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Moduli Of Certain Wild Covers Of Curves

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Abstract

A fine moduli space (see Chapter~\ref{secn&t} Definition~\ref{finemdli}) is constructed, for cyclic-by-p covers of an affine curve over an algebraically closed field k of characteristic $\mathsf{p}>0$. An intersection (see Definition~\ref{M}) of finitely many fine moduli spaces for cyclic-by-p covers of affine curves gives a moduli space for p'-by-p covers of an affine curve. A local moduli space is also constructed, for cyclic-by-p covers of $Spec(k((x)))$, which is the same as the global moduli space for cyclic-by-p covers of $\mathbb{P}^1-\{0\}$ tamely ramified over ∞ with the same Galois group. Then it is shown that a restriction morphism (see Lemma~\ref{res mor-2}) is finite with degrees on connected components $\textsf{p}$ powers: There are finitely many deleted points (see Figure 1) of an affine curve from its smooth completion. A cyclic-by-p cover of an affine curve gives a product of local covers with the same Galois group, of the punctured infinitesimal neighbourhoods of the deleted points. So there is a restriction morphism from the global moduli space to a product of local moduli spaces.

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ABSTRACT

MODULI OF CERTAIN WILD COVERS OF CURVES

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David Harbater

A fine moduli space (see Chapter 2 Definition 28) is constructed, for cyclic-by-p covers of an affine curve over an algebraically closed field k of characteristic $\mathsf{p} > 0$. An intersection (see Definition 51) of finitely many fine moduli spaces for cyclic-by-p covers of affine curves gives a moduli space for p'-by-p covers of an affine curve. A local moduli space is also constructed, for cyclic-by-p covers of $Spec(k((x)))$, which is the same as the global moduli space for cyclic-by-p covers of $\mathbb{P}^1 - \{0\}$ tamely ramified over ∞ with the same Galois group. Then it is shown that a restriction morphism (see Lemma 82) is finite with degrees on connected components p powers: There are finitely many deleted points (see Figure 1) of an affine curve from its smooth completion. A cyclic-by-p cover of an affine curve gives a product of local covers with the same Galois group, of the punctured infinitesimal neighbourhoods of the deleted points. So there is a restriction morphism from the global moduli space to a product of local moduli spaces.

Contents

Chapter 1

Introduction

The paper mainly generalizes the results in [H80] for p-groups to cyclic-by-p groups defined in Chapter 2 Definition 16 a. See Chapter 2 for notations and terminology below. Since [H80] is frequently cited, the statements of its main results are given in the Introduction.

In [H80], it is shown that (Theorem 1.2 [H80]) when P is a finite p-group, there exists a fine moduli space for pointed principal P-covers (see Chapter 2 Definition 19 b and c for the definition) of an affine curve over an algebraically closed field k of characteristic $\mathsf{p} > 0$, which is an ind affine space (see Definition 25 b). When P' is a finite group whose order is prime to p, there are only finitely many pointed principal P'-covers of an affine curve. The *wild case*, where p divides the order of the Galois group of the cover, and the *tame case*, where p does not divide the order of the Galois group of the cover, are very different. The fine moduli space for

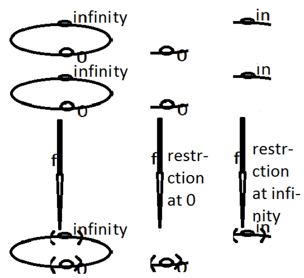


Figure 1.1: a trivial $\mathbb{Z}/2$-cover of $\mathbb{A}^1 - \{0\}$

pointed principal P-covers of $\mathbb{P}^1 - \{0\}$ gives a coarse moduli space for local pointed principal P-covers of $Spec(k((x)))$ (Proposition 2.1 [H80]). This is a special case of the next result, since the finite etale morphism there becomes an isomorphism here. Finally it is shown that a restriction morphism is finite etale (Proposition 2.7 [H80]), where the restriction morphism is described in the Abstract with the cyclic-by-p group there replaced by P here. The result can be interpreted as a local-global principle: Given a pointed P-local cover at each of the deleted points of the affine curve from its smooth completion, there are only finitely many global pointed P-covers of the affine curve, whose restrictions at the deleted points are the ones given.

In Figure 1, points 0 and ∞ are the deleted points of $\mathbb{A}^1 - \{0\}$ from its s-mooth completion $\mathbb{P}^1$. The infinitesimal neighborhood of 0 is $Spec(k((x)))$ and the infinitesimal neighborhood of ∞ is $Spec(k((x^{-1})))$. f gives a trivial $\mathbb{Z}/2$-cover of $\mathbb{A}^1 - \{0\}$. The restriction of the global cover at 0 is a trivial $\mathbb{Z}/2$-cover of the infinitesimal neighborhood. Similarly for ∞.

The fine moduli space for pointed principal P-covers of an affine curve in [H80] is constructed in an inductive way with the base case for $P = \mathbb{Z}/\mathsf{p}$.

Cyclic-by-p groups are the next simplest after p-groups in the wild case. In the local situation, the Galois group of a connected Galois cover of $Spec(k((x)))$ is a cyclic-by-p group when k is algebraically closed.

The fine moduli space for pointed principal cyclic-by-p covers of an affine curve is also constructed in an inductive way, using similar methods to those in the proof of Theorem 1.2 of [H80]. The fine moduli space is a disjoint union of finitely many ind affine spaces. Its relation to the the fine moduli space for pointed principal P-covers of an affine curve constructed in Theorem 1.2 [H80], is shown in Chapter 4 Lemma 47.

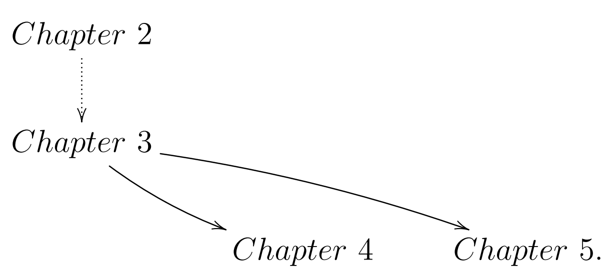
The next simplest groups after cyclic-by-p groups are p'-by-p groups defined in Chapter 2 Definition 16 a. A disjoint union of finitely many unions, of certain irreducible components in an intersection, of finitely many fine moduli spaces for cyclic-by-p covers of affine curves, gives a moduli space for p'-by-p covers of an affine curve.

Two local-global principle results similar to those in [H80] described above are obtained, based on the construction of the fine moduli space for pointed principal cyclic-by-p covers of an affine curve, again using similar methods to those in [H80].

Here is the structure of the paper.

In Chapter 2, notations and terminology are given, which are used throughout Chapters 3, 4, and 5 without explanation again. In Chapter 3, a fine moduli space for pointed principal G-covers of an affine curve (Theorem 44), where G is a cyclic-by-p group, is constructed. In Chapter 4, it is shown that a disjoint union of finitely many unions, of certain irreducible components in an intersection, of finitely many fine moduli spaces for cyclic-by-p covers of some affine curves, gives a moduli space for p'-by-p covers of an affine curve (Corollary 58). In Chapter 5, a global fine moduli space is constructed (Proposition 64) for cyclic-by-p covers of an affine curve at most tamely ramified over finitely many closed points, as well as a parameter space for local cyclic-by-p covers of $Spec(k((x)))$ (Proposition 75). Then it is shown that a restriction morphism is finite with degrees on connected components p powers, which is from the global moduli space to a product of the local parameter spaces (Proposition 83).

Leitfaden:



Similar work can be found in [K86] and [P02]. In [K86], Main Theorem 1.4.1 is essentially the version over a general field of characteristic $\mathsf{p} > 0$ of Proposition 76. In [P02], a configuration space $C(I, j)$ is constructed in 2.2, which is for

$I = \mathbb{Z}/\mathsf{p} \rtimes \mathbb{Z}/n$-covers of $Spec(k[[u^{-1}]])$ with jump j. This is related to the parameter space given in Proposition 75.

For the results in [H80] below, Gr is a finite group, P a finite p-group, k an algebraically closed field, (U_0, u_0) geometrically pointed $Spec(k((x)))$ and (U, u_g) a geometrically pointed affine curve, as defined in 2.0.2.

Let (S, s_0) be a pointed (see Definition 17 b and Definition 21 a) connected affine k-scheme. In Definition 21 b, when two pointed S-parameterized Gr-covers of U are equivalent is defined; two such covers are equivalent if they agree pulled back to a finite etale cover T of S. Since a pointed S-parameterized Gr-cover of U corresponds to a homomorphism $\tilde{\varphi} : \pi_1(S \times U, (s_0, u_g)) \to Gr$, the definition of an equivalence class of $\tilde{\varphi}$ is induced in Remark 22 a. Similarly for the local case; in Definition 66, the w-equivalence class of a pointed S-parameterized Gr-cover of $Spec(k((x)))$ is defined, which induces the definition of a w-equivalence class of a homomorphism $\tilde{\varphi} : \pi_1(S \times Spec(k((x))), (s_0, u_0)) \to Gr$.

With these terminology, the following definition can be given.

Definition 1. a. Define $F_{U,P}$ as the functor: $\mathcal{S}_1 \to$(Sets); $(S, s_0) \mapsto \{[\tilde{\varphi}]$, where $\tilde{\varphi} : \pi_1(S \times U, (s_0, u_g)) \to P$ is a group homomorphism and $[\tilde{\varphi}]$ is the equivalence class (see above or Remark 22 a) of $\tilde{\varphi}.\}$

b. Define $F^w_{U_0,P}$ as the functor: $\mathcal{S}_1 \to$(Sets); $(S, s_0) \mapsto \{[\tilde{\varphi}]^w$, where $\tilde{\varphi} : \pi_1(S \times U_0, (s_0, u_0)) \to P$ is a group homomorphism and $[\tilde{\varphi}]^w$ is the w-equivalence class (see above or Definition 66) of $\tilde{\varphi}$.}

See 2.0.5 for the definition of a fine moduli space. Theorem 2 means that $M_{U,P}$ represents the moduli functor $F_{U,P}$. The "direct limit of affine spaces" in Theorem 2 is called an ind affine space in the paper (See 2.0.4 Definition 25 b).

Theorem 2. *(Theorem 1.2, [H80]) There is a fine moduli space $M_{U,P}$ (denoted by M_G there with $G = P$) for pointed families of principal P-covers of U, namely a direct limit of affine spaces $\mathbb{A}_k^N$.*

The local case, moduli problem for P-covers of $Spec(k((x)))$, is simpler than the global case. A parallel construction to the one in the global case gives a coarse moduli space of pointed P-covers of $Spec(k((x)))$. Proposition 3 below means that $M_{\mathbb{P}^1-\{0\},P}$ represents the moduli functor $F^w_{U_0,P}$.

Proposition 3. *(Proposition 2.1, [H80]) The fine moduli space $M_{\mathbb{P}^1-\{0\},P}$ for pointed principal P-covers of $\mathbb{P}^1 - \{0\}$ is also a coarse moduli space for pointed principal P-covers of $Spec(k((x)))$, compatibly with the inclusion $Spec(k((x))) \subseteq \mathbb{P}^1 - \{0\}$.*

Proposition 4. *(Proposition 2.7, [H80]) Let $M_{U,P} \to \Pi_i M^l_{U_{0_i},P}$ be the restriction morphism described in the Abstract. It is an etale cover. Its degree is a power of p, and is equal to the number of pointed principal P-covers of the completion $\bar{U}$.*

Theorem 5. *(Theorem 1.12, [H80]) Let $M_{U,P}$ be the fine moduli space for pointed principal P-covers of (U, u_g). There is a natural action of $Aut(P)$ on $M_{U,P}$, and a dense open subset $M^0_{U,P}$ of $M_{U,P}$ parameterizing connected principal covers, such* *that* $\bar{M}^0_{U,P} := M^0_{U,P}/Aut(P)$ *is a fine moduli space for pointed families of Galois covers of* (U, u_g) *with group* P.

1.0.1 More detailed explanation

More detailed explanation and some basic facts with [S09] the main reference:

Given a finite group Gr and a curve U, how many principal Gr-covers (see Definition 19 b) of U are there? Is there a fine moduli space for these covers? That is to ask if there is a space M whose points parameterize all these covers, such that the covers vary naturally over the points of M.

First, definitions like covers are given, as well as some basic facts involved to answer the questions above.

The starting point is topological covers.

A continuous map between topological spaces $Y \xrightarrow{pr} X$ is a *topological cover*, if every point $x \in X$ has an open neighborhood U, such that $pr^{-1}(U)$ is $\coprod V_i$, a disjoint union of open subsets of Y, with each V_i homeomorphic to U under pr. Let $Gal(Y/X)$ be the group consisting of all homeomorphisms from Y to itself over X. The cover $Y \xrightarrow{pr} X$ is *Galois*, if Y is connected and $Gal(Y/X)$ acts simply transitively on $pr^{-1}(x)$ for every point $x \in X$. In this situation, $|Gal(Y/X)|$ is called the degree of the cover.

Example 6. Let S^1 be the unit circle in the complex plane given by $|z| = 1$. The map $S^1 \xrightarrow{pr} S^1$ given by $z \mapsto z^n$ is a Galois cover of degree n.

The theorem to classify topological Galois covers of a topological space X, is Theorem 7 whose statement involves some terminology given below. It is analogous to the fundamental theorem of Galois theory for finite Galois field extensions.

Let Fib_x be the functor: (Covers of X) $\to$ (Sets); $Y \mapsto Y_x$, where $Y \xrightarrow{pr} X$ is a cover and Y_x denotes the fiber $pr^{-1}(x)$.

For every point $x \in X$, the fundamental group $\pi_1(X, x)$ is defined as the group consisting of homotopy classes of loops based at x.

The group $\pi_1(X, x)$ has an action on $pr^{-1}(x)$ that is called the *monodromy action*: Under certain restrictions on X, given a point $y \in pr^{-1}(x)$, every loop α based at x has a unique lifting path in Y that starts at y. The action of the homotopy class of α sends y to the end point of the lifting path. Hence $pr^{-1}(x)$ is a left $\pi_1(X, x)$-set.

Theorem 7. *(Theorem 2.3.4 in [S09]) Let X be a connected and locally simply connected topological space. The functor Fib_x (defined under Example 6) induces an equivalence between the category (Covers of X) and the category of left $\pi_1(X, x)$-sets. Connected covers correspond to $\pi_1(X, x)$-sets with transitive actions and Galois covers to coset spaces of normal subgroups.*

Theorem 8 below is the algebraic analogue of Theorem 7.

Let k be an algebraically closed field, X a connected k-scheme, and x any geometric point of X.

A Gr-cover of X, as well as the pointed version, is defined in Definition 19 b and c; Fib_x and (Fet_X) in Definition 19 a; P is a finite p-group and (U, u_g) a geometrically pointed affine curve, as defined in 2.0.2. The definition of profinite groups can be found in Wikipedia or [S09].

Theorem 8. *(Theorem 5.4.2 in [S09])*

1. The group $\pi_1(X, x)$ is profinite, and its action on Y_x is continuous for every Y in (Fet_X).

2. The functor Fib_x induces an equivalence from (Fet_X) to the category of finite continuous left $\pi_1(X, x)$-sets. Here connected covers correspond to sets with transitive $\pi_1(X, x)$-action, and Galois covers to finite quotients of $\pi_1(X, x)$.

Corollary 9. *There is a natural bijection between the set of isomorphism classes of Gr-covers of X pointed over a fixed base point x_0, and the set of homomorphisms from $\pi_1(X, x_0)$ to Gr(, hence below a pointed Gr-cover is often identified with the homomorphism corresponding to it).*

Proof. Directly from Theorem 8. See also proof of Proposition 5.4.6 in [S09]. □

If Gr is abelian, the set of homomorphisms from $\pi_1(X, x_0)$ to Gr is a group. It may be identified with the etale cohomology group $H^1(X, Gr)$ ([SGA 1, XI 5] or [M13, Example 11.3]), or in terms of group cohomology with $H^1(\pi_1(X, x_0), Gr)$.

Now attempts begin to answer the two questions at the beginning of 1.0.1.

Proposition 10. *With the notations above Theorem 8. Let Gr be a finite group whose cardinality is prime to p. There are only finitely many pointed Gr-covers of*

(U, u_g), *up to isomorphism.*

Proof. Need to show that there are only finitely many group homomorphisms from $\pi_1(U, u_g)$ to Gr, by Corollary 9. Any homomorphism factors through $\pi_1^{(\mathsf{p}')}(U, u_g)$ since $|Gr|$ is prime to p. By Theorem 4.9.1 in [S09], $\pi_1^{(\mathsf{p}')}(U, u_g)$ is topologically finitely generated. The proposition follows. □

Hence the tame case is simple; the moduli space is a finite set with discrete topology. With the notations above Theorem 8. For the simplest wild case, how many pointed $\mathbb{Z}/\mathsf{p}$-covers are there of the pointed curve (U, u_g)? By Remark 20, base points do not matter here. By Artin-Schreier sequence (see the beginning of the proof of Theorem 32), they are all given by $z^{\mathsf{p}} - z = a$ with $a \in A$ and $U = Spec(A)$; two given by a and a' respectively are isomorphic if and only if $a - a' \in \wp(A) = \{a^p - a | a \in A\}$. Hence $\mathbb{Z}/\mathsf{p}$-covers of (U, u_g) are parameterized by $A/\wp(A)$. Theorem 1.2 in [H80] shows how to rearrange elements in $A/\wp(A)$ to build a fine moduli space for $\mathbb{Z}/\mathsf{p}$-covers of (U, u_g).

The method to rearrange elements in $A/\wp(A)$: Let $DivU$ be the divisor that is the sum of the deleted closed points of U from its smooth completion. Then elements in A have a k-vector space filtration: $H^0(U, DivU) \leq H^0(U, \mathsf{p}DivU) \leq H^0(U, \mathsf{p}^2 DivU) \leq ...$, which will give rise to the 0th piece, the 1st piece, the 2nd piece and so on, of the fine moduli space. Choose a basis L_n for $H^0(U, \mathsf{p}^n DivU)$ inductively such that $L_n \supset L_{n-1}$ and $L_n \supset (L_{n-1} - L_{n-2})^{\mathsf{p}}$ (see paragraph (32.1) for an analogue). It turns out that vectors in the k-linear span of $L_n - L_{n-1}$ correspond bijectively, to all the $\mathbb{Z}/\mathsf{p}$-covers of U given by $z^{\mathsf{p}} - z = a$ with $a \in H^0(U, \mathsf{p}^n DivU)$. Let $M_n = Spec(k[L_n^\vee - L_{n-1}^\vee])$, where $L_n^\vee$ consists of dual vectors of L_n. M_n is the n-th piece of the fine moduli space desired. The transition morphism from M_n to M_{n+1} is given by Frobenius (see the last two paragraphs of Theorem 32 for an analogue, or Remark 11 b). The fine moduli space is an ind affine space (see 2.0.4 and 2.0.5).

Remark 11. a. The inductive choice of $H^0(U, \mathsf{p}^n DivU)$'s basis L_n is crucial in the construction of the moduli space. An arbitrary choice of its basis can not construct a moduli space.

b. As described above, $M_{U,P}$ is an ind affine space, whose n-th piece is $M_n = Spec(k[L_n^\vee - L_{n-1}^\vee])$. The transition morphism $M_n \xrightarrow{f_n} M_{n+1}$ corresponds to a ring homomorphism $k[L_n^\vee - L_{n-1}^\vee] \leftarrow k[L_{n+1}^\vee - L_n^\vee] : f_n^*$. Suppose $L_n - L_{n-1} = \{l_{ni}\}$. Then $L_{n+1} - L_n = \{l_{n+1j}\} \supset \{l_{ni}^{\mathsf{p}}\}$. f_n^* sends $(l_{ni}^{\mathsf{p}})^\vee$ to $(l_{ni}^\vee)^{\mathsf{p}}$, ($(l_{ni}^{\mathsf{p}})^\vee$ is the dual of l_{ni}^{p}, an element in $k[L_{n+1}^\vee - L_n^\vee]$. $(l_{ni}^\vee)^{\mathsf{p}}$ is the pth-power of $l_{ni}^\vee$, an element in $k[L_n^\vee - L_{n-1}^\vee]$), and every $l_{n+1j}^\vee$ with l_{n+1j} not in $\{l_{ni}^{\mathsf{p}}\}$ to 0. Hence f_n can be decomposed into two parts

$$\begin{array}{ccc} M_n & \xrightarrow{f_{1n}} & \mathbb{M}_n \\ {\scriptstyle f_n}\downarrow & \swarrow{\scriptstyle \iota_n} & \\ M_{n+1}, & & \end{array}$$

where $\mathbb{M}_n = Spec(k[\{(l_{ni}^\vee)^{\mathsf{p}}\}])$ and ι_n is a closed embedding. f_{1n} is a bijection between closed points in M_n and $\mathbb{M}_n$.

c. What is the relationship between the ind affine space $M_{U,P}$ and $Spec(\varprojlim_n k[L_n^\vee - L_{n-1}^\vee])$? The projective limit over n's of $k[L_n^\vee - L_{n-1}^\vee]$'s consists of tuples with the form $(G_n(\{l_{ni}\}))_n$, where $G_n(\{l_{ni}\})$ is a polynomial in l_{ni}'s. The transition morphism given in b, would imply that any non constant entry in a tuple should have the infinite degree as a polynomial, a contradiction. Hence the projective limit consists of constants only and is the base field k. Therefore, $M_{U,P}$ is a much bigger space than the spectrum of the projective limit.

The case $P = \mathbb{Z}/\mathsf{p}$ is the base case of Theorem 2.

With the notations above Theorem 8.

See 2.0.5 for the definition of a fine moduli space. Theorem 2 means that $M_{U,P}$ represents the moduli functor $F_{U,P}$ (see Definition 1 a). The "direct limit of affine spaces" in Theorem 2 is called an ind affine space in the paper (See 2.0.4 Definition 25 b).

Theorem 12. *(Theorem 1.2, [H80]) There is a fine moduli space $M_{U,P}$ (denoted by M_G there with $G = P$) for pointed families of principal P-covers of U, namely a direct limit of affine spaces $\mathbb{A}_k^N$.*

For a general finite p-group P, how many pointed P-covers are there for (U, u_g)? Theorem 1.2 in [H80] builds a fine moduli space for these P-covers using induction based on the concrete $\mathbb{Z}/\mathsf{p}$ case: Choose an $H \approx \mathbb{Z}/\mathsf{p}$ in the center of P. Let $\bar{P} = P/H$. $M_{V,H}$ and $M_{V,\bar{P}}$ exist and are ind affine spaces, by the base case and the inductive hypothesis respectively. A $\tilde{\varphi} : \pi_1(S \times U, (s_0, u_g)) \to P$ can be decomposed into two parts $\bar{\tilde{\varphi}} : \pi_1(S \times U, (s_0, u_g)) \to \bar{P}$ and $\tilde{\eta}_0 : \pi_1(S \times U, (s_0, u_g)) \to H$. The decomposition process below would yield $M_{V,P} = M_{V,\bar{P}} \times M_{V,H}$, again an ind affine space. $\bar{\tilde{\varphi}}$ is $\tilde{\varphi}$ composed with $P \twoheadrightarrow \bar{P}$. And $\tilde{\eta}_0$ is the "quotient" of $\tilde{\varphi}$ "divided" by $\bar{\tilde{\varphi}}$. To get $\tilde{\eta}_0$, first lift a universal family representative (see Remark 34) $\tilde{\mu}_0 : \pi_1(M_{U,\bar{P}} \times U, (m, u_g)) \to \bar{P}$ on $M_{U,\bar{P}}$, to a $\tilde{\psi}_0 : \pi_1(M_{U,\bar{P}} \times U, (m, u_g)) \to P$, using the p-cohomological dimension of the fundamental group is ≤ 1 ([Serre], I Prop. 16). The group homomorphism $\bar{\tilde{\varphi}}$ determines a morphism from S to $M_{U,\bar{P}}$, since $M_{U,\bar{P}}$ is a fine moduli space, which induces from $\tilde{\psi}_0$ a lift $\tilde{\psi} : \pi_1(S \times U, (s_0, u_g)) \to P$ of $\bar{\tilde{\varphi}}$. Then $\tilde{\eta}$ can be defined as $\tilde{\varphi}\tilde{\psi}^{-1}$ (see paragraph (39.3) for an analogue).

The next simplest groups after p-groups in the wild case are cyclic-by-p groups. The method to build a fine moduli space for pointed cyclic-by-p covers of (U, u_g), is still to treat the concrete case of $H \rtimes_\rho \mathbb{Z}/n$ first, with (H, ρ) in the case of Theorem 32. In this case equations of covers can be written down using the Artin-Schreier sequence (see the beginning of the proof of Theorem 32). Then use induction to treat the general case.

The base case of pointed $H \rtimes_\rho \mathbb{Z}/n$-covers of (U, u_g): The idea is to focus on the upper p-part of a $H \rtimes_\rho \mathbb{Z}/n$-cover.

For any $H \rtimes_\rho \mathbb{Z}/n$-cover (W, w_g) of (U, u_g), take the pointed connected component (V, v_g) of the quotient $(W, w_g)/H$ and get $(V, v_g) \to (U, u_g)$ a $\mathbb{Z}/n'$-cover with $n'|n$, called the cyclic part of $(W, w_g) \to (U, u_g)$. Connected covers are easier to deal with and the *induced* $\mathbb{Z}/n$*-cover* (construction 3.5.2, p84, [S09]) of $(V, v_g) \to (U, u_g)$ is the original cover $(W, w_g)/H \to (U, u_g)$. $(W, w_g) \to (V, v_g)$ is called the p-part. Without loss of generality, assume $n' = n$ below.

Then figure out which pointed H-covers of (V, v_g) composed with $V \to U$ can give pointed $H \rtimes_\rho \mathbb{Z}/n$-covers of (U, u_g). These H-covers, denoted by $\mathbb{X}$, are "ρ-liftable pointed H-covers of (V, v_g)" defined around diagram (3.1). Identify a pointed H-cover of (V, v_g) with the group homomorphism corresponding to it $\varphi : \pi_1(V, v_g) \to H$. The pointed H-cover is ρ-liftable iff φ makes diagram (30.1) commutative, a fact used in the construction of Theorem 32.

Suppose $|H| = q$. Similarly to the case in [H80], all the H-covers of V can be given by $z^q - z = b$ with $b \in B$ and $V = Spec(B)$. The covers in $\mathbb{X}$ are given by b's in the kernel of some operator D, denoted by $KerD$ (see proof of Theorem 32). Then similarly rearrange elements in $KerD$ to build a fine moduli space (Theorem 32) for covers in $\mathbb{X}$, pointed ρ-liftable H-covers of (V, v_g). The fine moduli space is also an ind affine space.

The fine moduli space of the base case is the building block for the fine moduli space (Theorem 44) for pointed G-covers of (U, u_g) (G defined in Definition 16 c): Theorem 32$\Rightarrow$ Theorem 39$\Rightarrow$ Theorem 44. The first $\Rightarrow$ is an induction proof, similarly to the case of Theorem 2 with detailed description above. The second $\Rightarrow$ is to decompose a pointed G-cover $(W, w_g) \to (U, u_g)$ into its cyclic part and p-part, and to put the cyclic part of $(W, w_g) \to (U, u_g)$ and its p-part together. The decomposition process is given above for the base $H \rtimes_\rho \mathbb{Z}/n$ case.

Above is the global case. Below starts the local case.

The local case, moduli problem for pointed P-covers of $Spec(k((x)))$, is simpler than the global case. A parallel construction to the one in the global case gives a coarse moduli space of pointed P-covers of $Spec(k((x)))$.

Proposition 3 means that $M_{\mathbb{P}^1-\{0\},P}$ represents the moduli functor $F^w_{U_0,P}$ defined in Definition 1 b.

Proposition 13. *(Proposition 2.1, [H80]) The fine moduli space $M_{\mathbb{P}^1-\{0\},P}$ for pointed principal P-covers of $\mathbb{P}^1 - \{0\}$ is also a coarse moduli space for pointed principal P-covers of $Spec(k((x)))$, compatibly with the inclusion $Spec(k((x))) \subseteq \mathbb{P}^1 - \{0\}$.*

Combining global moduli spaces and local moduli spaces, there is a global-to-local restriction morphism.

As described in the Abstract and Figure 1, a global cover can give a product of local covers. Hence, there is a restriction morphism from the global moduli space to a product of local moduli spaces. The restriction morphism is finite etale. An ingredient in the proof is to compute the dimensions, of the n-th pieces of the source and the target of the restriction morphism, which turn out the same. Using this fact it can then be checked that the restriction morphism is finite etale.

Proposition 14. *(Proposition 2.7, [H80]) Let $M_{U,P} \to \Pi_i M^l_{U_{0_i},P}$ be the restriction* *morphism described in the Abstract. It is an etale cover. Its degree is a power of* p*, and is equal to the number of pointed principal* P*-covers of the completion* $\bar{U}$*.*

In the cyclic-by-p case, analogues of Proposition 3 (Proposition 76) and Proposition 4 (Proposition 83) also hold and are proved by similar methods.

The relationship between one of the main moduli spaces $M^{\rho\bullet}_{V,P}$ to $M_{V,P}$ (let $U{=}V$ in Theorem 2) in [H80] is given in Chapter 4. Each connected component (see Remark 26) of $M^{\rho\bullet}_{V,P}$ turns out to be "a closed subscheme" (see Remark 49) of $M_{V,P}$. This is shown by induction as well. In the base case when $P = H$ and (ρ, H) is in the case of Theorem 32, the "embedding morphism" (see Lemma 47) from $M^{\rho}_{V,H}$, a connected component of $M^{\rho\bullet}_{V,H}$, to $M_{V,H}$ can be given explicitly using the constructions of both moduli spaces (see Theorem 32 for the one of $M^{\rho}_{V,H}$; the one for $M_{V,H}$ is in the proof of Theorem 2 and similar).

An intersection of several such closed subschemes of $M_{V,P}$ can give a moduli space for p'-by-p covers of an affine curve (Corollary 58). The rest of the Introduction is devoted to Corollary 58 alone.

Given a p'-by-p group $P \rtimes_{\rho'} P'$, for each $p'_i \in P'$, a cyclic-by-p group $P \rtimes_{\rho'_i} \langle p'_i \rangle$ can be formed, with $\langle p'_i \rangle$ the subgroup generated by p'_i and $\rho'_i : \langle p'_i \rangle \to Aut(P)$ the restriction of ρ'.

Suppose $(V', v'_g) \to (U', u'_g)$ is a connected P'-cover. Let $(V'_{i'}, \overline{v'_{gi}})$ be the quotient of (V', v'_g) by $\langle p'_i \rangle$. The pointed $\langle p'_i \rangle$-cover $(V', v'_g) \to (V'_{i'}, \overline{v'_{gi}})$ is the counterpart of the pointed $\mathbb{Z}/n$-cover $(V, v_g) \to (U, u_g)$ in Theorem 39 of Chapter 3. Apply

Theorem 39 on $(V', v'_g) \to (V'_{i'}, \overline{v'_{gi}})$ and a fine moduli space $M^{\rho'_i,\bullet}_{V',P}$ for ρ'_i-liftable pairs of (V', v'_g) is got.

For every p'_i denote by $\{M^{\rho'_i}_{V',P,ij}\}$ the set of finitely many connected components of $M^{\rho'_i,\bullet}_{V',P}$. Denote by $(M^{\rho'_i}_{V',P,ij})_i$ a tuple of connected components indexed by i, an element in $\Pi_i\{M^{\rho'_i}_{V',P,ij}\}$. For each tuple $(M^{\rho'_i}_{V',P,ij})_i$ do their intersection in $M_{V',P}$, an intersection of closed subschemes. Then take the disjoint union of intersections belonging to different tuples intersecting furthermore the dense open $M^0_{V,P}$ of $M_{V,P}$. $M^0_{V,P}$ parameterizes connected covers.

Theorem 15. *(Theorem 1.12, [H80]) Let $M_{U,P}$ be the fine moduli space for pointed principal P-covers of (U, u_g). There is a natural action of $Aut(P)$ on $M_{U,P}$, and a dense open subset $M^0_{U,P}$ of $M_{U,P}$ parameterizing connected principal covers, such that $\bar{M}^0_{U,P} := M^0_{U,P}/Aut(P)$ is a fine moduli space for pointed families of Galois covers of (U, u_g) with group P.*

The disjoint union is denoted by $M^{0\rho'}_{V',P}$ (see Definition 51). Its closed points parameterize all connected pointed $P \rtimes_{\rho''_t} P'$-covers of (U, u_g) that factor through (V', v'_g) (see Proposition 57) with $\{P \rtimes_{\rho''_t} P'\}$ a finite set of groups related to $P \rtimes_{\rho'} P'$ (see Remark 53). Getting more groups than $P \rtimes_{\rho'} P'$ is because a connected pointed P-cover of (V', v'_g) that is ρ'_i-liftable for every i, does not necessarily have $P \rtimes_{\rho'} P'$ as its Galois group over U (see Remark 53). The case of disconnected covers is even more complicated and hence excluded here.

Take the union of all the finitely many irreducible components of $M^{0\rho'}_{V',P}$ to whose closed points $P \rtimes_{\rho'} P'$ belong to (see the end of the proof of Proposition 57). In other words, pick the irreducible components of $M^{0\rho'}_{V',P}$ that parameterize $P \rtimes_{\rho'} P'$-covers of (U, u_g) factoring through (V', v'_g), but leave out the rest irreducible components parameterizing $P \rtimes_{\rho''_t} P'$-covers for other $P \rtimes_{\rho''_t} P'$'s. A disjoint union taken over all possible (V', v'_g)'s of such unions of certain irreducible components of $M^{0\rho'}_{V',P}$, is a fine moduli space for pointed $P \rtimes_{\rho'} P'$-covers of (U, u_g) (see Corollary 58). An ER-equivalence is introduced to make the intersection argument work smoothly. Otherwise more technical work would be needed.

Chapter 2

Notations and Terminology

Terms and symbols are defined here that will be used in Chapters 3, 4, and 5 without being explained again.

2.0.2 General settings

Definition 16. a. Groups of the form $P \rtimes_\rho \mathbb{Z}/n$ are called *cyclic-by-*p *groups*, where p is a prime number, P a finite p-group and $\rho : \mathbb{Z}/n \to Aut(P)$ an action of $\mathbb{Z}/n$ on P with n and p coprime. Groups of the form $P \rtimes_{\rho'} P'$ are called p'*-by-*p *groups*, where P' is a finite group whose order is prime to p and $\rho' : P' \to Aut(P)$ an action of P' on P.

b. Let n_t be a factor of n, $x_t = n/n_t$, ι_{n_t} the embedding of $\mathbb{Z}/n_t$ into $\mathbb{Z}/n$ sending $\bar{1} \in \mathbb{Z}/n_t$ to $\bar{x}_t \in \mathbb{Z}/n$ and $\rho_{n_t} = \rho \circ \iota_{n_t}$.

c. Gr always represents an arbitrary finite group and G represents $P \rtimes_\rho \mathbb{Z}/n$.

Definition 17. a. Let k be an algebraically closed field of characteristic $\mathsf{p} > 0$ and fix a primitive n-th root of unity ζ_n in k. Write $U_0 = Spec(k((x)))$ and $\bar{U}_0 = Spec(k[[x]])$. Denote the fiber product $S \times_k X$ by $S \times X$, where S and X are k-schemes.

b. *Pointed* means geometrically pointed unless otherwise stated. A geometric point of a scheme X is a morphism from $Spec(\Omega)$ to X with Ω an algebraically closed field. *Curve* means a connected smooth integral affine 1 dimensional scheme of finite type over k.

c. Denote by $\mathcal{S}$ (resp. $\mathcal{S}_1$) the full subcategory of the category (Pointed k-schemes) of all pointed k-schemes, whose objects are connected affine pointed finite type k-schemes (resp. connected affine pointed k-schemes). Denote by $\mathcal{S}'$ (resp. $\mathcal{S}'_1$) the non pointed version of $\mathcal{S}$ (resp. $\mathcal{S}_1$).

d. (U, u_g) always represents a pointed curve.

Remark 18. The word "lift" has two meanings in the paper. The first meaning is to extend a group homomorphism whose domain is a fundamental group, to a group homomorphism with domain a bigger fundamental group. This meaning is given around diagram (3.1) in the definition of ρ-liftable. The second meaning is to lift a morphism $\bar{\phi}$ mapping to a quotient group $\bar{P}$, to some morphism ϕ mapping to the original group P:

$$\begin{array}{ccc} & & P \\ & \overset{\phi}{\nearrow} & \downarrow \\ \pi_1(U, u_g) & \xrightarrow[\bar{\phi}]{} & \bar{P}. \end{array}$$

2.0.3 Covers

Definition 19. a. Let X be a connected scheme and (Fet_X) the category of finite etale covers of X. For every point x of X (recall from Definition 17 b that a point on X always means a geometric point), the *fiber functor* Fib_x: $(Fet_X)\to$(Sets) sends a finite etale cover $Y \stackrel{pr}{\to} X$ to Y_x, the geometric fiber (pullback of Y to x) of Y at x. The fundamental group $\pi_1(X,x)$ is defined as the automorphism group of the fiber functor Fib_x. Then Y_x is a left $\pi_1(X,x)$-set.

b. A *principal Gr-cover* of a connected scheme X not necessarily over k, is a finite etale cover $Z \to X$ together with an embedding of Gr in the group $Aut(Z/X)$, such that Gr acts simply transitively (left group action) on every geometric fiber of $Z \to X$. A *Gr-cover* means a principal Gr-cover.

c. With the notations of b, Z is pointed over x_0 means Z is pointed at some point z_0 that maps to x_0 under $Z \to X$. Two pointed Gr-covers of (X, x_0) are isomorphic if there is an isomorphism between them:

$$\begin{array}{ccc} (Z, z_0) & \xrightarrow{f\simeq} & (Z', z'_0) \\ & \searrow \quad \swarrow & \\ & (X, x_0), & \end{array}$$

such that the triangle diagram commutes and the diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & Z' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{f} & Z' \end{array}$$

commutes for each $g \in Gr$.

Remark 20. a. There is a natural bijection between the set of isomorphism classes of Gr-covers of X pointed over a fixed base point x_0, and the set of homomorphisms from $\pi_1(X, x_0)$ to Gr, hence below a pointed Gr-cover is often identified with the homomorphism corresponding to it.

b. If Gr is abelian, the set of homomorphisms from $\pi_1(X, x_0)$ to Gr is a group. It may be identified with the etale cohomology group $H^1(X, Gr)$ (SGA 1, XI 5, or Example 11.3 in [M13]), or in terms of group cohomology with $H^1(\pi_1(X, x_0), Gr)$.

c. Let Gr be an abelian group and $W \to U$ be a Gr-cover of U. Then W pointed at a point w_g over u_g is isomorphic to, as pointed Gr-covers of (U, u_g), W pointed at any other point w'_g over u_g.

Definition 21. a. A point (s_0, v_g) on a fiber product $S \times V$ of k-schemes means a commutative diagram by the universal property of a fiber product:

$$\begin{array}{ccc} Spec(\Omega) & \xrightarrow{v_g} & V \\ {\scriptstyle s_0}\downarrow & & \downarrow \\ S & \longrightarrow & Spec(k), \end{array} \tag{2.1}$$

where Ω is some algebraically closed field.

b. A *pointed family of Gr-covers* of a pointed connected k-scheme X, parametrized by a pointed connected affine k-scheme S, means an equivalence class of pointed Gr-covers of $S \times X$, two being equivalent if they become isomorphic after being pulled back by some finite etale cover $(T, t_0) \to (S, s_0)$.

Remark 22. a. Two elements $\widetilde{\phi}$ and $\widetilde{\phi}'$ in $Hom(\pi_1(S \times U, (s_0, u_g)), Gr)$ are *equivalent* if their corresponding pointed Gr-covers of $(S \times U, (s_0, u_g))$ are equivalent. Denote the equivalence class of $\widetilde{\phi}$ by $[\widetilde{\phi}]$.

b. Using equivalence classes (see Definition 31, Definition 35, Definition 37, Definition 43), rather than isomorphism classes, a fine moduli space can be constructed. The definition of equivalence, using finite etale covers, arises naturally in the proof of Theorem 32: Assume for instance $e_\rho = 1$, then the last paragraph in the proof gives $F(S, s_0) = \frac{H^1(S \times U, \mathbb{F}_q)}{H^1(S, \mathbb{F}_q)}$. The equality holds because the definition of equivalence uses finite etale covers (see c). With the equality it can be shown that F is represented by an ind affine space.

c. If Gr is abelian, then the set of such pointed families may be identified with $H^1(S \times X, Gr)/H^1(S, Gr)$, where $H^1(S \times X, Gr)$ and $H^1(S, Gr)$ are standard etale cohomology groups. This is proved at the end of this subsection.

Definition 23. Suppose X is a connected scheme and x, x' two geometric points on X. A *chemin* $x' \to x$ means an isomorphism from the fiber functor Fib_x to the fiber functor $Fib_{x'}$ ([S09], Remark 5.5.3, p171). Since the fundamental group

$\pi_1(X, x)$ is defined as the automorphism group of the fiber functor Fib_x, a chemin $x' \to x : Fib_x \xrightarrow{i} Fib_{x'}$ induces an isomorphism $\pi_1(X, x) \xrightarrow{\simeq} \pi_1(X, x') : \alpha \mapsto i\alpha i^{-1}$.

Lemma 24. *If Gr is abelian, then the set of pointed families defined in Definition 21 b can be identified with $H^1(S \times X, Gr)/H^1(S, Gr)$, where $H^1(S \times X, Gr)$ and $H^1(S, Gr)$ are standard etale cohomology groups.*

Proof. By Example 11.3 in [M13], $H^1(X_{et}, Gr) = Hom(\pi_1(S \times X, (s_0, x)), Gr)$. Replace $H^1(S\times X, Gr)/H^1(S, Gr)$ by $Hom(\pi_1(S\times X, (s_0, x)), Gr)/Hom(\pi_1(S, s_0), Gr)$. A pointed family of Gr-covers of X parameterized by S can be identified with $[\widetilde{\phi}]$ (see Remark 22 a) with $\widetilde{\phi} : \pi_1(S \times X, (s_0, x)) \to Gr$ corresponding to a representative in the family. Denote $\{[\widetilde{\phi}] | \widetilde{\phi} : \pi_1(S \times X, (s_0, x)) \to Gr\}$ by A, and $Hom(\pi_1(S \times X, (s_0, x)), Gr)/Hom(\pi_1(S, s_0), Gr) = \{[\widetilde{\phi}]' | \widetilde{\phi} : \pi_1(S \times X, (s_0, x)) \to Gr\}$ by B, where $[\widetilde{\phi}]'$ denotes the coset containing $\widetilde{\phi}$ in the quotient group. Want to show that the two maps

$$A \to B; [\widetilde{\phi}] \mapsto [\widetilde{\phi}]' (24.1)$$

and

$$A \leftarrow B; [\widetilde{\phi}] \leftarrow [\widetilde{\phi}]' (24.2)$$

are both well defined, hence inverse to each other.

Suppose $\widetilde{\phi}_i : \pi_1(S\times X, (s_0, x))(i = 1, 2)$ are equivalent. Then by definition, there exists a connected finite etale cover $(T, t_0) \xrightarrow{f} (S, s_0)$ such that $\widetilde{\phi}_i$'s composed with

$\pi_1(T \times X, (t_0, x)) \xrightarrow{\tilde{f}_*} \pi_1(S \times X, (s_0, x))$ are the same. Define $\Delta\phi := \widetilde{\phi}_1 - \widetilde{\phi}_2$. For the map in (24.1) to be well defined, want to show $\Delta\phi \in Hom(\pi_1(S, s_0), Gr)$.

Denote by $i_{S*} : \pi_1(S, s_0) \to \pi_1(S \times X, (s_0, x))$ the homomorphism induced by $i_S : S \hookrightarrow S \times X$, and $pr_{S*} : \pi_1(S \times X, (s_0, x)) \to \pi_1(S, s_0)$ the homomorphism induced by $pr_S : S \times X \to S$. Then $pr_{S*} \circ i_{S*} = Id$. Define $\phi : \pi_1(S, s_0) \to Gr$ as $\Delta\phi \circ i_{S*}$.

$$\begin{array}{ccccc} \pi_1(T \times X, (t_0, x)) & \xrightarrow[\tilde{f}_*]{} & \pi_1(S \times X, (s_0, x)) & & \\ \downarrow{\scriptstyle pr_{T*}} & & {\scriptstyle pr_{S*}}\downarrow & \searrow{\scriptstyle \Delta\phi} & \\ \pi_1(T, t_0) & \xrightarrow{f_*} & \pi_1(S, s_0) & & \\ & & & \searrow{\scriptstyle \phi} & \\ & & & & Gr \end{array} \tag{24.3}$$

Want to show $\phi \circ pr_{S*} = \Delta\phi$, then $\Delta\phi \in Hom(\pi_1(S, s_0), Gr)$.

By taking Galois closure of T over S, assume $T \xrightarrow{f} S$ is Galois below. Then there is a short exact sequence for some θ'' after choosing an isomorphism from $Gal(T/S)$ to Gr:

$$1 \to \pi_1(T, t_0) \to \pi_1(S, s_0) \xrightarrow{\theta''} Gr \to 1.$$

Choose $s_i (1 \leq i \leq |Gr|)$ in $\pi_1(S, s_0)$ that maps to $g_i \in Gr$ under θ''. Denote by $s_{i\bullet}$ in the image of s_i under i_{S*}.

By its definition $\Delta\phi = 0$ on $\pi_1(T \times X, (t_0, x))$. Since $\pi_1(S \times X, (s_0, x))$ is generated by $\pi_1(T \times X, (t_0, x))$ and $s_{i\bullet}$'s, want to show that $\phi \circ pr_{S*} = 0$ on $\pi_1(T \times X, (t_0, x))$ and agrees with $\Delta\phi$ on $s_{i\bullet}$'s. Both can be verified using diagram (24.3).

Conversely want to show the map in (24.2) is well defined. Suppose $\widetilde{\phi}_1 - \widetilde{\phi}_2 = \phi \circ pr_{S*}$ for some $\phi \in Hom(\pi_1(S, s_0), Gr)$. Then ϕ gives a pointed Gr-cover (T, t_0) of (S, s_0) by Corollary 9. WLOG., assume T is connected. It is easy to verify that $\widetilde{\phi}_i$'s $(i = 1, 2)$ become the same pulled back to $T \times X$. □

2.0.4 Ind schemes

Definition 25. a. An *ind scheme* means, in the paper, a direct system of k-schemes $\{X_i\}$ indexed by natural numbers with transition k-morphisms $\{X_i \xrightarrow{x_i} X_{i+1}\}$.

b. An ind scheme is an *ind affine space*, if every X_i is an affine space $\mathbb{A}_k^{n_i}$.

Remark 26. Every moduli space M in the paper is a disjoint union of finitely many ind affine spaces, then each ind affine space is called *a connected component* of M. An ind affine space M can be viewed as a functor: $\mathcal{S}_1 \to$(Sets); $(S, s_0) \mapsto Hom(S, M)$. The disjoint union of finitely many ind affine spaces $\{M_i\}$ is the functor $\amalg M_i$: $\mathcal{S}_1 \to$(Sets); $(S, s_0) \mapsto \amalg_i Hom(S, M_i)$.

Definition 27. a. A *pre-morphism* from a k-scheme X to an ind scheme $\{X_m\}$ is the equivalence class of a k-morphism between schemes $g_{m_0} : X \to X_{m_0}$ for some m_0, where two morphisms g_{m_0} and g_{m_1} are equivalent if for some $m_2 \geq m_0, m_1$ the two composition morphisms $X \xrightarrow{g_{m_0}} X_{m_0} \to X_{m_2}$ and $X \xrightarrow{g_{m_1}} X_{m_1} \to X_{m_2}$ are the same.

b. A *pre-morphism* from an ind scheme $\{X_m\}$ with transition morphisms $\{x_m\}$ to another ind scheme $\{Y_m\}$ with transition morphisms $\{y_m\}$, is the equivalence class of a system of compatible k-morphisms $\{f_m | m \geq m_0\}$ between schemes with $f_m : X_m \to Y_{N_m}$. The system $\{f_m | m \geq m_0\}$ is compatible means that for every $m \geq m_0$, there exists an n_m such that the following diagram is commutative:

$$\begin{array}{ccccc} X_m & \xrightarrow{f_m} & Y_{N_m} & & \\ \downarrow{\scriptstyle x_m} & & & \searrow{\scriptstyle y_{N_m n_m}} & \\ X_{m+1} & \xrightarrow{f_{m+1}} & Y_{N_{m+1}} & \xrightarrow{y_{N_{m+1} n_m}} & Y_{n_m}, \end{array}$$

where $y_{N_m n_m}$ is the transition morphism from Y_{N_m} to Y_{n_m} and similarly for $y_{N_{m+1} n_m}$. Two compatible systems $\{f_m | m \geq m_0\}$ and $\{g_m | m \geq m_1\}$ are *equivalent*, if there exists an $m_2 \geq m_0, m_1$ such that for every $m \geq m_2$ the two morphisms f_m and g_m are equivalent in the sense of a. Every pre-morphism between ind schemes in the paper, in Lemma 47, Lemma 48, Lemma 81 and Lemma 82, can be given by a compatible system $\{f_m\}$ of the special form: $X_m \xrightarrow{f_m} Y_m$ and the following diagram commutes

$$\begin{CD} X_m @>{f_m}>> Y_m \\ @V{x_m}VV @VV{y_m}V \\ X_{m+1} @>{f_{m+1}}>> Y_{m+1}. \end{CD} \qquad (27.1)$$

c. In either a or b, a presheaf can be gotten. In b, the presheaf Pre is from the site of (ind schemes)$\times$(ind schemes) with etale topology to (sets); $(\{X_m\}, \{Y_m\}) \mapsto PreMorph(\{X_m\}, \{Y_m\})$. Let $sPre$ be the sheafification of Pre. A *morphism* between $\{X_m\}$ and $\{Y_m\}$ is an element in $sPre(\{X_m\}, \{Y_m\})$. Similarly for a. Since the construction is canonical, it suffices to check assertions on pre-morphisms.

d. In the cases of Lemma 47, Lemma 48, Lemma 81 and Lemma 82, the morphism given by $\{f_m\}$ in diagram (27.1), is *surjective* (resp. *finite, finite etale*), if there exists some natural number m_0 such that for every $m \geq m_0$ the k-morphism f_m is surjective (resp. finite, finite etale).

2.0.5 Fine moduli space

Definition 28. A *fine moduli space* M for a contravariant functor F from the category $\mathcal{S}_1$ to the category (Sets), is an ind scheme such that F is isomorphic to the functor $Hom(\bullet, M)$: $\mathcal{S}_1 \rightarrow$(Sets); $(S, s_0) \mapsto \{k$-morphisms from S to $M\}$.

Below is a list of moduli functors in the paper, with their rough meanings and places where they are defined.

List of moduli functors

$F_{U,P}$; the functor for pointed P-covers of (U, u_g) using equivalence classes; Definition 1 a

$F^w_{U_0,P}$; the functor for pointed P-covers of $(Spec(k((x))), u_0)$ using w-equivalence classes; Definition 1 b

$F^\rho_{V,H}$; the functor for pointed ρ-liftable H-covers of (V, v_g) using equivalence classes; Definition 31 and Definition 61

$F^{\rho\bullet}_{V,H}$; the functor for ρ-liftable pairs (of pointed H-covers) of (V, v_g) using equivalence classes; Definition 35

$F^{\rho\bullet}_{V,P}$; the functor for ρ-liftable pairs (of pointed P-covers) of (V, v_g) using equivalence classes; Definition 37

$F_{U,G}$; the functor for pointed G-covers of (U, u_g) using equivalence classes; Definition 43

$F^T_{U,G}$; the functor for pointed G-covers of $(U - T, u_g)$, at most tamely ramified over T consisting of finitely many closed points on U, using equivalence classes; Definition 59

$F^{\rho_{n_l}\bullet/T}_{V_l,P}$; the functor for ρ_{n_l}-liftable pairs (of pointed P-covers) of (V_l, v_l) using equivalence classes, with V_l a cover of U at most ramified over T consisting of finitely many closed points on U; Definition 60

$F^{w\rho\bullet}_{V_0,P}$; the functor for ρ-liftable pairs (of pointed P-covers) of (V_0, v_0) using w-equivalence classes; Definition 66

Comments about several subtle concepts are collected below for reference convenience.

Comments concerning ind schemes include Remark 26.

Comments concerning universal families include Remark 34, Definition 38, Remark 40, Remark 41, and Definition 50.

Comments concerning fine moduli spaces include Remark 22, Remark 33, Remark 26, Remark 41, Remark 42, Remark 49, and Definition 50.

2.0.6 Table of symbols

Below is a table of symbols, which are used in Chapters 3, 4 and 5 without explanation again after their definitions. It gives meanings of symbols and places where they are defined. "Beginning" of a section means beginning of the rest of the section below the introductory part.

Table of symbols

c; a fixed element in $\pi_1(U, u_g)$ that maps to $\bar{1}$ under θ; Chapter 3, beginning

c_i: similar to c

c'_i; a fixed element in $\pi_1(V'_i, \overline{v'_{gi}})$ that maps to p'_i under θ'_i; Chapter 4, beginning

H; an elementary abelian group of order a p-power; Lemma 29

$\{pr_i : (V_i, v_i) \to (U, u_g)\}$; the set of all connected pointed $\mathbb{Z}/n_i$-covers of (U, u_g) with n_i running over factors of n; Chapter 3, beginning

T; a finite set of closed points on U not including u_g; Chapter 5, beginning

U^0; $U - T$; Chapter 5, beginning

(U_0, u_0); pointed $Spec(k((x)))$; Chapter 5, Notation 65

(V, v_g); a fixed connected pointed $\mathbb{Z}/n$-cover of (U, u_g); Chapter 3, beginning

(V', v'_g); a fixed connected pointed P'-cover of (U, u_g); Chapter 4, beginning

$(V'_{i'}, \overline{v'_{gi}})$; quotient of (V', v'_g) by $\langle p'_i \rangle$; Chapter 4, beginning

$\{(V_l^0, v_l)\}$; the set of all connected pointed $\mathbb{Z}/n_l$-covers of (U^0, u_g) with n_l running over factors of n; Chapter 5, beginning

V_l; extension of V_l^0, by putting back in the closed points over $T \subset U$ to V_l^0, which are originally missing from V_l^0's smooth completion; Chapter 5, beginning

(V_{0t}, v_{0t}); a connected pointed $\mathbb{Z}/n_t$-cover of (U_0, u_0) with n_t a factor of n; Chapter 5, Notation 65

ρ; an action of $\mathbb{Z}/n$ on P; 2.0.2 Definition 16 a

ρ'; an action of P' on P; same as ρ above

$[\widetilde{\phi}]$; the equivalence class of $\widetilde{\phi}$; Remark 22

ρ_i'; an action of $\langle p_i' \rangle$ on P given by restriction of ρ'; Chapter 4 below Remark 45

θ; the group homomorphism $\pi_1(U, u_g) \to \mathbb{Z}/n$ corresponding to $(V, v_g) \to (U, u_g)$; Chapter 3, beginning

θ_i: similar to θ

θ'; the group homomorphism $\pi_1(U, u_g) \to P'$ corresponding to $(V', v_g') \to (U, u_g)$; Chapter 4, beginning

θ_i'; the group homomorphism $\pi_1(V_i', \overline{v_{gi}'}) \to \langle p_i' \rangle$ corresponding to $(V', v_g') \to (V_i', \overline{v_{gi}'})$; Chapter 4, beginning

Chapter 3

Existence of moduli space for cyclic-by-p covers

In Chapter 3, a fine moduli space that represents the functor $F_{U,G}$ defined above Theorem 44, for pointed G-covers of the pointed affine curve (U, u_g), where G is a cyclic-by-p group, is constructed. The construction is done in 3 steps: Theorem 32⇒Theorem 39⇒ Theorem 44. Theorem 32 is the base case of an induction, and Theorem 39 is the inductive step. Theorem 44 collects building blocks given in Theorem 39 to build the target fine moduli space.

As always, we follow notations and terminology defined in Chapter 2. For example G represents a cyclic-by-p group.

Here are necessary settings for Theorem 32.

Since $(n, \mathsf{p}) = 1$, for every factor n' of n, there are only finitely many connected pointed $\mathbb{Z}/n'$-covers of (U, u_g), up to isomorphism. See also Remark 20.

Denote these covers by $pr_i : (V_i, v_i) \to (U, u_g)$, for all n''s. For each i, $pr_i : (V_i, v_i) \to (U, u_g)$ is of some degree $n_i | n$ and corresponds to some surjective group homomorphism $\pi_1(U, u_g) \xrightarrow{\theta_i} \mathbb{Z}/n_i$; fix a $c_i \in \pi_1(U, u_g)$ that maps to $\bar{1} \in \mathbb{Z}/n_i$ under θ_i. Pick a $pr : (V, v_g) \to (U, u_g)$ that is a $\mathbb{Z}/n$-cover. Suppose it corresponds to $\pi_1(U, u_g) \xrightarrow{\theta} \mathbb{Z}/n$ with c the chosen element in $\pi_1(U, u_g)$ above. There is a short exact sequence of groups

$$1 \to \pi_1(V, v_g) \to \pi_1(U, u_g) \xrightarrow{\theta} \mathbb{Z}/n \to 1.$$

Let $Hom(\pi_1(V, v_g), P)$ be the set of group homomorphisms from $\pi_1(V, v_g)$ to P. A group homomorphism $\phi \in Hom(\pi_1(V, v_g), P)$ is *ρ-liftable*, if there exists a group homomorphism $\widehat{\phi}$ such that the diagram

$$\begin{array}{ccccc} \pi_1(V, v_g) & \xrightarrow{\phi} & P & & \\ \downarrow & & \downarrow & & \\ \pi_1(U, u_g) & \xrightarrow{\widehat{\phi}} & G & \xrightarrow{Q_P} & \mathbb{Z}/n \end{array} \tag{3.1}$$

commutes and the bottom horizontal arrow $\pi_1(U, u_g) \to \mathbb{Z}/n$ is θ, where Q_P is the projection map. We also say that *$\widehat{\phi}$ lifts ϕ*. There are two different meanings of "lift" in the paper (see Remark 18). In this situation, the pointed P-cover of (V, v_g) corresponding to ϕ is called a *pointed ρ-liftable cover of* (V, v_g). If $\widehat{\phi}(c) = (p, \bar{1})$ for some $p \in P$ then (ϕ, p) is called a *ρ-liftable pair*.

Let $(S, s_0) \in \mathcal{S}_1$. When (V, U) is replaced by $(S \times V, S \times U)$, similarly a *ρ-liftable* $\widetilde{\phi} \in Hom(\pi_1(S \times V, (s_0, v_g)), P)$ is defined; the pointed family of P-covers of V parameterized by S corresponding to $\widetilde{\phi}$ is called a *pointed ρ-liftable family*.

Denote by $c_\bullet$ the image of c under the group homomorphism $\pi_1(U, u_g) \to \pi_1(S \times U, (s_0, u_g))$ induced by $U \hookrightarrow S \times U$. Similarly a *$\rho$-liftable pair* $(\widetilde{\phi}, p)$ is defined. A *pointed ρ-liftable family pair* means a pair whose first entry is the pointed ρ-liftable family corresponding to $\widetilde{\phi}$ and the second entry p, for some $(\widetilde{\phi}, p)$ a ρ-liftable pair. The pair is also denoted by $([\widetilde{\phi}], p)$.

Below are two Lemmas for Theorem 32.

Irreducible linear representations of $\mathbb{Z}/n$ over the field $\mathbb{F}_\mathsf{p}$ correspond to the direct summands in $\mathbb{F}_\mathsf{p}[x]/(x^n - 1) = \oplus_i \mathbb{F}_\mathsf{p}[x]/(f_i(x))$, where $f_i(x)$'s are irreducible factors of $x^n - 1$ over $\mathbb{F}_\mathsf{p}$. The action of $\bar{1} \in \mathbb{Z}/n$ on $\mathbb{F}_\mathsf{p}[x]/(f_i(x))$ is multiplication by $[x]$, where $[x]$ means the equivalence class of x. Thus for a pair (H, ρ) in the case of Lemma 29, there is a group isomorphism $H \xrightarrow{\tau} \mathbb{F}_q$, where $q = \mathsf{p}^m$, such that the induced action of $\rho(-\bar{1})$ on $\mathbb{F}_q$ is the multiplication by some $e_\rho \in \mathbb{F}_q$ with $e_\rho^n = 1$.

Lemma 29. *Let $P = H = (\mathbb{Z}/\mathsf{p})^m$, an elementary abelian group, and suppose the action ρ on H is irreducible (i.e. ρ can not be an action on any subgroup of H). A* *group homomorphism $\phi \in Hom(\pi_1(V, v_g), H)$ is ρ-liftable iff for every $b \in \pi_1(V, v_g)$*

$$\phi(c^{-1}bc) = \rho(-\bar{1})(\phi(b)). \tag{$*$}$$

Moreover, if $\rho = 1$, there is only one $\widehat{\phi}$ that can lift ϕ, and in this case $\widehat{\phi}(c) = (n_{-1}\phi(c^n), \bar{1})$, where n_{-1} is a natural number such that $n_{-1}n \equiv 1 \pmod{\mathsf{p}}$. If $\rho \neq 1$, there is a set $\{\widehat{\phi}_h | h \in H\}$ consisting of $|H|$ elements that can all lift ϕ and in this case $\widehat{\phi}_h(c) = (h, \bar{1})$.

Proof. Only if : If there is a $\widehat{\phi}$ fitting in the diagram of (3.1), then $\phi(c^{-1}bc) = \widehat{\phi}(c)^{-1}\phi(b)\widehat{\phi}(c)$. Since $\widehat{\phi}(c) = (h, \bar{1})$ for some $h \in H$, $\phi(c^{-1}bc) = (h, \bar{1})^{-1}\phi(b)(h, \bar{1}) = \rho(-\bar{1})(\phi(b))$.

If : Suppose $(*)$ holds. For every element $h \in H$ define a map $\widehat{\phi}_h : \pi_1(U, u_g) \to H \rtimes_\rho \mathbb{Z}/n$ by $\widehat{\phi}_h(bc^i) = \phi(b)(h, \bar{1})^i$. The map is well defined since every element in $\pi_1(U, u_g)$ can be written uniquely in the form bc^i with $b \in \pi_1(V, v_g)$ and $0 \leq i \leq n-1$. Such $\widehat{\phi}_h$'s are not necessarily homomorphisms; they make the diagram commute. The map $\widehat{\phi}_h$ is a homomorphism iff $\phi(c^n) = (h, \bar{1})^n$. If $\rho = 1$, $(h, \bar{1})^n = (nh, \bar{0})$. Then there is a unique $h_0 = n_{-1}\phi(c^n) \in H$ such that $\widehat{\phi}_{h_0}$ is a homomorphism. If $\rho \neq 1$, the condition automatically holds since both sides equal 0. One can compute $(h, \bar{1})^n = 0$ using $\rho(-\bar{1})(h) = e_\rho h$. Hence for every $h \in H$, $\widehat{\phi}_h$ is a homomorphism. □

Here is the second lemma needed in the proof of Theorem 32.

Let $\underline{\sigma}$ be the automorphism in $Gal(V/U)$ corresponding to $\bar{1} \in \mathbb{Z}/n$. Since U

and V are affine, $U = Spec(A)$ and $V = Spec(B)$ for some rings A and B. Then $\underline{\sigma}$ corresponds to a ring automorphism $\sigma \in Gal(B/A)$.

Lemma 30. *A group homomorphism $\phi \in Hom(\pi_1(V, v_g), H)$ satisfies condition $(*)$ of Lemma 29 iff ϕ makes the diagram commutative:*

$$\begin{array}{ccc} \pi_1(V, v_g) & \xrightarrow{\phi} & H \\ \downarrow{\scriptstyle \underline{\sigma}_*} & & \downarrow{\scriptstyle \rho(-\bar{1})} \\ \pi_1(V, v_{g1}) & \xrightarrow[\phi_1]{} & H, \end{array} \tag{30.1}$$

where v_{g1} is the image of v_g under $\underline{\sigma}$, $\underline{\sigma}_*$ induced by $\underline{\sigma}$ and ϕ_1 induced from ϕ using any chemin $v_g \to v_{g1}$.

Proof. Since H is abelian, any chemin $v_g \to v_{g1}$ gives the same isomorphism $\pi_1(V, v_{g1}) \simeq \pi_1(V, v_g)$, thus induces the same ϕ_1 from ϕ.

Denote by Fib_{v0} (resp. Fib_{v1}) the fiber functor from (Finite etale covers of V) to (Sets) at v_g (resp. v_{g1}). Similarly denote by Fib_{u0} the fiber functor from (Finite etale covers of U) to (Sets) at u_g. Denote by PL_{VU} the pullback functor from (Finite etale covers of U) to (Finite etale covers of V) using $V \to U$. There are canonical isomorphisms i_0 from $Fib_{v0} \circ PL_{VU}$ to Fib_{u0} and i_1 from $Fib_{v1} \circ PL_{VU}$ to Fib_{u0}.

The element $c \in \pi_1(U, u_g)$ maps to $\bar{1}$ under θ, and $\bar{1} \in \mathbb{Z}/n$ corresponds to $\underline{\sigma} \in Gal(V/U)$, which sends v_g to v_{g1}. And since every finite etale cover of V composed with $V \to U$ is a finite etale cover of U, $c \in \pi_1(U, u_g)$ induces c_{01} a chemin $v_{g1} \to v_g$. So the first and the last squares of diagram (30.2) commute:

$$\begin{array}{ccccccc} Fib_{v0} \circ PL_{VU} & \xrightarrow{c_{01}} & Fib_{v1} \circ PL_{VU} & \xrightarrow{\underline{\sigma}_*(b)} & Fib_{v1} \circ PL_{VU} & \xrightarrow{c_{01}^{-1}} & Fib_{v0} \circ PL_{VU} \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} \\ Fib_{u0} & \xrightarrow{c} & Fib_{u0} & \xrightarrow{\pi_*(b)} & Fib_{u0} & \xrightarrow{c^{-1}} & Fib_{u0}. \end{array} \tag{30.2}$$

Since $\pi_*(b) = \pi_* \underline{\sigma}_*(b)$ for every $b \in \pi_1(V, v_g)$,

$$\begin{array}{ccc} \pi_1(V, v_g) & \xrightarrow{\underline{\sigma}_*} & \pi_1(V, v_{g1}) \\ & {\scriptstyle \pi_*}\searrow \quad \swarrow{\scriptstyle \pi_*} & \\ & \pi_1(U, u_g). & \end{array}$$

The middle square of diagram (30.2) commutes.

Hence the whole diagram (30.2) is commutative, which shows $\phi(c^{-1}bc) = \phi(c_{01}^{-1}\underline{\sigma}_*(b)c_{01})$. Since $\phi(c_{01}^{-1}\underline{\sigma}_*(b)c_{01}) = \phi_1(\underline{\sigma}_*(b))$, the lemma follows. $\square$

The two lemmas above are used to prove Theorem 32, the first step in the three step construction of the fine moduli space in Theorem 44.

Definition 31. Define $F^{\rho}_{V,H}$: $\mathcal{S}_1 \to$ (Sets) as the contravariant functor given by $F^{\rho}_{V,H}(S, s_0) = \{[\widetilde{\phi}] | \widetilde{\phi} : \pi_1(S \times V, (s_0, v_g)) \to H$ is ρ-liftable$\}$, the set of ρ-liftable families of H-covers of V parameterized by S pointed over (s_0, v_g).

Let $S = Spec(k)$ with s_0 determined by v_g using diagram (2.1). Then $F^{\rho}_{V,H}(S, s_0)$ is the set of all isomorphism classes of ρ-liftable pointed H-covers of (V, v_g).

Theorem 32. *Let H be an elementary abelian group $(\mathbb{Z}/\mathsf{p})^m$, $\rho : \mathbb{Z}/n \to Aut(H)$ an irreducible action of $\mathbb{Z}/n$ on H, and $V \to U$ as above in this Chapter. There is* *a fine moduli space $M^{\rho}_{V,H}$ representing $F^{\rho}_{V,H}$, the functor for isomorphism classes of pointed ρ-liftable H-covers of (V, v_g), which is an ind affine space.*

Proof. In the proof, we will pass between $\mathbb{F}_q$ and H freely using the isomorphism τ between them given above Lemma 29.

Let $F = F^{\rho}_{V,H}$.

The Artin-Schreier short exact sequence $0 \to \mathbb{F}_q \to \mathbb{G}_a \overset{\wp}{\to} \mathbb{G}_a \to 0$, where $\wp(f) = f^q - f$, yields $H^0(V, \mathcal{O}) \overset{\wp}{\to} H^0(V, \mathcal{O}) \to H^1(V, \mathbb{F}_q) \to 0$, where $0 = H^1(V, \mathcal{O})$. This is a short exact sequence of $\mathbb{F}_q$-vector spaces.

Let $\mathbb{X}$ be the subset of $Hom(\pi_1(V, v_g), H)$ that consists of all the isomorphism classes of pointed ρ-liftable H-covers of (V, v_g). Let ϕ be any element in $Hom(\pi_1(V, v_g), H)$. By Lemma 30, $\phi \in \mathbb{X}$ iff $\phi_1 \circ \underline{\sigma}_* = e_\rho \phi$. Let $\underline{\sigma}^* : H^1(V, \mathbb{F}_q) \to H^1(V, \mathbb{F}_q)$ be induced by $\underline{\sigma}$; it is a homomorphism of $\mathbb{F}_q$-vector spaces. Identify $H^1(V, \mathbb{F}_q)$ with $Hom(\pi_1(V, v_g), \mathbb{F}_q)$. By definition of $\underline{\sigma}^*$, $\underline{\sigma}^*(\phi) = \phi_1 \circ \underline{\sigma}_*$. So $\phi \in \mathbb{X}$ iff

$$\underline{\sigma}^*(\phi) = e_\rho \phi, \qquad (*1)$$

which shows that $\mathbb{X}$ is an $\mathbb{F}_q$-subspace of $H^1(V, \mathbb{F}_q)$.

There is a commutative diagram consisting of two short exact sequences of $\mathbb{F}_q$-

vector spaces with every symbol already defined above:

$$\begin{array}{ccccccc} B = H^0(V,\mathcal{O}) & \xrightarrow{\wp} & H^0(V,\mathcal{O}) & \xrightarrow{\pi} & H^1(V,\mathbb{F}_q) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\underline{\sigma}^*} & & \\ H^0(V,\mathcal{O}) & \xrightarrow{\wp} & H^0(V,\mathcal{O}) & \xrightarrow{\pi} & H^1(V,\mathbb{F}_q) & \longrightarrow & 0, \end{array} \tag{32.1}$$

which comes from a commutative diagram consisting of two Artin-Schreier short exact sequences of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{F}_q & \longrightarrow & \mathbb{G}_a & \xrightarrow{\wp} & \mathbb{G}_a & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{\sigma}_* F_q & \longrightarrow & \underline{\sigma}_*\mathbb{G}_a & \xrightarrow{\wp} & \underline{\sigma}_*\mathbb{G}_a & \longrightarrow & 0, \end{array}$$

where $\mathbb{F}_q \to \underline{\sigma}_*\mathbb{F}_q$ is induced by $V \stackrel{\sigma}{\to} V$ and similarly for $\mathbb{G}_a$.

Let $b \in B$. By the right square of diagram (32.1), $(*1)$ implies

$$\phi := \pi b \in \mathbb{X} \Leftrightarrow \sigma b \in e_\rho b + \wp B. \tag{$*2$}$$

Define $D = \sigma - e_\rho$: $B \to B$, an A-module endomorphism of B, where e_ρ acts on B by multiplication. Similarly to the proof of Theorem 1.2 in [H80], there is an exact sequence $KerD \stackrel{\wp}{\to} KerD \stackrel{\pi}{\to} \mathbb{X} \to 0$, of $\mathbb{F}_q$-vector spaces. (Denote the restriction of $\wp$ (resp. π) to $KerD$ also by $\wp$ (resp. π).)

(32.1) Now construct $M^\rho_{V,H}$ using the Ker short exact sequence above. Let $(KerD)_n = KerD \cap H^0(V, q^n Div_V)$, where $Div_V = \Sigma P_i$ the sum of all the closed points in $\overline{V} - V$ and $\overline{V}$ is the smooth completion of V. There is a k-vector space filtration $(KerD)_0 \leq (KerD)_1 \leq ... \leq (KerD)_n \leq ...$. Let $\mathbb{X}_n = \pi((KerD)_n)$. There is a short exact sequence $(KerD)_{n-1} \xrightarrow{\wp} (KerD)_n \xrightarrow{\pi} \mathbb{X}_n \to 0$ obtained from the similar one above. Inductively choose bases K_n of each $(KerD)_n$ as a finite dimensional k-vector space, such that K_{n+1} includes both K_n, and $\{f^q | f \in K_n - K_{n-1}$ and f is not in $k\}$. This is the way to choose bases inductively in a similar situation in the proof of Theorem 1.2 in [H80]. The restriction of π to the k-linear span $\langle K_i - K_{i-1}\rangle_k$ of $K_i - K_{i-1}$ is an isomorphism of $\mathbb{F}_q$-vector spaces $\langle K_i - K_{i-1}\rangle_k \xrightarrow{\pi} \mathbb{X}_i$, which gives a k-vector space structure to $\mathbb{X}_i$.

Let $(S, s_0) \in \mathcal{S}_1$ with $S = Spec(R)$. Similarly there is a commutative diagram consisting of two short exact sequences of $\mathbb{F}_q$-vector spaces:

$$\begin{array}{ccccccc} H^0(S \times V, \mathcal{O}) & \xrightarrow{\wp} & H^0(S \times V, \mathcal{O}) & \xrightarrow{\Pi} & H^1(S \times V, \mathbb{F}_q) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \hat{\sigma}} & & \downarrow{\scriptstyle \hat{\sigma}} & & \downarrow{\scriptstyle \underline{\hat{\sigma}}^*} & & \\ H^0(S \times V, \mathcal{O}) & \xrightarrow{\wp} & H^0(S \times V, \mathcal{O}) & \xrightarrow{\Pi} & H^1(S \times V, \mathbb{F}_q) & \longrightarrow & 0, \end{array}$$

where $\hat{\sigma}$ is an $R \otimes_k A$-module endomorphism: $R \otimes_k B \to R \otimes_k B$, $r \otimes b \mapsto r \otimes \sigma(b)$ and $\wp : r \otimes b \mapsto (r \otimes b)^q - r \otimes b$. Let $\widehat{D} = \hat{\sigma} - e_\rho$. As above, there is a short exact sequence of $\mathbb{F}_q$-vector spaces $Ker\widehat{D} \xrightarrow{\wp} Ker\widehat{D} \xrightarrow{\Pi} \widehat{\mathbb{X}} \to 0$, where $\widehat{\mathbb{X}}$ denotes $\{\widetilde{\phi} \in H^1(S \times V, \mathbb{F}_q) | \underline{\hat{\sigma}}^*(\widetilde{\phi}) = e_\rho \widetilde{\phi}\}$ and $\underline{\hat{\sigma}} \in Gal(S \times V / S \times U)$ corresponds to $\hat{\sigma}$. One can check that $Ker\widehat{D} = R \otimes_k KerD$.

If two S-parametrized P-covers of V pointed over (s_0, v_g) are equivalent, they are considered the same element in $F(S, s_0)$, by definition of F. Hence $F(S, s_0) =$

$\frac{\widehat{\mathbb{X}}+H^1(S,\mathbb{F}_q)}{H^1(S,\mathbb{F}_q)} = \frac{\widehat{\mathbb{X}}}{\widehat{\mathbb{X}}\cap H^1(S,\mathbb{F}_q)}$ (see Remark 22 b). The automorphism $\underline{\hat{\sigma}}$ of $S \times V$ does not change the S-factor, thus for any $\tilde{\phi} \in H^1(S, \mathbb{F}_q)$, $\underline{\hat{\sigma}}^*(\tilde{\phi}) = \tilde{\phi}$. If $e_\rho \neq 1$, $\widehat{\mathbb{X}} \cap H^1(S, \mathbb{F}_q) = 0$ and $F(S, s_0) = \widehat{\mathbb{X}} = \Pi(Ker\widehat{D}) = \Pi(R \otimes_k KerD)$. Let the transition map from $R \otimes_k \langle K_n - K_{n-1}\rangle_k$ to $R \otimes_k \langle K_{n+1} - K_n\rangle_k$ be Frobenius ($r \otimes b \mapsto (r \otimes b)^q$). Then $\varinjlim_n R \otimes_k \langle K_n - K_{n-1}\rangle_k$ is an $\mathbb{F}_q$-vector space. There is an $\mathbb{F}_q$-vector space isomorphism $\varinjlim_n R \otimes_k \langle K_n - K_{n-1}\rangle_k \xrightarrow{\Pi} \Pi(R \otimes_k KerD)$. Hence $F(S, s_0) = \varinjlim_n R\otimes_k\langle K_n - K_{n-1}\rangle_k$. Write out elements in $K_n - K_{n-1}$ as $\{\mathtt{k}_1, ..., \mathtt{k}_{d_K}\}$, whose dual vectors are $\{\mathtt{k}_1^\vee, ..., \mathtt{k}_{d_K}^\vee\}$. Define $\mathbb{A}(\langle K_n - K_{n-1}\rangle_k)$ as $Spec(k[\mathtt{k}_1^\vee, ..., \mathtt{k}_{d_K}^\vee])$. Now $R \otimes_k \langle K_n - K_{n-1}\rangle_k = Hom_k(\langle K_n - K_{n-1}\rangle_k^\vee, R) = Hom_k(S, \mathbb{A}(\langle K_n - K_{n-1}\rangle_k))$. Therefore $M^\rho_{V,H} :=$ the ind scheme $\{\mathbb{A}(\langle K_n - K_{n-1}\rangle_k)\}$, where the transition morphism between $\mathbb{A}(\langle K_n - K_{n-1}\rangle_k)$ and $\mathbb{A}(\langle K_{n+1} - K_n\rangle_k)$ is given by Frobenius, represents F.

If $e_\rho = 1$, then $F(S, s_0) = \frac{H^1(S\times U,\mathbb{F}_q)}{H^1(S,\mathbb{F}_q)}$. This is the case, if $H = \mathbb{Z}/\mathtt{p}$, of the base step in the proof of Theorem 1.2 in [H80]; the proof there also works for any elementary abelian group H. Hence F is represented by $M^\rho_{V,H} := M_{U,H}$, which is denoted by M_G there with $G = H$, an ind affine space with transition morphisms given by Frobenius as well. Since now G is a product $H \times \mathbb{Z}/n$, it can be derived directly that F is represented by $M_{U,H}$, □

Remark 33. a. By Theorem 32, for any pointed affine connected k-scheme (S, s_0), there is a bijection between $F(S, s_0)$ and $M^\rho_{V,H}(S)$, where the latter set is the set of k-morphisms from S to $M^\rho_{V,H}$.

b. Let $S = Spec(k)$ with s_0 determined by v_g using diagram (2.1). Then

$F(S, s_0)$, the set of all ρ-liftable pointed H-covers of (V, v_g), are in bijection with $M^{\rho}_{V,H}(S)$, the set of k-points of $M^{\rho}_{V,H}$, same as the set of closed points of $M^{\rho}_{V,H}$.

c. Let $M^{\rho}_{V,H,n} = Spec(k[\mathtt{k}_1^{\vee}, ..., \mathtt{k}_{d_K}^{\vee}])$ if $\rho \neq 1$. It is the *n-th piece* of $M^{\rho}_{V,H}$. Similarly the n-th piece of $M^{\rho}_{V,H}$ when $\rho = 1$ can be defined.

Remark 34. (Remark/Definition)

With the same notations of Theorem 32.

Let $M^{\rho}_{V,H,n}$ be the n-th piece of $M^{\rho}_{V,H}$ (see Remark 33 c). *A compatible system of H-covers of V over $M^{\rho}_{V,H}$* means a collection of covers $\{H\text{-covers } \widetilde{Z}_n \text{ of } M^{\rho}_{V,H,n} \times V \mid n \geq 1\}$ such that $\widetilde{Z}_n$ pulled back to $M^{\rho}_{V,H,n-1} \times V$ is isomorphic to $\widetilde{Z}_{n-1}$. Since H is abelian, given any point m on $M^{\rho}_{V,H,n}$, where we point $\widetilde{Z}_n$ over (m, v_g) does not matter by Remark 20.

A universal family representative over the moduli space $M^{\rho}_{V,H}$ means, a compatible system of H-covers $\{H\text{-covers } \widetilde{Z}_n \text{ of } M^{\rho}_{V,H,n} \times V \mid n \geq 1\}$ of V over $M^{\rho}_{V,H}$, which can be used to give the isomorphism of functors $Hom(\bullet, M^{\rho}_{V,H}) \xrightarrow{\simeq} F^{\rho}_{V,H} : \mathcal{S}_1 \rightarrow (Sets)$ given in Theorem 32 as follows: Sending a k-morphism f from a k-scheme S with $(S, s_0) \in \mathcal{S}_1$ to $M^{\rho}_{V,H}$ (see Definition 27 a), to the equivalence class of the pullback of $\widetilde{Z}_n$, using the morphism f, to $S \times V$ pointed arbitrarily over (s_0, v_g), is the isomorphism of functors $Hom(\bullet, M^{\rho}_{V,H}) \xrightarrow{\simeq} F^{\rho}_{V,H}$ given in Theorem 32. Since the $\widetilde{Z}_n$'s are compatible any n can be used.

It is derived from definitions that any two universal family representatives are equivalent. The equivalence class of a universal family representative is *the universal* *family over the moduli space $M^{\rho}_{V,H}$.* There must be a universal family representative: Let S be $M^{\rho}_{V,H,n}$. Identity morphism of $M^{\rho}_{V,H,n}$ determines a morphism $S \to M^{\rho}_{V,H}$. The morphism gives an equivalence class in $F^{\rho}_{V,H}(S,m)$ using $Hom(\bullet, M^{\rho}_{V,H}) \overset{\simeq}{\to} F^{\rho}_{V,H}$ given in Theorem 32, for any point m on S. Then ρ-liftable representatives in the equivalence class are candidates for the n-th element of a universal family representative. Use the same kind of argument as in Lemma 4.25 of [TY17], a compatible system of H-covers can be chosen.

If $\rho \neq 1$, a universal family representative over the moduli space $M^{\rho}_{V,H}$ can be given by $\{$the H-cover of $M^{\rho}_{V,H,n} \times V$ given by $z^q - z = \sum_{k_i \in K_n - K_{n-1}} k_i^{\vee} \otimes k_i \mid n \geq 1\}$, by the construction of $M^{\rho}_{V,H}$. The H-covers are compatible for different n's.

If $\rho = 1$, similarly a universal family representative over $M^{\rho}_{V,H}$ can be given explicitly: Replace $z^q - z = \sum_{k_i \in K_n - K_{n-1}} k_i^{\vee} \otimes k_i$ above by $z^q - z = \sum_{l_i \in L_n - L_{n-1}} l_i^{\vee} \otimes l_i$. L_n, an analogue of K_n, is the basis chosen inductively for A_n/k^+ in the proof of Theorem 1.2 in [H80]; here $A_n = H^0(U, q^n Div_U)$ with p there replaced by q and B_n there is denoted by L_n here.

The universal family representative over $M^{\rho}_{V,H}$ given above is *the canonical universal family representative over $M^{\rho}_{V,H}$.* The n-th H-cover in every other universal family representative over $M^{\rho}_{V,H}$, differs from the n-th H-cover in the canonical one by an element in $H^1(M^{\rho}_{V,H,n}, H)$, as shown in the last two paragraphs in the proof of Theorem 32.

The corollary below is a version of Theorem 32 for pairs, which will be used in the proof of Theorem 39.

Definition 35. With the same setting as in Theorem 32. Define $F^{\rho\bullet}_{V,H}\colon \mathcal{S}_1 \to (\text{Sets})$ as the contravariant functor given by $F^{\rho\bullet}_{V,H}(S,s_0) = \{([\widetilde{\phi}],h) \mid \widetilde{\phi} : \pi_1(S\times V,(s_0,v_g)) \to H$ and $(\widetilde{\phi},h)$ is a ρ-liftable pair$\}$, the set of ρ-liftable family pairs of H-covers of V parameterized by S, pointed over (s_0, v_g).

Let $S = Spec(k)$ with s_0 determined by v_g using diagram (2.1). Then $F^{\rho\bullet}_{V,H}(S,s_0)$ is the set of ρ-liftable pairs of V.

Corollary 36. *Under the same setting of Theorem 32, there is a fine moduli space* $M^{\rho\bullet}_{V,H}$ *representing* $F^{\rho\bullet}_{V,H}$*, the functor for* ρ*-liftable pairs of* V*. It is a disjoint union of finitely many copies of* $M^{\rho}_{V,H}$ *in Theorem 32.*

Proof. Let $M^{\rho\bullet}_{V,H}$ be $M^{\rho}_{V,H}$ if $\rho = 1$ and $\amalg_{h\in H} M^{\rho}_{V,H,h}$ if $\rho \neq 1$, where $M^{\rho}_{V,H,h}$ means a copy of $M^{\rho}_{V,H}$ indexed by h.

Let $\phi \in Hom(\pi_1(V,v_g),H)$ and (ϕ,h_0) be a ρ-liftable pair. The map ϕ_{h_0}, as defined in the proof of Lemma 29, is in fact a homomorphism. As stated in Lemma 29, if $\rho = 1$ then h_0 is the only element in H such that (ϕ,h_0) is a ρ-liftable pair. If $\rho \neq 1$ then for every $h \in H$ the pair (ϕ,h) is ρ-liftable. Using this fact and Theorem 32 $F^{\rho\bullet}_{V,H}$ is represented by $M^{\rho\bullet}_{V,H}$. □

By Corollary 36, $M^{\rho}_{V,H}$ is a connected component of the ind scheme $M^{\rho\bullet}_{V,H}$. See Remark 26.

Here is the 2nd step of the 3 step construction of the fine moduli space in Theorem 44. Let P be an arbitrary finite p-group now.

Definition 37. Let $F^{\rho\bullet}_{V,P}$: $\mathcal{S}_1 \to$ (Sets) be the contravariant functor given by $F^{\rho\bullet}_{V,P}(S,s_0) = \{([\widetilde{\phi}],p) \mid \widetilde{\phi} : \pi_1(S \times V, (s_0, v_g)) \to P$ and $(\widetilde{\phi}, p)$ is a ρ-liftable pair$\}$, the set of ρ-liftable family pairs of P-covers of V parameterized by S, pointed over (s_0, v_g).

Let $S = Spec(k)$ with s_0 determined by v_g using diagram (2.1). Then $F^{\rho\bullet}_{V,P}(S,s_0)$ is the set of ρ-liftable pairs of V.

Definition 38. A similar definition for pairs to a universal family representative over $M^{\rho}_{V,H}$ in Remark 34 will be given.

Assume there is an ind scheme $M^{\rho\bullet}_{V,P}$, consisting of finitely many connected components (see Remark 26), representing $F^{\rho\bullet}_{V,P}$ with an isomorphism between functors $Hom(\bullet, M^{\rho\bullet}_{V,P}) \xrightarrow{\simeq} F^{\rho\bullet}_{V,P}$.

Below $M^{\rho\bullet}_{V,P}$ is viewed as a scheme instead of an ind scheme (see Remark 41). Connected components of $M^{\rho\bullet}_{V,P}$ are denoted by $\{M^{\rho}_{V,P,j}\}$.

A system of universal family pair representatives over $M^{\rho\bullet}_{V,P}$, means a collection of a ρ-liftable pair $(\tilde{\phi}_{0j,m_j} : \pi_1(M^{\rho}_{V,P,j} \times V, (m_j, v_g)) \to P, p_{\tilde{\phi}_{0j,m_j}})$ for every base point m_j over each $M^{\rho}_{V,P,j}$, which can be used to give the isomorphism of functors $Hom(\bullet, M^{\rho\bullet}_{V,P}) \xrightarrow{\simeq} F^{\rho\bullet}_{V,P}$ as follows: Sending a k-morphism $S \xrightarrow{\mathfrak{c}} M^{\rho}_{V,P,j}$ with $(S,s_0) \in \mathcal{S}_1$ and s_0 mapped to m_j under $\mathfrak{c}$, to the pair $([\tilde{\phi}_{0j,m_j} \circ \tilde{\mathfrak{c}}_*], p_{\tilde{\phi}_{0j,m_j}})$ with $S \times V \xrightarrow{\tilde{\mathfrak{c}}} M^{\rho}_{V,P,j} \times V$ induced by $\mathfrak{c}$ and $\tilde{\mathfrak{c}}_*$ the homomorphism between fundamental groups induced by $\tilde{\mathfrak{c}}$, is the given isomorphism of functors $Hom(\bullet, M^{\rho\bullet}_{V,P}) \xrightarrow{\simeq} F^{\rho\bullet}_{V,P}$.

It can be derived from definition that any two $\tilde{\phi}_{0j,m_j}$ and $\tilde{\phi}'_{0j,m_j}$ in two different systems of universal family pair representatives over $M^{\rho\bullet}_{V,P}$ are equivalent and $p_{\tilde{\phi}_{0j,m_j}} = p_{\tilde{\phi}'_{0j,m_j}}$.

Similarly to Remark 34, there must be a system of universal family pair representatives over $M^{\rho\bullet}_{V,P}$.

Theorem 39. *With the notations above, there exists a fine moduli space* $M^{\rho\bullet}_{V,P}$ *representing* $F^{\rho\bullet}_{V,P}$*, the functor for* ρ*-liftable pairs of* V*. It is a disjoint union of finitely many ind affine spaces.*

Proof. Induct on $|P|$.

Let $F = F^{\rho\bullet}_{V,P}$.

(39.1) Take a minimal normal subgroup H of G inside $C(P)$, the nontrivial center of P. It is a product of copies of some simple group S. Hence $H \approx (\mathbb{Z}/p)^m$ for some $m \geq 1$. Let $\rho_0 : \mathbb{Z}/n \to Aut(H)$ be the $\mathbb{Z}/n$-action induced by ρ; ρ_0 is irreducible by the minimality of H. If $H = P$, then this is in the case of Corollary 36. Hence assume $H < P$ below. Let $\bar{\rho} : \mathbb{Z}/n \to Aut(\bar{P})$ be the $\mathbb{Z}/n$-action induced by ρ, where $\bar{P} = P/H$. By the inductive hypothesis and Corollary 36 respectively $\bar{M} := M^{\bar{\rho}\bullet}_{V,\bar{P}}$ and $M^0 := M^{\rho_0\bullet}_{V,H}$ exist.

Denote $F^{\bar{\rho}\bullet}_{V,\bar{P}}$ by $\bar{F}$.

It will be shown that $\bar{M} \times M^0$ is the moduli space desired.

(39.2) First need to lift a system of universal family pair representatives over $\bar{M}$. For every point $\bar{m}$ of $\bar{M}$, there is a $\bar{\rho}$-liftable pair $(\widetilde{\mu}_0 : \pi_1(\bar{M} \times V, (\bar{m}, v_g)) \to \bar{P}, \bar{p}_0)$ in the system, which is the counterpart of $(\tilde{\phi}_{0j,m_j} : \pi_1(M^{\rho}_{V,P,j} \times V, (m_j, v_g)) \to \bar{P}, p_{\tilde{\phi}_{0j,m_j}})$

in Definition 38. Denote by $c_\bullet$ the image of c under the group homomorphism $\pi_1(U, u_g) \to \pi_1(\bar{M} \times U, (\bar{m}, u_g))$ induced by $U \hookrightarrow \bar{M} \times U$. The pair gives a $\widehat{\widehat{\bar{\mu}}}_0$: $\pi_1(\bar{M} \times U, (\bar{m}, u_g)) \to \bar{P} \rtimes_{\bar{\rho}} \mathbb{Z}/n$ with $\widehat{\widehat{\bar{\mu}}}_0(c_\bullet) = (\bar{p}_0, \bar{1})$, similar to the diagram (3.1). As $\pi_1(\bar{M} \times U, (\bar{m}, u_g))$ has $cd_p \leq 1$ ([H80], p1101), $\widehat{\widehat{\bar{\mu}}}_0$ lifts (a different meaning of "lift", see Remark 18) to a $\widehat{\widehat{\psi}}_0 : \pi_1(\bar{M} \times U, (\bar{m}, u_g)) \to P \rtimes_\rho \mathbb{Z}/n$ ([Serre], I Prop. 16) with $\widehat{\widehat{\psi}}_0(c_\bullet) = (p_0, \bar{1})$, for some $p_0 \in P$ mapping to $\bar{p}_0 \in \bar{P}$, under the quotient map $P \twoheadrightarrow \bar{P}$. Denote the restriction of $\widehat{\widehat{\psi}}_0$ on $\pi_1(\bar{M} \times V, (\bar{m}, v_g))$ by $\widetilde{\psi}_0$.

(39.3) Then use the lift $\widetilde{\psi}_0$ obtained above to separate a ρ-liftable pair of $S \times V$ into two parts. Let $(S, s_0) \in \mathcal{S}_1$ and suppose $(\widetilde{\phi} : \pi_1(S \times V, (s_0, v_g)) \to P,\ p_1)$ is a ρ-liftable pair. Its quotient $(\bar{\widetilde{\phi}} : \pi_1(S \times V, (s_0, v_g)) \to \bar{P}, \bar{p}_1)$ is a $\bar{\rho}$-liftable pair. By the inductive hypothesis, the quotient pair corresponds to a morphism $\beta : S \to \bar{M}$. Denote $\beta \times Id_V$ by $\tilde{\beta}$. Denote the induced homomorphism $\pi_1(S \times V, (s_0, v_g)) \to \pi_1(\bar{M} \times V, (\beta(s_0), v_g))$ by $\tilde{\beta}_*$, and let $\widetilde{\psi} := \widetilde{\psi}_0 \circ \tilde{\beta}_*$ (letting $\bar{m}$ above be $\beta(s_0)$ here). Then define a "quotient homomorphism" $\widetilde{\eta} : \pi_1(S \times V, (s_0, v_g)) \to H$ by $\widetilde{\phi}\widetilde{\psi}^{-1}$: Since $\bar{M}$ is a fine moduli space for $\bar{F}$ that involves equivalence classes, $\bar{\widetilde{\phi}}$ and $\bar{\widetilde{\psi}}$ only agree pulled back to some finite etale cover T of S, by definition of $\bar{F}$. Pick a point t_0 on T mapping to s_0. Define $\widetilde{\eta}_T(a) = \widetilde{\phi}_T(a)\widetilde{\psi}_T(a^{-1})$ for every $a \in \pi_1(T \times V, (t_0, v_g))$, where $\widetilde{\phi}_T$ means $\widetilde{\phi}$ pulled back to T and similarly for $\widetilde{\psi}_T$. Actually $\widetilde{\eta}_T$ maps to H and the centrality of H in P implies that $\widetilde{\eta}_T$ is a homomorphism . Let $h_1 = p_1 p_0^{-1}$. Then $(\widetilde{\eta}_T, h_1)$ is a ρ_0-liftable pair and hence corresponds to a morphism $\alpha_T : T \to M^0$. By etale decent ([H80], p1109, second paragraph) α_T descends to a morphism

$\alpha : S \to M^0$. Hence get $(\alpha, \beta) : S \to M$, where $M = M^{\rho\bullet}_{V,P} := M^0 \times \bar{M}$.

It is straightforward to verify that the assignment $(\widetilde{\phi}, p_1) \rightsquigarrow (\alpha, \beta)$ is well defined on ρ-liftable family pairs (i.e. is independent of the choice of $\widetilde{\phi}$ in its equivalence class), and yields a bijection between $F(S, s_0)$ and $Hom(S, M)$. As the bijection is compatible with pullback, it follows that M represents F. □

Remark 40. Keeping track of universal family representatives in the inductive construction, a ρ-liftable universal family representative over (each connected component of) $M^{\rho\bullet}_{V,P} = M^0 \times \bar{M}$ can be given by the product of a ρ_0-liftable universal family representative over M^0 and a ρ-liftable lift of a $\bar{\rho}$-liftable universal family representative over $\bar{M}$, using the inclusions $M^0 \hookrightarrow M^0 \times \bar{M}$ and $\bar{M} \hookrightarrow M^0 \times \bar{M}$.

Remark 41. Since the moduli spaces are ind schemes, strictly speaking the argument above needs to be carried out for each n and check compatibility for different n's. The argument given above has the advantage of being more concise, which follows the way of presentation in Theorem 1.2 of [H80].

Remark 42. In the inductive proof of Theorem 39, (P, ρ) is said to *be decomposed* to (H, ρ_0) and $(\bar{P}, \bar{\rho})$. If $(\bar{P}, \bar{\rho})$ is not in the case considered in Theorem 32, then it can be further decomposed similarly. Repeat the inductive step in Theorem 39 until the last pair got is in the case of Theorem 32. The pairs got in the process are denoted by $(H_t, \rho_t)_t$, which are all in the case of Theorem 32 and the first of which is (H, ρ_0). Then $M^{\rho\bullet}_{V,P} = \Pi_t M^{\rho_t\bullet}_{V,H_t}$, which consists of finitely many connected components of the form $\Pi_t M^{\rho_t}_{V,H_t}$.

Here is the main theorem of this Chapter, on moduli for covers with a given cyclic-by-p Galois group.

Definition 43. Let $F_{U,G}$: $\mathcal{S}_1 \to$ (Sets) be the contravariant functor given by $F_{U,G}(S, s_0) = \{[\tilde{\phi}] \,|\, \tilde{\phi} : \pi_1(S \times U, (s_0, u_g)) \to G\}$, the set of families of G-covers of U parametrized by S, pointed over (s_0, u_g).

Let $S = Spec(k)$ with s_0 determined by u_g using diagram (2.1). Then $F_{U,G}(S, s_0)$ is the set of pointed G-covers of (U, u_g).

Theorem 44. *There exists a fine moduli space $M_{U,G}$ representing $F_{U,G}$, the functor for pointed G-covers of (U, u_g), which is a disjoint union of finitely many ind affine spaces.*

Proof. It will be shown that $F_{U,G}$ is isomorphic to $\amalg_{V_i} F_{V_i,P}^{\rho_{n_i},\bullet}$ (see 2.0.6 "Table of symbols" for V_i). The disjoint union of functors means taking disjoint union of sets, since the functors map to the category of sets. Hence it is represented by $\amalg_{V_i} M_{V_i,P}^{\rho_{n_i},\bullet}$, by Theorem 39.

First the left to right direction map is given in the isomorphism wanted.

Let $(S, s_0) \in \mathcal{S}_1$ and $(\widetilde{W}, \widetilde{w}_g) \to (S \times U, (s_0, u_g))$ be a pointed G-cover corresponding to some $\tilde{\tilde{\phi}} \in Hom(\pi_1(S \times U, (s_0, u_g)), G)$. Let $(\widetilde{W}_m, \overline{\widetilde{w}_g})$ be the pointed connected component of $\widetilde{W}/P$, a $(\mathbb{Z}/n')$-cover of $(S \times U, (s_0, u_g))$ with $(\mathbb{Z}/n')$ the order n' subgroup in $\mathbb{Z}/n$ for some $n'|n$. The diagram commutes:

$$\begin{array}{ccc} \pi_1(\widetilde{W}_m, \overline{\widetilde{w}_g}) & \xrightarrow{\widetilde{\phi}_m} & P \\ \big\downarrow & & \big\downarrow \\ \pi_1(S \times U, (s_0, u_g)) & \xrightarrow{\widetilde{\widetilde{\phi}}} & G. \end{array}$$

Let T be a connected component of the inverse image in $\widetilde{W}_m$ of $S \times \{u'_g\}$, where u'_g is any k-point on U. The fibers of $\widetilde{W}_m$ over k-points of U does not vary since the degree of the cover is prime to p. The k-scheme T is a finite etale cover of S and pick any base point t_0 that maps to s_0. The cover $\widetilde{W}_m$ pulled back to $T \times U$ is isomorphic to a disjoint union of copies of a product $T \times V_i$ for some V_i a $\mathbb{Z}/n_i$-cover of U:

$$\begin{array}{ccc} \coprod(T \times V_i, (t_0, v_i)) & \longrightarrow & (\widetilde{W}_m, \overline{\widetilde{w}_g}) \\ \big\downarrow & & \big\downarrow \\ (T \times U, (t_0, u_g)) & \longrightarrow & (S \times U, (s_0, u_g)), \end{array}$$

as $(\mathbb{Z}/n')$-covers of $T \times U$, using the canonical embedding ι_{n_i} of $\mathbb{Z}/n_i$ in $\mathbb{Z}/n$ given in Chapter 2 Definition 16 b.

Let $\widetilde{\phi}_T$ be the composition $\pi_1(T \times V_i, (t_0, v_i)) \to \pi_1(\widetilde{W}_m, \overline{\widetilde{w}_g}) \to P$ induced by $T \times V_i \to \widetilde{W}_m$. Let $c_{i\bullet}$ be the image of c_i under $\pi_1(U, u_g) \to \pi_1(S \times U, (s_0, u_g))$. Let p_0 be the first entry of $\widetilde{\widetilde{\phi}}(c_{i\bullet}) \in G = P \rtimes_\rho \mathbb{Z}/n$. Then $(\widetilde{\phi}_T, p_0)$ is a ρ_{n_i}-liftable pair. It corresponds to a morphism $\mathfrak{c}_T : T \to M^{\rho_{n_i}, \bullet}_{V_i, P}$. By etale decent again $\mathfrak{c}_T$ descends to a morphism $\mathfrak{c}_S : S \to M^{\rho_{n_i}, \bullet}_{V_i, P}$. The morphism $\mathfrak{c}_S$ corresponds to an element in $F^{\rho_{n_i}, \bullet}_{V_i, P}(S, s_0)$. In fact a morphism $\delta : F_{U,G} \to \amalg_{V_i} F^{\rho_{n_i}, \bullet}_{V_i, P}$ is got.

Conversely, suppose $(\widetilde{\phi}, p_0)$ is a ρ_{n_i}- liftable pair with $\pi_1(S \times V_i, (s_0, v_i)) \xrightarrow{\widetilde{\phi}} P$.

The diagram commutes:

$$
\begin{array}{ccccc}
\pi_1(S \times V_i, (s_0, v_i)) & \xrightarrow{\quad\widetilde{\phi}\quad} & P & & \\
\big\downarrow & & \big\downarrow & & \\
\pi_1(S \times U, (s_0, u_g)) & \xrightarrow{\widehat{\widetilde{\phi}}} & P \rtimes_{\rho_{n_i}} \mathbb{Z}/n_i & \xhookrightarrow{\widetilde{\iota_{n_i}}} & G,
\end{array}
$$

where $\widehat{\widetilde{\phi}}$ sends $c_{i\bullet}$ to $(p_0, \bar{1})$, and $\widetilde{\iota_{n_i}}$ is the group embedding induced by ι_{n_i}. Hence a pointed family of G-covers of U parametrized by S corresponding to $\widetilde{\iota_{n_i}} \circ \widehat{\widetilde{\phi}}$ is got. In fact a morphism $\gamma : \amalg_{V_i} F_{V_i,P}^{\rho_{n_i},\bullet} \to F_{U,G}$ is got, which is inverse to δ. $\square$

Chapter 4

Moduli for p'-by-p covers

In Chapter 4, it is shown that, given a pointed affine curve (U, u_g), an intersection of finitely many fine moduli spaces for cyclic-by-p covers of some affine curves gives a moduli space for p'-by-p covers of the curve (Corollary 58).

The next simplest groups after cyclic-by-p groups are p'-by-p groups. The first idea on how to get a moduli space for p'-by-p covers of (U, u_g), out of fine moduli spaces for cyclic-by-p covers of affine curves constructed in Chapter 3, is to intersect them.

The fine moduli spaces for cyclic-by-p covers of some affine curves, intersect in a fixed fine moduli space $M_{V',P,0}$ for some affine curve V', which is given first below in Remark 45.

Lemma 47 and Lemma 48 show how to embed a fine moduli space for cyclic-by-p covers of an affine curve in $M_{V',P,0}$. The first lemma is the base case for the induction in the proof of the 2nd lemma.

Then an intersection in $M_{V',P,0}$ gives a target moduli space $M^{0\rho'}_{V',P}$ in Definition 51. However, it is not a moduli space for covers of (U, u_g) with Galois group the p'-by-p group given, because pieces do not patch together well when P is not abelian (see Remark 53). It is a moduli space for something else; see Proposition 57. Similarly pieces may not patch together well for a disconnected P-cover. Therefore $M^{0\rho'}_{V',P}$ only contains connected covers. The moduli space for covers of (U, u_g) with Galois group the given p'-by-p group is a corollary of Proposition 57.

One final thing for the intersection idea to work, is to use a weaker definition of equivalence. A new ER-equivalence is introduced below in the definition of $F^{er,Gal/\rho'}_{V',P}$, the functor to present, and that of $M^{er0\rho'}_{V',P}$, a functor related to the moduli space $M^{0\rho'}_{V',P}$. Using ER-equivalence $F^{er,Gal/\rho'}_{V',P}$ and $M^{er0\rho'}_{V',P}$ are proven isomorphic in Proposition 57.

As always, we follow notations and terminology defined in Chapter 2.

First the space where intersections take place is given.

Let $(V', v'_g) \to (U, u_g)$ be a pointed connected P'-cover of (U, u_g), which corresponds to a surjective group homomorphism $\theta' : \pi_1(U, u_g) \to P'$.

Remark 45. Since P can be decomposed in different ways in the construction of $M_{V',P}$ (see proof of Theorem 1.2 in [H80]; see Remark 42 for a ρ-liftable version), there are different forms of $M_{V',P}$. Since they are all fine moduli spaces of $F_{V',P}$, it is derived from the definition that they are isomorphic. Fix a fine moduli space $M_{V',P,0}$ for $F_{V',P}$ below, where intersections take place.

Now the objects that intersect later are given.

Let $(V'_{i'}, \overline{v'_{gi}})$ be the quotient of (V', v'_g) by $\langle p'_i \rangle$, the subgroup generated by p'_i, and let $\rho'_i : \langle p'_i \rangle \to Aut(P)$ be the restriction of ρ'. There is a short exact sequence of groups:

$$1 \to \pi_1(V', v'_g) \to \pi_1(V'_i, \overline{v'_{gi}}) \xrightarrow{\theta'_i} \langle p'_i \rangle \to 1.$$

Let π'_{i*} be the homomorphism between fundamental groups induced by $\pi'_i : V'_i \to U$. The following diagram commutes:

$$\begin{array}{ccc} \pi_1(V'_i, \overline{v'_{gi}}) & \xrightarrow{\theta'_i} & \langle p'_i \rangle \\ {\scriptstyle \pi'_{i*}} \downarrow & & \downarrow {\scriptstyle \subset} \\ \pi_1(U, u_g) & \xrightarrow{\theta'} & P'. \end{array}$$

For every $p'_i \in P'$, fix a c'_i in $\pi_1(V'_i, \overline{v'_{gi}})$ that maps to p'_i under θ'_i. The pointed $\langle p'_i \rangle$-cover $(V', v'_g) \to (V'_{i'}, \overline{v'_{gi}})$ is the counterpart of the pointed $\mathbb{Z}/n$-cover $(V, v_g) \to (U, u_g)$ in Theorem 39 of Chapter 3. Apply Theorem 39 on $(V', v'_g) \to (V'_{i'}, \overline{v'_{gi}})$ and a fine moduli space $M^{\rho'_i, \bullet}_{V',P}$ for ρ'_i-liftable pairs of (V', v'_g) is got.

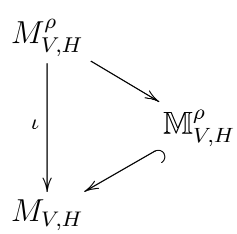
For every p'_i denote by $\{M^{\rho'_i}_{V',P,ij}\}$ the set of finitely many connected components of $M^{\rho'_i, \bullet}_{V',P}$. Denote by $(M^{\rho'_i}_{V',P,ij})_i$ a tuple of connected components indexed by i, an element in $\Pi_i \{M^{\rho'_i}_{V',P,ij}\}$. For each tuple $(M^{\rho'_i}_{V',P,ij})_i$ do their intersection in $M_{V',P,0}$, the way of which will be defined below. Then take the disjoint union of intersections belonging to different tuples. The disjoint union is almost $M^{0\rho'}_{V',P}$, the moduli space in Proposition 57.

Below are two lemmas to embed every $M^{\rho'_i}_{V',P,ij}$ in $M_{V',P,0}$ for intersection purpose.

The base case is for (ρ, H) in the case of Theorem 32. With the same setting as in Theorem 32. Let the morphism $M^{\rho}_{V,H} \xrightarrow{\iota} M_{V,H}$ be given by the canonical universal family representative over $M^{\rho}_{V,H}$ (see Remark 34). The morphism ι can be given explicitly by tracking the construction of both moduli spaces in Lemma 47.

Example 46. Here is an example that is a prototype for the morphism $M^{\rho}_{V,H} \to \mathbb{M}^{\rho}_{V,H}$ in the diagram of Lemma 47 below. The subring $k[X^{\mathsf{p}}]$ of $k[X]$ is also a polynomial ring. The inclusion $k[X^{\mathsf{p}}] \subset k[X]$ induces a bijection between closed points in $Spec(k[X])$ and those in $Spec(k[X^{\mathsf{p}}])$, given explicitly by $(X - \lambda) \leftrightarrow (X^{\mathsf{p}} - \lambda^{\mathsf{p}})$.

Lemma 47. *There is a closed subscheme $\mathbb{M}^{\rho}_{V,H}$ of $M_{V,H}$ which ι factors through and whose closed points are in bijection with those of $M^{\rho}_{V,H}$ under ι.*



Proof. Theorem 32, Remark 42 and the base step for induction in the proof of

Theorem 1.2 in [H80] are the references for this proof. Every fact used here can be found in one of the three places.

The explicit expression of ι on each n-th piece of $M^{\rho}_{V,H}$ (see Remark 33 c) will be given, using which the statements in the Lemma can be shown.

Denote by $M^{\rho}_{V,H,n}$ the n-th piece of $M^{\rho}_{V,H}$. The affine space $M^{\rho}_{V,H,n}$ can be identified with $Spec(k[K_n^{\vee} - K_{n-1}^{\vee}])$, where K_n, containing K_{n-1}, is the basis chosen for the k-vector space $(KerD)_n = KerD \cap H^0(V, q^n Div_V)$ in the proof of Theorem 32. Denote by $K_n^{\vee}$ the set of the dual's of vectors in K_n. Write out elements in $K_n - K_{n-1}$ as $\{\mathtt{k}_i, 1 \leq i \leq d_K\}$. Then $Spec(k[K_n^{\vee} - K_{n-1}^{\vee}]) = Spec(k[\mathtt{k}_1^{\vee}, ..., \mathtt{k}_{d_K}^{\vee}])$.

Similarly denote by $M_{V,H,n}$ the n-th piece of $M_{V,H}$. The affine space $M_{V,H,n}$ can be identified with $Spec(k[L_n^{\vee} - L_{n-1}^{\vee}])$, where L_n, containing L_{n-1}, is the basis chosen for $H^0(V, q^n Div_V)/k^+$. Denote by $L_n^{\vee}$ the set of the dual's of vectors in L_n. Write out elements in $L_n - L_{n-1}$ as $\{l_j, 1 \leq j \leq d_L\}$. The way to choose L_n is described in the base step for induction in the proof of Theorem 1.2 in [H80], analogous to the way to choose K_n. Only need to change the symbol U there to V, B_n there to L_n, and $A_n = H^0(U, q^n Div_U)$ there to $B_n = H^0(V, q^n Div_V)$. Recall that $U = Spec(A)$ and $V = Spec(B)$. A_n and B_n denote k-subspaces of A and B respectively.

Denote by ι_n the restriction of ι on $M^{\rho}_{V,H,n}$. The morphism ι_n maps every closed point in $M^{\rho}_{V,H,n}$ to the closed point in $M_{V,H,n}$ that represents the same pointed H-cover as it. Denote the k-algebra homomorphism that corresponds to ι_n by $\iota_n^* : k[L_n^{\vee} - L_{n-1}^{\vee}] \to k[K_n^{\vee} - K_{n-1}^{\vee}]$. It turns out that ι_n^* has the form: $l_j^{\vee} \mapsto$

$\Sigma_i \Sigma_{t\in X_j}(\lambda_{it}\mathtt{k}_i^{\vee})^{q_{tj}}$, where X_j some finite set, $\lambda_{it} \in k$ and q_{tj} is some p-power.

The form of ι_n^* is obtained as follows. All pointed H-covers of (V, v_g) can be given by elements in B using Artin-Schreier equations $z^q - z = b$ with $b \in B$. Elements in the k-linear span of $L_n - L_{n-1}$ give bijectively all the pointed H-covers of (V, v_g) that can be given by $z^q - z = b$ with $b \in B_n$. Every element $\sum_i \lambda_i \mathtt{k}_i$ in the k-linear span of $K_n - K_{n-1}$ is in B_n. Hence the pointed H-cover of (V, v_g) given by $z^q - z = \sum_i \lambda_i \mathtt{k}_i$ is isomorphic to the pointed H-cover of (V, v_g) given by $z^q - z = \sum_j \lambda'_j l_j$, for some unique $\sum_j \lambda'_j l_j$ in the k-linear span of $L_n - L_{n-1}$. The correspondence $\sum_i \lambda_i \mathtt{k}_i \leftrightarrow \sum_j \lambda'_j l_j$ is what is used to get the form of ι_n^*: A closed point in $Spec(k[K_n^{\vee} - K_{n-1}^{\vee}])$ has the form $(\mathtt{k}_1^{\vee} - \lambda_1, ..., \mathtt{k}_{d_K}^{\vee} - \lambda_{d_K})$. The maximal ideal represents the pointed H-cover of (V, v_g) given by $z^q - z = \sum_i \lambda_i \mathtt{k}_i$, pointed anywhere above v_g. There is a unique k-algebra homomorphism $k[L_n^{\vee} - L_{n-1}^{\vee}] \to k[K_n^{\vee} - K_{n-1}^{\vee}]$ such that the inverse image of $(\mathtt{k}_1^{\vee} - \lambda_1, ..., \mathtt{k}_{d_K}^{\vee} - \lambda_{d_K})$ is $(l_1^{\vee} - \lambda'_1, ..., l_{d_L}^{\vee} - \lambda'_{d_L})$, which represents the pointed H-cover of (V, v_g) given by $z^q - z = \sum_j \lambda'_j l_j$, for every closed point $(\mathtt{k}_1^{\vee} - \lambda_1, ..., \mathtt{k}_{d_K}^{\vee} - \lambda_{d_K})$ in $Spec(k[K_n^{\vee} - K_{n-1}^{\vee}])$. Hence the homomorphism is ι_n^*, by the definition of ι_n^*. It is left as an exercise to the reader to write out the precise formula of the homomorphism, which has the form given above.

Let $\mathbb{M}_{V,H,n}^{\rho} = Spec(Im\iota_n^*)$, which is a closed subscheme of $M_{V,H,n}$. After simplification by elimination $Im\iota_n^*$ turns out a polynomial ring $k[\{\mathtt{k}'_i, 1 \leq i \leq d_K\}]$, where $\mathtt{k}'_i$ is a sum of powers $\Sigma_{i\leq t\leq d_K}\mathtt{k}_t^{n_{it}}$ and n_{ii} is a p-power. Moreover for every i the polynomial ring $Im\iota_n^*$ contains a $\mathtt{k}_i^{q_i}$ with q_i a p-power. Similar to Example 46, ι_n

gives a bijection between the closed points of $M^{\rho}_{V,H,n}$ and those of $\mathbb{M}^{\rho}_{V,H,n}$.

The ι_n's for different n's are compatible. □

Here are some necessary settings to prove the 2nd lemma for embedding $M^{\rho'_i}_{V',P,ij}$ in $M_{V',P,0}$.

With the same setting as in Theorem 39. The ind scheme $M^{\rho;\bullet}_{V,P}$ consists of finitely many connected components $\{M^{\rho}_{V,P,j}\}$. For every j, the universal family (see Remark 38 for more precise terminology) over $M^{\rho}_{V,P,j}$ determines a morphism $M^{\rho}_{V,P,j} \xrightarrow{\iota} M_{V,P}$, since $M_{V,P}$ is the fine moduli space for $F_{V,P}$.

If $(\rho_t, H_t)_t$ is a decomposition of (ρ, P) (see Remark 42), then $M^{\rho;\bullet}_{V,P} = \Pi_t M^{\rho_t;\bullet}_{V,H_t}$ and $M_{V,P} = \Pi_t M_{V,H_t}$. Hence $M^{\rho}_{V,P,j}$ has the form $\Pi_t M^{\rho_t}_{V,H_t}$ for every j. The morphism ι can be given componentwise for each t.

Lemma 48. *With the notations above, the morphism* $M^{\rho}_{V,P,j} = \Pi_t M^{\rho_t}_{V,H_t} \xrightarrow{\iota} \Pi_t M_{V,H_t}$ *is given by* $\Pi_t \iota_t$*, where* $M^{\rho_t}_{V,H_t} \xrightarrow{\iota_t} M_{V,H_t}$ *is the morphism given in Lemma 47.*

Proof. Theorem 32, Remark 42 and the base step for induction in the proof of Theorem 1.2 in [H80] are the references for this proof. Every fact used here can be found in one of the three places.

Induct on $|P|$.

(48.1) The base case is done in Lemma 47. Moreover for every t since M_{V,H_t} is the fine moduli space for F_{V,H_t}, the canonical universal family representative over M_{V,H_t} given in 1.9 Rmk of [H80] pulled back to $M^{\rho_t}_{V,H_t}$ via ι_t, differs from the canonical universal family representative over $M^{\rho_t}_{V,H_t}$ given in Remark 34 by some element in $H^1(M^{\rho_t}_{V,H_t}, H)$, by tracking definitions. Lemma 47 shows that ι_t gives a bijection on closed points of $M^{\rho_t}_{V,H_t}$ and $\mathbb{M}^{\rho_t}_{V,H_t}$. Using this fact and the same kind of argument in Lemma 4.25 of [TY17], a universal family representative over M_{V,H_t} can be chosen such that it pulls back to the canonical universal family representative over $M^{\rho_t}_{V,H_t}$.

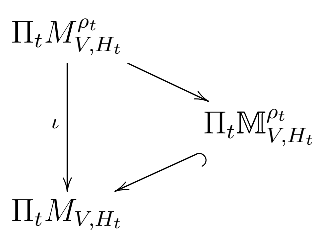
Below is the inductive step.

In Theorem 39 an H inside the center $C(P)$ of P is taken, and then the inductive process is carried out, which gives $M^{\rho\bullet}_{V,P}$ as $M^{\bar{\rho}\bullet}_{V,\bar{P}} \times M^{\rho_0\bullet}_{V,H}$. The notation $\Pi_t M^{\rho_t}_{V,H_t}$ means that (H, ρ_0) is denoted by (H_1, ρ_1) here and the inductive step there is carried out for some finite steps until the induction ends (see Remark 42). Thus a connected component of $M^{\bar{\rho}\bullet}_{V,\bar{P}}$ can be denoted by $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t}$ and $M_{V,\bar{P}}$ by $\Pi_{t\geq 2} M_{V,H_t}$.

(48.2) By inductive hypothesis, the universal family over $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t}$ determines a morphism $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t} \xrightarrow{\Pi_{t\geq 2}\iota_t} \Pi_{t\geq 2} M_{V,H_t}$ with ι_t given in Lemma 47; and there is a universal family representative over $\Pi_{t\geq 2} M_{V,H_t}$ such that it pulls back to a $\bar{\rho}$-liftable universal family representative over $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t}$. Then lift the $\bar{\rho}$-liftable universal family representative over $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t}$ as in the proof of Theorem 39, and pick any lift of the universal family representative over $\Pi_{t\geq 2} M_{V,H_t}$. Again using the same kind of argument in Lemma 4.25 of [TY17], the latter lift can be modified such that its pullback to $\Pi_{t\geq 2} M^{\rho_t}_{V,H_t}$ is the previous lift. (Strictly speaking, Definition 38 needs to be used and pairs should be dealt with, which however will make the proof unnecessarily longer.)

A ρ-liftable universal family representative over $\Pi_t M_{V,H_t}^{\rho_t}$ can be given by the product of a ρ_1-liftable universal family representative over $M_{V,H_1}^{\rho_1}$ and a ρ-liftable lift of a $\bar{\rho}$-liftable universal family representative over $\Pi_{t\geq 2} M_{V,H_t}^{\rho_t}$ (see Remark 40). A universal family representative over $\Pi_t M_{V,H_t}$ is a similar product. Paragraphs (48.1) and (48.2) together show that the pullback of some universal family representative over $\Pi_t M_{V,H_t}$ via $\Pi_t M_{V,H_t}^{\rho_t} \xrightarrow{\Pi_t \iota_t} \Pi_t M_{V,H_t}$ is a ρ-liftable universal family representative over $\Pi_t M_{V,H_t}^{\rho_t}$. By the definition of the fine moduli space $M_{V,P}$,this means that $\Pi_t \iota_t$ is the morphism ι determined by the universal family over $M_{V,P,j}^{\rho} = \Pi_t M_{V,H_t}^{\rho_t}$. $\square$

Remark 49. By Lemma 47 and Lemma 48, $\Pi_t \iota_t$ factors through $\Pi_t \mathbb{M}_{V,H_t}^{\rho_t}$, a closed subscheme of $M_{V,P}$.



Definition 50. If a $M_{V,P} = \Pi_t M_{V,H_t}$, a $M_{V,P,j}^{\rho} = \Pi_t M_{V,H_t}^{\rho_t}$ and their respective universal family representatives are constructed together, using the inductive process in the proof of Lemma 48. Then the $M_{V,P}$ is called *attached to* the $M_{V,P,j}^{\rho}$.

Definition 51. With the preparation of the two lemmas above, the moduli space $M_{V',P}^{0\rho'}$ for Proposition 57 can be defined. There is a closed subscheme image (like $\Pi_t \mathbb{M}_{V,H_t}^{\rho_t}$ in Remark 49) of $M_{V',P,ij}^{\rho_i'}$ in the $M_{V',P}$ attached (see Definition 50) to $M_{V',P,ij}^{\rho_i'}$. Using the isomorphism (see Remark 45) from the $M_{V',P}$ attached to

$M^{\rho'_i}_{V',P,ij}$, to the fixed $M_{V',P,0}$, the closed subscheme image has its isomorphic image in $M_{V',P,0}$, which is denoted by $\mathbb{M}^{\rho'_i}_{V',P,ij}$. Denote the morphism $M^{\rho'_i}_{V',P,ij} \to \mathbb{M}^{\rho'_i}_{V',P,ij}$ by $\iota_{\rho'_i,ij}$. Let $M^{\rho'}_{V',P} = \coprod_{(M^{\rho'_i}_{V',P,ij})_i} \bigcap_i \mathbb{M}^{\rho'_i}_{V',P,ij}$ (see between Remark 45 and Example 46 for the tuple $(M^{\rho'_i}_{V',P,ij})_i$). Let M^0 be the dense open subset of $M_{V',P,0}$ which parameterizes all connected pointed P-covers of (V', v'_g) ([H80], Theorem 1.12). Let $M^{0\rho'}_{V',P} = \coprod_{(M^{\rho'_i}_{V',P,ij})_i} (\bigcap_i \mathbb{M}^{\rho'_i}_{V',P,ij} \bigcap M^0)$.

Remark 52. What does the space $M^{0\rho'}_{V',P}$ parameterize?

Every closed point in $M^{0\rho'}_{V',P}$ represents a connected pointed P-cover $(W, w_g) \to (V', v'_g)$ corresponding to some homomorphism $\pi_1(V', v'_g) \xrightarrow{\phi} P$ that is ρ'_i-liftable for every i. In fact, for every i the closed point gives a ρ'_i-liftable pair (ϕ, p_i) for some $p_i \in P$, by the definition of $M^{0\rho'}_{V',P}$ and the fact that every point in $M^{\rho^\bullet}_{V,P}$ represents a ρ-liftable pair as shown in Theorem 39.

See between Remark 45 and Example 46 for c'_i and θ'. The cover $(V', v'_g) \to (V'_{i'}, \overline{v'_{gi}})$ corresponds to some homomorphism $\pi_1(V'_{i'}, \overline{v'_{gi}}) \xrightarrow{\theta'_i} \langle p'_i \rangle$ that maps c'_i to p'_i. There is a similar diagram as in (3.1) with $\widehat{\phi}_i(c'_i) = (p_i, p'_i)$:

$$\begin{array}{ccccc} \pi_1(V', v'_g) & \xrightarrow{\quad\phi\quad} & P & & \\ \big\downarrow & & \big\downarrow & & \\ \pi_1(V'_{i'}, \overline{v'_{gi}}) & \xrightarrow{\quad\widehat{\phi}_i\quad} & P \rtimes_{\rho'_i} \langle p'_i \rangle & \xrightarrow{\quad Q_P\quad} & \langle p'_i \rangle, \end{array}$$

where the composition of the bottom two arrows $\pi_1(V'_{i'}, \overline{v'_{gi}}) \to \langle p'_i \rangle$ is θ'_i.

Hence the cover $(W, w_g) \to (V'_{i'}, \overline{v'_{gi}})$ is a pointed $P \rtimes_{\rho'_i} \langle p'_i \rangle$-cover. Denote by γ_i the element in the Galois group of $W/V'_{i'}$ corresponding to $(1, p'_i)$ with 1 the identity of P. Denote by $\gamma_{p'_i}$ the automorphism in the Galois group of $V' \to U$ that corresponds to $p'_i \in P'$. Denote by γ_p the element in the Galois group of W/V' that corresponds to $p \in P$. γ_i lies over $\gamma_{p'_i}$, and satisfies $ord(\gamma_i) = ord(p'_i)$ and $\gamma_i \gamma_p {\gamma_i}^{-1} = \gamma_{\rho'(p'_i)(p)}$ for every $p \in P$. These three conditions are called condition $(***i)$.

So a closed point in $M^{0\rho'}_{V',P}$ gives a pair $((W, w_g) \to (V', v'_g), \{\gamma_i\})$ with the first entry a connected pointed P-cover of (V', v'_g) and the second entry a subset of the Galois group of W/U with cardinality $|P'|$, the i-th element of which satisfies condition $(***i)$. The set of such pairs is denoted by $Gal_{V'}/\rho'$.

Reading backwards the discussion above, every pair in $Gal_{V'}/\rho'$ has a unique closed point in $M^{0\rho'}_{V',P}$ which represents the pair. Hence there is a canonical bijection between closed points in $M^{0\rho'}_{V',P}$ and $Gal_{V'}/\rho'$.

Remark 53. There is a finite partition of closed points in $M^{0\rho'}_{V',P}$ by covers' Galois groups over U.

With the same notations as in the previous remark, the cover W/U is Galois by a group order counting argument.

Denote $Gal(W/V')$ by Γ_p, $Gal(W/U)$ by Γ, and $Gal(V'/U)$ by $\Gamma_{p'}$. The isomorphism $\Gamma_p \simeq P$ is already given since $(W, w_g) \to (V', v'_g)$ is a P-cover. Similarly for $\Gamma_{p'} \simeq P'$.

Fix a subgroup $\widehat{\Gamma_{p'}}$ in Γ which maps isomorphically to $\Gamma_{p'}$ under the canonical quotient map $\Gamma \twoheadrightarrow \Gamma_{p'}$. The existence of such a subgroup is given by Schur-

Zassenhaus since $(\mathsf{p}, |P'|) = 1$. Then Γ is canonically isomorphic to $\Gamma_p \rtimes \widehat{\Gamma_{p'}}$, an inner semiproduct.

The isomorphism $\Gamma_{p'} \simeq P'$ induces an isomorphism $\widehat{\Gamma_{p'}} \simeq P'$. Substituting Γ_p by P and $\widehat{\Gamma_{p'}}$ by P' in $\Gamma_p \rtimes \widehat{\Gamma_{p'}}$, an induced semiproduct $P \rtimes_{\rho''} P'$ and an induced isomorphism $\Gamma \approx P \rtimes_{\rho''} P'$ are got. The diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_p & \longrightarrow & \Gamma & \longrightarrow & \Gamma_{p'} & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow \approx & & \downarrow \simeq & & \\ 1 & \longrightarrow & P & \longrightarrow & P \rtimes_{\rho''} P' & \longrightarrow & P' & \longrightarrow & 1. \end{array} \tag{53.1}$$

For every $p' \in P'$, the action of $\rho'(p')$ on P differs from that of $\rho''(p')$ by the conjugation of some element $p_{p'} \in P$, since γ_i and its counterpart in $\widehat{\Gamma_{p'}}$ differ by some element in Γ_p.

When P is abelian, the two groups $P \rtimes_{\rho'} P'$ and $P \rtimes_{\rho''} P'$ are the same. The Galois group over U for any element in $Gal_{V'/\rho'}$ is $P \rtimes_{\rho'} P'$. If P is not abelian, the two groups $P \rtimes_{\rho'} P'$ and $P \rtimes_{\rho''} P'$ may not be the same. The Galois group can not be nailed down.

The action ρ'' that arises above motivates the definition of a finite set consisting of certain semidirect products. Define $Gp_{\rho'}$ as the finite set $\{P \rtimes_{\rho''_s} P' \mid$ for every p'_i there exists a (p_i, p'_i) in $P \rtimes_{\rho''_s} P'$ such that ord(p_i, p'_i)=ord(p'_i) and $(p_i, p'_i)p(p_i, p'_i)^{-1} = \rho'(p'_i)(p)\}$, where $\rho''_s : P' \to Aut(P)$ is an action of P' on P.

By the end of Remark 52, the closed points of $M^{0\rho'}_{V',P}$ are in canonical bijection with $Gal_{V'}/\rho'$. Every pair in $Gal_{V'}/\rho'$ gives a pointed $P \rtimes_{\rho''_s} P'$-cover (a similar diagram to diagram (3.1)):

$$\begin{array}{ccccc} \pi_1(V', v'_g) & \xrightarrow{\phi} & P & & \\ \big\downarrow & & \big\downarrow & & \\ \pi_1(U, u_g) & \xrightarrow{\widehat{\phi}} & P \rtimes_{\rho''_s} P' & \xrightarrow{Q_P} & P', \end{array} \tag{53.2}$$

for some ρ''_s using the process given above diagram (53.1), where the composition of the bottom two arrows is θ'. The group $P \rtimes_{\rho''_s} P'$ is said to *belong to the pair or belong to the closed point corresponding to the pair*. A different $P \rtimes_{\rho''_{s_1}} P'$ can belong to the same pair, if a different section $\widehat{\Gamma_{p'}}$ is chosen in the process. If two $P \rtimes_{\rho''_s} P'$ and $P \rtimes_{\rho''_{s_1}} P'$ belong to the same pair, then a similar diagram to (53.2)

$$\begin{array}{ccccc} \pi_1(V', v'_g) & \xrightarrow{\phi} & P & & \\ \big\downarrow & & \big\downarrow & & \\ \pi_1(U, u_g) & \xrightarrow{\widehat{\phi}_1} & P \rtimes_{\rho''_{s_1}} P' & \xrightarrow{Q_P} & P' \end{array}$$

is also commutative. It together with diagram (53.2) gives a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P & \longrightarrow & P \rtimes_{\rho''_{s_1}} P' & \longrightarrow & P' & \longrightarrow & 1 \\ & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \simeq}\big\downarrow & & {\scriptstyle =}\big\downarrow & & \\ 1 & \longrightarrow & P & \longrightarrow & P \rtimes_{\rho''_s} P' & \longrightarrow & P' & \longrightarrow & 1. \end{array}$$

Hence $P \rtimes_{\rho''_s} P'$ and $P \rtimes_{\rho''_{s_1}} P'$ are isomorphic extensions. Pick a representative (pick $P \rtimes_{\rho'} P'$ in its class) in each isomorphism class of extensions and denote the subset obtained in this way of $Gp_{\rho'}$ by $\bar{G}p_{\rho'}$. The set $Gal_{V'}/\rho'$ has a finite partition by elements in $\bar{G}p_{\rho'} = \{P \rtimes_{\rho''_t} P'\}$ by discussion above.

For any $P \rtimes_{\rho''_s} P' \in Gp_{\rho'}$ and any pointed $P \rtimes_{\rho''_s} P'$-cover $(W, w_g) \to (U, u_g)$ corresponding to some $\pi_1(U, u_g) \xrightarrow{\hat{\hat{\phi}}} P \rtimes_{\rho''_s} P'$, a pointed $P \rtimes_{\rho''_{si}} \langle p'_i \rangle$-cover $(W, w_g) \to (V'_{i'}, \overline{v'_{gi}})$ can be got for every i:

$$\begin{array}{ccccc}
\pi_1(V', v'_g) & \longrightarrow & P & \xrightarrow{=} & P \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\pi_1(V'_{i'}, \overline{v'_{gi}}) & \longrightarrow & P \rtimes_{\rho''_{si}} \langle p'_i \rangle & \xrightarrow{\simeq} & P \rtimes_{\rho'_i} \langle p'_i \rangle \\
\big\downarrow & & \big\downarrow & & \\
\pi_1(U, u_g) & \xrightarrow{\hat{\hat{\phi}}} & P \rtimes_{\rho''_s} P', & &
\end{array} \qquad (53.3)$$

where ρ''_{si} is the restriction of ρ''_s on $\langle p'_i \rangle$. The pointed $P \rtimes_{\rho''_{si}} \langle p'_i \rangle$-cover $(W, w_g) \to (V'_{i'}, \overline{v'_{gi}})$ is also a pointed $P \rtimes_{\rho'_i} \langle p'_i \rangle$-cover, as shown in the commutative diagram (53.3), where the group isomorphism $P \rtimes_{\rho''_{si}} \langle p'_i \rangle \to P \rtimes_{\rho'_i} \langle p'_i \rangle$ sends (p_i, p'_i) to $(1, p'_i)$ and every $p \in P$ to p. Then Remark 52 shows that $(W, w_g) \to (U, u_g)$ gives a pair in $Gal_{V'}/\rho'$ corresponding to some closed point in $M^{0\rho'}_{V',P}$, which can be used to discover the original $(W, w_g) \to (U, u_g)$ using diagram (53.3). If a closed point in $M^{0\rho'}_{V',P}$ is used to discover, using diagram (53.3), a pointed $P \rtimes_{\rho''_s} P'$-cover of (U, u_g) for some $P \rtimes_{\rho''_s} P'$ in $Gp_{\rho'}$, there are several possibilities for ρ''_s.

To define the functor $F^{er,Gal/\rho'}_{V',P}$ in Proposition 57, several new definitions are needed. The inclusion of polynomial rings $Im\iota^*_{tn} \subset k[\{\mathtt{k}_i, 1 \le i \le d_K\}]$ in the proof of Lemma 47 and the morphism $\Pi_t M^{\rho_t}_{V,H_t} \to \Pi_t \mathbb{M}^{\rho_t}_{V,H_t}$ in Remark 49 motivate the first two definitions given below respectively.

Definition 54. Let $k[X_1, ..., X_d]$ be a polynomial ring. Suppose $k[X'_1, ..., X'_d]$ is a subring where each X_i' is a sum of powers $\Sigma_{i \leq t \leq d} X_t^{n_{it}}$ with n_{ii} a p-power, such that the subring also contains $X_i^{\mathsf{p}^{l_i}}$ with some $l_i \geq 0$ for every i. Let $\mathcal{P}'$ be a polynomial ring with an injective k-algebra homomorphism $f : \mathcal{P}' \hookrightarrow k[X_1, ..., X_d]$. If f gives an isomorphism between $\mathcal{P}'$ and $k[X_1', ..., X_d']$, then $f : \mathcal{P}' \hookrightarrow k[X_1, ..., X_d]$ is an *R-extension.*

Let $\{\mathcal{P}_i \hookleftarrow \mathcal{P}_i'\}$ be a collection of finitely many R-extensions, with possibly different $\mathcal{P}_i$'s and $\mathcal{P}_i'$'s. Tensoring over k gives a morphism $Spec(\otimes_i \mathcal{P}_i) \to Spec(\otimes_i \mathcal{P}_i')$. For any $(S', s_0') \in \mathcal{S}$ and $(S', s_0') \xrightarrow{f} (Spec(\otimes_i \mathcal{P}_i'), x_0')$ a morphism in $\mathcal{S}$, the pullback $(S, s_0) \to (S', s_0')$ of $(Spec(\otimes_i \mathcal{P}_i), x_0) \to (Spec(\otimes_i \mathcal{P}_i'), x_0')$, for some x_0 mapping to x_0', is called *a morphism of type R*:

$$\begin{array}{ccc} S & \xrightarrow{f'} & Spec(\otimes_i \mathcal{P}_i) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & Spec(\otimes_i \mathcal{P}_i'). \end{array}$$

A morphism $(T, t_0) \to (S, s_0)$ in $\mathcal{S}$ is of *type ER*, if it can be decomposed into a finite sequence of finite etale covers and morphisms of type R.

Remark 55. The morphism $M^{\rho_i'}_{V',P,ij} \xrightarrow{\iota_{\rho_i',ij}} \mathbb{M}^{\rho_i'}_{V',P,ij}$ in Definition 51 is an example of the right column in the square diagram in Definition 54.

Below is the definition for the functor to present in Proposition 57, which is motivated by the discussion in Remark 52. Let $(S, s_0) \in \mathcal{S}$ and let $Gal_{(S,s_0)}$ be the set of T-parameterized P-covers $(\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v_g'))$ of V' pointed over (t_0, v_g') for some $(T, t_0) \to (S, s_0)$ of type ER with connected fibers over the closed points of T, such that the composition $(\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g)) \to (T \times U, (t_0, u_g))$ is Galois. For a pointed P-cover $(\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g))$, and an element $p \in P$, denote by $\tilde{\gamma}_p$ the automorphism in its Galois group that corresponds to p. Denote by $\tilde{\gamma}_{p'_i}$ the automorphism in the Galois group of $T \times V' \to T \times U$ that corresponds to $p'_i \in P'$. Let $Gal/\rho'_{(S,s_0)}$ be the set of pairs $((\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g)), \{\tilde{\gamma}_i\})$, where $(\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g))$ is in $Gal_{(S,s_0)}$ and $\tilde{\gamma}_i$ is in the Galois group of the cover $\widetilde{W} \to T \times U$ that lies over $\tilde{\gamma}_{p'_i}$ and satisfies $ord(\tilde{\gamma}_i) = ord(p'_i)$ and $\tilde{\gamma}_i \tilde{\gamma}_p \tilde{\gamma}_i^{-1} = \tilde{\gamma}_{\rho'(p'_i)(p)}$ for every $p \in P$. The set $Gal/\rho'_{(S,s_0)}$ is an S-parameterized version of $Gal_{V'/\rho'}$; by Remark 52, the set $F^{er,Gal/\rho'}_{V',P}(Spec(k), s_0)$, with s_0 determined by v'_g using diagram (2.1), is the set $Gal_{V'}/\rho'$. Two elements $((\widetilde{W}_j, \tilde{w}_{gj}) \to (T_j \times V', (t_{j0}, v'_g)), \{\tilde{\gamma}_{ji}\})$ $(j = 1, 2)$ in $Gal/\rho'_{(S,s_0)}$ are *ER-equivalent* if there exists a morphism $(T_d, t_{d0}) \to (S, s_0)$ of type ER, where (T_d, t_{d0}) also maps to (T_j, t_{j0}) $(j = 1, 2)$ in the category $\mathcal{S}$, such that the two pointed P-covers, together with $\{\tilde{\gamma}_{1i}\}$ and $\{\tilde{\gamma}_{2i}\}$, pulled back to T_d become isomorphic. Let $F^{er,Gal/\rho'}_{V',P}$ be the functor: $\mathcal{S} \to$ (Sets); $(S, s_0) \mapsto$ $\{ER$-equivalence classes of $((\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g)), \{\tilde{\gamma}_i\}) \in Gal/\rho'_{(S,s_0)}\}$.

Here is the last definition involved in the statement of Proposition 57. Two morphisms $T_j \xrightarrow{f_j} M^{0\rho'}_{V',P}$ (j =1 or 2, where $(T_j, t_{j0}) \to (S, s_0)$ is of type ER) are *ER-equivalent*, if there exists a morphism $(T_d, t_{d0}) \to (S, s_0)$ of type ER with (T_d, t_{d0}) also mapping to (T_j, t_{j0}) $(j = 1, 2)$ in the category $\mathcal{S}$, such that the f_j's pulled back to T_d are the same. Let $M^{er0\rho'}_{V',P}$ be the functor: $\mathcal{S} \to$ (Sets); $(S, s_0) \mapsto \{ER$-equivalence classes of $T \to M^{0\rho'}_{V',P}$, where $(T, t_0) \to (S, s_0)$ runs over all morphisms to S of type $ER\}$.

Remark 56. The two ER-equivalences in the definitions of functors $F^{er,Gal/\rho'}_{V',P}$ and $M^{er0\rho'}_{V',P}$ arise naturally in the proof of Proposition 57 based on the intersection idea.

Proposition 57. *With the same notations as above, the ind scheme $M^{0\rho'}_{V',P}$ is the moduli space for $F^{er,Gal/\rho'}_{V',P}$ in the sense that there exists an isomorphism between functors $F^{er,Gal/\rho'}_{V',P} \simeq M^{er0\rho'}_{V',P}$.*

Moreover, on each of the finitely many irreducible components of $M^{0\rho'}_{V',P}$, there is a unique $P \rtimes_{\rho''_t} P'$ in $\bar{G}p_{\rho'}$ which belongs to (defined in Remark 53) all the closed points. Conversely, for every $P \rtimes_{\rho''_t} P'$ in $\bar{G}p_{\rho'}$, there is an irreducible component, such that $P \rtimes_{\rho''_t} P'$ belongs to all the closed points of the component.

Proof. Proof of the first statement:

Let $(S, s_0) \in \mathcal{S}$ and $((\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g)), \{\gamma_i\})$ be a representative in an ER-equivalence class of $F^{er,Gal/\rho'}_{V',P}(S, s_0)$. Then $(\widetilde{W}, \tilde{w}_g) \to (T \times V'_{i'}, (t_0, \overline{v'_{gi}}))$ is Galois (see Remark 52). Letting γ_i correspond to $(1, p'_i) \in P \rtimes_{\rho'_i} \langle p'_i \rangle$, $(\widetilde{W}, \tilde{w}_g) \to (T \times V'_{i'}, (t_0, \overline{v'_{gi}}))$ is a pointed $P \rtimes_{\rho'_i} \langle p'_i \rangle$-cover. By the definition of $M^{\rho'_i,\bullet}_{V',P}$, the pointed $P \rtimes_{\rho'_i} \langle p'_i \rangle$-cover corresponds to a morphism $T \xrightarrow{\mathfrak{c}_i} M^{\rho'_i,\bullet}_{V',P}$. Since T is connected, the morphism $\mathfrak{c}_i$ lands in a connected component $M^{\rho'_i}_{V',P,ij}$ of $M^{\rho'_i,\bullet}_{V',P}$. Embedding $M^{\rho'_i}_{V',P,ij}$ in $M_{V',P,0}$ as in Remark 55, a morphism $T \xrightarrow{\hat{\mathfrak{c}}_i} M_{V',P,0}$ is got. The morphism $\hat{\mathfrak{c}}_i$ is the same as the morphism from T to $M_{V',P,0}$ determined by the pointed P-cover $(\widetilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g))$, using that $M_{V',P,0}$ is the fine moduli space for pointed families of P-covers of (V', v'_g) ([H80], Theorem 1.2). The above discussion applies for every i. Hence a morphism $T \xrightarrow{\hat{\mathfrak{c}}} M^{0\rho'}_{V',P}$ is got, by the definition of $M^{0\rho'}_{V',P}$.

Conversely, given $T \xrightarrow{\hat{\mathfrak{c}}} M^{0\rho'}_{V',P}$ for some $(T, t_0) \to (S, s_0)$ of type ER, a morphism $T \xrightarrow{\hat{\mathfrak{c}}} M_{V',P,0}$ is got by the definition of $M^{0\rho'}_{V',P}$ and the connectedness of T. For each i, there is a morphism $T \xrightarrow{\hat{\mathfrak{c}}_i} \mathbb{M}^{\rho'_i}_{V',P,ij}$, for some $\mathbb{M}^{\rho'_i}_{V',P,ij}$ (see Definition 51 for $\mathbb{M}^{\rho'_i}_{V',P,ij}$) that $\hat{\mathfrak{c}}$ factors through

$$\begin{array}{ccc} T & \xrightarrow{\hat{\mathfrak{c}}_i} & \mathbb{M}^{\rho'_i}_{V',P,ij} \\ & \searrow^{\hat{\mathfrak{c}}} & \big\downarrow \\ & & M_{V',P,0}. \end{array}$$

After pointing $M^{\rho'_i}_{V',P,ij} \xrightarrow{\iota_{\rho'_i,ij}} \mathbb{M}^{\rho'_i}_{V',P,ij}$ (see Definition 51) properly, the pullback morphism $(T_i, t_{i0}) \to (T, t_0)$ is of type R, by Remark 55 and the lower square of the diagram, in which both squares are pullbacks:

$$\begin{array}{ccc} T_{di} & \longrightarrow & M_{ij} \\ \downarrow & & \downarrow \\ T_i & \xrightarrow{\mathfrak{c}_i} & M^{\rho'_i}_{V',P,ij} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\hat{\mathfrak{c}}_i} & \mathbb{M}^{\rho'_i}_{V',P,ij}. \end{array}$$

The (ind) (see Remark 41) scheme M_{ij} in the upper right corner is a finite etale cover of $M^{\rho'_i}_{V',P,ij}$, such that a fixed universal family representative over $M_{V',P,0}$ pulled back to M_{ij}, is the same as the pullback to M_{ij} of a ρ'_i-liftable universal family representative over $M^{\rho'_i}_{V',P,ij}$ (see the discussion between Lemma 47 and Lemma 48).

The relationship between M_{ij} and $M^{\rho'_i}_{V',P,ij}$ is the same as that between T and S in the proof of Theorem 39. Hence the pullback $(T_{di}, t_{di0}) \to (T_i, t_{i0})$ is a finite etale cover.

The two diagrams together imply that the fixed universal family representative over $M_{V',P,0}$, which is a pointed P-cover of $(M_{V',P,0} \times V', (\hat{\mathfrak{c}}(t_0), v'_g))$, pulled back to T_{di} is ρ'_i-liftable. Let (T_d, t_{d0}) be the common pullback of the (T_{di}, t_{di0})'s over (T, t_0). The pullback to T_d of the fixed universal family representative over $M_{V',P,0}$ is ρ'_i-liftable for every i. Consider ρ'_i-liftable pairs, rather than merely ρ'_i-liftable covers, at some places in the discussion above. Then a pair (see Remark 52) $((\widetilde{W}, \tilde{w}_g) \to (T_d \times V', (t_{d0}, v'_g)), \{\gamma_i\})$ is got, a pointed P-cover together with $|P'|$ elements in $Gal(\widetilde{W}/T_d \times U)$, whose ER-equivalence class is in $F^{er,Gal/\rho'}_{V',P}(S, s_0)$.

The two maps are well defined for equivalence classes and inverse to each other.

Proof of the statements after "Moreover": Every component $\bigcap_i \mathbb{M}^{\rho'_i}_{V',P,ij} \bigcap M^0$ of $M^{0\rho'}_{V',P}$ is a dense open of $\bigcap_i \mathbb{M}^{\rho'_i}_{V',P,ij}$, which itself is an affine closed subscheme of $M_{V',P,0}$. Pick any irreducible component of $\bigcap_i \mathbb{M}^{\rho'_i}_{V',P,ij} \bigcap M^0$ and a covering of it consisting of connected affine open subsets of finite type over k, which are all dense and intersect each other. Denote any of the affine open subsets by $\mathbb{M}$ and apply the first statement proven above to $\mathbb{M}$. The inclusion of $\mathbb{M}$ in $M^{0\rho'}_{V',P}$, with any base point m_g, gives a pointed P-cover of $(\mathbb{M}' \times V', (m'_g, v'_g))$ for some $(\mathbb{M}', m'_g) \to (\mathbb{M}, m_g)$ of type-ER. The pointed P-cover satisfies a $\mathbb{M}'$-parameterized version of $(***i)$ for every i and thus gives a pointed $P \rtimes_{\rho''_t} P'$-cover of $(\mathbb{M}' \times U, (m'_g, u_g))$ for some unique $P \rtimes_{\rho''_t} P'$ in $\bar{G}p_{\rho'}$ (see Remark 52 and Remark 53). Hence for every closed point m (need to use chemins for base point issues) in $\mathbb{M}$, $P \rtimes_{\rho''_t} P'$ belongs to the pair in

$Gal_{V'/\rho'}$ that m corresponds. Then Remark 52 and Remark 53 suffice to give all the statements. $\square$

Denote the maximal union of irreducible components of $M^{0\rho'}_{V',P}$, to all of whose closed points $P \rtimes_{\rho'} P'$ belongs (see the last paragraph in the proof of Proposition 57), by $M_{U,V',P\rtimes_{\rho'}P'}$. Denote by $M_{U,P\rtimes_{\rho'}P'}$ the disjoint union of $M_{U,V',P\rtimes_{\rho'}P'}$'s over all possible (V', v'_g)'s pointed connected P'-covers of (U, u_g).

Define a functor $M^{er}_{U,P\rtimes_{\rho'}P'}$: $\mathcal{S} \to$ (Sets); $(S, s_0) \mapsto \{ER$-equivalence classes of $T \to M_{U,P\rtimes_{\rho'}P'}$, where $(T, t_0) \to (S, s_0)$ runs over all morphisms to S of type $ER\}$. Similarly to $M^{er0\rho'}_{V',P}$ and $M^{0\rho'}_{V',P}$ above.

Define a functor $F^{er}_{U,P\rtimes_{\rho'}P'}$: $\mathcal{S} \to$ (Sets); $(S, s_0) \mapsto \{ER$-equivalence classes of pointed $P \rtimes_{\rho'} P'$-covers $(\widetilde{W}, \tilde{w}_g) \to (T \times U, (t_0, u_g))$ whose fibers over closed points of T are all connected, where $(T, t_0) \to (S, s_0)$ runs over all morphisms to S of type $ER\}$. The definition of ER-equivalence classes here is obvious (see definition of the functor $F^{er,Gal/\rho'}_{V',P}$).

Corollary 58. *The functor $F^{er}_{U,P\rtimes_{\rho'}P'}$ is isomorphic to the functor $M^{er}_{U,P\rtimes_{\rho'}P'}$, which shows that $M_{U,P\rtimes_{\rho'}P'}$ is a moduli space for $P \rtimes_{\rho'} P'$-covers of (U, u_g).*

Proof. Directly from Proposition 57 and its proof. See also proof of Theorem 44. $\square$

Chapter 5

Local vs. global moduli

In Chapter 5, a fine moduli space (Proposition 64) for cyclic-by-p covers of an affine curve at most tamely ramified over finitely many closed points, is constructed. The new type of fine moduli space is obtained by modifying the proof for the previous global fine moduli space constructed in Theorem 44 Chapter 3, and is constructed in similar 3 steps. The new type of fine moduli space is the global side of a local-global principal Proposition 76. There is a different phenomenon for cyclic-by-p covers from that for p-covers. In [H80] the similar local-global principal for p-groups stated in Proposition 2.1 does not involve (tamely) ramified global covers; there the global covers are etale. The local-global principal Proposition 76 has a version over a general field of characteristic $\mathsf{p} > 0$, which is Main Theorem 1.4.1 in [K86].

A parameter space for local cyclic-by-p covers of $Spec(k((x)))$ is constructed in

Proposition 75, which is the local side of the local-global principal Proposition 76. The construction is also by modifying the one in Chapter 3 and has similar 3 steps.

Finally it is shown that a restriction morphism (a general case of the local-global principal Proposition 76 with the isomorphism there replaced by a finite morphism now) is finite, which is from the new type of global moduli space to a product of the local parameter spaces (Proposition 83), an analogue to Proposition 2.7 in [H80]. It is proved by a similar argument.

As always, we follow notations and terminology defined in Chapter 2. For example G represents a cyclic-by-p group.

Here are some necessary settings for the construction of the fine moduli space (Proposition 64).

Let T be a finite set of closed points on U not including u_g and $U^0 = U - T$. Denote by $\{(V_l^0, v_l)\}$ the set of all the finitely many connected pointed $\mathbb{Z}/n_l$-covers of (U^0, u_g), where n_l can be any factor of n. Let $(V_l, v_l) \to (U, u_g)$ be the extension of $(V_l^0, v_l) \to (U^0, u_g)$, obtained by putting back in some deleted closed points from the smooth completions of both curves.

Definition 59. Let $F_{U,G}^T$ be the functor: $\mathcal{S}_1 \to$(Sets), $(S, s_0) \mapsto$\{equivalence classes of possibly ramified G-covers $\widetilde{W} \to S \times U$ pointed over (s_0, u_g), where the restriction of $\widetilde{W}$ over $S \times U^0$ is a G-cover and $\widetilde{W} \to \widetilde{W}/P$ is finite etale\}. Let $S = Spec(k)$ with s_0 determined by u_g using diagram (2.1). Then $F_{U,G}^T(S, s_0)$ is the set of possibly ramified pointed G-covers $(W, w_g) \to (U, u_g)$ whose restriction W^0 over U^0 is a G-cover and $W \to W/P$ is finite etale.

A group homomorphism $\widetilde{\phi}^0 : \pi_1(S \times V_l^0, (s_0, v_l)) \to P$ *factors through* V_l if $\widetilde{\phi}^0 = (\pi_1(S \times V_l^0, (s_0, v_l)) \xrightarrow{(Id_S \times (V_l^0 \subset V_l))_*} \pi_1(S \times V_l, (s_0, v_l)) \xrightarrow{\widetilde{\phi}} P)$ for some $\widetilde{\phi}$.

Definition 60. Let $F^{\rho_{n_l}\bullet/T}_{V_l^0,P}$ be the functor: $\mathcal{S}_1 \to$(Sets), $(S, s_0) \mapsto \{([\widetilde{\phi}^0], p)$, where $(\widetilde{\phi}^0, p)$ is a ρ_{n_l}-liftable pair with $\widetilde{\phi}^0$ factoring through V_l.$\}$. Let $S = Spec(k)$ with s_0 determined by v_l using diagram (2.1). Then $F^{\rho_{n_l}\bullet/T}_{V_l^0,P}(S, s_0)$ is the set of ρ_{n_l}-liftable pairs of (V_l^0, v_l) the first entries of which can all extend to P-covers of V_l.

Below is the first of the 3 steps in constructing the fine moduli space in Proposition 64.

Definition 61. (Remark/Definition)

The cover $V_l \to U$ above is ramified at finitely many closed points on U. Hence the old definition of ρ-liftable can not apply here. New definition: A group homomorphism $\phi : \pi_1(V_l, v_l) \to H$ with (ρ, H) in the case of Theorem 32 is *ρ-liftable* if ϕ makes the diagram in Lemma 30 commutative. Using the new definition of ρ-liftable in the definition of $F^\rho_{V,H}$, Theorem 32 still holds and $F^\rho_{V,H}$ is presented by the same fine moduli space $M^\rho_{V,H}$.

The new definition of ρ-liftable is used below for $V_l \to U$.

Lemma 62. *With the notations above, suppose (ρ_{n_l}, P) is in the case of Theorem 32 and write H for P in this case. The functor $F^{\rho_{n_l}\bullet/T}_{V_l^0,H}$ for ρ_{n_l}-liftable pairs of (V_l^0, v_l)*

the first entries of which can all extend to H-covers of V_l, has a fine moduli space, a disjoint union of finitely many ind affine spaces.

Proof. It is to be shown that $F^{\rho_{n_l}\bullet/T}_{V_l^0,H} = \amalg_h F^{\rho_{n_l}}_{V_l,H}$, copies of $F^{\rho_{n_l}}_{V_l,H}$ indexed by h, where depending on (ρ_{n_l}, H) the h runs over H or it is just 1 (see Corollary 36). Suppose a ρ_l-liftable pair $(\widetilde{\phi}^0 : \pi_1(S \times V_l^0, (s_0, v_l)) \to H, h_0)$ with $\widetilde{\phi}^0$ factoring through V_l is a representative for an equivalence class $([\widetilde{\phi}^0], h_0)$ in $F^{\rho_{n_l}\bullet/T}_{V_l^0,H}(S, s_0)$. Since $(Id_S \times (V_l^0 \subset V_l))_*$ is surjective, $\widetilde{\phi}$ is also ρ_{n_l}-liftable (see Definition 61 and see the definition of "factoring through V_l" above for $\widetilde{\phi}$). The left to right map sends $([\widetilde{\phi}^0], h_0)$ to $[\widetilde{\phi}]$ in the copy of $F^{\rho_{n_l}}_{V_l,H}(S, s_0)$ indexed by h_0. The inverse map is obvious. By Theorem 32 (see also Definition 61), $F^{\rho_{n_l}}_{V_l,H}$ is represented by an ind affine space $M^{\rho_{n_l}}_{V_l,H}$. □

Here is the 2nd of the 3 steps in constructing the fine moduli space in Proposition 64.

Lemma 63. *The functor $F^{\rho_{n_l}\bullet/T}_{V_l^0,P}$ for ρ_{n_l}-liftable pairs of (V_l^0, v_l) the first entries of which can all extend to P-covers of V_l, has a fine moduli space, a disjoint union of finitely many ind affine spaces.*

Proof. The proof is by simply replacing with their obvious counterparts symbols in and slightly modifying the proof of Theorem 39.

Denote V_l by V and ρ_{n_l} by ρ in the proof. Replace $F^{\rho\bullet}_{V,P}$ with $F^{\rho\bullet/T}_{V^0,P}$, Corollary 36 with Lemma 62, $M^{\bar{\rho}\bullet}_{V,\bar{P}}$ with $M^{\bar{\rho}\bullet/T}_{V^0,\bar{P}}$, and $M^{\rho_0\bullet}_{V,H}$ with $M^{\rho_0\bullet/T}_{V^0,H}$.

In paragraph (39.2) of the proof of Theorem 39, since the first entry of some universal pair $(\widetilde{\mu}_0^0 : \pi_1(\bar{M} \times V^0, (\bar{m}, v_l)) \to \bar{P}, \bar{p}_0)$ can factor through V: $\widetilde{\mu}_0^0 = (\pi_1(\bar{M} \times V^0, (\bar{m}, v_l)) \to \pi_1(\bar{M} \times V, (\bar{m}, v_l)) \xrightarrow{\widetilde{\mu}_0} \bar{P})$ for some $\widetilde{\mu}_0$, lift $(\widetilde{\mu}_0, \bar{p}_0)$ first to get $(\widetilde{\psi}_0, p_0)$ with restriction $\widetilde{\psi}_0^0$ on $\bar{M} \times V^0$.

Paragraph (39.3) in the proof of Theorem 39 carries over with some obvious modification involving the property of factoring through V. □

Below is the last of the 3 steps in constructing the fine moduli space in Proposition 64.

Proposition 64. *There is a fine moduli space representing $F_{U,G}^T$, the functor for pointed G-covers of (U, u_g) tamely ramified over finitely many closed points T on U, which is a disjoint union of finitely many ind affine spaces.*

Proof. By the same argument for Theorem 44 with slight modification, $F_{U,G}^T = \amalg_{V_l} F_{V_l^0,P}^{\rho_{n_l}\bullet/T}$ which has a fine moduli space by Lemma 63. □

Above is the construction of the global side of the local-global principal Proposition 76, whose local side is the local parameter space in Proposition 75, constructed below in similar 3 steps.

Below are necessary settings of Proposition 68.

Notation 65. Recall $U_0 = Spec(k((x)))$ and point U_0 at u_0. Let (V_{0t}, v_{0t}) run over all the finitely many connected pointed $\mathbb{Z}/n_t$-covers of (U_0, u_0), where n_t can be any factor of n. Let V_0 be a connected $\mathbb{Z}/n$-cover of U_0 given by $k((x))[Y]/(Y^n - x) =$

$k((y))$ with $\bar{1} \in \mathbb{Z}/n$ acting on $k((y))$ as $y \mapsto \zeta_n y$. Since $\mathbb{Z}/n$ is abelian, V_0 can be pointed at any v_0 over u_0, by Remark 20.

Definition 66. Two pointed Gr-covers of $(S \times U_0, (s_0, u_0))$ with $(S, s_0) \in \mathcal{S}_1$ are *w-equivalent* if they become isomorphic pulled back to $(\widetilde{T}^0, \tilde{t}_0)$, which is the restriction over $S \times U_0$ of some finite etale cover $(\widetilde{T}, \tilde{t}_0) \to (S \times \bar{U}_0, (s_0, u_0))$.

Let $\tilde{\varphi}_i : \pi_1(S \times U_0, (s_0, u_0)) \to Gr$ (i=1,2) be two group homomorphisms. They are *w-equivalent* if their corresponding pointed Gr-covers of $(S \times U_0, (s_0, u_0))$ are w-equivalent. Denote the w-equivalence class of $\tilde{\varphi}_1$ by $[\tilde{\varphi}_1]^w$.

Let $F^{w\rho}_{V_0,P}$ be the functor: $\mathcal{S}_1 \to$(Sets), $(S, s_0) \mapsto${w-equivalence classes of ρ-liftable P-covers of $S \times V_0$ pointed over (s_0, v_0)}.

Remark 67. The definition of w-equivalence is taken from the 2nd paragraph in the proof of Proposition 2.1 in [H80], which is the right definition of equivalence in the local case to make the proof work.

Below is the building block needed in the first of the 3 steps in constructing the local parameter space in Proposition 75.

Proposition 68. *Let (V_0, v_0) be given in Notation 65. Suppose (ρ, H) is in the case of Theorem 32. Then there exists a fine moduli space $M^{w\rho}_{V_0,H}$ for $F^{w\rho}_{V_0,H}$, the functor for w-equivalence classes of pointed ρ-liftable H-covers of (V_0, v_0).*

Proof. The proof is parallel to that of Theorem 32.

Similarly to the proof of Theorem 32, start with a short exact sequence $k((y)) \xrightarrow{\wp} k((y)) \xrightarrow{\pi} H^1(V_0, H) \to 0$ given by the Artin-Schreier sequence. It can be simplified to $y^{-1}k[y^{-1}] \xrightarrow{\wp} y^{-1}k[y^{-1}] \xrightarrow{\pi} H^1(V_0, H) \to 0$ (68.1).

Denote by σ_0 the automorphism in $Gal(k((y))/k((x)))$ given by $y \mapsto \zeta_n y$. The action of $\rho(-\bar{1})$ on H is given by multiplication by some $e_\rho \in \mathbb{F}_q$ ($q = |H|$; see proof of Theorem 32). Similarly let D_0 be the $x^{-1}k[x^{-1}]$-module endomorphism $\sigma_0 - e_\rho$ of $y^{-1}k[y^{-1}]$.

Similarly extract from (68.1) an $\mathbb{F}_q$-vector space short exact sequence $KerD_0 \xrightarrow{\wp} KerD_0 \xrightarrow{\pi} \mathbb{X}_0 \to 0$, where $\mathbb{X}_0$ is the set of ρ-liftable pointed H-covers of (V_0, v_0). From the Ker exact sequence construct the fine moduli space $M^{w\rho_0}_{V_0,H}$ same as before, which is also an ind affine space. Choose basis K_{0i}, an analogue to K_i in the proof of Theorem 32, inductively for $i \in \mathbb{N}$. The affine space $Spec(k[K^\vee_{0i+1} - K^\vee_{0i}])$ can be identified with the $(i+1)$-th piece of $M^{w\rho}_{V_0,H}$; the transition morphism from $Spec(k[K^\vee_{0i} - K^\vee_{0i-1}])$ to $Spec(k[K^\vee_{0i+1} - K^\vee_{0i}])$ is given by Frobenius as before. Finally, with slight modification to the last two paragraghs of the proof of Theorem 32, $M^{w\rho}_{V_0,H}$ can be shown to represent $F^{w\rho}_{V_0,H}$. □

Remark 69. Similar to Remark 34, a canonical universal family representative over $M^{w\rho}_{V_0,H}$ can be given by $z^q - z = \sum_{\mathtt{k}_{0i} \in K_{0n} - K_{0n-1}} \mathtt{k}^\vee_{0i} \otimes \mathtt{k}_{0i}$ ($n \geq 1$). Precise description can be got by some obvious replacement of symbols in Remark 34.

Remark 70. The remark is the base case in the proof for the local-global principal Proposition 76. Let $\mathbb{A}^{1'} = Spec(k[x^{-1}])$. Suppose $\mathbb{A}^{1'}$ is pointed at a_g such that the map $U_0 \to \mathbb{A}^{1'}$ sends u_0 to a_g.

Let $V \to \mathbb{A}^{1'}$ be the $\mathbb{Z}/n$-cover given by $k[x^{-1}][Y^{-1}]/((Y^{-1})^n - x^{-1}) = k[y^{-1}]$ ramified at ∞, with $\bar{1} \in \mathbb{Z}/n$ acting as $y^{-1} \mapsto \zeta_n^{-1} y^{-1}$. Point V at v_g such that $V \to \mathbb{A}^{1'}$ sends v_g to a_g. Its restriction (pullback) at 0 gives (V_0, v_0') and let v_0 above be v_0':

$$\begin{array}{ccc} (V_0, v_0') & \longrightarrow & (V, v_g) \\ \downarrow & & \downarrow \\ (U_0, u_0) & \longrightarrow & (\mathbb{A}^{1'}, a_g). \end{array}$$

The constructions show that $M^{w\rho}_{V_0,H} = M^{\rho}_{V,H}$:

The short exact sequence $y^{-1}k[y^{-1}] \xrightarrow{\wp} y^{-1}k[y^{-1}] \xrightarrow{\pi} H^1(V_0, H) \to 0$ in the proof of Proposition 68, is similar to the one $k[y^{-1}] \xrightarrow{\wp} k[y^{-1}] \xrightarrow{\pi} H^1(V, H) \to 0$ for V in the proof of Theorem 32, after modding $k[y^{-1}]$ by k. The short exact sequence $KerD_0 \xrightarrow{\wp} KerD_0 \xrightarrow{\pi} \mathbb{X}_0 \to 0$ above, is similar to the short exact sequence $KerD \xrightarrow{\wp} KerD \xrightarrow{\pi} \mathbb{X} \to 0$ for V in the proof of Theorem 32 and $KerD_0 = KerD$. Then the constructions of the two moduli spaces out of the Ker short exact sequences are the same, which shows that $M^{w\rho}_{V_0,H}$ is the same ind scheme as $M^{\rho}_{V,H}$.

Moreover, there is a triangle compatibility diagram. For any pointed ρ-liftable H-cover (W, w_g) of (V, v_g) corresponding to some k-morphism $Spec(k) \xrightarrow{\mathfrak{c}_g} M^{\rho}_{V,H}$, its restriction (W_0, w_0) over (V_0, v_0) is a pointed ρ-liftable H-cover of V_0:

$$\begin{array}{ccc} (W_0, w_0) & \longrightarrow & (W, w_g) \\ \downarrow & & \downarrow \\ (V_0, v_0) & \longrightarrow & (V, v_g). \end{array}$$

The local cover corresponds to some k-morphism $Spec(k) \xrightarrow{\mathfrak{c}_0} M^{w\rho}_{V_0,H}$. The following diagram commutes:

$$\begin{array}{ccc} Spec(k) & \xrightarrow{\mathfrak{c}_0} & M^{w\rho}_{V_0,H} \\ & \searrow^{\mathfrak{c}_g} & \downarrow = \\ & & M^{\rho}_{V,H}. \end{array} \tag{70.1}$$

Remark 71. All the $\mathbb{Z}/n_t$-covers of U_0, where n_t can be any factor of n, correspond bijectively to all the $\mathbb{Z}/n_t$-covers of $\mathbb{A}^{1'}$ ramified at ∞, since these covers can be given by explicit equations of the type in Proposition 68 and that in Remark 70.

Below is the first of the 3 steps in constructing the local parameter space in Proposition 75 using the building block in Proposition 68.

Definition 72. For the local case, a *pointed ρ-liftable P-cover* of (V_0, v_0) is defined in the obvious similar way to the global case defined in diagram (3.1). Similarly for a *ρ-liftable pair.*

The k-points of an ind scheme M *parameterize* certain covers, if there is a bijection χ together given with M between the set of k-points of M and the set of these certain covers.

Lemma 73. *Suppose (ρ, H) is in the case of Theorem 32. There exists a parameter space $M^{p\rho,\bullet}_{V_0,H}$, a disjoint union of finitely many ind affine spaces, whose k-points parameterize (see Definition 72) all the ρ-liftable pairs of (V_0, v_0).*

Proof. Let $S = Spec(k)$ pointed at s_0 that is determined by v_0, using diagram (2.1). Since $M^{w\rho}_{V_0,H}$ represents $F^{w\rho}_{V_0,H}$, there is a bijection $\chi^{w\rho}_{V_0,H}$ between $F^{w\rho}_{V_0,H}(Spec(k), s_0)$ and $M^{w\rho}_{V_0,H}(Spec(k))$. $F^{w\rho}_{V_0,H}(Spec(k), s_0)$ is the set of pointed ρ-liftable H-covers of (V_0, v_0).

Let $M^{p\rho,\bullet}_{V_0,H} = \amalg_h M^{w\rho}_{V_0,H}$, an analogue of Corollary 36. Depending on (ρ, H), h runs over H or it is just 1. By the same kind of argument of Corollary 36, there is a bijection $\chi^{p\rho,\bullet}_{V_0,H}$ between the set of ρ-liftable pairs of pointed H-covers of (V_0, v_0), and $M^{p\rho,\bullet}_{V_0,H}(Spec(k))$. □

Here is the 2nd of the 3 steps in constructing the local parameter space in Proposition 75.

Lemma 74. *There exists a parameter space $M^{p\rho,\bullet}_{V_0,P}$, a disjoint union of finitely many ind affine spaces, whose k-points parameterize (see Definition 72) all the ρ-liftable pairs of (V_0, v_0).*

Proof. The proof is parallel to that of Theorem 39 but simpler. It simply replaces some symbols in and do a little modification to the proof of Theorem 39.

First of all there is no longer an F, instead there is $C^{\rho\bullet}_{V_0,P}$ the set of ρ-liftable pairs (of pointed P-covers) of (V_0, v_0).

In paragraph (39.1) replace $M^{\bar{\rho}\bullet}_{V,\bar{P}}$ by $M^{p\bar{\rho}\bullet}_{V_0,\bar{P}}$ and $M^{\rho_0\bullet}_{V,H}$ by $M^{p\rho_0\bullet}_{V_0,H}$.

In paragraph (39.2) replace V by V_0.

In paragraph (39.3), there is no longer an S. Replace every V by V_0, and v_g by v_0. Replace $(\widetilde{\phi}, p_1)$ by an element $(\varphi : \pi_1(V_0, v_0) \to P, p_1)$ in $C^{\rho\bullet}_{V_0,P}$ and $(\bar{\widetilde{\phi}}, \bar{p}_1)$ by $(\bar{\varphi} : \pi_1(V_0, v_0) \to \bar{P}, \bar{p}_1)$. Then replace β by a k-morphism $Spec(k) \xrightarrow{\mathfrak{c}_\beta} \bar{M}$ and $\tilde{\beta}_*$ by $\tilde{\mathfrak{c}}_{\beta*}$. There is no need for etale descent now and one directly gets a $\mathfrak{c}_\alpha : Spec(k) \to M^0$. Then replace $M^{\rho\bullet}_{V,P}$ by $M^{p\rho\bullet}_{V_0,P}$. Finally the assignment $\varphi \mapsto (\mathfrak{c}_\alpha, \mathfrak{c}_\beta)$ is a bijection between $C^{\rho\bullet}_{V_0,P}$ and $M(Spec(k))$, which gives the bijection $\chi^{p\rho,\bullet}_{V_0,P}$ desired. □

Here is the last of the 3 steps in constructing the local parameter space in Proposition 75.

Proposition 75. *There exists a parameter space $M^p_{U_0,G}$, a disjoint union of finitely many ind affine spaces, whose k-points parameterize (see Definition 72) all the pointed G-covers of (U_0, u_0).*

Proof. Let $M^p_{U_0,G} = \amalg_{V_{0t}} M^{p\rho_{n_t},\bullet}_{V_{0t},P}$, an analogue of Theorem 44. Using the argument of Theorem 44 with some obvious modification and Lemma 74, there is a bijection $\chi^p_{U_0,G}$ between k-points of $M^p_{U_0,G}$ and pointed G-covers of (U_0, u_0). □

Let $M^\infty_{\mathbb{A}^{1'},G}$, with $\mathbb{A}^{1'}$ defined in Remark 70, be the short hand notation for $M^{\{\infty\}}_{\mathbb{A}^{1'},G}$, the fine moduli space for $F^{\{\infty\}}_{\mathbb{A}^{1'},G}$ given by Proposition 64. Below is the local-global principal that involves the global moduli space in Proposition 64 and the local parameter space in Proposition 75.

Proposition 76. *The fine moduli space* $M^{\infty}_{\mathbb{A}^{1'},G}$*, is the same ind scheme as the parameter space* $M^{p}_{U_0,G}$*, compatibly with the inclusion of* U_0 *in* $\mathbb{A}^{1'}$ *(see diagram (70.1)).*

Proof. In the construction of both spaces, there are similar 3 steps to the global case, i.e. Theorem 32⇒Theorem 39⇒ Theorem 44. Hence the equality wanted will be proven in similar 3 steps. The bijections χ's given in Lemma 73, Lemma 74 and Proposition 75 will be used but not written out unnecessarily.

First both spaces have as building blocks an analogue of the moduli space in Theorem 32. Let V_0 and V be the same as in Remark 70, which shows that the local building block is the same as the global one and a triangle compatibility diagram holds. Then using the same kind of argument as in Corollary 36 $M^{p\rho\bullet}_{V_0,H} = M^{\rho\bullet}_{V,H}$ and a triangle compatibility diagram similar to that in Remark 70 holds. Moreover, by Remark 70, Remark 34, Corollary 36 and Remark 69, the canonical ρ-liftable universal family representative of H-covers of V_0 over each connected component of $M^{p\rho\bullet}_{V_0,H}$, is the restriction of the canonical ρ-liftable universal family representative of H-covers of V over the corresponding connected component of $M^{\rho\bullet}_{V,H}$, which is $M^{\rho\bullet/\{\infty\}}_{V^0,H}$ by Lemma 62. Here $U = \mathbb{A}^{1'}$, $T = \{\infty\}$, and V^0 is defined at the beginning of this Chapter.

Next $M^{p\rho\bullet}_{V_0,P} = M^{p\rho_0\bullet}_{V_0,H} \times M^{p\bar{\rho}\bullet}_{V_0,\bar{P}}$ and $M^{\rho\bullet/\{\infty\}}_{V,P} = M^{\rho_0\bullet/\{\infty\}}_{V,H} \times M^{\bar{\rho}\bullet/\{\infty\}}_{V,\bar{P}}$ given respectively in Lemma 74 and Lemma 63. By inductive hypothesis $M^{p\bar{\rho}\bullet}_{V_0,\bar{P}} = M^{p\bar{\rho}\bullet/\{\infty\}}_{V,\bar{P}}$, a triangle compatibility diagram similar to that in Remark 70 holds, and a $\bar{\rho}$-liftable universal family representative of pointed $\bar{P}$-covers of (V_0, v_0) over each connected component of $M^{p\bar{\rho}\bullet}_{V_0,\bar{P}}$, is the restriction of a $\bar{\rho}$-liftable universal family representative of pointed $\bar{P}$-covers of (V, v_g) over the corresponding connected component of $M^{\bar{\rho}\bullet/\{\infty\}}_{V,\bar{P}}$. (Strictly speaking, Definition 38 needs to be used and pairs should be dealt with, which however will make the proof unnecessarily longer.) A ρ-liftable lift of the previous representative can be got from the restriction of a ρ-liftable lift of the latter representative. By this fact and the paragraph above $M^{p\rho\bullet}_{V_0,P} = M^{\rho\bullet/\{\infty\}}_{V,P}$ and a triangle compatibility diagram similar to that in Remark 70 holds.

Finally by Remark 71, Proposition 64 and Proposition 75, the proposition follows and there is a triangle compatibility diagram similar to that in Remark 70. □

Corollary 77. *Any pointed G-cover of (U_0, u_0) extends uniquely to a pointed G-cover of $(\mathbb{A}^{1'}, a_g)$ which is tamely ramified at ∞.*

Proof. By the compatibility assertion in Proposition 76. □

Here are some necessary settings for the last result in Chapter 5, Proposition 83.

Notation 78. Let U_{0_i} be the spectrum of the fraction field of the complete local ring at the i-th closed point of $\bar{U} - U$, which is an infinitesimal neighborhood of that point. Let n' be a factor of n. Let $V_{0_in'}$ be a fixed connected $\mathbb{Z}/n'$-cover of U_{0_i}. All the connected $\mathbb{Z}/n'$-covers of U_{0_i} are isomorphic to $V_{0_in'}$ (see Remark 71); they only differ by the action of $\mathbb{Z}/n'$. Any two actions differ by an element in $Aut(\mathbb{Z}/n')$.

For any $(n_i)_i$, where n_i is a factor of n, there exists a possibly ramified connected $\mathbb{Z}/n$-cover V of U that may ramify at a finite set of closed points T on U, such that its ramification index at U_{0_i} is n_i. The cover V can be obtained as follows: Suppose $U = Spec(A)$ and denote the fraction field of A by K. Pick $a_0 \in A$, such that a_0 has poles $\sum_i N_i Q_i$, where Q_i is the i-th closed point of $\bar{U} - U$ and $N_i >> 0$ with $(N_i, n) = n/n_i$. By Riemann-Roch, such an a_0 exists. By adding a constant in k to a_0, $Y^n - a_0$ can be assumed an irreducible polynomial in $K[Y]$. Denote $K[Y]/(Y^n - a_0) = K(y)$ by F. The normalization of U in F gives V, which may ramify over the zeros of a_0 on U.

Suppose U_{0_i} is pointed at u_{0_i} and (U_{0_i}, u_{0_i}) maps to (U, u_{gi}). Choose v_{gi} such that $(V, v_{gi}) \to (U, u_{gi})$. Let the pointed connected component of V's restriction (pullback) over U_{0_i} be (V_{i0}, v_{i0}):

$$\begin{array}{ccc} (V_{i0}, v_{i0}) & \longrightarrow & (V, v_{gi}) \\ \downarrow & & \downarrow \\ (U_{0_i}, u_{0_i}) & \longrightarrow & (U, u_{gi}). \end{array}$$

Then V_{i0} is isomorphic to $V_{0_{i n_i}}$ that is one of those fixed above, as covers of U_{0_i}. Choose v_g such that $(V, v_g) \to (U, u_g)$ and chemins ω_i from v_{gi} to v_g that induce chemins ϖ_i from u_{gi} to u_g.

Let $U^0 = U - T$ and V^0 be V's restriction over U^0, same as the beginning of this Chapter. A ρ-liftable pointed P-cover of (V^0, v_g) gives a ρ_{n_i}-liftable (see proof of Theorem 44) pointed P-cover of (V_{i0}, v_{i0}) for each i using the following diagram:

$$\begin{array}{ccccccc}
\pi_1(V_{i0}, v_{i0}) & \longrightarrow & \pi_1(V^0, v_{gi}) & \xrightarrow{\tau_{\omega_i}} & \pi_1(V^0, v_g) & \longrightarrow & P \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\pi_1(U_{0_i}, u_{0_i}) & \longrightarrow & \pi_1(U^0, u_{gi}) & \xrightarrow{\tau_{\varpi_i}} & \pi_1(U^0, u_g) & \longrightarrow & P \rtimes_\rho \mathbb{Z}/n,
\end{array}$$

where τ_{ω_i} is the isomorphism induced by the chemin ω_i and similarly for τ_{ϖ_i}.

Here is a definition involved in the statement of Proposition 83.

Definition 79. For every (V_{0n_i}, v_{0i}) a degree n_i cover of (U_{0_i}, u_{0_i}), denote by $M^{p\rho_{n_i}}_{V_{0n_i},P}$ a connected component (see Remark 26) of $M^{p\rho_{n_i}\bullet}_{V_{0n_i},P}$.

Let i be the index for the i-th closed point of $\bar{U} - U$ and $(n_i)_i$ the same notation in Notation 78. A morphism from an ind scheme that is a disjoint union of finitely many ind affine spaces, to $\Pi_i M^p_{U_{0_i},G}$ is *essentially surjective*, if for any $(n_i)_i$ there is a connected component of the source ind scheme that maps surjectively (see Definition 27 d) to a connected component of the target ind scheme, whose i-th factor for each i is $M^{p\rho_{n_i}}_{V_{0n_i},P}$ for some (V_{0n_i}, v_{0i}) a degree n_i cover of (U_{0_i}, u_{0_i}).

Remark 80. The definition of essentially surjective is needed because: Suppose $(V_{i0})_i$ is a tuple whose i-th component is the restriction of a possibly ramified $\mathbb{Z}/n$-cover V of U and of degree n_i over U_{0_i}. The Galois actions of $\mathbb{Z}/n_i$'s on the V_{i0}'s are related to each other as shown in the example below. Thus not every tuple $(V_{0n_i})_i$ (same notation as in Definition 79) could be the image of the restrictions of some V. Hence the restriction morphism in Proposition 83 below is not surjective.

However in some sense it is surjective, which motivates the definition of essential surjectivity.

Suppose $\mathsf{p} = 3$. The $\mathbb{Z}/3$-cover of $U = Spec(k[x, x^{-1}])$, the affine line with 0 deleted, given by $V = Spec(k[x, x^{-1}][Y]/(Y^3 - x)) = Spec(k[x, x^{-1}][y])$ with $\bar{1} \in \mathbb{Z}/3$ acting on V over U as $y \mapsto \zeta_3 y$, has restrictions at 0 and ∞. At 0, its restriction is a $\mathbb{Z}/3$-cover of $Spec(k((x)))$ given by $V_{00} = Spec(k((x))[Y]/(Y^3 - x)) = Spec(k((x))[y])$ with $\bar{1} \in \mathbb{Z}/3$ acting on V_{00} over $Spec(k((x)))$ as $y \mapsto \zeta_3 y$. At ∞, its restriction is a $\mathbb{Z}/3$-cover of $Spec(k((x^{-1})))$ given by $V_{0\infty} = Spec(k((x^{-1}))[Y]/(Y^3 - x)) = Spec(k((x^{-1}))[y])$ with $\bar{1} \in \mathbb{Z}/3$ acting on $V_{0\infty}$ over $Spec(k((x^{-1})))$ as $y \mapsto \zeta_3 y$.

Changing the $\mathbb{Z}/3$ actions on the two local $\mathbb{Z}/3$-covers at 0 and ∞ above, the pair of local $\mathbb{Z}/3$-covers ($(Spec(k((x))[Y]/(Y^3 - x)) \to Spec(k((x))), \bar{1} : y \mapsto \zeta_3 y)$, $(Spec(k((x^{-1}))[Y]/(Y^3 - x)) \to Spec(k((x^{-1}))), \bar{1} : y \mapsto \zeta_3^{-1} y)$) got can not come from restrictions of a global $\mathbb{Z}/3$-cover of $Spec(k[x, x^{-1}])$.

Below is another ingredient involved in the statement of Proposition 83.

For any $(n_i)_i$, as shown in Notation 78, there exists a $V_{(n_i)_i}$ that may ramify at a finite set of closed points on U, denoted by $T_{V_{(n_i)_i}}$, such that its ramification index at U_{0_i} is n_i. Let $T = \cup_{(n_i)_i} T_{V_{(n_i)_i}}$.

The last ingredient involved in the statement of Proposition 83, the restriction morphism, is given in two steps in Lemma 81 and Lemma 82. First a map $\mathtt{r}$ involved in the statements of Lemma 81 and Lemma 82, is defined.

A pointed G-cover of (U^0, u_g) gives a local cover of (U_{0_i}, u_{0_i}) for each i: $\pi_1(U_{0_i}, u_{0_i}) \to \pi_1(U^0, u_{gi}) \xrightarrow{\tau_{\varpi_i}} \pi_1(U^0, u_g) \to G$. Thus there is a map $\mathtt{r}$ from the closed points (same as k-points) of $M^T_{U,G}$ (see Proposition 64), which parameterize certain pointed G-covers of (U^0, u_g), to the closed points of $\Pi_i M^p_{U_{0_i},G}$, which parameterize tuples each of which consists of covers indexed by i with the i-th entry a pointed G-cover of (U_{0_i}, u_{0_i}). Similarly there is a map $\mathtt{r}_0$ from the closed points of $M^\rho_{V,H}$ to those of $\Pi_i M^{p\rho_{n_i}}_{V_{i0},H}$.

Lemma 81. *Suppose (ρ, H) is in the case of Theorem 32. With the notations above, there is a restriction morphism $r_0 : M^\rho_{V,H} \to \Pi_i M^{p\rho_{n_i}}_{V_{i0},H}$ such that every closed point of $M^\rho_{V,H}$ maps to the same closed point of $\Pi_i M^{p\rho_{n_i}}_{V_{i0},H}$ under r_0 or $\boldsymbol{r}_0$.*

Proof. r_0 is given by giving for every i its i-th factor using $M^{p\rho_{n_i}}_{V_{i0},H} = M^{w\rho_{n_i}}_{V_{i0},H}$ is a fine moduli space.

Denote by $\widetilde{Z}$ the canonical ρ-liftable universal family representative of H-covers of V over $M^\rho_{V,H}$, which corresponds to, for every point m on $M^\rho_{V,H}$, some group homomorphism $\pi_1(M^\rho_{V,H} \times V, (m, v_g)) \to H$. The composition $\pi_1(M^\rho_{V,H} \times V_{i0}, (m, v_{i0})) \to \pi_1(M^\rho_{V,H} \times V, (m, v_{gi})) \xrightarrow{\tilde{\tau}_{\omega_i}} \pi_1(M^\rho_{V,H} \times V, (m, v_g)) \to H$, where $\tilde{\tau}_{\omega_i}$ is induced by ω_i similar to τ_{ω_i} given at the end of Notation 78, gives a pointed H-cover of $(M^\rho_{V,H} \times V_{i0}, (m, v_{i0}))$ the non pointed version of which is denoted by $\widetilde{Z}_0$. By Remark 20, base points here do not matter. It is the restriction (pullback) of $\widetilde{Z}$ to

$M^{\rho}_{V,H} \times V_{i0}$:

$$\begin{array}{ccc} \widetilde{Z}_0 & \longrightarrow & \widetilde{Z} \\ \downarrow & & \downarrow \\ M^{\rho}_{V,H} \times V_{i0} & \longrightarrow & M^{\rho}_{V,H} \times V. \end{array}$$

Since $M^{p\rho_{n_i}}_{V_{i0},H} = M^{w\rho_{n_i}}_{V_{i0},H}$ and $M^{w\rho_{n_i}}_{V_{i0},H}$ represents $F^{w\rho_{n_i}}_{V_{i0},H}$ by Proposition 68, there is a morphism $M^{\rho}_{V,H} \xrightarrow{r_{0i}} M^{p\rho_{n_i}}_{V_{i0},H}$ given by $\widetilde{Z}_0$. A different base point m' gives the same r_{0i}. Then define $r_0 := (r_{0i})_i$.

Now it is enough to verify that a closed point m' of $M^{\rho}_{V,H}$ maps to the same closed point under r_{0i} or $\mathtt{r}_{0i}$, where $\mathtt{r}_{0i}$ is the i-th factor of $\mathtt{r}_0$. Tracking definitions, $r_{0i}(m')$ represents the restriction to (V_{i0}, v_{i0}) of the pointed H-cover of (V, v_g) represented by m'. And $\mathtt{r}_i$ does the same thing by its definition. So r_{0i} and $\mathtt{r}_{0i}$ agree. $\square$

Lemma 82. *Let $G = H \rtimes_\rho \mathbb{Z}/n$ for some (ρ, H) in the case of Theorem 32. There is a restriction morphism $r : M^T_{U,G} \to \amalg_i M^p_{U_{0_i},G}$, such that every closed point of $M^T_{U,G}$ maps to the same closed point of $\amalg_i M^p_{U_{0_i},G}$ under r or $\boldsymbol{r}$, where $\boldsymbol{r}$ is defined above Lemma 81.*

Proof. By construction, $M^T_{U,G}$ and $\amalg_i M^p_{U_{0_i},G}$ are both a disjoint union of finitely many ind affine spaces. The morphism r will be given for each connected component of $M^T_{U,G}$.

Proposition 64 and Lemma 62 give $M^T_{U,G} = \amalg_{V_l} M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$ and $M^{\rho_{n_l}\bullet/T}_{V^0_l,H} = \amalg_h M^{\rho_{n_l}}_{V_l,H}$ respectively. A connected component of $M^T_{U,G}$ is of the form $M^{\rho_{n_l}}_{V_l,H}$.

Let the pointed connected component of V_l over U_{0_i} be V_{li0}, a $\mathbb{Z}/n_i$-cover of

U_{0_i}. Using Notation 78, r should map $M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$ to $\Pi_i M^{p\rho_{n_i}\bullet}_{V_{li0},H}$, since it is required to agree with the map $\mathtt{r}$ on closed points. Similarly the target connected component of each connected component of $M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$ under r can be identified. Denote by $M^{\rho_{n_l}}_{V_l,H}$ a connected component of $M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$, by $\Pi_i M^{p\rho_{n_i}}_{V_{li0},H}$ its target connected component, and by r_{lj} (suppose $M^{\rho_{n_l}}_{V_l,H}$ is the j-th component of $M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$) the restriction of r on $M^{\rho_{n_l}}_{V_l,H}$.

Finally let r_{lj} be the restriction morphism given in Lemma 81 for every index lj. One can check that the morphism r satisfies the requirement. □

With the preparation from Notation 78 to Lemma 82, the last result in Chapter 5 can be given.

Proposition 83. *Let $G = H \rtimes_\rho \mathbb{Z}/n$ for some (ρ, H) in the case of Theorem 32. The restriction morphism $M^T_{U,G} \xrightarrow{r} \Pi_i M^p_{U_{0_i},G}$ given in Lemma 82 is essentially surjective and finite. And the degrees of r on different connected components of $M^T_{U,G}$ are all powers of* p.

Proof. The proof follow the points of the proof of Proposition 2.7 in [H80]. A calculation of the dimensions of the n-th pieces of both source and target shows that they are the same. By this fact the map r restricted on each connected component of the source can be proven surjective. Then all the three statements follow.

With the same notations as in the proof of Lemma 82, denote a connected component of $M^{\rho_{n_l}\bullet/T}_{V^0_l,H}$ by $M^{\rho_{n_l}}_{V_l,H}$, whose n-th piece is $M^{\rho_{n_l}}_{V_l,H,n}$. Denote the connected component of $\Pi_i M^{p\rho_{n_i}\bullet}_{V_{li0},H}$, which $M^{\rho_{n_l}}_{V_l,H}$ maps to under r_{lj}, by $\Pi_i M^{p\rho_{n_i}}_{V_{li0},H}$, whose n-th piece is $\Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n}$.

For $n >> 0$, Riemann-Roch shows that the dimension of $M^{\rho_{n_l}}_{V_l,H,n}$ is at least that of $\Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n}$ for a subsequence $\{n_\mathsf{k}\}$ of $\mathbb{N}$: A simpler but similar computation is done in the 1st paragraph of the proof of Proposition 2.7 in [H80]. Here pass from V_l to U first using ramification indices and then do a similar computation to [H80]. Denote by $\Sigma_i d_{i0n}$ the dimension of $\Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n}$. For $n >> 0$ Riemann-Roch gives $\Sigma_i d_{i0n} = \Sigma_i(\lfloor \frac{q^n - i_0}{n_i} \rfloor - \lfloor \frac{q^{n-1} - i_0}{n_i} \rfloor)$ with some natural number i_0 between 0 and n_l. Denoted by d_n the dimension of $M^{\rho_{n_l}}_{V_l,H,n}$. For $n >> 0$ a similar computation gives $d_n = \lfloor \Sigma_i \frac{q^n}{n_i} + \delta \rfloor - \lfloor \Sigma_i \frac{q^{n-1}}{n_i} + \delta \rfloor$ for some $\delta \in \mathbb{Q}$. Since the remainder of q^n divided by n_i is periodic for $n \in \mathbb{N}$ there is a subsequence $\{n_\mathsf{k}\}$ of $\mathbb{N}$ such that $d_{n_\mathsf{k}} \geq \Sigma_i d_{i0n_\mathsf{k}}$.

r_{lj} is quasi finite of degree a p-power: The restriction of r_{lj} on the n-th piece of $M^{\rho_{n_l}}_{V_l,H}$ gives $M^{\rho_{n_l}}_{V_l,H,n} \xrightarrow{r_{ljn}} \Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n}$ using Lemma 81, which is in fact a homomorphism between $\mathbb{F}_\mathsf{p}$-vector spaces (The closed points of any moduli space involved here form a $\mathbb{F}_\mathsf{p}$-vector space, by the definitions of the moduli spaces.). Since there are, up to isomorphism, only finitely many pointed (etale) P-covers of the completion $X_l = \bar{V}_l$ [SGA1, X 2.12], the kernel of r_{ljn} is finite (and equal to this number when $n >> 0$). Thus every non empty fiber of r_{ljn} (hence of r_{lj}) contains the same finite number of points. This number is a power of p, being the cardinality of a $\mathbb{F}_\mathsf{p}$-vector space.

The 2nd paragraph in the proof of Proposition 2.7 in [H80] shows that since $d_{n_\mathsf{k}} \geq$

$\Sigma_i d_{i0n_{\mathsf{k}}}$ for every k large enough and $r_{ljn_{\mathsf{k}}}$ is quasi finite, the morphism $M^{\rho_{n_l}}_{V_l,H,n_{\mathsf{k}}} \to \Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n_{\mathsf{k}}}$ is surjective. Thus for every n large enough, using Lemma 81, the morphism $M^{\rho_{n_l}}_{V_l,H,n} \to \Pi_i M^{p\rho_{n_i}}_{V_{li0},H,n}$ is surjective. Hence the map r restricted on every connected component of $M^{\rho_{n_l}\bullet/T}_{V_l^0,H}$ maps surjectively to a connected component of $\Pi_i M^{p\rho_{n_i}\bullet}_{V_{li0},H}$.

Direct computation shows that the restriction morphism is finite. The choice of T shows that r is essentially surjective. $\square$

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