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Spectral Networks And Non-Abelianization For Reductive Groups

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Abstract

Non-abelianization was introduced in [16] as a way to study the moduli space of local systems of n-dimensional vector spaces on a Riemann surface X. This thesis, which is based on the forthcoming paper [23], explains how to generalize non-abelianization to the setting of G-local systems, for any reductive Lie group G. The main tool used to achieve this goal is a graph on X called a spectral network. These graphs have been introduced in [16] for groups of type A, and extended in [27] to groups of type ADE. We construct spectral networks for all reductive G, using a branched cover of X called a cameral cover, which is, in general, different from the spectral cover used in previous work on the subject. Our framework emphasizes the relationship between spectral networks and the trajectories of quadratic differentials, which provides a strategy to prove genericity results about spectral networks. Finally, we show how to associate, in an equivariant fashion, unipotent automorphisms called Stokes factors to edges of a spectral network. We define non-abelianization as a "cut and reglue" construction: we cut along the spectral network and reglue using the Stokes factors. Our construction, unlike the one in [16], does not rely on choices of trivializations for the local systems or for the branched cover.

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Matei Ionita

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Most of the mathematics that I learned at Penn was in the seminars that I attended, in which, aside from the nutritious mathematical content, I also learned to give talks which were, I hope, progressively less disastruous. As postdocs, Mauro and Justin organized the first of these seminars, and set a high bar for the next ones. Benedict and Sukjoo, as my office-mates and the recurrent characters in all seminars, heavily influenced the way

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Finally, I thank Andreea, Benedict and Nilay for the grunt work of reading an early draft of this thesis and giving helpful suggestions.

ABSTRACT

SPECTRAL NETWORKS AND NON-ABELIANIZATION FOR REDUCTIVE GROUPS

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Non-abelianization was introduced in [16] as a way to study the moduli space of local systems of n-dimensional vector spaces on a Riemann surface X. This thesis, which is based on the forthcoming paper [23], explains how to generalize non-abelianization to the setting of G-local systems, for any reductive Lie group G. The main tool used to achieve this goal is a graph on X called a spectral network. These graphs have been introduced in [16] for groups of type A, and extended in [27] to groups of type ADE. We construct spectral networks for all reductive G, using a branched cover of X called a cameral cover, which is, in general, different from the spectral cover used in previous work on the subject. Our framework emphasizes the relationship between spectral networks and the trajectories of quadratic differentials, which provides a strategy to prove genericity results about spectral networks. Finally, we show how to associate, in an equivariant fashion, unipotent automorphisms called Stokes factors to edges of a spectral network. We define non-abelianization as a "cut and reglue" construction: we cut along the spectral network and reglue using the Stokes factors. Our construction, unlike the one in [16], does not rely on choices of trivializations for the local systems or for the branched cover.

Contents

Chapter 1

Introduction

1.1 Local systems and non-abelianization

Given a Riemann surface X and a reductive Lie group G, the moduli space of local systems $\mathrm{Loc}_G(X)$ is an important topological invariant of X, widely used in geometry and physics. For example, if X is hyperbolic and $G = PSL_2(\mathbb{R})$, a certain subset of $\mathrm{Loc}_G(X)$ can be identified with Teichmüller space, which parametrizes complex structures on X, up to homeomorphisms isotopic to the identity. Moreover, $\mathrm{Loc}_G(X)$ can be identified complex-analytically with the moduli space of principal G-bundles with flat connection. In this guise they arise in physics, for example as classical solutions to Chern-Simons theory.

Fixing a basepoint $x \in X$, we can regard $\mathrm{Loc}_G(X)$ as the space of group homomorphisms from the fundamental group of X to G, modulo conjugation by G:

$$\mathrm{Loc}_G(X) \cong \mathrm{Hom}\big(\pi_1(X, x), G\big)/G. \tag{1.1.1}$$

In the case of abelian groups, such as a torus $T \cong (\mathbb{C}^*)^n$, the conjugation action of T

is trivial, and $\mathrm{Loc}_T(X)$ is a quotient of a product of copies of T by this trivial action.

For non-abelian G, it is harder to describe the effect of the quotient by the adjoint action, and, consequently, the structure of $\mathrm{Loc}_G(X)$. The purpose of this thesis is to construct "non-abelianization maps", which, modulo some details, are morphisms from a moduli space of T-local systems on a branched cover of X to $\mathrm{Loc}_G(X)$. Since T is abelian, the structure of the source space is well understood, and we can use the non-abelianization maps to probe the structure of the target space $\mathrm{Loc}_G(X)$.

Gaiotto, Moore and Neitzke first introduced non-abelianization for the cases $G = GL(n, \mathbb{C})$, $G = SL(n, \mathbb{C})$ in the paper [16], which builds up on their work on the $n = 2$ case in [17]. They further explored this topic in [18], where they also made a connection with the coordinates on $\mathrm{Loc}_G(X)$ constructed by Fock and Goncharov in [15]. Subsequently, other authors gave detailed constructions and new results in the cases $G = SL(2, \mathbb{C}), SL(2, \mathbb{R})$; see [14], [21], [29]. The author of this thesis, in joint work with Morrissey, generalize non-abelianization to arbitrary reductive G in the forthcoming paper [23]. The results we present in this thesis are based on sections 3-5 of loc. cit.

The main results are theorems 1.1.4 and 1.1.5 below. Before stating them, we discuss an example of non-abelianization in the case of $GL(2, \mathbb{C})$, to provide the reader with some intuition. For ease of exposition, in this example we work with the associated rank 2 vector bundles, rather than the $GL(2, \mathbb{C})$-principal bundles.

Example 1.1.1. Consider the double cover of the punctured affine line with a branch point at the origin:

$$\begin{aligned} \mathbb{A}^1_z \setminus \{0\} &\xrightarrow{\pi} \mathbb{A}^1_x \setminus \{0\}, \\ z &\longmapsto z^2. \end{aligned} \tag{1.1.2}$$

Let $\mathcal{L}$ be a local system of 1d vector spaces on $\mathbb{A}^1_z \setminus \{0\}$, with monodromy $m \in \mathbb{C}^*$ around the origin. The pushforward $\pi_*\mathcal{L}$ is a local system of 2d vector spaces on $\mathbb{A}^1_x \setminus \{0\}$, with fiber at $x \in \mathbb{A}^1_x \setminus \{0\}$ given by:

$$(\pi_*\mathcal{L})_x = \mathcal{L}_{\sqrt{x}} \oplus \mathcal{L}_{-\sqrt{x}}. \tag{1.1.3}$$

As we travel along a loop around $x = 0$, the two sheets of the covering map π are exchanged, so the monodromy of $\pi_*\mathcal{L}$ around this loop can be represented as a matrix:

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \tag{1.1.4}$$

where $ab = m$. This matrix representation is only well-defined up to the action by conjugation of $N_{GL(2)}$, the normalizer of the maximal torus of $GL(2)$. Indeed, $\pi_*\mathcal{L}$ has a natural action of $GL(2)$, and $N_{GL(2)} \subset GL(2)$ preserves the local decomposition of $\pi_*\mathcal{L}$ into line sub-bundles, as in equation 1.1.3. Each factor of $T_{GL(2)} \cong (\mathbb{C}^*)^2$ acts by scaling on one of the sub-bundles, and elements in the non-trivial $T_{GL(2)}$-coset of $N_{GL(2)}$ also exchange the two sub-bundles. The $N_{GL(2)}$-conjugacy class of M contains a unique representative of the form:

$$M' = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}. \tag{1.1.5}$$

The idea of non-abelianization is to modify the monodromy of $\pi_*\mathcal{L}$ around $x = 0$, in the hope of obtaining a local system with trivial monodromy around $x = 0$, which

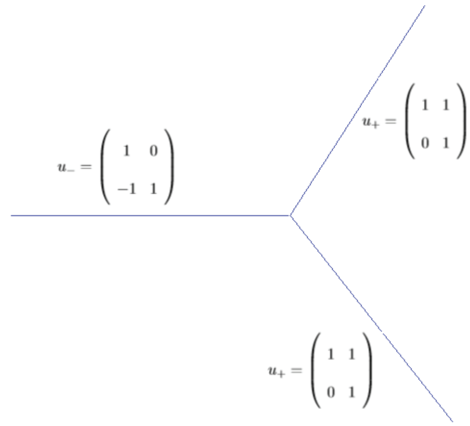

Figure 1.1: Trivalent graph with a unipotent automorphism associated to each edge.

would therefore extend to a local system on $\mathbb{A}^1_x$. The modification is done by cutting and re-gluing the local system along the edges of the trivalent graph $\mathcal{W}$ pictured in Figure 1.1. Concretely:

- consider the restriction $\pi_*\mathcal{L}|_{\mathbb{A}^1_x \setminus \mathcal{W}}$;
- re-glue this restriction across the edges of $\mathcal{W}$, using an unipotent automorphism called Stokes factor associated to each edge, as in Figure 1.1.

The resulting local system has monodromy around the origin:

$$\begin{aligned} M' u_+ u_- u_+ &= \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix}. \end{aligned} \tag{1.1.6}$$

In the particular case $m = -1$, we obtain a local system which extends to all of $\mathbb{A}^1_x$; we call this the non-abelianization $\mathrm{nonab}(\mathcal{L})$ of the original $\mathcal{L}$. The terminology is motivated by the passage from the abelian structure group $GL(1, \mathbb{C})$ to the non-abelian structure group $GL(2, \mathbb{C})$. The situation is summarized in the following diagram.

$$\begin{tikzcd}
\mathrm{Loc}_{\mathrm{Vect}_{1d}}(\mathbb{A}^1_z \setminus \{0\}) \arrow[r, "\pi_*"] & \mathrm{Loc}_{\mathrm{Vect}_{2d}}(\mathbb{A}^1_x \setminus \{0\}) \arrow[r, "\mathrm{reglue}"] & \mathrm{Loc}_{\mathrm{Vect}_{2d}}(\mathbb{A}^1_x \setminus \{0\}) \\
\mathrm{Loc}^{m=-1}_{\mathrm{Vect}_{1d}}(\mathbb{A}^1_z \setminus \{0\}) \arrow[u, hook] \arrow[rr, dashed, "\mathrm{nonab}"] & & \mathrm{Loc}_{\mathrm{Vect}_{2d}}(\mathbb{A}^1_x) \arrow[u, hook]
\end{tikzcd} \tag{1.1.7}$$

Remark 1.1.2. Equation 1.1.6 above is a calculation performed with matrices, but a more rigorous approach would involve working with $N_{GL(2)}$-conjugacy classes. In section 5.3.1, we will show how to map the monodromy of N-local systems to Stokes factors in an N-equivariant way, which gives a morphism between conjugacy classes modulo the adjoint action of N.

Remark 1.1.3. It may seem pointless to study rank 2 vector bundles on $\mathbb{A}^1$, as we did in Example 1.1.1, because all of them are equivalent to the trivial bundle. The point is that this example is a local model for computations we will do later, using Riemann surfaces X with more complicated topology.

We generalize the calculation done in Example 1.1.1 in two ways:

- rather than local systems on the affine line, we work with local systems on X°_D}, where X is a compact Riemann surface, D is a nonzero, reduced, effective divisor on X, and X°_D} is the oriented real blowup of X at D;
- rather than local systems of 2-dimensional vector spaces, we work with local systems of principal G-bundles, for any reductive algebraic group G.

We need appropriate generalizations of the tools used in Example 1.1.1:

- The double cover is generalized to a cameral cover $\pi : \tilde{X} \to X$ (Definition 2.3.5).
- The restriction of π away from 0 and ∞ is generalized by replacing π with the map induced on the oriented real blowup at the branch divisor P, the ramification divisor R, and the divisor at infinity D. The result is an unbranched covering $\pi^\circ : \tilde{X}^{\circ_{D+R}} \to X^{\circ_{D+P}}$;
- the trivalent graph $\mathcal{W}$ is generalized to a spectral network (Definition 4.1.10, 4.3.2);
- the condition $m = -1$ from diagram 1.1.7 is generalized to the S-monodromy condition (Section 5.2).

Let T denote a fixed choice of maximal torus of G, N the normalizer of T in G, and W the Weyl group. Pending precise definitions of the moduli spaces being used, we state the main theorems of this work. The first theorem relates certain T-local systems on $\tilde{X}^{\circ_{D+R}}$ with certain N-local systems on X°_{D+P}. It is a straightforward adaptation of (a special case of) the work of Donagi and Gaitsgory in [12].

Theorem 1.1.4. *There is an isomorphism of algebraic stacks:*

$$\mathrm{Loc}_T^{N,S}(\tilde{X}^{\circ_{D+R}}) \xrightarrow{\cong} \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}}). \tag{1.1.8}$$

The second theorem is a generalization of the re-gluing construction from Example 1.1.1.

Theorem 1.1.5. *The data of a spectral network on X provides a morphism of algebraic stacks:*

$$\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}}) \xrightarrow{\text{nonab}} \mathrm{Loc}_G(X^{\circ_D}). \tag{1.1.9}$$

Note that, in the codomain of the non-abelianization map, X is punctured only at the divisor at infinity D. The non-abelianized local systems extend to the branch divisor P of the covering map.

Remark 1.1.6. The composition of the maps in the two theorems provides a map:

$$\mathrm{Loc}_T^{N,S}(\tilde{X}^{\circ_{D+R}}) \longrightarrow \mathrm{Loc}_G(X^{\circ_D}). \tag{1.1.10}$$

The left-hand side is a moduli space of abelian objects, while the right hand-side is a moduli space of non-abelian objects. This justifies the terminology "non-abelianization". Moreover, let us choose a basepoint z and generators for the fundamental group:

$$\pi_1(\tilde{X}^{\circ_{D+R}}, z) \cong \left\langle a_1, b_1, \dots, a_{2g}, b_{2g}, c_1, \dots, c_d \,\middle|\, \prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1}) \prod_{j=1}^{d} c_j \right\rangle, \tag{1.1.11}$$

where g is the genus of $\tilde{X}$ and d the degree of the divisor $D+R$. Then sending each local system to its monodromy around a_i, b_i, c_j gives an isomorphism:

$$\mathrm{Loc}_T(\tilde{X}) \cong T^{2g+d-1}/T, \tag{1.1.12}$$

where the quotient is by the (trivial) diagonal action of T by conjugation.

Then the map in equation 1.1.10 relates a modified version (to account for N-shifting and the S-monodromy condition) of the explicit stack T^{2g+d-1}/T from 1.1.12 to the *a priori* complicated $\mathrm{Loc}_G(X^{\circ_D})$.

As further motivation for the study of non-abelianization, we mention a few conjectures related to the map in equation 1.1.10.

Conjecture 1.1.7. *The map* nonab *from Theorem 1.1.5, and consequently the composition in equation 1.1.10, is a local isomorphism.*

The paper [16] presents evidence for this conjecture in the case $G = GL(n, \mathbb{C})$, at the physical level of rigor. Their strategy is to show that:

1. the two moduli spaces have the same dimension;

2. the map nonab preserves a symplectic form that both moduli spaces are naturally equipped with; consequently, the maps induced by nonab on tangent spaces must be injective.

We attempted to follow this strategy in our setting, using shifted symplectic structures in derived geometry, but were not yet successful in carrying out the second step.

Conjecture 1.1.7 is known to be true for $G = SL(2, \mathbb{C})$, at the level of coarse moduli spaces, due to work such as [15], [17], [21], [14]. Whenever the conjecture holds, nonab can be seen as giving an étale coordinate chart on $\mathrm{Loc}_G(X^{\circ_D})$.

The next conjectures are about the relation between coordinate charts obtained from different non-abelianization maps. We will show in chapter 4 how to associate a graph $\mathcal{W}_b$ on X to each point in a dense open subset of the Hitchin base $\mathcal{B}(X, G, K_X(D))$ (Definition 2.2.5).

Conjecture 1.1.8. *There is a dense open susbet $U \subset \mathcal{B}(X, G, K_X(D))$, such that for each $b \in U$, the graph $\mathcal{W}_b$ is a spectral network, hence gives rise to a non-abelianization map.*

We give some evidence for this conjecture in section 4.2. In fact, the non-abelianization map only depends on the topology of $\mathcal{W}$, which is locally constant as b varies in the subset U.

Conjecture 1.1.9. *Upon traversing appropriate real codimension 1 loci in the Hitchin base, the coordinate charts on the coarse moduli space corresponding to* $\mathrm{Loc}_G(X^{\circ_D})$*, induced by the non-abelianization maps, undergo a cluster mutation.*

Conjecture 1.1.9 is proved in the case of $G = SL(2, \mathbb{C})$. In this case, the spectral networks are related to ideal triangulations of X. Crossing codimension 1 loci in $\mathcal{B}(X, SL(2, \mathbb{C}), K_X(D))$ then corresponds to "flips" and "pops" of these triangulations, which corresponds to cluster transformations of the coordinate charts. Different portions of this story are worked out in [17], [21], [8]; also in [24], [25] from the point of view of exact WKB analysis.

Moreover, the work of Fock and Goncharov in [15] provides étale coordinate charts for framed moduli spaces of G-local systems, for G a semisimple group with trivial center, using configurations of flags on ideal triangulations. Their coordinates agree with the ones coming from spectral networks under special circumstances (e.g. the "minimal spectral networks" of [18]). For more general spectral networks, we don't expect the coordinate charts to be of Fock-Goncharov type.

1.2 Quadratic differentials and spectral networks

Spectral networks are certain directed graphs on X, whose edges are labeled by extra data. They generalize the trivalent graph which was used in Example 1.1.1 to cut and re-glue local systems.

The easiest examples of spectral networks are in the case $G = SL(2, \mathbb{C})$, where, up to orientation and labels, they coincide with the critical trajectories of quadratic differentials.

(See [35] for the definitive classical text on quadratic differentials, or [8], [24] for modern points of view.)

Definition 1.2.1. Let X be a compact Riemann surface, and D an effective divisor. A **meromorphic quadratic differential** on X is a section ω of $(K_X(D))^{\otimes 2}$.

Definition 1.2.2. Let $\mathrm{Crit}(\omega)$ denote the critical points (zeros and poles) of ω. Then ω determines a real projective vector field V_ω on $X \setminus \mathrm{Crit}(\omega)$, which is defined by $\pm\sqrt{\omega}(V_\omega) \in \mathbb{R}$. The choice of square root of ω in this condition does not matter. The integral curves $\gamma : \mathbb{R} \to X \setminus \mathrm{Crit}(\omega)$, of this vector field satisfy:

$$\int_{t_0}^{t_1} \pm\sqrt{\omega(\gamma(s))} \in \mathbb{R} \tag{1.2.1}$$

for all $t_0 < t_1 \in \mathbb{R}$. The **trajectories of the quadratic differential** ω are the maximal leaves of this foliation.

Remark 1.2.3. Strictly speaking, the trajectories that we use in this paper are horizontal trajectories. For each $\theta \in [0, \pi)$, we could also consider trajectories with angle θ, defined by:

$$\int_{t_0}^{t_1} \pm\sqrt{\omega(\gamma(s))} \in e^{i\theta}\mathbb{R}. \tag{1.2.2}$$

Since we are only interested in the case $\theta = 0$, we omit the word "horizontal" without fear of confusion.

Example 1.2.4. Let $X = \mathbb{P}^1$, $D = \{3 \cdot \infty\}$, and $\omega(x) = x dx \otimes dx$. Then equation 1.2.1 becomes:

$$\int_{x_0}^{x_1} \pm\sqrt{x} dx = \pm\frac{2}{3}\left(x_1^{3/2} - x_0^{3/2}\right) \in \mathbb{R}. \tag{1.2.3}$$

Assume, first, that $x_0 \neq 0$. Then there exists a neighborhood $U \ni x_0$ such that the restriction to U of the map $x_1 \mapsto x_1^{3/2}$ is injective, for either choice of branch of the square root function. This means that U contains a unique trajectory of ω passing through x_0; this trajectory can be parametrized by:

$$x_1(t) = \left(x_0^{3/2} + t\right)^{2/3},\ t \in \mathbb{R}. \tag{1.2.4}$$

On the other hand, if $x_0 = 0$, any neighborhood $U \ni 0$ contains three trajectories which start at 0. These can be parametrized by:

$$x_{1,k}(t) = t\, e^{2\pi i k/3},\ t \in \mathbb{R}_+,\ k \in \{0, 1, 2\}. \tag{1.2.5}$$

Therefore, in a neighborhood of $x = 0$, the trajectories of ω are as shown in Figure 1.2.

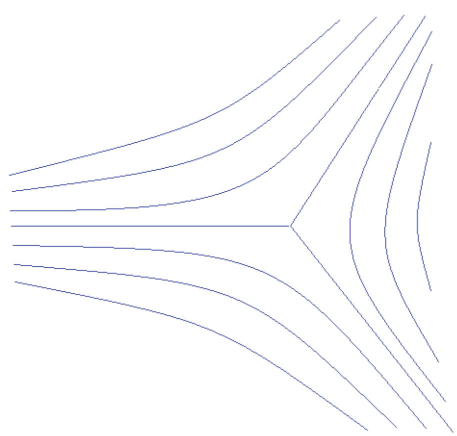

Figure 1.2: Trajectories in a neighborhood of $x = 0$.

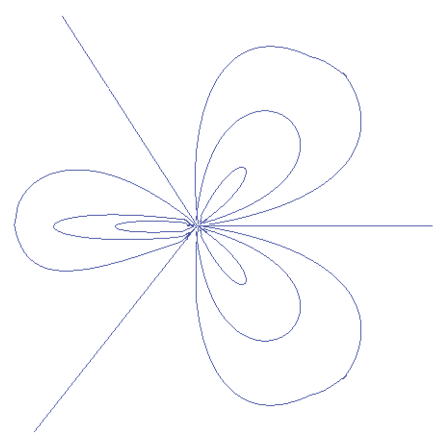

Figure 1.3: Trajectories in a neighborhood of $x = \infty$.

Finally, in a neighborhood of $x = \infty$, and using the local coordinate $y = x^{-1}$, the quadratic differential has a pole of order five:

$$\omega(y) = y^{-1} d(y^{-1}) \otimes d(y^{-1}) = y^{-5} dy \otimes dy. \tag{1.2.6}$$

The trajectory structure around $x = \infty$ is shown in Figure 1.3. However, in Chapters 4 and 5 we will work with quadratic differentials which only have poles of order two. The trajectory structure around poles of order two is described in Lemma 4.1.13.

Remark 1.2.5. Two trajectories of a given quadratic differential ω can only intersect at a zero or pole of ω. Indeed, equation 1.2.1 can be interpreted, up to scaling by $\mathbb{R}_+$, as a first order ODE for the integral curve γ. The Picard-Lindelöf theorem guarantees the existence and uniqueness of γ passing through a regular point of ω, up to reparametrization.

Of particular importance are the **critical trajectories**, which are, by definition, those starting from zeros of the quadratic differential.[1] The trivalent graph which was used for cutting and re-gluing local systems in Example 1.1.1 consists of the critical trajectories of the quadratic differential from Example 1.2.4. It turns out that the critical trajectories from Example 1.2.4 provide a local model for the critical trajectories of all quadratic differentials, around zeros whose multiplicity is 1. For example, in Figures 1.4 - 1.6, we use $X = \mathbb{P}^1$, $D = \{3 \cdot \infty\}$, and the quadratic differential is $\omega_-(x) = e^{-i\pi/5}(1 - x^2)dx^{\otimes 2}$, $\omega_0(x) = (1 - x^2)dx^{\otimes 2}$, $\omega_+(x) = e^{+i\pi/5}(1 - x^2)dx^{\otimes 2}$, respectively. See Figure 3 in [24] for more examples of critical trajectories.

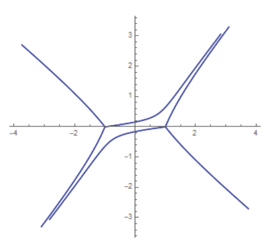

Figure 1.4: ω_-

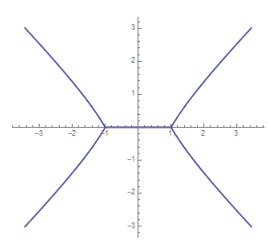

Figure 1.5: ω_0

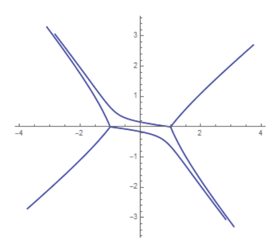

Figure 1.6: ω_+

[1]The set of critical trajectories is called "Stokes graph" in the literature on exact WKB analysis. We will stick to the terminology of [16] and call it a spectral network.

Remark 1.2.6. For reasons that will be clear later, we are interested in endowing trajectories with an orientation. Equation 1.2.1 doesn't seem to allow this, because any choice of square root of ω changes branch as we travel around a zero of ω, and the sign of the integral changes as a result.

To address this, let $\pi : \tilde{X} \to X$ be a double cover branched at the simple zeros of ω. This determines a global section $\sqrt{\omega} \in \Gamma\big(\tilde{X}, K_{\tilde{X}}(\pi^* D)\big)$, such that $\sqrt{\omega} \otimes \sqrt{\omega} = \pi^*(\omega)$.[2] Then we can modify the RHS of equation 1.2.1 to $\mathbb{R}_+$ instead of $\mathbb{R}$, and obtain a foliation of $\tilde{X}$ by unparametrized, oriented curves $\gamma : \mathbb{R} \to \tilde{X}$ satisfying:

$$\int_{t_0}^{t_1} \sqrt{\omega(\gamma(s))} \in \mathbb{R}_+. \tag{1.2.7}$$

Example 1.2.7. In the situation of Example 1.2.4, where $\omega = x dx \otimes dx$, the covering map $\pi : \tilde{X} \to X$ can be written in coordinates as $\pi(z) = z^2$. Then:

$$\pi^*(\omega)(z) = 4z^4 dz \otimes dz, \tag{1.2.8}$$

and $\sqrt{\omega}$ is:

$$\sqrt{\omega}(z) = 2z^2 dz. \tag{1.2.9}$$

Then, using the same reasoning as in Example 1.2.4, the oriented foliation determined by equation 1.2.7 is, in a neighborhood of 0, as in Figure 1.7.

Using the lessons learned in the previous discussion, we give an informal introduction to spectral networks. Precise definitions can be found in Chapter 4.

Previous work on spectral networks, such as the papers [16], [27], starts from a spectral curve (Definition 2.3.1), associated to a generic point $b \in \mathcal{B}(X, G, K_X(D))$ in the Hitchin

[2] In fact, there exist two such sections, corresponding to the two choices of square root. Each of them is globally defined on $\tilde{X}$.

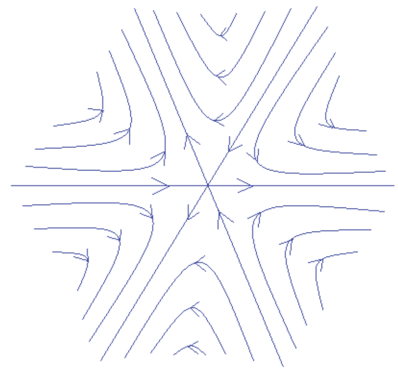

Figure 1.7: The oriented foliation determined by the quadratic differential $\omega(x) = xdx \otimes dx$ on the branched cover; compare to Figure 1.2.

base (Definition 2.2.5). We take a different approach and use a cameral curve (Definition 2.3.5) associated to b; this is a branched cover $\pi : \tilde{X}_b \to X$ equipped with a fiberwise action of the Weyl group W, which is free and transitive away from the ramification points. For every root α of $\mathfrak{g}$, denote by $s_\alpha \in W$ the reflection about the root hyperplane H_α.

Proposition 1.2.8 (See Proposition 4.1.3 for more precise version and proof)**.** *Let $b \in \mathcal{B}^{\diamondsuit}_R(X, G, K_X(D))$ and $\pi : \tilde{X}_b \to X$ the associated cameral curve, which is smooth (Proposition 2.3.6). For every root α of the Lie algebra $\mathfrak{g}$, we factor the projection π as:*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi_\alpha"'] \arrow[rr, bend left, "\pi"] & \tilde{X}/\langle s_\alpha \rangle \arrow[r, "p_\alpha"'] & X. \end{tikzcd} \tag{1.2.10}$$

Then b determines a quadratic differential $\omega_{b,\alpha}$ on $\tilde{X}/\langle s_\alpha \rangle$.

Therefore, for every root α, we obtain a foliation on $\tilde{X}/\langle s_\alpha \rangle$, by trajectories of $\omega_{b,\alpha}$. Pulling back the trajectories via π_α, we obtain an oriented foliation on $\tilde{X}$. A cameral network arises from the interplay of these oriented foliations as α varies.

Definition 1.2.9 (Important details ignored for now; see Definition 4.1.6)**.** The **WKB construction** $\tilde{\mathcal{W}}_b$ associated to $b \in \mathcal{B}_R^{\diamondsuit}(X, G, K_X(D))$ consists of Stokes curves, which are oriented curve segments on $\tilde{X}$, each labeled by a root of $\mathfrak{g}$, produced by the following algorithm.

- A primary Stokes curve is a critical oriented trajectory of one of the $\omega_{b,\alpha}$; its label is α.
- As mentioned in Remark 1.2.5, two distinct primary Stokes curves with the same label never intersect away from critical points. However, intersections between Stokes curves labeled by $\alpha \neq \pm\beta$ at some $x \in \tilde{X}$ do occur. In this case, for each $\gamma \in \Phi$ which is a linear combination of α, β with positive, integral coefficients, a secondary Stokes curve ℓ_γ starts at x; it is the unique leaf outgoing from x of the oriented foliation determined by $\omega_{b,\gamma}$. See Figure 4.1 for a local model of the intersection.
- Secondary Stokes curves are recursively created every time two or more of the existing Stokes curves intersect.

If $\tilde{\mathcal{W}}_b$ satisfies some acyclicity and finiteness conditions (Definitions 4.1.6, 4.1.10), we call it a **WKB cameral network**.

The Stokes curves are equivariant with respect to the W action on the covering $\pi : \tilde{X} \to X$, so they descend to a set of oriented curves on X. We call the resulting oriented graph on X the **spectral network**.

Remark 1.2.10. Our non-abelianization construction in Chapter 5 uses a recursive definition of Stokes factors associated to curves in the spectral network. For this recursive definition to make sense, our spectral networks are more restricted than those of Gaiotto, Moore and Neitzke. The restrictions we impose forbid, among other things, double walls such as the finite trajectory in Figure 1.5, or finite webs such as those in Figure 31 of [16]. In the particular case of $G = SL(2, \mathbb{C})$, a spectral network is the set of critical trajectories of a quadratic differential, together with some discrete data; our restrictions correspond to saddle-free differentials (Definition 4.2.5).

In Section 4.2, we prove some partial results in the direction of conjecture 1.1.8, which claims that our restrictions are satisfied for a subset of the Hitchin base which is open and dense in the classical topology. Our results, which rely on the relationship between quadratic differentials and WKB cameral networks, are:

- There is a dense, open subset of the Hitchin base, for which the WKB construction is saddle-free, i.e. lacks certain types of double walls. (Proposition 4.2.9)
- A saddle-free WKB construction has no dense Stokes curves, i.e. each Stokes curves ends at some $d \in D$. (Proposition 4.2.11.)
- Under the assumption that joints of the network accumulate only at points of D, the restriction of the WKB construction away from contractible neighborhoods of each $d \in D$ consists of finitely many Stokes curves. (Proposition 4.2.13.)

Remark 1.2.11. Apart from non-abelianization, physicists use WKB networks to understand the BPS spectrum of $N = 2$, $d = 4$ field theories of class S; see Section 3 of [16] and references therein. In particular, "finite webs" in WKB networks should correspond to BPS states in these theories. In the case of $G = SL(2, \mathbb{C})$, Theorem 1.4 in the paper [8] describes this more mathematically as a correspondence between finite-length trajectories of quadratic differentials and stable objects in a category of quiver representations.

1.3 Outline of the thesis

Chapter 2 provides background on local systems and Higgs bundles. The latter are strictly speaking not necessary for understanding non-abelianization – but they are important motivationally, and they offer a good setting to introduce cameral covers (Definition 2.3.5), which are essential for the rest of the paper.

Chapter 3 is devoted to statements about Lie groups that are useful for non-abelianization. We summarize some results about simple Lie groups and $\mathfrak{sl}_2$-triples. Then we address the construction of Stokes factors, which are generalizations of the unipotent automorphisms used for re-gluing in Example 1.1.1. For each Stokes curve labeled by a root α, the Stokes factor is an element of the 1-parameter subgroup $\exp(\mathfrak{g}_\alpha) \subset G$. In section 3.1 we state some technical lemmas which will evantually allow us to map the monodromy of a local system to Stokes factors of primary curves, in an equivariant fahsion. In section 3.2 we define 2d scattering diagrams (Definition 3.2.11), which are local models for cameral networks around intersections of Stokes curves. We use this framework to construct a map from Stokes factors of incoming curves to Stokes factors of outgoing curves (Theorem 3.2.21).

In chapter 4 we define cameral and spectral networks and discuss their properties. In section 4.1, we introduce the WKB construction (Definition 4.1.6), which draws a graph on the cameral cover $\tilde{X}_b$, using the data of a point in the Hitchin base $\mathcal{B}(X, G, \mathcal{L})$. We call the resulting graph a WKB cameral network (Definition 4.1.10) if it satisfies some acyclicity and finiteness conditions. In section 4.2 we conjecture (Conjecture 4.2.10) that these conditions are satisfied for a locus of $\mathcal{B}(X, G, \mathcal{L})$ which is dense and open in the classical topology. We provide some evidence in Proposition 4.2.9 and Proposition 4.2.13. Section 4.3 deals with the passage from cameral networks, which are objects on the cameral cover $\tilde{X}_b$, to spectral networks, which are objects on X.

With all the preliminary work in place, in Chapter 5 we state and prove the main results. Section 5.1 follows in the footsetps of Donagi and Gaitsgory, who gave in [12] a correspondence between certain T-bundles on the cameral cover and certain N-bundles on the base curve. We recall their definitions, and make the observation that their correspondence[3] goes through in the case of local systems. The result is a proof of Theorem 1.1.4. Section 5.3 then gives a construction of the non-abelianization map, hence a proof of Theorem 1.1.5. Leveraging the results of the previous chapters, this proof is a reasonably straightforward generalization of diagram 1.1.7 from Example 1.1.1. The main complication are the secondary Stokes curves, whose presence requires a recursive construction (Construction 5.3.6). Due to finiteness results for WKB cameral networks (Proposition 4.2.13), the recursion finishes after finitely many steps.

1.4 Conventions and notation

Lie theory: G is a reductive algebraic group over $\mathbb{C}$. We fix a maximal torus T, and let N denote its normalizer in G. $W \cong N/T$ is the Weyl group. The Lie algebras of G and T are $\mathfrak{g}$, $\mathfrak{t}$, respectively. The lower-case greek letters $\alpha, \beta, \gamma, \delta$ denote roots of $\mathfrak{g}$, and Φ the set of all roots. $\mathfrak{u}$ and U denote a nilpotent Lie subalgebra of $\mathfrak{g}$, and a unipotent Lie subgroup of G, respectively.

Geometry: X is a compact, closed Riemann surface, and D a reduced, effective,

[3] Specifically, we only need the unramified case of their corresponence.

non-zero divisor. $b \in \mathcal{B}(X, G, \mathcal{L})$ denotes a point in the Hitchin base for the group G and a line bundle $\mathcal{L}$; in Chapters 4 and 5, we are only interested in the case of meromorphic differential forms, $\mathcal{L} \cong K_X(D)$. $\pi : \tilde{X}_b \to X$ denotes the associated cameral cover. We denote by $P \subset X$, $R \subset \tilde{X}_b$ the branch and ramification divisor of π, respectively.

For E a reduced, effective divisor on X, we denote by X°_E} the oriented real blowup of X at every point in the support of E. $\tilde{X}^{\circ_E}$ is defined analogously.

Whenever we speak of trajectories of a quadratic differential, they are horizontal trajectories.

Chapter 2

Geometric background

2.1 Local systems

Definition 2.1.1. Let X be a Riemann surface, and G a reductive algebraic group over $\mathbb{C}$. A **G-local system** $\mathcal{E}$ on X is a locally constant sheaf of sets on X, together with a free, transitive, right G action on the stalk $\mathcal{E}_x$, for every $x \in X$. A **morphism of** G**-local systems** is a morphism of sheaves, equivariant with respect to the G-action.

Remark 2.1.2. Let $U \subset X$ be contractible, and let $x \in U$. Then, due to the locally constant requirement, the natural map $\Gamma(U, \mathcal{E}) \to \mathcal{E}_x$ is an isomorphism. The G-action on stalks therefore induces G-actions on the space of sections over every contractible open set.

Proposition 2.1.3. *Fix an effective, reduced (possibly zero) divisor D on X, and a basepoint $x \in X \setminus D$. Then there are equivalences of groupoids*[1] *between:*

[1] I.e. categories whose only morphisms are equivalences.

1. *for any fixed $x \in X$, homomorphisms $\pi_1(X \setminus D, x) \to G$, modulo the adjoint action of G;*

2. *G-local systems on $X \setminus D$;*

3. *principal G-bundles with flat connection on $X \setminus D$, which have tame singularities, in the sense that the connection has at most poles of order 1 at the punctures of X.*

Proof. These equivalences are well-known, so we only give a sketch of the argument.

To get from 1 to 2, take the constant local system on the universal cover of $X \setminus D$, and quotient by the action of $\pi_1(X \setminus D, x)$. This action is by deck transformations on the universal cover, and by the right action of the image of $\pi_1(X \setminus D, x)$ in G on the sections.

To get from 2 to 3, note that there exists a unique principal G-bundle on $X \setminus D$ up to isomorphism, whose transition functions, seen as elements of G, are the same as those of the local system. There is then a unique flat connection whose flat sections are the sections of the local system.

To get from 3 to 1, send each homotopy class of loops in $X \setminus D$ to the monodromy of the connection around the loop; this is well-defined up to the adjoint action of G. □

Remark 2.1.4. Perspective 1 from Proposition 2.1.3 makes it clear that the categories in question only depend on the topology of X, and not on a smooth, complex or algebraic structure.

Remark 2.1.5. The correspondence between (2) and (3) in Proposition 2.1.3 can be generalized to flat connections with poles of order higher than 1. The local system side of the correspondence then requires some extra data around the singularities; see [5]. We will not be concerned with this generalization in the present work.

For each of the three categories of Proposition 2.1.3 one can define a moduli stack of objects. (Or, by imposing an appropriate stability condition, a moduli space representable by a scheme.) The simplest construction is for homomorphisms from the fundamental group. Choose a basepoint x and generators for the fundamental group:

$$\pi_1(X,x) = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \middle| \prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) \prod_{j=1}^{r} c_j = \mathrm{id} \right\rangle. \tag{2.1.1}$$

Definition 2.1.6. The **character variety**, or **rigidified character stack**, is the subvariety $\mathrm{Char}^{\mathrm{rig}}(X,G) \subset G^{2g+r}$ cut out by the equation:

$$\prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) \prod_{j=1}^{r} c_j = \mathrm{id}. \tag{2.1.2}$$

The **character stack** is the quotient of the character variety by the action of G, induced from the diagonal action by conjugation on G^{2g+r}:

$$\mathrm{Char}(X,G) := [\mathrm{Char}^{\mathrm{rig}}(X,G)/G]. \tag{2.1.3}$$

Proposition 2.1.7. *There is an isomorphism of stacks:*

$$\mathrm{Loc}_G(X) \cong \mathit{Char}(X,G). \tag{2.1.4}$$

2.2 Higgs bundles

This section gives an introduction to the Hitchin moduli space and the Hitchin integrable systsem; these were introduced by Hitchin in [19] for the case of rank 2 vector bundles, and in [20] for the classical Lie groups. We state all definitions and results in the more general setting of principal G bundles.

Definition 2.2.1. For a principal G-bundle $\mathcal{E}$, the **adjoint bundle** is the vector bundle:

$$\mathrm{ad}(\mathcal{E}) := \mathcal{E} \times_G \mathfrak{g}. \tag{2.2.1}$$

Recall that the twisted product $\mathcal{E} \times_G \mathfrak{g}$ is the quotient of the product $\mathcal{E} \times \mathfrak{g}$ by the equivalence relation $(e \cdot g, x) \sim \big(e, \mathrm{ad}_g(x)\big)$, for all sections e of $\mathcal{E}$, $g \in G$, $x \in \mathfrak{g}$.

Definition 2.2.2. Let X be a compact Riemann surface, $\mathcal{L}$ a line bundle on X, and G a reductive algebraic group. (For applications in the subsequent chapters, $\mathcal{L}$ will be a bundle of meromorphic 1-forms with prescribed pole divisor.) A **G-Higgs bundle** on X with values in $\mathcal{L}$ is pair $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a principal G-bundle and φ is a section:

$$\varphi \in \Gamma\big(X, \mathrm{ad}(\mathcal{E}) \otimes \mathcal{L}\big). \tag{2.2.2}$$

We call φ a **Higgs field**.

Definition 2.2.3. The **Hitchin moduli space** is the moduli stack of G-Higgs bundles on X, twisted by $\mathcal{L}$. Formally:

$$\mathcal{M}_H(X, G, \mathcal{L}) := \mathrm{Map}_{St/X}\left(X, [\mathfrak{g}_{\mathcal{L}}/G]\right), \tag{2.2.3}$$

where $\mathfrak{g}_{\mathcal{L}} := \mathfrak{g} \times_{\mathbb{C}^*} \mathcal{L}$, the mapping stack is taken in the category of stacks over X, the square brackets denote a stack quotient, and this quotient is by the adjoint action of G on $\mathfrak{g}$.

Remark 2.2.4. We elaborate a bit on the formal definition 2.2.3. Consider the particular case of semisimple G, to avoid the posibility of infinite stabilizers. Post-composition with the map $\mathfrak{g}_{\mathcal{L}}/G \to BG$ gives a morphism:

$$\mathcal{M}_H(X, G, \mathcal{L}) \longrightarrow \mathrm{Map}(X, BG) \cong \mathrm{Bun}_G(X). \tag{2.2.4}$$

This maps a Higgs bundle to the underlying principal G-bundle. The fiber of this map over a point $\mathcal{E} \in \mathrm{Bun}_G$ is an element of $\Gamma(X, \mathrm{ad}(\mathcal{E}) \otimes \mathcal{L})$, i.e. a Higgs field. In the particular case $\mathcal{L} = K_X$, Serre duality provides a natural identification:

$$\Gamma\big(X, \mathrm{ad}(\mathcal{E}) \otimes K_X\big) \cong \Big(H^1\big(X, \mathrm{ad}(\mathcal{E})\big)\Big)^\vee = T^*_{\mathcal{E}} \mathrm{Bun}_G(X). \tag{2.2.5}$$

This means that $\mathcal{M}_H(X, G, K_X) \cong T^* \mathrm{Bun}_G(X)$.

Consider now the natural map from the stacky quotient $[\mathfrak{g}/G]$ to the categorical quotient $\mathfrak{g}/G := \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$, where the superscript denotes G-invariant polynomials. Due to the Chevalley restriction theorem, $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W$. This gives a map $[\mathfrak{g}/G] \to \mathfrak{t}/W$.

Definition 2.2.5. The **Hitchin map** is the map induced by post-composition with $[\mathfrak{g}/G] \to \mathfrak{t}/W$:

$$\mathcal{M}_H(X, G, \mathcal{L}) = \mathrm{Map}_{St/X}\left(X, [\mathfrak{g}_{\mathcal{L}}/G]\right) \xrightarrow{\text{Hitch}} \Gamma(X, \mathfrak{t}_{\mathcal{L}}/W). \tag{2.2.6}$$

We denote the right-hand side by $\mathcal{B}(X, G, \mathcal{L})$ and call it the **Hitchin base**.

Example 2.2.6. Consider the case $G = SL(2, \mathbb{C})$ and $\mathcal{L} = K_X(D)$, for an effective divisor D. Since $W \cong \mathbb{Z}_2$, any identification $\mathfrak{t} \cong \mathbb{C}$ implies that:

$$\mathcal{B}\big(X, SL(2, \mathbb{C}), K_X(D)\big) \cong \Gamma\big(X, K_X(D)/\mathbb{Z}_2\big) \cong \Gamma\big(X, (K_X(D))^{\otimes 2}\big). \tag{2.2.7}$$

In other words, the Hitchin base is the space of meromorphic quadratic differentials with divisor of poles D.

Example 2.2.7. In the case $G = GL(n, \mathbb{C})$, we have $W = S_n$, and $\mathbb{C}[\mathfrak{t}]^W$ is freely generated by the elementary symmetric polynomials of degrees $1 \leq d \leq n$. This identifies the Hitchin base:

$$\mathcal{B}(X, G, \mathcal{L}) \cong \bigoplus_{d=1}^{n} \Gamma(X, \mathcal{L}^{\otimes d}). \tag{2.2.8}$$

A Higgs field is a section $\varphi \in \mathrm{ad}(\mathcal{E}) \otimes \mathcal{L}$; for $x \in X$, $\varphi(x) \in \mathfrak{gl}_n \otimes \mathcal{L}_x$. The Hitchin map then sends a Higgs bundle to the coefficients of the characteristic polynomial of φ:

$$(\mathcal{E}, \varphi) \longmapsto \left(\operatorname{Tr}(\varphi^{\wedge d})\right)_{d=1}^{n}. \tag{2.2.9}$$

Theorem 2.2.8 ([19], [20], [13]). *In the case $\mathcal{L} = K_X$, the Hitchin map has the structure of an algebraically completely integrable system.*

Remark 2.2.9. The meaning of "algebraically completely integrable system" is that the generic fibers of the Hitchin map are abelian varieties, which are Lagrangian with respect to the natural symplectic structure on $\mathcal{M}_H(X, G, K_X) \cong T^* \operatorname{Bun}_G(X)$. The take-away is that the *a priori* complicated structure of the moduli stack $\mathcal{M}_H(X, G, K_X)$ can be understood in terms of:

- The Hitchin base $\mathcal{B}(X, G, \mathcal{L})$, which is an affine space. This a consequence of the Chevalley-Shephard-Todd theorem, which states that the ring of invariant polynomials $\mathbb{C}[\mathfrak{t}]^W$ is free over $\mathbb{C}$.

- The Hitchin fibers, which for generic $b \in \mathcal{B}(X, G, \mathcal{L})$ are abelian varieties, in fact isomorphic to moduli spaces of T-bundles on a branched cover of X, with some extra data and conditions (see Theorem 2.3.9). The passage from the Hitchin fibers to these abelian moduli spaces is called "abelianization".

More generally, if the total space of $\mathcal{L}$ has a Poisson structure (e.g. if $\mathcal{L} = K_X(D)$, the case of interest in our work), then $\mathcal{M}_H(X, G, \mathcal{L})$ has a Poisson structure, and the generic fibers of the Hitchin map are still abelian variaties, which are Lagrangian with respect to this Poisson structure (i.e. each fiber is Lagrangian inside some symplectic leaf). The discussion in Section 2.3 below will make this claim precise, and give the strategy of the proof. Before this, for the sake of completeness, we mention a result that relates the two moduli spaces discussed in this chapter.

Theorem 2.2.10 (Non-abelian Hodge theorem, [33], [34], [32])**.** *There is a real analytic diffeomorphism between the coarse moduli spaces of:*

- *G-local systems on X,*
- *G-Higgs bundles on X with vanishing Chern class,*

each satisfying a certain stability condition.

Note that Theorem 2.2.10 gives a diffeomorphism of coarse moduli spaces. In [32], Simpson leaves the stacky analogoue as an open question.

2.3 Spectral and cameral covers

The main ingredient in the proof of Theorem 2.2.8 is the construction, for every $b \in \mathcal{B}(X, G, \mathcal{L})$, of a branched cover of X, such that G-Higgs bundles in the fiber over b are related to either line bundles or T-bundles on the branched cover. For clarity, and following the historical order of events, we first introduce spectral curves for the case of $G = GL(n, \mathbb{C})$, and only afterwards cameral curves for general reductive G.

Recall from Example 2.2.7 that the Hitchin map for $GL(n, \mathbb{C})$ sends a Higgs bundle $(\mathcal{E}, \varphi)$ the coefficients of the characteristic polynomial of φ. Informally, a spectral curve parametrizes the eigenvalues of $\varphi(x)$, as $x \in X$ varies.

Definition 2.3.1 (Spectral curve, following [19], [20])**.** Let $b \in \mathcal{B}\big(X, GL(n,\mathbb{C}), \mathcal{L}\big)$, and let $f_b : \mathcal{L} \to \mathcal{L}^n$ be the corresponding characteristic polynomial:

$$f_b(\lambda) = \lambda^n + \sum_{d=1}^{n} \lambda^{n-d}(-1)^d \operatorname{Tr}(\varphi^{\wedge d}) \tag{2.3.1}$$

The **spectral curve** $\bar{X}_b$ is the subspace of the total space of $\mathcal{L}$, defined as the kernel of f_b. The projection $\pi_{\mathcal{L}} : \mathcal{L} \subset X$ induces a projection $\bar{\pi} : \bar{X}_b \to X$.

$$\begin{tikzcd} \bar{X}_b \arrow[r, hook] \arrow[dr, "\bar{\pi}"'] & \mathcal{L} \arrow[d, "\pi_{\mathcal{L}}"] \\ & X \end{tikzcd} \tag{2.3.2}$$

Moreover, the tautological section in $\Gamma(\mathcal{L}, \pi_{\mathcal{L}}^*\mathcal{L})$ restricts to a section $\lambda \in \Gamma(\bar{X}_b, \bar{\pi}^*\mathcal{L})$, which we will also call a tautological section.

Remark 2.3.2. For $x \in X$, evaluating the characteristic polynomial f_b at x gives a degree n polynomial $\mathcal{L}_x \to \mathcal{L}_x^{\otimes n}$. The fiber $\bar{\pi}^{-1}(x)$ consists of the distinct roots of this polynomial. For generic $b \in \mathcal{B}$, there are finitely many $x \in X$ where the n roots fail to be distinct; these are the ramification points of $\bar{\pi}$.

Proposition 2.3.3 ([3], section 3)**.** *There is a Zariski open subset $\mathcal{B}^{int}(X, GL(n), \mathcal{L})$ of $\mathcal{B}(X, GL(n), \mathcal{L})$ for which the spectral curve $\bar{X}_b$ is irreducible and reduced. If $\mathcal{L}^n$ admits a section whose divisor is not of the form mD, for m dividing n, then this open subset is nonempty.*

Proposition 2.3.4 ([3], Proposition 3.6)**.** *For $b \in \mathcal{B}^{int}(X, GL(n), \mathcal{L})$, the Hitchin fiber over b is isomorphic to the moduli space of rank 1, torsion-free sheaves on $\bar{X}_b$.*

For the smaller subset where $\bar{X}_b$ is actually smooth, rank 1, torsion-free sheaves are just line bundles, and we obtain an isomorphism between the Hitchin fiber over b and the abelian variety $\operatorname{Jac}(X_b)$. In this case, the Proposition is proved as follows.

Starting from a line bundle L on $\bar{X}_b$, $\bar{\pi}_* L$ is a rank n vector bundle on X, and the push-forward of the tautological section $\bar{\pi}_* \lambda \in \Gamma(\bar{X}_b, \bar{\pi}^* \mathcal{L})$ is a Higgs field on $\bar{\pi}_* L$.

Conversely, starting from a Higgs bundle (E, φ) on X, we define an eigenline bundle L on $\bar{X}_b$ as follows. Consider the sequence of vector bundles on $\bar{X}_b$:

$$\bar{\pi}^* E \xrightarrow{\bar{\pi}^* \varphi - \lambda} \bar{\pi}^* (E \otimes \mathcal{L}). \tag{2.3.3}$$

Then we define $L := \mathrm{Coker}(\bar{\pi}^* \varphi - \lambda) \otimes (\bar{\pi}^* \mathcal{L})^{-1}$.

The discussion of abelianization via spectral curves can be adapted to the setting of other classical groups $SL(n, \mathbb{C}), Sp(2n, \mathbb{C}), SO(n, \mathbb{C})$; see [20]. For a general reductive group G, one can choose a representation $\rho : G \to GL(n, \mathbb{C})$, and use this to define a spectral construction as above. But this comes with several disadvantages:

- in order to prove a result which does not depend on ρ, it becomes necessary to understand the interplay between spectral curves associated to different representations;
- spectral curves come with various "accidental singularities", see [10].

Donagi proposed a different approach in [10] and [11]. He introduced cameral covers, which, in an appropriate sense, dominate spectral curves associated to all representations of G. (See also related work by Faltings in [13] and Scognamillo in [31].)

Definition 2.3.5. Let $b \in \mathcal{B}(X, G, \mathcal{L})$, which determines the bottom horizontal morphism in the diagram below. Let the right vertical morphism be the natural projection. Then the **cameral cover** $\tilde{X}_b$ associated to b is the fiber product in the diagram.

$$\begin{tikzcd} \tilde{X}_b \arrow[r, "\tilde{b}"] \arrow[d, "\pi"'] & \mathfrak{t}_{\mathcal{L}} \arrow[d] \\ X \arrow[r, "b"] & \mathfrak{t}_{\mathcal{L}}/W \end{tikzcd} \tag{2.3.4}$$

Away from the ramification locus $\mathcal{R}$ of $\pi : \tilde{X}_b \to X$, the cameral cover is a principal W-bundle over X. We will exploit this W-action in the construction of cameral networks in Chapter 4. $\mathcal{R}$ is the locus in $\tilde{X}_b$ which is mapped by $\tilde{b}$ to the union of the root hyperplanes $\cup_{\alpha\in\Phi} H_\alpha$. (Since W acts with nontrivial stabilizer on this union.) Following Ngô in the paper [28], let $\mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$ denote the subset of the Hitchin base such that $\tilde{b}(\tilde{X}_b)$ intersects $\cup_{\alpha\in\Phi} H_\alpha \times_{\mathbb{C}^*} \mathcal{L}$ transversely. In other words, for all $b \in \mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$, all ramification points have order two.

Proposition 2.3.6 (Section 4.7 in [28])**.** *The locus $\mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$ is Zariski open in the Hitchin base, and nonempty if* $\deg(\mathcal{L}) > 2g$. *Moreover, $b \in \mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$ if and only if the cameral curve $\tilde{X}_b$ is smooth.*

The bound $\deg(\mathcal{L}) > 2g$ is not tight: for example, the result holds for $\mathcal{L} = K_X$, even though $\deg(K_X) = 2g - 2$.

Proposition 2.3.7 (Proposition 4.6.1 in [28])**.** *Assume that* $\deg(\mathcal{L}) > 2g$. *Then, for all $b \in \mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$, the cameral curve $\tilde{X}_b$ is connected.*

In fact, Ngô's Proposition 4.6.1 applies to a subset $\mathcal{B}^{\heartsuit}(X, G, \mathcal{L})$ which is larger than $\mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$.

Example 2.3.8. If $G = GL(n, \mathbb{C})$, then $W = S_n$, so the degree of the covers $\tilde{X}_b \to X$ is $n!$; compare this to the degree n spectral covers $\bar{X}_b \to X$. Whereas $\bar{X}_b$ parametrizes eigenvalues of the characteristic polynomial f_b, $\tilde{X}_b$ parametrizes orderings of the eigenvalues.

For generic enough b, and letting S_{n-1} be the stabilizer of one of the eigenvalues, we have:

$$\tilde{X}_b/S_{n-1} \cong \bar{X}_b. \tag{2.3.5}$$

In particular, for $n = 2$, the spectral and cameral curves are isomorphic. More interestingly, if $n = 3$ and two eigenvalues become equal at $x \in X$, then the local structure of the spectral and cameral curves are as in Figure 2.1. There is extra symmetry present in the cameral case.

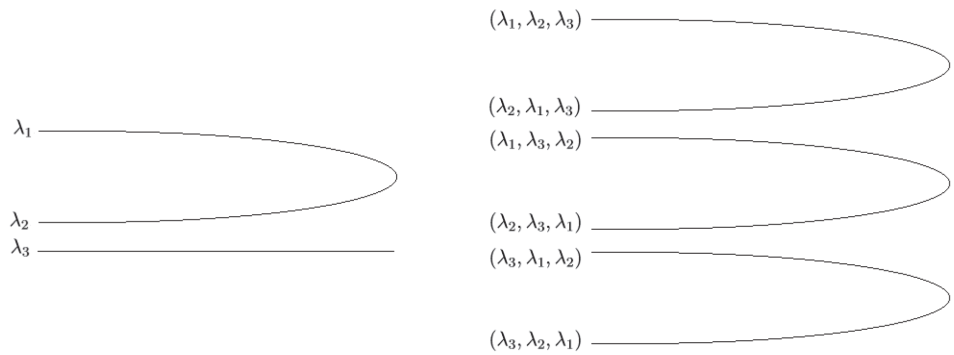

Figure 2.1: Preimage of a branch point of order 2, in the spectral (left) and cameral (right) curves.

The following theorem gives an analogue of the abelianization statement of Proposition 2.3.4.

Theorem 2.3.9 (Theorem 6.4 in [12])**.** *The fiber of the Hitchin map over $b \in \mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$ is isomorphic to the moduli space of weakly W-equivariant, N-shifted, R-twisted T-bundles on $\tilde{X}_b$.*

We do not define here the meaning of the terms "weakly W-equivariant", "N-shifted" or "R-twisted". The first two will be defined and used in Section 5.1; for the third, the reader can consult [12]. For the purposes of this section, the take-away is that there exists a moduli space of T-bundles on $\tilde{X}_b$, with appropriate extra data, which is isomorphic to the Hitchin fiber over b.

Chapter 3

Lie theoretic technicalities

3.1 Chevalley bases and $\mathfrak{sl}_2$-triples

This section is a collection of unoriginal results about the structure of reductive Lie algebras. We present and organize the specific material from this subject area that will be necessary in other sections.

Let $\mathfrak{g}$ be a simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{t}$, and let Φ denote the set of roots of $\mathfrak{g}$. This determines a root space decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{u}_\alpha. \tag{3.1.1}$$

Here the 1-dimensional root spaces $\mathfrak{u}_\alpha$ are the α-eigenspaces for the adjoint action of the Cartan.[1] We will make frequent use of the following relationship between root spaces and the Lie bracket.

[1] The root spaces are commonly denoted $\mathfrak{g}_\alpha$ in the literature. We use $\mathfrak{u}_\alpha$ instead, for compatibility with the discussion of nilpotent Lie algebras and unipotent groups in Section 3.2.

Lemma 3.1.1. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq -\beta$. Then $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha+\beta}$. Moreover, $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = 0$ if and only if $\alpha + \beta \notin \Phi$.*

Proof. Since the root spaces are 1-dimensional, so it suffices to consider the bracket $[e_\alpha, e_\beta]$ for some choice of nonzero $e_\alpha \in \mathfrak{u}_\alpha$ and $e_\beta \in \mathfrak{u}_\beta$. The Jacobi identity implies that, for all $h \in \mathfrak{t}$:

$$\begin{aligned}\big[h, [e_\alpha, e_\beta]\big] &= \big[[h, e_\alpha], e_\beta\big] + \big[e_\alpha, [h, e_\beta]\big] \\ &= \big[\alpha(h) \cdot e_\alpha, e_\beta\big] + \big[e_\alpha, \beta(h) \cdot e_\beta\big] \\ &= (\alpha + \beta)(h) \cdot [e_\alpha, e_\beta].\end{aligned}$$

Hence $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha+\beta}$.

It's clear then that if $\alpha + \beta \notin \Phi$, then $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = 0$. For the converse, see e.g. Theorem 6.44 in [26]. $\square$

Definition 3.1.2. We say that a basis of $\mathfrak{g}$ is **adapted to the root space decomposition** if it consists of a basis for $\mathfrak{t}$, together with one nonzero element from each of the root spaces $\mathfrak{u}_\alpha$.

In particular, there exist bases adapted to the root space decomposition, with respect to which the structure constants are particularly well behaved.

Definition 3.1.3. Choose a polarization $\Phi = \Phi_+ \coprod \Phi_-$ of the root system; this determines a set $\Phi_S \subset \Phi_+$ of simple roots. A **Chevalley basis** is a basis of $\mathfrak{g}$ compatible with the root space decomposition, consisting of the data:

- $\{h_\alpha\}_{\alpha \in \Phi_S}$, which form a basis for $\mathfrak{t}$;

- $\{e_\gamma\}_{\gamma\in\Phi}$;

such that the following conditions are satisfied. For all $\gamma \in \Phi$, write $\gamma = \sum_{\alpha\in\Phi_S} n_\alpha \alpha$, and define $h_\gamma = \sum_{\alpha\in\Phi_S} n_\alpha h_\alpha$. Then:

$$[h_\alpha, e_\gamma] = 2\frac{(\alpha,\gamma)}{(\alpha,\alpha)} e_\gamma, \tag{3.1.2}$$

$$[e_\alpha, e_{-\alpha}] = -h_\alpha \tag{3.1.3}$$

$$[e_\alpha, e_\gamma] = \begin{cases} 0 & \text{if } \alpha+\gamma \notin \Phi, \\ \pm(p_{\alpha,\gamma}+1)e_{\alpha+\gamma} & \text{if } \alpha+\gamma \in \Phi. \end{cases} \tag{3.1.4}$$

In condition 3.1.4, $p_{\alpha,\gamma}$ is defined as the largest integer such that $\alpha - p_{\alpha,\gamma}\gamma \in \Phi$.

Remark 3.1.4. Conditions 3.1.2 and 3.1.3 imply that $\{h_\alpha, e_\alpha, -e_{-\alpha}\}$ is an $\mathfrak{sl}_2$-triple, for every $\alpha \in \Phi$. We are using an uncommon sign convention in equation 3.1.3, which, in other sources, is $[e_\alpha, e_{-\alpha}] = h_\alpha$. This would imply that $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is an $\mathfrak{sl}_2$-triple, which looks like an aesthetically superior statement. However, we prefer our sign convention because it allows us to treat e_α and $e_{-\alpha}$ on an equal footing down the line.

Remark 3.1.5. According to Lemma 3.1.1, if $\alpha+\gamma \in \Phi$, then there must exist a constant $C_{\alpha,\gamma} \in \mathbb{C}^*$ such that $[e_\alpha, e_\gamma] = C_{\alpha,\gamma} \cdot e_{\alpha+\gamma}$. Then it's not hard to show that the constants must satisfy $C_{\alpha,\gamma}C_{-\alpha,-\gamma} = (p_{\alpha,\gamma}+1)^2$. The choice made in equation 3.1.4 is $C_{\alpha,\gamma} = C_{-\alpha,-\gamma} = \pm(p_{\alpha,\gamma}+1)$, which preserves the most symmetry between opposite roots. In particular, the constants are small integers.

- For $\mathfrak{g}$ of type ADE, the condition that $\alpha+\gamma \in \Phi$ makes all $p_{\alpha,\gamma} = 0$. Therefore $C_{\alpha,\gamma} = \pm 1$.

- For $\mathfrak{g}$ of type BCF, $p_{\alpha,\gamma} = 1$ if both roots are short, and 0 otherwise.

- For $\mathfrak{g}$ of type G_2, $p_{\alpha,\gamma} \in \{0,1,2\}$, depending on the angle between the roots.

To summarize Remarks 3.1.4 and 3.1.5, a Chevalley basis consists of $\mathfrak{sl}_2$-triples whose brackets are as simple as possible.

The following existence result was originally proved by Chevalley in [9], and a good exposition is given by Tao in the blog post [37].

Proposition 3.1.6 (Chevalley, [9])**.** *Every complex simple* $\mathfrak{g}$ *admits a Chevalley basis.*

Example 3.1.7. Let $\mathfrak{g} = \mathfrak{sl}_3$, and $\alpha, \beta \in \Phi_S$. We construct a Chevalley basis from the following basis of the Cartan:

$$h_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad h_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.1.5}$$

and the following basis vectors for the root spaces:

$$e_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.6}$$

$$e_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_{-\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad e_{-\alpha-\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.1.7}$$

The set of Chevalley bases for $\mathfrak{sl}_3$ is a torsor over the maximal torus $T_{SL(3,\mathbb{C})} \cong (\mathbb{C}^*)^2$. For any re-scaling of e_α, e_β by $A, B \in \mathbb{C}^*$, it is possible to re-scale $e_{\alpha+\beta}$ by AB, and $e_{-\alpha}, e_{-\beta}, e_{-\alpha-\beta}$ by $A^{-1}, B^{-1}, A^{-1}B^{-1}$, respectively, so that relations 3.1.2 – 3.1.4 are preserved.

Definition 3.1.8. Recall that a reductive Lie algebra $\mathfrak{g}$ is the direct sum of its simple sub-algebras and its abelian center. A Chevalley basis for $\mathfrak{g}$ is the data of a Chevalley basis for each simple summand, and an arbitrary basis for the center.

For the rest of the section, let $\mathfrak{g}$ be complex reductive and fix a Chevalley basis for $\mathfrak{g}$. Then each $\alpha \in \Phi$ determines an $\mathfrak{sl}_2$-triple, or equivalently a Lie algebra homomorphism $i_\alpha : \mathfrak{sl}_2 \to \mathfrak{g}$. Because $SL(2,\mathbb{C})$ is simply connected, Lie's theorems (Theorem 3.41 in [26]) provide a group homomorphism I_α such that the following diagram commutes.

$$\begin{tikzcd} SL(2,\mathbb{C}) \arrow[r, "I_\alpha"] & G \\ \mathfrak{sl}_2 \arrow[u, "\exp"] \arrow[r, "i_\alpha"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd} \tag{3.1.8}$$

Let $s_\alpha \in W$ denote the reflection about the root hyperplane H_α, and denote by p the quotient map $N \to T$. The Chevalley basis determines, for each $\alpha \in \Phi$, an element $n_\alpha \in N$; Lemma 3.1.11 below proves that $n_\alpha \in p^{-1}(s_\alpha)$.

$$n_\alpha := \exp\left[\frac{\pi}{2}(e_\alpha + e_{-\alpha})\right]. \tag{3.1.9}$$

Example 3.1.9. For $\mathfrak{g} = \mathfrak{sl}_2$, and the Chevalley basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad -f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{3.1.10}$$

we have:

$$n_{SL(2,\mathbb{C})} = \exp\left[\frac{\pi}{2}(e-f)\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1.11}$$

Due to diagram 3.1.8, $n_\alpha = I_\alpha(n_{SL(2,\mathbb{C})})$ is a characterization of n_α.

Remark 3.1.10. Sending $s_\alpha \mapsto n_\alpha$ does not, in general, give a section of the projection $p : N \to W$. Even in the case of $SL(2,\mathbb{C})$, we have $n_\alpha^2 = -\mathrm{id}$. For certain groups, including $SL(2,\mathbb{C})$, the normalizer short exact sequence below is not split.

$$1 \longrightarrow T \longrightarrow N \xrightarrow{p} W \longrightarrow 1 \tag{3.1.12}$$

Lemma 3.1.11. *n_α is an element of the T-coset $p^{-1}(s_\alpha)$.*

Proof. This follows from two observations:

1. $\mathrm{ad}_{n_\alpha}(h_\alpha) = -h_\alpha$. This is proved by an easy computation in the case of $\mathfrak{sl}_2$; then the general case follows by applying i_α. Note that I_α commutes with adjoint actions, by virtue of being a homomorphism. Taking a differential, we obtain that i_α also does.

2. $\mathrm{ad}_{n_\alpha}(h) = h$ for $h \in H_\alpha$. This is because of definition 3.1.9 and the fact that:

$$[h, e_{\pm\alpha}] = \pm\alpha(h)e_{\pm\alpha} = 0. \tag{3.1.13}$$

□

The next result is a generalization of equation 1.1.10, and will similarly be used to "cancel out" the monodromy of local systems around branch points.

Lemma 3.1.12. *The following identity holds in G:*

$$\exp(e_\alpha)\exp(e_{-\alpha})\exp(e_\alpha) = n_\alpha. \tag{3.1.14}$$

Proof. Apply the homomorphism I_α to:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1.15}$$

□

For every $\alpha \in \Phi$, the Killing form determines an orthogonal decomposition:

$$\mathfrak{t} = \mathfrak{t}_\alpha \oplus H_\alpha, \tag{3.1.16}$$

where $\mathfrak{t}_\alpha$ is a 1-dimensional subspace generated by the co-root $\alpha^\vee$, and H_α is the root hyperplane satisfying $\alpha(H_\alpha) = 0$.

Let $T_\alpha = \exp(\mathfrak{t}_\alpha)$ and $T_{H_\alpha} = \exp(H_\alpha)$. T_α is characterized as $I_\alpha(T_{SL(2)})$.

Lemma 3.1.13. *The multiplication homomorphism:*

$$T_\alpha \times T_{H_\alpha} \longrightarrow T \tag{3.1.17}$$

is surjective. Its kernel is trivial if I_α factors through $PSL(2,\mathbb{C})$, otherwise it is:

$$\big\{(\mathrm{id},\mathrm{id}), \big(I_\alpha(-\mathrm{id}_{SL(2)}), I_\alpha(-\mathrm{id}_{SL(2)})\big)\big\}. \tag{3.1.18}$$

Proof. Due to the orthogonal decomposition 3.1.16 at the Lie algebra level, multiplication $T_\alpha \times T_{H_\alpha} \to T$ is surjective. Its finite kernel is the intersection $T_\alpha \cap T_{H_\alpha}$ in T. Since $T_\alpha = I_\alpha(T_{SL(2,\mathbb{C})})$, and T_{H_α} is in the kernel of the character $\exp(\alpha)$, we need only analyze the diagram:

$$T_{SL(2)} \xrightarrow{I_\alpha} T \xrightarrow{\exp(\alpha)} \mathbb{C}^\times. \tag{3.1.19}$$

It follows that:

$$T_\alpha \cap T_{H_\alpha} = I_\alpha\big(\mathrm{Ker}(\exp(\alpha) \circ I_\alpha)\big). \tag{3.1.20}$$

Denote $\exp(\alpha_{SL(2)}) = \exp(\alpha) \circ I_\alpha$; so it suffices to show that $\mathrm{Ker}(\exp(\alpha_{SL(2)})) = \{\pm\mathrm{id}_{SL(2)}\}$. $\mathrm{Ker}(\exp(\alpha_{SL(2)}))$ consists of the elements of $T_{SL(2)}$ whose adjoint action fixes $e_{\alpha_{SL(2)}}$; the latter is either:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Either way, $\{\pm \mathrm{id}_{SL(2)}\}$ are the only elements of T_{SL_2} whose adjoint action fixes it. $\square$

Lemma 3.1.14. *The adjoint action of n_α satisfies:*

1. *For every $t \in T_\alpha$, $n_\alpha t n_\alpha^{-1} = t^{-1}$.*

2. *For every $t \in T_{H_\alpha}$, $n_\alpha t n_\alpha^{-1} = t$.*

Proof. An immediate consequence of Lemma 3.1.11. $\square$

We state two more results which are necessary in section 5.3.1. Their elementary proofs can be found in [23].

Lemma 3.1.15. *For all $t \in T$, $\mathrm{ad}_t(e_\alpha)$ is a scalar multiple of e_α. Moreover, all scalar multiples of e_α arise in this way.*

Lemma 3.1.16. *Let $n \in N$, and $[n] \in W$ its image in the Weyl group. Define $\alpha' = [n](\alpha)$. Then there exists some $t \in T_{\alpha'}$ such that:*

- $\mathrm{ad}_n(e_{\pm\alpha}) = \mathrm{ad}_t(e_{\pm\alpha'})$;
- $\mathrm{Ad}_n(n_\alpha) = \mathrm{Ad}_t(n_{\alpha'})$.

3.2 Scattering diagrams and Stokes factors

In this section we introduce 2D scattering diagrams (Definition 3.2.11), which are a local model for the intersections of Stokes curves that will appear in Chapters 4 and 5. Each ray in the scattering diagram is labeled by a root α of $\mathfrak{g}$, and decorated by an element of $\exp(\mathfrak{u}_\alpha)$ called a Stokes factor. The main goal of the section is to prove, in as much generality as possible, that the Stokes factors for incoming rays uniquely determine the Stokes factors for outgoing rays (Theorem 3.2.21).

We first make some definitions related to sets of roots.

Definition 3.2.1. We say that a set of roots $C \subset \Phi$ is **convex** if there exists a polarization $\Phi = \Phi_+ \coprod \Phi_-$ such that $C \subset \Phi_+$.

Equivalently, C is convex if it is contained in a strictly convex cone in $\mathfrak{t}^*$ with vertex at the origin.

Definition 3.2.2. Let $\{\alpha_1, \ldots, \alpha_j\}$ be a convex set of roots. Their **restricted convex hull** is the subset:

$$\mathrm{Conv}^{\mathbb{N}}_{\alpha_1,\ldots,\alpha_j} := \left\{ \gamma \in \Phi \middle| \gamma = \sum_{i=1}^{j} n_i \alpha_i,\ n_i \in \mathbb{N} \right\}. \tag{3.2.1}$$

For comparison, their **convex hull** is:

$$\mathrm{Conv}_{\alpha_1,\ldots,\alpha_j} := \left\{ \gamma \in \Phi \middle| \gamma = \sum_{i=1}^{j} n_i \alpha_i,\ n_i \in \mathbb{R}_+ \right\}. \tag{3.2.2}$$

In this paper we will mostly need the restricted convex hull.

The restricted convex hull is motivated by the following reformulation of Lemma 3.1.1.

Lemma 3.2.3. *Let $\alpha, \beta \in \Phi$, such that $\alpha \neq \pm\beta$. Then the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$ is spanned, as a vector space, by $\mathfrak{u}_\gamma$ with γ ranging over* $\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$.

Proof. Due to Lemma 3.1.1, $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = \mathfrak{u}_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$, and $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = 0$ otherwise. By recursive application of this result, we obtain that $\langle \mathfrak{u}_\alpha, \mathfrak{u}_\beta \rangle$ contains $\mathfrak{u}_\gamma$ if and only if $\gamma \in \mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$. □

Example 3.2.4. In root systems of type ADE, for every convex pair of roots $\{\alpha_1, \alpha_2\}$, $\mathrm{Conv}_{\alpha_1,\alpha_2} = \mathrm{Conv}^{\mathbb{N}}_{\alpha_1,\alpha_2}$. To see this, note that the restriction of the root system to the plane spanned by α_1 and α_2 is a root system of type $A_1 \times A_1$ or A_2. In both cases, the claim is obvious. (See Figures 3.1 and 3.2.)

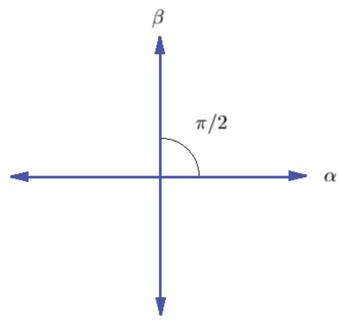

Figure 3.1: The root system $A_1 \times A_1$

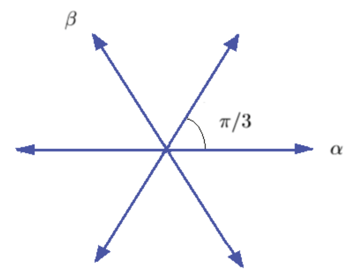

Figure 3.2: The root system A_2

Example 3.2.5. In a root system of type B2 (see Figure 3.3), let α_1, α_2 be orthogonal long roots. Then:

$$\mathrm{Conv}_{\alpha_1,\alpha_2} = \{\alpha_1, (\alpha_1+\alpha_2)/2, \alpha_2\},$$

$$\mathrm{Conv}^{\mathbb{N}}_{\alpha_1,\alpha_2} = \{\alpha_1, \alpha_2\}.$$

In Section 3.3 we give other explicit examples and computations, in the case of the planar root systems of Figures 3.1–3.4. In the meantime, we comment on the difference between Conv_C and $\mathrm{Conv}^{\mathbb{N}}_C$ for non-planar root systems.

Lemma 3.2.6. *For $\mathfrak{g}$ a simple Lie algebra of type A, and $C \subset \Phi_+$, $\mathrm{Conv}_C = \mathrm{Conv}^{\mathbb{N}}_C$.*

Proof. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots determined by the polarization Φ_+, and recall that Π is a basis for the root system; in particular, for any $\gamma \in \Phi_+$, there

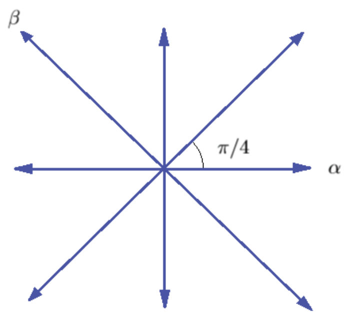

Figure 3.3: The root system B_2

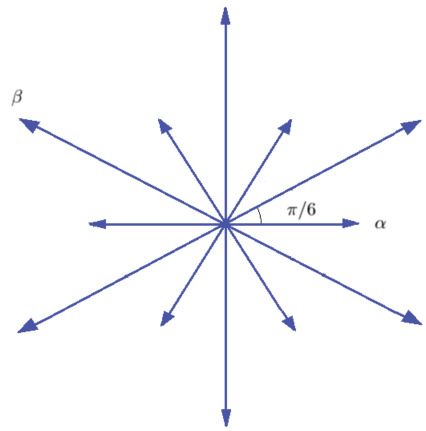

Figure 3.4: The root system G_2

is a unique expression:

$$\gamma = \sum_{k=1}^{n} a_k \alpha_k, \; a_k \in \mathbb{N}. \tag{3.2.3}$$

We need some facts about positive roots:

1. For any simple $\mathfrak{g}$, the support of $\gamma \in \Phi_+$, defined as those α_k for which the coefficient a_k in equation 3.2.3 is nonzero, is a connected subset of the Dynkin diagram. (See corollary 3 to Proposition VI.1.6.19 of [6].)

2. For $\mathfrak{g}$ of type A, all nonzero coefficients in 3.2.3 are equal to 1. To see this, we assume without loss of generality that Φ_+ corresponds to upper-triangular matrices in $\mathfrak{sl}_n$, so that the simple roots correspond to the entries immediately above the diagonal:

$$\begin{pmatrix} & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \\ & & & \end{pmatrix}. \tag{3.2.4}$$

Then the positive root γ_{ij}, corresponding to the root space of the elementary matrix E_{ij}, for $i < j$, satisfies $\gamma_{ij} = \sum_{k=i}^{j-1} \alpha_k$.

By the above facts, in the case of $\mathfrak{g}$ of type A, sending a positive root to its support gives a bijection between Φ_+ and discrete intervals $\overline{\{i,j\}} \subset \overline{\{1,n\}}$, where we define:

$$\overline{\{i,j\}} := [i,j] \cap \mathbb{Z}. \tag{3.2.5}$$

Moreover, this bijection maps the sum of roots to the union of discrete intervals.

Assume, then, that $C = \{\gamma_1, \ldots, \gamma_d\} \subset \Phi_+$, and $\gamma_0 \in \mathrm{Conv}_C$, i.e.:

$$\gamma_0 = \sum_{i=1}^{d} c_i \gamma_i, \;\; c_i \in [0, \infty). \tag{3.2.6}$$

For every i, let $I_i \subset \overline{\{1,n\}}$ denote the support of γ_i; then $I_i \subset I_0$, for every $i > 0$. We assume without loss of generality that $I_0 = \overline{\{1,n\}}$, otherwise we could restrict to the sub-root system generated by the support of γ_0.

We claim that there exist $\{i_j\}_{j=1}^{l}$ such that $I_0 = \coprod_{j=1}^{l} I_{i_j}$, from which it follows that $\gamma_0 = \sum_{j=1}^{l} \gamma_{i_j}$, so in particular $\gamma_0 \in \mathrm{Conv}_C^{\mathbb{N}}$. We prove this claim as follows.

- Choose i_1 be such that $1 \in I_{i_1}$ and $c_{i_1} \neq 0$. Such an index must exist, otherwise the simple root α_1 wouldn't be in the support of γ_0.

- Let end_1 denote the endpoint of I_{i_1}. Choose i_2 such that I_{i_2} starts at $\mathrm{end}_1 + 1$, and $c_{i_2} \neq 0$. Such an index must exist, otherwise the coefficient of α_1 in the basis expansion of γ_0 would be greater than the coefficient of α_2, contradicting the fact that both coefficients are equal to 1.

- Continue this process, terminating at step l, when the discrete interval I_{i_l} ends at

n. This must happen eventually, otherwise the simple root α_n wouldn't be in the support of γ_0.

□

It might be tempting, based on Lemma 3.2.6 and the previous examples, to conjecture that $\mathrm{Conv}_C = \mathrm{Conv}_C^{\mathbb{N}}$ for Lie algebras of type ADE. However, this is false as soon as we leave type A, as the following example shows.

Example 3.2.7. Consider the reduced root system of type D4, whose Dynkin diagram is shown in figure 3.5.

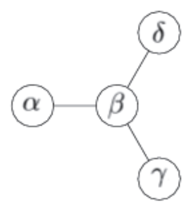

Figure 3.5: Dynkin diagram for D4, with simple roots labeled.

Beyond type A, fact 1 from the proof of Lemma 3.2.6 is still true, but fact 2 is not, i.e. supports of positive roots are still connected subsets of the Dynkin diagram, but the coefficients can be greater than 1. Consider the positive roots:

$$\begin{aligned} \gamma_0 &= \alpha + \beta + \gamma + \delta, \\ \gamma_1 &= \alpha, \\ \gamma_2 &= \gamma, \\ \gamma_3 &= \delta, \\ \gamma_4 &= \alpha + 2\beta + \gamma + \delta, \end{aligned} \tag{3.2.7}$$

and let $C = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Then:

$$\gamma_0 = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4), \tag{3.2.8}$$

which shows that $\gamma_0 \in \mathrm{Conv}_C$, but $\gamma_0 \notin \mathrm{Conv}_C^{\mathbb{N}}$.

Before we define 2D scattering diagrams, we associate unipotent groups to certain convex subsets of Φ.

Lemma 3.2.8. *Let $C \subset \Phi$ be a convex subset, closed under addition. (Equivalently, $C = \mathrm{Conv}_C^{\mathbb{N}}$.) Consider the Lie sub-algebra of $\mathfrak{g}$, spanned as a vector space by:*

$$\mathfrak{u}_C := \bigoplus_{\gamma \in C} \mathfrak{u}_\gamma. \tag{3.2.9}$$

Then $\mathfrak{u}_C$ is nilpotent.

Proof. Due to the convexity assumption, there exists a polarization $\Phi = \Phi_+ \coprod \Phi_-$ of the root system, such that $C \subset \Phi_+$. Recall that the Lie algebra:

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \tag{3.2.10}$$

is nilpotent. Since $\mathfrak{u}_C \subset \mathfrak{n}_+$, and Lie subalgebras of nilpotent Lie algebras are nilpotent, the claim follows. $\square$

Definition 3.2.9. For any $C \subset \Phi_+$ closed under addition, let $U_C := \exp(\mathfrak{u}_C)$ be the associated unipotent subgroup of G.

Remark 3.2.10. For any nilpotent Lie algebra $\mathfrak{u}$, the exponential map $\exp : \mathfrak{u} \to U$ is algebraic, because the Taylor series of the exponential is finite in this case. Therefore, all constructions in this section that involve the exponential map makes sense in the setting of algebraic groups. Moreover, for $\mathfrak{u}$ nilpotent, $\exp : \mathfrak{u} \to U$ is an isomorphism of schemes.

Definition 3.2.11. Let $C_{in} \subset \Phi$ be a convex set, and set $C_{out} = \mathrm{Conv}^{\mathbb{N}}_{C_{in}}$. An **undecorated 2D scattering diagram** is a finite collection of oriented rays in $\mathbb{R}^2$, starting or ending at $\{0\} \in \mathbb{R}^2$, together with the data of:

- a bijection between the set of incoming rays and C_{in} (we say that incoming rays are **labeled** by elements of C_{in});
- a bijection between the set of outgoing rays and C_{out}.

A **decorated 2D scattering diagram** is an undecorated 2d scattering diagram together with:

- For every ray with label α, an element $u_\alpha \in U_\alpha$ called the **Stokes factor**.

The Stokes factors are required to satisfy a constraint. The product taken over both incoming and outgoing Stokes factors, in clockwise order around the intersection point, is the identity:

$$\overrightarrow{\prod}_{\alpha \in C_{in} \coprod C_{out}} u_\alpha^{\pm 1} = \mathrm{id}. \tag{3.2.11}$$

Here $\overrightarrow{\prod}$ denotes the clockwise-ordered product, and the exponent accounts for orientation: it is -1 for incoming rays, and $+1$ for outgoing rays.

Definition 3.2.12. A **solution to an (undecorated) 2D scattering diagram** is a way to assign Stokes factors $u_\gamma \in U_\gamma$ to the outgoing half-lines, given arbitrary Stokes factors on the incoming rays, such that the result is a decorated 2D scattering diagram. Concretely, it is a morphism of schemes:

$$\prod_{\alpha \in C_{in}} U_\alpha \to \prod_{\gamma \in C_{out}} U_\gamma, \tag{3.2.12}$$

such that the product in 3.2.11, taken over the inputs and outputs of the morphism, is the identity.

Example 3.2.13. Let $\mathfrak{g} = \mathfrak{sl}_3$, and α, β be a choice of simple roots such that the root spaces are:

$$\begin{pmatrix} & \mathfrak{u}_{\alpha} & \mathfrak{u}_{\alpha+\beta} \\ \mathfrak{u}_{-\alpha} & & \mathfrak{u}_{\beta} \\ \mathfrak{u}_{-\alpha-\beta} & \mathfrak{u}_{-\beta} & \end{pmatrix}. \tag{3.2.13}$$

Figure 3.6 depicts an undecorated 2D scattering diagram, with incoming rays labeled by α, β and outgoing rays labeled by $\alpha, \alpha+\beta, \beta$.

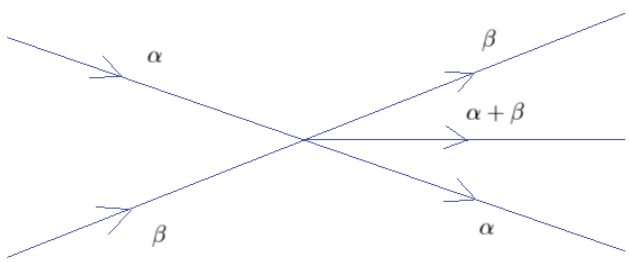

Figure 3.6: A 2D scattering diagram.

A solution for this 2D scattering diagram is a morphism:

$$\begin{aligned} U_{\alpha} \times U_{\beta} &\longrightarrow U_{\beta} \times U_{\alpha+\beta} \times U_{\alpha} \\ (u_{\alpha}, u_{\beta}) &\longmapsto (u'_{\beta}, u'_{\alpha+\beta}, u'_{\alpha}), \end{aligned} \tag{3.2.14}$$

such that:

$$u'_{\alpha} u'_{\alpha+\beta} u'_{\beta} u_{\alpha}^{-1} u_{\beta}^{-1} = \mathrm{id}. \tag{3.2.15}$$

Equivalently, we need to produce $u'_{\alpha}, u'_{\beta}, u'_{\alpha+\beta}$ such that $u'_{\alpha} u'_{\alpha+\beta} u'_{\beta} = u_{\beta} u_{\alpha}$. With the Chevalley basis for $\mathfrak{sl}_3$ from Example 3.1.7, let $u_{\alpha} = \exp(x e_{\alpha})$, $u_{\beta} = \exp(y e_{\beta})$, $u'_{\alpha} =$

$\exp(x' e_\alpha)$, $u'_\beta = \exp(y' e_\beta)$, $u'_{\alpha+\beta} = \exp(z' e_{\alpha+\beta})$. Then:

$$u_\beta u_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2.16}$$

$$u'_\alpha u'_{\alpha+\beta} u'_\beta = \begin{pmatrix} 1 & x' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x' & z' + x'y' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2.17}$$

whence we read off $x' = x$, $y' = y$, $z' = -xy$. In other words, the unique solution is:

$$u'_\alpha = u_\alpha, \tag{3.2.18}$$

$$u'_\beta = u_\beta, \tag{3.2.19}$$

$$u'_{\alpha+\beta} = u_\alpha^{-1} u_\beta u_\alpha u_\beta^{-1}. \tag{3.2.20}$$

In the rest of this section, we work towards a proof of existence and uniqueness of solutions to 2D scattering diagrams, which does not rely on explicit computations such as the ones in Example 3.2.13. The final result is Theorem 3.2.21.

Proposition 3.2.14. *Let $C \subset \Phi$ be a convex subset such that $C = \mathrm{Conv}_C^{\mathbb{N}}$. Then multiplication gives an isomorphism of schemes:*

$$\prod_{\gamma \in C} U_\gamma \xrightarrow{M} U_C, \tag{3.2.21}$$

for any ordering of the product on the left hand side.

Proof. We use the Baker-Campbell-Hausdorff formula:

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \dots\right), \tag{3.2.22}$$

where the dots indicate higher order iterated Lie brackets of X and Y. We only need this formula for the case when each of X, Y spans a root space of $\mathfrak{g}$. Due to Lemma 3.1.1 and the convexity assumption, there are only finitely many nonzero iterated Lie brackets in this case.

For all $\gamma \in C$, let $X_\gamma \in \mathfrak{u}_\gamma$. Applying the Baker-Campbell-Hausdorff formula iteratively, we obtain:

$$\prod_{\gamma \in C} \exp(X_\gamma) = \exp\left(\sum_{\gamma \in C}(X_\gamma + \mathrm{junk}_\gamma)\right), \tag{3.2.23}$$

where junk_γ is the sum of all iterated Lie brackets which belong to the root space $\mathfrak{u}_\gamma$.

It follows that we have a commutative diagram of schemes:

$$\begin{tikzcd} \prod_{\gamma\in C} U_\gamma \arrow[r, "M"] & U_C \\ \bigoplus_{\gamma \in C} \mathfrak{u}_\gamma \arrow[u, "\prod_{\gamma\in C}\exp"] \arrow[r, "m"] & \mathfrak{u}_C, \arrow[u, "\exp"'] \end{tikzcd} \tag{3.2.24}$$

where m is the map:

$$(X_\gamma)_{\gamma \in C} \longmapsto \sum_{\gamma \in C} X_\gamma + \mathrm{junk}_\gamma. \tag{3.2.25}$$

The vertical arrows are isomorphisms of schemes (because the Lie groups are unipotent), so it suffices to prove that m is invertible. Because we can compose m with the projections $\mathfrak{u}_C \to \mathfrak{u}_\gamma$, invertibility means recovering the input tuple $(X_\gamma)_{\gamma \in C}$ from the output tuple $(X_\gamma + \mathrm{junk}_\gamma)_{\gamma\in C}$. We will argue by induction on the height of $\gamma \in C$, so let us recall the concept of height of a root.

Since C is convex, there exists a polarization $\Phi = \Phi_+ \coprod \Phi_-$ such that $C \subset \Phi_+$. Let $\{\alpha_1, \ldots, \alpha_d\}$ denote the simple roots with respect to this polarization, and recall that the simple roots are a basis for the root system. Then all $\gamma \in C$ can be written uniquely as:

$$\gamma = \sum_{i=1}^{d} n_i \alpha_i,\ n_i \in \mathbb{N}. \tag{3.2.26}$$

Then we define the height of γ as $ht(\gamma) = \sum_{i=1}^{d} n_i$. In particular, $ht(\gamma_1) + ht(\gamma_2) = ht(\gamma_1 + \gamma_2)$.

Using Lemma 3.1.1, we obtain that, if $[X_\beta, X_\delta] \in \mathfrak{u}_\gamma$, then $ht(\beta), ht(\delta) < ht(\gamma)$. Generalizing, if an iterated Lie bracket involving X_β belongs to $\mathfrak{u}_\gamma$, then $ht(\beta) < ht(\gamma)$. In other words, junk_γ only depends on those X_β with $ht(\beta) < ht(\gamma)$.

The inductive argument is as follows. The base case is given by all γ of minimal height: for these, $\text{junk}_\gamma = 0$, and the composition of m with the projection $\mathfrak{u}_C \to \mathfrak{u}_\gamma$ recovers X_γ. For the inductive step, assume we know X_β for all $\beta \in C$ such that $ht(\beta) < ht(\gamma)$. These determine junk_γ, so we can recover X_γ uniquely from the output of 3.2.25. $\square$

Example 3.2.15. For $\mathfrak{g} = \mathfrak{sl}_3$, choose a polarization so that the positive root spaces correspond to upper-triangular matrices. We have the explicit formula:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c+ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \exp \begin{pmatrix} 0 & a & c+ab/2 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

The map $(a, b, c) \mapsto (a, b, c + ab/2)$ is clearly invertible.

Proposition 3.2.14 is really a statement about root spaces. If we use a basis for $\mathfrak{g}$ that is not adapted to the root space decomposition, then the result need not be true.

Example 3.2.16. Let $\mathfrak{g} = \mathfrak{sl}_3$, and choose the following basis for the Lie subalgebra of strictly upper triangular matrices:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.27}$$

Then:

$$\begin{pmatrix} 1 & a & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c & a-c+ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Elements of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.28}$$

with $y = -2$ and $x \neq -z$ form a codimension 1 locus not in the image of the multiplication map.

Corollary 3.2.17. *Consider a 2D scattering diagram where incoming rays are constrained to a sector of the plane with central angle $< \pi$, and the outgoing rays are constrained to the opposite sector. The situation is depicted in Figure 3.7. Then the scattering diagram has a unique solution.*

Proof. Due to the assumption about separation of incoming and outgoing rays, equation 3.2.11 has the form:

$$\overrightarrow{\prod}_{\alpha \in C_{in}} u_\alpha^{-1} \overrightarrow{\prod}_{\gamma \in C_{out}} u_\gamma = \mathrm{id}. \tag{3.2.29}$$

All factors in the first product are known, and all factors in the second product must be

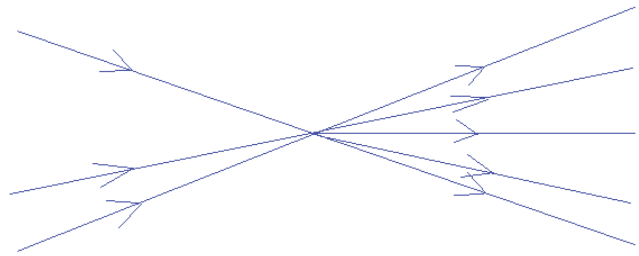

Figure 3.7: A 2D scattering diagram in which incoming and outgoing curves are restricted to opposite sectors.

determined. Let:

$$m_{in} : \overleftarrow{\prod}_{\alpha \in C_{in}} U_\alpha \to U_{C_{out}},$$
$$m_{out} : \overrightarrow{\prod}_{\gamma \in C_{out}} U_\gamma \to U_{C_{out}},$$

be the multiplication maps, where $\overrightarrow{\prod}$ denotes a clockwise-ordered product, and $\overleftarrow{\prod}$ a counterclockwise-ordered product. Then the solution to the 2D scattering diagram is the composition:

$$\overleftarrow{\prod}_{\alpha \in C_{in}} U_\alpha \xrightarrow{m_{in}} U_{C_{out}} \xrightarrow{m_{out}^{-1}} \overrightarrow{\prod}_{\gamma \in C_{out}} U_\gamma. \tag{3.2.30}$$

The map m_{out}^{-1} exists because of Proposition 3.2.14. $\square$

In Section 3.3, we will give explicit formulas for the composition $m_{out}^{-1} \circ m_{in}$, in the case of planar root systems.

It remains to generalize Corollary 3.2.17 by allowing incoming and outgoing rays in the 2D scattering diagram to be interspersed, as in Example 3.2.19. Note that the subset $\sum_{\gamma \in C_{out}} \mathbb{R}_+ \cdot \gamma \subset \mathfrak{t}^*$ is a cone with vertex at the origin. We define a face of the finite set C_{out} to be the intersection of C_{out} with a face of the cone $\sum_{\gamma \in C_{out}} \mathbb{R}_+ \cdot \gamma$, of any dimension between 1 and the dimension of the cone. We will prove the generalization of

Corollary 3.2.17 by induction on the dimension of the faces of C_{out}. Throughout, we use the notation $\Delta \subset C_{out}$ to denote a face of C_{out}.

Lemma 3.2.18. *Let $\Delta_f \subset \Delta$ be a face. The projection $p_f : \mathfrak{u}_\Delta \to \mathfrak{u}_{\Delta_f}$ with kernel $\oplus_{\gamma \in C_\Delta \setminus C_{\Delta_f}} \mathfrak{u}_\gamma$ is a morphism of Lie algebras.*

Consequently, there is a group homomorphism $U_\Delta \to U_{\Delta_f}$, which acts as the identity on U_{Δ_f} and sends every element of the form $\exp(X_\gamma)$ ($X_\gamma \in \mathfrak{u}_\gamma$), for $\gamma \notin \Delta_f$, to $\mathrm{id}_{U_{\Delta_f}}$.

Proof. We need to prove that $p_f\big([X_1, X_2]\big) = \big[p_f(X_1), p_f(X_2)\big]$, for all $X_1, X_2 \in \mathfrak{u}_\Delta$. We analyze two cases.

- If $X_1, X_2 \in \mathfrak{u}_{\Delta_f}$, then $[X_1, X_2] \in \mathfrak{u}_{\Delta_f}$, due to Lemma 3.1.1. So p_f leaves each of $X_1, X_2, [X_1, X_2]$ unchanged.

- If at least one of X_1, X_2 is not in $\mathfrak{u}_{\Delta_f}$, then, using Lemma 3.1.1 and the assumption that Δ_f is a face, either $[X_1, X_2] = 0$ or $[X_1, X_2] \notin \mathfrak{u}_{\Delta_f}$. The equality $p_f\big([X_1, X_2]\big) = \big[p_f(X_1), p_f(X_2)\big]$ holds with both sides equal to 0.

$\square$

The following easy example demonstrates our strategy for assigning Stokes factors to outgoing rays in 2D scattering diagrams, making use of Lemma 3.2.18. We then generalize and formalize this strategy in Theorem 3.2.21.

Example 3.2.19. Let $\mathfrak{g} = \mathfrak{sl}_4$, and consider the polarization such that the positive roots correspond to strictly upper triangular matrices. Let α, β, γ denote the simple roots.

Then the positive roots and their root spaces are:

$$\begin{pmatrix} 0 & \mathfrak{u}_\alpha & \mathfrak{u}_{\alpha+\beta} & \mathfrak{u}_{\alpha+\beta+\gamma} \\ 0 & 0 & \mathfrak{u}_\beta & \mathfrak{u}_{\beta+\gamma} \\ 0 & 0 & 0 & \mathfrak{u}_\gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2.31}$$

We consider the 2D scattering diagram pictured in Figure 3.8, with three incoming rays, labeled by the simple roots α, β, γ. Then $C_{out} = \mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta,\gamma} = \Phi_+$.[2]

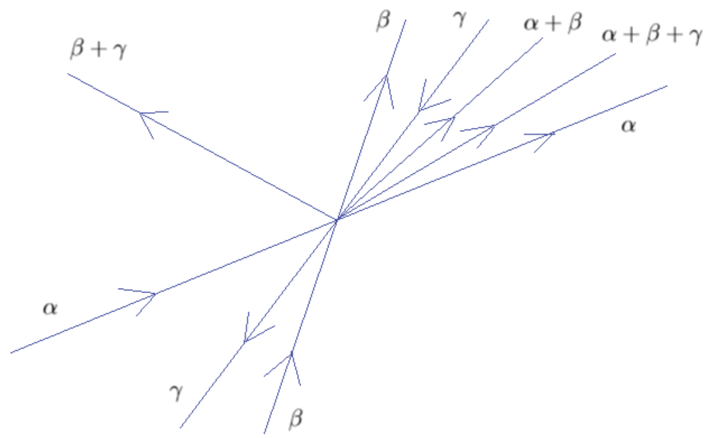

Figure 3.8: A 2D scattering diagram labeled by positive roots of A3.

Equation 3.2.11, which expresses the fact that the clockwise-ordered product of Stokes factors around the intersection is the identity, has the explicit form:

$$u_\alpha^{-1} u'_\gamma u_\beta^{-1} u'_\alpha u'_{\alpha+\beta+\gamma} u'_{\alpha+\beta} u_\gamma^{-1} u'_\beta u'_{\beta+\gamma} = \mathrm{id}, \tag{3.2.32}$$

where $u_\alpha, u_\beta, u_\gamma$ are the known incoming Stokes factors, and the elements with a prime are the outgoing Stokes factors which need to be determined.

[2]This is a toy example; we have not obtained it from a WKB construction.

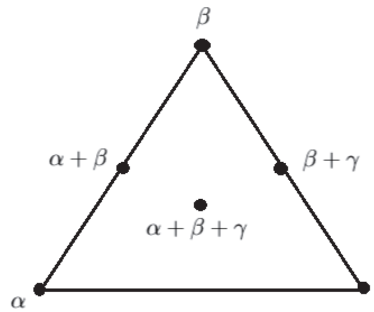

Figure 3.9: A cross-section of the cone Φ_+ in the root system A3.

Figure 3.9 shows a cross-section of the cone structure on C_{out}, with labels indicating 1-dimensional faces. Let Δ denote the 2D face spanned by α and β. The image of equation 3.2.32 under the projection $U_{C_{out}} \to U_\Delta$ is:

$$u_\alpha^{-1} u_\beta^{-1} u'_\alpha u'_{\alpha+\beta} u'_\beta = \text{id}. \tag{3.2.33}$$

Projecting further to the 1-dimensional face spanned by α, we obtain $u_\alpha = u'_\alpha$. Analogously, we obtain $u_\beta = u'_\beta$. Plugging these back into 3.2.33 gives:

$$u'_{\alpha+\beta} = u_\alpha^{-1} u_\beta u_\alpha u_\beta^{-1}. \tag{3.2.34}$$

As a sanity check, we would like to have $u'_{\alpha+\beta} \in U_{\alpha+\beta}$. The RHS of equation 3.2.34 is in the kernel of both projections $U_\Delta \to U_\alpha$ and $U_\Delta \to U_\beta$. The intersection of the two kernels is precisely $U_{\alpha+\beta}$.

The elements u'_γ, $u'_{\beta+\gamma}$ are determined analogously. Then $u'_{\alpha+\beta+\gamma}$ is the only remaining unknown in 3.2.32, so it is uniquely determined:

$$u'_{\alpha+\beta+\gamma} = u_\alpha^{-1} u_\beta u_\gamma^{-1} u_\alpha (u_\beta^{-1} u_\gamma u_\beta u_\gamma^{-1}) u_\beta^{-1} u_\gamma (u_\beta u_\alpha^{-1} u_\beta^{-1} u_\alpha). \tag{3.2.35}$$

Again, this element is in the intersection of the kernels of all face projections, which is

$U_{\alpha+\beta+\gamma}$.

Example 3.2.19 was made easy by the fact that, at every stage, there is at most a single root which doesn't lie on one of the faces. Clearly, we cannot expect this simplification in general: figures 3.3, 3.4 show that this fails for the root systems B2 and G2, respectively. In fact, it even fails for simply laced root systems beyond type A, as the following example shows.

Example 3.2.20. Consider the root system D4, with simple roots $\{\alpha, \beta, \gamma, \delta\}$ as labeled on the Dynkin diagram in Figure 3.5. The positive roots, ordered by the dimension of the smallest-dimensional face they lie on, are:

$$1 : \alpha, \beta, \delta, \gamma,$$

$$2 : \alpha + \beta, \beta + \gamma, \beta + \delta,$$

$$3 : \alpha + \beta + \gamma, \alpha + \beta + \delta, \beta + \gamma + \delta,$$

$$4 : \alpha + \beta + \gamma + \delta, \alpha + 2\beta + \gamma + \delta.$$

There are two roots in the interior of the 4-dimensional cone.

The following result generalizes the strategy of Example 3.2.19 appropriately. For convenience, we introduce some notation for an ordered product of unipotent elements. Let Δ be a face of C_{out} of arbitrary dimension. Then we define:

$$u_\Delta := \overrightarrow{\prod}_{\gamma \in (C_{in} \cap \Delta) \coprod (C_{out} \cap \Delta)} (u_\gamma)^{\pm 1} \tag{3.2.36}$$

as a product over all Stokes factors corresponding to roots $\gamma \in \Delta$, taken in clockwise order around the intersection, where the exponent is $+1$ for outgoing curves, and -1 for incoming curves. Note that equation 3.2.11, expressing the constraint that a solution to a 2D scattering diagram must satisfy, can be rewritten as $u_{C_{out}} = \mathrm{id}$.

Theorem 3.2.21. *Every 2D scattering diagram has a unique solution. Concretely, this means that there is a unique morphism of schemes:*

$$\prod_{\gamma \in C_{in}} U_\gamma \longrightarrow \prod_{\gamma \in C_{out}} U_\gamma$$

$$(u_\gamma)_{\gamma \in C_{in}} \longmapsto (u'_\gamma)_{\gamma \in C_{out}}$$

such that $u_{C_{out}} = \mathrm{id}$, *using the notation of equation 3.2.36. Moreover, for every input tuple* $(u_\gamma)_{\gamma \in C_{in}}$, $(u'_\gamma)_{\gamma \in C_{out}}$ *is the unique output tuple such that* $u_{C_{out}} = \mathrm{id}$.

Proof. The proof is by induction on the dimension of the faces Δ of C_{out}, and can be found in [23].

□

3.3 Explicit calculations for planar root systems

In this section we list a few explicit calculations, which exemplify the results of Section 3.2 in the case of planar root systems. Recall that there are only four planar root systems; they are depicted in Figures 3.1-3.4.

Lemma 3.3.1. *Choose a polarization of the planar root system, and let* α, β *be the simple roots determined by the polarization. If their lengths differ, let* α *be the shorter root. Then the restricted convex hull (see Definition 3.2.2) coincides with the convex hull, and is explicitly given by:*

1. $\mathrm{Conv}_{\alpha,\beta} = \{\alpha, \beta\}$ *in the* $A_1 \times A_1$ *case;*

2. $\mathrm{Conv}_{\alpha,\beta} = \{\alpha, \alpha+\beta, \beta\}$ *in the* A_2 *case;*

3. $\mathrm{Conv}_{\alpha,\beta} = \{\alpha, 2\alpha+\beta, \alpha+\beta, \beta\}$ *in the* B_2 *case;*

4. $\mathrm{Conv}_{\alpha,\beta} = \{\alpha, 3\alpha+\beta, 2\alpha+\beta, 3\alpha+2\beta, \alpha+\beta, \beta\}$ *in the* G_2 *case.*

Proof. Obvious from Figures 3.1-3.4; our convention on α and β agrees with the notation in the figures. □

Consequently, when two Stokes curves labeled by α and β intersect, the number of new Stokes curves produced is 0, 1, 2, 4, respectively.

Lemma 3.3.2. *In the setting of Lemma 3.3.1, let two Stokes curves labeled by* α *and* β *intersect, and consider the morphism of Corollary 3.2.17, which maps incoming Stokes factors to outgoing Stokes factors:*

$$U_\alpha \times U_\beta \longrightarrow \prod_{\gamma \in \mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}} U_\gamma. \tag{3.3.1}$$

Then, letting $x \in \mathfrak{u}_\alpha$, $y \in \mathfrak{u}_\beta$, *and using the short-hand notation:*

$$[x,y]^{[n]} := \Big[x, \big[x, \ldots, [x,y]\big]\Big],$$

the morphism has the following explicit form.

1. *In the* $A_1 \times A_1$ *case:*

$$(e^x, e^y) \longmapsto (e^y, e^x). \tag{3.3.2}$$

2. *In the* A_2 *case:*

$$(e^x, e^y) \longmapsto \big(e^y, \exp{[x,y]}, e^x\big). \tag{3.3.3}$$

3. In the B_2 case:

$$(e^x, e^y) \longmapsto \left(e^y, \exp[x,y], \exp\left(\tfrac{1}{2}[x,y]^{[2]}\right), e^x\right). \tag{3.3.4}$$

4. In the G_2 case:

$$\begin{gathered}(e^x, e^y)\\ \big\downarrow\\ \left(e^y, \exp[x,y], \exp\left(\tfrac{1}{6}\left[[x,y]^{[2]}, [x,y]\right]\right), \exp\left(\tfrac{1}{2}[x,y]^{[2]}\right), \exp\left(\tfrac{1}{6}[x,y]^{[3]}\right), e^x\right).\end{gathered} \tag{3.3.5}$$

Proof. Using Lemma 3.1.1, and the explicit description of $\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$ from Lemma 3.3.1, it's clear that the given elements live in the correct one-parameter subgroups U_γ, for $\gamma \in \mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$. It suffices, then, to prove that:

$$e^x e^y = e^y \exp[x,y] \exp\left(\frac{1}{6}\left[[x,y]^{[2]}, [x,y]\right]\right) \exp\left(\frac{1}{2}[x,y]^{[2]}\right) \exp\left(\frac{1}{6}[x,y]^{[3]}\right) e^x, \tag{3.3.6}$$

where we allow the possibility that some of the exponents are zero, in order to treat all four cases simultaneously.

Up to order 5 in $x, y \in \mathfrak{g}$, the Baker-Campbell-Hausdorff formula gives:

$$\begin{aligned}\log(e^x e^y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\left([x,[x,y]] + [y,[y,x]]\right) - \frac{1}{24}\Big[x, \big[y,[x,y]\big]\Big] -\\ -\frac{1}{720}\left(\Big[y,\Big[y,\big[y,[y,x]\big]\Big]\Big] + \Big[x,\Big[x,\big[x,[x,y]\big]\Big]\Big]\right) +\\ +\frac{1}{360}\left(\Big[y,\Big[x,\big[x,[x,y]\big]\Big]\Big] + \Big[x,\Big[y,\big[y,[y,x]\big]\Big]\Big]\right) +\\ +\frac{1}{720}\left(\Big[y,\Big[x,\big[y,[x,y]\big]\Big]\Big] + \Big[x,\Big[y,\big[x,[y,x]\big]\Big]\Big]\right) +\\ + \dots\end{aligned} \tag{3.3.7}$$

See, for example, Theorem 2 in II.6.6 of [6] for the general combinatorial formula, originally due to Dynkin.

Now let $x \in \mathfrak{u}_\alpha$, $y \in \mathfrak{u}_\beta$. Lemma 3.3.1 ensures that the only nonzero terms are:

$$\log(e^x e^y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,y]^{[2]} + \frac{1}{360}\big[[x,y]^{[2]},[x,y]\big]. \tag{3.3.8}$$

Similarly:

$$\begin{aligned} \log\left(e^y \exp{[x,y]} \exp\left(\frac{1}{6}\big[[x,y]^{[2]},[x,y]\big]\right)\right) =& y + [x,y] + \frac{1}{6}\big[[x,y]^{[2]},[x,y]\big], && (3.3.9) \\ \log\left(\exp\left(\frac{1}{2}[x,y]^{[2]}\right)\exp\left(\frac{1}{6}[x,y]^{[3]}\right)e^x\right) =& \frac{1}{2}[x,y]^{[2]} + \frac{1}{6}[x,y]^{[3]} + x + \\ & + \frac{1}{4}\big[[x,y]^{[2]},x\big] && (3.3.10) \\ =& x + \frac{1}{2}[x,y]^{[2]} - \frac{1}{12}[x,y]^{[3]}. \end{aligned}$$

Let z, w denote the right-hand sides of 3.3.9 and 3.3.10, respectively. Then we need to prove that $e^x e^y = e^z e^w$. The nonzero iterated Lie brackets of z, w are:

$$\begin{aligned} [z,w] &= -[x,y] - [x,y]^{[2]} - \frac{7}{12}\big[[x,y]^{[2]},[x,y]\big], \\ [z,w]^{[2]} &= \big[[x,y]^{[2]},[x,y]\big], \\ [w,z]^{[2]} &= [x,y]^{[2]} + [x,y]^{[3]} + \frac{1}{2}\big[[x,y]^{[2]},[x,y]\big], \\ \big[[w,z]^{[2]},[w,z]\big] &= \big[[x,y]^{[2]},[x,y]\big]. \end{aligned}$$

Then:

$$\begin{aligned} \log(e^z e^w) &= z + w + \frac{1}{2}[z,w] + \frac{1}{12}[z,w]^{[2]} + \frac{1}{12}[w,z]^{[2]} + \frac{1}{360}\big[[w,z]^{[2]},[w,z]\big] \\ &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,y]^{[2]} + \frac{1}{360}\big[[x,y]^{[2]},[x,y]\big]. \end{aligned}$$

This agrees with $\log(e^x e^y)$, as computed in 3.3.8. $\square$

Consider an intersection of k curves, labeled by a maximal planar, convex, set of roots, as in Figure 3.10.

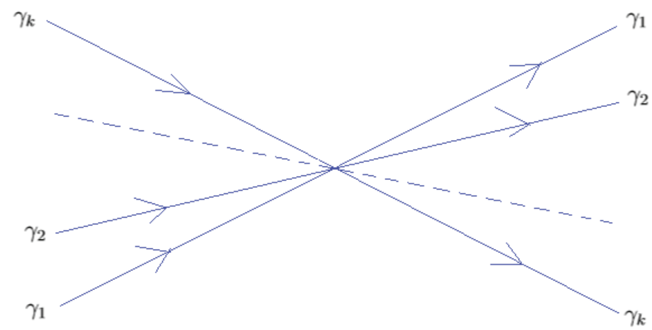

Figure 3.10: Intersection corresponding to maximal planar, convex set of roots.

Let $\gamma_1, \ldots, \gamma_k$ denote the order on $\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$ induced by the order of the incoming Stokes curves. Then the outgoing curves have the reverse order. So, in this case, Corollary 3.2.17 gives a morphism which commutes with the multiplication maps:

$$\begin{tikzcd} U_{\gamma_1} \times \cdots \times U_{\gamma_k} \arrow[r] \arrow[d, "\text{reverse}"'] & U_{\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}} \\ U_{\gamma_k} \times \cdots \times U_{\gamma_1} \arrow[ur] & \end{tikzcd} \tag{3.3.11}$$

We now explain how to obtain explicit formulas for the morphism reverse, by repeated application of Lemma 3.3.2.

We use the fact that, in the planar case we are dealing with, the order on outgoing Stokes curves is convex. This means that, if there are three outgoing curves labeled by $\gamma, \delta, \gamma + \delta$, then the one labeled by $\gamma + \delta$ lies within the sector bounded by the curves labeled by γ and δ. (See Lemma 4.1.17.) There are only two total orders on $\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$ satisfying this convexity property: they are $\{\gamma_1, \ldots, \gamma_k\}$ and its reverse $\{\gamma_k, \ldots, \gamma_1\}$, both considered in diagram 3.3.11. Since the order on $\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$ used on the RHS of Lemma 3.3.2 is convex, it must coincide with one of these two. We assume that it is $\{\gamma_k, \ldots, \gamma_1\}$; otherwise, we would describe reverse^{-1} instead.

We build the morphism reverse as a composition of "twisted transpositions" and "contractions", as defined below.

We define a **twisted transposition** to be an application of Lemma 3.3.2: we replace an element of a product of two neighboring factors, $U_{\gamma_i} \times U_{\gamma_j}$ such that $i < j$, with the tuple provided by the RHS of Lemma 3.3.2. The requirement that $i < j$ ensures that the resulting tuple is ordered correctly.

We define a **contraction** to be a multiplication map:

$$U_{\gamma_i} \times U_{\gamma_i} \to U_{\gamma_i}. \tag{3.3.12}$$

It is clear that, using finitely many twisted transpositions and contractions, we obtain a morphism which commutes with the multiplication maps as in diagram 3.3.11. This must agree with reverse, due to the uniqueness statement in Corollary 3.2.17.

Example 3.3.3 (Cecotti–Vafa Wall Crossing Formula). Consider a Lie algebra $\mathfrak{g}$ of ADE type. The restriction of the root system to the plane spanned by α and β is a root system of type $A_1 \times A_1$ or A_2. In the first case, $k = 2$, and we have $\mathrm{reverse}(e^{X_\alpha}, e^{X_\beta}) = (e^{X_\beta}, e^{X_\alpha})$, an honest transposition. In the second case, $k = 3$ and the commutativity of the diagram 3.3.11 is expressed by:

$$\exp(X_\alpha)\exp(X_{\alpha+\beta})\exp(X_\beta) = \exp\big(X_\beta\big)\exp\big(X_{\alpha+\beta} + [X_\alpha, X_\beta]\big)\exp\big(X_\alpha\big). \tag{3.3.13}$$

Example 3.3.4. In the B2 case, $k = 4$ and we have:

$$\begin{aligned}
&\exp(X_\alpha)\exp(X_{2\alpha+\beta})\exp(X_{\alpha+\beta})\exp(X_\beta) = \\
&= \exp\big(X_\beta\big)\times \\
&\quad \exp\big(X_{\alpha+\beta} + [X_\alpha, X_\beta]\big)\times \\
&\quad \exp\left(X_{2\alpha+\beta} + [X_\alpha, X_{\alpha+\beta}] + \frac{1}{2}[X_\alpha, X_\beta]^{[2]}\right)\times \\
&\quad \exp\big(X_\alpha\big).
\end{aligned} \tag{3.3.14}$$

Example 3.3.5. In the G2 case, $k = 6$ and we have:

$$\begin{aligned}
&\exp(X_\alpha)\exp(X_{3\alpha+\beta})\exp(X_{2\alpha+\beta})\exp(X_{3\alpha+2\beta})\exp(X_{\alpha+\beta})\exp(X_\beta) = \\
&= \exp\big(X_\beta\big)\cdot \\
&\quad \exp\big(X_{\alpha+\beta} + [X_\alpha, X_\beta]\big)\times \\
&\quad \exp\Bigg(X_{3\alpha+2\beta} + [X_{3\alpha+\beta}, X_\beta] + [X_{2\alpha+\beta}, X_{\alpha+\beta}] + \frac{1}{2}\big[[X_\alpha, X_\beta]^{[2]}, X_{\alpha+\beta}\big] + \\
&\qquad \frac{1}{6}\big[[X_\alpha, X_\beta], [X_\alpha, X_\beta]^{[2]}\big]\Bigg)\times \\
&\quad \exp\left(X_{2\alpha+\beta} + [X_\alpha, X_{\alpha+\beta}] + \frac{1}{2}[X_\alpha, X_\beta]^{[2]}\right)\times \\
&\quad \exp\left(X_{3\alpha+\beta} + [X_\alpha, X_{2\alpha+\beta}] + \frac{1}{6}[X_\alpha, X_\beta]^{[3]}\right)\times \\
&\quad \exp\big(X_\alpha\big).
\end{aligned} \tag{3.3.15}$$

Chapter 4

Cameral and spectral networks

Throughout this chapter and the next, we fix a compact Riemann surface X and a reduced, effective non-zero divisor D. Let $\mathcal{L}$ denote the line bundle $K_X(D)$.

Recall that $\mathcal{B}^{\diamondsuit}(X, G, \mathcal{L})$ is the set of all b such that all ramification points of the cameral cover $\pi : \tilde{X}_b \to X$ have order 2. Equivalently, due to Proposition 2.3.6, $\tilde{X}_b$ is smooth. We denote by $P \subset X$ the branch points, and by $R \subset \tilde{X}_b$ the ramification points of the map π.

4.1 The WKB construction

Definition 4.1.1. Let $b \in \mathcal{B}^{\diamondsuit}(X, G, K_X(D))$. For any $d \in D$, we can write a series expansion of b in a local coordinate x around d:

$$b(x) = \left(\sum_{i=-1}^{\infty} a_i x^i \right) dx, \quad a_i \in \mathfrak{t}/W. \tag{4.1.1}$$

We call any lift $\tilde{a}_{-1} \in \mathfrak{t}$ of a_{-1} a **residue** of b at d.[1]

We say that b **satisfies condition R** if for all $d \in D$, any residue of b at d lies in the complement of the root hyperplanes:

$$\tilde{a}_{-1} \in \mathfrak{t} \setminus \bigcup_{\alpha \in \Phi} H_\alpha. \tag{4.1.2}$$

(This condition is W-invariant, so if one residue satisfies it, then all of them do.)

We denote by $\mathcal{B}_R^\diamondsuit(X, G, K_X(D)) \subset \mathcal{B}^\diamondsuit(X, G, K_X(D))$ the subset of points b which satisfy condition R. It is a Zariski open set; its complement has complex codimension 1.

Remark 4.1.2. Recall the pullback diagram that defines a cameral cover:

$$\begin{tikzcd} \tilde{X}_b \arrow[r, "\tilde{b}"] \arrow[d, "\pi"] & \mathfrak{t}_{K_X(D)} \arrow[d] \\ X \arrow[r, "b"] & \mathfrak{t}_{K_X(D)}/W. \end{tikzcd} \tag{4.1.3}$$

The ramification points of the cover π are precisely the points $\tilde{b}^{-1}\big(\cup_{\alpha \in \Phi} H_\alpha \times_{\mathbb{C}^\times} K_X(D)\big)$. Therefore, condition R says that π is unramified at D.

Proposition 4.1.3. *Let $b \in \mathcal{B}_R^\diamondsuit\big(X, G, \mathcal{L}\big)$, and let $\pi : \tilde{X}_b \to X$ be the associated smooth cameral cover. For each $\alpha \in \Phi$, let $\tilde{X}_{b,\alpha} := \tilde{X}_b / \langle s_\alpha \rangle$, and consider the decomposition:*

$$\begin{tikzcd} \tilde{X}_b \arrow[r, "\pi_\alpha"'] \arrow[rr, bend left, "\pi"] & \tilde{X}_{b,\alpha} \arrow[r, "p_\alpha"'] & X. \end{tikzcd} \tag{4.1.4}$$

Then b determines a meromorphic quadratic differential on $\tilde{X}_{b,\alpha}$, with simple zeros at the branch points of π_α, and double poles at preimages of D.

[1] Residues do not depend on the local coordinate x, because they can be obtained from a contour integral of $\tilde{b}(z)$ around a preimage of d in $\tilde{X}_b$.

Proof. We have the following commutative diagram, in which the horizontal arrows in the middle column are pullbacks of differential forms; all squares are fiber products.

$$\begin{tikzcd}
\tilde{X}_b \arrow[r, "\tilde{b}"] \arrow[d, "\pi_\alpha"'] & \mathfrak{t}_{\pi^* K_X(D)} \arrow[r] \arrow[d] & \mathfrak{t}_{\pi_\alpha^*(K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D))} \arrow[r, "\alpha"] \arrow[d] & \pi_\alpha^*(K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D)) \arrow[d] \\
\tilde{X}_{b,\alpha} \arrow[r, "b_\alpha"] \arrow[d, "p_\alpha"'] & \mathfrak{t}_{p_\alpha^* K_X(D)}/\langle s_\alpha \rangle \arrow[r] \arrow[d] & \mathfrak{t}_{K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D)}/\langle s_\alpha \rangle \arrow[r, "\alpha"] & K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D)/\mathbb{Z}_2 \\
X \arrow[r, "b"] & \mathfrak{t}_{K_X(D)}/W & &
\end{tikzcd} \tag{4.1.5}$$

The composition of the middle horizontal arrows is an element of:

$$\Gamma\big(\tilde{X}_{b,\alpha}, K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D)/\mathbb{Z}_2\big) \cong \Gamma\big(\tilde{X}_{b,\alpha}, (K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D))^{\otimes 2}\big). \tag{4.1.6}$$

This is the desired quadratic differential. The branch points of π_α are $b_\alpha^{-1}\big(H_\alpha \times_{\mathbb{C}^*} K_X(D)\big)$, which coincide with the zeros of the quadratic differential. Since we started with $b \in \mathcal{B}_R^\diamondsuit\big(X, G, K_X(D)\big)$, $b_\alpha(\tilde{X}_{b,\alpha})$ intersects $H_\alpha \times_{\mathbb{C}^*} K_X(D)$ transversely, so all the zeros are simple. □

Definition 4.1.4. For each $\alpha \in \Phi$, we denote by:

$$\omega_{b,\alpha} \in \Gamma\big(\tilde{X}_{b,\alpha}, (K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D))^{\otimes 2}\big) \tag{4.1.7}$$

the quadratic differential obtained in Proposition 4.1.3. We denote by:

$$\chi_{b,\alpha} \in \Gamma\big(\tilde{X}_b, K_{\tilde{X}_{b,\alpha}}(p_\alpha^* D)\big) \tag{4.1.8}$$

the linear differential on $\tilde{X}_b$ obtained from composing the top horizontal arrows in diagram 4.1.5. It is immediate from the diagram that:

$$\chi_{b,\alpha} \otimes \chi_{b,\alpha} = \pi_\alpha^* \omega_{b,\alpha}. \tag{4.1.9}$$

Moreover, due to Condition R (Definition 4.1.1), $\omega_{b,\alpha}$ has a pole of order 2 at each $d \in D$.

Definition 4.1.5. Each $\chi_{b,\alpha}$ determines an oriented foliation on $\tilde{X}_b$, by curves $\gamma : \mathbb{R} \to \tilde{X}_b$ satisfying:

$$\int_{t_0}^{t_1} \chi_{b,\alpha}(\gamma(s)) \in \mathbb{R}_+. \tag{4.1.10}$$

The **oriented trajectories** of $\chi_{b,\alpha}$ are the maximal leaves of this foliation.

In Section 1.2, we already gave a teaser of how spectral networks arise from the interaction between these oriented trajectories for different $\alpha \in \Phi$. Below we make this precise. This is a generalization to arbitrary reductive G of the construction of WKB spectral networks in [16] (for $SL(n)$, $GL(n)$), [27] (type ADE), and of the Stokes graphs in the literature on exact WKB analysis, e.g. [4, 1, 24, 22, 36] (these correspond to networks for $SL(n)$, and $GL(n)$).

Definition 4.1.6. From the data determined by $b \in \mathcal{B}_R^{\diamondsuit}(X, G, K_X(D))$, we make the **WKB construction** $\tilde{\mathcal{W}}_b$, which consists of oriented curve segments on $\tilde{X}_b$ called **Stokes curves**, each labeled by an element of Φ. The curves are constructed algorithmically as follows.

- A primary Stokes curve is a critical oriented trajectory of one of the $\chi_{b,\alpha}$; its label is α. (We will prove in Lemma 4.1.15 that there are six primary Stokes curves starting from each ramification point.)

- Let $x \in \tilde{X}_b$ be any isolated intersection point of Stokes curves labeled by distinct roots in some subset $C_{in} \subset \Phi$. For each $\gamma \in \mathrm{Conv}_{C_{in}}^{\mathbb{N}}$, a secondary Stokes curve ℓ_γ starts at x; it is the unique leaf outgoing from x of the oriented foliation determined by $\chi_{b,\gamma}$. See Figure 4.1.

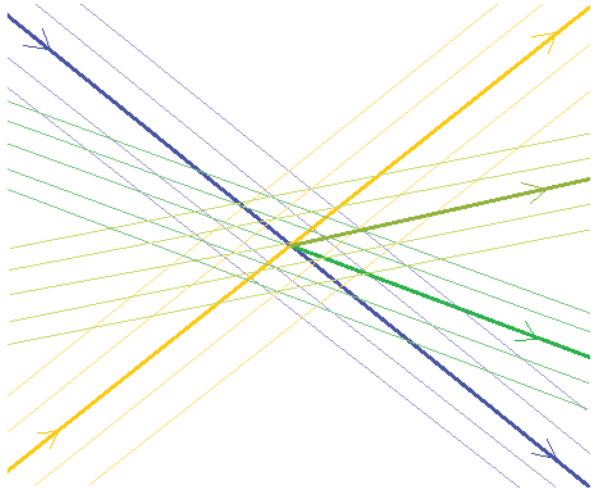

Figure 4.1: Stokes curves (solid) are segments of certain oriented trajectories (transparent). When Stokes curves labeled by α (blue) and β (yellow) intersect, oriented trajectories of χ_{b,γ_1}, χ_{b,γ_2}, for $\gamma_1, \gamma_2 \in \mathrm{Conv}^{\mathbb{N}}_{\{\alpha,\beta\}}$, are activated at the intersection point, and become Stokes curves labeled by γ_1, γ_2 (green, khaki, respectively).

- Secondary Stokes curves are recursively created every time two or more of the existing Stokes curves intersect.

Define the **joints** of $\tilde{\mathcal{W}}_b$ to be the intersection points of Stokes curves; we denote by J the set of joints.

Think of $\tilde{\mathcal{W}}_b$ as a directed graph, with vertex set $R \cup J$, and a directed edge for each connected component of $\tilde{\mathcal{W}}_b \setminus \{R \cup J\}$. We say that $\tilde{\mathcal{W}}_b$ is **admissible** if:

- no point in $\tilde{X}_b \setminus \pi^{-1}(D)$ is an accumulation point of the set J;
- $R \cap J = \emptyset$, i.e. Stokes curves do not run into ramification points;
- the directed graph corresponding to $\tilde{\mathcal{W}}_b$ is acyclic;

- for every joint, the set $C_{in} \subset \Phi$ of labels of the incoming Stokes curves is convex, in the sense of Definition 3.2.1. (Since $C_{out} = \mathrm{Conv}^{\mathbb{N}}_{C_{in}}$, convexity of C_{in} implies convexity of C_{out}.)

Remark 4.1.7. Any directed acyclic graph admits a total order on its vertices, such that for every edge starting at v and ending at w, $v < w$. This is an elementary result in the field of graph theory, where this order is called a **topological order**. Since we required admissible WKB constructions to be acyclic, we can make inductive arguments on the topologically ordered set of vertices.

Example 4.1.8. We give some examples of inadmissible WKB constructions, and spell out why we want to exclude them. The reason has to do with our algorithm for associating Stokes factors to Stokes curves, in the non-abelianization construction of Chapter 5.

Figure 4.2 depicts what is called a saddle trajectory in the literature on quadratic differentials; two critical trajectories with opposite orientations overlap. (Readers may be more familiar with the depiction in Figure 4.3, which is obtained by pushing forward the WKB construction via $\pi : \tilde{X}_b \to X$.) This is not admissible due to the $R \cap J = \emptyset$ condition. Stokes factors associated to the primary Stokes curves starting at $r \in R$ will eventually be used to cancel out the monodromy of local systems around the branch point $\pi(r)$, as demonstrated in Example 1.1.1. If we allowed Stokes curves to run into r, we would introduce new factors in equations such as 1.1.10. Dealing with these extra factors requires, at the very least, generalizations of our results on Stokes factors in Section 3.2.

Figure 4.4 shows an oriented cycle: a Stokes curve is about to enter a joint at which one of its ancestors was created. The methods of Section 3.2 allow us to assign Stokes

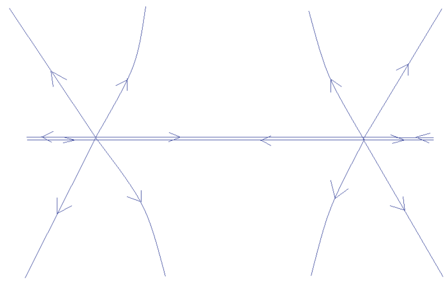

Figure 4.2: Saddle trajectory, cameral view.

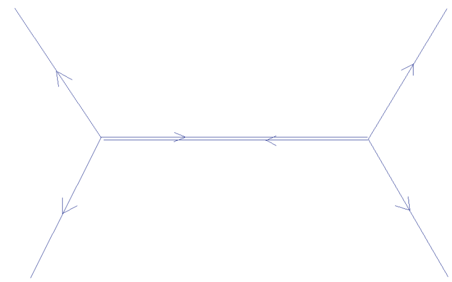

Figure 4.3: Saddle trajectory, spectral view.

factors to outgoing curves, based on the factors for incoming curves. Applying this in the presence of oriented cycles would lead to circular reasoning.

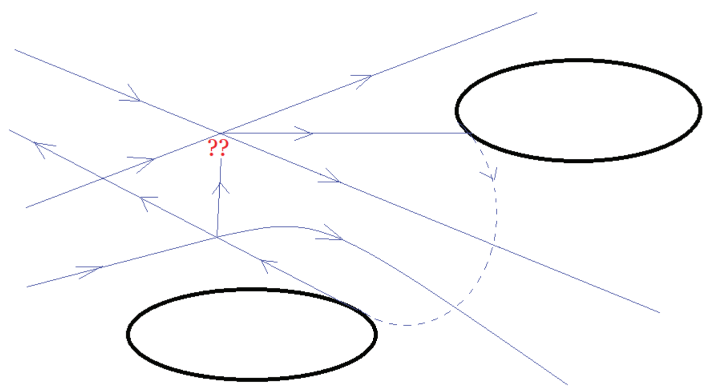

Figure 4.4: An oriented cycle (labels of curves not shown).

Example 4.1.9. The "finite webs" that [16] consider also break the admissibility condition. In Figures 4.5 - 4.6, we show finite webs for $\mathfrak{g} = \mathfrak{sl}_3$, where α, β are simple roots (see the root system A2 in Figure 3.2), and $\gamma = -\alpha - \beta$.

The webs are not admissible because there are Stokes curves running into ramification points, and because each double line creates an oriented cycle.

Definition 4.1.10. Let $\tilde{\mathcal{W}}_b$ be an admissible WKB construction. We call $\tilde{\mathcal{W}}_b$ a **WKB**

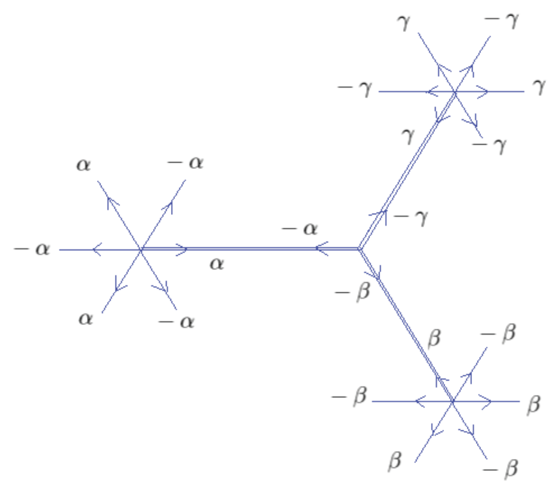

Figure 4.5: A finite web for $\mathfrak{sl}_3$; $\alpha + \beta + \gamma = 0$.

cameral network if there exists a subset $X' \subset X \setminus D$ such that:

- the inclusion map $X' \hookrightarrow X \setminus D$ is a homotopy equivalence;
- with $\tilde{X}'_b = \pi^{-1}(X') \subset \tilde{X}_b$, the restriction of $\tilde{\mathcal{W}}_b$ to $\tilde{X}'_b$ consists of finitely many Stokes curves.

In other words, WKB cameral networks are WKB constructions which have finitely many curves away from a neighborhood of the divisor at infinity D. We expect that generic $b \in \mathcal{B}^{\diamondsuit}_R(X, G, K_X(D))$ produce WKB cameral networks; we present some evidence for this assertion in Section 4.2. However, it would not be reasonable to expect finitely many Stokes curves for generic b, without first restricting away from a neighborhood of D, as in Definition 4.1.10; we will explain why in Example 4.1.14. In order to state this example, we need to introduce some results about the local structure of trajectories of quadratic differentials around poles. All results about quadratic differentials are classical, and we learned about them from the work of Strebel in [35]. Our only contribution is

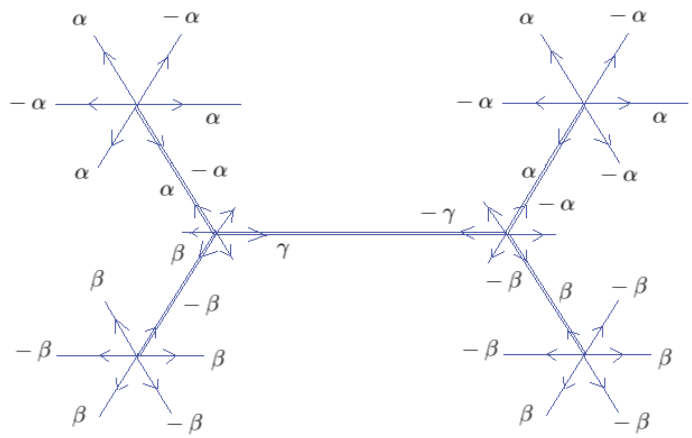

Figure 4.6: A more complicated finite web for $\mathfrak{sl}_3$; $\alpha + \beta + \gamma = 0$.

deducing the implications of these results for WKB cameral networks.

Lemma 4.1.11 (Section 6.3 in [35])**.** *Let $\mathcal{L} = K_X(D)$, and $\omega \in \Gamma\big(X, \mathcal{L}^{\otimes 2}\big)$ a meromorphic quadratic differential. For every $d \in D$, there exists a local coordinate t centered at d such that:*

$$\omega(t) = c_d \frac{dt \otimes dt}{t^2}. \tag{4.1.11}$$

Moreover, $c_d \in \mathbb{C}^\times$ is independent of the choice of coordinate t.

Proof. Expand ω in a power series around $d \in D$:

$$\omega(z) = \left(\sum_{i=-2}^{\infty} a_i z^i \right) dz \otimes dz. \tag{4.1.12}$$

Strebel, in Theorem 6.3 of [35], defines a holomorphic function $t = t(z)$, as an explicit power series in the coefficients a_i with non-zero radius of convergence, such that 4.1.11 holds. We have $c_d = a_{-2}$ – this coefficient is independent on the choice of coordinate, because it is determined by a contour integral of $\sqrt{\omega}$ around d. ∎

Definition 4.1.12. In the setting of Lemma 4.1.11, the **residue** of ω at $d \in D$ is[2]:

$$\mathrm{Res}_d(\omega) = \pm\sqrt{c_d}. \tag{4.1.13}$$

Lemma 4.1.13 (Theorem 7.2 in [35]; see also Figure 9 in [8])**.** *Let $d \in D$ be a pole of order 2 of ω. Then there exists a neighborhood of d where the trajectories of ω are the images, under a conformal mapping, of those in Figure 4.7, based on whether the residue* $\mathrm{Res}_d(\omega)$ *is real, imaginary, or generic.*

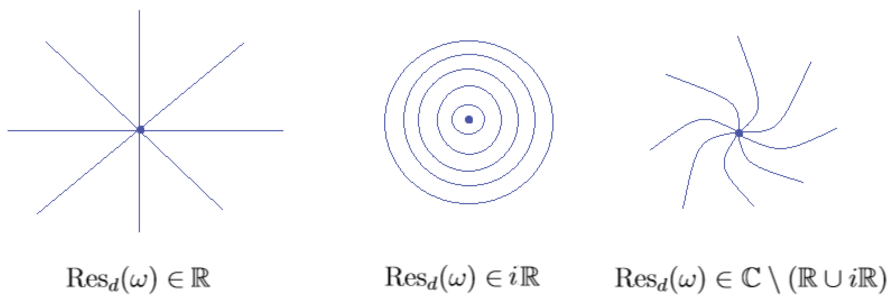

Figure 4.7: The trajectories of ω in a neighborhood of a double pole, based on the value of the residue.

In particular, if $\mathrm{Res}_d(\omega) \notin i\mathbb{R}$*, then the pole is an attractor for all trajectories which pass close enough to it.*

Proof. Working with the t coordinate from Lemma 4.1.11, a square root of the quadratic differential is:

$$\pm\sqrt{a_d}\frac{dt}{t}. \tag{4.1.14}$$

The trajectories $\gamma(s)$ are determined, up to reparametrization, by the equation:

$$\sqrt{a_d}\frac{d\gamma(s)}{\gamma(s)} \in \mathbb{R}^*, \tag{4.1.15}$$

[2]Our convention differs from that of [8] by a factor of $4\pi i$.

which has the solution:

$$\gamma(s) = \gamma(0) \exp\left(\mathbb{R}^* \sqrt{a_d} s\right). \tag{4.1.16}$$

If $\sqrt{a_d} \in \mathbb{R}$, then $\gamma(s)$ are radial rays. If $\sqrt{a_d} \in i\mathbb{R}$, then $\gamma(s)$ are circles centered at d. In the generic case $\sqrt{a_d} \notin \mathbb{R} \cup i\mathbb{R}$, $\gamma(s)$ are logarithmic spirals.

The transformation $z \mapsto t(z)$ is conformal. So, in any other coordinate, the trajectories are conformal images of the ones in Figures 4.7. $\square$

Example 4.1.14. Let $\tilde{\mathcal{W}}_b$ be a WKB construction, and consider two Stokes curves ℓ_α, ℓ_β labeled by $\alpha, \beta \in \Phi$, in a neighborhood of $d \in D$. Modulo orientation, they are the preimages in $\tilde{X}_b$ of trajectories of $\omega_{b,\alpha}$, $\omega_{b,\beta}$, respectively. So, assuming that the residues of $\omega_{b,\alpha}$, $\omega_{b,\beta}$ at d are generic and different, ℓ_α, ℓ_β are logarithmic spirals with different slopes[3], and one end converging to the pole. Assume that the orientation of ℓ_α, ℓ_β is towards the pole. Then ℓ_α, ℓ_β intersect infinitely many times as they spiral towards the pole. If it happens that $\alpha + \beta \in \Phi$, then each intersection spawns a new Stokes curve labeled by $\alpha + \beta$; the situation is depicted in Figure 4.8. Gaiotto, Moore and Neitzke noticed this accumulation of joints at a pole; see figure 17 in their paper [16].

In the remainder of this section, we give some details about the structure of WKB cameral networks locally around a ramification point or a joint on $\tilde{X}_b$.

Lemma 4.1.15. *Assume that $\tilde{\mathcal{W}}_b$ is a WKB cameral network, and let $r \in \tilde{X}_b$ be a ramification point of the covering map $\pi : \tilde{X}_b \to X$. Then there are six Stokes curves starting from r. Moreover, there exists $\alpha \in \Phi$, uniquely determined up to sign, such that the lines are labeled by α and $-\alpha$, in alternating fashion, as shown in Figure 4.9.*

[3] By "slope" of a logarithmic spiral we mean the phase of $\sqrt{a_d}$ from equation 4.1.16.

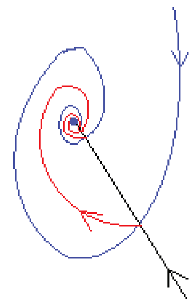

Figure 4.8: Two Stokes curves labeled by α, β (blue, black, respectively) spawn infinitely many curves labeled by $\alpha + \beta$ (red).

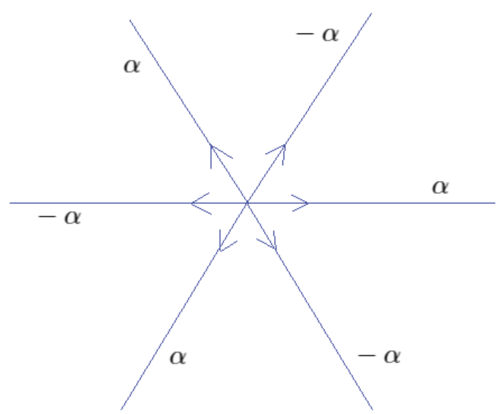

Figure 4.9: Stokes curves near a ramification point.

Proof. Due to admissibility, the only Stokes curves incident to r are the primary curves produced there. Up to orientation, they are the inverse images, by $\pi_\alpha : \tilde{X}_b \to \tilde{X}_{b,\alpha}$, of critical trajectories of the quadratic differential $\omega_{b,\alpha}$, for some $\alpha \in \Phi$. Specifically, we want the critical trajectories of $\omega_{b,\alpha}$ starting from a simple zero at $\pi_\alpha(r)$; this means $\tilde{b}(r) \in H_\alpha$. For $b \in \mathcal{B}^\diamondsuit(X, G, \mathcal{L})$, there is a unique hyperplane on which $\tilde{b}(r)$ lies – this determines α up to sign.

There are three critical trajectories of $\omega_{b,\alpha}$ starting at r, because of the computation done in Example 1.2.4. Pulling back to $\tilde{X}_b$ via the 2:1 covering map π_α, we obtain six Stokes curves, oriented as in Figure 1.7. Notice that only three of them are oriented out of r, thus giving Stokes curves labeled by α. The other three are oriented out of r when regarded as trajectories of $\omega_{b,-\alpha}$, so they are labeled by $-\alpha$. $\square$

Definition 4.1.16. Let $x \in J$ be a joint of a WKB cameral network $\tilde{\mathcal{W}}_b$. We say that $\tilde{\mathcal{W}}_b$ is **convexity-preserving at x** if the following condition holds. For each triple of outgoing Stokes curves labeled by $\alpha, \alpha+\beta, \beta \in \Phi$, denote by $v_\alpha, v_{\alpha+\beta}, v_\beta \in T_x\tilde{X}_b$, their tangent vectors at x, well-defined up to scaling by $\mathbb{R}_+$. Then $v_{\alpha+\beta}$ is contained in the cone spanned by v_α, v_β. See Figure 4.10.

More formally, the condition is that there exist $c_\alpha, c_\beta \in \mathbb{R}_+$ such that $v_{\alpha+\beta} = c_\alpha v_\alpha + c_\beta v_\beta$. The fullfillment of this condition clearly does not depend on rescaling the vectors by $\mathbb{R}_+$.

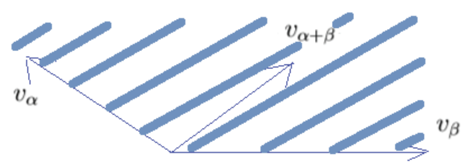

Figure 4.10: The tangent vector $v_{\alpha+\beta}$ is contained in the cone (shaded) spanned by v_α, v_β.

Lemma 4.1.17. *WKB cameral networks are convexity-preserving at every joint $x \in J$.*

Proof. For each $\gamma \in \{\alpha, \alpha+\beta, \beta\}$, saying that v_γ is tangent to a Stokes curve labeled by γ means that:

$$\gamma\big(\tilde{b}_x(v_\gamma)\big) \in \mathbb{R}_+, \tag{4.1.17}$$

with $\tilde{b}$ as in diagram 4.1.5, and $\tilde{b}_x : T_x\tilde{X}_b \to \mathfrak{t}$ its evaluation at x. Since the result of this lemma is independent of rescaling v_γ by $\mathbb{R}_+$, we may as well assume that:

$$\gamma\big(\tilde{b}_x(v_\gamma)\big) = 1. \tag{4.1.18}$$

Moreover, since $T_x\tilde{X}_b$ is 1-dimensional over $\mathbb{C}$, there exist $\eta_\alpha, \eta_\beta \in \mathbb{C}^*$ such that:

$$v_\alpha = \eta_\alpha v_{\alpha+\beta},$$

$$v_\beta = \eta_\beta v_{\alpha+\beta}.$$

Then:

$$\begin{aligned}
1 &= (\alpha+\beta)\big(\tilde{a}_x(v_{\alpha+\beta})\big)\\
&= \alpha\big(\tilde{a}_x(v_{\alpha+\beta})\big) + \beta\big(\tilde{a}_x(v_{\alpha+\beta})\big)\\
&= \eta_\alpha^{-1}\alpha\big(\tilde{a}_x(v_\alpha)\big) + \eta_\beta^{-1}\beta\big(\tilde{a}_x(v_\beta)\big)\\
&= \eta_\alpha^{-1} + \eta_\beta^{-1}.
\end{aligned}$$

Consequently, $\bar{\eta}_\alpha^{-1} + \bar{\eta}_\beta^{-1} = 1$. Then, using the fact that $\bar{\eta}\eta = |\eta|^2$, for all $\eta \in \mathbb{C}$:

$$\begin{aligned}
\frac{1}{|\eta_\alpha|^2}v_\alpha + \frac{1}{|\eta_\alpha|^2}v_\beta &= \frac{1}{|\eta_\alpha|^2}\eta_\alpha v_{\alpha+\beta} + \frac{1}{|\eta_\alpha|^2}\eta_\beta v_{\alpha+\beta}\\
&= (\bar{\eta}_\alpha^{-1} + \bar{\eta}_\beta^{-1})v_{\alpha+\beta}\\
&= v_{\alpha+\beta}.
\end{aligned}$$

So we can take $c_\alpha = 1/|\eta_\alpha|^2$ and $c_\beta = 1/|\eta_\beta|^2$. □

Remark 4.1.18. An immediate corollary of Lemma 4.1.17 is that, if the outgoing Stokes curves at some joint $x \in J$ ar labeled by a planar subset $C_{out} \subset \Phi$, then the ordering of these curves respects the ordering of the roots in the plane. For example, let α, β be the short and long root of $\mathfrak{g}_2$, respectively. Figure 4.11 depicts a possible pattern of outgoing Stokes curves, labeled by $C_{out} = \mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta}$; compare to the G_2 root system from Figure 3.4.

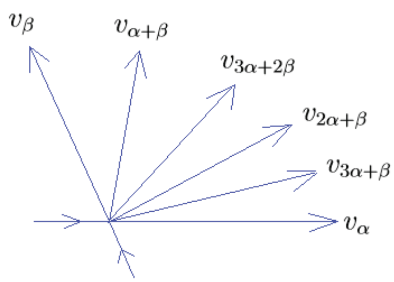

Figure 4.11: Ordering of outgoing Stokes curves respects the ordering of the planar set of labels.

When listing admissibility requirements in Definition 4.1.6, we included a convexity property of the set $C_{in} \subset \Phi$ of labels of incoming Stokes curves at every joint. This convexity assumption is needed for the results about Stokes factors from section 3.2, such as Theorem 3.2.21. We believe that the convexity property follows from the acyclicity property, which is another item on the list of admissibility requirements. But to prove this, we seem to need the following fact about root systems.

Conjecture 4.1.19. *Assume that $\{\gamma_i\}_{i=0}^k \subset \Phi_+$ is a subset of roots, positive for a choice of polarization Φ_+. Assume that they satisfy a linear relation:*

$$n_0\gamma_0 = \sum_{i=1}^{k} n_i\gamma_i, \tag{4.1.19}$$

with all $n_i \in \mathbb{N}$. Finally, assume that this relation is minimal, in the sense that, if there is a tuple $(m_0, \dots, m_k) \in \mathbb{N}^{k+1}$, with $m_i \leq n_i$ for all i, and:

$$m_0\gamma_0 = \sum_{i=1}^{k} m_i\gamma_i, \tag{4.1.20}$$

then $(m_0, \ldots, m_k) = (n_0, \ldots, n_k)$.

Then at least one of the coefficients in $\{n_i\}_{i=0}^k$ *is equal to 1.*

We can prove this conjecture in the following cases:

1. For root systems of type A, the proof of Lemma 3.2.6 shows that, in a minimal linear relation between positive roots, all coefficients are 1.

2. For the planar root systems B_2 and G_2, the result is obvious from Figures 3.3–3.4.

Lemma 4.1.20. *Let* $\tilde{\mathcal{W}}_b$ *be a WKB cameral network for a Lie algebra* $\mathfrak{g}$ *whose root system satisfies Conjecture 4.1.19. Let* $x \in J$ *be a joint. Then the set* C_{in} *of labels of the incoming Stokes curves at* x *is a convex subset of* Φ*.*

Proof. Assume that C_{in} is not convex, and let $C'_{in} \subset C_{in}$ be the smallest subset which is not convex. Then:

- For all $\gamma \in C'_{in}$, the set $C'_{in} \setminus \gamma$ is convex.

- By Definition 3.2.1, for all $\gamma \in C'_{in}$, $-\gamma \in \mathrm{Conv}_{C'_{in} \setminus \{\gamma\}}$.

We claim that there exists $\gamma_0 \in C'_{in}$, such that $-\gamma_0 \in \mathrm{Conv}^{\mathbb{N}}_{C'_{in} \setminus \{\gamma_0\}}$. As a consequence of the claim, there is an outgoing Stokes curve labeled by $-\gamma_0$ at the joint x. Its underlying curve, and the underlying curve of the incoming Stokes curve labeled by γ_0, are the same, but they have opposite orientations. This "double wall" creates an oriented cycle in $\tilde{\mathcal{W}}_b$, contradicting admissibility.

To prove the claim, choose any $\gamma_0 \in C'_{in}$, and let $\{\gamma_1, \ldots, \gamma_n\}$ be an ennumeration of $C'_{in} \setminus \{\gamma_0\}$. Then:

$$-\gamma_0 = \sum_{i=1}^{k} c_i \gamma_i, \; c_i \in [0, \infty). \tag{4.1.21}$$

In fact, we can take $c_i \in \mathbb{Q}$. This is because each γ_i is an integral linear combination of simple roots, so the c_i are a solution to a system of linear equations with integral coefficients. Then, multiplying 4.1.21 by the lowest common multiple of all denominators, we obtain a linear relation with integral coefficients:

$$n_0(-\gamma_0) = \sum_{i=1}^{k} n_i \gamma_i,\ n_i \in \mathbb{N}. \tag{4.1.22}$$

Moreover, by the convexity of $C'_{in} \setminus \{\gamma_0\}$, there is a polarization of the root system, such that $-\gamma_0$ and all other γ_i are positive.

Applying the result of Conjecture 4.1.19, and decreasing the coefficients in 4.1.22 if the relation is not already minimal, we obtain that one of the n_i is equal to 1. If $n_0 = 1$, then $-\gamma_0 \in \mathrm{Conv}^{\mathbb{N}}_{C'_{in} \setminus \{\gamma_0\}}$, and we are done. If one of the other n_j is 1, then rewrite the linear relation as:

$$-\gamma_j = \sum_{i \neq j} n_i \gamma_i, \tag{4.1.23}$$

and swap the indices of γ_j and γ_0. $\square$

Remark 4.1.21. In Section 9 of their paper [16], Gaiotto, Moore and Neitzke observe that certain applications, such as non-abelianization, make use of the topological structure of spectral networks, without relying on their analytic structure, given by the differential equations that prescribe the trajectories. They introduce "generic spectral networks" in [16], as objects with just as much structure as non-abelianization requires.

Motivated by this point of view, in [23] we introduced what we think is a minimal list of axioms for Stokes curves, in order to make non-abelianization possible. We called the resulting topological objects "abstract cameral networks".

4.2 How generic are WKB cameral networks?

We begin this section by reviewing some facts about the global structure of trajectories, following the book [35] by Strebel and the paper [8] by Bridgeland and Smith. Then we will use these facts to comment on the genericity of admissible WKB constructions and WKB cameral networks – see Conjecture 4.2.10 and Proposition 4.2.13 below.

Definition 4.2.1 (Section 3.4 in [8])**.** Let ω be a meromoprhic quadratic differential on a compact Riemann surface.

- The **finite critical points** of ω, denoted by $\mathrm{Crit}_{<\infty}(\omega)$, are the zeros and poles of order 1 of ω. [4]
- The **infinite critical points** of ω, denoted by $\mathrm{Crit}_{\infty}(\omega)$, are the poles of order ≥ 2 of ω.

We call a trajectory of ω:

1. a **saddle trajectory** if it approaches finite critical points at both ends (see Figure 4.3);
2. a **separating trajectory** if it approaches a finite critical point at one end, and an infinite critical point at the other;
3. a **closed trajectory** if it is a closed curve;
4. a **recurrent trajectory** if it is dense at every point in its closure;

[4] This definition is motivated by the fact that trajectory segments which run into a finite critical point have finite length, measured in the metric determined by ω. See section 5.3 in [35].

5. a **generic trajectory** if it approaches infinite critical points at both ends.

Proposition 4.2.2 (Sections 9-11 in [35])**.** *Let ω be a meromorphic quadratic differential on a compact Riemann surface. Then every trajectory of ω is of one of the 5 types listed above.*

Proposition 4.2.2 implies that, if a trajectory is dense at a regular point of ω, then it is recurrent, i.e. dense at every point in its closure. In this case, it must be a space-filling curve in some open subset of X.

Example 4.2.3. Consider the meromorphic quadratic differential on $\mathbb{P}^1$:

$$\omega(x) = \frac{x(x-4)}{(x-1)^2(x-2)^2(x-3)^2} dx \otimes dx. \tag{4.2.1}$$

It has simple zeros at $0, 4$ and double poles at $1, 2, 3$. Away from $\infty \in \mathbb{P}^1$, the trajectory structure of ω is sketched in Figure 4.12.

The solid curves are separating trajectories; there are three of them for each simple zero. They subdivide $\mathbb{P}^1$ into 3 regions, each of which is filled by generic trajectories having both ends on a double pole.

Example 4.2.4 (§12.1 in [35]). Let X be the compact torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Consider holomorphic quadratic differentials on X, $\omega(x) = f(x)dx \otimes dx$. Then $f(x)$ is a holomorphic function on the compact torus, hence equal to a constant c. Up to scaling by $\mathbb{R}^*$, tangent vectors to the trajectories are $c^{-1/2}\partial_x$. We consider two cases:

1. If $c \in \mathbb{R}_+$, then the trajectories are horizontal on $\mathbb{C}$, hence closed curves on the torus.

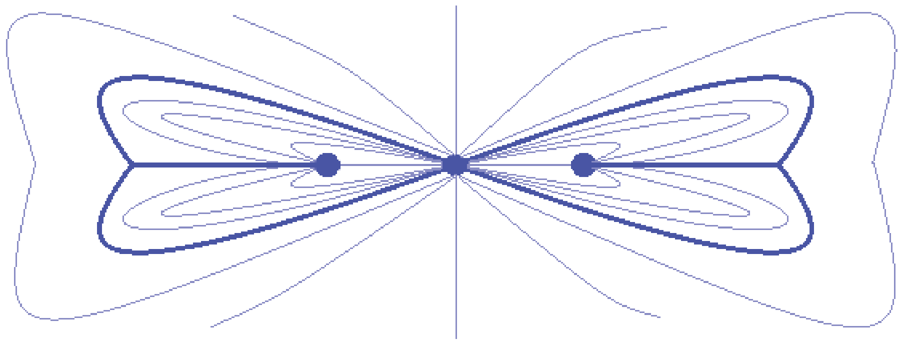

Figure 4.12: Separating trajectories (solid) and generic trajectories (transparent), for the quadratic differential from equation 4.2.1. The three blue circles are the double poles.

2. If $\Re(c^{-1/2})/\Im(c^{-1/2})$ is irrational, then the trajectories are space-filling curves, dense at every point of the torus.

Definition 4.2.5. We say that a quadratic differential is **saddle-free** if it has no saddle trajectories. We say that $b \in \mathcal{B}_R^{\diamondsuit}(X, G, \mathcal{L})$ is **saddle-free** if for all $\alpha \in \Phi$, the quadratic differential $\omega_{b,\alpha}$ induced by b on $\tilde{X}_{b,\alpha}$ is saddle-free.

The next results show that, under mild hypotheses on the number and type of critical points, saddle-free quadratic differentials form a dense open subset in the space of quadratic differentials; moreover, all their trajectories are either separating (finitely many) or generic (all others), as was the case in Example 4.2.3.

Proposition 4.2.6 (Sections 9-11 of [35]; Lemma 3.1 in [8]; Section 6.3 in [17])**.** *Let ω be a saddle-free meromorphic quadratic differential, with at least one finite and one infinite critical point. Then ω has no recurrent or closed trajectories.*

Proof. Strebel proves that closed trajectories never come alone, but in families whose union is open; he calls these ring domains (Theorem 9.4 in [35]). For example, the case

$\mathrm{Res}_d(\omega) \in i\mathbb{R}$ from Figure 4.7 depicts a ring domain. Similarly, recurrent trajectories span an open set called a spiral domain. Strebel then proves that the boundaries of both ring domains and spiral domains necessarily include a saddle trajectory, except for two situations (Corollary 2 to Theorem 11.2 in loc. cit.):

- A ring domain which contains all but finitely many points (e.g. a compact torus, or $\mathbb{P}^1 \setminus \{0, \infty\}$ can be foliated by circles). This is forbidden by the assumption that ω has at least one finite critical point.

- A spiral domain which is dense on the Riemann surface. This is forbidden by the existence of an infinite critical point. Indeed, due to Proposition 4.1.13, poles of order 2 have a neighborhood in which no trajectory can be recurrent. Strebel proves in Theorem 7.4 of [35] that an analogous statement holds for all poles of order ≥ 2.

□

Definition 4.2.7 (§2.5 in [8])**.** For $g, n \in \mathbb{N}$ such that $2g-2+n > 0$ let $\mathrm{Quad}(g, n)$ denote the moduli stack (orbifold) of meromorphic quadratic differentials ω on a Riemann surface of genus g with n punctures, such that:

- ω has a pole of order 2 at each puncture;

- ω has only simple zeros.

More precisely, let $\mathcal{M}_{g,n}$ denote the moduli stack of Riemann surfaces of genus g and n punctures, and let $\mathcal{H}_{g,n} \to \mathcal{M}_{g,n}/S_n$ denote the vector bundle whose fiber over $(X, \{p_1, \ldots, p_n\})$ is:

$$H^0\big(X, K_X^{\otimes 2}(2p_1 + \cdots + 2p_n)\big). \tag{4.2.2}$$

Then $\mathrm{Quad}(g,n)$ is the Zariski open substack of $\mathcal{H}_{g,n}$, consisting of sections with simple zeros that avoid the p_i.

Proposition 4.2.8 (Lemma 4.11 in [7]; see Theorem 1.4 in [2] for a statement involving higher-order poles). *The subset $U \subset \mathrm{Quad}(g,n)$ consisting of saddle-free quadratic differentials is open and dense in the classical topology. Moreover, the intersection of U with every orbit of S^1, acting on quadratic differentials as rescaling by $e^{i\theta}$, is dense in the orbit.*

Consequently, we can make the following statement about points in the Hitchin base.

Proposition 4.2.9. *The subset of saddle-free points in $\mathcal{B}_R^{\diamondsuit}(X,G,\mathcal{L})$ is open and dense in the classical topology.*

Proof. Diagram 4.1.5, provides, for each $\alpha \in \Phi$, a map:

$$\begin{aligned} \mathcal{B}_R^{\diamondsuit}\big(X,G,K_X(D)\big) &\xrightarrow{\omega_\alpha} \mathrm{Quad}(g_\alpha,n), \\ b &\longmapsto \omega_{b,\alpha}, \end{aligned} \tag{4.2.3}$$

where n is the degree of $p_\alpha^*(D)$, and g_α is the genus of $\tilde{X}_{b,\alpha}$; both numbers are constant as b varies in $\mathcal{B}_R^{\diamondsuit}(X,G,\mathcal{L})$. (Note that, using Propositions 2.3.6 and 2.3.7, $\tilde{X}_b$ is smooth and connected; then the same holds for $\tilde{X}_{b,\alpha}$.)

According to Proposition 4.2.8, saddle-free differentials in $\mathrm{Quad}(g_\alpha,n)$ form an open subset U_α. Then saddle-free points of $\mathcal{B}_R^{\diamondsuit}(X,G,\mathcal{L})$ also form the open subset:

$$U := \bigcap_{\alpha\in\Phi} \omega_\alpha^{-1}(U_\alpha). \tag{4.2.4}$$

It remains to prove that U is dense. Clearly the image of ω_α is a disjoint union of S^1 orbits: under $b \mapsto e^{i\theta}b$, we have $\omega_{b,\alpha} \mapsto e^{2i\theta}\omega_{b,\alpha}$. By Proposition 4.2.8, U_α is dense in each S^1 orbit. Therefore $\omega_\alpha^{-1}(U_\alpha)$ is dense in $\mathcal{B}_R^\diamondsuit(X,G,\mathcal{L})$. Taking the intersection over all $\alpha \in \Phi$, U is also dense. $\square$

Conjecture 4.2.10. *The subset of $b \in \mathcal{B}_R^\diamondsuit(X,G,\mathcal{L})$ such that $\tilde{\mathcal{W}}_b$ is admissible is open and dense in the classical topology.*

Proposition 4.2.9 is a partial result in the direction of this conjecture. For a full proof, we would need to argue that the following phenomena are also non-generic:

- oriented cycles such as the one in Figure 4.4;
- a Stokes curve ℓ running into a ramification point $r \in R$, with the label of ℓ not necessarily equal to the root α such that $\tilde{b}(r) \in H_\alpha$;
- joints accumulating at some point in $\tilde{X}_b \setminus \pi^{-1}(D)$.

Such arguments necessarily involve dealing with $\omega_{b,\alpha}$ for multiple α at the same time; we do not carry them out.

However, being saddle-free is enough to guarantee the following desirable property for the WKB construction.

Proposition 4.2.11. *If $b \in \mathcal{B}_0^\diamondsuit(X,G,\mathcal{L})$ is saddle-free, then the limit set of each Stokes curve ℓ in $\tilde{\mathcal{W}}_b$ consists of only two points:*

- *the finite critical point or joint where ℓ is created;*
- *the infinite critical point where ℓ ends.*

Proof. Each Stokes curve is a subset of a (not necessarily critical) trajectory of $\omega_{b,\alpha}$, for some α. By assumption, each $\omega_{b,\alpha}$ is saddle-free. Then Proposition 4.2.6 guarantees that all trajectories are either separating or generic. In particular, they all end at an infinite critical point. □

Putting together Propositions 4.2.9 and 4.2.11, we obtain:

Corollary 4.2.12. *There is a dense open subset of the Hitchin base, consisting of points b such that for all Stokes curves ℓ in the associated WKB construction $\tilde{\mathcal{W}}_b$, the limit set of ℓ consists of two critical points, one initial and one final.*

Finally, we prove that, under the admissibility hypotheses made in Definition 4.1.6, WKB constructions have finitely many Stokes curves away from a neighborhood of $\pi^{-1}(D)$.

Proposition 4.2.13. *Assume that $\tilde{\mathcal{W}}_b$ is admissible. Then $\tilde{\mathcal{W}}_b$ is a WKB cameral network.*

Proof. We must prove that the restriction of $\tilde{\mathcal{W}}_b$ away from a small neighborhood of $\pi^{-1}(D)$ contains finitely many Stokes curves. The strategy is to find a contractible neighborhood U_d of each $d \in D$, such that the (possibly infinitely many, as in Example 4.1.14) Stokes curves produced in U_d never leave U_d.

For admissible WKB constructions $\tilde{\mathcal{W}}_b$, and for all $\alpha \in \Phi$, the quadratic differentials $\omega_{b,\alpha}$ are saddle-free. From Proposition 4.2.6, $\omega_{b,\alpha}$ have no closed trajectories. Then, due to Lemma 4.1.13, we have $\mathrm{Res}_d(\omega_{b,\alpha}) \notin \mathbb{R}$ for all $d \in D$. Then there exists a contractible neighborhood $U_{\alpha,d}$ of d such that the trajectories of $\omega_{b,\alpha}$ passing through $U_{\alpha,d}$ have one end at d, and cross the boundary of $U_{\alpha,d}$ only once.

Define $U_d = \cap_{\alpha\in\Phi}\pi_\alpha^*(U_{d,\alpha}) \subset \tilde{X}_b$. The Stokes curves passing through U_d are inverse images by some π_α of the trajectories discussed in the previous paragraph. Therefore, they have one end at d, and pass through the boundary of U_d at most once. We want to prove that all of them are oriented towards d, so that they never exit U_d.

This is clear for primary Stokes curves – by definition, they are created at a ramification point of π, outside U_d, so they must end at d. The argument for secondary Stokes curves is by induction on the topologically-ordered set of joints (see Remark 4.1.7); its validity relies on the acyclicity condition for admissible WKB constructions. So let $\{x_n\}_{n\in\mathbb{N}}$ be a topologically-ordered ennumeration of J. The primary Stokes curves provide the base case for the induction. Assume, then, that all Stokes curves created at $\{x_1, \ldots, x_{n-1}\}$, for some $n \in \mathbb{N}$, are oriented towards d. We want to prove the same for x_n. Due to the topological order and the inductive hypothesis, the incoming Stokes curves at x_n are oriented towards d. Lemma 4.1.17 then guarantees that tangent vectors to the outgoing curves are contained in the real cone spanned by tangent vectors to the incoming curves. Therefore, the outgoing curves at x_n must also be oriented towards d.

It remains to prove that there cannot be infinitely many secondary Stokes curves produced outside of the opens U_d. Note that the subset:

$$\tilde{X}_b' := \tilde{X}_b \setminus \cup_{d\in D} U_d \tag{4.2.5}$$

is compact; as such, if there are infinitely many joints inside it, they must have an accumulation point in $\tilde{X}_b'$. Our hypothesis explicitly forbids this, so there must be finitely many joints inside $\tilde{X}_b'$. Since the number of secondary Stokes curves produced at a joint is bounded above by $|\Phi|$, we conclude that there are finitely many secondary Stokes curves produced inside $\tilde{X}_b'$.

The subsets U_d are contractible; by shrinking them if necessary, we can assume they are disjoint. Then the inclusion $\tilde{X}_b' \hookrightarrow \tilde{X}_b \setminus \pi^{-1}(D)$ induces a homotopy equivalence. $\square$

4.3 Equivariance and spectral networks

Ultimately, to perform non-abelianization we need networks on X, and not on $\tilde{X}_b$. In this section we describe the passage from WKB cameral networks, which are objects on $\tilde{X}_b$, to spectral networks, which are objects on X.

Lemma 4.3.1. *WKB cameral networks are W-equivariant, in the following sense. For every $w \in W$, and Stokes curve $\ell \subset \tilde{X}_b$ labeled by $\alpha \in \Phi$, the subset $w(\ell) \subset \tilde{X}_b$[5] is a Stokes curve labeled by $w(\alpha)$.*

Proof. The action of W on $\mathfrak{t}^* \ni \alpha$, is dual to that on $\mathfrak{t}$. Then, for every $w \in W$ and $r \in R$, $\tilde{b}\big(w(r)\big) \in H_{w(\pm\alpha)} \times_{\mathbb{C}^*} K_X(D)$. Therefore, the primary Stokes curves starting at $w(r)$ are labeled by $w(\alpha)$.

The W-equivariance of $\tilde{b}$ implies that:

$$\chi_{w\alpha} = (w(\alpha) \otimes 1) \circ \tilde{b} = (\alpha \otimes 1) \circ w\tilde{b} = w^* \chi_\alpha. \tag{4.3.1}$$

For the oriented trajectories, this means $w(\ell_\alpha) = \ell_{w(\alpha)}$.

Whenever ℓ_1, ℓ_2 intersect at $x \in J$, then $w(\ell_1), w(\ell_2)$ intersect at $w(x)$. The restricted convex hulls satisfy $\mathrm{Conv}^{\mathbb{N}}_{\alpha_{w(\ell_1),w(\ell_2)}} = w\big(\mathrm{Conv}^{\mathbb{N}}_{\alpha_{\ell_1,\ell_2}}\big)$. Therefore, the production of secondary Stokes curves is W-equivariant. $\square$

Definition 4.3.2. Let $\tilde{\mathcal{W}}_b$ be a WKB cameral network. The associated **WKB spectral network** $\mathcal{W}_b$ is the following union of oriented, labeled curves on X:

[5] When we write $w(\ell)$, we are using the action of the Weyl group on the cameral cover $\tilde{X}_b$.

- As oriented curves, $\mathcal{W}_b = \pi_* \tilde{\mathcal{W}}_b$.

- The label of $\ell \in \mathcal{W}_b$ is a locally constant section ψ_ℓ of $\mathrm{Hom}_W(\tilde{X}_b|_\ell, \Phi)$. It maps any preimage $\tilde{\ell}$ of ℓ to the root which labels $\tilde{\ell}$.

Remark 4.3.3. WKB cameral networks have six Stokes curves are created from each ramification point $r \in R$ (Lemma 4.1.15). The restriction of $\pi : \tilde{X}_b \to X$ to a small neighborhood of r has degree 2, so the W-equivariance proved in Lemma 4.3.1 ensures that the corresponding spectral network has three Stokes curves outgoing from each branch point. This agrees with the constructions of spectral networks in [16], [27].

To analyze the relationship between the labeling of our spectral networks and the labeling of spectral networks in [16], [27], we introduce the following definition.

Definition 4.3.4. Let $x \in X \setminus P$. A **W-framing** of the cameral cover $\pi : \tilde{X}_b \to X$ at x is the data of a local trivialization $\phi_x : \pi^{-1}(x) \cong W$.

Remark 4.3.5. A W-framing ϕ_x at any $x \in \ell$, for $\ell \in \mathcal{W}$, gives a mapping:

$$\begin{gathered} \mathrm{Hom}_W(\tilde{X}_b|_\ell, \Phi) \to \Phi, \\ \psi_\ell \mapsto \psi_\ell\big(\phi_x^{-1}(1_W)\big). \end{gathered} \tag{4.3.2}$$

Thus, each choice of trivialization gives a way to label curves in the spectral network by elements of Φ. If x is chosen close to a branch point p, then we can parallel transport a trivialization ϕ_x along a loop around p, to obtain a labeling of the 3 outgoing Stokes curves at p by $\alpha, -\alpha, \alpha$, for some $\alpha \in \Phi$. Note, however, that this process is necessarily discontinuous, because the cover has monodromy around p.

The spectral cover used by the earlier works [16], [27] is trivialized away from a system of branch cuts, and curves in the spectral network are labeled by roots of $\mathfrak{g}$. The map in equation 4.3.2 gives the relationship between our labels and their labels. This identifies our spectral networks with those of loc. cit.

In the remainder of this section, we present examples of spectral networks.

Example 4.3.6. If $G = SL(2,\mathbb{C})$, then the Hitchin base $\mathcal{B}\big(X, SL(2,\mathbb{C}), K_X(D)\big)$ is the set of meromorphic quadratic differentials with poles of order at most 2 at D (Example 2.2.6). Then the spectral network consists only of critical trajectories of the quadratic differential, three for each branch point. They are oriented away from the branch point.

Let $X = \mathbb{P}^1$, $D = 4 \cdot \infty$, and $\omega \in \Gamma\big(X, (K_X(D))^{\otimes 2}\big)$ be:

$$\omega(x) = (x+1)x(x-1)dx \otimes dx. \tag{4.3.3}$$

The resulting spectral network is shown in Figure 4.13.

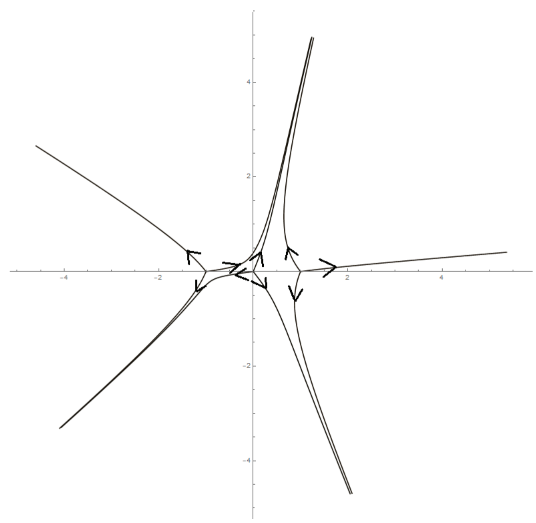

Figure 4.13: An A1 spectral network.

Example 4.3.7. Recall from Lemma 3.3.1 that, for Lie algebras of type A, given any two roots α, β, the restricted convex hull satisfies:

$$\mathrm{Conv}^{\mathbb{N}}_{\alpha,\beta} \subset \{\alpha, \alpha+\beta, \alpha\}. \tag{4.3.4}$$

See Figure 3.2. As such, there is at most one new Stokes curve generated at each joint.

Consider $G = SL_3(\mathbb{C})$, which has the root system A_2; $X = \mathbb{P}^1$; $D = \infty$. Take $b \in \mathcal{B}_R^{\diamondsuit}(X, G, \mathcal{L})$ corresponding to the characteristic polynomial:

$$f(\lambda, x) = \lambda^3 + 3\lambda + 2ix = 0. \tag{4.3.5}$$

The discriminant of f:

$$-4 \cdot 3^3 - 27(2ix)^2 \tag{4.3.6}$$

vanishes at $x = \pm 1$. These are the branch points of the associated cameral cover. We choose a pre-image r_{+1}, r_{-1} of each branch point in the cameral cover. Let $\pm\alpha, \pm\beta \in \Phi$ denote the roots such that $\tilde{b}(r_{+1}) \in H_\alpha$, $\tilde{b}(r_{-1}) \in H_\beta$. The cameral network has 6 primary Stokes curves starting at r_{+1}, labeled by $\pm\alpha$, and 6 starting at r_{-1}, labeled by $\pm\beta$. There are two pairs of curves labeled by α, β which intersect on $\tilde{X}_b$, producing curves labeled by $\alpha + \beta$; by equivariance, two analogous intersections happen for pairs $w(\alpha), w(\beta)$, for all $w \in W$.

Pushing forward the network to X, we obtain the result in Figure 4.14.

This example first appeared in [4], which was, to the best of our knowledge, the first source to mention new Stokes curves at the intersections of old ones. The context was WKB analysis of solutions to differential equations.

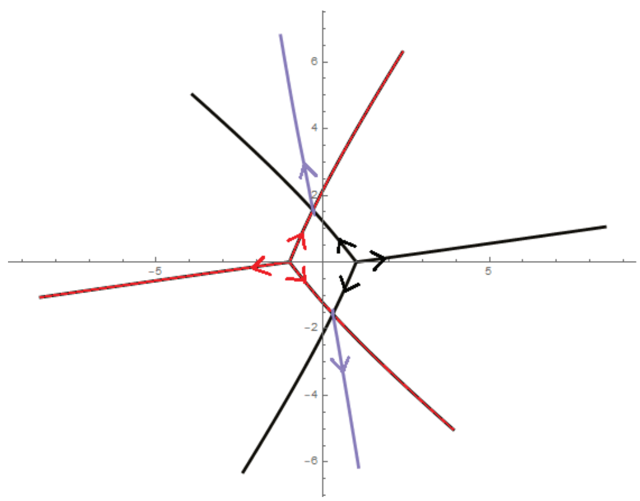

Figure 4.14: The BNR spectral network. Black is $w(\alpha)$, red is $w(\beta)$, purple is $w(\alpha+\beta)$.

Chapter 5

The non-abelianization map

In this chapter we introduce the moduli spaces involved in non-abelianization, and provide the non-abelianization construction. Rather than local systems on the closed Riemann surface X, we will consider local systems on certain oriented real blowups[1] of X. If E is a reduced, effective divisor on X, then X°_E} denotes the oriented real blowup at every point in the support of E. Specifically, in this chapter we will consider:

- $E = D$, where D is the divisor at infinity;
- $E = D + P$, where P is the branch divisor of the cover π;
- $E = D + P + J$, where J are the joints of a spectral network.

Throughout the chapter, we work with a fixed $b \in \mathcal{B}_R^{\diamondsuit}(X, G, K_X(D))$, and we suppress b from the notation for the cameral cover, which we denote by $\tilde{X}$. We will also use $\tilde{X}^{\circ_{D+R}}$, where R is the ramification divisor of π, and we omit the distinction between D and its

[1]Concretely, an "oriented real blowup at $x \in X$" means replacing x with a boundary circle S^1_x, which inherits an orientation from the orientation of X.

preimage $\pi^{-1}(D)$. (Due to Condition R, introduced in Definition 4.1.1, π is unramified over D.) The restriction $\pi^\circ : \tilde{X}^{\circ_{D+R}} \to X^{\circ_{D+P}}$ of π is unramified, hence a principal W-bundle.

In section 5.1 we introduce N-shifted, weakly W-equivariant T-local systems on $\tilde{X}^{\circ_{D+R}}$ and show their equivalence to those N-local systems on $X^{\circ_{D+P}}$ which extend the cameral cover $\tilde{X}$. In section 5.2, we introduce a restriction on the monodromy of these T- and N-local systems, which we call the S-monodromy condition. Finally, in section 5.3 we provide the non-abelianization construction, giving a proof of Theorem 1.1.5.

5.1 From T-local systems to N-local systems

In the case $G = GL(n, \mathbb{C})$, the spectral construction described in Section 2.3 works by pushing forward a line bundle $\mathcal{L}$ on the n-sheeted spectral cover $\bar{\pi} : \bar{X} \to X$, to obtain a rank n vector bundle on X. Away from the branch points of the cover, we obtain a reduction of structure of $\bar{\pi}_*(\mathcal{L})$ to a $N_{GL(n,\mathbb{C})}$-bundle, by considering automorphisms which preserve the following decomposition, for $x \in X$ not a branch point:

$$\pi_*(\mathcal{L})|_x \cong \bigoplus_{\bar{x} \in \bar{\pi}^{-1}(x)} \mathcal{L}_{\bar{x}}. \tag{5.1.1}$$

$T_{GL(n,\mathbb{C})} \cong (\mathbb{C}^*)^n$ acts by scaling on each summand, while elements in non-trivial cosets of $N_{GL(n,\mathbb{C})}$ also permute the factors.

In [12], Donagi and Gaitsgory give an analogue of this for reductive algebraic groups G, using the cameral cover $\tilde{X}$ instead of the spectral cover $\bar{X}$. In the special case of an unramified cover such as $\pi^\circ : \tilde{X}^{\circ_{D+R}} \to X^{\circ_{D+P}}$, their work relates certain N-bundles on $X^{\circ_{D+P}}$ to "N-shifted, weakly W-equivariant" T-bundles on $\tilde{X}^{\circ_{D+R}}$. In this section, we introduce these ideas in the setting of local systems, where the unramified case of [12] applies with no significant modifications. The result is Theorem 5.1.7.

In a nutshell, the strategy is to:

- Take the direct image $\pi_*^\circ(\mathcal{L})$ of a T-local system on $\tilde{X}^{\circ_{D+R}}$.

- Find conditions on $\mathcal{L}$, such that $\pi_*^\circ(\mathcal{L})$ is an N-local system.

Lemma 5.1.1. *Let $\mathcal{L} \in \mathrm{Loc}_T(\tilde{X}^{\circ_{D+R}})$. Let $\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)$ denote the sheaf of automorphisms of the direct image $\pi_*^\circ(\mathcal{L})$, seen as a sheaf on $X^{\circ_{D+P}}$ with the classical topology. Then its sections over an open set $U \subset X^{\circ_{D+P}}$ are:*

$$\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)(U) = \left\{(w,\phi)\middle| w \in W, \phi : w^*\mathcal{L}|_{(\pi^\circ)^{-1}(U)} \xrightarrow{\cong} \mathcal{L}|_{(\pi^\circ)^{-1}(U)}\right\}. \tag{5.1.2}$$

Proof. Let $\tilde{U}_i \subset (\pi^\circ)^{-1}(U)$ be a homeomorphic preimage of U. Since π° has the structure of a W-local system, for each $1 \leq i,j \leq |W|$, there is a unique $w_{ij} \in W$ such that $w_{ij}(\tilde{U}_i) = \tilde{U}_j$. An element of $\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)(U)$ is equivalent to:

- A choice of $w \in W$, such that $w^*(\mathcal{L}|_{\tilde{U}_1})$ is a sheaf over $\tilde{U}_l$, for some l.

- An isomorphism $\phi_1 : w^*\mathcal{L}|_{\tilde{U}_1} \cong \mathcal{L}|_{\tilde{U}_l}$.

Then, for values of j other than 1, $w^*(\mathcal{L}|_{\tilde{U}_j})$ is a sheaf over $ww_{1j}w^{-1}(\tilde{U}_l)$, and ϕ_1 determines an isomorphism:

$$\phi_j := (ww_{1j}w^{-1})^*(\phi_1) : w^*(\mathcal{L}|_{\tilde{U}_j}) \cong \mathcal{L}|_{ww_{1j}w^{-1}(\tilde{U}_l)}. \tag{5.1.3}$$

The set of all ϕ_l provides ϕ as in 5.1.2. □

In order for $\pi_*^\circ(\mathcal{L})$ to be an N-local system, we need an identification of $\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)$ with the constant sheaf N. Such an identification does not exist for arbitrary $\mathcal{L}$; we need to impose a constraint on $\mathcal{L}$ (weak W-equivariance) and equip it with extra data (N-shifting).

Definition 5.1.2 (cf. Definition 5.7 of [12]). We call a T-local system $\mathcal{L}$ on $\tilde{X}^{\circ D+R}$ **weakly W-equivariant** if for each U, the projection $\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big) \to W$ is surjective.

Let $\mathcal{H}om_{lc}(\tilde{X}^{\circ D+R}, T)$ denote the sheaf of locally constant morphisms $\tilde{X}^{\circ D+R} \to T$. Then $\mathcal{A}ut_{X^{\circ D+P}}(\mathcal{L})$ is part of a sequence of locally constant sheaves on $X^{\circ D+P}$, which is exact on the right if and only if $\mathcal{L}$ is weakly W-equivariant:

$$1 \longrightarrow \mathcal{H}om_{lc}(\tilde{X}^{\circ D+R}, T) \longrightarrow \mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big) \longrightarrow W \longrightarrow 1. \tag{5.1.4}$$

Defining this sequence uses the fact that T is abelian, so we can globally identify automorphisms of $\mathcal{L}$ on $\tilde{X}^{\circ D+R}$ with locally constant maps $\tilde{X}^{\circ D+R} \to T$.

Remark 5.1.3. The short exact sequence 5.1.4 splits if and only if $\mathcal{L}$ is W-equivariant. In this case, we obtain an identification of $\mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)$ with the constant sheaf $T \rtimes W$. This is not quite what we want, because N can be a non-split extension of W by T. For example, $N \cong T \rtimes W$ for the groups $GL(n,\mathbb{C})$, $SL(2n+1,\mathbb{C})$, $SO(n,\mathbb{C})$, but not for $SL(2n,\mathbb{C})$ (see, for example, the introduction to [12]).

For general G, we need the following definition.

Definition 5.1.4 (cf. §6.2 in [12]). An **N-shifted, weakly W-equivariant T-local system** on $\tilde{X}^{\circ D+R}$ is a weakly W-equivariant T-local system $\mathcal{L}$ on $\tilde{X}^{\circ D+R}$, together with a map $\gamma : N \to \mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)$ making the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\mathrm{diag}"] & N \arrow[r] \arrow[d, "\gamma"] & W \arrow[r] \arrow[d, equal] & 1 \arrow[d] \\
1 \arrow[r] & \mathcal{H}om_{lc}(\tilde{X}^{\circ D+R}, T) \arrow[r] & \mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big) \arrow[r] & W \arrow[r] & 1.
\end{tikzcd} \tag{5.1.5}$$

Definition 5.1.5. We define $\mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}})$ to be the moduli stack of N-shifted, weakly W-equivariant T-local systems on $\tilde{X}^{\circ_{D+R}}$. A more formal definition, in terms of fiber products of stacks, can be found in [23]. We do not reproduce it here, because it is not necessary for the understanding of the rest of this chapter.

We will prove that, if $\mathcal{L} \in \mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}})$, then $\pi_*^\circ(\mathcal{L})$ is an N-local system on $X^{\circ_{D+P}}$. The following definition specifies which N-local systems arise in this way.

Definition 5.1.6. Recall that $\pi^\circ : \tilde{X}^{\circ_{D+R}} \to X^{\circ_{D+P}}$ is a W-local system. An N-local system on $\tilde{X}^{\circ_{D+P}}$ which **extends** π° (or **extends the cameral cover**, when the particular cameral cover is clear from context) is an N-local system $\mathcal{E}$, together with the data of an isomorphism:

$$\mathcal{E}/T \cong \tilde{X}^{\circ_{D+R}}. \tag{5.1.6}$$

We denote by $\mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}})$ the moduli space of N-local systems extending the cameral cover. More formally, the relevant moduli space is:

$$\mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}}) := \mathrm{Loc}_N(X^{\circ_{D+P}}) \times_{\mathrm{Loc}_W(X^{\circ_{D+P}})} \{\tilde{X}^{\circ_{D+R}}\}. \tag{5.1.7}$$

Theorem 5.1.7. *Then there is an equivalence of categories between:*

1. *Weakly W-equivariant, N-shifted T-local systems $\mathcal{L}$ on $\tilde{X}^{\circ_{D+R}}$;*

2. *N-local systems which extend the cameral cover $\tilde{X}^{\circ_{D+R}}$.*

This gives an isomorphism of stacks:

$$\mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}}) \cong \mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}}). \tag{5.1.8}$$

Proof. Let $\mathcal{L}$ be a weakly W-equivariant, N-shifted T-local system on $\tilde{X}^{\circ D+R}$, and define $\mathcal{E} = \pi_*^\circ(\mathcal{L})$. Then the map $\gamma : N \to \mathrm{Aut}(\mathcal{E})$ allows us to consider $\mathcal{E}$ as an N-local system. Due to Diagram 5.1.4, there is an isomorphism of W-local systems:

$$\mathcal{E}/T \cong \tilde{X}^{\circ D+R}. \tag{5.1.9}$$

Conversely, let $\mathcal{E}$ be an N-local system on $X^{\circ D+P}$ which extends π°. Then $\mathcal{E} \to \mathcal{E}/T \cong \tilde{X}^{\circ D+R}$ is a T-local system $\mathcal{L}$ on $\tilde{X}^{\circ D+R}$, on which the T-action comes from the inclusion $T \subset N$. It is clear that $\pi_*^\circ(\mathcal{E}) \cong \mathcal{L}$; the identification $N \cong \mathrm{Aut}(\mathcal{E})$ provides the map $\gamma : N \to \mathcal{A}ut\big(\pi_*^\circ(\mathcal{L})\big)$, required by diagram 5.1.4.

The above constructions work in families, and their application to the universal families over the two stacks yields the stated isomorphism. □

Being N-shifted, weakly W-equivariant imposes non-trivial monodromy constraints on the T-local system. We give an example below, and we will elaborate on this point in the next section.

Example 5.1.8. Consider $G = SL(2)$, $X = \mathbb{P}^1_x$, $D = \{\infty\}$ and $\tilde{X} = \mathbb{P}^1_z$, with the map $\tilde{X} \to X$ given by $z \mapsto z^2$. Then $\mathrm{Loc}_N^{\tilde{X}}(X^{\circ D+P}) \cong (nT)/N$, where $n \in N$ is a representative of the T-coset $N \setminus T$, and the map takes a local system to its monodromy around 0, well-defined up to the adjoint action of N.

Now let $\mathcal{L} \in \mathrm{Loc}_T^N(\tilde{X}^{\circ D+R})$. Then the monodromy of $\mathcal{L}$ around 0 is M^2, where M is the monodromy of $\pi_*^\circ(\mathcal{L})$ around 0. By the previous paragraph, $M \in (nT)/N$; for any such element, a quick computation shows that $M^2 = -\mathrm{id}$.

Remark 5.1.9. In their version of non-abelianization in [16], Gaiotto, Moore and Neitzke notice that they cannot work with honest local systems on the spectral cover, and consider "twisted local systems" instead. The monodromy constraint for $\mathrm{Loc}_T^N(\tilde{X}^{\circ D+R})$ that we obtained in Example 5.1.8 is a manifestation of the same phenomenon: we need to work with T-local systems which have constrained, nontrivial monodromy around ramification points.

5.2 The S-monodromy condition

We could attempt to define non-abelianization as a map[2]:

$$\mathrm{Loc}_N^{\tilde{X}}(X^{\circ D+P}) \longrightarrow \mathrm{Loc}_G(X^{\circ D+P}). \tag{5.2.1}$$

However, this is not satisfactory: we would like the resulting G-local systems to extend past the branch divisor P. To ensure this, we will construct a map from an appropriate subspace of the source moduli space.

$$\begin{tikzcd} ? \arrow[r, dashed, "\mathrm{nonab}"] \arrow[d, dashed, hook] & \mathrm{Loc}_G(X^{\circ D}) \arrow[d, hook] \\ \mathrm{Loc}_N^{\tilde{X}}(X^{\circ D+P}) & \mathrm{Loc}_G(X^{\circ D+P}) \end{tikzcd} \tag{5.2.2}$$

The goal of this section is to introduce the unknown space in the diagram above.

Definition 5.2.1. We say that $\mathcal{E} \in \mathrm{Loc}_N^{\tilde{X}}(X^{\circ D+P})$ **satisfies the S monodromy condition** if, for each $p \in P$, the monodromy of $\mathcal{E}$ around S_p^1 is contained in:

$$\left(\coprod_{\alpha \in \Lambda_p} n_\alpha T_\alpha \right) /N, \tag{5.2.3}$$

where Λ_p denotes the W-orbit of roots α such that $b(p) \in H_\alpha \times_{\mathbb{C}^*} K_X(D)$.

For Definition 5.2.1 to make sense, $\coprod_{\alpha \in \Lambda_p} n_\alpha T_\alpha$ must be preserved by the adjoint action of N. This is shown in Lemma 3.1.16.

[2] See Remarks 5.3.7, 5.3.8.

Definition 5.2.2. We denote by $\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$ the moduli space of N-local systems which extend the cameral cover π° and satisfy the S-monodromy condition.

More formally, we define:

$$\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}}) := \mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}}) \times_{(\prod_{p\in P} N/N)} \prod_{p\in P} \left(\left(\coprod_{\alpha\in\Lambda_p} n_\alpha T_\alpha\right)/N\right), \tag{5.2.4}$$

where the map $\mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}}) \to \prod_{p\in P} N/N$ is restriction of local systems to the boundary circles, and Λ_p is the W-orbit of $\alpha \in \Phi$ such that $b(p) \in H_\alpha \times_{\mathbb{C}^*} K_X(D)$.

We provide a further point of view on the S-monodromy condition, which will be useful in section 5.3.

Remark 5.2.3. By Lemma 3.1.14, conjugation by T preserves the subset $n_\alpha T_\alpha$ of $n_\alpha T$, so we obtain a map:

$$n_\alpha T_\alpha/T \to n_\alpha T/T. \tag{5.2.5}$$

Let $\mathcal{E} \in \mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}})$, and let $p \in P$. In the presence of a W-framing (Definition 4.3.4) at x_p, for some x_p on the boundary circle S_p^1, the monodromy of the cameral cover $\mathcal{E}/T$ around S_p^1 is identified with $s_\alpha \in W$, for some root α. Then, using the notation of section 3.1, the monodromy of $\mathcal{E}$ around S_p^1 is identified with an element of $n_\alpha T/T$. The S-monodromy condition for N-local systems is the requirement that this element be in the image of the map 5.2.5.

Example 5.2.4. If $G = SL(2)$ or $PSL(2)$, then the homomorphism $I_\alpha : SL(2) \to G$ is surjective; in particular, $n_\alpha T_\alpha = n_\alpha T$. Therefore $\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}}) = \mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}})$.

Remark 5.2.5. For more general G, the S-monodromy condition has actual content. Recall the comparison of spectral and cameral covers for $G = GL(n,\mathbb{C})$, and in particular Figure

2.1, which compares the ramification of the two covers in the case $n = 3$. Note that, for the spectral cover, not all preimages of the branch point are ramification points. It turns out that:

- N-shifted, weakly W-equivariant T-local systems on the cameral cover induce $\mathbb{C}^*$ local systems on the spectral cover;
- the T-local system satisfies the S-mondromy condition if any only if the $\mathbb{C}^*$ local system on the spectral cover has trivial monodromy at unramified preimages of branch points.

See [23] for more details.

Remark 5.2.6. We would also like to define a S-monodromy condition for weakly W-equivariant, N-shifted T-local systems, in such a way that it corresponds to the S-monodromy condition for N-local systems under the isomorphism of Theorem 5.1.7.

Let $\mathcal{L}$ be an N-shifted, weakly W-equivariant, T-local system on $\tilde{X}^{\circ_{D+R}}$. For every $r \in R$, we can canonically identify the monodromy of $\mathcal{L}$ around the boundary circle S^1_r with an element $t_r \in T$ – this makes sense because T is commutative. Since we work with cameral covers associated to $b \in \mathcal{B}^{\diamondsuit}(X, G, K_X(D))$, there is a unique root hyperplane H_{α_r} such that $\tilde{b}(r) \in H_{\alpha_r} \times_{\mathbb{C}^*} K_X(D)$. The calculation in example 5.1.8 shows that, if $\pi_*^{\circ}(\mathcal{L}) \in \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$, then:

$$t_r = I_{\alpha_r}(-1), \tag{5.2.6}$$

where we are using the map $I_{\alpha_r} : \mathbb{C}^\times \cong T_{\alpha_r} \to T$ from diagram 3.1.8. (Note that α_r is only well-defined up to a sign, but the element $I_{\alpha_r}(-1) \in T$ does not depend on this sign.)

This determines the underlying T-bundle of $\mathcal{L}$, but not the data of the N-shift. For example, say that $G = GL(3,\mathbb{C})$, and the underlying T-bundle of $\mathcal{L}$ has monodromy:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.7}$$

around a ramification point r. Then, depending on the N-shifting data, the monodromy of $\pi_*^\circ(\mathcal{L})$ around $\pi(r)$ can be in either of the following distinct N-conjugacy classes.

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.2.8}$$

However, only the first of these is in $\coprod_{\alpha\in\Lambda_{\pi(r)}} n_\alpha T_\alpha$.

Definition 5.2.7. Consider the restriction $\mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}}) \to \prod_{r\in R} T/T$ of local systems to the boundary circles, and for all $r \in R$, choose $\alpha_r \in \Phi$ so that $\tilde{b}(r) \in H_{\alpha_r} \times_{\mathbb{C}^*} K_X(D)$. We denote by $\mathrm{Loc}_T^{N,S}(\tilde{X}^{\circ_{D+R}})$ the connected component of:

$$\mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}}) \times_{(\prod_{r\in R} T/T)} \prod_{r\in R} I_{\alpha_r}(-1)/T, \tag{5.2.9}$$

consisting of weakly W-equivariant, N-shifted T-local systems $\mathcal{L}$, such that $\pi_*^\circ(\mathcal{L}) \in \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$.

As a consequence of Theorem 5.1.7 and the definitions in this section, we obtain:

Theorem 5.2.8. *There is a commutative diagram:*

$$\begin{tikzcd} \mathrm{Loc}_T^{N,S}(\tilde{X}^{\circ_{D+R}}) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{Loc}_T^N(\tilde{X}^{\circ_{D+R}}) \arrow[d, "\cong"] \\ \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}}) \arrow[r, hook] & \mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}}) \end{tikzcd} \tag{5.2.10}$$

5.3 From N-local systems to G-local systems

The main result of this section is:

Theorem 5.3.1. *The data of a spectral network $\mathcal{W}$ determines a morphism of algebraic stacks:*

$$\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ D+P}) \xrightarrow{\text{nonab}} \mathrm{Loc}_G(X^{\circ D}). \tag{5.3.1}$$

Proof. Recall that a spectral network comes equipped with a subset $X' \subset X$, homotopy equivalent to $X^{\circ D}$, such that the restriction of the network to X' consists of finitely many Stokes curves (Definition 4.1.10). Since local systems are topological objects, there are isomorphisms $\mathrm{Loc}_G(X^{\circ D}) \cong \mathrm{Loc}_G(X')$, $\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ D+P}) \cong \mathrm{Loc}_N^{\tilde{X},S}(X'^{\circ P})$. Then it suffices to give a proof in the case $X' = X \setminus D$; otherwise we could replace $X^{\circ D}$ by X' everywhere.

The proof strategy is an extension of the reasoning used in Example 1.1.1. Concretely, we define nonab as the composition of the morphisms in the following diagram:

$$\begin{tikzcd}
\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ D+P}) \arrow[r, "S"] \arrow[drr, dashed, "\text{nonab}"'] & \mathrm{Aut}_{\mathcal{W},G}(X^{\circ D+P+J}) \arrow[r, "\text{reglue}"] & \mathrm{Loc}_G(X^{\circ D+P+J}) \\
& & \mathrm{Loc}_G(X^{\circ D}) \arrow[u, hook]
\end{tikzcd} \tag{5.3.2}$$

We now give an informal description of these stacks and morphisms, and we will spell out the details throughout the rest of the section.

- $\mathrm{Aut}_{\mathcal{W},G}(X^{\circ D+P+J})$ (Definition 5.3.3) is the moduli stack of G-local systems $\mathcal{E}_G$ on $X^{\circ D+P+J}$, equipped with the extra data of a unipotent automorphism of $\mathcal{E}|_c$, for every Stokes curve segment c. Specifically, c is a connected component of $\mathcal{W}\backslash(P \cup J)$.

- The morphism S (Construction 5.3.6) sends an N-local system $\mathcal{E}$ to the induced G-local system $\mathcal{E} \times_N G$, and equips each Stokes curve segment c with a Stokes factor

$S_c \in \mathrm{Aut}(\mathcal{E} \times_N G|_c)$.

- The morphism reglue (Definition 5.3.5) cuts a G-local system $\mathcal{E}_G$ along each Stokes curve segment c in $\mathcal{W}$, and reglues it using the automorphism S_c.

- For each joint $x \in J$, the oriented product of Stokes factors S_c for segments c incident to x is the identity (Lemma 5.3.14). Therefore, the image of reglue $\circ S$ consists of local systems with trivial monodromy around joints of the network, so reglue $\circ S$ factors through $\mathrm{Loc}_G(X^{\circ D+P})$.

- Moreover, for each branch point $p \in P$, the product of Stokes factors S_c for segments c incident to x cancels out the monodromy of $\mathcal{E} \times_N G$ around p (Lemma 5.3.11). Therefore, reglue $\circ S$ factors through the substack $\mathrm{Loc}_G(X^{\circ D})$, giving the morphism nonab.

□

Let us fill in the missing definitions and constructions.

Definition 5.3.2. Let $\mathcal{W}$ be a spectral network. A **Stokes curve segment** c is a connected component of $\mathcal{W} \setminus (P \cup J)$.

$\mathcal{W}$ has finitely many Stokes curves and joints, so there are only finitely many Stokes curve segments.

Definition 5.3.3. Informally, we define the stack $\mathrm{Aut}_{\mathcal{W},G}(X^{\circ D+P+J})$ to be the moduli stack parametrizing G-local systems $\mathcal{E}_G$ on $X^{\circ D+P+J}$, together with a section S_c of the locally constant sheaf $\mathrm{Aut}(\mathcal{E}_G)|_c$, for each Stokes curve segment c. A formal definition, using fiber products of stacks, can be found in [23].

In Construction 5.3.6, we will also make use of the following variation.

Definition 5.3.4. $\mathrm{Aut}^{\tilde{X}}_{\mathcal{W},N,G}(X^{\circ_{D+P+J}})$ is the moduli stack of N-local systems extending the cameral cover $\tilde{X}$, together with the data of an automorphism of the induced G-local system over each Stokes curve segment of $\mathcal{W}$. More formally, it is the fiber product in the diagram:

$$\begin{tikzcd} \mathrm{Aut}^{\tilde{X}}_{\mathcal{W},N,G}(X^{\circ_{D+P+J}}) \arrow[r] \arrow[d] & \mathrm{Loc}^{\tilde{X}}_{N}(X^{\circ_{D+P+J}}) \arrow[d, "\mathrm{ind}"] \\ \mathrm{Aut}_{\mathcal{W},G}(X^{\circ_{D+P+J}}) \arrow[r, "\mathrm{forget}"] & \mathrm{Loc}_{G}(X^{\circ_{D+P+J}}). \end{tikzcd} \tag{5.3.3}$$

Above, ind induces a G-local system from an N-local system, and forget maps a G-local system with automorphisms to the underlying G-local system.

Next, we prepare to define the regluing morphism. Consider the decomposition of $X^{\circ_{D+P+J}}\backslash\mathcal{W}$ into connected components:

$$X^{\circ_{D+P+J}} \setminus \mathcal{W} = \cup_{k\in\mathcal{K}} V_k. \tag{5.3.4}$$

For each $k \in \mathcal{K}$, let $\overline{V}_k$ be the closure of $V_k \subset X^{\circ_{D+P+J}}$. Let $\iota_k : \overline{V}_k \to X^{\circ_{D+P+J}}$ be the inclusions, and $\iota = \coprod_{k\in cK} \iota_k$.

Each Stokes curve segment c separates two distinct $\overline{V}_{k_1(c)}$ and $\overline{V}_{k_2(c)}$. We define $c_1 = \iota^{-1}_{k_1(c)}(c)$ and $c_2 = \iota^{-1}_{k_2(c)}(c)$. The orientations on X and c give a preferred normal direction to c; we may assume without loss of generality that this normal direction points out of $\overline{V}_{k_1(c)}$ and into $\overline{V}_{k_2(c)}$.

Definition 5.3.5. We define:

$$\mathrm{reglue} : \mathrm{Aut}_{\mathcal{W},G}(X^{\circ_{D+P+J}}) \to \mathrm{Loc}_G(X^{\circ_{D+P+J}}) \tag{5.3.5}$$

as follows. Letting $\mathcal{E}$ denote a G-local system on $X^{\circ_{D+P+J}}$, and $S_c \in \mathrm{Aut}(\mathcal{E})|_c$, for all Stokes curve segments c, we define:

$$\mathcal{E}' := (\iota^*\mathcal{E})/\left\{\iota^*_{k_1(c)}\mathcal{E}|_{c_1} \xrightarrow{S_c} \iota^*_{k_2(c)}\mathcal{E}|_{c_2}\right\}. \tag{5.3.6}$$

This gives a G-local system on $(\cup_k V_k')/\{c_1 \sim c_2\} \cong X^{\circ_{D+P+J}}$, and by the universal property this defines the map in equation 5.3.5.

Next, we construct the morphism S:

Construction 5.3.6. Recall the acyclicity assumption on WKB constructions made in Definition 4.1.6. Spectral networks satisfy the same acyclicity property; this can be proved by a simple lifting argument. As explained in Remark 4.1.7, this endows the set of joints of $\mathcal{W}$ with a total order $J = \{x_1, \dots, x_n\}$, such that for each Stokes curve segment oriented from x_i to x_j, we have $i < j$. Define, then, an increasing filtration $F_\bullet(\mathcal{W} \setminus J)$ on the set of Stokes curve segments, such that:

- $F_0(\mathcal{W} \setminus J)$ contains those Stokes curve segments starting from branch points.
- $F_j(\mathcal{W} \setminus J)$ contains those Stokes curve segments starting at ramification points or joints x_i with $i \leq j$.

We first construct a map:

$$\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P+J}}) \xrightarrow{S^0} \mathrm{Aut}^{\tilde{X}}_{F_0(\mathcal{W}\setminus J),N,G}(X^{\circ_{D+P+J}}), \tag{5.3.7}$$

meaning that we only assign automorphisms to Stokes curve segments which start from branch points.

Fix $p \in P$; then there are three Stokes curve segments c_1, c_2, c_3 starting from p (Remark 4.3.3). The monodromy of $\mathcal{E} \in \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P+J}})$ around the boundary circle S^1_p takes values in:

$$\left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right)/N, \tag{5.3.8}$$

where $\Lambda \subset \Phi$ is an orbit of the W-action on roots, and the quotient is by the adjoint action of N. We will use maps:

$$n_\alpha T_\alpha \xrightarrow{S_{\pm\alpha}} U_{\pm\alpha}, \tag{5.3.9}$$

where U_α is the 1-parameter subgroup of G obtained as the exponential of the root space $\mathfrak{u}_\alpha$, as in Section 3.2; we postpone a definition of $S_{\pm\alpha}$ until Definition 5.3.9. Lemma 5.3.11 then shows that $S_{\pm\alpha}$ satisfy an equivariance property, so they induce maps of stacks:

$$\left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right)/N \xrightarrow{S_\pm} \left(\coprod_{\alpha\in\Lambda} U_\alpha\right)/N. \tag{5.3.10}$$

The automorphisms assigned to c_1, c_2, c_3, called **Stokes factors**, are defined as the composition:

$$\begin{tikzcd}
\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P+J}}) \arrow[d, "\mathrm{Mon}_{S^1_p}"'] & & \mathrm{Aut}(\mathcal{E}\times_N G)|_{c_1 \coprod c_2 \coprod c_3} \\
\left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right)/N \arrow[r, "{(S_\pm, S_\pm, S_\pm)}"] & \left(\coprod_{\alpha\in\Lambda} U_{\pm\alpha}\right)^3/N \arrow[r] & G^3/G \arrow[u, "\cong"']
\end{tikzcd} \tag{5.3.11}$$

where the sign is chosen for each of c_1, c_2, c_3 to match the label of the Stokes curve.

Doing this for each branch point provides the desired map:

$$\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P+J}}) \xrightarrow{S^0} \mathrm{Aut}^{\tilde{X}}_{F_0(\mathcal{W}\setminus J),N,G}(X^{\circ_{D+P+J}}). \tag{5.3.12}$$

We now lift S^0 to a map S. We work inductively, by assuming we are given a map S^i as in the diagram below, such that for all $c \in F_i(\mathcal{W}\setminus J)$, the Stokes factors S_c are in U_α/T, when using a W-framing (Definition 4.3.4) such that the Stokes curve segment c is labelled by α. We show how to construct a map S^{i+1} such that the diagram commutes.

$$\begin{tikzcd} & \mathrm{Aut}^{\tilde{X}}_{F_{i+1}(\mathcal{W}\setminus J),N,G}(X^{\circ_{D+P+J}}) \arrow[d, "\text{forget}"] \\ \mathrm{Loc}^{\tilde{X},S}_N(X^{\circ_{D+P+J}}) \arrow[ur, "S^{i+1}"] \arrow[r, "S^i"'] & \mathrm{Aut}^{\tilde{X}}_{F_i(\mathcal{W}\setminus J),N,G}(X^{\circ_{D+P+J}}) \end{tikzcd} \tag{5.3.13}$$

The base step S^0 is provided in equation 5.3.12. The inductive step of obtaining S^{i+1} from S^i works by picking a W-framing at the joint x_{i+1}, and applying the N-equivariant map from Lemma 5.3.14. This provides the Stokes factors S_c, for all Stokes curve segments c outgoing from x_{i+1}, and hence the lift S^{i+1}.

Because the number of joints is finite, there is some integer N, such that $F_N(\mathcal{W}) = \mathcal{W}$. Then we define:

$$\begin{tikzcd} \mathrm{Loc}^{\tilde{X},S}_N(X^{\circ_{D+P+J}}) \arrow[r, "S^N"'] \arrow[rr, bend left, "S"] & \mathrm{Aut}^{\tilde{X}}_{\mathcal{W},N,G}(X^{\circ_{D+P+J}}) \arrow[r] & \mathrm{Aut}_{\mathcal{W},G}(X^{\circ_{D+P+J}}), \end{tikzcd} \tag{5.3.14}$$

which completes the construction.

Before proving the remaining lemmas about Stokes factors, we make some remarks about a possible extension of Theorem 5.3.1.

Remark 5.3.7. It would be desirable to define an extended morphism nonab$'$ like in the diagram below, whose restriction to $\mathrm{Loc}^{\tilde{X},S}_N(X^{\circ_{D+P}})$ is nonab.

$$\begin{tikzcd} \mathrm{Loc}^{\tilde{X},S}_N(X^{\circ_{D+P}}) \arrow[r, "\text{nonab}"] \arrow[d, hook] & \mathrm{Loc}_G(X^{\circ_D}) \arrow[d, hook] \\ \mathrm{Loc}^{\tilde{X}}_N(X^{\circ_{D+P}}) \arrow[r, dashed, "\text{nonab}'"] & \mathrm{Loc}_G(X^{\circ_{D+P}}) \end{tikzcd} \tag{5.3.15}$$

For example, this would fit well in the framework of derived symplectic geometry. Let $\mathcal{C}_D = \prod_{d\in D}[\mathcal{C}_d/G]$ denote a product of conjugacy classes of G, modulo the adjoint action of G. Let $\mathrm{Loc}_G^{\mathcal{C}_D}(X^{\circ_D}) \to \mathrm{Loc}_G(X^{\circ_D})$ denote the substack of local systems whose monodromy around each $d \in D$ is constrained to be in $[\mathcal{C}_d/G]$. Then we can write $\mathrm{Loc}_G^{\mathcal{C}_D}(X^{\circ_D})$ as a Lagrangian intersection, in the sense of [30]. This is displayed in the following diagram, where the horizontal arrows are evaluation of the monodromy around boundary circles.

$$\begin{tikzcd}
\mathrm{Loc}_G^{\mathcal{C}_D}(X^{\circ_D}) \arrow[r] \arrow[d, hook] & \prod_{d\in D}[\mathcal{C}_d/G] \times \prod_{p\in P}[1_G/G] \arrow[d, hook] \\
\mathrm{Loc}_G(X^{\circ_{D+P}}) \arrow[r] & \prod_{d\in D}[G/G] \times \prod_{p\in P}[G/G]
\end{tikzcd} \tag{5.3.16}$$

Similarly, we can write $\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$ as a Lagrangian intersection. Then we could interpret non-abelianization as a morphism between Lagrangian intersections. This opens up the possibility of using the formalism of shifted symplectic structures to prove that nonab is a symplectomorphism.

Remark 5.3.8. A naive attempt to construct nonab$'$ as in 5.3.15 fails, for the following reason. We would need to extend the maps $S_{\pm\alpha}$ from equation 5.3.9 to maps $S'_{\pm\alpha}$ fitting in the following diagram.

$$\begin{tikzcd}
n_\alpha \cdot T \arrow[dr, dashed, "S'_{\pm\alpha}"] & \\
n_\alpha \cdot T_\alpha \arrow[u, hook] \arrow[r, "S_{\pm\alpha}"'] & U_{\pm\alpha}
\end{tikzcd} \tag{5.3.17}$$

If there exists a projection $T \to T_\alpha$, we can obtain $S'_{\pm\alpha}$ as a composition of $S_{\pm\alpha}$ with the projection $n_\alpha \cdot T \to n_\alpha \cdot T_\alpha$. However, as shown in Lemma 3.1.13, the natural map $T_\alpha \times T_{H_\alpha} \to T$ sometimes has a kernel of order 2. In this case, the domain of definition of $S'_{\pm\alpha}$ cannot be $n_\alpha \cdot T$, but a double cover thereof. Consequently, the domain of definition of nonab$'$ must be a finite cover of $\mathrm{Loc}_N^{\tilde{X}}(X^{\circ_{D+P}})$. It would not be difficult to define this finite cover and the map nonab$'$, but we do not carry this out.

5.3.1 Equivariant assignment of Stokes factors

We now state and prove some lemmas used in Construction 5.3.6.

- First, in Lemma 5.3.11, we provide an equivariant map from the monodromy of an object of $\mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$ to the Stokes factors used in diagram 5.3.11.
- Then, in Lemma 5.3.14, we provide an equivariant map from the Stokes factors of incoming curves, to the Stokes factors of outgoing curves, at each joint of the network.

Fix a branch point p. Temporarily, fix also a W-framing of the W-local system $\pi^\circ : \tilde{X}^{\circ_{D+R}} \to X^{\circ_{D+P}}$ at any point $x_p \in S^1_p$ (Definition 4.3.4). Under the identification given by the framing, and due to the S-monodromy condition, the monodromy of $\mathcal{E} \in \mathrm{Loc}_N^{\tilde{X},S}(X^{\circ_{D+P}})$ around S^1_p takes values in $n_\alpha \cdot T_\alpha/T$. (See Remark 5.2.3.)

Definition 5.3.9. Let $e_{\pm\alpha} \in \mathfrak{u}_{\pm\alpha}$ be Chevalley basis elements (Definition 3.1.3). We define the morphisms:

$$\begin{aligned} n_\alpha \cdot T_\alpha &\xrightarrow{S_{\pm\alpha}} U_{\pm\alpha} \\ n_\alpha t_\alpha &\longmapsto (u_{\pm\alpha}), \end{aligned} \tag{5.3.18}$$

where:

$$u_{\pm\alpha} = \mathrm{Ad}_{t_\alpha^{-1/2}} \exp(-e_{\pm\alpha}). \tag{5.3.19}$$

The choice of square root $t_\alpha^{1/2}$ is irrelevant for the adjoint action $\mathrm{Ad}_{t_\alpha^{-1/2}}$.

Remark 5.3.10. Definition 5.3.9 is phrased in terms of a choice of Chevalley basis. Due to the T-equivariance which we will prove in Lemma 5.3.11, different choices of Chevalley basis actually produce the same map. However, this statement would be false if we allowed an arbitrary basis adapted to the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

Concretely, since root spaces are 1-dimensional, and since all scalar multiples of $e_{\pm\alpha}$ can be obtained from the adjoint action of T_α on $e_{\pm\alpha}$ (see Lemma 3.1.15), in a new basis we have:

$$e'_\alpha = \mathrm{ad}_{t_1} e_\alpha$$

$$e'_{-\alpha} = \mathrm{ad}_{t_2} e_{-\alpha}$$

for some $t_1, t_2 \in T_\alpha$. Now, if we assume that $\mathrm{ad}_{t_1} = \mathrm{ad}_{t_2}$, then the relation:

$$n_\alpha = \exp(e_\alpha) \exp(e_{-\alpha}) \exp(e_\alpha) \tag{5.3.20}$$

shows that $n'_\alpha = \mathrm{ad}_{t_1}(n_\alpha)$. Thus, T-equivariance implies that the definition of $S_{\pm\alpha}$ using the new basis:

$$\mathrm{Ad}_t\, n'_\alpha \mapsto \mathrm{Ad}_t \exp(e'_{\pm\alpha}) \tag{5.3.21}$$

agrees with the definition of $S_{\pm\alpha}$.

A Chevalley basis is constrained to satisfy:

$$[e_\alpha, e_{-\alpha}] = -h_\alpha, \tag{5.3.22}$$

where h_α is fixed. Therefore, in two distinct Chevalley bases, it must be the case that $[e_\alpha, e_{-\alpha}] = [e'_\alpha, e'_{-\alpha}]$. This forces $\mathrm{ad}_{t_1} = \mathrm{ad}_{t_2}$, justifying the assumption we had made in the previous paragraph.

Outside the setting of Chevalley bases, there would be no reason for ad_{t_1} and ad_{t_2} to be related, and different bases for $\mathfrak{g}$ would give different maps in Definition 5.3.9.

Lemma 5.3.11. *For all $t_\alpha \in T_\alpha$, and $u_{\pm\alpha}$ as in Definition 5.3.9, the following relation holds in G:*

$$n_\alpha t_\alpha u_\alpha u_{-\alpha} u_\alpha = \mathrm{id}. \tag{5.3.23}$$

Moreover, $S_{\pm\alpha}$ are equivariant with respect to the adjoint action of T. As such, they descend to morphisms of stacks:

$$n_\alpha \cdot T_\alpha / T \longrightarrow U_{\pm\alpha}/T, \tag{5.3.24}$$

where the action of T is by conjugation.

Proof. In light of Lemma 3.1.14 applied to a square root $t_\alpha^{1/2}$:

$$n_\alpha t_\alpha^{1/2} n_\alpha^{-1} = t_\alpha^{-1/2}, \tag{5.3.25}$$

the relation $n_\alpha t_\alpha = \mathrm{Ad}_{t_\alpha^{-1/2}} n_\alpha$ holds. Then we can write the maps $S_{\pm\alpha}$ in the manifestly T_α-equivariant form:

$$\mathrm{Ad}_{t_\alpha^{-1/2}} n_\alpha \mapsto \mathrm{Ad}_{t_\alpha^{-1/2}} \exp(-e_{\pm\alpha}). \tag{5.3.26}$$

Due to Lemma 3.1.14, the adjoint action of T_{H_α} on $e_{\pm\alpha}$ and n_α is trivial, so that $S_{\pm\alpha}$ are actually T-equivariant.

It remains to prove equation 5.3.23:

$$\begin{aligned}
n_\alpha t_\alpha u_\alpha u_{-\alpha} u_\alpha &= \mathrm{Ad}_{t_\alpha^{-1/2}} n_\alpha \, \mathrm{Ad}_{t_\alpha^{-1/2}} \exp(-e_\alpha) \, \mathrm{Ad}_{t_\alpha^{-1/2}} \exp(-e_{-\alpha}) \, \mathrm{Ad}_{t_\alpha^{-1/2}} \exp(-e_\alpha) \\
&= \mathrm{Ad}_{t_\alpha^{-1/2}} \Big(n_\alpha \exp(-e_\alpha) \exp(-e_{-\alpha}) \exp(-e_\alpha) \Big) \\
&= \mathrm{Ad}_{t_\alpha^{-1/2}} \left(n_\alpha n_\alpha^{-1} \right) \\
&= \mathrm{id}
\end{aligned}$$

In the calculation above, the third equality follows from Lemma 3.1.12. ∎

Next, we remove the W-framing, but work at the same branch point p. The monodromy of $\mathcal{E}$ around S^1_p canonically takes values in:

$$\left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right)/N, \tag{5.3.27}$$

where $\Lambda \subset \Phi$ is a W-orbit. In this setting, we can refine Lemma 5.3.11 as follows.

Lemma 5.3.12. *Let Λ denote an orbit of the W action on Φ. Let $S_\pm$ be the coproduct of the maps $S_{\pm\alpha}$ of Lemma 5.3.11, ranging over $\alpha \in \Lambda$:*

$$S_\pm : \left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right) \to \coprod_{\alpha\in\Lambda} U_{\pm\alpha}. \tag{5.3.28}$$

Then $S_\pm$ are N-equivariant, so they induce maps of stacks:

$$\left(\coprod_{\alpha\in\Lambda} n_\alpha T_\alpha\right)/N \to \left(\coprod_{\alpha\in\Lambda} U_{\pm\alpha}\right)/N. \tag{5.3.29}$$

Proof. In light of Lemma 3.1.14, for every $\alpha \in \Phi$ and $t_\alpha \in T_\alpha$, we have $n_\alpha t_\alpha = \mathrm{Ad}_{t_\alpha^{-1/2}} n_\alpha$, This means that we can parametrize $n_\alpha T_\alpha$ by $\mathrm{Ad}_{t_\alpha^{-1/2}} n_\alpha$, where t_α ranges over T_α. Moreover, we can parametrize $\coprod_\alpha n_\alpha T_\alpha$ by $\mathrm{Ad}_n n_\alpha$, where α is fixed and n ranges over N.

Let $[n]$ denote the image of n under the projection to W, and $\alpha' = [n](\alpha)$. Then, according to Lemma 3.1.16, there exists $t_0 \in T_{\alpha'}$ such that:

- $\mathrm{ad}_{n_0}(e_{\pm\alpha}) = \mathrm{ad}_{t_0}(e_{\pm\alpha'})$;
- $\mathrm{Ad}_{n_0}(n_\alpha) = \mathrm{Ad}_{t_0}(n_{\alpha'})$.

Then $S_\pm(\mathrm{Ad}_n n_\alpha)$ is the following composition, which is manifestly N-equivariant.

$$\begin{tikzcd}
\mathrm{Ad}_n n_\alpha \arrow[d, equal, "\text{Lem. 3.1.16}"'] & \mathrm{Ad}_n \exp(-e_\alpha) \arrow[d, equal, "\text{Lem. 3.1.16}"] \\
\mathrm{Ad}_{t_0}(n_{\alpha'}) \arrow[r, mapsto, "\text{Eq. 5.3.26}"] & \mathrm{Ad}_{t_0} \exp(-e_{\alpha'})
\end{tikzcd} \tag{5.3.30}$$

□

We move on to the equivariant map between incoming and outgoing Stokes factors at a joint x of $\mathcal{W}$. Temporarily fix a trivialization $\phi_x : \mathcal{E}_x \cong N$ of the fiber of the N-local system over the joint. Then ϕ_x determines a W-framing at x, which determines a labeling of the Stokes curves segments incident to x by roots (see Remark 4.3.5). Moreover, ϕ_x determines an isomorphism:

$$\mathrm{Aut}(\mathcal{E} \times_N G)|_x \cong G. \tag{5.3.31}$$

Due to the inductive procedure in Construction 5.3.6, we assume that we already have Stokes factors for the incoming Stokes curves at x; furthermore, we can assume that, for an incoming curve labeled by $\gamma \in \Phi$, the image of the Stokes factor under 5.3.31 is an element $u_\gamma \in U_\gamma$. This discussion, together with the convexity result in 4.1.20, proves:

Lemma 5.3.13. *Under a trivialization $\phi_x : \mathcal{E}_x \cong N$, the structure of the spectral network in the neighborhood of the joint x is locally modeled by an undecorated 2D scattering diagram.*

According to Theorem 3.2.21, the scattering diagram has a unique solution, i.e. there is a unique morphism of schemes:

$$\begin{aligned} \prod_{\gamma \in C_{in}} U_\gamma &\longrightarrow \prod_{\gamma \in C_{out}} U_\gamma \\ (u_\gamma)_{\gamma \in C_{in}} &\longmapsto (u'_\gamma)_{\gamma \in C_{out}} \end{aligned} \tag{5.3.32}$$

such that the product $u_{C_{out}}$ (equation 3.2.36) is the identity in G. We interpret the relation $u_{C_{out}} = \mathrm{id}$ as saying that the re-glued local system has no monodromy around the joint x.

Then, we define the outgoing Stokes factors as the preimages of $(u'_\gamma)_{\gamma \in C_{out}}$, under the identification 5.3.31 determined by the trivialization ϕ_x. It remains to show that this definition does not depend on ϕ_x. This follows from the next lemma.

Lemma 5.3.14. *Let Λ be an orbit of the action of W on convex, ordered subsets of Φ. For every $C_{in} \in \Lambda$, define $C_{out} := \mathrm{Conv}^{\mathbb{N}}_{C_{in}}$. Consider the coproduct, ranging over $C_{in} \in \Lambda$, of the morphisms given by Theorem 3.2.21:*

$$\coprod_{C_{in} \in \Lambda} \prod_{\gamma \in C_{in}} U_\gamma \longrightarrow \coprod_{C_{in} \in \Lambda} \prod_{\gamma \in C_{out}} U_\gamma. \tag{5.3.33}$$

It is N-equivariant with respect to the adjoint action of N, acting diagonally on the factors of the product, but permuting factors of the coproduct. As a consequence, it descends to a morphism of stacks:

$$\left(\coprod_{C_{in} \in \Lambda} \prod_{\gamma \in C_{in}} U_\gamma \right) /N \to \left(\coprod_{C_{in} \in \Lambda} \prod_{\gamma \in C_{out}} U_\gamma \right) /N. \tag{5.3.34}$$

Proof. For every $C_{in} \in \Lambda$, Theorem 3.2.21 asserts that there is a unique tuple $(u'_\gamma)_{\gamma \in C_{out}}$, such that the product $u_{C_{out}}$ from equation 3.2.36 is the identity. This takes the explicit form

$$\overrightarrow{\prod}_{\gamma \in C_{in} \coprod C_{out}} u_\gamma^{\pm 1} = \mathrm{id}, \tag{5.3.35}$$

where the product is ordered clockwise around the joint, and the exponent is -1 for incoming curves and $+1$ for outgoing curves. Applying Ad_n to this equation, and using the uniqueness of the solution, the result follows. $\square$

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