. Both of stochastic processes are discretized using the Tauchen method for computation.

Investment stage policy

I assume a large-scale investment could be made only after \(\bar{s} > 0\) investment stages are completed, and accelerating completion of stages takes time and costs. \(\bar{s}\) could be interpreted as a number of bureaucratic steps in capital budgeting, such as meetings of the
board of directors for the large-scale investment decisions.\textsuperscript{19} From now on, I describe the model without time subscript for the simpler notation. Instead, a future period’s allocation is marked with a prime. Without a prime, the variable is for the current period. Due to the recursive nature of the problem, my model can be fully characterized without time index.\textsuperscript{20}

In the beginning of a period, a firm is given with the number of completed stages $s$. I assume $s$ takes discrete nonnegative integer value.\textsuperscript{21} If $s = \bar{s}$, the firm reached at the completion period of the large investment. A manager chooses the number of stages $b > 0$ to process within the current period. $s' = s + b$ is the number of total stages completed by the end of the current period. $b = 0$ implies no change in the given stage $s$. After a large investment is made, I assume the stage starts again from stage 1. Thus, the future stage $\tilde{s}'$ is equivalent to $s'$ such that $\tilde{s}' \equiv s'$ (mod $\bar{s}$).\textsuperscript{22}

Completion of one stage per period does not incur a cost. However, completion of multiple stages in a period entails convexly increasing cost, which I name as acceleration cost, specified in the following form:

\[
\text{(Acceleration Cost)} \quad acc(s', s, k) := \mathbb{I}\{s' > s + 1\} \left(\frac{\mu^a}{2} (s' - s - 1)^2\right) k^2
\]

where $s'$ is the targeted future stage, and $\mu^a$ is the acceleration cost parameter. The timing of capital adjustment in extensive margin has been only implicitly determined in the models with fixed cost. In contrast, firms that are subject to acceleration cost explicitly determine the optimal timing of lumpy investment. If a firm faces higher acceleration cost, a firm’s nimble capital adjustment is costly. Thus, it becomes less sensitive to surrounding economic environment such as interest rate changes. To capture large firms’ inelastic

\textsuperscript{19} In a framework of the optimal internal capital allocation studied in Malenko (2019), this could be understood as an auditing process for large-scale investment project.
\textsuperscript{20} With time index, the notation in the model can become highly complicated due to coexistence of calendar time and planning horizon.
\textsuperscript{21} All the results are unaffected in the choice of discrete or continuous stage assumptions.
\textsuperscript{22} According to this notation, $s = 0$ is equivalent to $s = \bar{s}$.
capital adjustment observed from data, I assume that 1) acceleration cost convexly increases over the size of a firm, and 2) hazard rate decreases over firm size. Therefore, large firms face large acceleration cost. The hazard rate is explained later more in detail.

Firms that face extremely high acceleration cost will behave as if they are strictly bound by timing constraints. For these firms, the acceleration cost imposes a similar restriction as time-to-build or time-to-plan constraints (Kydland and Prescott, 1982).

**Stage-contingent investment**

When the stage is incomplete, $s' \leq \bar{s}$, I assume a firm can invest/disinvest only a small portion of the owning capital stock, following Khan and Thomas (2008) and Winberry (2018). This is also a similar setup as Malenko (2019), where the optimal allocation of capital within a firm follows a threshold rule. According to the paper, divisional managers are allocated with a discretionary account below a threshold in the optimal budgeting. Large-scale investments beyond the threshold needs to be audited by headquarters. The costly auditing process is equivalent to the acceleration cost in this paper.

A firm’s capital stock evolves in the following law of motion if $s' \leq \bar{s}$:

$$k' = (1 - \delta)k + I, \quad I \in \Omega(k) := [-vk, vk]$$

where investment entails a convex adjustment cost $c(k, I) = \frac{\mu}{2} \left( \frac{I}{k} \right)^2 k$ as in Winberry (2018). The convex adjustment cost is considered to mitigate intensive-margin elasticity of firm-level investment to the interest rate change. Note that investment is restricted to $[-vk, vk]$, and $0 < v < \delta$. In this setup, a firm’s capital stock does not reach a steady-state, and a firm’s investment follows $(S, s)$ rule in the optimal policy.
When the stage is complete, \( s' > \bar{s} \), a firm can make a large-scale investment or disinvestment. Thus, a firm’s capital stock evolves in the following law of motion if \( s' > \bar{s} \):

\[
k' = (1 - \delta)k + I, \quad I \in (-\infty, \infty)
\]

where investment entails a convex adjustment cost \( c(k, I) = \frac{\mu}{2} \left( \frac{I}{k} \right)^2 k \).

### Hazard function

I introduce hazard function that determines exit rate for firms. According to Clementi and Palazzo (2016), the exit rate exponentially decreases as a firm grows older. In my model, age is not explicitly considered as a state variable. However, by introducing an exponentially decreasing hazard function over firm size proxied by capital stock \( k \), old firms are large on average, consistent with the empiric observation. On top of this, I assume that exiting firms are replaced by the same new firms. This is to purely focus on lumpy investments’ role on aggregate fluctuations without heterogeneous entry and exit over business cycle. The assumed functional form of hazard function \( h \) is as follows:

\[
h(k) := \bar{h} \ast \left( 1 + \frac{1}{\exp(k)} \right)
\]

where \( \bar{h} \) is the parameter that determines the entire level of exit rate. I calibrate this parameter by matching the average exit rate 6.2% in Clementi and Palazzo (2016).

#### 1.2.2 Timing decision for large-scale investment: intensive margin in the extensive margin

Given the acceleration cost, a firm’s timing decision for large-scale investment becomes substantially different from the one in the model with fixed cost. In the latter, firm-level large-scale investment is a binary decision to make it today or not. In contrast, in the
model of acceleration cost, there is an additional dimension: an intensive margin in the extensive margin. On top of the decision on whether to make a large-scale investment today or not (extensive margin), a firm needs to decide how further to go with respect to investment stages (intensive margin in the extensive margin).

If a firm processes multiple stages today, a firm can reach a better stage in the future for a large-scale capital adjustment. However, the trade-off is convexly increasing acceleration cost. Regardless of whether a firm makes a large-scale investment today or not, this decision is necessary in every period. This captures the long-run horizon of investment plans consistent with the survey results in section 1.1.5.

The investment timing decision can be summarized as the following problem. To capture the core mechanism, I assume the small-scale investment is zero ($v = 0$) and the hazard rate is zero in this formulation. Given a firm’s value function $J(k, z, s)$ where $k$ is capital stock; $z$ is firm-level idiosyncratic productivity; and $s$ is the number of stages completed, the investment timing decision is as follows:

$$
\max \left\{ \max_{s > \bar{s}} \left[ -I^* - \frac{1}{1 + r} \mathbb{E} J \left( k(1 - \delta) + I^*, z', s' \mod \bar{s} \right) \right], \right. \\
\left. \max_{\bar{s} \leq s} \left[ -\frac{1}{1 + r} \mathbb{E} J \left( k(1 - \delta), z', s' \mod \bar{s} \right) \right] \right\}
$$

where $I^*$ stands for the optimal large-scale investment; $r$ and $w$ are interest rate and wage.

In this formulation, a firm first decides whether to make a large-scale investment today ($s' > \bar{s}$) or not ($s' \leq \bar{s}$). This decision problem is the choice between payoffs from the first and the second line. Then, the firm needs to decide how many stages to process given the tradeoff between acceleration cost and the value gain from future investment stage. In this decision, an acceleration of investment stage does not give a flow payoff to the firm in the next period. Instead, it guarantees a better capital adjustment stage in the next period. In this regard, the model with acceleration cost captures a firms’ long-run preparation steps for investments.
Therefore, the nature of a firm’s problem is starkly distinguished from the problem in the models with fixed cost. In the models with fixed cost, firms determine whether to make a large-scale investment in the current period or not. In this decision making, large-scale investment does not require a preparation step, so firms respond more sensitively in extensive margin to interest rate changes.

Specifically, in the acceleration cost model, the spike ratio for firms greater than a size threshold $\bar{k}$, is as follows:

$$\text{SpikeRatio}(\bar{k}) = \int \mathbb{I}\{s'(k, z, s) > \bar{s}\} \frac{\mathbb{I}\{k > \bar{k}\} I(k, z, s)}{\Phi(k > \bar{k})} > 0.2 \right\} d\Phi$$

where the first indicator function specifies the condition that firms complete the whole investment stages. The second indicator specifies the firm size requirement, and the third indicator is for investment size requirement. The given distribution of each firm’s individual state $(k, z, s)$ is denoted by $\Phi$. For brevity, I define $M(k, z, s)$ as the product of the last two indicator functions.

Firms that satisfy the condition in the first indicator function $s' > \bar{s}$ can be categorized into two groups: 1) firms that are ready for a large-scale investment ($s = \bar{s}$) and 2) firms that accelerate the stages for a large-scale investment ($s' < \bar{s}$ and $s' > \bar{s}$).

$$\text{SpikeRatio}(\bar{k}) = \int \left( \mathbb{I}\{s = \bar{s}\} + \mathbb{I}\{s < \bar{s}\} \mathbb{I}\{s'(k, z, s) > \bar{s}\} \right) M(k, z, s)d\Phi$$

For the notational brevity, I denote the expected value when a firm makes a large-scale investment as $\mathbb{E}J$ and the expected value when a firm does not make a large-scale invest-

---

23. Consistent with the empirical section, I define spike ratio as the fraction of firms making investment greater than 20% of existing capital stock.
ment as $\mathbb{E}J'$. For firms that accelerate for their large-scale investment, the marginal benefit of acceleration is greater than the marginal cost from the acceleration:

$$\frac{1}{1+r} (\mathbb{E}J - \mathbb{E}J') - I^* > \frac{\mu_{acc}}{2} (\bar{s} - s)k^2$$

The marginal benefit $\frac{1}{1+r} (\mathbb{E}J - \mathbb{E}J')$ is greater than the marginal cost $\frac{\mu_{acc}}{2} (\bar{s} - s)k^2$.

Therefore, the spike ratio could be formulated as follows:

$$SpikeRatio(k) = \int \left( \mathbb{I}\{s = \bar{s}\} + \mathbb{I}\{s < \bar{s}\} \mathbb{I}\left\{ \frac{1}{1+r} (\mathbb{E}J - \mathbb{E}J') - I^* > \frac{\mu_{acc}}{2} (\bar{s} - s)k^2 \right\} \right) Md\Phi$$

The first term in the bracket is invariant over the contemporaneous interest change. As $k$ increases, a mass of firms that accelerate decreases. It is because large capital stock $k > \bar{k}$ makes the marginal cost of acceleration greater than the marginal benefit for a great portion of the firms. Therefore, for large firms, the spike ratio is dominantly driven by firms that are already at the last stage for their lumpy investments, so it is highly interest-inelastic in the model with acceleration cost. On the other hand, the spike ratio in the fixed cost model becomes highly responsive to the interest rate change regardless of the size as formulated in Appendix A.5.

### 1.2.3 Nonlinear size effect on interest-inelasticity

In the model, the acceleration cost is assumed to convexly increase in the size of a firm’s capital stock. In this section, I study whether this convexity assumption is empirically supported from the data.

Figure 6 illustrates the stationary distribution of the capital stocks in the model and interest-inelasticity in the thick curve. Due to convexly increasing acceleration cost in size,
the interest-inelasticity of a firm’s investment timing convexly increases. Hence, the model predicts that medium sized firms and small sized firms are not distinguishable in terms of their interest-inelasticity. This is an empirically testable model implication. So, I set two cutoffs $\bar{k}_0$ and $\bar{k}_1$ in the capital distribution to define small and medium firms. Specifically, I set $\bar{k}_0$ as the 50th percentile and $\bar{k}_1$ as the 80th percentile of capital distribution for each two-digit NAICS industry. Thus, small firms are the firms holding capital stock $k$ such that $k < \bar{k}_0$, and medium sized firms are the firms holding capital stock $k$ such that $\bar{k}_0 < k < \bar{k}_1$.

![Figure 6: Capital distribution and interest-inelasticity in the model](image)

Then I run the same VAR analysis as in the section 1.1.2 for small and medium size firms for an interest rate shock. Figure 7 plots the impulse responses of the spike ratio of small and medium firms. Both of the firms display significant drops in the spike ratios, and the difference between two responses are statistically insignificant. The responses are starkly different from the inelastic response of the large firms as shown in Figure 3. From this evidence, I claim the acceleration cost’s convexity in capital size is empirically supported.
1.2.4 Firm’s problem: recursive formulation of baseline model

A firm is given with capital $k$, an idiosyncratic productivity $z$, and the number of completed stages $s$ in the beginning of a period. Also, they are given with the knowledge on the contemporaneous distribution of firms $\Phi$ and the aggregate TFP level $A$. For each period, firm determines investment level $I$, labor demand $l_d$, and when to make a large investment by choosing next period’s investment stage $s'$. A manager of a firm can decide either to get closer to the larger investment period ($s' > s$) or delay ($s' = s$) the process. A firm’s problem is formulated in the following recursive form:

$$J(k, z, s; \Phi, A) = \pi(z, k; \Phi, A) + \max \{ $$

$$\max_{s' > s, I} \{-I - c(k, I) - acc(s', s, k)w(\Phi, A) + \frac{1 - h(k)}{1 + r(\Phi, A)} \mathbb{E}J(k', z', s' (\text{mod } 5); \Phi', A') \},$$

$$\max_{s \leq s' \leq 5, I \in \Omega(k)} \{-I^c - c(k, I^c) - acc(s', s, k)w(\Phi, A) + \frac{1 - h(k)}{1 + r(\Phi, A)} \mathbb{E}J(k^c', z', s'; \Phi', A') \} \}$$

Figure 7: Impulse response of spike ratio for small and medium firms
(Operating Profit) $\pi(z, k; \Phi, A) := \max_{l_d} z A k^a l_d^\gamma - w(\Phi, A) l_d$ ($l_d$: labor demand)

(Convex Adjustment Cost) $c(k, I) := \frac{\mu^I}{I} \left( \frac{1}{k} \right)^2 k$

(Acceleration Cost) $acc(s', s, k) := \left[ \mathbb{I}\{s' > s + 1\} \left( \frac{\mu^a}{2} (s' - s - 1)^2 \right) \right] k^2$

(Constrained Investment) $I^* \in \Omega(k) := [-kv, kv]$ ($v < \delta$)

(Aggregate Law of Motion) $\Phi' := H(\Phi, A), A' = G_A(A)$ (AR(1) process)

(Hazard rate) $h(k) := \bar{h} \ast \left( 1 + \frac{1}{\exp(k)} \right)$

(Idiosyncratic Law of Motion) $z' = G_z(z)$ (AR(1) process)

where $J$ denotes the value function of a firm; $l_d$ is a labor demand; $w$ is wage; $r$ is real interest rate; $c(k, I)$ is a convex adjustment cost, and $acc(s', s)$ is an acceleration cost. $z$ and $A$ are idiosyncratic and aggregate productivities, respectively. The prime in superscript of each variable indicates that the variable is for the next period.

1.2.5 Household

A stand-in household is considered. The household consumes, supplies labor, and saves. In the beginning of a period, the household is given with wealth level $a$, information on the contemporaneous distribution of firms $\Phi$, and the aggregate TFP level $A$. The household problem is as follows:

$$V(a; \Phi, A) = \max_{c, a', l_H} \log(c) - \eta l_H + \beta \mathbb{E} A' V(a'; \Phi', A')$$

s.t. $c + \frac{a'}{1+r(\Phi, A)} = w(\Phi, A) l_H + a$

$G(a, \Phi) = \Phi'$

$G_A(A) = A'$

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where $V$ is the value function of the household; $a$ is a current saving level; $\Phi$ is a distribution of firms; $A$ is an aggregate productivity; $c$ is consumption; $a'$ is a future saving level; $l_H$ is labor supply; $w$ is wage, and $r$ is real interest rate. Household is holding the equity of firms as their asset. Following Bachmann et al. (2013) and Khan and Thomas (2008), I assume labor supply is indivisible.

### 1.2.6 Cyclical competitive equilibrium

I define cyclical competitive equilibrium that conceptually extends conventional stationary recursive competitive equilibrium. This equilibrium includes aggregate allocations’ stationary cycle as a possible equilibrium outcome. When the length of stationary cycle’s period is one, the cyclical competitive equilibrium collapses to a stationary recursive competitive equilibrium. A stationary endogenous cycle is one theoretical possibility an acceleration cost model can lead to when a group of firms’ lumpy investment timings are independent from idiosyncratic stochastic process. The stationary cycle in cyclical competitive equilibrium will be studied in Section 1.5. I provide a version without aggregate uncertainty. The extension to a stochastic version is not different from an extension of the stationary competitive equilibrium to the recursive competitive equilibrium. The cyclical competitive equilibrium is defined as follows.

**Definition 1** (Cyclical competitive equilibrium).

$\left(g_c, g_a, g_{lH}, g_k, g_l, g_b, V, J, G, r, w, \Phi, n^*\right)$ are cyclical general equilibrium if

1. $g_c, g_a, g_{lH}, V : \mathbb{R} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, solve the household’s problem. Note that $\mathcal{D}$ is a set of all probability measures $\Phi$ defined on the cartesian product of the sigma algebras $\mathcal{K} \times \mathcal{Z} \times \mathcal{S}$ generated from $(\mathcal{K}, \mathcal{Z}, \mathcal{S})$.

2. $g_k, g_l, J : \mathcal{K} \times \mathcal{Z} \times \mathcal{S} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, $g_b : \mathcal{K} \times \mathcal{Z} \times \mathcal{S} \times \mathbb{R} \to \{0, 1, 2, \ldots\}$ solve a firm’s problem.
3. Define $g_s : K \times Z \times S \times D \times R \rightarrow S$, s.t. $g_s(k, z, s; \Phi) = s + g_b(k, z, s; \Phi)$.

$$(g_k, g_s)(\Phi)(k, z, s) := \int_{K \times Z \times S} \left( \int_{\mathbb{Z}} \Gamma_{z,s} dz' \right) \mathbb{I}\{g_k(k, z, s) \in K\} \mathbb{I}\{g_s(k, z, s) \in S\} d\Phi(k, z, s)$$

for any set $(k, z, s)$ in the $\sigma$-algebra $(K, Z, S)$ generated from the domains $(K, Z, S)$ and $(g_k, g_s)^n(\Phi) = (g_k, g_s)((g_k, g_s)^{n-1}(\Phi))$, for any $n \in \{1, 2, 3, \ldots\}$, and $\Phi \in \mathcal{D}$.

There exist $n^* \geq 1$, and $\Phi_0 \in \mathcal{D}$, s.t. $(g_k, g_s)^{n^*}(\Phi_0) = \Phi_0$.

And define $\Phi_n := (g_k, g_s)^n(\Phi_0)$ for $n \in \{0, 1, 2, \ldots, n^* - 1\}$.

4. Market Clearing: for $\forall n \in \{0, 1, 2, \ldots, n^* - 1\}$

   (Labor Market) $g_{!H}(a; \Phi_n) = \int g_id\Phi_n$

   (Equity Market) $a = \int J(k, z, s; \Phi_n)d\Phi_n$

5. Consistency Condition:

   $\Phi' = G(a, \Phi) = (g_k, g_s)(\Phi)$, for $\forall \Phi \in \mathcal{D}$

It is worth to note that the length of the equilibrium cycle $n^*$ is an endogenous equilibrium object in the definition. When $n^* = 1$, the equilibrium allocations are at a stationary point. If $n^* > 1$, the equilibrium allocations form a stationary cycle. The markets are required to clear for entire $n^*$ periods within a cycle.

For convenient computation, I use a technique in Khan and Thomas (2008) that solves a firm’s problem with normalized value function $\bar{J}$ instead of $J$, where $\bar{J}(\cdot; \Phi, A) := p(\Phi, A)J(\cdot; \Phi, A)$ and $p(\Phi, A) = u'(c(\Phi, A))$. Then, wage and real interest are simultaneously determined by dynamics of $p(\Phi, A)$. Therefore, $p(\Phi, A)$ is the only price to be computed in the outer loop.
Under the aggregate uncertainty, stochastic general equilibrium is hard to compute due to two problems: 1) infinite dimension of state variable $\Phi$, and 2) nonlinear dynamics in aggregate allocations and prices. Due to the latter concern, the celebrated algorithm of Krusell and Smith (1998) is not helpful in the computation of stochastic general equilibrium.

To overcome this difficulty, I use a computation method called repeated transition method which I am concurrently developing in Lee (2020a). This method can solve heterogeneous agent model under aggregate uncertainty without relying on parametric form of the law of motion. I elaborate the method in section 1.4.4.

### 1.3 Calibration

The core parameters to be calibrated are acceleration cost and adjustment cost parameters. All the parameters are set at the level that matches simulated moments with target moments except for the parameters of firm-level idiosyncratic productivity process. I fix non-core parameters at the reasonable level consistent with the literature. The labor supply parameter $\eta$ is set at the level that gives labor participation rate around 60%. The fixed parameters are summarized in Table A.6.3.
### Table 6: Fitted Moments

<table>
<thead>
<tr>
<th>Target Moments</th>
<th>Data</th>
<th>Model</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Persistence of inaction periods</td>
<td>0.88</td>
<td>0.88</td>
<td>Compustat data</td>
</tr>
<tr>
<td>Average inaction periods (years)</td>
<td>5.98</td>
<td>5.54</td>
<td>Compustat data</td>
</tr>
<tr>
<td>Cross-sectional average of $i_t/k_t$ ratio (%)</td>
<td>10.4</td>
<td>9.9</td>
<td>Zwick and Mahon (2017)</td>
</tr>
<tr>
<td>Cross-sectional dispersion of $i_t/k_t$ (s.d.)</td>
<td>0.16</td>
<td>0.15</td>
<td>Zwick and Mahon (2017)</td>
</tr>
<tr>
<td>Cross-sectional average spike rate (%)</td>
<td>14.4</td>
<td>17.7</td>
<td>Zwick and Mahon (2017)</td>
</tr>
<tr>
<td>Cross-sectional average hazard rate (%)</td>
<td>6.20</td>
<td>7.23</td>
<td>Clementi and Palazzo (2016)</td>
</tr>
<tr>
<td>Autocorrelation of $Y_t$</td>
<td>0.94</td>
<td>0.90</td>
<td>NIPA data (Annual)</td>
</tr>
<tr>
<td>$sd(i_t)/sd(Y_t)$</td>
<td>1.98</td>
<td>1.79</td>
<td>NIPA data (Annual)</td>
</tr>
</tbody>
</table>

First, I estimated parameters for firm-level idiosyncratic productivity process outside of the model from Compustat data. The detailed steps for firm-level TFP estimation is explained in Appendix A.3. Using the estimated firm-level TFP, I run a pooled autoregression, and the parameters of the autoregressive process are $\rho = 0.55$ and $\sigma = 0.18$. In the computation, the idiosyncratic process is discretized using the Tauchen method with 7 grid points.

Then, I calibrate parameters from stationary equilibrium allocations. Matching the average inaction period of 5.98 years, I calibrate the required number of stages $s = 4$. From the average persistence of inaction duration 0.88, the acceleration cost parameter is set as $\mu^{acc} = 0.052$.

Zwick and Mahon (2017) summarizes statistics on firm-level investment rates using IRS data. I used the empirical moments reported in Appendix B.1 in Zwick and Mahon (2017) as the target moments for investment rates. From the average investment rate 14.4%,

---

24. Persistence of inaction duration is autoregression coefficients of inaction duration obtained from U.S. Compustat data.
I set $\mu^I = 0.580$. From the average spike rate ($\%$), the small investment range parameter $\nu = 0.030$ is calibrated. The hazard rate parameter $\bar{h} = 0.0565$ is identified from average exit rate, and the level is matched to 6.2% as studied in Clementi and Palazzo (2016). I found this cross-sectional parameter setup gives a close match in an untargeted moment: the cross-sectional dispersion of investment rate.

Aggregate moments are matched in the dynamic stochastic general equilibrium. From the autocorrelation of output obtained from BEA data, I calibrate the autocorrelation parameter of the aggregate TFP process $\rho_A = 0.8145$. From the volatility of private domestic investment relative to output volatility, I set the aggregate TFP volatility parameter $\sigma_A = 0.027$. Based on these parameters, the aggregate TFP process is discretized using the Tauchen method with 5 grid points.

The fitted moments are summarized in Table 6, and the fitted parameters are reported in Table 7.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^{acc}$</td>
<td>Baseline acceleration cost</td>
<td>0.052</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>Investment completion stage</td>
<td>4</td>
</tr>
<tr>
<td>$\mu^I$</td>
<td>Baseline adjustment cost</td>
<td>0.580</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Small investment range</td>
<td>0.030</td>
</tr>
<tr>
<td>$h$</td>
<td>Hazard rate</td>
<td>0.0565</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>Standard deviation of aggregate TFP shock</td>
<td>0.027</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>Persistence of aggregate TFP</td>
<td>0.8145</td>
</tr>
</tbody>
</table>

Table 7: Calibrated Parameters

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25. Spike rate ($\%$) is the percentage of firms making lumpy investments.
1.4 Quantitative analysis

1.4.1 Echo effects in post-shock period

In this section, I quantitatively analyze state-dependent impulse response of aggregate investments to an aggregate TFP shock in the general equilibrium framework.

A novel feature of the acceleration cost model is that impulse response of aggregate investment displays echoes in the post shock periods in general equilibrium. Figure 8 plots impulse responses of aggregate investment in the baseline model for both partial and general equilibrium (panel (a)); its growth in general equilibrium (panel (b)); and impulse response of average investment stages in general equilibrium (panel (c)). The impulse response is obtained from nonlinear method that computes transition path from a shock period to the stationary period as described in Boppart et al. (2018). All the responses are expressed in terms of percentage deviations from the steady-state level.

As can be seen from panel (a), general equilibrium effect only partially dampens the response of aggregate investment. Upon impact, aggregate investment drops by 8.3% in partial equilibrium and drops by 6.6% in general equilibrium. Thus, the factor price decreases contemporaneous response by only 20% (≈ 100 * (1 – 6.6/8.3)). In contrast, in models with fixed cost, general equilibrium effect dampens the contemporaneous response by around 6 folds (≈ 83%). This difference is due to large firms’ inelasticity to interest rate fluctuations in the acceleration cost model.

In the post-shock period, aggregate investment gradually recovers to the stationary level. Along the recovery path, there are both trend of recovery and oscillation around the trend. I refer to the oscillation as echo effect. The magnitude of the oscillation in aggregate investment decays overtime, and its magnitude ranges from -6.6% to 2.2% of the stationary level. As shown in panel (b) and (c), the impulse responses of aggregate

26. Certainty equivalence is assumed in the impulse response by the nature of MIT shock.
27. Strong general equilibrium effects in models with fixed cost are compared in Figure 13
investment growth rate and average investment stages also display echo effects. For both of the responses, the lowest is at the fifth period from the shock after a shock period. This lowest point is the timing where an economy becomes the most fragile to another negative aggregate TFP shock. This will be studied more in detail in the next section.

On impact of a negative aggregate shock, firms that are ready to adjust capital stock in extensive margin tend to delay their adjustment to escape from low aggregate TFP. Therefore, the aggregate TFP shock synchronizes firms’ large-scale investment timings. Against this synchronized investment timing, there are two mitigating forces that spread

---

28. If a positive aggregate TFP shock hits the economy, firms that have not considered large-scale investment in the shock period newly launch or accelerate their projects to utilize high aggregate TFP level.
out the timings back to the stationary equilibrium distribution. The first is factor price, and the second is stochastic mean reversion. The first force works by making firms’ investments costlier when more firms are investing together. In the acceleration cost model, large firms that face high acceleration cost become insensitive to the first force because large acceleration cost already strongly constrained these firms’ investment timing. Thus, there is only little room for factor price to affect further the constrained investment timing. Therefore, interest rate dynamics does not fully mitigate the synchronized investment timings among large firms.

The second force, stochastic mean reversion flattens the synchronized timings by spreading out the distribution of lengths of inaction periods. Depending on the mixing rate implied by the idiosyncratic stochastic process, the distribution of lumpy investment timings quickly or slowly move back to stationary distribution. In other words, the speed of convergence in the law of large numbers is the key condition to determine whether synchronized timings of lumpy investments can persist or not. In the calibrated acceleration cost model, the average persistence of inaction periods are as high as in the observed level in the data ($\approx 0.9$). Thus, the timings of lumpy investments revert back to stationary distribution slowly.

For the decomposition analysis, I define large firms as the top 20% largest firms. Under this approach, 26.7% of total capital belong to large firms. Compustat space covers around half of the total U.S. private fixed investment, and around 60% of capital stocks in Compustat data are from large firms defined in the empirical section. So, around 30% of total capital stock in the U.S. belongs to the large firms. Therefore, the definition of large firms as the top 20% largest firms is consistent with the definition in the empirical analysis.\textsuperscript{29} Figure 9 visualizes heterogeneous echo effect for large and small firms in the impulse responses of average investment stages (panel (a)) and aggregate investment (panel (b)). Heterogeneous echo effects under the alternative definitions of large firms are

\textsuperscript{29} The model does not capture the thick tail of the firm distribution observed in the data.
reported in Figure A.7.4 (Top 30%) and Figure A.7.5 (Top 40%). Despite the difference in the magnitudes, the qualitative results stay unchanged over the different proxies.

The echo effect is mostly driven by large firms. Panel (a) shows that large firms’ investment timings are persistently synchronized in the post-shock period. By around 25 years later from the shock, the synchronization is mitigated, showing the flattened path of average investment stages. Small firms barely shows persistent synchronization due to fast mean-reversion of inaction duration.

![Graph](image_url)

(a) Average investment stage  
(b) Aggregate investment

Figure 9: Heterogeneous echo effects for large and small firms

Panel (b) shows that large firms’ aggregate investment bounce up quickly right after the aggregate shock. Then, the aggregate investment of the large firms display oscillation ranging from -6% to 5.7% deviation from the steady state. On the contrary, small firms’ aggregate investment slowly recovers to the steady-state level without much oscillation.

1.4.2 Fragility after a surge of lumpy investments

In this section, I study how differently aggregate investments respond to a TFP shock depending on the aggregate state. When an aggregate TFP shock hits the model economy, there arise echoes of the shock in aggregate investment during the post-shock period. Then I hit the economy with another aggregate TFP shock separately at each period on the
recovery path. For each experiment, I control the magnitude of aggregate TFP shock to equalize the level of aggregate TFP at the shock period across the experiments.\textsuperscript{30}

Figure 10 compares different impulse responses of aggregate investment depending on where the economy is located at the time of shock. The response is strongest when the negative aggregate TFP shock hits the economy right after the surge of aggregate investment; the aggregate investment drops by $7.5\%$. The response is weakest if a shock arrives at the surge of lumpy investments; the aggregate investment responds by $5.8\%$. Therefore, the response of aggregate investment is stronger by $29\%$ ($\approx 100 \times (7.5 - 5.8)/5.8$) when a shock hits after the surge of lumpy investments than when it does at the surge.

![Figure 10: State-dependent impulse response of aggregate investment](image)

I make the same experiment for large and small firms, separately. Figure 11 visualizes the state-dependent impulse response of aggregate investment for large firms (panel (a)) and small firms (panel (b)). Large firms’ immediate response is stronger for a shock after the surge than for a shock at the surge by around $9.7\%$. In contrast, small firms’ response is stronger only by $0.5\%$ for the same comparison. Regardless of where the economy is located, small firms’ timings of large-scale investment are strongly smoothed out by real

\textsuperscript{30} Specifically, I set the magnitude to set the aggregate TFP at shock period equivalent to one-standard-deviation drop from stationary level.
interest rate. Thus, they display almost constant contemporaneous investment sensitivity to the aggregate productivity shock.

![Graph](image)

**Figure 11: Heterogeneous endogenous effect**

Table 8 summarizes the state-dependent contemporaneous impulse responses of investments. The $i^{th}$ column reports the conditioning states of the $i^{th}$ period from the initial aggregate TFP shock and the responses when a TFP shock arrives at the $i^{th}$ period. The first row represents the lagged aggregate investment expressed in terms of percentage deviations from the steady-state level; The second row represents the lagged aggregate investment of large firms in terms of percentage deviations from the steady-state level; The third row represents the contemporaneous impulse response of aggregate investment; The fourth row represents the contemporaneous impulse response of aggregate investment of large firms; And the fifth row represents the contemporaneous impulse response of aggregate investment of small firms. The contemporaneous impulse response of aggregate is the largest at the fifth period after the initial aggregate TFP shock. The fourth period display the smallest contemporaneous impulse response.

<table>
<thead>
<tr>
<th></th>
<th>$t=+2$</th>
<th>$t=+3$</th>
<th>$t=+4$</th>
<th>$t=+5$</th>
<th>$t=+6$</th>
<th>$t=+7$</th>
<th>$t=+8$</th>
<th>$t=+9$</th>
<th>$t=+10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta I_{i-1}$ (%)</td>
<td>-6.6</td>
<td>-4.8</td>
<td>-2.7</td>
<td>-2.0</td>
<td>-3.2</td>
<td>-1.5</td>
<td>-1.6</td>
<td>-0.9</td>
<td>-0.9</td>
</tr>
<tr>
<td>$\Delta I_{large,i-1}$ (%)</td>
<td>-6.5</td>
<td>0.7</td>
<td>2.8</td>
<td>5.2</td>
<td>-5.7</td>
<td>-2.6</td>
<td>-1.3</td>
<td>0.8</td>
<td>-4.7</td>
</tr>
<tr>
<td>$\Delta I_{response,i-1}$ (%)</td>
<td>-7.3</td>
<td>-6.0</td>
<td>-5.8</td>
<td>-7.5</td>
<td>-6.2</td>
<td>-6.5</td>
<td>-6.2</td>
<td>-6.4</td>
<td>-6.4</td>
</tr>
<tr>
<td>$\Delta I_{large response,i}$ (%)</td>
<td>-2.6</td>
<td>-1.5</td>
<td>0.6</td>
<td>-9.1</td>
<td>-7.0</td>
<td>-5.7</td>
<td>-3.9</td>
<td>-9.0</td>
<td>-7.6</td>
</tr>
<tr>
<td>$\Delta I_{small response,i}$ (%)</td>
<td>-8.0</td>
<td>-6.6</td>
<td>-6.7</td>
<td>-7.2</td>
<td>-6.1</td>
<td>-6.6</td>
<td>-6.5</td>
<td>-6.0</td>
<td>-6.3</td>
</tr>
</tbody>
</table>

Table 8: Summary of state-dependent impulse responses
The core mechanism of the state-dependent responsiveness of aggregate investment is in the large firms’ heterogeneous investment decision. There are two situations in which a large firm makes a lumpy investment in a period $t$:

1. Firm enters period $t$ with only one stage remaining ($s = 5$)

2. Firm enters period $t$ with more than one stage remaining ($s < 5$)

If a negative aggregate TFP shock hits the economy, large firms at situation (1) still invest, while large firms at situation (2) do not. Therefore, the state-dependent responsiveness is crucially determined by the fraction of large firms that have only one stage remaining for a large-scale investment, $s = 5$. Then, I define an investment fragility measure as follows:

$$\text{Investment fragility} := \frac{\sum_{\text{Large}} 1\{s < 5\}}{\sum_{\text{Large}} 1\{s \leq 5\}}$$

The investment fragility measure is the fraction of firms that are not in the last stage for a lumpy investment. In the model, a negative aggregate TFP shock has greater effect when the investment fragility is higher. And the investment fragility becomes high after a surge of lumpy investments.

Figure 12 plots large firms’ marginal distributions of stages at different states of the model economy. It is worth to note that the stages are determined one period before the aggregate shock hits the economy by each firm’s inter-temporal stage policy. Therefore, the contemporaneous stage distribution is an exogenous condition to an aggregate shock even if they share the same time index. As can be seen from the most right bars in the graph, at the surge of lumpy investments, many firms are at the last stage 5 in the beginning of the period. Therefore, despite a negative aggregate shock, large firms make large-scale investments, and this results in the weak response of aggregate investments to the negative shock. The investment fragility is 90.1% at the surge of lumpy investments. On the other
hand, after the surge of lumpy investments, the least fraction of firms are at the last stage. The investment fragility is 93.3% at the surge of lumpy investments.

Then, I decompose the total contemporaneous change of the aggregate investment into exogenous component and the endogenous component. The endogenous component is from the direct effect from aggregate TFP shock. The endogenous component accounts for all the other remaining variation unaccounted by the direct effect.

Aggregate investment responds differently to an aggregate TFP shock depending on the aggregate states of the economy. Therefore, we can write aggregate investment $I_t$ as a function of TFP level $A_t$ and a vector of sufficient statistics of the aggregate state of the economy $X_t$:

$$I_t = I_t(A_t, X_t)$$
Then, the total variation in the aggregate investment $I_t$ after an aggregate TFP shock can be further decomposed into exogenous component and the endogenous component.

$$\Delta \log I_t(A_t, X_t) \equiv \left( \frac{\partial \log I_t}{\partial \log A_t} \right) \Delta \log A_t + \left( \frac{\partial \log I_t}{\partial \log X_t} \right) \Delta \log X_t \quad \text{(Exogenous component)}$$

$$\quad + \text{Direct effect}$$

$$\quad + \left( \frac{\partial \log I_t}{\partial \log X_t} \right) \Delta \log X_t \quad \text{(Endogenous component)}$$

In the identity above, exogenous component is driven purely by exogenous direct variation in aggregate TFP; and the endogenous component indicates all other variations residualized after the exogenous variation.

From the experiments of the TFP shocks hitting the economy at different conditioning states, variations in the conditioning states ($\Delta \log X_t$) and the contemporaneous total change of the aggregate investment ($\Delta \log I_t(A_t, X_t)$) are available. The exogenous component is directly from the TFP shock and it is common across all observations as the shock magnitude is controlled.

Then, I non-parametrically approximate the conditioning states $X_t$ using the following measures:

1. Lagged aggregate investment $I_{t-1}$
2. Investment fragility $S_t$

Hence,

$$X_t = X_t(I_{t-1}, S_t)$$

---

31. To obtain enough number of samples, I utilize the variations from three rounds of the TFP shock experiments. Specifically, the second round gives 10 observations, and the third round gives 10 observations for each 10 observations of the second round. Thus, I obtain total $10 + 10 \times 10 = 110$ observations.

32. For robustness check, I use average investment stages $\overline{I}_{t-1}$ instead of $S_t$. The result stays unchanged for this alternative choice.
The variation in the conditioning states $X_t$ can be non-parametrically approximated by the variation in $I_{t-1}$ and $S_t$:

$$\Delta \log X_t = \chi(\Delta \log I_{t-1}, \Delta \log S_t)$$

From the decomposition equation,

$$\Delta \log I_t(A_t, X_t) = \left( \frac{\partial \log I_t}{\partial \log A_t} \right) \Delta \log A_t + \left( \frac{\partial \log I_t}{\partial \log X_t} \right) \Delta \log X_t$$

$$= \left( \frac{\partial \log I_t}{\partial \log A_t} \right) \Delta \log A_t + \tilde{\chi}(\Delta \log I_{t-1}, \Delta \log S_t)$$

Based on the equation above, I run the non-parametric regression of aggregate investment variation $\Delta \log I_t(A_t, X_t)$ on $I_{t-1}$ and $S_t$ to identify the direct effect which is the intercept term in the regression. The intercept is estimated as $-6.4$, and the $R^2$ is around $96\%$. From this high $R^2$, I confirm $I_{t-1}$ and $S_t$ non-parametrically approximate $X_t$ well.

The nonlinear effect amplifies or mitigates the direct effect based on the conditioning states. From the total change we have after surge of lumpy investments and at the surge of lumpy investments, the following decomposition is obtained:

After a surge of lumpy investment, the total change of aggregate investment could be decomposed as

$$100\% \ (7.5\%) = 85\% \ (-6.4\%) + 15\% \ (-1.1\%)$$

At the surge of lumpy investment, the total change of aggregate investment could be decomposed as

$$100\% \ (-5.8\%) = 110\% \ (-6.4\%) - 10\% \ (+0.6\%)$$
where the numbers in the bracket indicate the absolute effect in percentage.

From this decomposition analysis, I conclude that aggregate investment responds to an aggregate TFP shock substantially differently depending on the conditioning states of the economy. After the surge of lumpy investments from large firms (by 5.2%), the negative aggregate shock effect is amplified by 15% through the nonlinear endogenous channel. This is the upper bound of the amplification effect among the simulated responses. On the other hand, at the surge of lumpy investment, the negative aggregate shock effect is diminished by 10% through the nonlinear endogenous channel. I found this is the lower bound of the amplification effect among the simulated responses.

1.4.3 Comparison with other models

The timing synchronization upon an aggregate TFP shock is not a unique feature of the acceleration cost model; timing synchronization also happens in the models with fixed cost. However, in those models, firms’ capital adjustment timing is highly elastic to factor prices. Therefore, in the post-shock period, firms have a strong tendency of not making large-scale investment together with other firms. By flexibly adjusting their investment timings, these firms spread out their lumpy investment schedules to have no lumpiness in the response of aggregate investments. Due to this flexibility allowed in the model, the persistence in the length of inaction periods are significantly ($\approx 0.70$) lower than the level observed from the data.

Figure 13 compares the impulse responses of aggregate investment in three different models including a non-lumpy frictionless investment model, Gourio and Kashyap (2007), and Khan and Thomas (2008). For the computation of these models, I use the parameters reported in Khan and Thomas (2008). For the non-uniform fixed cost distribution in Gourio and Kashyap (2007), I use a truncated normal distribution with the mean matched
Figure 13: Strong general equilibrium effect on aggregate investment in models with fixed cost: irrelevance results

to the uniform distribution in Khan and Thomas (2008). I used a small standard deviation (=0.001) for the highly concentrated mass around the mean.  

The first row of Figure 13 (panel (a),(b)) plots impulse responses of aggregate investment in the frictionless model. When the factor price is not considered (panel (a)),

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33. The result for Gourio and Kashyap (2007) is robust over other parameter choices that give concentrated distributions of fixed cost.
investment drops by more than 200% of steady-state level. However, in general equilibrium (panel (b)), simultaneous drop in the interest rate in the shock period incentivize firms to make more investments in the shock period; this reduces the response of the aggregate investment by more than 10 folds. In the second row (panel (c),(d)), the impulse response of aggregate investment in Gourio and Kashyap (2007) are added. As pointed out by the authors, fixed cost plays an important role to smooth the reactions of aggregate investments to the aggregate TFP shock in partial equilibrium (panel (c)). The aggregate investment drops by around 100% compared to the steady-state level, and smoothly recover. On the recovery path, aggregate investment forms a smooth hump before it converges to steady-state level due to synchronized lumpy investment timings in partial equilibrium. However, when the factor prices are considered (panel (d)), the impulse response of aggregate investment in Gourio and Kashyap (2007) becomes similar to frictionless model due to strong general equilibrium effect. In general equilibrium, the initial response of aggregate investment is dampened by around 6 folds due to factor price fluctuations. As shown in the third row of Figure 13 (panel (e),(f)), Khan and Thomas (2008) model results in similar impulse responses to that of Gourio and Kashyap (2007) in general equilibrium.34

Motivated from empirical findings on pro-cyclical sensitivity of aggregate investment to an aggregate shock, Bachmann et al. (2013) suggests a model with fixed cost with maintenance and replacement cost. Figure 14 compares the impulse responses in the acceleration cost model (panel (a)) and Bachmann et al. (2013) (panel (b)). The flattening effect from general equilibrium is substantially smaller in these two models as shown from the small difference between partial and general equilibrium response.

34. Due to the concentrated fixed cost distribution, Gourio and Kashyap (2007) has a stronger smoothing effect in partial equilibrium than Khan and Thomas (2008).
However, the implied persistence of inaction duration in Bachmann et al. (2013) does not achieve the level observed from the data because investment spikes beyond 20% of existing capital stock are modeled to happen more sparsely in general equilibrium to result in significantly lower persistence. Therefore, this model captures interest-inelastic investment spikes while it cannot capture firms’ persistent inaction patterns. So the impulse response does not feature echo effect.

### 1.4.4 Business cycle analysis

In this section, I analyze business cycle characteristics implied by the dynamic stochastic general equilibrium from the acceleration cost model, and compare these with the results in Khan and Thomas (2008) (hereafter, KT). There are two computational hurdles in this exercise: 1) curse of dimensionality in aggregate state variable and 2) nonlinearity in the aggregate dynamics.

In KT, due to strong general equilibrium effect, the true dynamics of aggregate capital stocks closely follows log-linear prediction rule. So, the dynamic stochastic general equilibrium is obtained by tracking only one moment as in the algorithm suggested by
However, as shown from the previous section, aggregate fluctuations implied by the acceleration cost model is highly nonlinear. Therefore, to use Krusell and Smith (1998) algorithm, more moments need to be considered potentially in nonlinear form in the predicted law of motions at large computational cost.

To overcome this difficulty, I use another algorithm, named as the repeated transition method to solve the acceleration cost model under aggregate uncertainty concurrently developed in Lee (2020a). In the algorithm, I update an agent’s prediction rule for aggregate states repeatedly from transition dynamics on a single simulated path until the prediction rule converges to the simulation. This method does not rely on parametric assumption on the predicted law of motions for the future aggregate states; market clearing prices, expected future aggregate states, and value functions on the transition path are explicitly computed. Then, I back out the prediction rule implied by the fitted outcomes on the sample path and check the validity from the out-of-sample simulation paths. I leave the detailed explanation on the algorithm to Lee (2020a). The length of simulated path is 1,000 periods. I use histogram method for transition of the cross-sectional distribution of firms following Young (2010).

Figure 15: Comparison on dynamics of investment rates ($I_t/K_t$) in business cycle

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35. Khan and Thomas (2003) found that there is no difference in the approximated dynamics of aggregate states between tracking one central moment and tracking two central moments of the partitioned state distribution.
Figure 15 plots simulated aggregate investment rates \((I_t/K_t)\) obtained from the baseline model (solid line), KT (dashed line), and simulated aggregate TFP path (dotted line) for different aggregate shocks. Panel (a) is based on the calibrated aggregate TFP shock in this paper, and the TFP shock in panel (b) is from KT. The aggregate investment rates are expressed in terms of percentage deviations from the steady-state level. The baseline results plotted in panel (a) and (b) are separately obtained from applying the repeated transition method to each of simulated paths from the two different aggregate shocks.\(^{36}\) The law of motions in KT are obtained from the exact replication of the paper following their computation methodology explained in the paper.

In panel (b), the shock effect in aggregate investment rate of the baseline decays slower than KT. High persistence in lengths of inaction periods contributes to this slowly decaying shock effect. Also, the investment rate is less responsive to a shock in the baseline model than in KT. The acceleration cost makes large firms insensitive to exogenous shocks, weakening the responsiveness of aggregate investment rate. Due to nonlinearity in aggregate dynamics in the acceleration cost model, the baseline result includes echo effects after a jump (drop) in the TFP, as shown in the zig-zag patterns.

Under the calibrated TFP shock in panel (a), the implied dynamics of aggregate investment rate becomes closer between baseline and KT. However, the baseline result still features higher persistence and lower responsiveness than KT.

Table 9 summarizes the business cycle statistics for the simulated allocations in comparison with the statistics in the macro-level data at annual frequency. The data other than employment is from National Income and Product Accounts data (NIPA Table 1.1.5). Employment \((L_t)\) (not an hour) is from Current Employment Statistics.\(^{37}\) The sample period covers from 1955 to 2018. I use private domestic investments as investment \((I_t)\). All

\(^{36}\) For the other parameters, I use the same parameters as calibrated in this paper.

\(^{37}\) Both of the models assumed an indivisible labor supply in the household’s utility.
variables are real at annual frequency, and I linearly detrend these variables after taking log.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Baseline</th>
<th>Baseline + KT</th>
<th>KT (2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( corr(Y_t, Y_{t-1}) )</td>
<td>0.941</td>
<td>0.900</td>
<td>0.867</td>
<td>0.825</td>
</tr>
<tr>
<td>( corr(I_t, I_{t-1}) )</td>
<td>0.742</td>
<td>0.788</td>
<td>0.755</td>
<td>0.685</td>
</tr>
<tr>
<td>( corr(I_t, Y_t) )</td>
<td>0.795</td>
<td>0.924</td>
<td>0.939</td>
<td>0.873</td>
</tr>
<tr>
<td>( corr(L_t, Y_t) )</td>
<td>0.898</td>
<td>0.819</td>
<td>0.718</td>
<td>0.805</td>
</tr>
<tr>
<td>( corr(C_t, Y_t) )</td>
<td>0.978</td>
<td>0.989</td>
<td>0.993</td>
<td>0.899</td>
</tr>
<tr>
<td>( sd(I_t)/sd(Y_t) )</td>
<td>1.976</td>
<td>1.792</td>
<td>1.651</td>
<td>3.079</td>
</tr>
</tbody>
</table>

Table 9: Business cycle statistics

The numbers in the first column of Table 9 are the statistics from the data; the second is from the calibrated baseline model; the third is from the baseline model computed with an aggregate TFP shock used in KT; and the last column is from the result in KT. In both of the baseline and the baseline combined with KT shock, the autocorrelations of investment are higher than KT, and they are closer to the level observed from the data. High persistence in inaction periods implied by the acceleration cost model contributes to high persistence in aggregate investments. Correlations between employment and output, and between consumption and output are better captured in the acceleration cost model, while the correlation between investment and output are explained better in KT. Also, the relative volatility of aggregate investments in the acceleration cost model is closer to the level in the data.

1.5 A theoretical limit case: permanent echo and endogenous business cycle

In this section, I explore a theoretical possibility the acceleration cost model can lead to: a permanent echo after an aggregate TFP shock forms synchronization of spiky investment timings across firms.
In the empirical distribution of lengths of inaction periods, there are firms with extremely persistent inaction periods and that are inelastic to interest rate changes. These firms could be understood as having strict investment timing policies for their investments that is almost invariant over time. In this section, I model these highly persistent inaction duration as a nature of investment technologies of large firms. Large firms’ inaction periods are modeled to have a strict persistence at one.

Then, if timings of these firms are synchronized, the echo effects from these firms will not be muted by factor-prices. Also, idiosyncratic stochastic force does not flatten the co-movements of these firms’ lumpy investments. Therefore, the echoes of aggregate TFP will last forever. Depending on how investment timings among these firms are synchronized, the endogenous fluctuations in the permanent echo can take various patterns.

I compute the cyclical competitive equilibrium with a synchronized initial distribution that might have been initialized by some large aggregate TFP shocks in the prior history. I mute aggregate TFP fluctuations to solely focus on endogenous component of aggregate fluctuations. Without exit and entry, large and small firms are assumed to be permanently separated for simplicity.

I assume heterogeneous parameters for adjustment cost and acceleration cost. Specifically, for a firm with type $j \in \{\text{small}, \text{large}\}$, the investment technologies are modelled as follows:

\[
\begin{align*}
(\text{Convex Adjustment Cost}) & \quad c(k, I, j) := \frac{\mu^f_j}{2} \left( \frac{I}{k} \right)^2 k \\
(\text{Acceleration Cost}) & \quad \text{acc}(s', s, j) := \left[ \mathbb{I}\{s' > s + 1\} \left( \frac{\mu^a_j}{2} (s' - s - 1)^2 \right) \right]
\end{align*}
\]

38. Around 3% of firms have identical inaction periods across years.
39. Event analysis shown in Figure 4 also points out large firms’ lumpy investments are almost unaffected by idiosyncratic forces. Therefore, the speed of convergence in the law of large numbers is extremely low for large firms.
40. The full formulation of heterogeneous large and small firms’ problem is available in Appendix A.9.
Note that differently from the baseline model, acceleration cost now does not depend on the size of capital stock. Instead, I assume ex-ante heterogeneous investment technology characterized by different parameters. When I bring the model to fit into the data, I obtain following strict orders between parameters:

\[ \mu_{a \text{ large}} > \mu_{a \text{ small}}, \mu_{I \text{ large}} < \mu_{I \text{ small}} \]

The larger acceleration cost is needed for large firms to capture large firms’ interest-inelasticity and more persistent inaction duration. The smaller convex adjustment cost is to match the fact that large firms are greater in size than small firms.\(^{41}\)

If a large firms’ acceleration cost is large enough, a firm might not choose to accelerate its investment stage at all and stick to one-stage-per-period rule despite the fluctuations in the factor prices and idiosyncratic productivities. In this case, the firm’s capital adjustment policy will follow semi-deterministic \((S, s)\) cycle: adjustment timing in extensive margin is deterministic while the intensive margin stochastically changes depending on the factor price and idiosyncratic productivity realizations. The following proposition formally states the existence of such large acceleration cost parameter that warrants semi-deterministic \((S, s)\) cycle of large firms.

**Proposition 1** (Isolated stage policy).

Given an idiosyncratic productivity process \(G_z(z)\) with a bounded support \(Z\), there exists \(\mu_{G_z} > 0\) such that

\[ \mu_{a \text{ large}}^l \geq \mu_{G_z} \implies s'(k, z, s, \text{large}; \Phi, A) = s'(s, \text{large}) \text{ for } \forall (k, z, s) \in (K, Z, S) \]

where \((K, S)\) denotes the domains of capital and investment stages, respectively.

\(^{41}\) If \(\mu_{I \text{ large}}^l \geq \mu_{I \text{ small}}^l\) holds, large firms’ size become smaller than small firms. This is because large firms make less frequent capital adjustment in extensive margin \( (\mu_{a \text{ large}}^l > \mu_{a \text{ small}}^l)\), and the size of adjustment is smaller for large firms due to larger convex adjustment cost. This is counterfactual in that large firms are bigger firms than small firms on average.
Proof. See Appendix A.11.1.

It is worth to note that the threshold of large acceleration cost $\mu_{G_z}$ is specific to idiosyncratic stochastic process $G_z$. If $z$ can take an extreme value with positive probability, investment stage policy $\delta'$ depends on shock realizations. However, if the idiosyncratic productivity process has a bounded support, there exists a sufficiently large level of acceleration cost parameter that makes investment stage policy independent from the shock process and interest rate fluctuations. Hereafter, given $G_z$ with a bounded support $Z$, I assume $\mu_{\text{large}} > \mu_{G_z}$.

Figure 16: Large firms’ semi-deterministic $(S,s)$ cycle and capital adjustment in intensive margin

Figure 16 shows large firms’ semi-deterministic $(S,s)$ capital adjusting rule. In this exercise, heterogeneous firms are given with different level of capital stocks at period 0, and the trajectory of each firm’s optimal level of capital stocks is tracked over time. I use $\bar{s} = 4$ as in the calibration of baseline parameters. Panel (a) highlights the deterministic extensive-margin rule for capital adjustment. Due to large acceleration cost, large firms follow one-stage-per-period rule, and this makes capital stocks jumps up in every $\bar{s}$ periods, regardless of idiosyncratic productivity realizations and interest rate fluctuations. However, the magnitude of jumps changes depending on the idiosyncratic productivity realizations.
as shown in the single firms’ capital adjusting rule in panel (b). Thus, the capital adjusting rule follows a semi-deterministic \((S, s)\) rule.

Large firms’ semi-deterministic \((S, s)\) capital adjusting rule leads to a stationary capital cycle after aggregation in general equilibrium. This is because the initially synchronized firm’s adjusting timings permanently stay synchronized without being mitigated by either factor prices or stochastic forces. This could be understood as a limit case of baseline model where stochastic mean reverting forces gradually flatten the echo effect.

Specifically, if the initial distribution of large firms’ investment timings is non-uniform, there will be a stationary cycle of aggregate investments. This class of initial distributions is formally defined as follows:

**Definition 2 (Class of synchronized distributions).**

Given \((K, Z, S)\), \(\mathcal{D}\) denotes a set of all probability measures \(\Phi\) defined on the cartesian product of the sigma algebras \(K \times Z \times S\) generated from \((K, Z, S)\). Define a partition \(\{\mathcal{D}_0, \mathcal{D}_1\}\) of \(\mathcal{D}\) as follows:

\[
\mathcal{D}_1 := \{\Phi \in \mathcal{D} \mid \text{for } \forall s \in S, \int_{K \times Z \times \{s\} \times \{y, o\}} d\Phi (k, z, s, j; \Phi, A) = \frac{1}{s}\}, \quad \mathcal{D}_0 := \mathcal{D} \setminus \mathcal{D}_1
\]

The partition \(\mathcal{D}_0\) is a class of firm distributions that support stationary cycle of aggregate investments once they become an initial distribution. In Proposition 2, I show that if large firms’ investment stage policy is independent from price fluctuations and idiosyncratic productivity shocks, and initial distribution belongs \(\mathcal{D}_0\), there does not exist a stationary recursive competitive equilibrium. In Corollary 1, I show that under the same condition, the cyclical competitive equilibrium with \(n^* > 1\) exists. Before the theoretical results, I define an implied sequence of distributions which is useful for throughout the theoretical statements.

---

42. See Definition 3.
Definition 3 (Implied sequence of distributions).

Given firms’ policy $k’, s’, and an initial distribution $\Phi_0$, I define the implied sequence of distributions as $\{\Phi_t\}_0^\infty$ such that

$$(\Phi_{t+1})(K, Z, S, j; \Phi, A) := \int_{K \times Z \times S} \left( \int_Z \Gamma_{z,z'}dz' \right) \mathbb{1}\{k'(k, z, s, j; \Phi, A) \in K\} \mathbb{1}\{s'(k, z, s, j; \Phi, A) \in S\} d\Phi_t(k, z, s, j; \Phi, A)$$

for any set $(K, Z, S)$ in the $\sigma$-algebra $(K, Z, S)$ generated from the domains $(K, Z, S)$.

It is worth to note that hazard rate, type transition, and new entry are only implicitly considered because type distribution is assumed to stay the same after replacement for simplicity.

Proposition 2 (Breaking the law of large numbers).

If $\mu^a_{\text{large}} \geq \overline{\mu}_{G_z}$, and $\Phi_0 \in D_0$, the implied sequence of distributions $\{\Phi_t\}_0^\infty$ does not have a limit point.

Proof. See Appendix A.11.2. 

Therefore, Proposition 2 states that there does not exists a stationary recursive competitive equilibrium. Then, I show there exists cyclical competitive equilibrium in the following corollary.

Corollary 1 (Endogenous stationary cycle).

Given $\mu^a_{\text{large}} \geq \overline{\mu}_{G_z}$ and the initial distribution $\Phi_0 \in D_0$, for $\forall \epsilon > 0$, there is a sufficiently large $\tau \in \{1, 2, 3, ...\}$ such that the implied sequence of distributions $\{\Phi_t\}_{t=0}^\infty$ satisfies following property:

$$||\Phi_{t+\tau} - \Phi_t||_{\sup} < \epsilon, \text{ for } \forall \tau > \tau$$
Proof. See Appendix A.11.3.

From Corollary 2, I show that the synchronized distribution $D_0$ includes nearly all possible distributions of state variables. Thus, under the perfect isolation of large firms’ stage policy from price fluctuations and idiosyncratic shock process, a stationary cycle arises in almost every initial distribution. Therefore, any slight synchronization of lumpy investment timings resulting from an aggregate TFP shock will lead to aggregate fluctuations due to a permanent echo.

**Corollary 2** (Commonness of aggregate cycles).

Consider a non-degenerate atomless distribution $\Psi$ defined on $\sigma$-algebra $\mathcal{D}$ generated from $\mathcal{D}$, where $\mathcal{D}$ is the support of $\Psi$. Then, $\Psi(D_1) = 0$, and $\Psi(D_0) = \Psi(\mathcal{D}) = 1$.

Proof. See Appendix A.11.4.

Given these theoretical results, I compute the cyclical competitive equilibrium using the parameters reported in Table A.9.4. I set the fraction of total large and small firms at the level where large firms hold the half of total capital stocks in the economy. These parameters give similar cross-sectional moments to the target moments in the baseline.

For the stationary cycle, important parameters to be specified are the initial distribution on completed investment stages for large firms. The distribution could be estimated from data by matching the fraction of firms making large investments. For the computation exercise, I use $\phi_S = (0.2211, 0.2412, 0.2613, 0.2764)$ as an initial synchronized distribution of investment stages for large firms. As theory predicts, the initial distribution of completed stages is not mixed in the stationary cycle, and it moves in the circular pattern to make

---

43. This is based on the summary statistics reported in Table 1.
44. Small firms’ initial distribution converges to an ergodic distribution following the law of large numbers. Therefore, initial distribution does not have to be specified for small firms.
45. Small firms’ initial distribution is not specified. It is because small firms’ stage distribution converges to ergodic distribution regardless of the initial distribution.
aggregate fluctuations. For the other parameters, I use the same parameters as in the calibrated baseline model.

The cyclical competitive equilibrium requires market clearing for the whole periods within a cycle. Computing market clearing prices for the entire cycle is a difficult task because firms’ inter-temporal policies are sensitive to price rankings across periods. For this, I introduce a novel algorithm that solves the cyclical competitive equilibrium by preserving relative rankings of prices over the convergence path. I describe the details of the computation method in Appendix A.

Figure 17 plots the endogenous fluctuations in the marginal distributions of large and small firms’ logged capital stocks in the cyclical competitive equilibrium. Small firms’ capital distribution shows little fluctuations, while large firms’ capital distribution dramatically fluctuate endogenously without reliance on any exogenous aggregate forces.

![Endogenous fluctuations in capital distributions for large and small firms](image)

Figure 17: Endogenous fluctuations in capital distributions for large and small firms

Along these aggregate fluctuations in the state distributions, aggregate allocations also move forming a stationary cycle in general equilibrium. Figure 18 plots the time path of aggregate allocations in the cyclical competitive equilibrium. As I set the required stages for large-scale investment $\bar{s} = 4$, the length of a period in the stationary cycle is also four periods. In the endogenous cycle, aggregate investment ($i$), employment ($l$), and real interest rate ($r$) are pro-cyclical, and aggregate capital stocks ($k$), consumption ($c$), and wage ($w$) are counter-cyclical.
These endogenous aggregate fluctuations in the stationary cycle are permanent echoes from large aggregate TFP shock that might have happened in the prior history that is not specified in the model. The initiation mechanism of these endogenous cycle is checked by impulse response of the economy to the large aggregate shocks such as a negative aggregate TFP shock during the Great Depression; According to Ohanian (2001), aggregate TFP dropped by around 18% during the Great Depression.

Figure 19 shows permanent echoes in the aggregate investments after a sudden 18% drop in the aggregate TFP. In this exercise, I assume the economy’s investment stages were uniformly distributed (unsynchronized) before the shock. Then, after the large aggregate shock, the firms’ investment timings are synchronized, and it generates a permanent echo in the economy.

---

46. Thus, the aggregate allocations are at the stationary competitive equilibrium.
To sum up, large firms’ extreme inelasticity to interest rate and extreme persistence in inaction durations lead to a permanent echo that generates endogenous aggregate fluctuations. This is a limit case of the baseline model which features decaying echo in the post shock periods. When a large shock hits the economy, the acceleration cost model predicts that an echo effect will not decay in the short run both in decaying echo and permanent echo setup. Specifically, a permanent echo model could be potentially used to analyze short run business cycle after a large aggregate TFP shock such as the Great Depression or COVID-19 pandemic.

In the short run business cycle analysis, a permanent echo model is particularly useful because the initial distribution of investment stages is a free parameter that can be estimated; the model could be fitted into large cross-sectional data. This characteristic is unique among general equilibrium models that studies aggregate fluctuations.

However, there are limitations in this theoretical argument: permanent echoes are difficult to detect empirically from data because the endogenous stationary cycle is not a response to any impulse. Another difficulty is that permanent echo patterns are subject to change depending on the arrival of different aggregate TFP shocks. Thus, I leave this
endogenous cycle as a theoretical possibility an acceleration cost model can lead to, with a possibility to be used in short run business cycle analysis in future researches.

1.6 Empirical evidence from aggregate-level data

General equilibrium in acceleration cost model features echoes in aggregate investment after an aggregate TFP shock. This is due to interest-inelastic firm-level lumpy investments and high persistence in the length of inaction periods. These two characteristics are based on micro-level observations from the U.S. Compustat data. In this section, I show the echo effect is empirically supported in the macro-level data.

Figure 20 plots time series of the growth rate of investment in non-residential structures in manufacturing industry from 1935 to 2014 (thick solid line). The data is from BEA (NIPA Table 5.4.1, line 14). According to Ohanian (2001), the aggregate TFP has dropped around 18% during the Great Depression. Thus, if there were interest-inelastic firms with persistent inaction periods, there must have been large echoes in the post-crisis period according to acceleration cost model prediction.

Testing whether certain fluctuations are from echo effects or from stochastic shock process is a demanding task. However, there is a clear difference between fluctuations from two sources: echo effects results in deterministic periodicity across the humps, while stochastic shocks lead to random periodicity. Therefore, the key to detect echo effects from a time-series hinges on the existence of deterministic periodicity.

After the Great depression in 1933, the growth rate in non-residential structures for manufacturing industry has fluctuated dramatically, as shown from the solid line in Figure 20. To statistically test the deterministic periodicity in these fluctuations, I use Fisher’s

47. Deterministic periodicity can happen in stochastic process with probability zero.
\(g\)-test, following Wichert et al. (2004). Fisher’s \(g\)-test tests deterministic periodicity in a time-series \(X_t\) by fitting the series into the following functional form:

\[
X_t = \beta \cos(\omega t + \phi) + \epsilon_t
\]

where \(\beta > 0\), \(\omega \in (0, \pi)\), \(\phi \sim U(-\pi, \pi]\), and \(\epsilon_t\) is a serially uncorrelated noise which is assumed to be independent from \(\phi\). And the null hypothesis \(H_0\) is as follows:

\[
H_0 : \beta = 0
\]

I apply this test to two sub-periods: 40 years right after the crisis (1933~1972) and the

Figure 20: Echo effect in investments of manufacturing industry after the Great Depression recent 40 years (1973~2012). I refer to the former period as echo period, and the latter as non-echo period. As reported in Table 10, the large fluctuations after the Great Depression, featured a significant deterministic periodicity, and the length of a period is around 4.7 years.\(^{48}\) However, the deterministic periodicity disappears in the recent years, which can be explained by decaying echo in the acceleration cost model.\(^{49}\) The dashed line in Figure 20 displays fitted time-series in Fisher’s \(g\)-test. During the echo period, fluctuations from

\(^{48}\) This result is robust over choices of sample periods and over test specifications such as extended \(g\)-test and likelihood-based tests.
\(^{49}\) I apply the same \(g\)-test to the model-generated investment growth rate fluctuations in the post-shock period plotted in Figure 8. The estimated duration of deterministic period is 4.17 years, and it is statistically significant.
the data and fitted series share the timings of ups and downs. However, this does not hold in the non-echo period. Consistently, Figure A.9.8 shows significant jump in the spectral density at the frequency of four to five years in the echo period. In the non-echo period, the spectral density does not display a peak. Table A.9.5 reports the serial correlation in the residuals. For echo periods, there was no significant serial correlation in the residuals. Thus, it validates the test result that is based on the assumption of serially uncorrelated errors.

<table>
<thead>
<tr>
<th></th>
<th>Echo period (1933 ~ 1972)</th>
<th>Non-echo period (1973 ~ 2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated period</td>
<td>4.706</td>
<td>5.714</td>
</tr>
<tr>
<td>p-value</td>
<td>0.024</td>
<td>0.165</td>
</tr>
</tbody>
</table>

Table 10: Periodicity testing: echo after the Great depression

Additionally, I apply the same test procedure to oil industry. I use the nonresidential fixed investment growth data from BEA (NIPA Table 5.4.1, line 20). The large aggregate TFP shock of interest is the oil crisis at 1979.\textsuperscript{50} Similar to the previous test, I divide the time-series into two sub-periods: 25 years right after the crisis (1933~1972) as echo period and 25 years prior to the crisis (1973~2012) as non-echo period.

![Figure 21: Echo effect in investments of oil industry after the oil crisis](image)

\textsuperscript{50} Specifically, I test investment growth in non-residential structures for industries of mining exploration, shafts, and wells.
As can be seen from the solid line in Figure 21, after the oil crisis, investment growth in non-residential structures featured larger fluctuations compared to pre-crisis period. According to the test results reported in Table 11, the investment growth during the post-crisis periods features significant deterministic periodicity. However, in the period prior to crisis, there was no deterministic periodicity.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated period</td>
<td>3.333</td>
<td>25.000</td>
</tr>
<tr>
<td>p-value</td>
<td>0.030</td>
<td>0.820</td>
</tr>
</tbody>
</table>

Table 11: Periodicity testing: echo after the oil crisis in 1979

Next, I document evidence on echo effects from VAR analysis on aggregate investment. Specifically, using VAR, I study whether there are nonlinear responses to an output shock that has a similar pattern as echo effects from the macro-level data. For the VAR on macro-level investment data, I include HP-filtered real GDP and HP-filtered investments in non-residential structures for manufacturing industry from National Income and Product Accounts data (NIPA), in the stated order.\(^5\) AIC criterion is used for the choice of optimal lags \((p = 4)\) in the regression.

![Figure 22: Impulse response of non-residential structure investment from NIPA data](image)

---

\(^5\) All the variables are at an annual frequency. In the Hodrick-Prescott filter, I use 6.25 as a smoothing parameter following Ravn and Uhlig (2002).
Figure 22 plots impulse responses of investments in non-residential structures for manufacturing industries from BEA (Fixed Asset Accounts Table 4.8, line 11 and line 15). As can be seen from the figure, nonlinear echo effects are present in the impulse responses of the investments of manufacturing industries. This evidence supports the echo component in the response of the aggregate investment to an aggregate shock.

1.7 Conclusion

This paper studies how interest-inelastic lumpy investments at the firm level affect the business cycle. From empirical analysis, I show large firms’ lumpy investments are inelastic to interest rate changes, and their inaction durations are highly persistent across periods. Then I develop a real business cycle model with heterogeneous firms that captures these two empirical facts. In the model, the aggregate investments display nonlinear impulse response to aggregate TFP shocks in general equilibrium. Specifically, there arise echoes of aggregate TFP shock in aggregate investment in the post-shock period. This is because synchronized timings of lumpy investments across large firms persistently synchronized over time due to weak flattening forces from factor prices and stochastic mean reversion. The endogenous fluctuations in large firms’ lumpy investments generate a cycle of relaxation and contraction in the large firms’ investment rate. After the high concentration, the economy becomes fragile to a negative aggregate TFP shock. The aggregate investment responds 29% stronger after the surge of large firms’ lumpy investments than a shock at the surge of lumpy investments. Then I decompose the total response into exogenous effect and endogenous effect. The endogenous effect accounts for up to 15% of aggregate investment response.

The acceleration cost model gives a theoretical framework that explains the endogenous business cycle when large firms’ lumpy investments are perfectly inelastic to factor prices and idiosyncratic shock in the extensive margin. The resulting stationary cycle in the
cyclical competitive equilibrium is a limit case of the baseline model’s decaying echoes in the post-shock periods.

For the echo effects and the state-dependent responsiveness of the aggregate investments, the key state variable is the fraction of large firms that are ready to make lumpy investments. A rise in this key state variable makes an economy fragile to a negative TFP shock in the following period. Therefore, this paper’s findings point out the necessity of a state-contingent stabilization policy based on micro-level observations.

Also, the acceleration cost model provides a meaningful monetary policy implication. According to the model, the fraction of large firms ready to make large-scale investment fluctuates. This implies the efficacy of monetary policy through the interest rate channel would also fluctuate. If there are a great number of large firms that are at the last stage for their large-scale investment, the economy will respond strongly to the monetary policy through the interest rate channel. On the other hand, if only a few firms are ready for large-scale investment, the monetary policy will not effectively work. I leave the optimal monetary policy design under the presence of interest-inelastic firms to future research.
Chapter 2

Top Income Inequality and the Business Cycle

This paper studies how the changes in the income distribution affected the aggregate fluctuations in the U.S. economy. One of the most notable changes in the income distribution of the U.S. during the recent 30 years is the rising top income inequality. In 1984, the top 1% and 0.1% income group’s income share was around 12% and 5%. In 2014, their income share marked around 19% and 9% of total income in the U.S, respectively. Over the same period, the business cycle statistics in the U.S. have dramatically changed. Among the changes, lower aggregate output volatility and the negative correlation between hours and labor productivity are conspicuous.

The main link between the two macroeconomic changes is the rising business income. The rising pass-through businesses have substantially contributed to the rise in the top income inequality. In 1984, 28% of top income earners’ income and 25% of top 0.1% income earners’ income were business income. 30 years later, 34% of top 1% income earners’ income and 37% of top 0.1% income earners’ income is from the pass-through businesses.

52. The numbers are from distributional national accounts (DINA) of Piketty et al. (2018).
These rising pass-through businesses generate significant changes in aggregate fluctuations because pass-through businesses are more financially constrained than C-corporations. Pass-through businesses are mostly owned by a single owner or a closed group such as a family. Therefore, their equity financing channel is limited by the nature of the closed ownership. This makes the pass-through businesses rely more on debt financing. Then, they tend to display less sensitivity to an aggregate productivity shock in the data. Instead, they sensitively respond to an aggregate financial shock. Using the SOI Integrated Business Data, I document that pass-through businesses display the explained patterns over the business cycle compared to the C-corporations.

To quantify how the rising top income inequality driven by pass-through businesses affects the business cycle, I develop a heterogeneous-household business cycle model that can capture the endogenous pass-through business formation and labor supply decision. In the model, the pass-through businesses face a financial constraint (maximal debt level), and the wealth of the owning household determines the constraint level. Then, I calibrate the model without an aggregate uncertainty separately for the 2010s and the early 1980s based on the empirical moments in the income distribution of the corresponding periods. Then, using the disciplined model, I study how the same exogenous fluctuations in aggregate productivity and aggregate financial condition affect the business cycles of the two different periods differently.

I additionally calibrate the model based on the same income distribution moments as in the 1980s, but I target the top income group share at the level of the 2010s. This counterfactual economy enables the quantification of two important channels through which the cross-sectional changes in the income affect the business cycle. The first channel is the top income inequality channel. By comparing the aggregate fluctuations in the 1980s with the counterfactual economy, I quantitatively analyze how the increased top income inequality affects the business cycle without changes in the income sources’ composition. The second channel is through the composition of income sources of the top income
earners. This channel is analyzed by the comparison between the economy in the 2010s and the counterfactual economy.

According to the calibrated model, top income inequality driven by the pass-through businesses dramatically changes the productivity-driven aggregate fluctuations. First, rising pass-through business dampens the output volatility by around 50% compared to the economy of the 1980s and 24% compared to the counterfactual economy. Pass-through businesses are more financially constrained than C-corporations. Therefore, their binding allocations do not sensitively respond to the aggregate TFP fluctuations. And the greater weight on pass-through businesses leads to a substantial dampening effect in the aggregate output. An increase in top income inequality also contributes to the output volatility change through the wealth effect in the saving behavior. However, this effect is only marginal compared to the channel of the composition of top income earners’ income source.

Second, the rising pass-through businesses make the labor productivity and the labor hours more negatively correlated. This is because the pass-through businesses’ low output volatility leads to a large invariant component in the aggregate output. This invariant part in the aggregate output generates an intercept effect in the relationship between aggregate output and the labor hours. Thus, the labor productivity and the labor hours become negatively correlated in the economy of the 2010s. I verify this effect through the data decomposition analysis. In the data, the negative correlation between the labor productivity and the labor hours becomes substantially flattened once the intercept effect from the pass-through businesses is removed from the aggregate output. On the other hand, in the economy of the 1980s and counterfactual economy, this effect is only negligible.

Related literature This paper is directly related to three strands of literature. The first is the literature that studies cross-sectional changes in the income and wealth distribution

53. The insensitivity of pass-through businesses are conditional on their operation (intensive-margin). However, the exit and entry of pass-through (extensive-margin) is more volatile than those of C-corporations. This is true in the data and also captured in the model.
in the economy. Rios-Rull and Kuhn (2016) and Piketty et al. (2018) have documented that the top income inequality has sharply increased in the U.S. Cooper et al. (2016) and Smith et al. (2019) document that the rising top income inequality is dominantly driven by pass-through businesses.\textsuperscript{54} Relatedly, Hubmer et al. (2020) argue the rising inequality in wealth has been substantially driven by wealth returns. This paper builds upon these facts investigated already in the literature. Given that the top income inequality has been strongly driven by pass-through businesses, I study how this trend has affected the aggregate fluctuations through a lens of a business cycle model with heterogeneous households. The stationary equilibrium of the model closely follows Quadrini (2000), Cagetti and De Nardi (2006), and Quadrini and Rios-Rull (2015). However, the model includes fluctuations in the aggregate TFP to explore the business cycle implications of the rising top income inequality.

The second strand of related literature is about the recent changes in the business cycle. Stock and Watson (2002) documents that the output volatility has dramatically dampened since the mid-1980s. This change has been referred to as the Great Moderation. According to Gali and Gambetti (2009), the Great Moderation has been accompanied by multi-dimensional changes in the business cycle. Especially, they document the volatility of hours relative to output has increased, and the correlation between labor productivity and hours has become negative. In the perspective of the Real Business Cycle models, the low correlation between labor productivity and labor hours had been a puzzling observation even before the Great Moderation started. It is because the aggregate productivity fluctuations make these two allocations co-move in the same direction in the RBC models. This has been one of the main rationales for the claim that RBC models cannot capture the realistic business cycle. Regarding this, by considering the extensive-margin labor supply of heterogeneous households, Chang and Kim (2007) capture realistic co-movements of allocations, including labor hours and labor productivity from the aggregate TFP fluctua-

\textsuperscript{54} The literature has not reached a consensus on whether to categorize the business income as a labor income or capital income. Many papers treat the business income as capital income.
tions. However, the recent negative correlation between these two allocations documented by Gali and Gambetti (2009) is substantially different from the low correlation that is close to zero in the existing models with aggregate TFP fluctuations. My paper introduces both endogenous labor supply and endogenous occupation choice in each household’s problem. Due to the financially constrained nature of pass-through businesses, realistic aggregate fluctuations happen in the model even when only aggregate TFP fluctuations are considered. On top of that, the quantitative analysis result shows that the entire economy becomes more sensitive to aggregate financial shocks. This gives a possible explanation of why the financial crisis has been more disastrous than the recessions before the crisis.

This paper is also related to the literature that studies how the business cycle affects agents in the economy differently across various dimensions. Castañeda et al. (2003) study how each income-level groups’ income share changes over the business cycle. They document that low-income groups’ income share is highly pro-cyclical over the business cycle, and the top income groups’ income share is acyclical over the business cycle. Kwark and Ma (2021) claims that the top income group’s income share is acyclical over the business cycle due to the endogenous change in the number of entrepreneurs in the top income group. Instead of studying how the business cycle affects the cross-section of the households’ income, my paper studies the relationship in a reverse direction. In this regard, the spirit of my paper follows Krueger et al. (2016), which studies how cross-sectional variations in the economy affect aggregate fluctuations. Especially, I argue that on top of the changes in the cross-section of the income distribution, the changes in the income source strongly affect the aggregate fluctuations in the economy.

**Roadmap** Section 2.1 explores empirical facts. Section 2.2 develops a business cycle model with heterogeneous households where endogenous labor supply and occupation choice are allowed. In Section 2.3, I explain calibration used for this model. Using the model under the calibrated parameters, Section 3.4 quantitatively analyze how the top
income earner’s pass-through business affect the business cycle. Section 3.5 concludes. Proofs and other detailed figures and tables are included in the appendices.

2.1 Empirical facts

2.1.1 Rising top income inequality and the pass-through businesses

Figure 1 plots the time-series of the top income group’s income share and the composition of the income source over the 60 years from 1957 to 2016. Panel (a) is for top 1% income group, and panel (b) is for top 0.1% income group. The data is pre-tax income from Piketty et al. (2018), which I refer to as PSZ hereafter. The top income share has shown fast growth over the last 30 years in the sample. In 1984, the top 1% and 0.1% income group’s income share was around 12% and 5%. 30 years later, their income share has become around 19% and 9% of total income in the U.S, respectively.

![Figure 1](image1.png)

(a) Top 1%

(b) Top 0.1%

Figure 1: Rising top income share and business income (from PSZ data)

Table 1 summarizes the annual growth rate of the income share of top income groups for two periods: 1) from 1960 to 1984 and 2) from 1985 to 2016. As reported in the first column, income inequality among the top 10% group has shown a declining trend during the first periods. However, the income inequality among the top 10% group has risen sharply in the following periods, as can be seen from the second column. The growth rate displays a monotonously increasing pattern along with the ranking of the income
group. Especially, the top 0.01% income group’s growth rate of income share (2.53%) was substantially higher than the other groups.

The earlier period’s diminishing income inequality was largely contributed by the shrinking labor income inequality, as reported in the fifth column. Top 0.5% income groups’ labor income share has declined faster than lower-ranked top income groups’ labor income share. In the recent rising income inequality among top income groups, the pass-through business has played a dominant role (the fourth column). The pass-through business growth rate has been greater than any top income group’s gross income share growth rate.

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Pass-Through</th>
<th>Labor Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre84</td>
<td>Post85</td>
<td>Pre84</td>
<td>Post85</td>
</tr>
<tr>
<td>Top 10%</td>
<td>0.02</td>
<td>0.76</td>
<td>1.22</td>
</tr>
<tr>
<td>Top 5%</td>
<td>-0.21</td>
<td>0.95</td>
<td>1.69</td>
</tr>
<tr>
<td>Top 1%</td>
<td>-0.48</td>
<td>1.32</td>
<td>1.6</td>
</tr>
<tr>
<td>Top 0.5%</td>
<td>-0.57</td>
<td>1.49</td>
<td>2.1</td>
</tr>
<tr>
<td>Top 0.1%</td>
<td>-0.45</td>
<td>1.87</td>
<td>2.61</td>
</tr>
<tr>
<td>Top 0.01%</td>
<td>-0.32</td>
<td>2.53</td>
<td>2.74</td>
</tr>
</tbody>
</table>

Table 1: Income share growth rate by top income group (from PSZ data)

The entire pass-through businesses have shown steady growth since 1980. Figure 2 plots the time-series of profit (panel (a)) and sales (panel (b)) for pass-through businesses (solid line) and C-corporations (dashed line).\textsuperscript{55} In the early 1980s, the total profit of pass-through businesses were only 20% of the entire profit in the U.S. economy. Then, by the end of the 1990s, the pass-through business has earned more profit than the rest of businesses, including C-corporations. In sales, pass-through businesses still take less portion than C-corporations do. However, pass-through businesses’ sales growth (6.8%, annual) was dominantly faster than C-corporations (2.4%, annual).

\textsuperscript{55} The data is from SOI Tax Stats - Integrated Business Data. The profits are “Net Income (less Deficit)” before tax, and the sales are “Business receipts.”
2.1.2 Changes in the business cycle

In this section, I summarize the facts about the changes in the cyclical behaviors of the aggregate allocations in the U.S. economy. Table 2 reports the major changes in the business cycle statistics. The data is the quarterly frequency and obtained from BEA. The first row reports the statistics in the period from 1947 Q1 to 1984 Q4. The second column reports the statistics for the period from 1985 Q1 until 2017 Q4. The third and fourth column is model-implied statistics from the standard RBC model and the heterogeneous-agent model with endogenous labor supply (Chang and Kim, 2007). All the variables are logged and HP-filtered with a smoothing parameter at 1600.

It has been vastly documented that the output volatility has been dramatically reduced since the mid-1980s. The first row reports that logged-output volatility has been reduced by almost half in the recent period. In the second row, the correlation between labor productivity and hours is reported. In the earlier period, the correlation is 0.18, which is much lower than standard RBC model’s statistics (0.93, the third column). By including the extensive-margin labor supply decision, Chang and Kim (2007) capture the low correlation between labor productivity and hours.

Gali and Gambetti (2009) have documented that the correlation between labor productivity and hours has become negative in recent years, as computed in the second column.

---

56. The data on the labor hour is from Cociuba et al. (2018).
of the second row (-0.46). Also, as in the third and fourth rows in the table, the volatilities of productivity and hours relative to output volatility have significantly increased. These changes are hardly captured in the existing real business cycle models. Gali and Gambetti (2009) argue that the non-real aggregate shock accounts for these changes in the business cycle statistics.

This paper shows that these changes are well-explained by the changes in the productivity-driven aggregate fluctuations induced from the cross-sectional changes in the economy. Especially, the rising pass-through businesses play a key role in capturing the changes in the business cycle. The next section analyzes the cyclical characteristics of the pass-through businesses compared to the C-corporations.

### 2.1.3 Cyclical characteristics of pass-through businesses

In this section, I compare the differences in the characteristics between the pass-through businesses and the C-corporations. I use sector-level balance sheet data from SOI Integrated Business Data. The data is the annual frequency, and covers from 1995 until 2016. Table 3 reports the volatilities of logged balance sheet items for C-corporations and pass-through businesses. The volatilities of C-corporations’ non-financial allocations and cash holdings are substantially greater than the pass-through businesses’. However, the debt-to-asset ratio of pass-through businesses is around three-times more volatile than the C-corporations’ debt-to-asset ratio.

---

57. The exogenous aggregate productivity process is assumed to stay the same.

---

<table>
<thead>
<tr>
<th></th>
<th>Pre-1984</th>
<th>Post-1985</th>
<th>RBC</th>
<th>Hetero. Labor supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>sd(Y)</td>
<td>2.59</td>
<td>1.23</td>
<td>1.60</td>
<td>1.28</td>
</tr>
<tr>
<td>Corr(Y/H,H)</td>
<td>0.18</td>
<td>-0.46</td>
<td>0.93</td>
<td>0.23</td>
</tr>
<tr>
<td>std(Y/H)/std(Y)</td>
<td>0.91</td>
<td>1.12</td>
<td>0.99</td>
<td>0.68</td>
</tr>
<tr>
<td>std(H)/std(Y)</td>
<td>0.79</td>
<td>1.20</td>
<td>0.44</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Table 2: Changes in business cycle statistics
Table 3: Volatilities of C-corporations and pass-through businesses

<table>
<thead>
<tr>
<th></th>
<th>C-Corp</th>
<th>Pass-Through</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma(\log(\text{Investment})) )</td>
<td>103.54</td>
<td>46.31</td>
</tr>
<tr>
<td>( \sigma(\log(\text{Profit})) )</td>
<td>1.42</td>
<td>0.56</td>
</tr>
<tr>
<td>( \sigma(\log(\text{Value-Added})) )</td>
<td>0.38</td>
<td>0.27</td>
</tr>
<tr>
<td>( \sigma(\log(\text{Labor})) )</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>( \sigma(\text{Debt/Asset}) )</td>
<td>0.56</td>
<td>1.54</td>
</tr>
<tr>
<td>( \sigma(\text{Cash/Asset}) )</td>
<td>7.86</td>
<td>3.87</td>
</tr>
</tbody>
</table>

Table 4: Cyclicality of C-corporations and pass-through businesses

<table>
<thead>
<tr>
<th></th>
<th>C-Corp</th>
<th>Pass-Through</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{corr}(\log(\text{Investment}),\log(Y)) )</td>
<td>0.64</td>
<td>0.80</td>
</tr>
<tr>
<td>( \text{corr}(\log(\text{Profit}),\log(Y)) )</td>
<td>0.57</td>
<td>0.75</td>
</tr>
<tr>
<td>( \text{corr}(\log(\text{Value-Added}),\log(Y)) )</td>
<td>0.68</td>
<td>0.81</td>
</tr>
<tr>
<td>( \text{corr}(\log(\text{Labor}),\log(Y)) )</td>
<td>0.92</td>
<td>0.76</td>
</tr>
<tr>
<td>( \text{corr}(\text{Debt/Asset},\log(Y)) )</td>
<td>0.13</td>
<td>-0.07</td>
</tr>
<tr>
<td>( \text{corr}(\text{Cash/Asset},\log(Y)) )</td>
<td>-0.34</td>
<td>-0.40</td>
</tr>
</tbody>
</table>

Table 4 reports the correlation coefficients between logged balance sheet items and the logged aggregate output. The balance sheet items of C-corporations and pass-through businesses share similar cyclical patterns except for the debt-to-asset ratio. The debt-to-asset ratio is weakly pro-cyclical for C-corporations, while it is weakly counter-cyclical for pass-through businesses. For both production sectors, investment, profit, value-added, and labor are all pro-cyclical allocations, and the cash-to-asset ratio is the counter-cyclical allocation.

Pass-through businesses and C-corporations display significantly different behavior during the recession. And the difference varies by the different recessions. There are two recessions in the sample periods: one is the dot-com bubble crash at 2001, and the other is the financial crisis in 2008. Table 5 reports the balance sheet items’ deviation from the trend by production sector and by different recessions.\footnote{58}{The deviation from the trend is normalized by the level of the trend. The trend is obtained from the HP-filter.} During the dot-com bubble...
crash, the impact of the recession on the investment and the value-added were much smaller in pass-through businesses than in C-corporations. However, the debt-to-asset ratio responded more strongly in pass-through businesses than C-corporations. However, during the financial crisis, the recession impacted the pass-through businesses’ investment stronger than it did the C-corporations’ investment. The value-added has responded at a similar rate between the two sectors. The debt-to-asset ratio of pass-through businesses increased by a substantially greater rate than C-corporations’ ratio during the financial crisis.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Δlog(Investment)</td>
<td>-377.43</td>
<td>-59.93</td>
<td>-124.08</td>
<td>-150.80</td>
</tr>
<tr>
<td>Δlog(Value-Added)</td>
<td>-0.60</td>
<td>-0.37</td>
<td>-0.69</td>
<td>-0.63</td>
</tr>
<tr>
<td>Δ(Debt/Asset)</td>
<td>0.46</td>
<td>1.05</td>
<td>1.30</td>
<td>4.18</td>
</tr>
</tbody>
</table>

Table 5: Different responses to two recessions: Dot-com bubble crash vs. Financial crisis

Figure 3 plots the time-series of each sector’s balance sheet items. The dashed line represents C-corporations, and the solid line represents pass-through businesses. As reported in Table 3, C-corporations’ real allocations such as investment, profit, value-added, and labor expenditure and cash-to-asset ratio are more volatile than pass-through businesses’ allocations. However, the debt-to-asset ratio is more volatile in pass-through businesses than in C-corporations.

The investment rate (panel (a)) of C-corporations significantly dropped during the dot-com bubble crash. During the same period, the pass-through businesses barely display any decrease in the rate. Profit, value-added, and labor expenditure show similar pro-cyclical patterns over the business cycle, and C-corporations’ profit features substantially greater volatility than pass-through businesses. In contrast, the debt-to-asset ratio is more volatile and counter-cyclical in pass-through businesses than in C-corporation.
In this section, I summarize the cyclical characteristics of pass-through businesses and discuss the reason for their specific behaviors over the business cycle.

To summarize the empirical facts about the cyclical characteristics of pass-through businesses:
1. Pass-through businesses feature low volatility in real allocations while their debt-to-asset ratio is highly volatile.

2. During the dot-com bubble crash, which is more associated with a negative productivity shock than the financial crisis, pass-through businesses’ allocations are not strongly affected.

3. During the financial crisis, all the real allocations showed a significant drop in the level, while the debt-to-asset ratio has dramatically increased.

A hypothesis that can coherently explain all the facts above is that pass-through businesses are financially constrained firms. Financially constrained firms cannot flexibly adjust their allocations against the real shocks as their original allocations were at the constrained level. Therefore, their allocations feature less volatility over the aggregate productivity fluctuations. However, the financially-constrained firms’ allocations strongly respond to the fluctuations in the financing conditions. Figure 4 plots the deb-to-capital ratio of pass-through businesses and C-corporations.\(^{59}\) As shown in Figure 3, the debt-to-asset ratio has increased during the recessions, but debt-to-capital has dramatically decreased during the recessions. Given the capital is a collateralizable asset, deterioration in the financing condition would decrease the debt-to-capital ratio. And the rise in the debt-to-asset ratio is driven by a shrink in the non-debt assets in the balance sheet.

Therefore, the hypothesis of financially constrained pass-through businesses is well supported by the observed cyclical patterns in the balance sheet items. Consistent with the evidence, pass-through businesses are severely constrained by the equity financing channel by the nature of closed ownership structure. These firms are extensively owned by a family or a small group of owners, so they cannot easily liquidate their ownerships to external investors. Therefore, their financing naturally tends to rely more on debt financing rather than equity financing. This makes pass-through businesses become debt constrained.

\(^{59}\) The capital is depreciable assets reported in the balance sheet. I interpret this capital as a tangible capital that can be collateralized.
In this paper, I study a business cycle model where heterogeneous households decide labor supply and occupation. In the model, there are two production sectors: pass-through businesses and C-corporations. The critical difference between these two sectors is the pass-through businesses are financially constrained. Through the lens of the model, I study how the cross-sectional changes in the economy affect the aggregate fluctuations given the exogenous aggregate shocks fixed.

2.2 Model

2.2.1 Household

A measure one of a continuum of ex-ante homogenous households is considered. Time is discrete, and households live forever. Each household consumes, supplies labor, and saves
wealth which could be flexibly used for production in the C-corporation sector or in the pass-through business. Household’s temporal utility takes the following form:

\[ \log(c_t) - h_t \xi \quad (h_t \in \{0, \bar{h}\}, \quad \xi \sim_{iid} Unif([0, \xi_0])) \]

where \( \bar{h} = 1/3 \) is the full-time working hours and \( \xi \) is the labor disutility drawn from \( U([0, \xi_0]) \).

At the beginning of each period, households are given the wealth level \( a_t \), managerial ability \( z_t \), and labor efficiency \( x_t \). Managerial ability and labor efficiency \( x_t \) follow Markov processes specified as follows:

\[
\begin{align*}
\ln(\ln(z_{t+1})) &= \rho_z \ln(\ln(z_t)) + \sigma_z \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \tilde{N}(0, 1) \\
\ln(\ln(x_{t+1})) &= \rho_x \ln(\ln(x_t)) + \sigma_x \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \tilde{N}(0, 1)
\end{align*}
\]

where \( \tilde{N}(0, 1) \) represents a folded standard normal distribution.\(^{60}\) In the computation, I discretize each process using seven grid points based on the Tauchen Method. In the discretization, the double-logged grid points \( \{\ln(\ln(z_t))\}_{t=1}^{7} \), for \( \zeta \in \{x, z\} \) are equally spaced in the interval of \( ([0, 5\sigma_\zeta/(1 - \rho_\zeta^2)^{1/2}]) \).

Given labor efficiency \( x_t \) and wage \( w_t \), a household earns labor income \( x_t w_t \bar{h} \) if it becomes a worker. If they do not work, they make zero labor income. Lastly, if they choose to become an entrepreneur, they earn a pass-through business profit, which will be specified in the next section.

Owning a pass-through business is assumed to incur the same labor disutility as the full-time worker’s labor disutility. Households are subject to borrowing constraint \( a_{t+1} \geq 0 \) as in the standard incomplete market models (Aiyagari, 1994).

\( ^{60} \) For a random variable \( X \), the following equivalence holds: \( X \sim \tilde{N}(0, 1) \iff |X| \sim N(0, 1) \).
Pass-through business

Household earns a pass-through business profit when it operates a business by combining own managerial ability, capital, and labor. Specifically, the production function takes the following Cobb-Douglas form:

$$z_t A_t \left( \frac{k_t^{1-a} l_t}{d_t} \right)^\gamma$$

where $z_t$ is the given managerial ability; $A_t$ is an aggregate TFP; $k_t$ is capital stock; $l_{d,t}$ is labor demand; $\gamma$ is the parameter that governs the span of control. The span-of-control parameter plays a crucial role in determining the thickness of the tail in the entrepreneurs’ income distribution. Aggregate productivity shock $A_t$ follows following AR(1) stochastic process:

$$\ln(A_{t+1}) = \rho_A \ln(A_t) + \sigma_A \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, 1)$$

The stochastic process $\{A_t\}_{t=1}^\infty$ is discretized into three states using the Tauchen method. In the discretization, the logged aggregate productivity grid points $\{\ln(A_t)\}_{i=1}^3$ are equally spaced in the interval $([-\sigma_A/(1 - \rho_A^2)_{1/2}, \sigma_A/(1 - \rho_A^2)_{1/2}])$.

Following the literature that studies occupation choice of entrepreneurs (Buera and Shin, 2013; Cagetti and De Nardi, 2006; Quadrini, 1999, 2000; Evans and Jovanovic, 1989), I assume entrepreneurs are subject to a financing constraint that limits the size of capital borrowing to a fraction of given wealth $a_t$. Specifically, I assume $k_t \leq a_t / \lambda$, where $\lambda > 0$ is the parameter that determines the level of financial constraint. Thus, $\lambda \to 0$ implies there is no financial constraint.

Entrepreneur maximizes profit using the production function explained above. Given rental rate $r_t$ and wage $w_t$, business profit $\pi_t$ of an entrepreneur with wealth $a_t$ and managerial ability $z_t$ is

$$\pi_t := \pi(a_t, z_t) = \max_{k_t \leq a_t / \lambda, l_{d,t}} z_t A_t \left( \frac{k_t^{1-a} l_t}{d_t} \right)^\gamma - w_t l_{d,t} - (r_t + \delta) k_t$$

88
Tax function

Following the literature studying progressivity of income taxation (Benabou, 2002; Heathcote et al., 2017; Holter et al., 2019; Luduvice, 2020), I assume the following parametric tax function:

$$\tau(y) = (y - \theta_0 y^{1-\theta_1})/y$$

where $\tau(y)$ is the tax rate for a household with income level $y$. In the literature, capital income and labor income tax functions are often differently treated to capture the actual tax policy. However, as this paper focuses on the U.S., where the interest income and the labor income are taxed at the same rate, I do not distinguish capital income tax and labor income tax.

Tax revenue at time $t$ is spent out as a uniform lump-sum subsidy $T_t$.

$$\int \tau(y_t(a, z, x, j)) y_t(a, z, x, j) d\Phi_t(a, z, x, j) = T_t$$

where $a$ is a wealth level; $z$ is an idiosyncratic managerial ability; $x$ is an idiosyncratic labor efficiency; $j$ is an occupation type; $\Phi_t$ is the distribution of $(a, z, j)$ at time $t$.

2.2.2 C-corporation sector

The C-corporation sector is separately introduced on top of the pass-through business sector. The main difference between the C-corporation and the pass-through business is in the taxation of the profit. A pass-through business is a type of legal entity where the flow of income is regarded as the owner’s individual income. From this definition, my model assumes the owner has the claim for the whole operating profit.\(^{61}\)

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\(^{61}\) Among pass-through businesses, some entities such as partnerships also split the operating profits due to the multiple ownerships. For this, I regard each partner as operating a separate business even though they share the same business.
In contrast, there are non-pass-through entities, where each of the shareholders holds a claim for only a part of the whole profit. C-corporation takes the dominant portion of this class of business. I model this sector as perfectly competitive and assume all the revenues are expensed out to factor costs. Specifically, the C-corporation sector’s production function takes the following Cobb-Douglas form:

\[ F(A_t, K_t, L_t) = A_t K_t^a L_t^{1-a} \]

where \( A_t \) is aggregate TFP as defined above; \( K_t \) is capital stock used in the C-corporation sector; \( L_t \) is aggregate hours to be spent on the C-corporation sector.

Given Aggregate productivity \( A_t \), wage \( w_t \), and rental rate \( r_t \), the C-corporation sector’s problem is described as follows:

\[
\max_{K_t, L_t} A_t K_t^a L_t^{1-a} - w_t L_t - r_t K_t
\]

Input factors in the C-corporation sector and input factors in the pass-through business sector are assumed to be perfectly substitutable. Thus, in general equilibrium, the price of input factors is determined at the level where each factor’s supply meets the combined factor demand of C-corporation and private businesses.

In the literature, several different assumptions have been imposed on the competitive input market. Quadrini (2000) modeled the intermediation sector, which charges extra cost for households’ borrowing.\(^6\) Thus, the public production sector’s financing cost is lower than households’ financing cost in the model. In Cagetti and De Nardi (2006), all the labor demand is from the non-entrepreneurial production sector. This paper’s input market is closely following Kwark and Ma (2021). In the dynamic stochastic general equilibrium,

---

\(^6\) In Buera and Shin (2013), the input market is also competitive, but there is no separate public production sector.
the interplay between C-corporations and pass-through sectors through the factor price channel builds the core dynamics of aggregate allocations and the entry of businesses.

2.2.3 Occupation choice

Given managerial ability $z_t$, labor efficiency $x_t$, wealth $a_t$, aggregate productivity $A_t$, wage $w_t$ and rental rate $r_t$, a household willing to supply labor, decides on the occupation between entrepreneur and labor worker. As both occupations share the same disutility of labor, the occupation choice between two is determined from the income level one can earn using his/her wealth and ability. Therefore, the decision problem could be expressed in the following static form:

$$
\max \{ \pi_t, w_t x_t l_t, w_t x_t h_t \} = \max \{ \max_{k_t \leq a_t / \lambda} z_t A_t \left( \frac{k_t^{1-a}}{k_t^{1-a}} \right)^{\gamma} - w_t l_t - (r_t + \delta) k_t, w_t x_t h_t \}
$$

A household becomes an entrepreneur if pass-through business income exceeds the factor income he would get from working and capital rent. I formally state the condition when the household decides to become an entrepreneur.

**Proposition 3. (Occupation choice threshold)**

Given $(z_t, x_t, A_t, w_t, r_t)$, there exists $\bar{a}_t \in [0, \infty]$ such that a household decides to become an entrepreneur if

$$
a_t \geq \bar{a}_t = \pi(z_t, x_t; A_t, w_t, r_t)
$$

**Proof.** See Appendix B.1.1.

Proposition 3 states that if a household holds a wealth level beyond a certain threshold, it becomes an entrepreneur. In the proof of the proposition, the key step is to find

63. The occupation choice is a static problem. However, persistent idiosyncratic ability processes and smooth changes in wealth warrant infrequent occupation change in the model. The theoretical predictions are consistent with the model assuming pre-commitment to the occupation before the contemporaneous idiosyncratic abilities are realized (Bohacek, 2006).
$\bar{z}_t = \bar{z}(z_t, x_t, A_t, w_t, r_t)$ such that only if $z_t > \bar{z}_t$, the wealth level $\bar{a}$ exists, where the household indifferent between two occupations. Thus, the following equation holds:

$$x_t w_t \bar{h} = (z_t, A_t)^{-\frac{1}{1-\sigma}} (1 - \gamma (1 - a)) \left( \frac{\gamma (1 - a)}{w_t} \right)^{\frac{1}{1-\sigma}} \left( \frac{\bar{a}}{\lambda} \right)^{\frac{\sigma \gamma}{(1-\sigma)}} - (r_t + \delta) \left( \frac{\bar{a}}{\lambda} \right)$$

This $\bar{z}_t$ has an important implication for entrepreneurship choice because $z_t < \bar{z}_t$ implies the household cannot become an entrepreneur regardless of the wealth level. In other words, there exists a minimum requirement of managerial ability for entrepreneurship that cannot be complemented by large wealth. This is formally stated in the following Corollary 3.

**Corollary 3. (Minimum requirement for managerial ability)**

Given $(x_t, A_t, w_t, r_t)$, there exists $\bar{z}_t = \bar{z}(x_t; A_t, w_t, r_t)$ such that a household with $z_t < \bar{z}_t$ cannot become an entrepreneur at any wealth $a_t > 0$.

**Proof.** See Appendix B.1.2.

If managerial ability $z_t$ is highly persistent, there exists a diverging saving motivation between households with $z_t < \bar{z}$ and households with $z_t > \bar{z}$. For low $z_t$ households, cumulated wealth can be used for own business only in the far future. Therefore, large-scale saving is not as appealing as it is to high $z_t$ households. From the comparative statics around the stationary equilibrium, I numerically check that higher persistence in $z_t$ intensifies income and wealth inequality in the equilibrium. In other words, persistence in labor efficiency intensifies wealth inequality induced from a different saving motivation based on the heterogeneous wealth return, as in Fagereng et al. (2020). Also, for $z_t > \bar{z}_t$, $\bar{a}_t$ decreases in $z_t$. Thus, households with better managerial ability can become an entrepreneur with relatively lower wealth. This theoretical prediction is consistent with the well-known results in the literature (Quadrini, 2009; Bohacek, 2006). Corollary 4 formally states this as follows:
Corollary 4. (Monotonicity of threshold)

Given \((z_t, x_t, A_t, w_t, r_t)\), for \(z_t > z_t\),

\[
\tilde{z}_t > z_t \implies \tilde{\pi}(\tilde{z}_t, x_t, A_t, w_t, r_t) < \tilde{\pi}(z_t, x_t, A_t, w_t, r_t)
\]

Proof. See Appendix B.1.3. \(\blacksquare\)

2.2.4 Recursive formulation

I define the following set of value functions:

\[
\{V, V_E, V_W, V_N\}
\]

where \(V\) is an interim value function; \(V_E\) is a value function of an entrepreneur; \(V_W\) is a value function of a worker; \(V_N\) is a value function of a non-worker.\(^{64}\)

\[
V_j(a, z, x; A, \Phi) = \max_{c, a'} \log(c) + \beta E V(a', z', x'; A, \Phi')
\]

s.t.

\[
c + a' = T(A, \Phi) + a + \left(\pi(a, z; A, \Phi)I\{j = E\}\right) + \left(w(A, \Phi) x h I\{j = W\} + ar(A, \Phi)\right) (1 - \tau(a, z; A, \Phi))
\]

\[
a' \geq 0, \quad j \in \{E, W, N\}, \quad \Phi' = G_\Phi(\Phi, A),
\]

\[
\log(A') = \rho_A \log(A) + \sigma_A, \quad \log(z') = \rho_z \log(z) + \sigma_z, \quad x' \sim \pi(x'|x), x' \in [x_0, x_1, x_2]
\]

\(A\) is the aggregate TFP following AR(1) process; \(\Phi\) is the distribution of households’ individual states. When a household is an entrepreneur \((j = E)\), it earns business profit \(\pi\); when a household is a worker \((j = W)\), it earns labor income \(w(A, \Phi)z h\).

\(^{64}\) Labor disutility is considered when the interim value function \(V\) is defined.
The interim value function $V$ is defined as follows:

$$V(a, z; A, \Phi) := \int_0^\xi \max \{ \max \{ V_E(a, z; A, \Phi), V_W(a, z; A, \Phi) \} - \bar{h}_\xi, V_N(a, z; A, \Phi) \} \left( \frac{1}{\xi} \right) d\xi$$

The occupation choice between entrepreneur and worker has a closed-form characterization, explained in the previous section. For the labor supply decision, choice-specific labor disutility shock smoothens the value function around the indifference point. Thus, $V$ can be interpolated smoothly without concern about the kink point.

### 2.2.5 Equilibrium

I assume factor markets are competitive. Thus, both pass-through businesses and C-corporations can use a unit of labor and capital stock at the same prices for each input. The clearing condition for the capital market is as follows:

$$\int k(a, z; A, \Phi) \mathbb{I} \{ a \geq \bar{a}(a, z; A, \Phi) \} d\Phi(a, z, h) + K(A, \Phi) = \int ad\Phi(a, z, h)$$

The clearing condition for labor market is as follows:

$$\int l_d(a, z; A, \Phi) \mathbb{I} \{ a \geq \bar{a}(a, z; A, \Phi) \} d\Phi(a, z, h) + L(A, \Phi) = \int z\bar{h} \mathbb{I} \{ h = \bar{h} \} \mathbb{I} \{ a < \bar{a}(a, z; A, \Phi) \} d\Phi(a, z, h)$$

Based on the market clearing conditions above, I formally define recursive competitive equilibrium as follows:
Definition 4. (Recursive Competitive Equilibrium)

\((g_c, g_a, g_{Occ}, V, V_E, V_W, V_N, \tilde{K}, \tilde{L}, w, r, G_F)\) are recursive competitive equilibrium if

1. \(g_c, g_a, g_k, g_l, V, V_E, V_W, V_N : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}^\infty) \to \mathbb{R}\), solve the household’s problem. Note that \(\mathbb{R}^\infty\) is a set of all distributions of individual state variables \((a, z, x)\).

2. \(\tilde{K}, \tilde{L} : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}\) solves C-corporation’s problem.

3. Market clearing: \(w, r : (\mathbb{R} \times \mathbb{R}^\infty) \to \mathbb{R}\) are set to satisfy

   \[
   \int k(a, z, x; A, \Phi)I\{a \geq \pi(a, z, x; A, \Phi)\}d\Phi + \tilde{K}(A, \Phi) = \int a d\Phi \\
   \int l_d(a, z, x; A, \Phi)I\{a \geq \pi(a, z, x; A, \Phi)\}d\Phi + \tilde{L}(A, \Phi) = \int x\pi I\{h = \pi\}I\{a < \pi(a, z, x; A, \Phi)\}d\Phi
   \]

4. Lump-sum subsidy:

   \[
   T(A, \Phi) = \int \left(\pi(a, z, x; A, \Phi)I\{j = E\} + w(A, \Phi)x\pi I\{j = W\} + ar(A, \Phi)\right)\tau(a, z, x; A, \Phi)d\Phi
   \]

5. Consistency condition: the law of motion of \(\Phi\) is consistent with the household’s inter-temporal saving policy \(g_a\)

Due to the non-trivial market-clearing condition and a lump-sum subsidy, the standard Krusell and Smith (1998) algorithm is not easily applicable to the recursive competitive equilibrium computation. Also, due to heterogeneous labor demand sensitivity between path-through business and C-corporation, aggregate labor dynamics become highly non-linear. For this problem, I use the repeated transition method, which I concurrently developed in Lee (2020a). In this method, aggregate allocations’ expected dynamics,
including market-clearing prices, are directly calculated on the simulated path during the
iterative computation. Also, the method does not rely on a parametric assumption on
the law of motions of aggregate states. Thus, the recursive competitive equilibrium can
be computed accurately despite the non-trivial market-clearing conditions and nonlinear
dynamics.

2.3 Calibration

The model is calibrated to match key moments in the data. For basic parameters such as a
discount factor \( \beta \) and a depreciation rate \( \delta \), I fixed them following the standard level in the
literature. Those parameters are reported in Table B.3.3.

The model has been calibrated separately for the early 1980s and the 2010s periods.
Additionally, the model is calibrated for a counterfactual economy where the top income
shares are identical to the 2010’s economy, while factor incomes drives the top income
inequality.\(^{65}\)

I calibrate the model parameters in two steps. The first step is calibrating model
parameters to match cross-sectional moments of the stationary equilibrium with the
data counterparts. The target moments and corresponding parameters for the 2010s are
summarized in Table 1. The other period calibration results are summarized in Table B.2.1
and Table B.2.2. The second step is to calibrate the parameters that govern the aggregate
TFP process based on the stochastic dynamic general equilibrium outcomes. The TFP
process is calibrated using the cross-sectional parameters fitted to the 2010s.

The employment-to-population ratio is targeted at 58.9\%, and this moment identifies
the labor disutility parameter \( \xi \). The target moment is from the U.S. Bureau of Labor
Statistics. The debt-to-asset ratio of pass-through business is from the IRS SOI Integrated
Business Data. This moment identifies the parameter of the financial constraint, \( \lambda \). At

\(^{65}\) Specifically, the shares of business income out of top income groups’ total income are fixed at the level of
the 1980s.
the stationary equilibrium, almost all pass-through businesses are financially constrained. However, over the business cycle, this constraint binds occasionally. The target level of the value-add ratio between pass-through businesses and C-corporations is also from IRS SOI Integrated Business Data. This moment identifies the pass-through specific productivity level $A$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Target Moments</th>
<th>Data</th>
<th>Model</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>Employment/Population</td>
<td>58.9</td>
<td>57.5</td>
<td>1.25</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Debt/Asset of path-through businesses</td>
<td>45.6</td>
<td>45.6</td>
<td>0.544</td>
</tr>
<tr>
<td>$A$</td>
<td>Value-Add ratio between path-through and C-corp.</td>
<td>75.3</td>
<td>66.5</td>
<td>0.352</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Top 10% income share</td>
<td>46.3</td>
<td>62.1</td>
<td>0.945</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>Top 1% income share</td>
<td>19.0</td>
<td>21.8</td>
<td>0.688</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>Top 0.1% income share</td>
<td>8.9</td>
<td>7.8</td>
<td>0.173</td>
</tr>
<tr>
<td>$\rho_x$</td>
<td>Top 10% business income share</td>
<td>20.9</td>
<td>31</td>
<td>0.902</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>Top 1% business income share</td>
<td>33.2</td>
<td>33.4</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>Top 0.1% business income share</td>
<td>37.1</td>
<td>35.9</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Target moments of the economy of the 2010s

The next moments are related to the top income inequality in the economy. Top income 0.1%, 1%, and 10% earners’ income shares of total income is targeted at 8.9%, 19.0%, and 46.3% based on the PSZ distributional national accounts (DINA). Also, the business income share is matched for each top income group. The span-of-control parameter $\gamma$ and the parameters of idiosyncratic labor shock process $\rho$ and $\sigma$ are jointly identified by matching the target moments. Especially, the span-of-control parameter governs the business income distribution’s thickness of the right tail. In the calibrated results, this parameter $\gamma$ has been dramatically changed between the 1980s and 2010s. This is consistent with the fact that the rising top income inequality has been driven by the thickened tail of business income distribution. However, the counterfactual economy’s span-of-control parameter stays the same as in the 1980s.

Then, I calibrate the aggregate TFP process. I first compute the Solow residual of the production sector and then fit the time series into the AR(1) process. As the aggregate production side of the economy is composed of two sectors, Solow residual follows a
different process than the model TFP shock process. I calibrate this AR(1) process to have the auto-correlation of 0.95 and the standard deviation of 0.007 following Kydland and Prescott (1982).

The calibrated aggregate TFP process is as follows:

\[
TFP_{t+1} = 0.87 \times TFP_t + \epsilon \quad \epsilon \sim N(0, 0.003)
\]

Under the TFP process above, the Solow residual’s autocorrelation becomes 0.949 and the shock volatility is 0.008.

2.4 Quantitative analysis

In this section, I quantitatively analyze how the model economy behaves over the business cycle.

2.4.1 Business cycle analysis

Table 7 reports the correlations of aggregate allocations that display significant differences across different cross-sectional calibrations. The first column is baseline outcome; the second is based on the early 1980’s calibration; the third is based on the counterfactual calibration; the fourth and the fifth columns are about pre-1984 and post-1985 periods, respectively.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Baseline</th>
<th>1980</th>
<th>CF</th>
<th>Pre84</th>
<th>Post85</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Corr}(\log(Y_{t-1}), \log(Y_t)))</td>
<td>0.960</td>
<td>0.986</td>
<td>0.972</td>
<td>0.841</td>
<td>0.886</td>
</tr>
<tr>
<td>(\text{Corr}(\log(Y_t/H_t), \log(H_t)))</td>
<td>-0.526</td>
<td>0.224</td>
<td>-0.073</td>
<td>-0.218</td>
<td>-0.526</td>
</tr>
<tr>
<td>(\text{Corr}(\log(1+r_t), \log(Y_t)))</td>
<td>0.357</td>
<td>-0.690</td>
<td>-0.247</td>
<td>-0.058</td>
<td>0.164</td>
</tr>
</tbody>
</table>

Table 7: Time-series correlations
Under the same calibrated productivity fluctuations, the aggregate output displays lower autocorrelation than in the early 1980s and in the counterfactual economy. This is consistent with the changes observed in the data. The relationship between the labor hours and labor productivity is well captured. As the business income drives the top income inequality (a change from the early 1980s to the baseline), the correlation between the labor hours and labor productivity becomes substantially negative. However, if the factor income drives the top income inequality (CF), the correlation does not drop as starkly as in the baseline case. The correlation between real gross interest rate and the output increases in recent years, and this is the consistent change with the observed patterns in the data.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& \text{Baseline} & 1980 & \text{CF} & \text{Pre84} & \text{Post85} \\
\hline
\sigma(\log(Y_t)) & 0.010 & 0.020 & 0.013 & 0.020 & 0.011 \\
\sigma(\log(Y_t/H_t))/\sigma(\log(Y_t)) & 1.068 & 0.952 & 0.998 & 0.612 & 0.654 \\
\hline
\end{array}
\]

Table 8: Time-series volatilities

Table 8 reports the volatility of allocations that display substantial differences across different cross-sectional calibrations. Each column indicates the same model as Table 7. Even if the same aggregate productivity process is assumed, the output volatility drops by 50% in the baseline compared to the one in the early 1980s. Also, the counterfactual economy displays a large drop in output volatility. The relative volatility of the labor productivity has increased in the 2010s compared to the 1980s, and this change is well-captured by the cross-sectional variations in the model. However, the level of the relative volatility of labor hours is dramatically different from the observed level. This is possibly due to a lack of non-technological shocks in the model, which might be prevalent in reality.

2.4.2 Intercept effect and empirical evidence

According to the computation results, the aggregate TFP fluctuation leads to the co-movement of labor hours and labor productivity in the opposite direction. In this section, I
analyze why the correlation becomes negative in the economy with top income inequality driven by pass-through businesses.

The aggregate output (GDP), $Y^A$, can be decomposed into pass-through businesses’ output ($y$) and C-corporations’ output ($Y$). Then, the following equations are immediate from the first order condition of the C-corporation sector:

$$Y^A = y + Y$$
$$= y + \frac{w(A, \Phi)}{(1 - \alpha)} L$$
$$= y + \frac{w(A, \Phi)}{(1 - \alpha)} (L^A - \ell_d)$$
$$= \left( y - \frac{w(A, \Phi)}{(1 - \alpha)} \ell_d \right) + \frac{w(A, \Phi)}{(1 - \alpha)} L^A$$

where $\ell_d$ denotes the labor demand of pass-through businesses and $L$ denotes the labor demand of C-corporations. In the model, labor hour is an affine function of labor demand.

$$h_d = \bar{h} \ast \ell_d, \quad H^A = \bar{h} \ast L^A$$

This leads to the following decomposition:

$$Y^A = \left( y - \frac{w(A, \Phi)}{\bar{h}(1 - \alpha)} \ell_d \right) + \frac{w(A, \Phi)}{\bar{h}(1 - \alpha)} H^A$$

Then, the pass-through businesses’ output and labor hours generate the intercept effect in the relationship between aggregate output and hours. This generates a negative correlation between labor productivity and hours:

$$\text{Productivity} = \frac{\left( y - \frac{w(A, \Phi)}{\bar{h}(1 - \alpha)} \ell_d \right)}{H^A} + \frac{\frac{w(A, \Phi)}{\bar{h}(1 - \alpha)}}{H^A} H^A$$
The negative relationship is illustrated in Figure 6. If an hour increases, the slope of the ray that passes the origin and the coordinate of hours and output decreases due to the presence of intercept in the graph. And the intercept moves only marginally over the business cycle as the pass-through businesses are financially constrained.

![Figure 5: Intercept effect in the relationship between output and hours](image)

The passes-through business allocations \(y_d\) and \(h_d\) are observable together with the aggregate allocations and the wage. Using the data, I decompose the labor productivity as in the equation above. 66 Figure 6 shows the scatter plot of labor productivity and labor hours with the intercept term (panel (a)) and without the intercept term (panel (b)). With the intercept term, labor hours and productivity display negative correlation over the business cycle. However, when the intercept term is removed, the relationship between labor hours and productivity becomes flat, consistent with the near-zero correlation

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66. \(y_d\) is computed by the value-add of pass-through businesses, and \(w(A, \Phi)h_d\) is directly from the labor expenditure in the balance sheet. Sector-level allocations of pass-through businesses and C-corporations do not exactly add up to the aggregate-level allocations. Therefore, I decomposed aggregate allocations based on the weight between passes-through businesses and C-corporations.
Figure 6: Mitigated negative relationship between productivity and hours after removing the intercept between the two allocations in the 1980s before the pass-through businesses has risen. This result verifies the presence of the intercept effect from the pass-through businesses.

2.5 Conclusion

In this paper, I study how top income inequality driven by pass-through business affects the business cycle through a lens of a heterogeneous-household business model with endogenous labor supply and occupation choice.

According to the quantitative analysis based on the calibrated model, the top income inequality driven by pass-through business affects the productivity-driven aggregate fluctuations in two ways. First, the output volatility is reduced due to the rising importance of financially constrained pass-through businesses. Second, pass-through businesses generate an intercept effect in the aggregate output and labor hours relationship, leading to a negative correlation between labor productivity and labor hours. Also, when the pass-through businesses drive the top income inequality, the economy becomes more sensitive to a financial shock.
The paper’s quantitative theory provides a useful tool to understand how the changes in the cross-section of the households’ income sources affect the business cycle. At the same time, it gives an analytical framework to analyze how a fiscal policy affects the business cycle through the changes in the cross-sectional income distribution. For example, fiscal policies that support small businesses, such as the Tax Reform Act of 1986, have boosted pass-through businesses’ entry. This change affects aggregate fluctuations due to the financially constrained nature of pass-through businesses. I leave the quantitative analysis on how the fiscal policy changes have affected the business cycle through the cross-sectional changes to future research.
Chapter 3

Aggregate Uncertainty and the Repeated Transition Method

In this paper, I introduce an algorithm that solves a heterogeneous agent model with aggregate uncertainty that is free from the law of motion specification. I name the algorithm as repeated transition method.

Under the rational expectation, heterogeneous agents are aware of the true law of motion in the aggregate states and make a correct prediction on the future aggregate state. In contrast, there is no specific form of the law of motion known to a researcher. And it is computationally costly to track the evolution of a distribution that is an infinite-dimensional object. To overcome this problem, Krusell and Smith (1998) suggested a log-linear prediction rule of the finite number of moments of the individual state distribution as an approximation to the true law of motion. Afterward, numerous research papers in the literature have found this prediction rule gives a surprisingly accurate approximation to the true law of motion in the broad class of heterogeneous agent models with aggregate uncertainty.
Still, there are macroeconomic environments where the log-linear rule does not apply. A dynamics of aggregate allocations subject to occasionally binding constraints are an example of such cases (Fernandez-Villaverde et al., 2020). Also, history dependence in the investment dynamics, as in Lee (2020b), makes it difficult to approximate the true law of motion using the log-linear specification. According to Krusell and Smith (1997) and Krusell and Smith (1998), these problems can be handled by tracking more moments of the state distribution. However, the functional form of the prediction rule and selection of the moments still remain as an open-ended problem.

The repeated transition method overcomes these problems by recursive approximation to the evolution of true state distributions on a single simulated path of aggregate shocks (in-sample simulation). The method does not depend on the parametric form of the prediction rule because the market-clearing prices and the expected allocations are directly computed at each point on the path. Once the approximation is completed, I estimate the best-fitting non-parametric/parametric law of motion from the in-sample simulation. Using this law of motion, I extrapolate the stochastic dynamics of allocations over the out-of-sample simulated paths of the aggregate shocks. Lastly, I check the validity of the law of motion by comparing the model’s solution over the out-of-sample simulated paths of the aggregate shocks based on the estimated law of motion and the extrapolated aggregate allocations.

The key step in the repeated transition method is to build a correct time-specific expected future value function in each period by combining value functions in the simulation history that share the same aggregate states. For example, if an economy is located at time $t$, for each possible aggregate shock realization $s \in S$ in $t + 1$, I find a period $\tau_s$ in the simulation history where the aggregate states are the closest to the aggregate state of period $t + 1$, including the aggregate shock $s$. Then, I combine the value functions from these periods $\{\tau_s\}_{s \in S}$ to construct the expected value function at $t + 1$. Theoretically, if the simulation path is infinitely long, there exists the period $\tau_s$ where the aggregate allocations
are perfectly identical to period $t+1$ with an aggregate shock realization $s$ with probability one. Therefore, the true expected value function can be constructed from this approach. However, in practice, due to the finite length of the simulation path, often there is no exact period $\tau_s$ in the simulation history that shares the same aggregate allocations including a shock realization $s$ as in period $t+1$. Therefore, I approximate the expected value function by interpolating value functions from periods that closely mimics period $t+1$ for each aggregate shock realization.

The repeated transition method builds upon the method utilizing perfect-foresight impulse response suggested by Boppart et al. (2018). In the paper, aggregate allocations’ impulse responses are obtained from the transition dynamics induced from MIT shocks to the steady-state distribution. Then, the law of motion of aggregate allocations is locally approximated around the steady-state. Therefore, the method assumes certainty equivalence between the expected deterministic path and the expected path when the uncertainty is present. In contrast, the repeated transition method does not assume certainty equivalence and globally solves the model. And it directly computes aggregate allocations and market-clearing prices in each period on the simulation path without specifying the law of motion.

The repeated transition algorithm outperforms the algorithm of Krusell and Smith (1997) in models with non-trivial market-clearing conditions and nonlinear aggregate dynamics in terms of accuracy and computation time. However, for the models with log-linear aggregate dynamics without a non-trivial market-clearing condition, such as the model of Krusell and Smith (1998), the repeated transition method does not work as fast as Krusell and Smith (1998) algorithm.

Using the repeated transition method, I study a business cycle implication of corporate cash holdings in a heterogeneous-firm business cycle model. In the model, firms face a convex external financing cost, so they have a precautionary motivation to hoard cash. Cash is assumed to be an internal asset of a firm. Thus, it is not traded across firms.
and discounted at a different rate than the interest rate in the equity market. The rate is exogenously given as a parameter in the model. Due to these features of cash, the dynamics of aggregate cash holdings in the model become highly nonlinear; there is no general equilibrium force to flatten the dynamics of aggregate cash holdings. On top of this nonlinearity, the market-clearing condition in the model is not trivial, as in Khan and Thomas (2008). Despite these difficulties in computation, the repeated transition method solves the model efficiently and accurately.

In the model, lagged aggregate cash holding significantly lowers the consumption volatility. This model prediction is empirically supported by consumption heteroskedasticity on lagged cash holding, and this empirical pattern is observed only after the early 1980s.67 The fact that the corporate cash holding has dramatically increased after the early 1980s partly explains why conditional heteroskedasticity is observed only after the early 1980s. Then I show that the smoothing effect of cash holding on consumption occurs only when the negative aggregate productivity shock hits the economy. This validates the model’s main mechanism where cash holding gives insurance to households’ dividend income against the negative productivity shock.

**Roadmap** Section 3.1 explains the repeated transition method based on the model in Krusell and Smith (1998). Section 3.2 compares the computation results of the repeated transition method with other methods existing in the literature. Section 3.3 introduces a heterogeneous-firm business cycle model where firms save cash. Section 3.4 discusses the business cycle implication of corporate cash holdings predicted by the model compared to the observations from the data. Section 3.5 concludes. Other detailed figures and tables are included in appendices.

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67. The result is robust over other choices of the cutoff year around 1980.
3.1 Repeated transition method

3.1.1 A model for algorithm introduction: Krusell and Smith (1998)

I explain the repeated transition method based on the heterogeneous agent model with aggregate uncertainty in Krusell and Smith (1998). In this section, I briefly introduce the basic environment of the model.

A measure one of ex-ante homogenous households consumes and saves. At the beginning of a period, a household is given wealth $a_t$ and an idiosyncratic labor supply shock $z_t$. Households are aware of the distribution of households $\Phi_t$, the aggregate productivity shock $A_t$, and how the aggregate states evolve in the future $G(A_t, \Phi_t)$. The idiosyncratic shock and the aggregate shock follow the stochastic Markov processes elaborated in Krusell and Smith (1998). Households are subject to a borrowing constraint $a_{t+1} \geq 0$, as in the standard incomplete market model. I close the model by introducing a representative firm producing output from a constant returns-to-scale production function. The recursive formulation of the model is as follows:

(Household) \[ v(a,s;S,\Phi) = \max_{c,a'} \quad \log(c) + \beta \mathbf{E}(v(a',s';S',\Phi')) \]
\[ \text{s.t.} \quad c + a' = w(S,\Phi)z(s) + (1 + r(S,\Phi))a \]
\[ a' \geq 0, \quad \Phi' = G(\Phi, S) \]

(Production sector) \[ \max_{K,L} \quad A(S)K^aL^{1-a} - w(S,\Phi)L - (r(S,\Phi) + \delta)K \]

(Market clearing) \[ \hat{K}(S,\Phi) = \int a d\Phi(a,z;S,\Phi) \]
\[ \hat{L}(S,\Phi) = \int z d\Phi(a,z;S,\Phi) \]

(Shock processes) \[ \mathbb{P}(s',S'|s,S) = \pi_{sS,s'S'}, \quad s,s' \in \{u,e\}, \quad S,S' \in \{B,G\} \]
\[\pi := \begin{bmatrix}
\pi_{uB,uB} & \pi_{uB,eB} & \pi_{uB,uG} & \pi_{uB,eG} \\
\pi_{eB,uB} & \pi_{eB,eB} & \pi_{eB,uG} & \pi_{eB,eG} \\
\pi_{uG,uB} & \pi_{uG,eB} & \pi_{uG,uG} & \pi_{uG,eG} \\
\pi_{eG,uB} & \pi_{eG,eB} & \pi_{eG,uG} & \pi_{eG,eG}
\end{bmatrix} = \begin{bmatrix}
0.525 & 0.350 & 0.03125 & 0.09375 \\
0.035 & 0.84 & 0.0025 & 0.1225 \\
0.09375 & 0.03125 & 0.292 & 0.583 \\
0.0099 & 0.1151 & 0.0245 & 0.8505
\end{bmatrix}\]

where \(s = u\) means unemployed idiosyncratic state, \(z = 0\). \(s = e\) means employed idiosyncratic state, \(z = 1\). \(S = B\) indicates a bad aggregate state, \(A = 0.99\). \(S = G\) indicates a good aggregate state, \(A = 1.01\).

### 3.1.2 Algorithm

I simulate a single path of aggregate TFP shocks, \(A = \{A_t\}_{t=0}^T\) from the aggregate transition matrix \(\pi^A\). The aggregate transition matrix is as follows:

\[\pi^A = \begin{bmatrix}
\pi_{B,B} & \pi_{B,B} \\
\pi_{G,B} & \pi_{G,B}
\end{bmatrix} = \begin{bmatrix}
0.875 & 0.125 \\
0.125 & 0.875
\end{bmatrix}\]

For the brevity of notation, I define a price vector \(p_t := (w_t, r_t)\). The repeated transition method is based on the following statements:

1. If the true time-specific prices, \(\{p_t\}_{t=0}^T\) are known, the true dynamic path of value function \(\{v_t\}_{t=0}^{T-BurnIn}\) can be approximated by solving the problem from backward starting from \(t = T\) with an initial guess \(v_{T}^0\).

2. If the true time-specific value functions, \(\{v_t\}_{t=0}^{T}\) are known, optimal inter-temporal policy functions, \(\{a_{t+1}\}_{t=0}^{T-1}\) can be obtained. Then, the true dynamic path of distribu-

---

68. To make an agent correctly expect a one-period-ahead value function for each future shock realization, I use an interpolation method which will be explained later.
tion $\{\Phi_t\}_{t=\text{BurnIn}}^T$ can be obtained by evolving the initial guess $\Phi_0^0$ forward using the optimal inter-temporal policy functions, $\{\alpha_t\}_{t=0}^{T-1}$.

3. If true dynamic paths of the value functions and the distributions, $\{v_t, \Phi_t\}_{t=0}^T$ are known, the time-specific prices, $\{p_t\}_{t=0}^T$ can be obtained from the market-clearing conditions.

By these three statements, I can approximate the true allocations, $(p_t, v_t, \Phi_t)_{t=\text{BurnIn}}^{T-\text{BurnIn}}$ from the following simulation chain:

1. Given $n$th guess on the price vector, $\{p_t^{(n)}\}_{t=0}^T$, compute $\{\tilde{v}_t^*\}_{t=0}^T$.

2. Given the value functions $\{\tilde{v}_t^*\}_{t=0}^T$, obtain the inter-temporal policy functions, and compute $\{\tilde{\Phi}_t^*\}_{t=0}^T$.

3. Given $\{\tilde{v}_t^*, \tilde{\Phi}_t^*\}_{t=0}^T$, compute $\{\tilde{p}_t^*\}_{t=0}^T$ from the market-clearing conditions.

4. Evaluate the following criterion.

$$\sup_{\text{BurnIn} \leq t \leq T - \text{BurnIn}} \|p_t^{(n)} - \tilde{p}_t^*\|_\infty < tol$$

If the criterion is not satisfied, update the guess $\{p_t^{(n+1)}\}_{t=0}^T$.

The detailed algorithm is explained below with the pseudo code. The convergence of the algorithm hinges on the stability of the equilibrium transition path. If the stability is not guaranteed, then the repeated transition method does not work. One important thing to note is the algorithm uses only a single simulated path of the aggregate shocks as if they are given parameters. I call this path to be fitted as in-sample path.

Once the algorithm converges, the approximated law of motion is parametrically or non-parametrically estimated from the simulated path (in-sample). Then, using the estimated law of motion, the stochastic equilibrium allocations for the out-of-sample paths are obtained.
The pseudo code for the repeated transition method is as follows:

1. Discretize the aggregate TFP shock process by $S$ states.\(^{69}\)

2. Simulate a path of aggregate TFP shocks, $A = \{A_t\}_{t=0}^T$. This path is going to be repeatedly used in the following steps without any change. This is the in-sample path.

3. Guess on the paths of the prices, the value functions, and the state distributions, $\{\tilde{p}^{(n)}_t, \tilde{v}^{(n)}_t, \tilde{\Phi}^{(n)}_t\}_{t=0}^T$.

4. Solve the model backward in the following sub-steps:
   
   \begin{enumerate}
   \item Make a partition $\tilde{T}(s)$ of simulation paths grouped by the realized aggregate TFP level: $\tilde{T}(s) = \{\tau|A_{\tau} = A_s\} \subseteq \{0, 1, 2, \ldots, T\}$ for $s \in \{1, 2, 3, \ldots, S\}$.
   \item For each $s \in \{1, 2, 3, \ldots, S\}$ find $\{\omega_{j,s}|\sum \omega_{j,s} = 1, j \in \tilde{T}(s)\}$ such that $\tilde{\Phi}^{(n)}_{t+1} = \sum_{j \in \tilde{T}(s)} \omega_{j,s} \tilde{\Phi}^{(n)}_j$. That is, find a set of weights that interpolates $\Phi^{(n)}_{t+1}$ using $\{\tilde{\Phi}^{(n)}_j|j \in \tilde{T}(s)\}$. The uniqueness of the weight set $\{\omega_{j,s}\}$ is not guaranteed. However, for the given model, I track only the first moment $\tilde{k}_{t+1}$ in the spirit of Krusell and Smith (1998). Therefore, I can find the unique $\{\omega_{1,s}, \omega_{2,s}\}$ such that $\tilde{k}_{t+1} = \tilde{k}_{1,1}\omega_{1,s} + \tilde{k}_{2,2}\omega_{2,s}$, and $\omega_{1,s} + \omega_{2,s} = 1$ with $\{\tilde{k}_{1,1}, \tilde{k}_{2,2}\}$ being the nearest combination to $\tilde{k}_{t+1}$. I set $\omega_{m,s} = 0$ for $m \notin \{1, 2\}$.
   \item Approximate the true future value function $E_v_{t+1}(\cdot, \cdot) := E_v(\cdot, \cdot; A_{t+1}, \Phi_{t+1})$ as follows:
   
   $E_v_{t+1}(\cdot, \cdot) \equiv E \left[ \sum_{j \in \tilde{T}(s)} \tilde{\phi}^{(n)}_j(\cdot, \cdot)\omega_{j,s} \right] = E \left[ \tilde{\phi}^{(n)}_{1,1}(\cdot, \cdot)\omega_{1,s} + \tilde{\phi}^{(n)}_{2,2}(\cdot, \cdot)\omega_{2,s} \right]
   $ \end{enumerate}

In this step, the value function is linearly interpolated. As the value function is the most smooth object in the equilibrium allocations, this step incurs only a

\(^{69}\) This discretization step is unnecessary. However, for practical illustration, I describe the pseudo code based on the discretization.
small approximation error if the elements of \( \{ K_j | j \in \bar{T}(s) \} \) are closely located to each other.

(d) Solve the problem for given \( t \). Then I obtain the solution \( \{ \bar{a}_{t+1}^t, \bar{\nu}_{t+1}^t \} \)

(e) If \( t > 0 \), update \( t' = t - 1 \) and go back to step 4a. If the algorithm arrives at \( t = 0 \), the time specific inter-temporal policy functions \( \{ \bar{a}_{t+1}^t \}_{t=0}^T \) are obtained. Also, the sequence of implied value functions \( \{ \bar{\nu}_{t+1}^t \}_{t=0}^T \) are obtained.

5. Using \( \{ \bar{a}_{t+1}^t \}_{t=0}^T \), simulate the distribution for \( t = 1, 2, 3, \ldots T \) starting from \( \hat{\Phi}_0^{(n)} \). From the simulation, I can get an implied sequence of state distributions \( \{ \bar{\Phi}_t^* \}_{t=0}^T \), where the initial distribution satisfies \( \bar{\Phi}_0^* = \hat{\Phi}_0^{(n)} \).\(^70\)

6. Using \( \{ \bar{\Phi}_t^* \}_{t=0}^T \), all the aggregate allocations over the whole path such as \( \{ \bar{K}_t^* \}_{t=0}^T \) can be obtained. Using the market-clearing condition, compute \( \{ \bar{p}_t^* \}_{t=0}^T \).\(^71\)

7. From the obtained prices, check the distance between the implied sequence of \( \{ \bar{p}_t^* \}_{t=0}^T \) and the guess on the prices over the whole path \( \{ \hat{p}_t^{(n)} \}_{t=0}^T \) using the following metric:

\[
\sup_{\text{BurnIn} \leq t \leq T - \text{BurnIn}} ||\bar{p}_t^* - \hat{p}_t^{(n)}||_\infty < \text{tol}
\]

Note that the distance is measured from simulations except for the burn-in periods. If the distance is smaller than the tolerance level, the algorithm is converged. Otherwise, make the following updates on the guess:\(^72\)

---

\(^70\) In this step, I use the non-stochastic simulation method (Young, 2010).

\(^71\) This step directly computes market-clearing prices even for a model with non-trivial market-clearing conditions. In Section 3.2, I use this algorithm to solve Khan and Thomas (2008) where the marginal value of consumption needs to be computed in the external loop of the model due to the non-trivial market-clearing condition. I found this technique significantly saves computation time. Further discussion on the computational gain is in Section 3.2.

\(^72\) In highly nonlinear aggregate dynamics, I have found that the log-convex combination updating rule marginally dominates the standard convex combination updating rule in terms of convergence speed. The log-convex combination rule is as follows:

\[
\log(\bar{p}_{t+1}^{(n+1)}) = \log(\bar{p}_t^{(n)})\psi_1 + \log(\bar{p}_t^{(n)})(1 - \psi_1)
\]
\[ p_t^{(n+1)} = \hat{p}_t^{(n)} \psi_1 + \hat{p}_t^* (1 - \psi_1) \]

\[ \bar{v}_t^{(n+1)} = \bar{v}_t^{(n)} \psi_2 + \bar{v}_t^* (1 - \psi_2) \]

\[ \Phi_t^{(n+1)} = \Phi_t^{(n)} \psi_3 + \Phi_t^* (1 - \psi_3) \]

for \( \forall t \in \{0, 1, 2, 3, ..., T\} \).

With the updated guess \( \{ \hat{p}_t^{(n+1)}, \bar{v}_t^{(n+1)}, \Phi_t^{(n+1)} \}_{t=0}^T \), go to step 4.

\( (\psi_1, \psi_2, \psi_3) \) are the control parameters of convergence speed in the algorithm. If \( \psi_i \) is high, then the algorithm conservatively updates the guess, leading to slow convergence speed. If the equilibrium dynamics are almost linear due to strong general equilibrium effect as in Krusell and Smith (1998), I found setting \( \psi_i \) around 0.8 guarantees convergence without much sacrifice in the convergence speed. However, if a model is highly nonlinear, as in the baseline model in Section 3.3, the convergence speed needs to be controlled to be much slower than the one in the linear models. This is because the nonlinearity can lead to a sudden jump in the realized allocations during the iteration if a new guess is too dramatically changed from the last guess. Potentially, heterogeneous updating rule \( \psi_i \neq \psi_j (i \neq j) \) might be helpful. However, without a particular reason to do so, I assume homogenous weights throughout whole computations in this paper.

Note that in step 4c, I assume the value function’s local linearity in aggregate states is given. If the local linearity of value function is significantly violated at an aggregate state, the approximation breaks down, so another approximation is needed.\(^73\) However, this is not a common concern in the broad class of problems because the value function is generally smooth and locally linear along the aggregate states, while the policy functions might not be the case.

\(^73\) If there is a kink point in the value function along the individual states, the algorithm can be modified to include backward steps in the endogenous grid method.
After the equilibrium allocations are computed over the in-sample path $A$, I estimate the implied law of motion from the in-sample allocations. The law of motion can potentially take any nonlinear form. Then, using the fitted law of motion, equilibrium allocations are computed over out-of-sample paths of simulated aggregate shocks.

### 3.2 Computation validation

This section compares the equilibrium allocations obtained from the repeated transition method and the method in Krusell and Smith (1998). In the computation, parameters are set as in the benchmark model in Krusell and Smith (1998) without idiosyncratic shocks in the patience parameter $\beta$. For both of the algorithms, I stopped when the largest absolute difference between the simulated average capital stock and the expected average capital stock is less than $10^{-6}$.

In the converged solution, the mean squared difference in the solutions between the repeated transition method and Krusell and Smith (1998) algorithm is around $2 \times 10^{-4}$. It takes around 30 minutes for the repeated transition method to converge under the convergence speed parameter $\psi_1 = \psi_2 = \psi_3 = 0.8$, while it takes around 20 mins for Krusell and Smith (1998) algorithm.\textsuperscript{74} The convergence speed might change depending on the updating weight.

Figure 1 plots the expected path (Predicted) and the simulated path (Realized) of aggregate capital $K_t$ obtained from the repeated transition method and the simulated path from Krusell and Smith (1998). As can be seen from all three lines hardly distinguished from each other, the repeated transition method computes almost identical equilibrium allocations as Krusell and Smith (1998) algorithm at a slower speed. This is because the model in Krusell and Smith (1998) features linear dynamics of aggregate capital. Thus,

\textsuperscript{74} This computation is done in 2015 MacBook Pro laptop with a 2.2 GHz quad-core processor.
their algorithm with the log-linear law of motion can compute the solution both fast and accurately.

Figure 1: Computed dynamics in aggregate wealth (Krusell and Smith, 1998)

However, the repeated transition method outperforms Krusell and Smith (1997) algorithm when the market-clearing condition is non-trivial, as in the model of Khan and Thomas (2008). This is because the non-trivial market-clearing condition requires an extra loop to find an exact market-clearing condition in each iteration.

I solve Khan and Thomas (2008) using both the repeated transition method and the Krusell and Smith (1998) algorithm with an external loop for non-trivial market-clearing condition. I stopped the iteration when the following criterion is satisfied:

$$\max\left\{ \sup_t \{||p_t^{(n)} - p_t^{(n+1)}||\}, \sup_t \{||K_t^{(n)} - K_t^{(n+1)}||\} \right\} < 10^{-6}$$

Figure 2 plots the dynamics of price $p_t$ and aggregate capital stock $K_t$ computed from the repeated transition method and Krusell and Smith (1998) algorithm. For the allocations computed from the repeated transition method, both the predicted value and the realized values are reported. As shown from the figure, all three lines display almost identical dynamics of the price and the aggregate allocations. The mean squared difference in the solutions between the repeated transition method and Khan and Thomas (2008) is less than $10^{-5}$.

In the computation of repeated transition method, I use $\psi_1 = \psi_2 = \psi_3 = 0.9$ for speed of convergence. The reason for using this conservative updating rule is because the model in Khan and Thomas (2008) features a strong general equilibrium effect; dramatic updates in the price might lead to divergence. The repeated transition method took around 20 minutes to converge on average, while Krusell and Smith (1998) algorithm converged in around 30 minutes on average. The convergence speed might change depending on the updating weight.

![Figure 2: Computed dynamics in aggregate capital stocks (Khan and Thomas, 2008)](image)

In the next section, I will compare the algorithm performance between the recursive transition method and Krusell and Smith (1998) algorithm using a model with nonlinear dynamics. The previous comparisons were made for linear aggregate dynamic models, where Krusell and Smith (1998) algorithm can make a successful approximation to true aggregate dynamics. However, in nonlinear models, the accurate approximation might
be hard to achieve for Krusell and Smith (1998) algorithm, while the repeated transition method successfully makes a convergence between predicted allocations and realized allocations.

3.3 Baseline model

In this section, I introduce a heterogeneous-firm business cycle model to study the role of corporate cash holdings on aggregate consumption.

There is a continuum of measure one of ex-ante homogenous firms that hoard cash and produces business outputs. For simplicity of the model, I assume all the firms are homogenous in terms of their capital stocks normalized at one. And the depreciation rate is assumed to be zero. At the beginning of each period, a firm $i$ is given with a cash holding $c_{a_{i,t}}$ and an idiosyncratic productivity level $z_{i,t}$. All firms rationally expect the future and are aware of the full distribution of each firm-level allocation and the law of motion of the aggregate states.

The business output is produced by the following Cobb-Douglas production function:

$$f(n_{i,t}, z_{i,t}; A_t) = z_{i,t} n_{i,t}^\gamma A_t$$

where $n_{i,t}$ is a labor demand; $\gamma < 1$ is a span of control parameter; a capital stock $k_{i,t}$ is normalized as just unity; $z_{i,t}$ and $A_t$ are idiosyncratic and aggregate productivities, respectively. Each firm needs to pay a fixed operation cost $\xi > 0$ in each period.

The idiosyncratic and aggregate productivity shock processes, $\{z_{i,t}\}, \{A_t\}$ are specified as follows:

$$z_{i,t+1} = \rho_z z_{i,t} + \epsilon_{i,t+1}, \quad \epsilon_{i,t+1} \sim_{i.i.d} N(0, \sigma_z)$$
$$A_{t+1} = \rho_A A_t + \tilde{\epsilon}_{t+1}, \quad \tilde{\epsilon}_{t+1} \sim_{i.i.d} N(0, \sigma_A)$$
For computation, both of the shock processes are discretized by the Tauchen method.\textsuperscript{76}

A firm earns operating profit and decides how much to distribute as a dividend $d_{i,t}$ to equity holders (a representative household). The remaining part in the operating profit after dividend payout is used to adjust cash holding, $ca_{t+1}/(1 + r_{ca}) - ca_t$. The future cash holding is discounted at an internal discount rate $r_{ca} > 0$ as cash is not traded in the market across the firms. $r_{ca}$ is an exogenous parameter and assumed to be lower than market interest rate $r_t$. Cash holding level is assumed to be non-negative $ca_t \geq 0$. Thus, the model imposes a standard incomplete market assumption as in Aiyagari (1994).

If a dividend is determined to be negative, then a firm is issuing equity, which incurs extra pecuniary cost $C(d_{i,t})$ (Jermann and Quadrini, 2012; Riddick and Whited, 2009). This equity issuance cost is specified as follows:

$$C(d_{i,t}) := \frac{\mu}{2} \mathbb{I}\{d_{i,t} < 0\} \sigma_{d_{i,t}}^2$$

Thus, the net dividend is $d_{i,t} - \frac{\mu}{2} \mathbb{I}\{d_{i,t} < 0\} \sigma_{d_{i,t}}^2$. It is worth noting that this net dividend function belongs to $C^1$ class as it smoothly changes the slope at $d_{i,t} = 0$ without a kink.

If there is no equity financing cost, holding cash is not the desired option for an equity holders because it is more expensive than receiving the dividend $\left(\frac{1}{1+r_{ca}} > \frac{1}{1+r_t}\right)$. However, due to the presence of equity financing cost, a firm has a precautionary motivation to hoard cash. They save cash for the case when their business is in trouble (low $z_t$ or low $A_t$), and by doing so, they reduce equity financing costs in their difficult times.

In the corporate finance literature, there has been a rich set of empirical evidence for corporates’ dividend smoothing behavior (Leary and Michaely, 2011; Bliss et al., 2015). Especially, Leary and Michaely (2011) empirically showed that cash-rich firms smoothen their dividend significantly more than the others.

\textsuperscript{76} I use three grid points where each neighboring points are apart by one standard deviation around the mean for both processes.
The recursive formulation of a firm’s problem is as follows:

\[
J(ca, z; A, \Phi) = \max_{ca',d} \left( d - C(d) + \frac{1}{1 + r(A, \Phi)} \mathbb{E}(J(ca', z'; A', \Phi')) \right) \\
\text{s.t.} \quad d + \frac{ca'}{1 + rca} = \pi(z; A, \Phi) + ca \\
ca' \geq 0, \quad \Phi' = G(\Phi, A)
\]

(Operating profit) \( \pi(z; A, \Phi) := \max_n zAn^\gamma - w(A, \Phi)n - \zeta \)

(Equity issuance cost) \( C(d) := \frac{\mu}{2} \mathbb{1}(d < 0)d^2 \)

where \( J \) is the value function of a firm; \( ca \) and \( z \) are cash holding and idiosyncratic productivity as an individual state variable; \( A \) is the aggregate productivity; \( \Phi \) is the distribution of the individual state variables; \( w \) and \( r \) are wage and interest rate which are functions of aggregate state variables \((A, \Phi)\).

I close the model by introducing a stand-in household that holds equity as wealth and saves on equity. The household consumes and supplies labor and rationally expects the future aggregate states. The income sources of the household are labor income and dividend from equity holding.

The recursive formulation of the representative household’s problem is as follows:

\[
V(a; \Phi, A) = \max_{c,a',l_H} \log(c) - \eta l_H + \beta \mathbb{E}A'V(a'; \Phi', A') \\
\text{s.t.} \quad c + \frac{a'}{1 + r(\Phi, A)} = w(\Phi, A)l_H + a \\
G(a, \Phi) = \Phi', \quad G_A(A) = A'
\]

where \( V \) is the value function of the household; \( a \) is wealth; \( c \) is consumption; \( a' \) is a future saving level; \( l_H \) is labor supply; \( w \) is wage, and \( r \) is the real interest rate. The household is holding the equity of firms as their wealth.
The recursive competitive equilibrium is defined based on the following market-clearing conditions:

(Labor market) \[ l_H(A, \Phi) = \int n(c_a, z ; A, \Phi) d\Phi \]

(Equity market) \[ a(A, \Phi) = \int f(c_a, z ; A, \Phi) d\Phi \]

The model does not assume a centralized market for cash holding. Therefore, \( r^{ca} \) is not endogenously determined at the market. This is a realistic assumption as a firm’s cash holding is not tradable across firms. I interpret this setup as the cash holding return is determined by each firm’s idiosyncratic financing status independently from the centralized capital market condition. \( r^{ca} \) is the average level of the idiosyncratic financing cost.

### 3.4 Quantitative analysis

In this section, I quantitatively analyze the recursive competitive equilibrium allocations computed from the repeated transition method. For easier computation, I first normalize the firm’s value function by contemporaneous consumption \( c_t \) following Khan and Thomas (2008). I define the consumption good price \( p_t := 1/c_t \), so the normalized value function is \( \tilde{J}_t = p_t J_t \). From the intra-temporal and inter-temporal optimality conditions of households, I have \( \omega_t = \eta p_t \) and \( r_t = p_{t+1}/p_t \). Thus, \( p_t \) is the only price to characterize the equilibrium. The following analysis will focus on the dynamics of \( p_t \) and the aggregate cash holdings (the first moment of the distribution of cash holding).

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77. For simplicity, the model is abstract from the heterogeneity in the financing cost.
3.4.1 Calibration

The model's key parameters are the external financing cost parameter $\mu$ and the operating cost parameter $\zeta$. The external financing cost is identified from the aggregate-level corporate cash holding-to-consumption ratio. In the moment calculation, the aggregate cash holding is obtained from the Flow of Funds. Consumption is from the National Income and Product Accounts (NIPA). In the model, as $\mu$ increases, the corporate cash holding-to-consumption ratio increases due to increasing precautionary motivation. The key identifying moment of the operating cost parameter is the dispersion of the cash holdings among corporates. For this, I use the time-series average of the cross-sectional standard deviation of cash holding normalized by the cross-sectional average of the cash holding. As operating cost increases, the dispersion of cash increases in the model. Additionally, labor disutility cost $\eta$ is calibrated to have a representative household spend a third of its hours on the labor supply. The calibrated results are summarized in Table 1. The other fixed parameters are summarized in Appendix C.1.1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Target Moments</th>
<th>Data</th>
<th>Model</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>Corporate cash holding/Consumption</td>
<td>17.7</td>
<td>17.9</td>
<td>0.33</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Avg. of $sd(Cash_{i,t})/mean(Cash_{i,t})$</td>
<td>1.5</td>
<td>1.5</td>
<td>0.42</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Labor supply hours</td>
<td>0.33</td>
<td>0.37</td>
<td>12.3</td>
</tr>
</tbody>
</table>

Table 1: Calibration target and parameters

3.4.2 Nonlinear business cycle

Using the repeated transition method, I compute the recursive competitive equilibrium allocations over the simulated path of aggregate shocks. In the algorithm, the interpolation of the value function (step 4b) is based on the first moment of the cash distributions (the

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78. The detailed definition of aggregate cash holding is available in Appendix C.1.2.
79. In this ratio, the consumption includes both durable and non-durable consumptions.
80. To rule out extreme outliers, I winsorize the cash holdings distribution at the top 90th percentile.
aggregate cash holding level) following Krussell and Smith (1998) (hereafter, KS algorithm). The aggregate cash holding level follows highly nonlinear dynamics in the computed outcome because the general equilibrium effect does not strongly affect each firm’s cash holding demand. The price of cash holding is $r^c$ which is exogenously determined in the model because the cash holding is not allowed to be traded across the firms. In the setup where the cash is traded across the firms, the opportunity cost of cash holding ($r_t - r_t^{ct}$) shrinks close to zero. So, the aggregate cash holding is predicted to be higher on average in the alternative setup. I check this point using the computed result from the prototype KS algorithm instead of the repeated transition method.\(^81\)

\[\text{Figure 3: Aggregate fluctuations in the economy}\]

Figure 3 plots a part of the simulated path of consumption good price and aggregate corporate saving obtained from both the repeated transition method and the prototype. The solid line plots expected allocations in the repeated transition method, and dash-dotted line plots simulated allocations in the repeated transition method. The dashed line represents the dynamics of the allocations in the prototype KS algorithm. As can be seen from the aggregate corporate saving in the right-hand side figure, the average

\(^81\). The prototype refers to the method of tracking the first moment of the state distribution, and the predicting prices based on the first moment as in Khan and Thomas (2008).
corporate saving is higher in the KS algorithm than the repeated transition method. This is because the prototype KS algorithm assumes log-linearity in the law of motion of aggregate corporate saving and assumes that internal cash holding is linearly affected by the real interest rate.

To determine which prediction is the correct approximation to the true dynamics, I first evaluate the goodness of fitness $R^2$ and mean-squared error between expected dynamics and simulated dynamics on the newly simulated shock path (out-of-sample path). KS algorithm immediately gives the parametric form of the law of motion after the algorithm converges. In contrast, the repeated transition method gives the sequence of allocations which requires an extra step to fit the sequences into a parametric/non-parametric law of motion.

The repeated transition method gives $R^2$ of 0.9999 and mean squared error of $10^{-6}$ for both consumption good price and aggregate cash holding dynamics. On the other hand, the KS algorithm gives the following law of motion and goodness of fitness:\textsuperscript{82}

\begin{align*}
\log(CA_{t+1}) &= -0.8238 + 0.9755 \times \log(CA_t), \quad \text{if } A_t = A_1, \text{ and } R^2 = 0.9788, \text{ MSE } = 1.0464 \\
\log(CA_{t+1}) &= -2.0397 + 0.2963 \times \log(CA_t), \quad \text{if } A_t = A_2, \text{ and } R^2 = 0.5532, \text{ MSE } = 0.6598 \\
\log(CA_{t+1}) &= -0.1787 + 0.8332 \times \log(CA_t), \quad \text{if } A_t = A_3, \text{ and } R^2 = 0.9854, \text{ MSE } = 0.0098 \\
\log(p_t) &= 2.5741 - 0.0008 \times \log(CA_t), \quad \text{if } A_t = A_1, \text{ and } R^2 = 0.5470, \text{ MSE } = 0.0000 \\
\log(p_t) &= 2.5508 - 0.0009 \times \log(CA_t), \quad \text{if } A_t = A_2, \text{ and } R^2 = 0.3410, \text{ MSE } = 0.0000 \\
\log(p_t) &= 2.5221 - 0.0042 \times \log(CA_t), \quad \text{if } A_t = A_3, \text{ and } R^2 = 0.8974, \text{ MSE } = 0.0000 \\
\end{align*}

The log-linear rule of the prototype KS algorithm relies on the prices’ smoothing effect on the dynamics of aggregate allocations. For example, when there is a surge of cash holding demand, the price of cash holding goes up to mitigate the surge, and vice versa for the case of decreasing cash holding demand. In numerous applications in the literature,\textsuperscript{82} the aggregate productivity shock is discretized by three grid points.

\textsuperscript{82} The aggregate productivity shock is discretized by three grid points.
this flattening force from the general equilibrium has been proved to be powerful enough to guarantee the log-linear specification as the true law of motion of aggregate variable. One example is Khan and Thomas (2008) where the micro-level lumpiness is smoothed out by real interest rate dynamics. However, in the baseline model of this paper, the general equilibrium effect is missing for the cash holding demand. Thus, the log-linear prediction rule fails to capture the true law of motion in the recursive competitive equilibrium.

On top of the nonlinearity, there is another feature in the model that makes the prototype KS algorithm cannot simply address: there is a non-trivial market-clearing condition with respect to consumption good price $p_t$. Krusell and Smith (1997) suggested an algorithm to solve this problem by considering an external loop in the algorithm that solves market-clearing price $p_t$ in each iteration. This algorithm is known to successfully solve the log-linear models with non-trivial market-clearing conditions such as Khan and Thomas (2008). However, due to the extra loop in each iteration, the algorithm entails high computation cost. In the repeated transition method, the price and allocations are explicitly computed at each point on the simulated path in every iteration. Therefore, the method does not require an extra loop for computing market-clearing price, so it saves great amount computation time. In the baseline model, computation time is reduced by factor of $2.83$.

3.4.3 Discussion: Model prediction and empirical evidence

In this section, I analyze the role of corporate cash holdings on the aggregate fluctuations using the baseline model and support the model prediction from the empirical evidence.

To investigate the role of the corporate cash holding on consumption dynamics, I analyze how the consumption volatility changes over the average lagged cash holding

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83. The KS algorithm takes around one hour to compute a converged solution when the simulation length is $T = 500$ and the cross-sectional grid of cash holding is 50 points. However, in the repeated transition method, it takes only around 30 minutes to make a convergence. For the fair comparison, the initial guess of the KS algorithm is from the log-linear relationship implied in the initial guess of the repeated transition method.
level. First, I residualize the aggregate consumption time-series by the recent four lagged consumptions after taking a log.

\[
\log(C_t) = \rho_1 \log(C_{t-1}) + \rho_2 \log(C_{t-2}) + \rho_3 \log(C_{t-3}) + \rho_4 \log(C_{t-4}) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma)
\]

Then, I run the regression of the logged absolute-valued residuals on the average lagged cash holdings for the periods with \(\Delta \log(C_t) > 0\) (positive consumption growth) and \(\Delta \log(C_t) < 0\) separately (negative consumption growth).

\[
\log(\hat{\sigma}_t) = \rho \log(\overline{\text{Cash}_{t-1}}) + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta)
\]

\[
\text{s.t. } \overline{\text{Cash}_{t-1}} = \frac{1}{4} \sum_{i=1}^{4} \text{Cash}_{t-i}
\]

Table 2 reports the regression results. The residual standard deviation is negatively correlated with the average lagged cash holding in the periods with the negative consumption growth. Conversely, the residual standard deviation is positively correlated with the average lagged cash holding in the periods with the positive consumption growth. The volatility of consumption decreases by 1.1% when the lagged aggregate cash holding increases by 1% for the periods with the negative consumption growth. This relationship is visualized by a scatter plot in Figure 4.

Therefore, the aggregate cash holding gives a consumption buffer against a negative aggregate shock by smoothing the dividend stream in the simulated data. I support this model prediction from the macro-level data. The data is the quarterly frequency and covers from 1951 to 2018. Consumption and the total dividend of the corporate sector are from BEA National Income and Product Accounts (NIPA); the aggregate cash holding and the total asset holding are obtained from the Flow of Funds. I normalize the aggregate cash

84. The residualized consumptions are normalized by the unconditional standard deviation of the residuals.
Table 2: Heteroskedasticity of consumption conditional on average lagged cash holding in the model

<table>
<thead>
<tr>
<th></th>
<th>Neg. (1)</th>
<th>Pos. (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>log(Cash_{t-1}) (%)</td>
<td>1.075*** (0.286)</td>
<td>1.694*** (0.337)</td>
</tr>
<tr>
<td>Constant</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Observations</td>
<td>197</td>
<td>204</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.068</td>
<td>0.111</td>
</tr>
</tbody>
</table>

Note: * p < 0.1; ** p < 0.05; *** p < 0.01

Figure 4: Scatter plot of logged residual standard deviation and average lagged cash holding conditional on $\Delta \log(C_t) < 0$

holding and dividend by the total asset holding. The aggregate consumption is detrended by HP-filter with a smoothing parameter at 1600.

Table 3 reports the regression results of conditional heteroskedasticity, using the empirical counterparts of the model variables. First, the consumption is residualized using the autoregressive process up to the fourth order. The residualized consumption

---

85. As in the model counterpart, the residualized consumptions are normalized by unconditional standard deviation of the residuals.
is regressed on average lagged normalized cash and dividend separately for pre-1980 periods and post-1980 periods. The reason for separating the two periods is because the corporate cash holding has increased dramatically after 1980, which made pre-1980 and post-1980 periods starkly different in terms of the size of corporate cash holdings.86

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>( \hat{s}_t ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash(_{t-1}) (%)</td>
<td>0.558 (0.448)</td>
</tr>
<tr>
<td>Dividends(_{t-1}) (%)</td>
<td>0.828 (0.607)</td>
</tr>
<tr>
<td>Constant</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>107</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Note: *p<0.1; **p<0.05; ***p<0.01

Table 3: Sensitivity of consumption to aggregate TFP shock contingent on corporate cash holdings

As can be seen from Table 3, the residualized consumption display heteroskedasticity conditional on aggregate cash holding during the post-1980 periods. The greater the lagged aggregate cash holding is, the weaker responsiveness consumption displays to an exogenous aggregate shock. The same interpretation can be made to the aggregate dividend as well. These empirical results are consistent with the model prediction.

However, the model diverges from the data when it comes to the pre-1980 periods. The possible explanation for this result is that before 1980, corporate cash holding was not large enough to play an important role in dividend smoothing. Therefore, an increase in cash holding did not help consumption smoothing in the pre-1980 periods.

Figure 5 plots the scatter plot of the residualized consumption’s standard deviation as a function of lagged aggregate cash holding (panel (a) and (b)), and as a function of lagged

86. The result is robust over other choices of the cutoff year around 1980.
aggregate dividend (panel (c) and (d)) separately for pre-1980 and post-1980 periods. A significant negative relationship is observed from the post-1980 periods.

I further investigate whether it is a negative aggregate shock or a positive aggregate shock that drives the conditional heteroskedasticity of consumption. Here I use variation in the Solow residual (TFP) as an aggregate shock. The TFP time-series is fitted into $AR(1)$ process to obtain the innovation in TFP, and I group observations into the positive innovation period and the negative innovation period based on the sign of TFP innovation in each period. Then, I run the following regression:

$$\frac{\Delta C_t}{C_t} = \beta_0 + \beta_1 \text{TFP Innovation}_t + \beta_2 \text{TFP Innovation}_t \times \text{Cash}_{t-1} + X_t + \epsilon_t$$
where $X_t$ is a vector of control variables including Cash_t and Dividend_t; TFP innovation_t is normalized by its standard deviation. The coefficient of interest is $\beta_2$. If cash holding buffers consumption response, the sign of $\beta_2$ would be negative.

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>$\Delta C_t / C_t$ (%) before 1980</th>
<th>$\Delta C_t / C_t$ (%) after 1980</th>
</tr>
</thead>
<tbody>
<tr>
<td>TFP Innovation_t (s.d.%)</td>
<td>$-0.001$</td>
<td>$0.010^*$</td>
</tr>
<tr>
<td></td>
<td>$(0.005)$</td>
<td>$(0.005)$</td>
</tr>
<tr>
<td>TFP Innovation_t $\times$ Cash_{t-1} (%)</td>
<td>$0.115$</td>
<td>$-0.090$</td>
</tr>
<tr>
<td></td>
<td>$(0.088)$</td>
<td>$(0.102)$</td>
</tr>
<tr>
<td>Control</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Constant</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>59</td>
<td>53</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.264</td>
<td>0.409</td>
</tr>
</tbody>
</table>

Table 4: Sensitivity of consumption: cash

Table 4 reports the regression coefficients, $\beta_1$ and $\beta_2$, with standard errors in the bracket. As can be seen from the third column of the table, the significant consumption smoothing effect is observed only for negative TFP innovation during post-1980 periods. A similar result is obtained when the TFP innovation term interacts with the lagged dividend, as reported in Table 5. Therefore, I conclude that the model prediction of the consumption smoothing effect of corporate cash holding towards the negative aggregate shock is empirically supported from the data.

### 3.5 Conclusion

This paper develops and introduces a novel algorithm to solve heterogeneous-agent models with aggregate uncertainty, which I name as repeated transition method. This method iteratively updates agents’ expectations on the future path of aggregate states from the transition dynamics on a single path of simulated shocks. The algorithm runs until the expected path converges to the simulated path. In each iteration, market-clearing prices
Table 5: Sensitivity of consumption: dividend

and aggregate allocations are explicitly computed at each period on the simulation path. Therefore, the method does not rely on a parametric form of the law of motion or an external loop for non-trivial market-clearing conditions.

Then, I introduce a heterogeneous-firm business cycle model where firms face a convex external financing cost and hoard cash out of precautionary motivation. Using the model, I study the business cycle implication of corporate cash holding. Cash is assumed to be an internal asset of a firm; thus, not traded across firms; and discounted at a different rate than the real interest rate in the equity market. The model features highly nonlinear dynamics of aggregate cash holdings due to the absence of general equilibrium force on the aggregate cash holding. I found the repeated transition method solves the problem more efficiently and more accurately than the existing global methods. The model predicts that the more outstanding corporate cash holding lowers the consumption volatility. This model prediction is supported by macro-level evidence of consumption heteroskedasticity conditional on the lagged aggregate cash holding.
Appendix A

Appendix for Chapter 1

A.1 Appendix: tables and figures

A.1.1 Conditional heteroskedasticity: Regression result

<table>
<thead>
<tr>
<th></th>
<th>Large</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_t )</td>
<td>( \hat{s}_{t-1} %)</td>
<td>( \hat{s}_{t-1} %)</td>
</tr>
<tr>
<td>(1)</td>
<td>0.068***</td>
<td>0.032</td>
</tr>
<tr>
<td>(2)</td>
<td>(0.025)</td>
<td>(0.020)</td>
</tr>
<tr>
<td>Constant</td>
<td>-0.454</td>
<td>-0.174</td>
</tr>
<tr>
<td>(1)</td>
<td>(0.396)</td>
<td>(0.510)</td>
</tr>
<tr>
<td>Observations</td>
<td>35</td>
<td>35</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.185</td>
<td>0.068</td>
</tr>
<tr>
<td>Adjusted ( R^2 )</td>
<td>0.161</td>
<td>0.040</td>
</tr>
</tbody>
</table>

Note: *\(p<0.1\); **\(p<0.05\); ***\(p<0.01\)

Table A.1.1: Increasing sensitivity of aggregate investments along with the large firms’ investment spike
Specifically, $\text{spike}_{t-1}$ is defined as follows:

$$
\text{spike}_{t-1} := \frac{1}{J} \sum_{j=0}^{J-1} \text{SpikeRatio}_{t-1-j}
$$

$$
\text{SpikeRatio}_t := \frac{\text{#Extensive-margin adjustment}_t}{\text{#Firms}_t}
$$

where $J$ is the number of past years to be includes. In the reported result, I use $J = 3$. The result is robust over $J = 1, 2, 4$. 
A.1.2 Survey result: Inelasticity of investments to interest rate change

Q1. By how much would your borrowing costs have to decrease to cause you to initiate, accelerate, or increase investment projects next year?

Q2. By how much would your borrowing costs have to increase to cause you to delay or stop investment projects next year?

<table>
<thead>
<tr>
<th>Change in interest rate</th>
<th>Plan changing firms (Q1)</th>
<th>Plan changing firms (Q2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5%</td>
<td>3%</td>
<td>6%</td>
</tr>
<tr>
<td>1%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td>2%</td>
<td>8%</td>
<td>16%</td>
</tr>
<tr>
<td>3%</td>
<td>5%</td>
<td>11%</td>
</tr>
<tr>
<td>More than 3%</td>
<td>11%</td>
<td>20%</td>
</tr>
<tr>
<td>No change</td>
<td>68%</td>
<td>37%</td>
</tr>
</tbody>
</table>

Table A.1.2: CFO survey results (Sharpe and Suarez, 2014): inelasticity to interest rate changes
A.2 Appendix: Monetary policy shock

Figure A.2.1: One-year moving average monetary policy shock: March 1990 ~ December 2009
A.3 Appendix: Firm-level TFP estimation

I estimate firm-level TFP following Ackerberg et al. (2015). The estimation is based on the following model specification:

\[
\log(\text{ValueAdd}_{i,t}) = \bar{\pi} + a \log(\text{Capital}_{i,t-1}) + \gamma \log(\text{Employment}_{i,t}) + \text{TFP}_{i,t} + \epsilon_{i,t}
\]

\[
\text{MaterialExpense}_{i,t} = f(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{TFP}_{i,t})
\]

Then, I assume the following assumptions:

- Production, material expenditure and idiosyncratic TFP shocks are all realized simultaneously.

- Before the realization of the idiosyncratic TFP, a firm receives an idiosyncratic TFP signal (sTFP\(_{i,t}\)): a firm determines labor demand based on the signal. The idiosyncratic TFP follows a Markov process conditional on the signal of idiosyncratic TFP (P(TFP\(_{i,t}\)|sTFP\(_{i,t}\))).

- The idiosyncratic TFP signal follows a Markov process conditional on the past realization of the idiosyncratic TFP (P(sTFP\(_{i,t}\)|TFP\(_{i,t-1}\))).

- The function \(f\) is invertible with respect to TFP\(_{i,t}\).

Then, the original model becomes

\[
\log(\text{ValueAdd}_{i,t}) = \bar{\pi} + a \log(\text{Capital}_{i,t-1}) + \gamma \log(\text{Employment}_{i,t})
+ f^{-1}(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{MaterialExpense}_{i,t}) + \epsilon_{i,t}
\]

\[
= g(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{MaterialExpense}_{i,t}) + \epsilon_{i,t}
\]

Then, the original model becomes

\[
\log(\text{ValueAdd}_{i,t}) = \bar{\pi} + a \log(\text{Capital}_{i,t-1}) + \gamma \log(\text{Employment}_{i,t})
+ f^{-1}(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{MaterialExpense}_{i,t}) + \epsilon_{i,t}
\]

\[
= g(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{MaterialExpense}_{i,t}) + \epsilon_{i,t}
\]
Then, I run a non-parametric regression of logged value-add on the capital, employment and material expenses to obtain $\hat{g}(\text{Capital}_{i,t-1}, \text{Employment}_{i,t}, \text{MaterialExpense}_{i,t})$. Using the predicted value $\hat{g}$, I estimate $\alpha$ and $\gamma$ from the following conditional moment condition:

$$\mathbb{E}(\hat{g}(\alpha, \gamma)|\text{Capital}_{i,t-1}, \text{Employment}_{i,t-1}) = 0$$

where $\hat{g}(\alpha, \gamma) = \text{TFP}_{i,t} - \mathbb{E}(\text{TFP}_{i,t}|\text{TFP}_{i,t-1})$.

Specifically, $\hat{\text{TFP}}_{i,t}(\hat{\alpha}, \hat{\gamma}) = \hat{g} - \hat{\alpha}\log(\text{Capital}_{i,t-1}) - \hat{\gamma}\log(\text{Employment}_{i,t})$. I obtain $\hat{\xi}_{i,t}$ from the residuals of AR(1) regression of $\hat{\text{TFP}}_{i,t}(\hat{\alpha}, \hat{\gamma})$. The empiric analogue of the conditional moment is

$$\frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \frac{\hat{\xi}_{i,t} \cdot \text{Capital}_{i,t-1}}{\hat{\xi}_{i,t} \cdot \text{Employment}_{i,t-1}} \right) = 0$$

Each of the variables are obtained from firm-level balance sheet information in U.S. Compustat data combined with wage data by industry from the Current Employment Statistics (CES) survey. I join two datasets by matching the first two-digit NAICS codes. Specifically, each variable is defined as follows:

- **ValueAdd** = Sale - Material Expense
- **Material Expense** = Total Expense - Wage $\times$ Firm-level Employment
- **Total expense** = Sale - Operating Income Profit Before Depreciation (OIBDP)
- **Capital** is obtained from applying perpetual inventory methods to the first available capital stock entry (PPEGT). Firm $i$’s net real investment at period $t$ is computed from $I_{i,t} - \delta k_{i,t-1} := (\text{PPENT}_{i,t} - \text{PPENT}_{i,t-1}) / p_t$, where $p_t$ is nonresidential fixed investment deflator available from National Income and Product Accounts data (NIPA Table 1.1.9, line 9). I assume $\delta = 0.1$ (annual) to get gross real investment at
firm level. All the results stay robust over other reasonable choices of depreciation rates.
A.4 Appendix: Event analysis using different idiosyncratic TFP measures

Figure A.4.2: Event study: sensitivity to idiosyncratic TFP innovation based on the Solow residuals

Figure A.4.3: Event study: sensitivity to idiosyncratic TFP innovation based on the method of Olley and Pakes (1996)
A.5 Spike ratio in the model with fixed cost

In the model with fixed cost (Khan and Thomas, 2008), a firm’s lumpy investment decision is characterized by a threshold rule $\xi^* = \xi^*(k, z)$:

$$I^*(k, z) = \begin{cases} I(k, z) & \text{if } \xi \leq \xi^*(k, z) \quad \text{(Unconstrained)} \\ I^c(k, z) & \text{if } \xi > \xi^*(k, z) \quad \text{(Constrained)} \end{cases}$$

where the fixed cost $\xi \sim_{i.i.d} G(\xi)$, and

$$\xi^* = \frac{1}{1 + r(A, \Phi)} \mathbb{E}(I + (1 - \delta)k, z'; A', \Phi') - \frac{1}{1 + r(A, \Phi)} \mathbb{E}(I^c + (1 - \delta)k, z'; A', \Phi') - (I - I^c)$$

where $I$ denotes the unconstrained investment, and $I^c$ denotes constrained (small-scale) investment. I denote the value function in the first term as $J$ and the second as $J^c$.

For firms greater than a size threshold $\bar{k}$, the spike ratio is

$$\text{SpikeRatio}(\bar{k}) = \int \mathbb{I}\{\xi < \xi^*(k, z)\} \mathbb{I}\{k > \bar{k}\} \mathbb{I}\left\{\frac{I(k, z)}{k} > 0.2\right\} d\xi d\Phi$$

Under the assumption that $\xi$ follows a uniform distribution ($\xi \sim \text{Unif}[0, \xi] = G$),

$$\text{SpikeRatio}(\bar{k}) = \int \left(\frac{\xi^*(k, z)}{\xi}\right) \mathbb{I}\{k > \bar{k}\} \mathbb{I}\left\{\frac{I(k, z)}{k} > 0.2\right\} d\Phi$$

Thus, the spike ratio is strongly affected by the interest rate changes.
A.6 Fixed parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm Side Fundamentals</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Capital share</td>
<td>0.3</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Labor share</td>
<td>0.6</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Depreciation rate</td>
<td>0.09</td>
</tr>
<tr>
<td>Household Side</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.96</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Labor supply parameter</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Table A.6.3: Fixed Parameters
A.7 Appendix: heterogeneous echo effects with alternative definitions of large firms

![Graph](image)

(a) Average investment stages  
(b) Investment

Figure A.7.4: Heterogeneous echo effects: large firms are top 30%

![Graph](image)

(a) Average investment stages  
(b) Investment

Figure A.7.5: Heterogeneous echo effects: large firms are top 40%

In the original definition where large firm are defined as top 20% largest firms, large firms take 26.7% of total capital. If large firm are defined as top 30% largest firms, then large firms take 38.5% of total capital. If large firm are defined as top 40% largest firms, then large firms take 49.6% of total capital.
A.8 Appendix: heterogeneous nonlinear effects with alternative definitions of large firms

In the original definition where large firm are defined as top 20% largest firms, large firms take 26.7% of total capital. If large firm are defined as top 30% largest firms, then large firms take 38.5% of total capital. If large firm are defined as top 40% largest firms, then large firms take 49.6% of total capital.
A.9  Appendix: Heterogeneous large and small firms’ problem

A firm of type $j \in \{\text{large, small}\}$ solves the following problem:

$$J(k, z, s, j; \Phi, A) = \pi(z, k; \Phi, A) + \max \{ \max_{s' \geq s, I} \{ -I - c(k, I) - \text{acc}_j(s', s)w(\Phi, A) + \frac{1 - h}{1 + r(\Phi, A)} \mathbb{E} J(k', z', s' \pmod{z}, j', \Phi', A') \}, \max_{s \leq s' \leq z, I \in \Omega(k)} \{ -I^c - c(k, I^c) - \text{acc}_j(s', s)w(\Phi, A) + \frac{1 - h}{1 + r(\Phi, A)} \mathbb{E} J(k^c, z', s', j'; \Phi', A') \} \} $$

(Operating Profit) $\pi(z, k; \Phi, A) := \max_{l_d} zAk^\mu l_d^\mu - \bar{w}(\Phi, A)l_d$ ($l_d$: labor demand)

(Convex Adj. Cost) $c(k, I) := \frac{\mu^l}{2} \binom{I}{k}^2$

(Acceleration Cost) $\text{acc}_j(s', s) := \left[ \mathbb{I}\{s' \geq s + 1\} \left( \frac{\mu^a}{2} (s' - s - 1)^2 \right) \right]$  

($\mu^a_{\text{large}} > \mu^a_{\text{small}}, \mu^l_{\text{large}} < \mu^l_{\text{small}}$)

(Constrained Investment) $I^c \in \Omega(k) := [-kv, kv]$  ($v < \delta$)

(Agg. Law of Motion) $\Phi' := H(\Phi, A), \ G_A(A) = A'$ (AR(1) process)

### A.9.1 Parameters for heterogeneous large and small firms’ problem

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^a_{\text{large}}$</td>
<td>Large firms’ acceleration cost</td>
<td>0.45</td>
</tr>
<tr>
<td>$\mu^l_{\text{large}}$</td>
<td>Large firms’ adjustment cost</td>
<td>2.50</td>
</tr>
<tr>
<td>$\mu^a_{\text{small}}$</td>
<td>Small firms’ acceleration cost</td>
<td>0.18</td>
</tr>
<tr>
<td>$\mu^l_{\text{small}}$</td>
<td>Small firms’ adjustment cost</td>
<td>3.50</td>
</tr>
<tr>
<td>$v_{\text{large}}$</td>
<td>Small investment range</td>
<td>0.01</td>
</tr>
<tr>
<td>$v_{\text{small}}$</td>
<td>Small investment range</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table A.9.4: Parameters in heterogeneous large and small firms’ problem
A.9.2 Testing serial correlation of residuals from fitting harmonic functions into the data

<table>
<thead>
<tr>
<th></th>
<th>Echo (Manuf.)</th>
<th>Non-echo (Manuf.)</th>
<th>Echo (Oil)</th>
<th>Non-echo (Oil)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_t$</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>$\varepsilon_{t-1}$</td>
<td>0.205</td>
<td>0.136</td>
<td>0.063</td>
<td>0.601***</td>
</tr>
<tr>
<td></td>
<td>(0.152)</td>
<td>(0.161)</td>
<td>(0.181)</td>
<td>(0.173)</td>
</tr>
<tr>
<td>Constant</td>
<td>-2.433</td>
<td>-0.312</td>
<td>-1.385</td>
<td>0.055</td>
</tr>
<tr>
<td></td>
<td>(7.186)</td>
<td>(2.342)</td>
<td>(2.365)</td>
<td>(1.273)</td>
</tr>
<tr>
<td>Observations</td>
<td>39</td>
<td>39</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>R²</td>
<td>0.047</td>
<td>0.019</td>
<td>0.006</td>
<td>0.355</td>
</tr>
<tr>
<td>Adjusted R²</td>
<td>0.021</td>
<td>-0.008</td>
<td>-0.040</td>
<td>0.326</td>
</tr>
</tbody>
</table>

*Note:* *p* < 0.1; **p** < 0.05; ***p*** < 0.01

Table A.9.5: Serial correlations in residuals

Table A.9.5 reports the autoregression results of residuals from fitting a harmonic function into the investment growth data used in Table 10 and 11. The harmonic function is specified as follows:

\[ X_t = \beta \cos(\omega t + \phi) + \varepsilon_t \]

where $\beta > 0$, $\omega \in (0, \pi)$, $\phi \sim U(-\pi, \pi)$, and $\varepsilon_t$ is a serially uncorrelated noise which is assumed to be independent from $\phi$. The function is nonlinerarly fitted into the data following Li (2010). For this estimation, I used R package “ptest” (Lai and McLeod, 2016).
A.9.3 Spectral densities of growth rates of non-residential fixed investment

Figure A.9.8: Spectral densities of growth rates of non-residential fixed investment for manufacturing and oil industries

Spectral densities are estimated with modified Daniell smoothing parameter of 5.
A.10 Appendix: Computation Methodology

A.10.1 Computation for cyclical competitive equilibrium I

To compute the cyclical competitive equilibrium, I take the following steps:

1. Set a capital grid \( K_0 \), stage grid \( S_0 = \{1, 2, 3, ..., \bar{s}\} \), and a macro stage grid \( T = \{1, 2, 3, ..., n^*\} \). For the capital grid \( K_0 \), note that the maximum and minimum values need to be distant enough to cover the entire closed capital domain \( K \) which will be obtained endogenously. Discretize the autoregressive productivity process \( z \), to get the unconditional productivity support \( Z \).

2. Guess the number of periods within a cycle \( n^* \) and the corresponding number of price bundles \( \{r_{\tau}, w_{\tau}\}_{\tau=1}^{n^*} \).

3. Solve a firm’s problem using a value function iteration.

4. Come up with an initial distribution \( \Phi_0 \) that has \( K_0 \times Z \times S_0 \) as a support of the distribution.

5. Make \( \Phi_0 \) evolve based on the policy functions from step 3 and transition rule of the autoregressive productivity process \( z \) to get \( \{\Phi_i\}_{i=0}^{M} \) until there exists \( M > N \geq 0 \) such that \( ||\Phi_M - \Phi_N|| < tol \). \((M - N)\) is the implied length of the cycle in the solution given the initial guess.

6. Calculate \( error_1 \) such that \( error_1 = |(M - N) - n^*| \). If \( error_1 > 0 \), then go back to step 2 to start over with another initial guess for \( n^* \). Otherwise, go the next step.
7. Compute the implied price bundles \((r_{\text{implied}}, t, w_{\text{implied}}, t)\) from the inter-temporal and intra-temporal optimality conditions of the household using the endogenous aggregate allocation \(\{c_t\}^M\) and optimal labor supply policy \(\{l_t\}^M\):

\[
\text{(Inter-temporal)} \quad E \left( \beta \frac{u_c(c_t, l_t)}{u_c(c_{t+1}, l_{t+1})} \right) = 1 + r_{\text{implied}, t}
\]

\[
\text{(Intra-temporal)} \quad - \frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = w_{\text{implied}, t}
\]

8. Calculate \(\text{error}_2 = \max \{||r_{\text{implied}, t} - r||_{\sup}, ||w_{\text{implied}, t} - w||_{\sup}\}\), where \(r_{\text{implied}, t}\) and \(r\) are vectors of the implied real interest rates and guessed real interest rates, respectively, and \(w_{\text{implied}, t}\) and \(w\) are vectors of the implied real wage and guessed real wage, respectively. If \(\text{error}_2 > \text{tol}\) then go back to step 2 to start over with another initial guess for \(\{r_t, w_t\}^n\). Otherwise, the solutions obtained from step 3, \(n^* = M - N\), and prices \(\{r_t, w_t\}^n\) are the cyclical competitive equilibrium.

Note that in the baseline computation where firms are not heterogeneous in terms of the length of the firm-level \((S, s)\) cycle, the equilibrium length \(n^*\) of the aggregate level cycle and the firm-level cycle \(S\) are identical. Therefore, with an initial guess \(n^* = S\), step 6 becomes unnecessary under the calibrated parameters because \(\text{error}_1 = 0\) always holds.

However, in the potential application of the model for firms with the heterogeneous cycle lengths, \(n^*\) might become different from \(S\). In this case, if the heterogeneity persists without shuffling across the firms, e.g. permanently different groups of firms with heterogeneous cycle lengths, \(n^* = \text{l.c.m.}(S_1, S_2, ..., S_G)\) is the correct guess for the aggregate cycle length, where \(G\) indicates the number of different groups.
A.10.2 Computation for cyclical competitive equilibrium II: a practical approach

The computation algorithm explained in Appendix A is implementable, but mathematical packages often fail to obtain a convergent solution to the fixed point due to the high sensitivity of the solution to the relative price levels across the periods. This issue becomes easier to understand, when compared to the stationary equilibrium case.

For example, suppose there is a stationary equilibrium, and the equilibrium real interest rate is \( r^* = 0.04 \). Let the initial guess for the real interest rate be \( r_{\text{guess}} = 0.05 \), which leads to an implied level of real interest rate \( r_{\text{implied}} = 0.03 \). During the approximation, it always holds that if \( r_{\text{implied}} > r_{\text{guess}} \), then \( r^* > r_{\text{guess}} \). This let the solver pick the next guess \( r'_{\text{guess}} < r_{\text{guess}} \), i.e. \( r'_{\text{guess}} = 0.038 \), and by iterating these steps, the fixed point solution is obtained by the convergence.

However, consider a cyclical competitive equilibrium with a cycle length \( n^* = 2 \). In this case, the initial guess needs to be the real interest rates for two periods in a cycle. Suppose the equilibrium real interest rates are \( (r_{1}^*, r_{2}^*) = (0.04, 0.045) \), and initial guess is \( (r_{\text{guess},1}, r_{\text{guess},2}) = (0.045, 0.047) \) which leads to implied real interest levels of \( (r_{\text{implied},1}, r_{\text{implied},2}) = (0.0361, 0.0362) \). Then, the prediction error is greater for the second period, even if the ranking of the guessed prices are correct across the periods. Then, a solver might recognize the current guess for the second period price is too high compared to the guess for the first period price. If it happens, for the next guess for the prices, a solver may choose to use a price bundle that has a flipped ranking of prices across the periods such as \( (r'_{\text{guess},1}, r'_{\text{guess},2}) = (0.0442, 0.0432) \). In this case, the implied price jumps dramatically due to the flipped ranking because many firms change their investment decision in the extensive margin to utilize the price gain in the second period. These occasional jumps in the prediction errors make the solver fail to achieve a converged solution.
In the stationary equilibrium case, flipped price ranking across the periods are not an issue because there is only one price, which always preserves the monotone relationship among a guessed price, an implied price, and the fixed point price. However, in the cyclical competitive equilibrium, the flipped ranking is a challenging issue for the computation as it leads to a completely different implied equilibrium cycle due to the investment changes in the extensive margin.

To overcome this problem, I introduce another simple method for the computation which makes the guessed prices slowly and steadily converge to the equilibrium prices without flipping the ranking. The new method is implemented simply by changing step 2 and 8 in the previous method. I elaborate the new steps for the method as follows:

- **Step 2**: Guess the number of periods within a cycle $n^*$. As an initial guess for the price bundles, consider a constant sequence of prices, that is $\{r^*_t, w^*_t\}_{t=1}^{n^*}$, s.t. $r^*_t = \bar{r}$, and $w^*_t = \bar{w}$, where $\bar{r}$ and $\bar{w}$ are taken to be large enough to be greater than any of possible equilibrium price levels.

- **Step 8**: Calculate $\text{error}^2 = \max\{||r_{\text{implied}} - r||_{\text{sup}}, ||w_{\text{implied}} - w||_{\text{sup}}\}$, where $r_{\text{implied}}$ and $r$ are vectors of the implied real interest rates and guessed real interest rates, respectively, and $w_{\text{implied}}$ and $w$ are vectors of the implied real wage and guessed real wage, respectively. If $\text{error}^2 > \text{tol}$ then go back to step 2 to start over with the specific initial guess $\{r'_t, w'_t\}_{t=1}^{n^*}$ such that $r'_t = \omega r^*_t + (1 - \omega) r_{\text{implied}, t}$ and $w'_t = \omega w^*_t + (1 - \omega) w_{\text{implied}, t}$, where $\omega$ is a price convergence parameter. If the price convergence parameter is close to 1, the prices converge slower while the convergence of the solution is more certainly guaranteed. I use the $\omega = 0.95$ for the assured convergence. If $\text{error}^2 \leq \text{tol}$, the solutions obtained from step 3, $n^* = M - N$, and prices $\{r^*_t, w^*_t\}_{t=1}^{n^*}$ are the cyclical competitive equilibrium.

For example, suppose the initial guess for the real interest rate is $(\bar{r}, \bar{r}) = (0.06, 0.06)$ for a cyclical competitive equilibrium with a cycle length $n^* = 2$. Suppose it leads to
an implied real interest rate level \((r_{\text{implied},1}, r_{\text{implied},2}) = (0.03, 0.032)\). Then, if the price convergence parameter \(\omega = 0.95\), the next guess for the prices is \((r'_1, r'_2) = 0.95 \ast (r, r) + 0.05 \ast (0.03, 0.032) = (0.0585, 0.0586)\). Here the ranking of the prices is determined by the first iteration of the algorithm, and the ranking is likely to persist through the convergence. The ranking persistence is stronger for a higher price convergence parameter \(\omega\) which gives slower but more certain convergence, and vice versa.

### A.11 Appendix: Proofs for the theoretical results

#### A.11.1 Proof for Proposition 1

**Proposition 1** (Isolated stage policy).

Given an idiosyncratic productivity process \(G_z(z)\) with a bounded support \(Z\), there exists \(\mu_{G_z} > 0\) such that

\[
\mu_{\text{large}} \geq \mu_{G_z} \implies s'(k, z, s, \text{large}; \Phi, A) = s'(s, \text{large}) \text{ for } \forall (k, z, s) \in (K, Z, S)
\]

where \((K, S)\) denotes the domains of capital and investment stages, respectively.

**Proof.**

Define \(\zeta(\mu_{\text{large}}) := \sup_{s' \in \{1, 2, \ldots, \tau\}, k' \in K} \{\frac{1}{\text{w}r(k, A)} \mathbb{E}[J(k', z', s', \text{large}; \Phi', A'; \mu_{\text{large}})]\}\). Note that the equilibrium value function \(J\) is a weakly decreasing function of cost parameter \(\mu_{\text{large}}\). Thus, \(\zeta(\mu_{\text{large}})\) is also weakly decreasing in \(\mu_{\text{large}}\).

\[
\text{Acc}(s', s; \mu_{\text{large}}) = \left[\mathbb{I}\{s' > s + 1\} \left(\frac{\mu_{\text{large}}}{2} (s' - s - 1)^2\right)\right]
\]

If \(s' > s + 1\), \(\text{Acc}(s', s; \mu_{\text{large}}) \geq \frac{\mu_{\text{large}}}{2}\). Therefore, if \(\exists \mu_{G_z} > 0\) such that \(\frac{\mu_{G_z}}{2} > \zeta(\bar{\mu}_{G_z})\), optimal stage policy is always one-stage-per-period rule only if \(\mu > \mu_{G_z}\). This is because
$s'(k, z, s, \text{large}; \Phi, A) = s'(k, s, \text{large}) = s + 1 \mod \bar{s}$.

So, it is sufficient to show $\exists \mu_G > 0$ such that $\frac{\mu_G}{2} > \zeta(\mu_G)$.

Suppose $\exists \mu_G > 0$ such that $\frac{\mu_G}{2} > \zeta(\mu_G)$. For $\forall \mu_G > 0$, $\frac{\mu_G}{2} \leq \zeta(\mu_G)$.

As $\zeta(\mu_G) < \infty$, $\exists N < \infty$ such that $N > \zeta(\mu_G)$. Then, define $M := \max\{\frac{\mu_G}{2}, N\} + \epsilon$.

Hence,

$$\zeta(\mu_G) < M \leq \zeta(2M)$$

This implies

$$\zeta(\mu_G) < \zeta(2M) \text{ and } \mu_G < 2M$$

This contradicts $\zeta(x)$ is weakly decreasing in $x$.

Therefore, $\exists \mu_G > 0$ such that $\frac{\mu_G}{2} > \zeta(\mu_G)$.

\[\tag*{\blacksquare}\]

### A.11.2 Proof for Proposition 2

**Proposition 2** (Breaking the law of large numbers).

If $\mu_{\text{large}} \geq \mu_G$, and $\Phi_0 \in \mathcal{D}_0$, the implied sequence of distributions $\{\Phi_t\}_0^\infty$ does not have a limit point.

**Proof.**

Suppose there exists a limit point $\Phi^*$, such that

$$(\Phi^*)(K, Z, S, j; \Phi^*, A) := \int_{K \times Z \times S} \left( \int_{Z} \Gamma_{z, z'} dz' \right) 1\{\hat{\beta}'(k, z, s, j; \Phi^*, A) \in K\}
1\{s'(k, z, s, j; \Phi^*, A) \in S\} d\Phi^*(k, z, s, j; \Phi^*, A)$$
for any set \((K, Z, S)\) in the \(\sigma\)-algebra \((K, Z, S)\) generated from the domains \((K, Z, S)\).

By the isolation proposition, for \(\forall \tilde{s} \in S\)

\[
(\Phi^*)(K, Z, \tilde{s}, \text{large}; \Phi^*, A)
\]

\[
= \int_{K \times Z \times S} \left( \int_Z \Gamma_{z', z} dz' \right) I\{\tilde{k}'(k, z, s, \text{large}; \Phi^*, A) \in K\}
\]

\[
I\{\tilde{s}'(s) = \tilde{s}\} d\Phi^*(k, z, s, \text{large}; \Phi^*, A)
\]

\[
= \int_{K \times Z \times S} \left( \int_Z \Gamma_{z', z} dz' \right) I\{\tilde{k}'(k, z, s, \text{large}; \Phi^*, A) \in K\}
\]

\[
I\{s = \tilde{s} - 1 \,(\text{mod} \, \tilde{s})\} d\Phi^*(k, z, s, \text{large}; \Phi^*, A)
\]

\[
= (\Phi^*)(K, Z, \tilde{s} - 1 \,(\text{mod} \, \tilde{s})\, \text{large}; \Phi^*, A)
\]

\[
= (\Phi^*)(K, Z, \tilde{s} - 2 \,(\text{mod} \, \tilde{s})\, \text{large}; \Phi^*, A)
\]

\[
= (\Phi^*)(K, Z, \tilde{s} - 3 \,(\text{mod} \, \tilde{s})\, \text{large}; \Phi^*, A)
\]

\[
= \ldots
\]

Thus, \(\Phi^* \in \mathcal{D}_1\). Therefore, it is sufficient to show that for \(\forall \Phi_t \in \{\Phi_t\}_{t=1}^\infty\),

\[
\Phi_t \in \mathcal{D}_0 \implies \Phi_{t+1} \in \mathcal{D}_0
\]

From the same step as (1), we get

\[
(\Phi_{t+1})(K, Z, \tilde{s}, \text{large}; \Phi_{t+1}, A) = (\Phi_t)(K, Z, \tilde{s} - 1 \,(\text{mod} \, \tilde{s})\, \text{large}; \Phi_t, A), \text{ for } \forall \tilde{s} \in S
\]

\(\Phi_t \in \mathcal{D}_0\) implies \(\exists s^* \in S\) such that,

\[
\int_{K \times Z \times \{s^*\} \times \{y, a\}} d\Phi_t (k, z, s^*, j; \Phi_t, A) \neq \frac{1}{\tilde{s}}
\]

So,

\[
\int_{K \times Z \times \{s^*+1\} \times \{y, a\}} d\Phi_{t+1} (k, z, s^* + 1, j; \Phi_{t+1}, A) \neq \frac{1}{\tilde{s}}
\]
Therefore, $\Phi_t \in \mathcal{D}_0 \implies \Phi_{t+1} \in \mathcal{D}_0$. 

A.11.3 Proof for Corollary 1

**Corollary 1** (Aggregate endogenous cycle under the persistent shock).

Given $\mu^a_{\text{large}} \geq \overline{\mu}_G$, and the initial distribution $\Phi_0 \in \mathcal{D}_0$, for $\forall \epsilon > 0$, there is a sufficiently large $\overline{\tau} \in \{1, 2, 3, \ldots\}$ such that the implied sequence of distributions $\{\Phi_\tau\}_{\tau=0}^\infty$ satisfies following property:

$$||\Phi_{\tau+\bar{\tau}} - \Phi_\tau||_{\text{sup}} < \epsilon, \text{ for } \forall \tau > \overline{\tau}$$

**Proof.**

The strategy of proof is to utilize the law of large numbers that gives convergence of conditional joint distribution of $(k, z)$ given $s$. For the notational brevity, the aggregate state variables are now omitted. By the isolation propositions,

$$(\Phi_{\tau+1})(k, z, s) = \int_{\mathcal{Z}} \int_{\mathcal{S}} \left( \int_{\mathcal{Z}} \Gamma_{z, z'} \int_{\mathcal{S}} \mathbb{I}\{ \hat{k}'(k, z, s) \in K \} \mathbb{I}\{ \hat{s}'(s) \in S \} d\Phi_\tau(k, z, s) \right) d\Phi_{\tau+1}(k, z, s)$$

Let $\phi_{s, \tau}$ denote the marginal density of $s$ for the distribution $\Phi_\tau$.

$$\phi_{s, \tau}(s) := \int_{\mathcal{Z} \times \{s\} \times \{y, o\}} \phi_{s, \tau}(k, z, s, j; \Phi_\tau, A)$$

From the exactly same derivation as the equations (1), the marginal density of $s$ satisfies the following property:

$$\phi_{s, \tau}(s) = \phi_{s, \tau+\overline{\tau}}(s)$$ (2)
i) for $S = \bar{s} \in 1 = 2, 3, ..., \bar{s},$

\[
(\Phi_{r+1})(K, Z, \bar{s}) = \int_{K \times Z \times S} \left( \int_{Z} \Gamma_{z, z'} dz' \right) \mathbb{I}\{\hat{k}'(k, z, s) \in K\} \mathbb{I}\{\bar{s} = s + 1 \text{ (mod } \bar{s})\} d\Phi_{r}(k, z, s) \\
= \int_{K \times Z} \left( \int_{Z} \Gamma_{z, z'} dz' \right) \mathbb{I}\{\hat{k}'(k, z, \bar{s} - 1 \text{ (mod } \bar{s})\} \in K\} d\Phi_{r}(k, z, \bar{s} - 1 \text{ (mod } \bar{s}))
\]

(3)

and it is known that

\[
(\Phi_{r+1})(K, Z|\bar{s}) \ast \phi_{s,r+1}(\bar{s}) = (\Phi_{r+1})(K, Z, \bar{s}) \\
\phi_{s,r+1}(\bar{s}) = \phi_{s,r}(\bar{s} - 1 \text{ (mod } \bar{s}))
\]

Dividing both sides of the equation (3) by $\phi_{s,r+1}(\bar{s}),$

\[
(\Phi_{r+1})(K, Z|\bar{s}) = \int_{K \times Z} \left( \int_{Z} \Gamma_{z, z'} dz' \right) \mathbb{I}\{\hat{k}'(k, z, \bar{s} - 1 \text{ (mod } \bar{s})\} \in K\} d\Phi_{r}(k, z|\bar{s} - 1 \text{ (mod } \bar{s}))
\]

This is equivalent to

\[
(\Phi_{r+1})(K, Z|s + 1 \text{ (mod } \bar{s})) = \int_{K \times Z} \left( \int_{Z} \Gamma_{z, z'} dz' \right) \mathbb{I}\{\hat{k}'(k, z, s) \in K\} d\Phi_{r}(k, z|s)
\]

Let $\Phi_{r|s}$ denote the joint distribution of $(k, z)$ conditional on $s.$

Define a transition operator $\Lambda_{s}$ such that

\[
\Lambda_{s}(\Psi)(K, Z) := \int_{K \times Z} \left( \int_{Z} \Gamma_{z, z'} dz' \right) \mathbb{I}\{\hat{k}'(k, z, s) \in K\} d\Psi(k, z), \text{ for } \forall \Psi \text{ measure on } K \times Z
\]

Then,

\[
(\Phi_{r+1}|s + 1 \text{ (mod } \bar{s}))(K, Z) = \Lambda_{s}(\Phi_{r|s})(K, Z)
\]
By applying the transition $\bar{s} - 1$ times additionally,

$$(\Phi_{\tau+\bar{s}}|_s)(K, Z) = \Lambda_s^{(\bar{s})}(\Phi_{\tau}|_s)(K, Z)$$  \hspace{1cm} (4)$$

The equation above holds for $\forall s \in S$.

Define a transition operator $T_s$ as

$$T_s(\Psi)(K, Z) := \Lambda_s^{(\bar{s})}(\Psi)(k, z, s), \text{ for } \forall \Psi \text{ measure on } K \times Z$$

Hence,

$$(\Phi_{\tau+T}|_s)(K, Z) = T_s(\Phi_{\tau}|_s)(K, Z)$$  \hspace{1cm} (5)$$

Note that this transition preserves the conditioning state variable $s$. By infinitely applying the transition $T_s$ to $\Phi_{\tau}|_s$, under the mild regularity conditions, the law of large numbers gives the following result:\textsuperscript{87}

$$\exists \Phi^*_s \text{ such that } (\Phi^*_s)(K, Z) = \lim_{n \to \infty} (T_s)^n(\Phi_{\tau}|_s)(K, Z), \text{ for } \forall (K, Z) \in (K \times Z)$$

Then, $\Phi^*_s$ is a fixed point of the transition $T_s$ such that

$$(\Phi^*_s)(K, Z) = T_s(\Phi^*_s)(K, Z)$$

All the convergent sequence is Cauchy sequence in metric space. Thus, we can find $\tau^*_s$ such that for $\forall \tau > \tau^*_s$

$$||\Phi_{\tau+T}|_s(K, Z) - \Phi_{\tau}|_s(K, Z)||_{\sup} < \epsilon, \text{ for } \forall (K, Z) \in (K \times Z)$$  \hspace{1cm} (6)$$

\textsuperscript{87}. As the transition relies on capital policy that only depends on the stochastic process $z$, and $k$, convergence of distribution of $z$ to the ergodic distribution makes the whole joint distribution of $(k, z)$ converges as well. This is the result coming from the law of large numbers, but I do not directly prove the convergence in this paper.
Then, define
\[ \tau^+ = \sup_{s \in S} \tau^+_s \]

For \( \forall (k, z, s) \in (K \times Z \times S) \) and \( \forall \tau > \tau^+ \),
\[
\left\| \Phi_{\tau+T}(k, z, s) - \Phi_\tau(k, z, s) \right\|_{\sup} = \left\| \int_s \Phi_{\tau+T}|s(K, Z)\phi_s,\tau+T(s)ds - \int_s \Phi_\tau|s(K, Z)\phi_s,\tau(s)ds \right\|_{\sup} \\
= \left\| \int_s \Phi_{\tau+T}|s(K, Z)\phi_s,\tau(s)ds - \int_s \Phi_\tau|s(K, Z)\phi_s,\tau(s)ds \right\|_{\sup}, \text{ from (2)} \\
\leq \int_s \left\| \Phi_{\tau+T}|s(K, Z) - \Phi_\tau|s(K, Z) \right\|_{\sup} \phi_s,\tau(s)ds \\
< \int_s \epsilon \phi_s,\tau(s)ds \\
\leq \epsilon
\]

Therefore, the proof is completed.

\[ \Box \]

A.11.4 Proof for Corollary 2

**Corollary 2** (Commonness of aggregate cycles).

Consider a non-degenerate atomless distribution \( \Psi \) defined on \( \sigma \)-algebra \( \mathcal{D} \) generated from \( \mathcal{D} \), where \( \mathcal{D} \) is the support of \( \Psi \). Then, \( \Psi(\mathcal{D}_1) = 0 \), and \( \Psi(\mathcal{D}_0) = \Psi(\mathcal{D}) = 1 \).

**Proof.**

\( \mathcal{D}_1 \) is a set of all distributions of which marginal distribution of \( x \) is a uniform distribution. Out of all possible marginal distribution of \( x \), \( \mathcal{D}_1 \) represents a singleton. Therefore, \( \Psi(\mathcal{D}_1) = 0 \). Because \( \mathcal{D}_0 = \mathcal{D} \setminus \mathcal{D}_0 \), \( \Psi(\mathcal{D}_0) = 1 \)

\[ \Box \]
Appendix B

Appendix for Chapter 2

B.1 Appendix: Proofs for the theoretical results

B.1.1 Proof for Proposition 3

Proposition 3. (Occupation choice threshold)

Given \((z_t, x_t, A_t, w_t, r_t)\), there exists \(\bar{a}_t \in [0, \infty)\) such that a household decides to become an entrepreneur if

\[
a_t \geq \bar{a}_t = \pi(z_t, x_t; A_t, w_t, r_t)
\]

Proof.

As the problem is static, I omit the time subscript for simplicity in the notation.

Suppose the financial constraint is not binding. Then, from the first-order conditions,

\[
\begin{align*}
l^* &= (zA_x)^{\frac{\alpha}{r + \delta}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\gamma}{1-\gamma}} \left( \frac{1 - \alpha}{w} \right)^{\frac{1}{1-\gamma}} \\
k^* &= (zA_x)^{\frac{\alpha}{r + \delta}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{1 - \gamma + \gamma}{1-\gamma}} \left( \frac{1 - \alpha}{w} \right)^{\frac{(1-\gamma)}{1-\gamma}} \\
p^* &= (zA_x)^{\frac{1}{1-\gamma}} (1 - \gamma)^{\frac{1}{1-\gamma}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\gamma}{1-\gamma}} \left( \frac{1 - \alpha}{w} \right)^{\frac{(1-\gamma)}{1-\gamma}}
\end{align*}
\]
There exists $\exists = \exists(x, A, w, r) > 0$ such that if $z < \exists$, even unconditionally optimal profit is less than the labor income $wx\tilde{h}$. Thus, it satisfies the following equation:

$$\pi^*(\exists(x, A, w, r), A, w, r) = wx\tilde{h}$$

$$\iff (\exists(x, A, w, r)A)^\frac{1}{1-\gamma} (1 - \gamma)^{\frac{1}{1-\gamma}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{r + \delta}} \left( \frac{1 - \alpha}{w} \right)^{\frac{(1-\alpha)\gamma}{r + \delta}} = wx\tilde{h}$$

$$\iff \exists(x, A, w, r)A = \frac{1}{A} \left( \frac{wx\tilde{h}}{M(w, r)} \right)^{1-\gamma}$$

where $M(w, r) := (1 - \gamma)^{\frac{1}{1-\gamma}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{r + \delta}} \left( \frac{1 - \alpha}{w} \right)^{\frac{(1-\alpha)\gamma}{r + \delta}}$

Therefore, if $z < \exists$, there is no finite wealth level that makes a household choose to become an entrepreneur. Thus,

$$\text{If } z < \exists, \quad a(z, x, A, w, r) = \infty$$

Now suppose $z \geq \exists$.

When the financial constraint is binding ($k^c = \frac{a}{\lambda}$), the constrained optima of the labor demand, $l^c$, and the profit, $\pi^c$, are as follows from the first-order conditions:

$$l^c(a, z, A, w, r) = \left( \frac{zA(1-a)\gamma}{w} \right)^{\frac{1}{1-(1-\alpha)\gamma}} \left( \frac{a}{\lambda} \right)^{\frac{\gamma}{1-(1-\alpha)\gamma}}$$

$$\pi^c(a, z, A, w, r) = (zA)^{\frac{1}{1-(1-\alpha)\gamma}} (1 - (1 - \alpha)\gamma)^{\frac{(1-\alpha)\gamma}{1-(1-\alpha)\gamma}} \left( \frac{1 - \alpha}{w} \right)^{\frac{(1-\alpha)\gamma}{1-(1-\alpha)\gamma}} \left( \frac{a}{\lambda} \right)^{\frac{(1-\alpha)\gamma}{1-(1-\alpha)\gamma}} (r + \delta) \left( \frac{a}{\lambda} \right)$$

Now we need to show there exists $\bar{a} \geq 0$ such that

$$a \geq \bar{a} (z, x, A, w, r) \iff \pi^c(\bar{a}(z, x, A, w, r), z, A, w, r) \geq wx\tilde{h}$$

I prove this in the following steps:
i) 

\[ \pi^c(0, z, A, w, r) = 0 < w x h \]
\[ \pi^c(k^*(z, A, w, r), z, A, w, r) \geq w x h \]

The second inequality is from the fact that \( \pi^c \) is maximized at \( k^* \) and \( z \geq \bar{z} \). \( z \geq \bar{z} \) implies there exists a wealth level where becoming an entrepreneur is better than working as a labor, and \( k^* \) is the optimal level of capital.

If the second weak inequality’s equality holds, then define \( \pi = k^* \). Otherwise, there exists at least one wealth level \( \bar{a} \geq 0 \) that satisfies

\[ \pi^c(\bar{a}, z, A, w, r) = w x h \]

due to intermediate value theorem. In other words, the left hand side is a continuous function of \( a \).

ii) Now, we will prove \( \pi^c(a, z, A, w, r) \) is strictly increasing in \( a \) over the interval \([0, k^*]\).

This step is to prove that \( \bar{a} \) found above is the unique crossing point between LHS and RHS. From the characterization of \( \pi^c \) above, the following is immediate:

\[ \frac{\partial \pi^c(a, z, A, w, r)}{\partial a} \bigg|_{a=0} = \infty \]
\[ \frac{\partial \pi^c(a, z, A, w, r)}{\partial a} \bigg|_{a=k^*} = 0 \]
\[ \frac{\partial^2 \pi^c(a, z, A, w, r)}{\partial a^2} < 0, \quad a \in (0, \infty) \]
The first fact is from the Inada condition of Cobb-Douglas function. Therefore,

\[ \frac{\partial \pi^c(a, z, A, w, r)}{\partial a} \text{ strictly decreases in } a \text{ until } a \text{ reaches } k^*. \] Therefore, \( \frac{\partial \pi^c(a, z, A, w, r)}{\partial a} > 0 \) for \( a \in (0, k^*) \). Therefore, there exists the unique \( \bar{a} = \bar{a}(z, x, A, w, r) \) such that

\[ \pi^c(\bar{a}(z, x, A, w, r), z, A, w, r) = wx\bar{h} \]

iii) \( \pi^c \) is strictly increasing in \( a \in (0, k^*) \). Therefore,

\[ a \geq \bar{a}(z, x, A, w, r) \iff \pi^c(\bar{a}(z, x, A, w, r), z, A, w, r) \geq wx\bar{h} \]

\[ \blacksquare \]

### B.1.2 Proof for Corollary 3

**Corollary 3.** (Minimum requirement for managerial ability)

Given \((x_t, A_t, w_t, r_t)\), there exists \( \bar{z}_t = \bar{z}(x_t; A_t, w_t, r_t) \) such that a household with \( z_t < \bar{z}_t \) cannot become an entrepreneur at any wealth \( a_t > 0 \).

**Proof.**

From the proof of Proposition 3,

\[ \pi^e(\bar{z}(x, A, w, r), A, w, r) = wx\bar{h} \]

\[ \iff (\bar{z}(x, A, w, r)A)^{1-(1-\gamma)}(1-\gamma)^{-\gamma} \left( \frac{a}{r+\delta} \right)^{\frac{\gamma^2}{r+\delta}} \left( \frac{1-a}{w} \right)^{\frac{\gamma(1-a)}{r+\delta}} = wx\bar{h} \]

\[ \iff \bar{z}(x, A, w, r)A = \frac{1}{A} \left( \frac{wx\bar{h}}{\mathcal{M}(w, r)} \right)^{1-\gamma} \]

where \( \mathcal{M}(w, r) := (1-\gamma)^{\gamma} \left( \frac{a}{r+\delta} \right)^{\frac{\gamma^2}{r+\delta}} \left( \frac{1-a}{w} \right)^{\frac{\gamma(1-a)}{r+\delta}} \)
Therefore, if \( z < \bar{z} \), the unconditionally optimal profit is less than the labor income \((\pi^* < wx)\). Thus, a household with \( z < \bar{z} \) would not choose to become an entrepreneur at any wealth \( a > 0 \).

**B.1.3 Proof for Corollary 4**

**Corollary 4.** (Monotonicity of threshold)

Given \((z_t, x_t, A_t, w_t, r_t)\), for \( z_t > \bar{z}_t \),

\[
\bar{z}_t > z_t \implies \pi(\bar{z}_t, x_t, A_t, w_t, r_t) < \pi(z_t, x_t, A_t, w_t, r_t)
\]

**Proof.**

From the proof of Proposition 3, \( \bar{a} = \bar{a}(z, x, A, w, r) \) satisfies

\[
\pi^c(\bar{a}, z, A, w, r, z, A, w, r) = (zA)^{(1-\gamma)/(1-\alpha)} (1 - (1 - \alpha)\gamma) A^{\gamma/(1-\gamma)} \left( \frac{1 - \alpha}{w} \right)^{(1-\alpha)/(1-\gamma)} \left( \frac{\bar{a}}{\bar{a}} \right)^{\alpha/(1-\alpha)} - (r + \delta) \left( \frac{\bar{a}}{\bar{a}} \right) = wx
\]

Suppose \( \tilde{z} > z \). Define \( \tilde{a} := \pi(\tilde{z}, x, A, w, r) \).

It can be easily checked that \( \pi \) strictly increasing in \( z \). Thus,

\[
\pi^c(\tilde{a}, \tilde{z}, A, w, r) > \pi^c(\bar{a}, z, A, w, r)
\]

And the following equation holds:

\[
\pi^c(\tilde{a}, \tilde{z}, A, w, r) = \pi^c(\bar{a}, z, A, w, r) = wx
\]

Suppose \( \tilde{a} \geq \bar{a} \). Then, as \( \pi^c \) is strictly increasing in \( a \),

\[
\pi^c(\tilde{a}, \tilde{z}, A, w, r) > \pi^c(\bar{a}, \tilde{z}, A, w, r) > \pi^c(\bar{a}, z, A, w, r)
\]

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which contradicts \( \pi^c(\bar{a}, \tilde{z}, A, w, r) = \pi^c(\bar{a}, z, A, w, r) \).

Therefore,

\[ \tilde{z} > z \implies \bar{a} = \pi(\tilde{z}, x, A, w, r) < \pi(z, x, A, w, r) \]

\[ \blacksquare \]

### B.2 Appendix: Other Calibration

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Target Moments</th>
<th>Data</th>
<th>Model</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi )</td>
<td>Employment/Population</td>
<td>58.7</td>
<td>57.4</td>
<td>1.27</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Debt/Asset of path-through businesses</td>
<td>83.2</td>
<td>82.8</td>
<td>0.13</td>
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<tr>
<td>( \Lambda )</td>
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<td>11.0</td>
<td>9.1</td>
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<td>( \gamma )</td>
<td>Top 10% income share</td>
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<td>49.4</td>
<td>0.9</td>
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<td>11.4</td>
<td>15.2</td>
<td>0.706</td>
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<td>( \sigma_z )</td>
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<td>4.3</td>
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</tr>
<tr>
<td>( \rho_x )</td>
<td>Top 10% business income share</td>
<td>18.5</td>
<td>8.9</td>
<td>0.701</td>
</tr>
<tr>
<td>( \sigma_x )</td>
<td>Top 1% business income share</td>
<td>27.1</td>
<td>26.2</td>
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</tr>
<tr>
<td>Top 0.1% business income share</td>
<td>24.1</td>
<td>31.7</td>
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Table B.2.1: Fixed parameters: Early 1980s

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Target Moments</th>
<th>Data</th>
<th>Model</th>
<th>Level</th>
</tr>
</thead>
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<tr>
<td>( \xi )</td>
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<td>59</td>
<td>1.16</td>
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<tr>
<td>( \lambda )</td>
<td>Debt/Asset of path-through businesses</td>
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<td>84.3</td>
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<td>Value-Add ratio between path-through and C-corp.</td>
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<td>11.7</td>
<td>0.265</td>
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<td>Top 10% income share</td>
<td>46.3</td>
<td>51.1</td>
<td>0.9</td>
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<tr>
<td>( \rho_z )</td>
<td>Top 1% income share</td>
<td>19.0</td>
<td>16.6</td>
<td>0.701</td>
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<tr>
<td>( \sigma_z )</td>
<td>Top 0.1% income share</td>
<td>8.9</td>
<td>5.5</td>
<td>0.162</td>
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<td>0.728</td>
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Table B.2.2: Fixed parameters: Counterfactual
## B.3 Appendix: Fixed parameters

<table>
<thead>
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<th>Parameters</th>
<th>Description</th>
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<td>$\alpha$</td>
<td>Capital share</td>
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<tr>
<td>$\beta$</td>
<td>Discount factor</td>
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<td>$\delta$</td>
<td>Depreciation rate</td>
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</tr>
<tr>
<td>$\bar{h}$</td>
<td>Labor hour</td>
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<td>$\theta_0$</td>
<td>Tax level parameter</td>
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<td>$\theta_1$</td>
<td>Tax progressivity parameter</td>
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</table>

Table B.3.3: Fixed parameters
Appendix C

Appendix for Chapter 3

C.1 Appendix: Parameters and Tables

C.1.1 Fixed Parameters

The fixed parameters are set at the following levels:

- (Span of control) \( \gamma = 0.7; \)
- (Corporate saving technology) \( \rho^{ca} = 0.038; \)
- (Idiosyncratic shock persistence) \( \rho_z = 0.90; \)
- (Idiosyncratic shock volatility) \( \sigma_z = 0.053; \)
- (Aggregate shock persistence) \( \rho_A = 0.95; \)
- (Aggregate shock volatility) \( \sigma_A = 0.007; \)
- (Household’s discount factor) \( \beta = 0.985; \)

These fixed parameters are chosen at a reasonable level based on the literature.
C.1.2 Definition: Aggregate cash holding from the Flow of Funds

The aggregate cash holding is defined as sum of following items in the Flow of Funds:

- (FL103091003) Foreign deposits
- (FL103020000) Checkable deposits and currency
- (FL103030003) Time and savings deposits
- (FL103034000) Money market fund shares
- (LM103064203) Mutual fund shares
- (FL102051003) Security repurchase agreements
- (FL103069100) Commercial paper
- (LM103061103) Treasury securities
## C.1.3 Cash holding and dividend

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Dividends$_t$ (%)</th>
</tr>
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<tbody>
<tr>
<td>Neg.</td>
<td>Pos.</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Cash$_{t-1}$ (%)</td>
<td>0.095***</td>
</tr>
<tr>
<td></td>
<td>(0.011)</td>
</tr>
<tr>
<td>TFP Control</td>
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<td>Constant</td>
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<td>Observations</td>
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<td>R$^2$</td>
<td>0.395</td>
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</tbody>
</table>

*Note:* *p* < 0.1; **p** < 0.05; ***p** < 0.01

Table C.1.1: Correlation between dividend and cash
Bibliography


