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Decay Properties Of Multilinear Oscillatory Integrals

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Abstract

In this thesis, we study the following multilinear oscillatory integral introduced by Christ, Li, Tao and Thiele \cite{CLTT}

\begin{equation}

I_{\lambda}(f_1,...f_n)=\int_{\mathbb{R}^m}e^{i\lambda P(x)}\prod_{j=1}^n f_j(\pi_j(x))\eta(x)dx,

\end{equation}

where $P:\mathbb{R}^m\to\mathbb{R}$ is a real-valued measurable function, η is a compactly supported smooth cutoff function. Each π_j is a surjective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^{k_j}$, where $1\le k_j\le m-1$.

Each $f_j:\mathbb{R}^{k_j}\rightarrow \mathbb{C}$ is a locally integrable function with respect to Lebesgue measure on $\mathbb{R}^{k_j}$.

In Chapter 2, we first introduce the nondegeneracy degree along with the nondegeneracy norm defined in \cite{CLTT} to characterize the nondegeneracy condition of the phase function. In the same chapter, we will summarize some powerful tools that can help to simplify the problem and introduce the idea of a special geometric structure called ``separation''.

There are three results in this thesis. The first proves trilinear oscillatory integrals with nondegenerate polynomial phase always have the decay property. The second one extends the one-dimensional case whose phase function has large nondegeneracy degree. The third result deals with the case where every linear mapping preserves the direct sum decomposition.

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ABSTRACT

DECAY PROPERTIES OF MULTILINEAR OSCILLATORY INTEGRALS

Zhen Zeng

Philip T. Gressman

In this thesis, we study the following multilinear oscillatory integral introduced by Christ, Li, Tao and Thiele [7]

$$I_\lambda(f_1, ...f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j(\pi_j(x))\eta(x)dx, \tag{0.0.1}$$

where $P : \mathbb{R}^m \to \mathbb{R}$ is a real-valued measurable function, η is a compactly supported smooth cutoff function. Each π_j is a surjective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^{k_j}$, where $1 \leq k_j \leq m - 1$. Each $f_j : \mathbb{R}^{k_j} \to \mathbb{C}$ is a locally integrable function with respect to Lebesgue measure on $\mathbb{R}^{k_j}$.

In Chapter 2, we first introduce the nondegeneracy degree along with the nondegeneracy norm defined in [7] to characterize the nondegeneracy condition of the phase function. In the same chapter, we will summarize some powerful tools that can help to simplify the problem and introduce the idea of a special geometric structure called "separation".

There are three results in this thesis. The first proves trilinear oscillatory integrals with nondegenerate polynomial phase always have the decay property. The second one extends the one-dimensional case whose phase function has large non-

degeneracy degree. The third result deals with the case where every linear mapping preserves the direct sum decomposition.

Contents

Chapter 1

Introduction

Oscillatory integrals have long been an essential part of harmonic analysis and have been a powerful tool in many central questions of mathematics. The most commonly seen oscillatory integral is Fourier transform, which has a wide range of applications in partial differential equations, physics and signal processing, see [9], [17], [19]. One of the most fundamental questions about oscillatory integrals is the asymptotic behavior of them under certain conditions. For example, if f is an L^1 function on $\mathbb{R}^n$, the Riemann-Lebesgue lemma implies the Fourier transform $\hat{f}$ is a continuous bounded function on $\mathbb{R}^n$, which vanishes at infinity. However, if $f \in L^2(\mathbb{R}^n)$, then $\hat{f}$ can be any function in $L^2(\mathbb{R}^n)$ and it does not necessarily tend to 0 at infinity.

Many mathematicians have investigated settings in which oscillatory integrals may have certain decay, and they have achieved fruitful result. See [14], [16], [20] for reference.

1.1 Oscillatory integrals of the first kind

We now introduce oscillatory integrals of the first kind, in the terminology of Stein[20], which are defined as below and we want to characterize the asymptotic behavior of these integrals for large positive λ:

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx, \tag{1.1.1}$$

where ϕ is a real-valued smooth function(the phase), and ψ is complex-valued, smooth, and compactly supported. See [20].

One tool to deal with it is the *principle of nonstationary phase*, which roughly speaking asserts that (1.1.1) is rapidly decreasing in λ whenever ϕ is smooth and nonstationary (that is, $\nabla\phi$ does not vanish).

Proposition 4 (Stein[20], pp.341) (*principle of nonstationary phase*) *Let ϕ and ψ be smooth functions so that ψ has compact support, and $\nabla\phi \neq 0$ for all x on supp ψ. Then*

$$I(\lambda) \leq C_{N,\psi,\phi}\lambda^{-N}$$

as $\lambda \to \infty$ for all $N \geq 0$.

Proof. For a function $f \in C^\infty$, we define the operator

$$L(f) = \frac{1}{i\lambda} a \frac{df}{dx} \tag{1.1.2}$$

and its transpose

$$L^T(f) = -\frac{1}{i\lambda}\frac{d}{dx}(af),$$

with

$$a(x) = \frac{1}{\phi'(x)}.$$

So if $f, g \in C^\infty$ then integration by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} L(f)g &= \int_{-\infty}^{\infty} fL^T(g) + \Big[\frac{a(x)g(x)f(x)}{i\lambda}\Big]_{-\infty}^{\infty} \\ &= \int_{-\infty}^{\infty} fL^T(g) + \Big[\frac{g(x)f(x)}{i\lambda\phi'(x)}\Big]_{-\infty}^{\infty}.\end{aligned}$$

If in addition $g \in C_0^\infty$, then we have

$$\int_{-\infty}^{\infty} L(f)g = \int_{-\infty}^{\infty} fL^T(g).$$

Also, this operator is useful here because $L(e^{i\lambda\phi}) = e^{i\lambda\phi}$ and then $L^N(e^{i\lambda\phi}) = e^{i\lambda\phi}$ for all $N \in \mathbb{N}$. Thus

$$I(\lambda) = \int_{\mathbb{R}} L^N(e^{i\lambda\phi(x)})\psi(x)dx = \int_{\mathbb{R}} e^{i\lambda\phi(x)}(L^T)^N(\psi(x))dx.$$

Now for each N, $(L^T)^N(\psi(x))$ is $(-\frac{1}{i\lambda})^N$ times a function that is continuous and supported in $supp(\psi)$. This function is then integrable and does not depend on λ. So we get

$$\Big|I(\lambda)\Big| \leq c_N\lambda^{-N},$$

where for each N the constant c_N depends on the phase and the amplitude but not on λ. Hence as λ goes to infinity, the decay of the integral is very fast and is in fact as fast as the decay of the Fourier transform mentioned above. □

If stationary points do exist, things may become complicated. However, in one dimension, where $n = 1$, even if we do not know the information of $\nabla\phi$, we may still obtain an estimate for $\int_a^b e^{i\lambda\phi(x)}dx$. Given $|\frac{d^k\phi(x)}{x^k}|$ is bounded away from 0 for some $k \geq 2$ by the following van der Corput lemma, which is one of the most fundamental results in this area. Notice that $\nabla\phi(x)$ being bounded away from 0 is not enough to guarantee the decay of the integral in this case, one can refer to [20] for counterexamples.

Proposition 2 (Stein[20], pp.332) (*van der Corput lemma*) *Let $k \in \mathbb{N}$. Let $I \subset \mathbb{R}$ be an interval and suppose that $\phi : I \to \mathbb{R}$ satisfies $|\phi^{(k)}(x)| \geq 1$ for $x \in I$. Then for $\lambda \in \mathbb{R}$,*

$$\Big|\int_I e^{i\lambda\phi(x)}dx\Big| \leq C_k|\lambda|^{-\frac{1}{k}},$$

provided, in addition when $k = 1$, $\phi'(x)$ is monotone on I. The constant C_k is independent of ϕ and I.

Proof. We use the same operator 1.1.2 but now when we do the integration by parts we get

$$\begin{aligned} I_1(\lambda) &= \int_a^b L(e^{i\lambda\phi(x)})dx \\ &= \int_a^b e^{i\lambda\phi(x)}L^T(1)dx + \Big[\frac{e^{i\lambda\phi(x)}}{i\lambda\phi'(x)}\Big]_a^b. \end{aligned}$$

The second term is obviously bounded by $\frac{2}{\lambda}$ and the first term is bounded by

$$\int_a^b \Big|L^T(1)\Big|dx = \frac{1}{\lambda}\int_a^b \Big|\frac{d}{dx}(\frac{1}{\phi'(x)})\Big| \tag{1.1.3}$$

and since $\phi'(x)$ is monotonic and continuous, $\frac{d}{dx}(\frac{1}{\phi'(x)})$ does not change sign. Then (1.1.3) is

$$\frac{1}{\lambda}\Big|\int_a^b \frac{d}{dx}(\frac{1}{\phi'(x)})dx\Big| = \frac{1}{\lambda}\Big|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\Big| \leq \frac{1}{\lambda}\Big|\frac{1}{\phi'(b)}\Big|,$$

where the last inequality holds because $\phi'(a)$ and $\phi'(b)$ have the same sign, and this is bounded by $\frac{1}{\lambda}$. Putting the terms together, we get the result. □

In higher dimensions, Carbery, Christ and Wright [3] give an analogue of van der Corput lemma as below.

Lemma 1.1.1. *Let $\beta = (\beta_1, ...\beta_n) \neq 0$ be a multi-index, and suppose that at least one of its entries β_j is greater than or equal to two. Then there exist $\epsilon > 0$ and $C < \infty$, depending only on β and on n, such that for any integrable $u : Q \to \mathbb{R}$ satisfying $D^\beta u \geq 1$ on Q in the sense of distributions, for all $\lambda \in \mathbb{R}$, the oscillatory integral $I(\lambda) = \int_Q e^{i\lambda u(x)}dx$ satisfies*

$$|I(\lambda)| \leq C|\lambda|^{-\epsilon}.$$

1.2 Oscillatory integrals of the second kind

Oscillatory integrals of the second kind, which are known as oscillatory integral operators, are given the following form

$$T_\lambda f(\xi) = \int_{\mathbb{R}^m} e^{i\lambda\phi(x,\xi)} f(x)\psi(x,\xi)dx.$$

Hörmander [12] gives a characterization of such operators when the Hessian of ϕ is nonvanishing in the support of the cutoff ψ.

Theorem 1.1 (Hörmander [12]) *Assume $\eta(x, y)$ is a smooth cut-off function supported in a neighborhood of 0 and $S(x, y)$ is a real-valued smooth function in $\mathbb{R}^m \times \mathbb{R}^m$ such that*

$$\left| \det \frac{\partial^2 S}{\partial x \partial y} \right| \geq 1 \tag{1.2.1}$$

for all $(x, y) \in$ supp η with $\lambda > 1$. For $1 \leq p \leq 2$, $1/p + 1/p' = 1$, one has

$$||T(f)||_{L^{p'}} \leq C|\lambda|^{-m/p'} ||f||_{L^p}. \tag{1.2.2}$$

Here

$$T(f)(x) = \int e^{i\lambda S(x,y)} f(y)\eta(x, y)dy.$$

Proof. The statement is obvious when $p = 1$ so in view of the interpolation argument, it suffices to prove it when $p = 2$. In the proof we may assume f has small support. We have to estimate

$$||T(f)||^2 = \int \int \tilde{\eta}(y, z) f(y)\bar{f}(z)dydz,$$

where $\tilde{\eta}(y, z) = \int e^{i\lambda(S(x,y)-S(x,z))}\eta(x, y)\eta(x, z)dx$.

When y and z are close to a given point and $(x, y) \in \text{supp}(\eta)$ we have

$$|\partial/\partial x(S(x, y) - S(x, z))| = |S''_{xy}(y - z)| + O(|y - z|^2) \geq c|y - z|.$$

So if k is any positive integer, k partial integrations give

$$|\tilde{\eta}(y, z)| \leq C_k(1 + \lambda|y - z|)^{-k}.$$

If $k = m + 1$, it follows that $\int |\tilde{\eta}(y,z)|dy < C\lambda^{-m}$, $\int |\tilde{\eta}(y,z)|dz < C\lambda^{-m}$. Hence $||Tf||_{L^2}^2 \leq C\lambda^{-m}||f||_{L^2}^2$ and the theorem is proved. $\square$

The situation where $\frac{\partial^2 S}{\partial x \partial y}$ vanishes at some point is more tricky. Just like van der Corput lemma, we may need some extra assumptions on other derivative. If there exists $\alpha \geq 1$, $\beta \geq 1$, but $(\alpha, \beta) \neq (1,1)$ and $\frac{\partial^{\alpha+\beta} S}{\partial x^\alpha \partial y^\beta} \neq 0$ on the support of η, estimates like 1.2.2 still holds. Detailed work can be found in [3]. When $S(x,y)$ is analytic, we do not need extra assumption. See [15].

When $p = q = 2$, the inequality (1.2.2) can be written in this form

$$\Big| \int e^{i\lambda S(x,y)} g(x) f(y) \eta(x,y) dy \Big| \leq C|\lambda|^{-m/2} ||f||_{L^2} ||g||_{L^2}.$$

So the above theorem actually obtains an estimate of a bilinear oscillatory integral. In the next section, we will give a more general formulation of estimating the asymptotic behavior of multilinear oscillatory integrals.

1.3 Multilinear oscillatory integrals

In [7], Christ, Li, Tao and Thiele initialize the study of a rather general multilinear functionals of the form

$$I_\lambda(f_1, ... f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j(\pi_j(x)) \eta(x) dx, \tag{1.3.1}$$

where $P : \mathbb{R}^m \to \mathbb{R}$ is a real-valued measurable function, η is a compactly supported smooth cutoff function. Each π_j is a surjective linear transformation from $\mathbb{R}^m$ to

$\mathbb{R}^{k_j}$, where $1 \leq k_j \leq m-1$. Each $f_j : \mathbb{R}^{k_j} \to \mathbb{C}$ is a locally integrable function with respect to Lebesgue measure on $\mathbb{R}^{k_j}$. This integral is well-defined if in addition all f_j belongs to L^∞. And the question is, under what conditions this integral has rapid decay? Here rapid decay means there exists some $\epsilon > 0$ such that

$$|I_\lambda(f_1, ...f_n)| \leq C(1+|\lambda|)^{-\epsilon} \prod_{j=1}^{n} ||f_j||_{L^\infty} \tag{1.3.2}$$

holds for every λ and every f_j lying in $L^\infty(\mathbb{R}^{k_j})$.

Notice that when $n = 0$, that is, the number of function is 0, we are dealing with $\int e^{i\lambda P(x)}\eta(x)dx$, which is oscillatory integrals of the first kind. Indeed, (1.3.2) with L^∞-bounds on the functions f_j is equivalent to $|\int e^{i\lambda\phi(x)}\eta(x)dx| \leq C|\lambda|^{-\epsilon}$ uniformly for all phase functions of the form $\phi = P - \sum_j h_j \circ \pi_j$, where the h_j are arbitrary real-valued measurable functions, one can refer to [3]. When $n = 2$, for certain exponents, oscillatory integrals of the second kind can be viewed as an example of the bilinear case of (1.3.1) with $\mathbb{R}^{2m} = \{(x,y)\}$ and $\pi_1 : (x,y) \to (x)$, $\pi_2 : (x,y) \to (y)$. So the two kinds of oscillatory integrals introduced before can be included into this framework.

In [7], the authors focus on the situation where the phase function is a polynomial of degree less than or equal to some constant d. They successfully prove that when $k_j = m-1$ for all j and when $k_j = 1$ for all j under some restrictions, to characterize the decay property of the above oscillatory integrals (1.3.1), it suffices to check certain nondegeneracy conditions of the phase function P as well as some geometric and dimensional conditions of the linear projections $\{\pi_j\}$. Despite the great progresses they make, the general cases have not been thoroughly explored.

After the work of Christ, Li, Tao and Thiele[7], Christ and Silva[8] study a rather special trilinear integrals of the form

$$I_\lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^{2m}} e^{i\lambda P(x,y)} f_1(x) f_2(y) f_3(x+y) \eta(x,y) dxdy \tag{1.3.3}$$

where P is a polynomial of degree less or equal to a constant d and $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m = \{(x,y)\}$ and give a characterization of such integrals by the nondegeneracy conditions of the phase function. But the more general cases are still left to be investigated.

Other results in this area include the oscillatory integral operator defined by Phong, Stein, and Sturm [16], which is of the form

$$I(\lambda) = \int_D e^{i\lambda\phi(x_1,...x_m)} f_1(x_1)...f_m(x_m) dx_1...dx_m,$$

where the phase function $\phi(x_1, ...x_m)$ is a polynomial. And D is a subset of the unit ball in $\mathbb{R}^m$. They have obtained some precise results on the exponent in the decay estimate, which is phrased in terms of the reduced Newton polyhedron.

1.4 Application

The estimate of oscillatory integrals can be used to get the corresponding sublevel set estimate. To be more precise, if a real-valued measurable function P satisfies (1.3.2) for all functions $f_j \in L^\infty(\mathbb{R}^{k_j})$, then there is an upper bound for the measures of the sublevel sets of the form

$$|\{y \in B : |P(y) - \sum_{j=1}^{n} g_j(\pi_j(y))| < \epsilon\}| \leq C\epsilon^{\delta}$$

uniformly for all measurable functions g_j. See [6]. And the sublevel set estimates of certain real-valued functions turn out to have close connections with some combinatorial problems arising in extremal graph theory. See [3] for more details. Understanding the decay property of various oscillatory integrals can certainly help us deal with many problems in mathematics. Hörmander discovers Theorem 1.1(Hörmander [12]) in order to simplify the proof of Carleson and Sjölin [4] that deals with the necessary and sufficient conditions for certain function to be a multiplier.

Other important application includes the restriction problem in harmonic analysis. It asks the question that what are the exponents q such that the Fourier transform of an $L^q(\mathbb{R}^m)$ function g can be meaningfully restricted to a given hypersurface S, in the sense that the map $g \to \hat{g}|_S$ can be continuously defined from $L^q(\mathbb{R}^m)$ to $L^1(S, d\sigma)$ with σ is the surface measure of S. It turns out that the variable coefficient (Hömander) setting of the problem is exactly dealing with the oscillatory integral operator

$$T_\lambda f(x) = \int e^{i\phi(x,y)} f(y) dx$$

with some specific analytic phase function and $||f||_{L^\infty} \leq 1$. The problem now becomes characterizing the range of q such that the bound $||T_\lambda||_q \leq c\lambda^{-\frac{m}{q}}$ holds. One can refer to [1] and [2].

Another important application is singular integrals. In [18], Ricci and Stein study the operator $T(f)(x) = \int_{\mathbb{R}^m} e^{iP(x,y)} K(x-y)f(y)dy$ where K is a standard Calderón-Zygmund kernel, that is, K is Lipschitz continuous except at the origin, $K(r\gamma) = r^{-m}K(\gamma)$ for all $r > 0$ and $\gamma \neq 0$, and $\int_S K d\sigma = 0$, where σ denotes surface measure on the unit sphere S. They have shown T is bounded on L^p for $1 < p < \infty$. The bilinear and multilinear analogue of it has been studied in [7] by combining previously known results for nonoscillatory singular integral operators with estimates for nonsingular oscillatory integrals.

Chapter 2

Goals and tools

2.1 Goals and conventions

In this thesis, we study the multilinear oscillatory integral initially introduced by Christ, Li, Thiele and Tao [7]:

$$I_\lambda(f_1, ...f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(X)} \prod_{j=1}^{n} f_j(\pi_j(X))\eta(X)dX. \qquad (2.1.1)$$

Here $\lambda \in \mathbb{R}$ is a parameter, $P : \mathbb{R}^m \to \mathbb{R}$ is a polynomial of degree less or equal to some given constant d, $m \geq 2$, $\eta \in C_0^\infty(\mathbb{R}^m)$ is a compactly supported smooth cutoff function. Each π_j is a surjective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^{k_j}$, where $1 \leq k_j \leq m - 1$. We assume $f_j \in L^\infty(\mathbb{R}^{k_j})$ and each f_j has support in a specified compact set $B_j \subset \mathbb{R}^{k_j}$. In this case, the integral is well-defined.

Definition 2.1.1. We say $\{P, \{\pi_j\}_{j=1}^n\}$ has power decay property in $\mathbb{R}^m$ on an open set $U \subset \mathbb{R}^m$, where P is a measurable, real-valued function, each π_j is a surjective linear map from $\mathbb{R}^m$ to V_j, if for any smooth cutoff function η defined on U, there exists ϵ independent of η and a constant C depends on the support and C^4 norm of η such that

$$|I_\lambda(f_1, ...f_n)| \leq C(1+|\lambda|)^{-\epsilon} \prod_{j=1}^{n} ||f_j||_{L^\infty} \tag{2.1.2}$$

for all $\lambda \in \mathbb{R}$ and all $f_j \in L^\infty(\mathbb{R}^{k_j})$.

In this thesis, we will focus on the case where the phase function is a polynomial. Though (2.1.2) may also hold for other exponents $||f_j||_{L^{p_j}}$, we will only assume all $||f_j||_{L^\infty}$ are finite. Since $|I_\lambda(f_1, ...f_n)| \leq C||f_k||_{L^1} \prod_{j\neq k} ||f_j||_{L^\infty}$ uniformly in λ for any k, if (2.1.2) holds, by the interpolation argument, the decay estimate for $\prod_j ||f_j||_{p_j}$ also holds for various p_j. So it suffices to consider this extreme formulation.

Notice that if $P = \sum_{j=1}^{n} p_j \circ \pi_j$ for some function p_j, then $f_j = e^{-i\lambda p_j(\pi_j(x))} \in L^\infty(\mathbb{R}^{k_j})$ gives $e^{i\lambda P(X)} \prod_{j=1}^{n} f_j(\pi_j(X)) = 1$ so that (2.1.1) becomes constant and thus has no decay. So a necessary condition for I_λ having power decay is that P cannot be decomposed in this way, which leads to the following definition in [7].

Definition 2.1.2. A polynomial P is said to be nondegenerate relative to a collection of surjective linear mappings $\{\pi_i\}$ if P cannot be expressed as a sum of $p_j \circ \pi_j$ where each p_j is a polynomial.

The goal of this thesis is to see under what conditions the power decay property is equivalent to the nondegeneracy of the phase function and what other assumptions may also lead to the power decay property of (2.1.1).

The regularity condition $\eta \in C^4$ is rather arbitrary. Here we will use the big O notation. If $f(x) = O(g(x))$ for $x \to \infty$, it means there exists constants M, x_0 such that $|f(x)| \leq M|g(x)|$ for $x \geq x_0$. If η is merely Hölder continuous then for any $s < \infty$, η may be decomposed as a smooth function whose C^s norm is $O(|\lambda|^{C\delta})$ plus a remainder which is $O(|\lambda|^{-\delta})$ in supremum norm. If (2.1.2) holds for all $\eta \in C_0^s$ with a constant C which is $O(||\eta||_{C^s})$, then it follows from the decomposition, with $\delta = \epsilon/2C$, that it continuous to hold for all Hölder continuous η.

If for some $1 \leq j \leq n$, π_j^1, π_j^2 are two surjective linear mappings with identical nullspaces and ranges of equal dimensions, then there is some invertible linear transformation L such that $\pi_j^2 = L \circ \pi_j^1$. So for every function f defined on the range space of $(\pi_j^2(x))$, there exists a function $g = f \circ L$ defined on the range space of π_j^1 such that for every $x \in \mathbb{R}^m$, $g(\pi_j^1(x)) = f \circ L(\pi_j^1(x)) = f(\pi_j^2(x))$. Therefore, if (2.1.2) holds for the collection of mappings $\{\pi_1, ...\pi_j^1..., \pi_n\}$, it also holds for the collection $\{\pi_1, ...\pi_j^2..., \pi_n\}$. So we can assume without loss of generality that each π_j has distinct nullspace and we may equivalently speak of nondegeneracy relative to a collection of subspaces $\{V_j\}_{j=1}^n$ of $\mathbb{R}^m$ where $V_j = \text{nullspace}(\pi_j)$. Similarly, we can also assume that there is no index i and j such that $\text{nullspace}(\pi_i) \subset \text{nullspace}(\pi_j)$. If so, there exists a surjective linear transformation from the range space of π_i to the range space of π_j such that $\pi_j = L \circ \pi_i$. So for every function f_j defined on the range space of π_j, there exists a function $g_i = f_j \circ L$ defined on the range space of π_i such that $g_i(\pi_i(x)) = f_j(\pi_j(x))$. In this case, we say $f_j(\pi_j(x))$ is absorbed into

$g_j(\pi_i(x))$.

So we may equivalently say $\{P, \{V_j\}_{j=1}^n\}$ has the power decay property in $\mathbb{R}^m$, if (2.1.2) holds true for any linear mappings π_j with nullspaces equal to V_j. In this paper, we will just consider the case where the phase function P is a polynomial of bounded degree d.

2.2 Nondegeneracy

In this section, we want to discuss some ways to characterize nondegeneracy.

In [7], the authors define the nondegeneracy norm which is one way to characterize nondegeneracy. Let $\mathcal{P}(d)$ be the vector space of all polynomials in $\mathbb{R}^m$ of degree at most d. Given d, fix a norm $||\cdot||_{P_d}$ on the finite dimensional vector space $\mathcal{P}(d)$. The nondegeneracy norm $||\cdot||_{nd(\pi_i,1\le i\le n)}$ of P with respect to $\{\pi_j\}$ is defined to be $\inf ||P - \sum_j p_j \circ \pi_j||_{P_d}$ where the infimum is taken over all real-valued polynomials of degree no greater than d. If there is no ambiguity, we may write $||\cdot||_{nd(\pi_i,1\le i\le n)}$ as $||\cdot||_{nd}$. Since the space of polynomials with degree at most d is finite dimensional, the infimum defining the relative norm is actually attained by some polynomial p_j. Thus P is either degenerate or the nondegeneracy norm is strictly positive.

We say a family P_α of real-valued polynomials of bounded degrees is uniformly nondegenerate relative to a collection of surjective linear map $\{\pi_i\}$ if $inf_\alpha ||P_\alpha - \sum_j p_j^\alpha \circ \pi_j||_{P_d} \ge c > 0$ with c a uniform constant.

Similarly, we say a collection of surjective linear map $\{\pi_j\}$ has the uniform power decay property if the power decay property holds with uniform constant C, ϵ, for any family of real-valued polynomials of bounded degrees which are uniformly nondegenerate relative to $\{\pi_j\}$.

The following lemma in [7] suggests that nondegeneracy is a property of every homogeneous part of the phase function.

Lemma 2.2.1. *A polynomial P is nondegenerate relative to $\{\pi_j\}$ if and only if at least one of its homogeneous summands is nondegenerate.*

This inspires the idea of using the highest degree of the nondegenerate homogeneous part to characterize nondegeneracy. In this thesis, we introduce the following definition.

Definition 2.2.2. The nondegeneracy degree of a polynomial P relative to a collection of $\{V_j\}$ is defined to be the highest degree of the nondegenerate homogeneous part of P. If P is degenerate with respect to $\{V_j\}$, the nondegeneracy degree of P is defined to be 0.

Given a homogeneous polynomial, the following lemma in [7] provides a way to distinguish nondegenerate polynomials from degenerate ones.

Lemma 2.2.3. *Let P be a homogeneous polynomial of some degree d. Then P is nondegenerate relative to a finite collection of surjective linear mappings $\{\pi_j\}$ on $\mathbb{R}$, if and only if there exists a constant coefficient partial differential operator $\mathcal{L}$, homogeneous of degree d, such that $\mathcal{L}(P) \neq 0$ but $\mathcal{L}(p_j \circ \pi_j) = 0$ for every polynomial $p_j : V_j \to \mathbb{C}$ of degree d, where V_j denote the range space of π_j.*

Example: In $\mathbb{R}^2$, assume $P(x,y) = xy$ and $\pi_1(x,y) = x$, $\pi_2(x,y) = y$. We pick $\mathcal{L} = \frac{\partial^2}{\partial x \partial y}$, then $\frac{\partial^2 P}{\partial x \partial y} = 1$. But for any polynomial $p_1(\pi_1(x,y)) = p_1(x)$, $p_2(\pi_2(x,y)) = p_2(y)$, $\frac{\partial^2 p_1}{\partial x \partial y} = 0$ and $\frac{\partial^2 p_2}{\partial x \partial y} = 0$. So xy is nondegenerate relative to $\{\pi_1, \pi_2\}$.

Notice that in the above example, $\frac{\partial}{\partial x}$ annihilates any polynomial of one variable y and $\frac{\partial}{\partial y}$ annihilates any polynomial of one variable x. So $\frac{\partial^2}{\partial x \partial y}$ which is the composition of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ annihilates any degenerate polynomials. This idea can be extended to the following definition in [7].

Definition 2.2.4. A polynomial P is said to be simply nondegenerate relative to $\{\pi_i : 1 \leq i \leq n\}$ if there is a differential operator L of the form $L = \prod_{i=1}^{n}(w_i \cdot \nabla)$, with each $w_i \in \ker(\pi_i)$, such that $L(P)$ does not vanish identically.

Remark: It is not difficult to see that simply nondegeneracy implies nondegeneracy. However, the following “light cone” example in [6] shows the converse does not always hold.

Example 2.2.5. Define $P : \mathbb{R}^3 \to \mathbb{R}$ to be $P(x_1, x_2, x_3) = x_3^2$. Fix an arbitrary large positive integer N. For $j \in \{1, 2...N\}$ choose nonzero unit vectors $v_i = (v_j^1, v_j^2, v_j^3) \in \mathbb{R}^3$, none of which is a scalar multiple of another, all satisfying

$$(v_j^3)^2 = (v_j^1)^2 + (v_j^2)^2.$$

Define $\pi_j(x) = x_1 v_j^1 + x_2 v_j^2 + x_3 v_j^3$. For any $f_j \in L^\infty(\pi_j(\mathbb{R}^3))$, the operator $L = \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ annihilates $f_j \circ \pi_j$ for all j, but does not annihilate P. Therefore P

is nondegenerate relative to $\{\pi_j, 1 \le j \le N\}$. But it is not simply nondegenerate since any simply nondegeneracy polynomial should have degree at least N.

The following theorem in [7] shows simple nondegeneracy implies power decay property.

Theorem 2.2.6. *Any simply nondegenerate polynomial has the power decay property in every open set. More precisely, let $d \in \mathbb{N}$. Let $L = \prod_{j=1}^{n}(w_j \cdot \nabla)$, where each $w_j \in \ker(\pi_j)$ is a unit vector. Then there exist $C < \infty$ and $\epsilon > 0$ such that for any real-valued polynomial of degree at most d such that $\max_{|x|\le 1} |L(P)(x)| \ge 1$,*

$$|I_\lambda(f_1, ...f_n)| \le C(1 + |\lambda|)^{-\epsilon} \prod_j ||f_j||_{L^\infty}$$

for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$.

Detailed proof can be found in [7]. The above theorem can also be generalized to the case where the phase is a smooth function with the simply nondegeneracy is defined as $L(S(x))$ does vanish to infinite order. See [11].

From the definition of simple nondegeneracy, one can see that every $w_i \cdot \nabla$ annihilate polynomials defined on one subspace. It turns out that by the same proof in [7], the definition of simple nondegeneracy could be extended to more general case. That is, if there is a differential operator L of the form $L = \prod_{i=1}^{t}(w_i \cdot \nabla)$, for some t, such that each $w_s \cdot \nabla$ annihilates polynomials defined on π_j for $j \in I_s$, with $\cup_s I_s = \{1, 2, ...n\}$ and $\max_{|x|\le 1} |L(P)(x)| \ge 1$, then

$$|I_\lambda(f_1, ...f_n)| \le C(1 + |\lambda|)^{-\epsilon} \prod_j ||f_j||_{L^\infty}$$

holds for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$. We will also see in Chapter 4 that when the subspaces all have one dimensional range space and lie in general position, we may only need to use nondegeneracy degree to characterize nondenegeneracy.

2.3 First tool: Elimination of codimension one range space

The codimension one range space in the title refers to the range space of some surjective linear mapping with dimension $m-1$ if it is defined on $\mathbb{R}^m$. In this section, we will show the result in [7] that to investigate the power decay property of a collection of surjective linear mappings $\{\pi_j\}$, it suffices to consider the collection without those mappings whose range space is codimension one. This property can help to reduce the multilinearity of the problem.

Theorem 2.3.1. *Let $\{\pi_j\}$ be any finite collection of surjective linear mappings on $\mathbb{R}^m$, and let $\{l_i\}$ be any finite collection of surjective linear mappings on $\mathbb{R}^m$ whose nullspace is of dimension one. If $\{\pi_j\}$ has the uniform power decay property, then so does $\{\pi_j\} \cup \{l_i\}$.*

Proof. It suffices to prove this in the case where a single linear mapping l is given. We now assume $\{\pi_i\}$ has the uniform power decay property and we want to show $\{\pi_i, l\}$ also has the uniform power decay property. Choose coordinates of $\mathbb{R}^m$ such that $l(x', x_m) = l(x', 0)$. Let P be any nondegenerate polynomial relative to the augmented collection $\{\pi_i, l\}$. It is no loss of generality to assume that $\ker(l)$ is not contained in any $\ker(\pi_i)$, for any such linear mappings may be deleted from $\{\pi_i\}$ without affecting the nondegeneracy of P.

$$I_\lambda(f_1, ...f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} f_1(\pi_1(x))...f_{n-1}(\pi_{n-1}(x)) f_n(l(x))\eta(x)dx. \tag{2.3.1}$$

Let $T(f_1, ...f_{n-1})(x') = \int e^{i\lambda P(x)} f_1(\pi_1(x))...f_{n-1}(\pi_{n-1}(x))\eta(x)dx_m$.

$$\begin{aligned}(2.3.1) &= \int T(f_1, ...f_{n-1})(x') f_n(l(x', 0))dx' \\ &\leq ||T(f_1, ...f_{n-1})||_{L^2} ||f_n||_{L^2}.\end{aligned} \tag{2.3.2}$$

Here

$$\begin{aligned}||T(f_1, ...f_{n-1})||_{L^2} &= \int e^{i\lambda(P(x)-P(z,x_m))} \prod_{i=1}^{n-1} f_i(\pi_i(x))\overline{f_i(\pi_i(z, x_m))} \\ &\quad \eta(x)\overline{\eta(z, x_m)}dx' dx_m dz \\ &= \int \Big(e^{i\lambda(P(x)-P(x'+z,x_m))} \prod_{i=1}^{n-1} f_i(\pi_i(x))\overline{f_i(\pi_i(x' + z, x_m))} \\ &\quad \eta(x)\overline{\eta(x' + z, x_m)}dx' dx_m\Big) dz \\ &= \int G(z)dz\end{aligned} \tag{2.3.3}$$

where

$$G(z) = \int e^{i\lambda(P(x)-P(x'+z,x_m))} \prod_{i=1}^{n-1} f_i(\pi_i(x))\overline{f_i(\pi_i(x' + z, x_m))} \eta(x)\overline{\eta(x' + z, x_m)}dx' dx_m$$

We want to show $P(x) - P(x' + z, x_m)$ is nondegenerate relative to $\{\pi_i\}$ for most z. To do this, we want to show the polynomial $\frac{\partial P}{\partial x_m}$ is nondegenerate relative to $\{\pi_i\}$.

If it is true, then there exists $z \in \mathbb{R}^m$ for which $P_z(x) = P(x', x_m) - P(x', x_m + z)$ is nondegenerate relative to $\{\pi_i\}$. For if not, since $\partial P/\partial x_m$ can be reconstructed from $\{P_{z_i}\}$ for a suitable finite collection of points z_i, degeneracy of each P_{z_i} implies degeneracy of $\partial P/\partial x_m$.

Consider the quotient space $\mathcal{P}$ of all polynomials of degrees not exceeding the degree of P modulo the subspace of all such polynomials which are degenerate relative to $\{\pi_i\}$, and equip it with some inner product structure. Then $||P_z||^2_{nd}$ is a polynomial in z which does not vanish identically. Since

$$\begin{aligned}
&||P_z||^2_{nd}\\
=&||P(x', x_m) - P(x', x_m + z)||^2_{nd}\\
=&|| - z \cdot \frac{\partial P}{\partial x_m}(x', x_m) + \frac{1}{2} z^2 \cdot \frac{\partial^2 P(x', x_m)}{\partial x_m^2} \ldots\\
&+ \frac{(-1)^d}{d!} z^d \cdot \frac{\partial^d P(x', x_m)}{\partial x_m^d} ||^2_{nd}\\
=&\sum_{h=1}^{d} \sum_{t=1}^{d} \frac{(-1)^{(h+t)}}{h!t!} z^{h+t} \langle \frac{\partial^h P(x)}{\partial {x_m}^h}, \frac{\partial^t P(x)}{\partial {x_m}^t} \rangle.
\end{aligned}$$

Hence there exist $C, \delta \in \mathbb{R}^+$ such that for any ball B of fixed finite radius and for any $\epsilon > 0$,

$$|\{z \in B : ||P_z||^2_{nd} < \epsilon\}| \leq C\epsilon^\delta. \tag{2.3.4}$$

Thus

$$(2.3.3) = \int_A G(z)dz + \int_{A^c} G(z)dz. \tag{2.3.5}$$

Here $A = \{z \in B : ||P_z||^2_{nd} \geq \epsilon\}$, $A^c = \{z \in B : ||P_z||^2_{nd} < \epsilon\}$, B is some compact set that z is supported on, which follows that η is compactly supported. For $z \in A$, the first term can be deal with by the hypothesis that $\{\pi_i\}$ has uniform power decay property. For $z \in A^c$, the second term can be dealt with by (2.3.4). So we have

$$(2.3.5) \leq C(|\epsilon^{1/2}\lambda|^{-\delta} + |\epsilon|^{\delta}) \prod_{i=1}^{n-1} ||f_i||_{L^\infty}. \tag{2.3.6}$$

Let $\epsilon = |\lambda|^{-\delta'}$ for some $\delta' < 2$, then we have (2.3.1) has power decay.

To establish the uniform decay property, fix $c \in (1, \infty)$ and a degree d. Let P be any polynomial of degree at most d whose norm, in the quotient space of polynomials of degree at most d modulo polynomials of degrees at most d that are degenerate relative to $\{\pi_i, l\}$, lies in $[c^{-1}, c]$. Since this quotient space is a finitely dimensional vector space, P thus belongs to a compact subset. Together with the above reasoning, this implies (2.3.4) holds with a constant depending on d and c but not on P.

So now it is only left to show polynomial $\frac{\partial P}{\partial x_m}$ is nondegenerate relative to $\{\pi_i\}$. If not, then there exists a polynomial decomposition $\frac{\partial P}{\partial x_m} = \sum_i q_i \circ \pi_i$. Since $\ker(l)$ is not contained in any $\ker(\pi_i)$, there exist nonzero vectors v_i in the range space V_i of π_i such that $\partial(f \circ \pi_i)/\partial x_m = (v_i \cdot \nabla f) \circ \pi_i$ for all functions $f : V_i \to \mathbb{R}$. Since $0 \neq v_i \in V_i$, there exist polynomials Q_i such that $v_i \cdot \nabla Q_i = q_i$. Consequently, $\partial(Q_i \circ \pi_i)/\partial x_m = q_i \circ \pi_i$, and hence $\tilde{P} = \sum Q_i \circ \pi_i$ satisfies $\partial(P - \tilde{P})/\partial x_m = 0$. Thus $P - \sum_i Q_i \circ \pi_i$ is a function of x_m alone, contradicting the hypothesized nondegeneracy of P.

□

From the above theorem, we can get the following corollary.

Corollary 2.3.2. *Any collection of surjective linear mappings* $\{\pi_j\}_{i=1}^n$ *whose range space has codimension one for each* i *has the uniform power decay property.*

Remark: in $\mathbb{R}^2$, the range space of any surjective linear mapping that we are interested in this problem is of dimension either 0 or 1. If it is 0, it is a constant map. If it is dimension one, it is also codimesion one so that linear mapping can be removed from the collection. The problem in $\mathbb{R}^2$ will finally be deduced to the oscillatory integrals of the first type that we have introduced in chapter 1. However, in higher dimensions, the range space is not necessarily of codimension one so things may be more complicated.

2.4 Second tool: Elimination of common subspace

Let us recall the example below introduced in [6]. In the bilinear case, if the mapping $x \to (\pi_1(x), \pi_2(x))$ of $\mathbb{R}^2$ to $\mathbb{R}^1 \times \mathbb{R}^1$ is a bijection, $I_\lambda(f_1, f_2)$ can be written as

$$\int e^{i\lambda P(x,y)} f(x)g(y)\eta(x,y)dxdy. \tag{2.4.1}$$

A necessary and sufficient condition for it to have the power decay property is $|\frac{\partial^2 P}{\partial x \partial y}|$ does not vanish identically, see [7]; equivalently, P is not a sum of one function of x plus another function of y. Since if $P = p_1(x) + p_2(y)$, $\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 p_1}{\partial x \partial y} + \frac{\partial^2 p_2}{\partial x \partial y} = 0$. Now consider the integral $\int e^{i\lambda P(x,y,z)} f(x,z)g(y,z)\eta(x,y,z)dxdydz$. Again, such an integral has power decay property if and only if there exists $\frac{\partial^2 P(x,y,z)}{\partial x \partial y}$ does not vanish identically, see [7]. For any fixed z, the integration becomes (2.4.1), the result of the previous case gives a bound

$$C\min(1, |Q(z)|^{-1}|\lambda|^{-\delta})||f(\cdot, z)||_{L^\infty}||g(\cdot, z)||_{L^\infty}$$

for some $\delta > 0$ and some polynomial Q which does not vanish identically. The power decay property follows by integration with respect to z.

One can see from this example that under the above assumptions, the problem can be reduced to a lower dimensional case by fixing the "common" subspaces z. It turns out that this observation can be extended to the following "separation" structure.

2.4.1 Separation

One of the most important observations of this thesis is that if $\{\pi_j\}$ preserving certain decomposition of $\mathbb{R}^m$, this property may help divide the problem of checking the the power decay property of oscillatory integrals (2.1.1) into two subproblems defined on subspaces. The definition is as below.

Definition 2.4.1. $\{\pi_i\}_{i=1}^n$ is said to preserve the direct sum decomposition of $\mathbb{R}^m = T_1 \oplus T_2$, where T_1, T_2 are subspace of $\mathbb{R}^m$, if $\pi_i(\mathbb{R}^m) = \pi_i(T_1) \oplus \pi_i(T_2)$ for all i.

In this case, it may look like π_j acts on T_1 and T_2 "separately". We denote $m_1 = \dim T_1$, $m_2 = \dim T_2$.

One example is the following integral which appears in [8],

$$I_\lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^4} e^{i\lambda P(x_1,y_1,x_2,y_2)} f_1(x_1, y_1) f_2(x_2, y_2) f_3(x_1 + x_2, y_1 + y_2)$$

$$\eta(x, y)dxdy \quad (2.4.2)$$

with $T_1 = \{(x_1, 0, x_2, 0)|x_1, x_2 \in \mathbb{R}\}$ and $T_2 = \{(0, y_1, 0, y_2)|y_1, y_2 \in \mathbb{R}\}$. We can show this integral has power decay property by the separation structure in Chapter 5.

One application of the separation property is to reduce dimensions.

Lemma 2.4.2. *Assume* $\mathbb{R}^m = C \oplus R$ *and* $\{\pi_i\}$ *preserves the direct sum decomposition, here* C *and* R *are subspaces of* $\mathbb{R}^m$*,* C *means "common" and* R *means "remaining". If* $\ker \pi_i \cap C = \{0\}$ *for* $1 \leq i \leq n$*, and* $\{\ker(\pi_i) \cap R\}$ *has uniform power decay property in* T_2*, then* $\{\ker(\pi_i)\}$ *has uniform power decay property in* $\mathbb{R}^m$*.*

In the example in [6], we have $\pi_1 : (x, y, z) \to (x, z)$ and $\pi_2 : (x, y, z) \to (y, z)$, so the corresponding kernels are $\{(0, y, 0)\}$ and $\{(x, 0, 0)\}$. Denote $\mathbb{R}^3 = C \oplus R$ where $C = \{(0, 0, z)\}$, $R = \{(x, y, 0)\}$, then $\ker \pi_i \cap C = \{0\}$ for $i = 1, 2$. Now $\ker(\pi_1) \cap R = \{(0, y, 0)\}$, $\ker(\pi_2) \cap R = \{(x, 0, 0)\}$. Denote $\pi_1' : (x, y, 0) \to (x, 0, 0)$ and $\pi_2' : (x, y, 0) \to (0, y, 0)$, then the corresponding kernels in R are exactly $\ker(\pi_1) \cap R$ and $\ker(\pi_2) \cap R$. As suggested by the above lemma, it suffices to check if the integral $\int e^{i\lambda P(x,y,0)} f(x, 0, 0) g(0, y, 0) \eta(x, y, 0) dxdy$ has the uniform power decay property. Denote $f'(x) = f(x, 0, 0)$, $g'(y) = g(0, y, 0)$, $P'(x, y) = P(x, y, 0)$ and $\eta'(x, y) = \eta(x, y, 0)$, the problem now becomes (2.4.1).

2.4.2 Proof of lemma 2.4.2

Proof. Adopt coordinate in $\mathbb{R}^m$ such that $C = \{(x,0)\}$, $R = \{(0,y)\}$.

Assume $\{\ker(\pi_i) \cap R\}$ has uniform power decay property in R. Once a norm $||\cdot||$ is fixed, without loss of generality, we can assume $P(x,y)$ is a polynomial nondegenerate relative to $\{\pi_i\}$ whose nondegeneracy norm is 1. For any fixed ϵ,

$$\begin{aligned}
&\int_{\mathbb{R}^m} e^{i\lambda P(x,y)} \prod_{i=1}^{n} f_i(\pi_i(x,y))\eta(x,y)dxdy \\
=&\int_A \Big(\int e^{i\lambda P^x(y)} \prod_{i=1}^{n} f_i^x(\pi_i(0,y))\eta^x(y)dy\Big)dx \\
+&\int_{A^c} \Big(\int e^{i\lambda P^x(y)} \prod_{i=1}^{n} f_i^x(\pi_i(0,y))\eta^x(y)dy\Big)dx
\end{aligned}$$

where $P^x(y) = P(x,y)$, $f_i^x(\pi_i(0,y)) = f_i(\pi_i(x,y))$, $\eta^x(y) = \eta(x,y)$. $A = \{x \in B : ||P^x||_{nd(\ker(\pi_i)\cap T_2, 1\le i\le n)} \le \epsilon\}$.

By the assumption

$$\begin{aligned}
&\int_{A^c} |\int e^{i\lambda P^x(y)} \prod_{i=1}^{n} f_i^x(\pi_i(0,y))\eta^x(y)dy|dx \\
\le& c|\lambda\epsilon|^{-\delta_1} \int_{A^c} \prod_{i=1}^{n} ||f_i^x||_{L^\infty} dx \\
\le& c|\lambda\epsilon|^{-\delta_1} \prod_{i=1}^{n} ||f_i||_{L^\infty}.
\end{aligned}$$

So it suffices to show there exists uniform constants c, δ such that $|A| \le c\epsilon^\delta$, here P^x is viewed as a polynomial of y. B is some given compact set in C, which may depend on the support of η. The nondegeneracy norm is taken relative to $\{\ker(\pi_i) \cap R\}$, which is defined as $\inf ||P(y) - \sum_i p_i(\pi_i(0,y))||$, where each p_i is a polynomial.

Write $P(x,y) = \sum_{i \le d} \sum_{k_1+...+k_{m_1}=i} P_{k_1,...k_{m_1}}(y_1,...y_{m_2}) x_1^{k_1}...x_{m_1}^{k_1}$. If, for any fixed $k_1,...k_{m_1}$, $P_{k_1,...k_{m_1}}(y_1,...y_{m_2}) = q^1_{k_1,...k_{m_1}}(\pi_1(0,y)) + ... + q^n_{k_1,...k_{m_1}}(\pi_n(0,y))$. Since $\ker \pi_i \cap C = \{0\}$ for $1 \le i \le n$ and π_i is surjective, $\dim(\pi_i(C)) = \dim(C)$. if $(e_1,...e_{m_1})$ is a basis in C, $\{\pi_i(e_j)\}$ will be a basis of $\pi_i()$, and it differ from the underly basis of $\pi_i(T_1)$ by some invertible linear transformation. For any polynomial $r[x]$ defined on C, there is a corresponding polynomial r^i defined on $\pi_i(C)$ such that $r[x] = r^i[\pi_i(x,0)]$. So there exists polynomials $r^i_{k_1,...k_{m_1}}[\pi_i(x,0)]$ such that $x_1^{k_1}...x_{m_1}^{k_{m_1}} = r^i_{k_1,...k_{m_1}}[\pi_i(x,0)]$. Then

$$
\begin{aligned}
P(x,y) =& \sum_{i \le d} \sum_{k_1...k_{m_1}} (q^1_{k_1...k_{m_1}}(\pi_1(0,y)) + ... + q^n_{k_1...k_{m_1}}(\pi_n(0,y)))x_1^{k_1}...x_{m_1}^{k_1} \\
=& \sum_{i \le d} \sum_{k_1+...+k_{m_1}=i} (q^1_{k_1,...k_{m_1}}(\pi_1(0,y))R^1_{k_1,...k_{m_1}}[\pi_i(x,0)] + ... \\
& + q^n_{k_1,...k_{m_1}}(\pi_n(0,y))R^n_{k_1,...k_{m_1}}[\pi_i(x,0)]).
\end{aligned}
$$

Notice for every k, $1 \le k \le n$, π_k preserve the direct sum, if we fix any $z \in \pi_k(\mathbb{R}^m)$, since π_k is surjective, there exists a pair (x,y) such that $\pi_k(x,y) = z$, then $z = \pi_k(x,0) \oplus \pi_k(0,y)$ is a unique decomposition. We also notice that $q^k_{k_1,...k_{m_1}}$ is a polynomial defined on $\pi_k(R)$ and $R^k_{k_1,...k_{m_1}}$ is a polynomials defined on $\pi_k(C)$, so $q^1_{k_1,...k_{m_1}}(\pi_1(0,y))r^1_{k_1,...k_{m_1}}[\pi_i(x,0)]$ is a polynomial defined on $\pi_k(x,y)$. This shows $P(x,y)$ can be represented by a sum of polynomials defined on $\{\pi_k\}_{1 \le k \le n}$ respectively, which contradicts the assumption that $P(x,y)$ is nondegenerate. So there exists $k_1,...k_{m_1}$ such that $P_{k_1,...k_{m_1}}(y_1,...y_{m_2})$ is nondegenerate relative to $\{\pi_k(0,y)\}_{k=1}^n$.

Among all $P_{k_1,...k_{m_1}}(y_1,...y_{m_2})$ that are nondegenerate relative to $\{\pi_k(0,y)\}_{i=1}^n$, we can always pick up one, denote it as $P_{k'_1,...k'_{m_1}}$, such that its nondegeneracy degree is the highest. This $P_{k'_1,...k'_{m_1}}$ may not be unique, but that's fine. Once we fix this $P_{k'_1,...k'_{m_1}}$, let N be the nondegeneracy degree of it, write $P_{k'_1,...k'_{m_1}} = \sum_{i\leq d} P^i_{k'_1,...k'_{m_1}}$ where $P^i_{k'_1,...k'_{m_1}}$ is its homogeneous part of degree i. Without loss of generality, we can assume $P^i_{k_1,...k_{m_1}}$ vanish for all $k_1,...k_{m_1}$ when $i \geq N$. By the assumption, $P^N_{k'_1,...k'_{m_1}}$ is nondegenerate relative to $\{\pi_k(0,y)\}_{1\leq k\leq n}$. So by the lemma 2.2.3, there exists a constant coefficient differential operator L whose symbol is a y polynomial of degree N such that $L(P^N_{k'_1,...k'_{m_1}}) \neq 0$ but $L(q_{k'_1,...k'_{m_1}}(\pi_k(0,y))) = 0$ for all polynomials $q_{k'_1,...k'_{m_1}}$ and all $1 \leq k \leq n$. Then

$$
\begin{aligned}
L(P(x,y)) &= L(\sum_i \sum_{k_1+...+k_{m_1}=i} P_{k_1,...k_{m_1}}(y_1,...y_{m_2})x_1^{k_1}...x_{m_1}^{k_{m_1}}) \\
&= \sum_i \sum_{k_1+...+k_{m_1}=i} L(P_{k_1,...k_{m_1}}(y_1,...y_{m_2}))x_1^{k_1}...x_{m_1}^{k_{m_1}}.
\end{aligned}
$$

Since N is the highest degree among all $P_{k_1,...k_{m_1}}$ and L is a constant coefficient operator of degree N, $L(P_{k_1,...k_{m_1}})$ is either 0 or a nonzero constant. Let $a_{k_1,...k_{m-1}} = L(P_{k_1,...k_{m_1}})$, then $L(P(x,y)) = \sum_i \sum_{k_1+...+k_{m_1}=i} a_{k_1...k_{m_1}} x_1^{k_1}...x_{m_1}^{k_{m_1}}$. And there exists at least one $a_{k_1...k_{m_1}}$ which is nonzero. Without loss of generality, for any $1 \leq k \leq n$

$$
\begin{aligned}
&L(q_k(\pi_k(x,y))) = L(q_k(\pi_k(x,0) + \pi_k(0,y))) \\
=&L(\sum_i \sum_{k_1+...k_{m_1}=i} q_{k_1...k_{m_1}}(\pi_k(0,y))\pi_k^{k_1}(x_1,0...0,0)...\pi_k^{k_{m_1}}(0,...0,x_{m_1},0)) \\
=&\sum_i \sum_{k_1+...k_{m_1}=i} Lq_{k_1...k_{m_1}}(\pi_k(0,y))\pi_k^{k_1}(x_1,0...0,0)...\pi_k^{k_{m_1}}(0,...0,x_{m_1},0) \\
=&0.
\end{aligned}
$$

The set A of x such that $\sum_i \sum_{k_1+...+k_{m_1}=i} a_{k_1...k_{m_1}} x_1^{k_1}...x_{m_1}^{k_{m_1}} = 0$ is of measure zero. For any fixed $x \in B\backslash A$, P^x is nondegenerate relative to $\{\pi_k(0,y)\}$.

Fix $c \in (1,\infty)$. Let P be a polynomial of bounded degree whose nondegeneracy norm relative to $\{\pi_k(x,y)\}_{k=1}^n$ lies in $[c^{-1},c]$. Since the quotient space of polynomial of bounded degree modulo degenerate polynomials is finite dimensional, P belongs to a compact set. Without loss of generality, we can assume the quotient space is equipped with some inner product structure. $||P^x||^2_{nd(\pi_i(0,y),1\leq i\leq n)}$ is a polynomial of x and it is not identically 0. So there exists a constant $c(P)$ and δ such that $|\{x \in B : ||P^x||_{nd(\pi_i(0,y),1\leq i\leq n)} \leq \epsilon\}| \leq c(P)\epsilon^\delta$. Then we can pick a uniform constant c that depends on c and d such that $|\{x \in B : ||P^x||_{nd(\pi_i(0,y),1\leq i\leq n)} \leq \epsilon\}| \leq c\epsilon^\delta$ hold for all P with $c^{-1} \leq ||P||_{nd} \leq c$. □

2.5 Third tool: Elimination of intersection of nullspaces

Let's consider the following example:

$$I_\lambda = \int_{\mathbb{R}^3} e^{i\lambda P(x,y,z)} f(x)g(y)\eta(x,y,z)dxdydz. \tag{2.5.1}$$

Here P is a polynomial, and f,g are two functions such that $||f||_\infty < \infty$, $||g||_\infty < \infty$. Notice that we have dealt with integrals of the form

$$I_\lambda = \int_{\mathbb{R}^2} e^{i\lambda P(x,y)} f(x)g(y)\eta(x,y)dxdy, \tag{2.5.2}$$

which is the oscillatory integral of the second kind in Chapter 1. If $P(x, y, z)$ is indeed a polynomial of x, y, then the problem of checking power decay property of (2.5.1) can be reduced to lower dimensions (2.5.2). We can do so because the span of (x) and (y) is not $\mathbb{R}^3$, that is, the intersection of $\ker(\pi_1)$ and $\ker(\pi_2)$ is not 0. Inspired by this observation, we might want to think if we can always reduce the problem to lower dimensions if the intersection of the nullspaces of the linear mappings are nonzero. First denote $l : \mathbb{R}^3 \to \mathbb{R}^2$ as $(x, y, z) \to (x, y)$. There are two cases that we want to consider.

If $||P||_{nd(l)}$ is bounded below by some constant, since $\ker(l) \subset \ker(\pi_1)$ and $\ker(l) \subset \ker(\pi_1)$, if we view $f(x)g(y)$ a function of x, y, then (2.5.1) is reduced to checking the power decay property of the following integral

$$I_\lambda = \int_{\mathbb{R}^3} e^{i\lambda P(x,y,z)} h(x, y)\eta(x, y, z)dxdydz, \tag{2.5.3}$$

which is essentially the oscillatory integral of the second type and we know (2.5.3) has power decay property.

If $||P||_{nd(l)}$ is actually very small, this suggests that P is "almost" a polynomial of x, y. To be more precise, P can be written as $P(x, y, z) = P_1(x, y) + r(x, y, z)$ for some polynomials P_1, r with $||r||$ very small. And $||P_1(x, y)||_{nd} \geq ||P(x, y, z)||_{nd} - ||r(x, y, z)||$, so $||P_1||_{nd}$ is bounded from below.We will give detailed proof later but essentially, $||r^z||$ is also small for most fixed z in the support of η and the problem is reduced to check the power decay property of the following integral

$$
\begin{aligned}
I_\lambda &= \int_{\mathbb{R}^3} e^{i\lambda P(x,y,z)} f(x)g(y)\eta(x,y,z)dxdydz \\
&= \int (\int e^{i\lambda(P_1(x,y)+r^z(x,y))} f(x)g(y)\eta(x,y,z)dxdy)dz.
\end{aligned} \tag{2.5.4}
$$

The result of the oscillatory integrals of the second kind gives a bound of (2.5.4):

$$C \min(1, ||P_1(x,y) + r^z(x,y)||_{nd(\pi_1,\pi_2)}^{-\delta} |\lambda|^{-\delta}) ||f||_{L^\infty} ||g||_{L^\infty}$$

for some $\delta > 0$. And $||P_1(x,y) + r^z(x,y)||_{nd(\pi_1,\pi_2)}$ is bounded from below since r^z is essentially very small.

The above example can be extended to the following lemma.

Lemma 2.5.1. *If* $\cap_{i=1}^n \ker(\pi_i) = N$, *where* $\dim(N) > 0$ *and let* $\mathbb{R}^m = N \oplus R$, *here* N *denote the intersection of the nullspaces, and* R *means "remaining". Then* $\{\ker(\pi_i) \cap R\}$ *has the uniform power decay property implies* $\{\ker(\pi_i)\}$ *has the uniform power decay property.*

Remark: here we need to assume there is no i such that $\ker(\pi_i) \cap R$ is 0. But since in this thesis, we assume there is no $k \neq j$ such that $\ker(\pi_k) \subset \ker(\pi_j)$. If $\ker(\pi_i) \cap R$ is 0, that means $\ker(\pi_i) \subset \ker(\pi_j)$ for $j \neq i$, which contradicting our assumption.

2.5.1 Proof of lemma 2.5.1

Proof. Without loss of generality, we can assume $||P||_{nd(\{\pi_i\}_{j=1}^n)} = 1$. Choose coordinate of $\mathbb{R}^m$ such that $\mathbb{R}^m = \{(x,y)\}$ where $N = \{(x,0)\}$ and $R = \{(0,y)\}$. Then

$f_i(\pi_i(x,y)) = f_i(\pi_i(0,y))$. Let $l : \mathbb{R}^m \to R$ be the projection from $\mathbb{R}^m$ to R. Notice $\ker(l) \subset \ker(\pi_i)$, the function $\prod_{i=1}^n f_i(\pi_i(0,y))$ can be viewed as a function defined on R. There are two cases.

The first case is $||P||_{nd(l)} > c_1$ for some constant c_1 that will be picked later, the problem becomes the case of oscillatory integrals of the second kind

$$\int_{\mathbb{R}^m} e^{i\lambda P(x,y)} h(y)\eta(x,y)dxdy. \tag{2.5.5}$$

We know in this case, (2.5.5) has the power decay estimate:

$$(2.5.5) \le (1 + |\lambda c_1|)^{-\delta} ||h||_{L^\infty}. \tag{2.5.6}$$

Denote $h = \prod_{i=1}^n f_i$, one can check $|| \prod_{i=1}^n f_i||_{L^\infty} \le \prod_{i=1}^n ||f_i||_{L^\infty}$, so we can get the power decay property of the original integral.

The second case is $||P||_{nd(l)} \le c_1$. Then $P(x,y) = P_1(y) + r(x,y)$ with $||r|| \le c_1$. The integral becomes

$$\begin{aligned} I_\lambda &= \int_{\mathbb{R}^m} e^{i\lambda P(x,y)} \prod_{i=1}^n f_i(\pi_i(x,y))\eta(x,y)dxdy \\ &= \int (\int e^{i\lambda(P_1(y)+r^x(y))} \prod_{i=1}^n f_i(\pi_i(0,y))\eta(x,y)dy)dx. \end{aligned} \tag{2.5.7}$$

We now want to show $||r^x|| \le c_2||r(x,y)||$ for some constant c_2 for any x in a compact set depending on the support of η.

Consider the space $\mathcal{T}$ of all polynomials of degree less than or equal to d equipped with the norm $||\cdot||$ on it. It is a finite dimensional space, so the set of polynomials in this space such that the norm of it is less than or equal to c_1 is compact. For any fixed x, define a operator $\mathcal{L}_x$ on this space such that $\mathcal{L}_x(r(x,y)) = r^x(y)$. $\mathcal{L}_x$ is a continuous operator for each x (since every norm is equivalent in finite dimensional space, if we take the l_2 norm of the polynomial space, we can check that $\mathcal{L}_x$ is a continuous operator for each x). Especially, the norm of this operator can be viewed as a continuous function on the compact set that x is defined. So there is a uniform constant c_2 such that $||r^x|| \leq c_2||r(x,y)|| \leq c_1c_2$ for every x in the given compact set. Since $||P||_{nd} = 1$, if $||r|| \leq c_1$, then $||P_1||_{nd} \geq 1 - c_1$. So when fix x, $||P_1(y) - r^x(y)||_{nd} \geq 1 - c_1 - c_1c_2$. By the hypothesis, $\{\ker(\pi_i) \cap R\}$ has uniform power decay property. The result of the oscillatory integrals of the second kind gives a bound of (2.5.7):

$$\begin{aligned} &C \min(1, ||P_1(x,y) + r^z(x,y)||^{-\delta}_{nd(\pi_i)}|\lambda|^{-\delta}) \prod_{i=1}^{n} ||f_i||_{L^\infty} \\ &\leq C \min(1, ||1 - c_1c_2||^{-\delta}_{nd(\pi_i)}|\lambda|^{-\delta}) \prod_{i=1}^{n} ||f_i||_{L^\infty}. \end{aligned} \tag{2.5.8}$$

Combine the two cases (2.5.6) and (2.5.8), we can conclude the result.

□

2.6 Fourth tool: λ-uniformity

In [7] and [13], a powerful tool called λ-uniformity is introduced to deal with the estimation of oscillatory integrals. This concept is inspired by a notion of uniformity employed by Gowers [10]. Fix a bounded ball $B \subset \mathbb{R}^k$. Let τ be a small quantity to be chosen later, let $|\lambda| \geq 1$ and consider any function $f \in L^2(\mathbb{R}^k)$ supported in

B.

Definition 2.6.1. A function $f \in L^2(B)$ is λ-nonuniform if there exists a polynomial $q(t)$, $t \in \mathbb{R}^k$, of degree bounded by d and a scalar c such that

$$||f - ce^{iq(t)}||_{L^2(B)} \leq (1 - |\lambda|^{-\tau})||f||_{L^2(B)}. \tag{2.6.1}$$

Otherwise, f is said to be λ-uniform.

This notion depends on the parameters d, τ. Once they are fixed, f is λ-uniform is equivalent to $|\int f(t)e^{-iq(t)}dt| \leq |\lambda|^{-\tau/2}||f||_{L^2(B)}$. Since if f_1 is λ-nonuniform, there exists a polynomial q of degree at most d such that $|\int f(t)e^{-iq(t)}dt| > |\lambda|^{-\tau/2}||f||_{L^2(B)}$ Let $c = \langle f, e^{iq}\rangle$.

$$\begin{aligned}||f - \langle f, e^{iq}\rangle e^{iq}||^2_{L^2} =&||f||^2_{L^2} - ||\langle f, e^{iq}\rangle||^2_{L^2} && (2.6.2)\\ =&(1 - \frac{|\langle f, e^{iq}\rangle|^2}{||f||^2_{L^2}})||f||^2_{L^2} && (2.6.3)\\ <&(1 - |\lambda|^{-\tau})||f||^2_{L^2} && (2.6.4)\end{aligned}$$

From the above argument, we can also see that once $||f||_{L^2}$ is bounded by some constant, $c = \langle f, e^{iq}\rangle$ is majorized by a uniform constant independent of q.

In [7], the authors use the λ-uniformity tool to prove that when every π_i has one-dimension range space and the number of functions is not too large, nondegeneracy of the phase function can imply the corresponding oscillatory integral has power decay.

In the setting of [7], each π_j is an orthogonal projection from $\mathbb{R}^m$ to a linear subspace V_j in $\mathbb{R}^m$ which is of same dimension k. They introduce the concept of the general position condition is to be any subcollection of $\{V_j\}_{j=1}^n$ of cardinality $t \geq 1$ spans a subspace of dimension $\min(kt, m)$.

In this thesis, we might want to extend the concept to linear spaces that are not necessarily a subspace of $\mathbb{R}^m$ and may not have the same dimension. To achieve this, we use $\ker(\pi_j)$ instead. We say a collection of distinct linear subspaces $\{A_i\}_{i\in I}$ of $\mathbb{R}^m$ lie in general position if

$$\dim(\cap_{j\in I_0} A_j) = \max\{0, \sum_{j\in I_0} \dim(A_j) - (|I_0| - 1)m\}$$

for any subset I_0 of the index set I.

A collection of surjective linear map $\{\pi_i\}$ is said to satisfy the general position condition if $\{\ker(\pi_i)\}$ satisfies the general position defined as above.

Given the inner product structure in $\mathbb{R}^m$, denote U_i to be the orthogonal complement of $\ker(\pi_i)$. In the case where π_j is indeed an orthogonal projection from $\mathbb{R}^m$ to $U_j \subset \mathbb{R}^m$, these two definitions coincide. Since $\text{span}(U_{j_1}, ...U_{j_t})^\perp = U_{j_1}^\perp \cap ... \cap U_{j_t}^\perp$. $\dim(\text{span}(U_{j_1}, ...U_{j_t})) = m - \dim(\text{span}(U_{j_1}, ...U_{j_t})^\perp) = m - \dim(\cap_{1\leq i\leq t} \ker(\pi_{ji})) = \min\{m, tm - \sum_{1\leq i\leq t} \dim(\ker(\pi_{ji}))\} = \min\{m, \sum_{1\leq i\leq t} \dim(U_{j_i})\}$ for any subset $\{j_1, ...j_t\} \subset \{1, 2...n\}$. In [5], the author also define the general position condition that be more abstract. But when the range space of every linear mapping has dimension one, these definitions are all compatible.

Theorem 2.6.2. *Suppose that $n < 2m$. Then any family $\{\pi_j, 1 \leq j \leq n\}$ of linear surjective maps whose range spaces are one-dimensional lying in general position* *has the uniform power decay property. Moreover, under these hypotheses,*

$$|I_\lambda(f_1, ...f_n)| \le C(1 + |\lambda|)^{-\epsilon} \prod_j ||f_j||_{L^2}$$

holds for all polynomials P of bounded degree which are uniformly nondegenerate with respect to $\{\pi_j\}$, for all functions $f_j \in L^2(\pi_j(\mathbb{R}^m))$, with uniform constants $C, \epsilon \in \mathbb{R}^+$.

Proof. Given the cutoff function η, we can assume without loss of generality that each f_j is supported in some $B_j \subset \mathbb{R}$. We assume $||f_j||_{L^2} \le 1$ for all j and $|\lambda|$ exceed some sufficiently large constant.

The proof divides into two parts depends on whether f_i is λ-uniform. Define $A(\lambda)$ to be the best constant such that

$$|I_\lambda(f_1, ..., f_n)| \le A(\lambda) \prod_j ||f_j||_{L^2}.$$

If f_1 is λ-nonuniform, let c, q satisfy (2.6.1). Then

$$|I_\lambda(f_1 - ce^{iq}, ..., f_n)| \le A(\lambda)(1 - |\lambda|^{-\tau}).$$

Where $|c|$ is majorized by an absolute constant. Notice $\{\pi_i\}_{i=2}^n$ still satisfies the general position condition, so by the induction on the number of functions, we have

$$|I_\lambda(ce^{iq}, f_2...f_n)| \le C|I_\lambda(e^{iq}, f_2, ...f_n)| \le C|\lambda|^{-\sigma},$$

for certain $C, \sigma \in (0, \infty)$. Combine the two terms, we have

$$|I_\lambda(f_1...f_n)| \le A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}.$$

Since $A(\lambda)$ is the best constant, we should have $A(\lambda) \leq A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}$. This implies $A(\lambda) \leq |\lambda|^{\tau-\sigma}$. Once we pick $\tau < \sigma$, we then have $A(\lambda) \leq |\lambda|^{-\epsilon}$ for some $\epsilon > 0$.

Now consider the case where f_1 is λ-uniform. We may assume that n is strictly larger than m since the theorem is already known in a more precise form for the case $n = m$. See [16], [14]. Consider $W_1 = \bigcap_{i=2}^m \ker(\pi_i)$, $W_2 = \bigcap_{i>m} \ker(\pi_i)$. By the general position condition,

$$\dim(\bigcap_{i=2}^m \ker(\pi_i)) = \max\{0, \sum_{i=2}^m \dim(\ker(\pi_i)) - (m-2)m\} = 1$$

$$\dim(\bigcap_{i>m} \ker(\pi_i)) = \max\{0, \sum_{i>m} \dim(\ker(\pi_j)) - (n-m-1)m\} = 2m - n > 0$$

What's more, by the general position condition, the intersection of $\bigcap_{i=2}^m \ker(\pi_i)$ and $\bigcap_{i>m} \ker(\pi_i)$ is $\{0\}$. We can adopt coordinates $(t_1, t_2, y) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^{m-2}$ such that $\{(t_1, 0, 0)\} = \bigcap_{i=2}^m \ker(\pi_i)$, $\{(0, t_2, 0)\} \subset \bigcap_{i>m} \ker(\pi_i)$. Denote $t = (t_1, t_2)$.

Since $\prod_{j=2}^m f_j \circ \pi_j(t_1, t_2, y) = \prod_{j=2}^m f_j \circ \pi_j(t_1, 0, 0) + \prod_{j=2}^m f_j \circ \pi_j(0, t_2, y) = \prod_{j=2}^m f_j \circ \pi_j(0, t_2, y)$, we can denote $F_1^y(t_2) = \prod_{j=2}^m f_j \circ \pi_j(t, y)$. By the same argument, $F_2^y(t_1) = \prod_{j=m+1}^m f_j \circ \pi_j(t, y)$ and let l be the linear mapping such that $G^y(l(t)) = f_1 \circ \pi_1(t, y)$, $\eta^y(t) = \eta(t, y)$. We have

$$\int ||F_1^y||_{L^2}^2 ||G^y||_{L^2}^2 dy = C \prod_{j=2}^m ||f_j||_{L^2}^2$$

$$\int ||F_2^y||_{L^2}^2 dy = C \prod_{j>m} ||f_j||_{L^2}^2$$

and consequently,

$$\int ||F_1^y||_{L^2} ||G^y||_{L^2} ||F_2^y||_{L^2} dy \leq C(|\lambda|^{1-\rho})^{-\tilde{\rho}} \prod_j ||f_j||_{L^2}$$

For each y, consider

$$I_\lambda^y = \int_{\mathbb{R}^2} e^{i\lambda P^y(t)} F_1^y(t_2) F_2^y(t_1) G^y(l(t)) \eta^y(t) dt.$$

There are two kinds of y.

1. The "bad" parameters y such that P^y can be decomposed in the form $P^y(t) = Q_1(t_1)+Q_2(t_2)+Q_3(l(t))+R(t)$ where Q_j are real-valued polynomials of degree at most d on $\mathbb{R}$ and $||R|| \leq |\lambda|^{-\rho}$, ρ will be specified later.

2. The "good" parameters y such that $||R|| > |\lambda|^{-\rho}$ under the above decomposition.

For the good parameters, $|\lambda|^\rho P^y$ is at least a fixed positive distance from the span of all polynomials $Q_1(t_1) + Q_2(t_2) + Q_3(l(t))$, so we can apply theorem 2.3.1 and it suffices to show $\{\pi_1\} \cup \{\pi_i\}_{i>m}$ has uniform decay property, which is proved by the induction hypothesis. Now the nondegeneracy norm of the phase function λP^y is at least $|\lambda \cdot \lambda^{-\rho}| = |\lambda|^{1-\rho}$ and by the induction hypothesis, there exists some constant $\tilde{\rho}$ such that

$$|I_\lambda^y| \leq C(|\lambda|^{1-\rho})^{-\tilde{\rho}} ||F_1^y||_{L^2} ||F_2^y||_{L^2} ||G^y||_{L^2}$$

So

$$\int |I_\lambda^y| dy \leq C(|\lambda|^{1-\rho})^{-\tilde{\rho}} \prod_j ||f_j||_{L^2}$$

For those bad parameter y, let $\tilde{F_1^y}(t_2) = F_1^y(t_2) e^{i\lambda Q_2(t_2)}$, $\tilde{F_2^y}(t_1) = F_2^y(t_1) e^{i\lambda Q_1(t_1)}$, $\tilde{G}(s) = G^y(s) e^{i\lambda Q_3(s)}$, $\tilde{\eta}^y(t) = \eta^y(t) e^{i\lambda R(t)}$. By the equality

$$\tilde{\eta}^y(t_1,t_2) = \int e^{2\pi(t_1\cdot\xi_1+t_2\cdot\xi_2)}\hat{\tilde{\eta}}^y(\xi_1,\xi_2)d\xi_1 d\xi_2,$$

$$\tilde{G^y}(l(t)) = \int e^{2\pi l(t)\cdot\xi_0}\hat{\tilde{G}}^y(\xi_0)d\xi_0,$$

If we denote $l:(t_1,t_2)\to c_1t_1+c_2t_2$ for some constant c_1,c_2, and let $\xi_1' = -c_2\xi_0-\xi_2$, $\xi_2' = -c_1\xi_0-\xi_1$, then

$$\tilde{F_1^y}(t_2) = \int e^{2\pi\xi_1'\cdot t_2}\hat{\tilde{F}}_1^y(\xi_1')d\xi_1',$$

$$\tilde{F_2^y}(t_1) = \int e^{2\pi\xi_2'\cdot t_1}\hat{\tilde{F}}_2^y(\xi_2')d\xi_2',$$

So we have

$$\begin{aligned}|I_\lambda^y| &= \Big|\int_{\mathbb{R}^2}\tilde{F_1^y}(t_2)\tilde{F_2^y}(t_1)\tilde{G^y}(l(t))\tilde{\eta}^y(t)dt\Big| \\ &= \Big|\int_{\mathbb{R}^3}\hat{\tilde{F}}_1^y(-c_2\xi_0-\xi_2)\hat{\tilde{F}}_2^y(-c_1\xi_0-\xi_1)\hat{\tilde{G}}^y(\xi_0)\hat{\tilde{\eta}}^y(\xi_1,\xi_2)d\xi_0d\xi_1d\xi_2\Big|.\end{aligned} \tag{2.6.5}$$

Notice λR is a polynomial of degree at most d and is $O(|\lambda|^{1-\rho})$ on the support of $\xi\in\mathbb{R}^2$. By the principle of non-stationary phase, we have

$$|\hat{\tilde{\eta}}^y(\xi)| \le C_N|\lambda|^{(1-\rho)N}(1+|\xi|)^{-N}$$

for $\xi=(\xi_1,\xi_2)$ and any $N\in\mathbb{N}$. Here C_N depends N and the C^N norm of $\tilde{\eta^y}$, which is majorized by some constants times C^N norm of η if supp(η) is fixed. Without loss of generality, we can assume $N=4$, then (2.6.5) $\le C||\hat{\tilde{F}}_1^y||_{L^\infty}||\hat{\tilde{F}}_2^y||_{L^1}||\hat{\tilde{G}}^y||_{L^\infty}$.

By the same argument, we also have $|I_\lambda^y|\le C||\hat{\tilde{F}}_1^y||_{L^1}||\hat{\tilde{F}}_2^y||_{L^\infty}||\hat{\tilde{G}}^y||_{L^\infty}$. Making use of a simply interpolation argument and Plancherel's theorem, we get $|I_\lambda^y|\le C||\hat{\tilde{F}}_1^y||_{L^2}||\hat{\tilde{F}}_2^y||_{L^2}||\hat{\tilde{G}}^y||_{L^\infty}\le C|\lambda|^{4(1-\rho)-\tau}||\tilde{F_1^y}||_{L^2}||\tilde{F_2^y}||_{L^2}\le C|\lambda|^{4(1-\rho)-\tau}\prod_{j=2}^n||f_j||_{L^2}$. So we can just pick $\rho\in(0,1)$ close enough to 1 such that $4(1-\rho)-\tau=-\epsilon<0$.

Combine, we get

$$A(\lambda) \leq \max(|\lambda|^{-\epsilon}, |\lambda|^{-(1-\rho)\tilde{\rho}}, A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}).$$

So we can pick τ and the corresponding ρ such that $A(\lambda) \leq |\lambda|^{-\epsilon'}$ for some $\epsilon' > 0$.

□

Remark: This result partially solves the "light cone" problem when the number of function is ≤ 5. We should notice that when the dimension of the range space is ≥ 2, 2.3.1 can no longer been used. However,the "splitting" argument in the next section will help to solve this problem.

2.6.1 Splitting

In this thesis, we will first generalize λ-uniformity to the following lemma.

Lemma 2.6.3. *Assume $f, g_1 ... g_n$ are functions : $V \to \mathbb{C}$, where V is a vector space with an inner product structure $\langle , \rangle$ where $\eta : V \to \mathbb{C}$ is a smooth function with compact support. If $\langle f, \eta \prod_{i=1}^{n} g_i \rangle = c||f||_{L^2} \prod_{i=1}^{n} ||g_i||_{L^\infty}$, with $||\eta \prod_{i=1}^{n} g_i||_{L^2} \leq \prod_{i=1}^{n} ||g_i||_{L^\infty} \leq 1$. Then $||f - a\eta \prod_{i=1}^{n} g_i||_{L^2} \leq (1 - \frac{1}{2}|c|^2 \prod_{i=1}^{n} ||g_i||_{L^\infty})||f||_{L^2}$, where $a = c||f||_{L^2}$.*

Proof.

$$\begin{aligned}
&\langle f - a\eta \prod_{i=1}^{n} g_i, f - a\eta \prod_{i=1}^{n} g_i \rangle \\
&= \langle f, f\rangle - \langle f, a\eta \prod_{i=1}^{n} g_i\rangle - \langle a\eta \prod_{i=1}^{n} g_i, f\rangle + \langle a\eta \prod_{i=1}^{n} g_i, a\eta \prod_{i=1}^{n} g_i\rangle \qquad (2.6.6)\\
&= ||f||_{L^2}^2 - \bar{a}\langle f, \eta \prod_{i=1}^{n} g_i\rangle - a\langle \eta \prod_{i=1}^{n} g_i, f\rangle + |a|^2 ||\eta \prod_{i=1}^{n} g_i||_{L^2}^2
\end{aligned}$$

Plug in $a = c||f||_{L^2}$ and $\langle f, \eta \prod_{i=1}^{n} g_i\rangle = c||f||_{L^2} \prod_{i=1}^{n} ||g_i||_{L^\infty}$, we get $\bar{a}\langle f, \eta \prod_{i=1}^{n} g_i\rangle = a\langle \eta \prod_{i=1}^{n} g_i, f\rangle = |c|^2||f||_{L^2}^2 \prod_{i=1}^{n} ||g_i||_{L_\infty}$. We should also notice that $||\eta \prod_{i=1}^{n} g_i||_{L^2}^2 \leq || \prod_{i=1}^{n} g_i||_{L^\infty}^2 \leq || \prod_{i=1}^{n} g_i||_{L^\infty}$ since $|| \prod_{i=1}^{n} g_i||_{L^\infty} \leq 1$.

$$\begin{aligned}
(2.6.6) &\leq ||f||_{L^2}^2 - 2|c|^2||f||_{L^2}^2 \prod_{i=1}^{n} ||g_i||_{L_\infty} + |c|^2||f||_{L^2}^2 \prod_{i=1}^{n} ||g_i||_{L_\infty}\\
&= (1 - |c|^2 \prod_{i=1}^{n} ||g_i||_{L_\infty})||f||_{L^2}^2\\
&\leq (1 - |c|^2 \prod_{i=1}^{n} ||g_i||_{L_\infty} + \frac{1}{4}|c|^4 \prod_{i=1}^{n} ||g_i||_{L_\infty}^2)||f||_{L^2}^2\\
&= (1 - \frac{1}{2}|c|^2 \prod_{i=1}^{n} ||g_i||_{L_\infty})^2||f||_{L^2}^2.
\end{aligned}$$

This gives $||f - a\eta \prod_{i=1}^{n} g_i||_{L^2} \leq (1 - \frac{1}{2}|c|^2 \prod_{i=1}^{n} ||g_i||_{L^\infty})||f||_{L^2}$. □

From the above lemma, given η and $c' > 0$, if $\langle f, \eta \prod_{i=1}^{n} g_i\rangle = c||f||_{L^2} \prod_{i=1}^{n} ||g_i||_{L^\infty}$ with $|c| > |c'|$, we say f splits into $a\eta \prod_{i=1}^{n} g_i$, since in this case $||f - a\eta \prod_{i=1}^{n} g_i||_{L^2} \leq (1 - \frac{1}{2}|c'|^2 \prod_{i=1}^{n} ||g_i||_{L_\infty})||f||_{L^2}$.

M. Christ develops the idea of "splitting" in [5] to reduce dimensions. He also defines the general position condition in an implicit way and may be different from the previous definition. However, if the kernel of each linear map is of codimension 1, these definitions are compatible.

Let $V_\alpha = \ker(\pi_\alpha)$. The main result of [5] is the following

Theorem 2.6.4. *If a finite family of subspaces $\{V_\alpha\}$ of $\mathbb{R}^m$ of codimensions $k_\alpha \in [1, m-1]$ is in general position and satisfies*

$$2 \max_\beta k_\beta + \sum_\alpha k_\alpha \leq 2m. \tag{2.6.7}$$

then $\{V_\alpha\}$ has the uniform power decay property.

The coefficient of 2 in 2.6.7 is unnatural and the proof still applies in many cases with a smaller quantity. For example, when $k_\alpha = 1$, the hypothesis 2.6.7 reduces to $n \leq 2m - 1$.

The following lemma in [8] could be proved by the above idea. It is also proved by an alternate method, see [8].

Lemma 2.6.5. *Let $\{\pi_j : 1 \leq j \leq 3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2m} \to \mathbb{R}^m$ which lie in general position. Then*

$$|\Lambda(\lambda)(f_1, f_2, f_3)| \leq (|\lambda| ||P||_{nd})^{-\epsilon} \prod_{j=1}^{3} ||f_j||_{L^2}$$

holds for all polynomials P: $\mathbb{R}^{2m} \to \mathbb{R}^m$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R}^m)$, with constants C, ϵ depend only on m, d, η. It is no loss of generality to restrict attention to the case where $\mathbb{R}^{2m}$ is identified with $\mathbb{R}^m_x \times \mathbb{R}^m_y$, and $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $\pi_3(x, y) = x + y$.

Chapter 3

First result: Trilinear case

We already discuss about some results of the case where the number of functions does not exceed 2, see [7], [14]. The general case of higher multilinearity has not been thoroughly investigated. In this thesis, we will deal with the very general trilinear case. Especially, we do not require the linear mappings to satisfy the general position condition. Here is the result.

Theorem 3.0.1. *Let* $\{\pi_i\}_{i=1}^{3}$ *be a collection of three surjective linear mappings from* $\mathbb{R}^m$ *to* V_i *where* $1 \leq \dim(V_i) \leq m-1$*. Then*

$$|I_\lambda(f_1, f_2, f_3)| \leq C(1 + |\lambda|||P||_{nd})^{-\epsilon} \prod_{j=1}^{3} ||f_j||_{L^\infty} \tag{3.0.1}$$

holds for all $f_j \in L^\infty(\pi_j(\mathbb{R}^m))$ *with uniform constant* C, ϵ*.*

Remark: though we do not require any geometric property of linear mappings, we may still assume there are no i, j in $\{1, 2, 3\}$ and $i \neq j$ such that $\ker(\pi_i) \subset$

$\ker(\pi_j)$. Since if so, up to a linear transformation, we may view f_j as a function defined on the range space of π_i. This will reduce the trilinear case to bilinear case, which has been discussed in Chapter 1.

3.1 First reduction: Get rid of the common space

By theorem 2.4.2, it suffices to assume

$$span(\ker(\pi_1), \ker(\pi_2), \ker(\pi_3)) = \mathbb{R}^m.$$

If not, denote span($\ker(\pi_1), \ker(\pi_2)$, $\ker(\pi_3)$) as R. And let C be the subspace of $\mathbb{R}^m$ such that $\mathbb{R}^m = C \oplus R$. Then $\ker(\pi_i) \cap C = 0$ by the definition of direct sum decomposition. By applying theorem 2.4.2, it reduces to check if $\{\ker(\pi_i) \cap R\}$ has the power decay property. In this case, by the definition of R, $\ker(\pi_i) \cap R = \ker(\pi_i)$, $i = 1, 2, 3$, so span$(\ker(\pi_1) \cap R, \ker(\pi_2) \cap R, \ker(\pi_3) \cap R) = R$. We assume there is no $i \neq j$ such that $\ker(\pi_i) \subset \ker(\pi_j)$, so for every $1 \leq i \leq 3$, $0 \subsetneq \ker(\pi_i) \cap R \subsetneq R$.

3.2 Second reduction: Get rid of the common intersection of nullspaces

By theorem 2.5.1, it suffices to consider the case where $\ker(\pi_1) \cap \ker(\pi_2) \cap \ker(\pi_3) = 0$.

3.3 Third reduction: Get rid of the intersection of any two nullspaces

We claim it suffices to prove the case where $\ker(\pi_i) \cap \ker(\pi_j) = 0$ when $i \neq j$.

If $W_1 = \ker(\pi_1) \cap \ker(\pi_2)$ with $\dim(W_1) > 0$. From the previous section, we can assume $W_1 \cap \ker(\pi_3) = 0$. By choosing coordinate properly, we can assume $\mathbb{R}^m = \{(x, y, z)\}$ and $W_1 = \{(x, 0, 0)\}$, $\ker(\pi_3) = \{(0, y, 0)\}$. Here $\{(0, 0, z)\}$ is the complement of $\text{span}(W_1, \ker(\pi_3))$. It is possible that this complement is $\{0\}$. If it is indeed $\{0\}$, we can identify π_3 as $(x, y) \to (x)$, $\pi_i(x, y) = \pi_i(0, y)$, $i = 1, 2$. (3.0.1) can be written as

$$\int_{\mathbb{R}^m} e^{i\lambda P(x,y)} f_1(\pi_1(0, y)) f_2(\pi_2(0, y)) f_3(x) \eta(x, y) dx dy \tag{3.3.1}$$

$$= \int (\int e^{i\lambda P^y(x)} f_3(x) \eta^y(x) dx) f_1(\pi_1(0, y)) f_2(\pi_2(0, y)) dy. \tag{3.3.2}$$

By the splitting lemma, we can replace $f_3(x)$ by $e^{ip(x)}$ for some polynomial $p(x)$, and replace $\eta(x)$ with $\eta'(x)$ for some smooth function $\eta'(x)$ of compact support. Then the problem is reduced to the bilinear case.

If the complement has dimension greater than 0, we can identify π_3 with the projection $(x, y, z) \to (x, z)$. $\pi_i(x, y, z) = \pi_i(0, y, z)$, $i = 1, 2$. (3.0.1) can be written as

$$\int_{\mathbb{R}^m} e^{i\lambda P(x,y,z)} f_1(\pi_1(0, y, z)) f_2(\pi_2(0, y, z)) f_3(x, z) \eta(x, y, z) dx dy dz \tag{3.3.3}$$

$$= \int (\int e^{i\lambda P^y(x,z)} f_1^y(\pi_1(0, 0, z)) f_2^y(\pi_2(0, 0, z)) f_3(x, z) \eta(x, y, z) dx dz) dy \tag{3.3.4}$$

where $P^y(x,z) = P(x,y,z)$, $f_i^y(\pi_i(0,0,z)) = f_i(\pi_i(0,y,z))$ for $i = 1,2$. For fixed y, by the splitting lemma, one can replace $f_3(x,z)$ with $f_3'(z)$, and η with some η'. The phase function $\lambda P(x,y,z) - Q(x,z)$ for any polynomial Q is still nondegenerate relative to $\{\ker_i\}$, $i = 1,2,3$, with the same nondegeneracy norm. Now π_3 is replaced by the projection $(x,y,z) \to (z)$.

Notice now $\ker(\pi_1) \cap \ker(\pi_2) \cap \ker(\pi_3) = W_1$, by the same method of last section, the problem is reduced to check whether $\{\ker(\pi_i) \cap R\}$, $i = 1,2,3$, has uniform power decay property in the ambient space $R = \{(0,y,z)\}$. Notice in R, $(\ker(\pi_1) \cap R) \cap (\ker(\pi_2) \cap R) = 0$. We should also check that $\ker(\pi_i) \cap R \neq 0$. Actually, if there is any $i \neq j$ such that $\ker(\pi_i) \subset \ker(\pi_j)$, the problem can be reduced to the bilinear case so we can just assume there is no $i \neq j$ such that $\ker(\pi_i) \subset \ker(\pi_j)$. From this, we have $W_1 \subsetneq \ker(\pi_i)$, $i = 1,2$. So $\ker(\pi_i) \cap R$ has dimension greater than 0 for $i = 1,2,3$.

By this argument, we can assume without loss of generality that $\ker(\pi_1) \cap \ker(\pi_2) = 0$. Repeating the same argument, one can then assume $\ker(\pi_i) \cap \ker(\pi_j) = 0$ when $i \neq j$.

3.4 Fourth reduction: Any two nullspaces span the ambient space

In this section, we will show it suffices to consider the case that

$$span(\ker(\pi_i), \ker(\pi_j)) = \mathbb{R}^m.$$

when $i \neq j$. If not, without loss of generality, we can assume

$$span(\ker(\pi_1), \ker(\pi_2)) \subsetneq \mathbb{R}^m.$$

By the previous section, we can conclude

$$span(\ker(\pi_1), \ker(\pi_2)) = \ker(\pi_1) \oplus \ker(\pi_2).$$

Let $\ker(\pi_1)$ is spanned by a set of vectors $\{y\}$, $\ker(\pi_2)$ is spanned by a set of vectors $\{z\}$. If $\text{span}(\ker(\pi_1), \ker(\pi_2)) \subsetneq \mathbb{R}^m$, since

$$span(\ker(\pi_1), \ker(\pi_2), \ker(\pi_3)) = \mathbb{R}^m,$$

there exists a set of vectors $\{x\}$ linearly independent of $\{y, z\}$ such that $\{x\} \subset \ker(\pi_3)$ and $\mathbb{R}^m$ is spanned by $\{x, y, z\}$. Without loss of generality, by choosing coordinate properly, one can identify π_1 as the projection $(x, y, z) \to (x, z)$, π_2 as the projection $(x, y, z) \to (x, y)$, $\pi_3(x, y, z) = \pi_3(0, y, z)$. The integral (3.0.1) can be written as

$$\int_{\mathbb{R}^m} e^{i\lambda P(x,y,z)} f_1(x, z) f_2(x, y) f_3(\pi_3(0, y, z))\eta(x, y, z)dxdydz \tag{3.4.1}$$

$$= \int (\int e^{i\lambda P^y(x,z)} f_1(x, z) f_2^y(x) f_3^y(\pi_3(0, 0, z))\eta(x, y, z)dxdz)dy \tag{3.4.2}$$

where

$$P^y(x,z) = P(x,y,z), f_2^y(z) = f_2(y,z), f_3^y(\pi_3(0,0,z)) = f_3(\pi_3(0,y,z)).$$

By splitting lemma, $f_1(x,z)$ can be replaced by $f_1'(z)g_1(x)$ for some f_1', g_1, while we can denote $f_2'(x,y) = g_1(x)f_2(x,y)$. η is replaced by a corresponding η'. $P'(x,y,z) = P(x,y,z) - Q_1(x,z)$ for some polynomial Q_1. The integral becomes

$$\int_{\mathbb{R}^m} e^{i\lambda P'(x,y,z)} f_1'(z)f_2'(x,y)f_3(\pi_3(0,y,z))\eta'(x,y,z)dxdydz$$

By the same argument, $f_2'(x,y)$ can be replaced by $f_2''(y)$. The integral becomes

$$\int_{\mathbb{R}^m} e^{i\lambda P''(x,y,z)} f_1'(z)f_2''(y)f_3(\pi_3(0,y,z))\eta''(x,y,z)dxdydz$$

Here π_1 is replace by the projection $L_1 : (x,y,z) \to (z)$, π_2 is replace by the projection $L_2 : (x,y,z) \to (y)$. Here the phase function $P''(x,y,z)$ is still nondegenerate relative to $\{L_1, L_2, \pi_3\}$ with nondegeneracy norm no less than 1.

Now we should notice that $\ker(\pi_1)\cap\ker(\pi_2)\cap\ker(\pi_3) = \{(x,0,0)\}$. By the same technique applied before, if we denote $R \cong \{(0,y,z)\}$, the problem is reduced to show the corresponding linear mappings $\{\ker(\pi_3)\cap R, \ker(L_1)\cap R, \ker(L_2)\cap R\}$ have power decay property. In this case,

$$span(\ker(L_1)\cap R, \ker(L_2)\cap R) = R.$$

One should also check that $(\ker(L_1)\cap R)\cap(\ker(L_2)\cap R) = 0$. And $(\ker(L_1)\cap R)\cap(\ker(\pi_3)\cap R) = 0$. Since $\ker(\pi_1) = \{(x,y,0)\}$ and by the assumption of the previous section, $\ker(\pi_1)\cap\ker(\pi_3) = \{(x,0,0)\}$. We also have $(\ker(L_2)\cap R)\cap(\ker(\pi_3)\cap R) = 0$

by the same reasoning. That means after this reduction, the linear mappings we are dealing with still satisfy the assumption of the previous section.

Without loss of generality, we still use $\{\ker(\pi_i)\}_{i=1}^3$ to denote the kernel of the linear mappings. Now we check the previous assumption still holds, that is, $\ker(\pi_i) \cap \ker(\pi_j) = 0$ for $i \neq j$ and what's more, span$(\ker(\pi_1), \ker(\pi_2))$=$\mathbb{R}^m$.

Let $span(\ker(\pi_2), \ker(\pi_3)) = W_2$. By exactly the same argument, we can set $\ker(\pi_2) = \{(0, y, 0)\}$, $\ker(\pi_3) = \{(0, 0, z)\}$ and $\{(x, 0, 0)\} \subset \ker(\pi_1)$ while $\mathbb{R}^m = \{(x, y, z)\}$. If we identify π_2 with $(x, y, z) \to (x, z)$, π_3 with $(x, y, z) \to (x, y)$, and let $\pi_1(x, y, z) = \pi_1(0, y, z)$, by the splitting lemma, it suffices to replace π_2 with the linear mapping $L_2 : (x, y, z) \to (z)$, and replace π_3 with $L_3 : (x, y, z) \to (y)$. Now $\ker(\pi_1) \cap \ker(L_2) \cap \ker(L_3) = \{(x, 0, 0)\}$. Let $R = \{(0, y, z)\}$. The problem is reduced to check $\{\ker(\pi_i) \cap R\}_{i=1}^3$ has the power decay property. One can check $\{\ker(\pi_1) \cap R, \ker(L_2) \cap R, \ker(L_3) \cap R\}$ still satisfy the condition that $(\ker(\pi_1) \cap R) \cap (\ker(L_j) \cap R) = 0$, $i = 2, 3$ and $(\ker(L_2) \cap R) \cap (\ker(L_3) \cap R) = 0$. What's more, one also need to check that $span(\ker(\pi_1) \cap R, \ker(L_2) \cap R) = R$. Here $R = \{(0, y, z)\}$, so $\ker(L_2) \cap R = \{(0, y, 0)\}$. Since $\mathbb{R}^m = \{(x, 0, 0)\} \oplus \{(0, y, z)\}$, there exists a subspace U of the space $\{(0, y, z)\}$ such that $\ker(\pi_1) = \{(x, 0, 0)\} \oplus U$. And $\ker(\pi_1) \cap R = U$. We already know $span(\ker(\pi_1), \ker(\pi_2)) = \mathbb{R}^m$, this implies $span(U, \{(0, y, 0)\}) = \{(0, y, z)\}$. And this shows $span(\ker(\pi_1) \cap R, \ker(L_2) \cap R) = R$. So we can repeat the above argument till $\ker(\pi_i)$ and $\ker(\pi_j)$ span the ambient space for any pair of $i \neq j$.

3.5 End of Proof

By the previous assumption, we can assume

$$span(\ker(\pi_i), \ker(\pi_j)) = \mathbb{R}^m$$

for $i \neq j$ and $\ker(\pi_i) \cap \ker(\pi_j) = 0$. That is,

$$\dim(\ker(\pi_1)) + \dim(\ker(\pi_2)) = m$$

$$\dim(\ker(\pi_2)) + \dim(\ker(\pi_3)) = m$$

$$\dim(\ker(\pi_1)) + \dim(\ker(\pi_3)) = m$$

So we must have every π_i is a surjective linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^k$ where $m = 2k$, $i = 1, 2, 3$. If one can check, $\{\ker(\pi_i)\}_{i=1}^{3}$ actually satisfy the general position condition in [7] and [8]. By the result below in [8], we can conclude $\{\pi_i\}$, $i = 1, 2, 3$ has uniform power decay property.

Lemma 3.5.1. *Let $\{\pi_j : 1 \leq j \leq 3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2m} \to \mathbb{R}^m$ which lie in general position. Then*

$$|I(\lambda)(f_1, f_2, f_3)| \leq (|\lambda| ||P||_{nd})^{-\epsilon} \prod_{j=1}^{3} ||f_j||_{L^2}$$

holds for all polynomials P: $\mathbb{R}^{2m} \to \mathbb{R}^m$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R}^m)$, with constants C, ϵ depend only on m, d, η.

3.6 Transverse splitting condition

Theorem 3.0.1 can be used to deal with some multiliear cases. Besides the general position condition, we want to introduce another kind of special structure of the surjective linear mappings that also has been discussed in the work of Christ [5] though his notation might be more abstract.

Definition 3.6.1. A set of linear mappings $\{\pi_i\}_{i\in I} : \mathbb{R}^m \to \mathbb{R}^{k_i}$ with nullspace V_i is said to satisfy the transverse splitting condition if they satisfies general position condition and, without loss of generality, if assuming $\dim(V_1) = \min_{i\in I}\dim(V_i)$, $I\backslash\{1\}$ can be divided into two disjoint groups I_1, I_2 such that:

$$(m - \dim(V_1)) + \sum_{j\in I_1}(m - \dim(V_j)) \le m,$$

$$(m - \dim(V_1)) + \sum_{j\in I_2}(m - \dim(V_j)) \le m.$$

Remark: one example of the above definition is the one dimensional range space case where the number of functions does not exceed $2m-1$ and those linear mappings lie in general position. Actually, theorem 2.6.2 can be extended to the following corollary.

Corollary 3.6.2. *Any collection of surjective linear mappings $\{\pi_i\}$ defined on $\mathbb{R}^m$ that satisfies the transverse splitting condition has the uniform power decay property.*

Proof. Let $V_i = \ker(\pi_i)$, $1 \le i \le n$. Without loss of generality, assuming $\dim(V_1) \le \dim(V_2) ... \le \dim(V_n)$, such that $I\backslash\{1\}$ can be divided into two disjoint groups I_1,I_2

such that:

$$(m - \dim(V_1)) + \sum_{j\in I_1}(m - \dim(V_j)) \le m,$$
$$(m - \dim(V_1)) + \sum_{j\in I_2}(m - \dim(V_j)) \le m.$$

Let Σ_1, Σ_2 be the linear mapping such that $\ker(\Sigma_1) = \cap_{i\in I_1}\ker(\pi_i)$ and $\ker(\Sigma_2) = \cap_{i\in I_2}\ker(\pi_i)$. Notice by the general position condition,

$$\begin{aligned}
&\dim(\mathrm{span}(\ker(\pi_1), \ker(\Sigma_1))) \\
=&\dim(\ker(\pi_1)) + \dim(\ker(\Sigma_1)) - \dim(\ker(\pi_1) \cap \ker(\Sigma_1)) \\
=&\dim(\ker(\pi_1)) + \dim(\ker(\Sigma_1)) - \max\{0, \dim(\ker(\pi_1)) \\
&+ \dim(\ker(\Sigma_1)) - m\} \\
=&\dim(\ker(\pi_1)) + \max\{0, \sum_{j\in I_1}\dim(\ker(\pi_j)) - (|I_1| - 1)m\} \\
=&m.
\end{aligned}$$

The same argument shows $\dim(\mathrm{span}(\ker(\pi_1), \ker(\Sigma_2))) = m$.

If f_1 is λ-uniform,

$$\begin{aligned}
I_\lambda(f_1, ...f_n) &= \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j(\pi_j(x))\eta(x)dx \\
&= \int_{\mathbb{R}^m} e^{i\lambda P(x)} f_1(\pi_1(x))G_1(\Sigma_1(x))G_2(\Sigma_2(x))\eta(x)dx.
\end{aligned} \tag{3.6.1}$$

Here $G_1(\Sigma_1(x)) = \prod_{i\in I_1} f_i(\pi_i)$, $G_2(\Sigma_2(x)) = \prod_{i\in I_2} f_i(\pi_i)$.

There are two cases:

1. P is nondegenerate relative to $\{\pi_1, \Sigma_1, \Sigma_2\}$. And the nondegeneracy norm is $\ge |\lambda|^{-\epsilon}$. This case is proved by theorem 3.0.1.

2. P's nondegeneracy norm relative to $\{\pi_1, \Sigma_1, \Sigma_2\}$ is $\leq |\lambda|^{-\epsilon}$.

In case 2, let π_i^*, Σ_i^* be the adjoint of π_i, Σ_i. That is, $\langle \pi_i(x), y\rangle = \langle x, \pi_i^*(y)\rangle$ for any $x \in \mathbb{R}^m$ and any $y \in V_i$. Define $l_1 : V_1 \times V_2 \to \mathbb{R}^m : (\xi_1, \xi_2) \to \pi_1^*(\xi_1) + \Sigma_1^*(\xi_2)$. We want l_1 to be linear and injective. Linearity is trivial. We just need to show injectivity. That is $\pi_1^*(\xi_1) + \Sigma_1^*(\xi_2) = 0$ implies $\xi_1 = \xi_2 = 0$. We will show $\mathrm{span}(\ker(\pi_1), \ker(\Sigma_1)) = \mathbb{R}^m$ can guarantee this. Notice $\ker(\pi_i) = \mathrm{im}(\pi_i^*)^\perp$ for $1 \leq i \leq n$. So $\mathrm{span}(\ker(\pi_1), \ker(\Sigma_1)) = \mathbb{R}^m$ implies $\mathrm{span}(\mathrm{im}(\pi_1^*)^\perp, \mathrm{im}(\Sigma_1^*)^\perp) = \mathbb{R}^m$. For $\mathrm{span}(\mathrm{im}(\pi_1^*)^\perp, \mathrm{im}(\Sigma_1^*)^\perp) = \mathrm{span}(\mathrm{im}(\pi_1^*) \cap \mathrm{im}(\Sigma_1^*))^\perp$, which gives $\mathrm{im}(\pi_1^*) \cap \mathrm{im}(\Sigma_1^*) = 0$, so $\pi_1^*(\xi_1) + \Sigma_1^*(\xi_2) = 0$ implies $\pi_1^*(\xi_1) = \Sigma_1^*(\xi_2) = 0$. Since π_i is surjective, we have π_1^*, Σ_1^* is injective , which gives $\xi_1 = \xi_2 = 0$. The same argument shows $l_2 : V_1 \times V_3 \to \mathbb{R}^m : (\xi_1, \xi_3) \to \pi_1^*(\xi_1) + \Sigma_2^*(\xi_3)$ is also injective.

$P(x) = P_1(\pi_1(x)) + Q_1(\Sigma_1(x)) + Q_2(\Sigma_2(x)) + R(x)$ where $||R||_{nd} \leq |\lambda|^{-\epsilon}$. Let $\tilde{f}_1 = f_1 \cdot e^{i\lambda P_1(\pi_1(x))}$, $\tilde{G}_i = G_i \cdot e^{i\lambda Q_i(\pi_1(x))}$, $1 \leq i \leq 2$, $\tilde{\eta} = \eta \cdot e^{i\lambda R(x)}$,

(3.6.1) $= \int \hat{\tilde{f}}_1(\xi_1)\hat{\tilde{G}}_1(\xi_2)\hat{\tilde{G}}_2(\xi_3)\check{\tilde{\eta}}(\pi_1^*(\xi_1) + \Sigma_1^*(\xi_2) + \Sigma_2^*(\xi_3))d\xi_1 d\xi_2 d\xi_3$.

It suffices to show $\sup_{\xi_3} \int\int |\check{\tilde{\eta}}(\pi_1^*(\xi_1) + \Sigma_1^*(\xi_2) + \Sigma_2^*(\xi_3))| d\xi_1 d\xi_2 < C|\lambda|^{1-\rho}$, which is guaranteed by the injectivity of $l : V_1 \times V_2 \to \mathbb{R}^m : (\xi_1, \xi_2) \to \pi_1^*(\xi_1) + \Sigma_1^*(\xi_2)$.

If f_1 is λ-nonuniform, the problem is reduced to $I_\lambda(f_2, ...f_n)$. By the induction hypothesis, $\{\pi_i\}_{i=2}^n$ still satisfies the transverse splitting condition. Keep repeating the above argument till the number of function is ≤ 3, by theorem 3.0.1, we know the integral has the uniform power decay property. □

Chapter 4

Second result: Generalization of one dimensional case

In this chapter, we will discuss a generalization of the one dimensional range space case. In [7], the authors already discuss about the case where $n \leq 2m-1$. However, higher multilinearity case has not been studied yet. Essentially, if a phase function is "very" nondegenerate, like it is simply nondegenerate or has very high nondegeneracy degree, then the corresponding oscillatory integral has the power decay property no matter what the structure of the linear mappings might be. Or, if the linear mappings satisfy the transverse splitting condition for instance, then the phase function being nondegenerate is sufficient to ensure the power decay property of the oscillatory integral. The second result in the thesis, which is listed below, provides a good trade off between the nondegeneracy degree of the phase function and the structure of the linear mappings.

Theorem 4.0.1. *$\{\pi_i\}_{i=1}^n$ is a collection of surjective linear mappings defined on $\mathbb{R}^m$ whose range space is one-dimensional for each i. Assume $\{\pi_i\}$ satisfies the general position condition. If there is an integer $k \leq d-1$ such that $n \leq k(m-1)+2$ and P is a polynomial with nondegeneracy degree $\geq k+1$ relative to $\{\pi_i\}_{i=1}^n$. Then we have*

$$|I_\lambda(f_1, ...f_n)| \leq C(1 + ||\lambda \sum_{i \geq k+1} P_i||_{nd})^{-\epsilon} \prod_{j=1}^n ||f_j||_{L^\infty} \tag{4.0.1}$$

for all $f_j \in L^\infty(\pi_j(\mathbb{R}^m))$ with uniform constant C, ϵ.

Sketch of the proof:

The proof of the theorem is by induction and divided into several steps. First we introduce some helper mappings. For a fixed $1 \leq s \leq m-2$, we define a linear surjective mapping Σ_s on $\mathbb{R}^m$ such that $\ker(\Sigma_s) = \ker(\pi_1(x)) \cap ... \cap \ker(\pi_s(x))$. For each s, there are two cases:

1. P has small nondegenerate norm relative to $\{\pi_i, \Sigma_s\}$.

2. P has large nondegenerate norm relative to $\{\pi_i, \Sigma_s\}$.

The question is, how we define "small" and "large" as above? Actually, the threshold of "small" and "large" is different for each s. By choosing proper threshold for each s, we can guarantee either the second case happens or if the first case happens, we can keep going to $s+1$ or we can prove the oscillatory integral has power decay property. We call Σ_s the helper mapping.

For $s = 1$, the helper mapping has the same kernel of π_1. So it is just $\{\pi_i\}_{i=1}^n$ and the second case holds. We then assume for cases k less than or equal to s, the second case hold, and we now consider case $s + 1$.

If the first case holds, P is essentially a polynomial defined on the complement of $\ker(\Sigma_{s+1})$. By choosing coordinate properly, we can assume $\pi_i(x_1, ...x_m) = \pi_i(0, ...0, x_i, 0..0)$ for $1 \leq i \leq m$. And $\prod_{i=1}^{s+1} f_i(\pi_i(x))$ can be viewed as function defined on the complement of $\ker(\Sigma_{s+1})$. By the TT^* method, essentially the integral becomes

$$\begin{aligned} I_\lambda &= \int e^{i\lambda P(x)} f_1(\pi_1(x_1)) f_2(\pi_2(x_2))...f_m(\pi_m(x_m))...f_n(\pi_n(x))\eta(x)dx \\ &= \int G(x_1, ..x_s, x_{s+2}..x_m) f_1(\pi_1(x_1))...f_s(\pi_s(x_s)) f_{s+2}(\pi_{s+2}(x_{s+2}))... \\ &\quad f_m(\pi_m(x_m))dx_1...dx_s dx_{s+2}...dx_m. \end{aligned}$$

Here

$$\begin{aligned} &G(x_1, ..., x_s, x_{s+2}, ..., x_m) \\ &= \int e^{i\lambda P(x)} f_{s+1}(\pi_{s+1}(x_{s+1})) f_m(\pi_m(x_m))...f_n(\pi_n(x))\eta(x)dx_{s+1}. \end{aligned}$$

We will show the new phase function $P(x_1, ...x_m) - P(x_1, ...x_s + y_s, ...x_m)$ is nondegenerate relative to $\{\pi_{s+1}, \pi_{m+1}, ...\pi_n\}$. We will choose a proper linear differential operator that vanishes on all degenerate polynomials but not the new phase function that we are dealing with. Basically, this differential operator is a product of the normal vector, similarly defined as the one in simple nondegeneracy 2.2.6. To prove the new phase function is nonvanishing under this differential operator, we need the nondegeneracy degree of P to satisfy the assumption in the theorem.

If it is the second case, we keep going to consider $s+2$ and repeat the above process. Finally we will reach the case where $s = m-1$. Notice the helper mapping's range space has codimension one and it turns out that it can be solved by the same codimension one argument as in 2.3.1.

4.1 First step: Induction settings

We prove theorem 4.0.1 by induction on k.

By theorem 2.3.1, we can assume $m \geq 3$. Without loss of generality, we also assume $||\sum_{i\geq k+2} P_i||_{nd(\pi_i, 1\leq i\leq n)} = 1$.

When $k = 1$ and $n \leq m+1$, it is concluded by theorem 2.1 in [7] that

$$|I_\lambda(f_1, ...f_n)| \leq C(1 + ||\lambda P||_{nd})^{-\epsilon} \prod_{j=1}^{n} ||f_j||_{L^\infty} \tag{4.1.1}$$

Since the space of all polynomials of degree at most d is a finite dimensional space, every norm defined on it is equivalent. So we have

$$||P||_{nd(\pi_i, 1\leq i\leq n)} \geq c|| \sum_{i\geq k+2} P_i||_{nd(\pi_i, 1\leq i\leq n)}$$

with c an absolute constant depends only on the choice of norm. So

$$(4.1.1) \leq C'(1 + ||\lambda \sum_{i\geq 2} P_i||_{nd})^{-\epsilon} \prod_{j=1}^{n} ||f_j||_{L^\infty}$$

for some constant C'.

Assume the theorem holds for any $1 \leq t \leq k$. We now deal with the case $t = k+1$. Now fix P. By the assumption, the highest-degree nondegenerate homogeneous part P_i of P has degree $i \geq k+2$. If $n \leq k(m-1)+2$, by the same argument, we have $||\sum_{i\geq k+1} P_i||_{nd(\pi_i, 1\leq i\leq n)} \geq c||\sum_{i\geq k+2} P_i||_{nd(\pi_i, 1\leq i\leq n)}$ with c an absolute constant depends only on the choice of norm. By the induction hypothesis, (4.0.1) holds true. So we can write $n = k(m-1)+2+i$, $1 \leq i \leq m-1$.

Let $W_s = \ker(\pi_1(x)) \cap ... \cap \ker(\pi_s(x))$ and denote Σ_s be any projection defined on $\mathbb{R}^m$ with nullspace W_s, $1 \leq s \leq m-1$. For any given fixed P and fixed $1 \leq s \leq m-1$, we will show one of these cases must be true:

1. $||\sum_{i\geq k+2} P_i||_{nd(\pi_i, \Sigma_s)} < \epsilon_s$. Here the nondegeneracy norm is taken relative to $\{\pi_i, \Sigma_s\}$, $1 \leq i \leq n$. Then (4.0.1) holds true.

2. $||\sum_{i\geq k+2} P_i||_{nd(\pi_i, \Sigma_s)} \geq \epsilon_s$.

Let $\epsilon_1 = 1$. We will continue to pick up ϵ_s, $2 \leq s \leq m-1$. The way of picking up ϵ_s will be specified later but now we can assume $\epsilon_i > \epsilon_{i+1}$, $1 \leq i \leq m-2$.

When $s = 1$, $W_s = \ker(\pi_1(x))$. $\{\pi_1, ...\pi_n, \Sigma_s, P\}$ is the same as $\{\pi_1, ...\pi_n, P\}$, So $||\sum_{i\geq k+2} P_i||_{nd(\pi_i, 1\leq i\leq n, \Sigma_1)} \geq \epsilon_1$ holds true.

Assume we know for $1 \leq u \leq s \leq m-2$, case (2) holds true. We now consider $u = s+1$.

4.2 Second step: Phase function has small non-degeneracy norm relative to helper mappings

If $||\sum_{i\geq k+2} P_i||_{nd(\pi_i,\Sigma_{s+1})} < \epsilon_{s+1}$ is the case, that is, the phase function has small nondegeneracy norm relative to helper mappings.

We write $P = \sum_{i=0}^{d} P_i$, where P_i is the homogeneous part of degree i, then we have

$$P_i = Q_i(\Sigma_{s+1}(x)) + R_i(x) + \sum_{j=1}^{n} p_{ij}(\pi_j(x)), \tag{4.2.1}$$

for $i \geq k+2$, with $||\sum_{i\geq k+2} R_i(x)|| < \epsilon_{s+1}$.

By the induction hypothesis of s, there exists some Q_i with $i \geq k+2$ such that Q_i is nondegenerate relative to $\{\pi_1, ...\pi_n, \Sigma_s\}$ since $\epsilon_s > \epsilon_{s+1}$. Let Q_r has the highest degree among those homogeneous part Q_i that is nondegenerate with respect to $\{\pi_1, ...\pi_n, \Sigma_s\}$. Then $r \geq k+2$. Without loss of generality, we can also write $Q_i = Q_i(\Sigma_s(x)) + \sum_{j=1}^{n} p_{ij}(\pi_j(x))$ for $i > r$. Without loss of generality, $\sum_{j=1}^{n} p_{ij}(\pi_j(x))$ could be assigned to f_j. To summarize it, we write

$$P(x) \;=\; \sum_{i<k+2} P_i(x) \;+\; \sum_{i>r} Q_i(\Sigma_s(x)) \;+\; \sum_{k+2\leq i\leq r} Q_i(\Sigma_{s+1}(x)) \;+\; \sum_{k+2\leq i} R_i(x).$$

Notice $||\sum_{i\geq k+2} Q_i(\Sigma_{s+1}(x)) + \sum_{i\geq k+2} R_i(x)||_{nd(\pi_i,1\leq i\leq n,\Sigma_s)} \geq \epsilon_s$ and $||\sum_{i\geq k+2} R_i(x)|| < \epsilon_{s+1}$, which implies

$$||\sum_{k+2\leq i\leq r} Q_i(\Sigma_{s+1}(x))||_{nd(\pi_i,1\leq i\leq n,\Sigma_s)} \geq \epsilon_s - \epsilon_{s+1}. \tag{4.2.2}$$

By the general position condition, denote I be any subset of $\{1, 2...n\}$ with cardinality m, $\dim(\cap_{i\in I}\ker(\pi_i)) = 0$, $\dim(\cap_{i\in I\setminus k}\ker(\pi_i)) = 1$ for any $k \in I$. Without loss of generality, let $I = \{1, 2...m\}$. Pick any nonzero vector $e_k \in \cap_{i\in I\setminus k}\ker(\pi_i)$, since it is one-dimensional, the vector we pick is unique up to a scale. We claim these m vectors are linearly independent. If not, without loss of generality, suppose $e_1 \in \text{span}\{e_2, ...e_m\}$. By definition, $e_i \in \ker(\pi_1(x))$ for any $i \in \{2, ...m\}$. So $\text{span}\{e_2, ...e_m\} \in \ker(\pi_1(x))$. Thus $e_1 \in \ker(\pi_1(x))$, which implies $e_1 \in \cap_{i\in I}\ker(\pi_i)$. But $\cap_{i\in I}\ker(\pi_i) = 0$, contradicting the assumption that e_1 is a nonzero vector. So these m vectors is a basis of $\mathbb{R}^m$ and we can have the corresponding coordinate. Notice for any fixed k with $1 \leq k \leq m$, $e_j \in \ker(\pi_k)$ for $j \neq k$ and $1 \leq j \leq m$. Let $x = (x_1, ...x_m) \in \mathbb{R}^m$, we have $\pi_k(x) = \pi_k(0, .., 0, x_k, 0, ..0) + \pi_k(x_1, ..., x_{k-1}, 0, x_{k+1}...x_m) = \pi_k(0, .., 0, x_k, 0, ..0)$. To make the notation easier to read, we will write $\pi_k(x_k)$ as $\pi_k(0, .., 0, x_k, 0, ..0)$ for $1 \leq k \leq m$. Actually, $\text{span}_{i\in I\setminus k}\{e_i\} \in \ker(\pi_k)$ and has dimension $m - 1$. So $\text{span}_{i\in I\setminus k}\{e_i\} = \ker(\pi_k)$. $W_s = \ker(\pi_1(x)) \cap ... \cap \ker(\pi_s(x)))$, so $\Sigma_s(x) = \Sigma_s(x_1, ...x_s, 0..0)$. We will also denote $\Sigma_s(x_1, ...x_s) = \Sigma_s(x_1, ...x_s, 0..0)$. Then

$$
\begin{aligned}
I_\lambda &= \int e^{i\lambda P(x)} f_1(\pi_1(x_1)) f_2(\pi_2(x_2))...f_m(\pi_m(x_m))...f_n(\pi_n(x))\eta(x)dx \\
&= \int G(x) e^{i\lambda \sum_{i>r} Q_i(\Sigma_s(x))} f_1(\pi_1(x_1))...f_s(\pi_s(x_s)) f_{s+2}(\pi_{s+2}(x_{s+2}))... \\
&\quad f_m(\pi_m(x_m)) dx_1...dx_s dx_{s+2}...dx_m.
\end{aligned}
$$

Here

$$G(x_1,...,x_s,x_{s+2},...,x_m)$$
$$= \int e^{i\lambda(\sum_{i<k+2} P_i + \sum_{k+2\le i\le r} Q_i + \sum_{i\ge k+2} R_i)} f_{s+1}(\pi_{s+1}(x_{s+1}))$$
$$f_m(\pi_m(x_m))...f_n(\pi_n(x))\eta(x)dx_{s+1}.$$

By Cauchy-Schwartz inequality,

$$I_\lambda \le (\int |G(x_1,...,x_s,x_{s+2},...,x_m)|^2 dx_1...dx_s dx_{s+2}...dx_m)^{\frac{1}{2}} \prod_{i\ne s+1} ||f_i||_{L_2}.$$

We want to show for some $\epsilon > 0$,

$$\int |G(x_1,...,x_s,x_{s+2},...,x_m)|^2 dx_1...dx_s dx_{s+2}...dx_m)^{\frac{1}{2}}$$
$$\le C(1+|\lambda|)^{-\epsilon} \prod_{i\in\{s+1,m+1,...n\}} ||f_i||_{L^\infty}.$$

We write $J(x) = \sum_{k+2\le i\le r} Q_i(\Sigma_{s+1}(x)) + \sum_{k+2\le i} R_i(x) + \sum_{i\le k+1} P_i(x)$.

$$\int | \int e^{i\lambda J(x)} f_{s+1}(\pi_{s+1}(x_{s+1})) f_{m+1}(\pi_{m+1}(x))...f_n(\pi_n(x))\eta(x)dx_{s+1}|^2$$
$$dx_1...dx_s dx_{s+2}...dx_m)^{\frac{1}{2}}$$
$$=(\int \Big(\int e^{i\lambda(J(x_1,...x_m) - J(x_1,...x_{s+1}+y_{s+1},...x_m))} {F_{s+1}}^{y_{s+1}}(\pi_{s+1}(x_{s+1}))$$
$$F_{m+1}^{y_{s+1}}(\pi_{m+1}(x))...F_n^{y_{s+1}}(\pi_n(x))\eta^{y_{s+1}}(x)dx_1...dx_s dx_{s+1}...dx_m\Big) dy_{s+1})^{\frac{1}{2}}$$
$$\le(\int |I_{y_{s+1}}| dy_{s+1})^{\frac{1}{2}}.$$

Here

$$F_k^{y_{s+1}}(\pi_k(x)) = f_k(\pi_k(x))\bar{f}_k(\pi_k(x_1,...x_s,x_{s+1}+y_{s+1},x_{s+2},...x_m)$$

for $k = s+1, m+1, ...n$.

$$\eta^{y_{s+1}}(x) = \eta(x)\bar{\eta}(x_1,...x_s,x_{s+1}+y_{s+1},x_{s+2},...x_m).$$

$$I_{y_{s+1}} = \int e^{i\lambda(J(x_1,...x_m)-J(x_1,...x_{s+1}+y_{s+1},...x_m))} {F_{s+1}}^{y_{s+1}}(\pi_{s+1}(x_{s+1}))$$

$$F_{m+1}^{y_{s+1}}(\pi_{m+1}(x))...F_n^{y_{s+1}}(\pi_n(x))\eta^{y_{s+1}}(x)dx_1...dx_s dx_{s+1}...dx_m.$$

The phase function becomes

$$J(x_1,...x_m) - J(x_1,...x_{s+1}+y_{s+1},...x_m)$$

$$= \sum_{k+2\le i\le r} (Q_i(\Sigma_{s+1}(x)) - Q_i(\Sigma_{s+1}(x_1,...x_{s+1}+y_{s+1},...x_m)))$$

$$+ \sum_{i\ge k+2} (R_i(x) - R_i(x_1,...x_{s+1}+y_{s+1},...x_m))$$

$$+ \sum_{i\le k+1} (P_i(x) - P_i(x_1,...x_{s+1}+y_{s+1},...x_m)).$$

Define $H^{z_{s+1}}(x) = J^{z_{s+1}}_{i\ge k+2}(x) = J_{i\ge k+2}(x_1,...x_m) - J_{i\ge k+2}(x_1,...x_{s+1}+z_{s+1},...x_m)$. We consider the quotient space of all polynomials of degree $\le d-1$ modulo sum of the subspace of all degenerate polynomials relative to $\{\pi_{s+1},\pi_{m+1},...\pi_n\}$ and the space of polynomial of degree $\le k+1$. If this quotient space has an inner product structure and the norm is induced by the inner product structure then

$||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}$ is a polynomial of z^{s+1} since

$$
\begin{aligned}
&||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}\\
=&||\sum_{j\geq k+1}(J_{i\geq k+2}(x_1,...x_m)\\
&-J_{i\geq k+2}(x_1,...x_{s+1}+z_{s+1},...x_m))_j||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}\\
=&||\sum_{j\geq k+1}-z_{s+1}(\frac{\partial J_{i\geq k+2}}{\partial x_{s+1}}(x_1,...x_m))_j+\frac{1}{2}\sum_{j\geq k+1}z^2_{s+1}(\frac{\partial^2 J_{i\geq k+2}}{\partial {x_{s+1}}^2})_j...\\
&+\frac{(-1)^d}{d!}\sum_{j\geq k+1}z^d_{s+1}(\frac{\partial^d J_{i\geq k+2}}{\partial {x_{s+1}}^d})_j||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}\\
=&\sum_{h=1}^{d}\sum_{t=1}^{d}\frac{(-1)^{(h+t)}}{h!t!}z^{h+t}_{s+1}\langle\sum_{j\geq k+1}(\frac{\partial^h J_{i\geq k+2}}{\partial {x_{s+1}}^h})_j,\sum_{j\geq k+1}(\frac{\partial^t J_{i\geq k+2}}{\partial {x_{s+1}}^t})_j\rangle.
\end{aligned}
$$

Now we will show for proper ϵ_s, $||\sum_{i\geq k+1}(\frac{\partial J}{\partial x_{s+1}})_i||_{nd(\pi_i,i=s+1,m+1...n)}$ could be greater than a constant $c>0$.

Consider the vector space T consisting of polynomials of the form $\sum_{i=1}^n p_i(\pi_i(x))+p_0(\Sigma_{s+1}(x))$ whose degree is $\leq d$. Let L_1 be the space of polynomials of the form $\sum_{i=1}^n p_i(\pi_i(x))+p_0(\Sigma_{s+1}(x))$ whose degree is $\leq k+1$ and L_2 be the space consisting of polynomials of the form $\sum_{i=1}^n p_i(\pi_i(x))+p_0(\Sigma_s(x))$ whose degree is $\leq d$. Let L denote the quotient space of T modulo L_1+L_2. Then L is a finite dimensional space with a norm $||\sum_{i\geq k+2}P_i||_{nd(\pi_i,1\leq i\leq n,\Sigma_s)}$ for any polynomial P defined on it.

Let H be the space consisting of all polynomials of degree $\leq d$ modulo the sum of polynomials of the form $\sum_{i=m+1}^n p_i(\pi_i(x))+p_{s+1}(\pi_{s+1}(x))$ and polynomials of degree $\leq k$. H is also finite dimensional with the norm $||\sum_{i\geq k+1}P_i||_{nd(\pi_i,1\leq i\leq n)}$.

We now define an operator $h:L\to H$ by $h(l)=\frac{\partial l}{\partial x_{s+1}}$. This is well-defined. Since if $t_1,t_2\in T$ with $\tilde{t_1}=\tilde{t_2}$ in L, then $t_1-t_2=\sum_{i=1}^n q_i(\pi_i(x))+q_0(\Sigma_s(x))+$

$p(\Sigma_{s+1}(x))$ for some polynomials p, q_i, $0 \leq i \leq n$ with degree of p is $\leq k+1$. So $\frac{\partial(t_1-t_2)}{\partial x_{s+1}} = \sum_{i=m+1}^{n} q_i'(\pi_i(x)) + q_{s+1}'(\pi_i(x)) + \frac{\partial p(\Sigma_{s+1}(x))}{\partial x_{s+1}}$, for some polynomials q_i', and the degree of $\frac{\partial p(\Sigma_{s+1}(x))}{\partial x_{s+1}}$ is $\leq k$, which is 0 in the quotient space H. Also, h is linear and continuous. If we define the function h_0 on L by $h_0(l) = ||h(l)||_H = ||\sum_{i\geq k+1}(\frac{\partial l}{\partial x_{s+1}})_i||_{nd(\pi_i, i=s+1, m+1\ldots n)}$. Then h_0 is continuous.

Let A be the compact set $\{l : c \leq ||\sum_{i\geq k+2} l_i||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} \leq C\}$ on L, where C is an absolute constant ≥ 2, c will be chosen later and $c \leq 1$. h_0 is a continuous function on L, So it attains minimum in A. What's more, since $h_0(al) = |a|h_0(l)$ where $a \in \mathbb{R}$ is a scalar, if $h_0(l^c) = inf_{l\in A} h_0(l)$, we must have $||\sum_{i\geq k+2} l_i^c||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} = inf_{l\in A}||\sum_{i\geq k+2} l_i||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} = c$. If we let $c = \epsilon_s - \epsilon_{s+1}$, then for any $P \in A$, there exists a constant C_1 independent of P such that $h_0(P) \geq C_1(\epsilon_s - \epsilon_{s+1})$. Notice $||\sum_{k+2\leq i\leq r} Q_i(\Sigma_{s+1}(x))||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} \geq \epsilon_s - \epsilon_{s+1}$, so we have $h_0(\sum_{k+2\leq i\leq r} Q_i(\Sigma_{s+1}(x))) \geq C_1(\epsilon_s - \epsilon_{s+1})$.

It is left to show this constant C_1 is nonzero. To make the proof easier to read, we will postpone it to last step.

Consider the space consisting of polynomials of degree $\leq d$ modulo degenerate polynomials relative to $\{\pi_i, 1 \leq i \leq n, \Sigma_s\}$. It is also a finite dimensional space. Notice $||\sum_{i\geq k+2} R_i|| \leq \epsilon_{s+1}$, so

$$||\sum_{i\geq k+2} R_i||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} \leq \epsilon_{s+1}.$$

By the same argument, we see $||\sum_{i\geq k+1}(\frac{\partial R}{\partial x_{s+1}})_i||_{nd(\pi_i, i=s+1, m+1\ldots n)} \leq C_2\epsilon_{s+1}$ for some constant C_2 independent of R.

So now we can pick ϵ_{s+1}, $1 \leq s \leq m-3$, such that $C_1(\epsilon_s - \epsilon_{s+1}) \geq 2C_2\epsilon_{s+1}$. Then $||\sum_{i\geq k+1}(\frac{\partial J}{\partial x_{s+1}})_i||_{nd(\pi_i, i=s+1,m+1...n)} \geq C_2\epsilon_{s+1} > 0$ relative to $\{\pi_i, i = s+1, m+1...n\}$.

From the above argument, we see $||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}$ is not identically 0 and it is of bounded degree as a polynomial of z_{s+1}. So there exists C, δ such that for any compact set B in $\mathbb{R}$ of bounded radius, for any $\epsilon > 0$,

$$|A| = |\{z_{s+1} \in B : ||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)} < \epsilon\}| \leq C(J)\epsilon^{\delta(J)}. \tag{4.2.3}$$

Since $||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)}$ is a polynomial of bounded degree, $\delta(J)$ could be a uniform constant independent of J. We now want to prove the constant $C(J)$ is uniform.

Consider the quotient space T_1 of polynomials of degree bounded by d modulo the sum of degenerate polynomials relative to $\{\pi_i, 1 \leq i \leq n, \Sigma_s\}$ and polynomials of degree $\leq k+1$. The set of polynomials in T_1 such that $c_1 \geq ||P_{i\geq k+2}||_{nd(\pi_i, 1\leq i\leq n, \Sigma_s)} \geq c_2$ is compact, for any constants $c_2 \leq c_1$.

Define a function l on T_1 by $l(P) = ||P_{i\geq k+2}||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})}$, then l is a well-defined and continuous function on T_1 since if $l(P_1) = l(P_2)$ then $\sum_{i\geq k+2}(P_1 - P_2)_i$ is degenerate relative to $\{\pi_i, \Sigma_s\}$ which is also 0 under the norm $||\cdot||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})}$ and $|||\sum_{j\geq k+2}(P_1)_j||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})} - ||\sum_{j\geq k+2}(P_2)_j||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})}| \leq ||\sum_{j\geq k+2}(P_1 - P_2)_j||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})} \leq ||\sum_{j\geq k+2}(P_1 - P_2)_j||_{nd(\pi_i,1\leq i\leq n,\Sigma_s)}$. So the set of P in T_1 such that $l(P) \leq c_3$ is closed for any c_3. Thus the subset A_1 in T_1 with $A_1 = \{P : c_2 \leq ||P_{i\geq k+2}||_{nd(\pi_i,1\leq i\leq n,\Sigma_s)} \leq c_1, ||P_{i\geq k+2}||_{nd(\pi_i,1\leq i\leq n,\Sigma_{s+1})} \leq c_3\}$ is compact in T_1. The polynomial J lies in A_1. Once we define an inner product structure on the space V of polynomials of degree bounded by d, for any subspace W of it, we can identify V/W with $W^{\perp}$. Thus $\langle \sum_{j\geq k+1}(\frac{\partial^h J_{i\geq k+2}}{\partial {x_{s+1}}^h})_j, \sum_{j\geq k+1}(\frac{\partial^t J_{i\geq k+2}}{\partial {x_{s+1}}^t})_j\rangle$ should vary continuously with respect to J for every h, t. So the constant $C(J)$ defined on 2.1.1 is continuous with respect to J. Thus there is a uniform constant C such that $|\{z_{s+1} \in B : ||H^{z_{s+1}}_{i\geq k+1}||^2_{nd(\pi_{s+1},\pi_{m+1},...\pi_n)} < \epsilon\}| \leq C\epsilon^{\delta}$.

This means for any $\epsilon > 0$ except for a set of measure $\lesssim \epsilon^{\delta}$, the nondegeneracy norm of $H^{z_{s+1}}_{i\geq k+1} > \epsilon$ with respect to $\{\pi_{s+1}, \pi_{m+1}, ...\pi_n\}$ for nonzero z_{s+1}. By the induction hypotheses, the number of function is $n - m + 1 \leq k(m - 1) + 2$.

$$\begin{aligned}|I_{z_{s+1}}(F_{s+1}, F_{m+1}...F_n)| &\leq C(1 + |\lambda\epsilon|)^{-\delta'}||F^{z_{s+1}}_{s+1}||_{L^\infty} \prod_{j=m+1}^{n} ||F^{z_{s+1}}_j||_{L^\infty} \\ &\leq C(1 + |\lambda\epsilon|)^{-\delta'}||f_{s+1}||^2_{L^\infty} \prod_{j=m+1}^{n} ||f_j||^2_{L^\infty},\end{aligned}$$

for some $\delta' > 0$. Since $\eta(x)$ is compactly supported in some bounded set B. z_{s+1} is also bounded by some compact set B_{s+1}. If $\epsilon \sim |\lambda|^{-\tau}$ for some $\tau < 1$, this gives

$$\begin{aligned}LHS &\leq (\int_{A\cap B_{s+1}} |I_{y_{s+1}}|dy_{s+1})^{\frac{1}{2}} + (\int_{A^c\cap B_{s+1}} |I_{y_{s+1}}|dy_{s+1})^{\frac{1}{2}} \\ &\leq C(1 + |\lambda|)^{-\epsilon'/2} \prod_{i\in\{s+1,m+1,...n\}} ||f_i||_{L^\infty}\end{aligned}$$

for some ϵ'.

4.3 Third step: Nonzero lower bound

In this section we will prove C_1 is nonzero. That is, for any nonzero polynomial Q in L, we want to prove $\frac{\partial Q}{\partial x_{s+1}}$ is nonzero in H. If we write $Q = \sum_{k+2\leq i\leq r} Q_i$, where

Q_r is nonzero in L, $r \leq d$, then it suffices to show $\frac{\partial Q_r}{\partial x_{s+1}}$ is nonzero in H since Q_r is $Q's$ highest degree term. Without loss of generality, we write $Q_r = Q_r(\Sigma_{s+1})$.

If not, $\frac{\partial Q_r \circ \Sigma_{s+1}}{\partial x_{s+1}}$ can be written as $p_{s+1}(\pi_{s+1}(x_{s+1})) + p_{m+1}(\pi_{m+1}(x)) + ...p_n(\pi_n(x))$ for some polynomials $p_{s+1}, p_{m+1}, ...p_n$. Given the inner product structure and the coordinate we already choose, for every $m+1 \leq i \leq n$, since $V_i = \ker(\pi_i)$ is of dimension $m-1$, we can always find its orthogonal complement U_i which is spanned by one nonzero vector u_i. This u_i is unique up to a scale. $\pi_i(\mathbb{R}^m) = \pi_i(V_i) + \pi_i(U_i) = \pi_i(U_i)$. Notice by the general position condition, $\pi_i(0, ...0, x_{s+1}, 0, ...0)$ is not identically zero for $m+1 \leq i \leq n$. Since $\{(0, ...0, x_{s+1}, 0, ...0) | x_{s+1} \in \mathbb{R}\} = \cap_{t \in \{1,2...s,s+2,...m\}} \ker(\pi_t)$ and $\cap_{t \in \{1,2...s,s+2,...m,i\}} \ker(\pi_t) = 0$. This means if we write $u_i = a_1^i e_1 + ...a_m^i e_m$, then $a_{s+1}^i \neq 0$ for $m+1 \leq i \leq n$. So there exists some polynomials q_{s+1}, q_{m+1},...q_n, q such that $Q_r \circ \Sigma_{s+1}(x_1, ...x_{s+1}) = q_{s+1}(\pi_{s+1}(x_{s+1})) + q_{m+1}(\pi_{m+1}(x)) + ...q_n(\pi_n(x)) + q(x_1, x_2, ...x_s, x_{s+2}, ...x_m)$. For $q_{m+1}, ...q_n$, there are $n - m = (k-1)(m-1) + i + 1$ many polynomials, $1 \leq i \leq m-1$. For every fixed $1 \leq j \leq k-1$, $\dim(\cap_{j(m-1)+2 \leq i \leq j(m-1)+m} \ker(\pi_i)) = 1$. Notice $\cap_{j(m-1)+2 \leq i \leq j(m-1)+m} \ker(\pi_i) = (\text{span}_{j(m-1)+2 \leq i \leq j(m-1)+m}(u_i))^{\perp}$, so $\text{span}_{j(m-1)+2 \leq i \leq j(m-1)+m}(u_i)$ is a $m-1$ dimensional subspace. So there exists l_j which is a normal vector to this subspace.

Let $L_j = l_j \circ \nabla$. Then $L_j(u_i) = 0$ for $j(m-1)+2 \leq i \leq j(m-1)+m$. Thus $L_j q_{j(m-1)+t}(\pi_{j(m-1)+t}(x)) = L_j q_{j(m-1)+t}(\pi_{j(m-1)+t}(x_{j(m-1)+t})) = 0$ for $2 \leq t \leq m$, here $x_{j(m-1)+t}$ is the orthogonal projection of x to $U_{j(m-1)+t}$. Since $Q_r = Q_r \circ \Sigma_{s+1}(x_1, ...x_{s+1})$, $q_{s+1} = q_{s+1}(\pi_{s+1}(x_{s+1}))$ and $1 \leq s \leq m-2$, we have $\frac{\partial}{\partial x_m} Q_r \circ \Sigma_{s+1} =$

0, $\frac{\partial}{\partial x_m} q_{s+1}(\pi_{s+1}(x_{s+1})) = 0$ and $\frac{\partial}{\partial x_{s+1}} q(x_1, ...x_s, x_{s+2}, ...x_n) = 0$.

For the rest polynomials, notice $n = k(m-1) + 2 + i$, $1 \leq i \leq m-1$, there are $n - (k-1)(m-1) - m = 1 + i$ many of them. $\text{span}_{k(m-1)+2 \leq t \leq k(m-1)+1+i}(u_t)$ has dimension i. We can always find a normal vector l_k of this subspace but l_k is not orthogonal to u_n. Since if u_n is orthogonal to every normal vector of $\text{span}_{k(m-1)+2 \leq t \leq k(m-1)+1+i}(u_t)$, we have $u_n \in \text{span}_{k(m-1)+2 \leq t \leq k(m-1)+1+i}(u_t)$. But this contradicts the general position condition. Similarly, we have $L_k = l_k \circ \nabla$. $L_k q_{k(m-1)+t}(\pi_{k(m-1)+t}(x)) = 0$, $2 \leq t \leq 1 + i \leq m$, but

$$L_k q_{k(m-1)+i+2}(\pi_{k(m-1)+i+2}(x)) \neq 0$$

for any $q_{k(m-1)+i+1}(\pi_{k(m-1)+i+1}(x)$ of degree ≥ 1.

Consider $L = \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_{s+1}} \prod_{j=1}^{k} L_j$, then $Lq_n(\pi_n(x))$ must be 0. By the general position condition, u_n is not orthogonal to any l_j, $1 \leq j \leq k$ or e_{s+1}, e_m, which means degree of q_n is $\leq k+1$. Same argument shows $\deg(q_j) \leq k+1$, $m+1 \leq j \leq n$. Since $Q_r \circ \Sigma_{s+1}$ is a homogeneous polynomial with degree $r \geq k+2$, if we just see the highest degree part, we get

$$Q_r(\Sigma_{s+1}(x_1, ...x_{s+1})) = p_{s+1}(\pi_{s+1}(x_{s+1})) + q(x_1, ...x_s, x_{s+2}, ...x_m),$$

let $x_i = 0$ for $s + 2 \leq i \leq m$, we see

$$Q_r(\Sigma_{s+1}(x_1, ...x_{s+1})) = p_{s+1}(\pi_{s+1}(x_{s+1})) + q(x_1, ...x_s, 0, ..0),$$

$q(x_1, ...x_s, 0, ..0)$ is a polynomial defined on $K_s = \{(x_1, ...x_s, 0...0) | x_i \in \mathbb{R}, 1 \leq i \leq s\}$.

If we denote π'_s to be the orthogonal projection from $\mathbb{R}^m$ to K_s, then the nullspace of it is W_s. So there exists a polynomial q' such that $q'(\Sigma_s(x)) = q(x_1, ...x_s, 0, ..0)$, which contradicts the assumption that $Q_r \circ \Sigma_{s+1}$ is nondegenerate relative to $\{\pi_1, ...\pi_n, \Sigma_s\}$. So $\frac{\partial Q_r}{\partial x_{s+1}}$ must be nondegenerate with respect to $\{\pi_{s+1}, \pi_{m+1}, ...\pi_n\}$. If $||\sum_{i\geq k+2} P_i||_{nd(\pi_i,\Sigma_{s+1})} < \epsilon_{s+1}$, this gives C_1 is nonzero and finish the proof.

4.4 Third step: Phase function has large nondegenerate norm relative to helper mappings

If $||\sum_{i\geq k+2} P_i||_{nd(\pi_i,\Sigma_{s+1})} \geq \epsilon_{s+1}$, that is, P has relative large nondegenerate norm relative to $\{\pi_i, \Sigma_{s+1}\}$, we keep repeating the above argument until $s+1 = m-1$. Now the problem becomes dealing with the following integral

$$I_\lambda = \int e^{i\lambda P(x)} F_1(\Sigma_{m-1}(x)) f_m(\pi_m(x_m))...f_n(\pi_n(x))\eta(x)dx \tag{4.4.1}$$

with $||\sum_{i\geq k+2} P_i||_{nd(\pi_i, m\leq i\leq n, \Sigma_{m-1})} \geq \epsilon_{m-1}$.

Let $H^{z_m} = P(x_1, ...x_m) - P(x_1, ...x_{m-1}, x_m + z_m)$. Let r be the highest degree of P such that P_r is nondegenerate relative to $\{\pi_i, m \leq i \leq n, \Sigma_{m-1}\}$. As previous, it suffices to show $\frac{\partial P_r}{\partial x_m}$ is nondegenerate relative to $\{\pi_i, m \leq i \leq n\}$. If not, $\frac{\partial P_r}{\partial x_m} = \sum_{i=m}^n q_i(\pi_i(x))$ for some polynomials $q_i, m \leq i \leq n$. So

$$P_r(x) = \sum_{i=m}^n p_i(\pi_i(x)) + p_0(x_1, ...x_{m-1})$$

for some polynomials $p_0, p_i, m \le i \le n$, contradicting the assumption that P_r is nondegenerate relative to $\{\pi_i, m \le i \le n, \Sigma_{m-1}\}$. By the same argument as above, we know (4.4.1) has power decay property.

Chapter 5

Third result: Separation structure

In chapter 2, we introduce the concept of separation structure, that is, there exists subspaces $T_1, T_2 \subset \mathbb{R}^m$ such that $\mathbb{R}^m = T_1 \oplus T_2$ and $\{\pi_i\}_{i=1}^n$ preserves the direct sum decomposition, namely, $\pi_i(\mathbb{R}^m) = \pi_i(T_1) \oplus \pi_i(T_2)$. Denote $m_1 = \dim T_1$, $m_2 = \dim T_2$.

Under this condition, we say a polynomial $P(x, y)$ is of bidegree (i, j) if it has degree i when viewed as a polynomial of x and has degree j when viewed as a polynomial of y. We can write $P = \sum_i \sum_j P_{ij}(x_1, ...x_{m_1}, y_1, ...y_{m_2})$, where every P_{ij} is of bidegree (i, j). we call every P_{ij} as a homogeneous part of P with bidegree (i, j). In this case, we have a similar result as Lemma 2.2.1.

Lemma 5.0.1. *Assume $\{\pi_k\}_{k=1}^n$ preserve the direct sum decomposition. A polynomial P is nondegenerate relative to $\{\pi_k\}_{k=1}^n$ if and only if at least one of its homogeneous part is nondegenerate. That is, a homogeneous polynomial of bidegree* *(i,j) is degenerate if and only if it may be expressed as* $\sum_k p_k \circ \pi_k$ *where each* p_k *is a homogeneous polynomial of the same bidegree.*

Proof of lemma 5.0.1. If $P_{ij} = \sum_k p_{ijk} \circ \pi_k$ for every i, j, then P is obviously degenerate.

If there is a (i, j) such that $P_{ij} \neq \sum_k p_{ijk} \circ \pi_k$, for any polynomial p_{ijk} defined on V_k, we want to show P cannot be expressed as a sum of $P_k \circ \pi_k$, $1 \leq k \leq n$, for any polynomial P_k.

Consider the pairing of the vector space consisting of all polynomials of bidegree (i, j) and the constant coefficient differential operator of bidegree (i, j), that is, its symbol is a polynomial of bidegree (i, j). This pairing is nondegenerate. Thus the dual space of polynomials of bidegree (i, j) can be canonically identified with the vector spaces of all such differential operators.

Consider the space consisting of all degenerate polynomials of bidegree (i, j). It is a subspace of the space consisting of all polynomials of bidegree (i, j). So there exists a constant coefficient differential operator L_{ij} of bidegree (i, j) such that $L_{ij}(p_{ijk}(\pi_k)) = 0$ for all $1 \leq k \leq n$ and all polynomials p_{ijk} defined on V_k with bidegree (i, j) but $L_{ij}(P_{ij}) \neq 0$. Define a linear functional L as $L(p) = L_{ij}(p)(0)$ for any polynomial p of degree $\leq d$. Then $L(p_k(\pi_k)) = 0$, $1 \leq k \leq n$, for any polynomial p_k. Since $L(p_k(\pi_k)) = \sum_t \sum_s L(p_{tsk}(\pi_k))$ for homogeneous polynomial p_{tsk} of bidegree (t, s). If $(t, s) \neq (i, j)$, $L_{ij}(p_{tsk}(\pi_k))(0) = 0$ for any polynomial p_{tsk}. If $(t, s) = (i, j)$, by the definition of L_{ij}, $L_{ij}(p_{ijk}(\pi_k))(0) = 0$. But $L(P) \neq 0$, since

$L_{ij}(P_{ij}) \neq 0$, which means P is nondegenerate relative to $\{\pi_k\}$. $\square$

One example of oscillatory integrals that has the separation structure is the following:

$$\int_{\mathbb{R}^4} e^{i\lambda P(x_1,x_2,y_1,y_2)} f_1(x_1,y_1)f_2(x_2,y_2)f_3(x_1+x_2,y_1+y_2)f_4(x_1,y_1+y_2)$$
$$f_5(x_2,y_1+y_2)\eta(x,y)dxdy, \quad (5.0.1)$$

which can be viewed as a generalization of the example treated in [8]. Indeed, if we see the slice of x and y separately, we can find that these two integrals consist of the same "component" in each slice.

To be more precise, when restricted to x, we only have three linear mappings $(x_1,x_2) \to (x_1)$, $(x_1,x_2) \to (x_2)$ and $(x_1,x_2) \to (x_1+x_2)$. When restricted to y, we also only have three linear mappings, $(y_1,y_2) \to (y_1)$, $(y_1,y_2) \to (y_2)$, $(y_1,y_2) \to (y_1+y_2)$. This observation leads to the question that if there is a general approach to deal with the oscillatory integrals consisting of the same components when they are equipped with the separation structure. The following third result in this thesis is trying to answer the above question.

Theorem 5.0.2. *If $\{\pi_i, P\}$ satisfies the conditions below:*

1. *There exists T_1, T_2 such that $\mathbb{R}^m = T_1 \oplus T_2$ and $\{\pi_i\}$ preserve the direct sum decomposition.*

2. *$\{\ker(\pi_i) \cap T_1\}$ satisfies the transverse splitting condition in T_1.*

3. $\{\ker(\pi_i) \cap T_2\}$ *has uniform power decay property in* T_2*.*

4. P *is nondegenerate relative to* $\{\pi_i\}$ *with some nondegeneracy bidegree* (k_1, k_2)*,* $k_1 \geq 2$*.*

Then $\{\pi_i, P\}$ *has power decay property.*

Remark: in (5.0.1), if we define $T_1 = \{(x, 0)\}$, $T_2 = \{(0, y)\}$, then $\{\ker(\pi_i) \cap T_1\}$ is $\{(x_1, 0, 0), (0, x_2, 0), (x_1, -x_1, 0)\}$ and $\{\ker(\pi_i) \cap T_2\}$ is $\{(0, y_1, 0), (0, 0, y_2), (0, y_1, -y_1)\}$. Actually one can check they both satisfy the transverse splitting condition thus have the uniform power decay property, see 3.6.2. We just need to require P has nondegeneracy bidegree (i, j) where either i or j is greater than 1. If we already P is nondegenerate, this requirement actually only excludes the case that $i = j = 1$.

5.1 Proof of Theorem 5.0.2

Let $\{e_1, ...e_{m_1}\}$ be a basis of T_1 and $\{E_1, ...E_{m_2}\}$ be a basis of T_2. Any point $z \in \mathbb{R}^m$ can be represented by $(x_1, ...x_{m_1}, y_1, ...y_{m_2})$ where x denotes the coefficient of e and y denotes the coefficient of E. Since we have a basis of $\mathbb{R}^m$, we then have a corresponding coordinate. And $\pi_i(x, y) = \pi_i(x, 0) \oplus \pi_i(0, y)$.

Without loss of generality, we can reset the index as

$$I_\lambda = \int e^{i\lambda P(x,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))...$$

$$f_{t1}(\pi_{t1}(x,y))...f_{tk_t}(\pi_{tk_t}(x,y))\eta(x,y)dxdy \quad (5.1.1)$$

with $\sum_{j=1}^{t} k_j = n$, $1 \le t$. Here $\ker \pi_{ij} \cap T_1$ are identical for $1 \le j \le k_i$.

Denote $A_i = \ker \pi_{ij} \cap T_1$. Let Σ_i be any linear map with nullspace $A_i \oplus \{0\}$. There are two cases.

(a) P is nondegenerate relative to $\{\pi_{ij}, \Sigma_s\}$, $1 \le i, s \le t$, $1 \le j \le k_i$.

(b) P is degenerate relative to $\{\pi_{ij}, \Sigma_s\}$, $1 \le i, s \le t$, $1 \le j \le k_i$.

5.1.1 First step: Reduction to lower dimensions

For case (a), notice $\ker(\Sigma_i) \subset \ker(\pi_{ij})$. Denote

$$F_i(\Sigma_i(x,y)) = \prod_{j=1}^{k_i} f_{ij}(\pi_{ij}(x,y))$$

for $1 \le i \le t$. We just need to show $\{\Sigma_s, P\}$ has power decay property, that is,

$$|\int e^{i\lambda P(x,y)} F_1(\Sigma_1(x,y))...F_t(\Sigma_t(x,y))\eta(x,y)dxdy| \le C|\lambda|^{-\epsilon} \prod_{j=1}^{t} ||F_j||_{L^\infty} \quad (5.1.2)$$

holds for all $F_j \in L^\infty(\Sigma_j(\mathbb{R}^m))$. Since

$$\prod_{j=1}^{t} ||F_j||_{L^\infty} \le C \prod_{i=1}^{t} \prod_{j=1}^{k_i} ||f_{ij}||_{L^\infty}$$

for some constant C only depends on the support of η, if we can show (5.1.2) holds, then we have I_λ has the power decay property.

Notice $\ker(\Sigma_i) \cap T_2 = \{0\}$, $1 \le i \le t$, by lemma 2.4.2, it suffices to show $\{\ker(\Sigma_i) \cap T_1\}$ has uniform power decay property in T_1. By the definition of Σ_i, $\ker(\Sigma_i) \cap T_1 = \ker(\pi_{ij}) \cap T_1$, $1 \le j \le k_i$. By corollary 3.6.2 and condition (2), we know $\{\ker(\Sigma_i) \cap T_1\}$ has uniform power decay property in T_1, thus (5.1.1) holds.

5.1.2 Second step: Transverse splitting case

For case (b), since $\ker(\Sigma_i) \subset \ker(\pi_{ij})$, $1 \le i \le t$, $1 \le j \le k_i$, without loss of generality, we can write $P = P_1(\Sigma_1(x,y)) + ...P_t(\Sigma_t(x,y))$, where P_i, $0 \le i \le t$ are polynomials.

At least one of $P_i \circ \Sigma_i$ is nondegenerate relative to $\{\pi_{ij}\}$, $1 \le j \le k_i$ with some nondegeneracy bidegree (k_1, k_2), $k_1 \ge 2$. Otherwise P will be degenerate to $\{\pi_{ij}\}$, or with only nondegeneracy bidegree (k_1, k_2), $k_1 \le 1$. Without loss of generality, we can assume $P_1 \circ \Sigma_1$ is nondegenerate relative to $\{\pi_{1j}\}$, $1 \le j \le k_1$, with some nondegeneracy bidegree (k_1, k_2), $k_1 \ge 2$. Notice η is compactly supported, so any slice of the support set is also compact. We can pick $\eta_i(\Sigma_i(x,y))$ to be the bump function which equals 1 on the slice of the support of η and vanishes outside of a neighborhood of that slice. Then $\eta(x,y)$ can be written as $\prod_{i=1}^{t} \eta_i(\Sigma_i(x,y))\eta(x,y)$, where $\eta_i(\Sigma_i(x,y))$ is a smooth function with compact support whose L^∞ norm is bounded by an absolute constant. The integral (5.1.1) becomes

$$\begin{aligned} &\int \tilde{F}_1(\Sigma_1(x,y))...\tilde{F}_t(\Sigma_t(x,y))\eta(x,y)dxdy \\ =& \int (\int \tilde{F_1}^y(\Sigma_1(x,0))...\tilde{F_t}^y(\Sigma_t(x,0))\eta^y(x)dx)dy, \end{aligned} \tag{5.1.3}$$

where

$$\tilde{F}_i^y(\Sigma_i(x,0)) = \tilde{F}_i(\Sigma_i(x,y)) = e^{iP_i(\Sigma_i(x,y))}\prod_{j=1}^{k_i} f_{ij}(\pi_{ij}(x,y))\eta_i(\Sigma_i(x,y))$$

for $1 \le i \le t$.

By condition (2), $\{\ker(\Sigma_i)\cap T_1\}$ has the transverse splitting condition in T_1. That is, if we denote $I = \{1, 2, ...t\}$ and $\dim(\ker(\Sigma_a) \cap T_1) = \min_{i\in I} \dim(\ker(\Sigma_i) \cap T_1)$, then there is a partition I_1, I_2 of $I\backslash\{a\}$ such that

$$(m_1 - \dim(V_a)) + \sum_{j\in I_1}(m_1 - \dim(V_j)) \le m_1,$$

$$(m_1 - \dim(V_a)) + \sum_{j\in I_2}(m_1 - \dim(V_j)) \le m_1.$$

Here $V_j = \ker(\Sigma_j) \cap T_1$.

If $a \ne 1$, we can assume $1 \in I_1$. Since $\dim(\ker(\Sigma_a) \cap T_1) = \min_{i\in I} \dim(\ker(\Sigma_i) \cap T_1)$, if $I_1' = \{a\} \cup I_1\backslash\{1\}$, then I_1', I_2 is a partition of $I\backslash\{1\}$ such that

$$(m_1 - \dim(V_1)) + \sum_{j\in I_1'}(m_1 - \dim(V_j)) \le m_1,$$

$$(m_1 - \dim(V_1)) + \sum_{j\in I_2}(m_1 - \dim(V_j)) \le m_1$$

still holds. Without loss of generality, we denote $I_1' = \{2, ...s\}$, $I_2 = \{s + 1, ...t\}$.

Let $U_1 = \ker\Sigma_2 \cap ... \cap \ker\Sigma_s$ and $U_2 = \ker\Sigma_{s+1} \cap ... \cap \ker\Sigma_t$. Let l_1 be any linear map defined on $\mathbb{R}^{m_1}$ with nullspace U_1 and l_2 be any linear map in $\mathbb{R}^{m_1}$ with nullspace U_2. Let $G_1(l_1(x,y)) = \prod_{i=2}^{s} \tilde{F}_i(\Sigma_i(x,y))$, $G_2(l_2(x,y)) = \prod_{i=s+1}^{t} \tilde{F}_i(\Sigma_i(x,y))$.

Using the same argument as in lemma 3.6.2, denote $\tilde{F}_1^y(\Sigma_1(x,0)) = \tilde{F}_1(\Sigma_1(x,y))$,

${G_1}^y(l_1(x,0)) = G_1(l_1(x,y))$, $G_2^y(l_2(x,0)) = G_2(l_2(x,y))$. Then we have

$$\begin{aligned}(5.1.3) = &\int (\int {\tilde{F}_1}^y(\Sigma_1(x,0)) {G_1}^y(l_1(x,0)) {G_2}^y(l_2(x,0)) \eta^y(x) dx) dy \\ \leq C &\int ||\hat{\hat{\tilde{F}}}{}_1^y||_{L^\infty} ||\hat{G}_1^y||_{L^2} ||\hat{G}_2^y||_{L^2} dy\end{aligned} \tag{5.1.4}$$

for some constant C only depends on η. We should notice that since η is compactly supported, we can always assume G_i is compactly supported for for $i = 1, 2$. For any fixed y,

$$\begin{aligned}||\hat{G}_1^y||_{L^2} = &||G_1^y||_{L^2} \\ = &\int |G_1(l(x,y))|^2 dx \\ \leq &C \prod_{i=2}^{s} ||\tilde{F}_i(\Sigma_i(x,y))||_{L^\infty} \leq C \prod_{i=2}^{s} \prod_{j=1}^{k_i} ||f_{ij}||_{L^\infty}.\end{aligned}$$

Apply the same argument to G_2, we have

$$(5.1.4) \leq C \int ||\hat{\hat{\tilde{F}}}{}_1^y||_{L^\infty} dy \cdot \prod_{i=2}^{t} \prod_{j=1}^{k_i} ||f_{ij}||_{L^\infty}.$$

In the next section, we will show $\int ||\hat{\hat{\tilde{F}}}{}_1^y||_{L^\infty} dy \leq c|\lambda|^{-\delta} \prod_{j=1}^{k_1} ||f_{1j}||_{L^\infty}$ for some $\delta > 0$.

5.1.3 Third step: Phase function with a special form

For any $\epsilon > 0$, by the definition of L^∞, we can pick a $\theta(\lambda, y, \epsilon)$ such that

$$\int ||\hat{\hat{\tilde{F}}}{}_1^y||_{L^\infty} dy \leq \int |\int e^{iP_1(\Sigma_1(x,y))} \prod_{j=1}^{k_1} f_{1j}(\pi_{1j}(x,y)) e^{i\Sigma_1(x,0)\cdot\theta(\lambda,y,\epsilon)}$$

$$\eta_1(\Sigma_1(x,y)) d\Sigma_1(x,0)| dy + C\epsilon. \tag{5.1.5}$$

Notice the oscillatory integral on the right hand side has a phase function with a special form. Though we do not have any information of θ, when fixed y, the phase function is still a polynomial of x.

Denote $m_1 - \dim(\ker(\Sigma_1) \cap T_1) = s$. Without loss of generality, we can assume $\ker(\Sigma_1) \cap T_1 = \{(0,..0, x_{s+1}, ...x_{m_1}, 0..0) | x_i \in \mathbb{R}, s+1 \le i \le m_1\}$. Since $\ker(\Sigma_1) \cap T_1 = \ker \pi_{1j} \cap T_1$ for $1 \le j \le k_1$, up to a linear transformation, we can identify $\Sigma_1(x,y)$ with the projection $(x_1, ...x_{m_1}, y) \to (x_1, ...x_s, y)$, $\pi_{1j}(x,0) = (x_1, ...x_s, 0)$ for $1 \le j \le k_1$. So we deal with

$$
\begin{aligned}
(5.1.5) = \int | \int & e^{i\lambda P_1(x_1,...x_s,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))\eta_1(x,y) \\
& e^{i\sum_{1\le j\le s} x_j \cdot \theta_j(\lambda,y,\epsilon)} dx_1...dx_s | dy \\
= \int & e^{i\lambda P_1(x_1,...x_s,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))\eta_1(x,y) \\
& e^{i\sum_{1\le j\le s} x_j \cdot \theta_j(\lambda,y,\epsilon)} g(y) dx_1...dx_s dy. \quad (5.1.6)
\end{aligned}
$$

Here $||g||_{L^\infty} = 1$. Since η_1 is compactly supported, we can assume g is compactly supported. Denote P_{1,k_1k_2} as the homogeneous part of P_1 with bidegree (k_1, k_2). Notice $e^{i\lambda \sum_{k_2} \sum_{k_1 \le 1} P_{1,k_1k_2}(x,y)}$ can be decomposed into $e^{i\sum_{j=1}^{s} x_j \cdot \theta_j(\lambda,y,\epsilon)} g(y)$. So without loss of generality, we can assume $P_1 = \sum_{k_2} \sum_{k_1 \ge 2} P_{1,k_1k_2}$.

We now consider the case where $s \ge 2$. If $s = 1$, one can jump to the fourth step. Let L_r be the projection from $(x_r, ...x_s, y)$ to $(x_{r+1}, ...x_s, y)$, $1 \le r \le s-1$. Every time we'll check if P_1 is nondegenerate relative to $\{L_r, \pi_{1j}\}$, $1 \le j \le k_1$. We'll show $r = 1$ as an example. There are two cases:

(I) $\sum_{k_2}\sum_{k_1\geq 2} P_1(x_1,...x_s,y)$ is nondegenerate with respect to

$\{L_1,\pi_{1j}\}$, $1\leq j\leq k_1$.

(II) $\sum_{k_2}\sum_{k_1\geq 2} P_1(x_1,...x_s,y)$ is degenerate with respect to $\{L_1,\pi_{1j}\}$, $1\leq j\leq k_1$.

For case (I), we just need to show

$$|I_\lambda| = |\int e^{i\lambda P_1(x_1,...x_s,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))\eta_1(x,y)e^{ix_1\cdot\theta_1(\lambda,y,\epsilon)}$$
$$G(x_2...x_s,y)dx_1...dx_s dy|$$
$$\leq C|\lambda|^{-\epsilon}\prod_{i=1}^{k_1}||f_{1i}||_{L^\infty}||G||_{L^\infty}$$

holds true for any function f_{1i}, G such that $||f_{1j}||_{L^\infty}\leq 1$ and $||G||_{L^\infty}\leq 1$. Notice

$$|I_\lambda| \leq \Big(\int|\int e^{i\lambda P_1(x_1,...x_s,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))\eta_1(x,y)$$
$$e^{ix_1\cdot\theta_1(\lambda,y,\epsilon)}dx_1|^2 dx_2...dx_s dy\Big)^{\frac{1}{2}}\cdot||G||_{L^2}.$$

We just need to deal with the following

$$\int|\int e^{i\lambda P_1(x_1,...x_s,y)} f_{11}(\pi_{11}(x,y))...f_{1k_1}(\pi_{1k_1}(x,y))\eta_1(x,y)$$
$$e^{ix_1\cdot\theta_1(\lambda,y,\epsilon)}dx_1|^2 dx_2...dx_s dy$$
$$=\int\Big(\int e^{i\lambda P_1^z(x,y)}\prod_{j=1}^{k_1} f_{1j}^z(\pi_{1j}(x,y))\eta_1^z(x,y)r^z(y)dxdy\Big)dz.$$

where $P_1^z(x,y) = P_1(x,y) - P_1(x_1+z,x_2,...x_s,y)$,

$f_{1j}^z(\pi_{1j}(x,y)) = f_{1j}(\pi_{1j}(x,y))\bar{f_{1j}}(\pi_{1j}(x_1+z,x_2,...x_s,y))$, for $1\leq j\leq k_1$, $\eta_1^z(x,y) = \eta_1(x,y)\bar{\eta}_1(x_1+z,x_2,...x_s,y)$, $r^z(y) = e^{-i\lambda z\cdot\theta_1(\lambda,y,\epsilon)}$.

We will use the notation Λ_2 as the projection from $\mathbb{R}^{s+m_2} \to \mathbb{R}^{m_2} : (x_1, ...x_s, y) \to (y)$. We want to show $||P_1^z(x,y)||_{nd(\pi_{1j},\Lambda_2,1\le j\le k_1)} \ge C|\lambda|^{-\rho'}$ as a function of z except for a set of measure $C|\lambda|^{-\rho''}$. By the same argument as in theorem 2.3.1 and the assumption that P is nondegenerate relative to $\{x_2, ...x_s, y\}$, it suffices to show $\frac{\partial P_1(x,y)}{\partial x_1}$ is nondegenerate relative to $\{\pi_{1j}, \Lambda_2\}$, $1 \le j \le k_1$. To prove this, assume it is degenerate, then there exists polynomials q_i, $0 \le i \le k_1$ such that $\frac{\partial P_1(x,y)}{\partial x_1} = q_0(y) + \sum_{i=1}^{k_1} q_i(\pi_{1i}(x,y))$. Then $P_1(x,y) = Q_0(x_2...x_s, y) + x_1 \cdot q_0(y) + \sum_{i=1}^{k_1} Q_i(\pi_{1i}(x,y))$, for some polynomials Q_i, $0 \le i \le k_1$, which contradicts the assumption that P_1 is nondegenerate relative to $\{\pi_{1j}, L_1\}$, $1 \le j \le k_1$ with some nondegneracy bidegree (k_1, k_2), $k_1 \ge 2$.

So now it suffices to show $\{\pi_{1j}, \Lambda_2\}$, $1 \le j \le k_1$ has uniform power decay. For fixed z, there are two cases.

1. $r^z(y)$ splits into $e^{iR(y)} \prod_{j=1}^{k_1} \pi_{1j}(0,y)$ given $|\lambda|^{-\sigma}$.

2. $r^z(y)$ does not split into $e^{iR(y)} \prod_{j=1}^{k_1} \pi_{1j}(0,y)$ given $|\lambda|^{-\sigma}$.

For case 1, since $ker(\pi_{1j}) \subset ker(\pi_{1j} \circ \Lambda_2)$, any polynomial defined on $\pi_{1j} \circ \Lambda_2(\mathbb{R}^m)$ can be "absorbed" into some polynomial defined on $\pi_{1j}(\mathbb{R}^m)$. By the definition of nondegeneracy norm, we have

$$||P_1^z(x,y) - R(y)||_{nd(\pi_{1j},1\le j\le k_1)} \ge ||P_1^z(x,y)||_{nd(\Lambda_2,\pi_{1j},1\le j\le k_1)}.$$

Thus the problem is reduced to show $\{\pi_{1j}\}_{j=1}^{k_1}$ has uniform power decay, which is given by condition 3.

If it is case 2, we have

$$
\begin{aligned}
&|\int e^{i\lambda P_1^z(x,y)}\prod_{j=1}^{k_1} f_{1j}^z(\pi_{1j}(x,y))\eta_1^z(x,y)r^z(y)dxdy| \\
\leq &\int|\int e^{i\lambda P_1^{x,z}(y)}\prod_{j=1}^{k_1} f_{1j}^{x,z}(\pi_{1j}(0,y))\eta_1^{x,z}(y)r^z(y)dy|dx \\
\leq &C|\lambda|^{-\sigma}\prod_{i=1}^{k_1}||f_{1i}||_{L^\infty}.
\end{aligned}
$$

The last inequality is proved by the assumption that $r^z(y)$ does not split into $e^{iR(y)}\prod_{j=1}^{k_1}\pi_{1j}(0,y)$ given $|\lambda|^{-\sigma}$.

So whether $r^z(y)$ splits, $\{\pi_{1j},\Lambda_2\}$, $1\leq j\leq k_1$ always has uniform power decay property.

5.1.4 Fourth step: Reduction to the codimension one case

We now consider case (II), that is, $\sum_{k_2}\sum_{k_1\geq 2}P_1(x_1,...x_s,y)$ is degenerate with respect to $\{L_1,\pi_{1j}\}$, $1\leq j\leq k_1$. Without loss of generality, we can assume

$$\sum_{k_2}\sum_{k_1\geq 2}P_1(x_1,...x_s,y)=\sum_{k_2}\sum_{k_1\geq 2}P_1(x_2,...x_s,y)$$

$$
\begin{aligned}
(5.1.6)=\int\Big(\int e^{i\lambda\sum_{k_2}\sum_{k_1\geq 2}P_1(x_2,...x_s,y)}\prod_{i=1}^{k_1}f_{1i}^{x_1}((x_2,...x_s,0)+\pi_{1i}(0,y))\\
\eta_1^{x_1}(x_2...x_s,y)e^{i\sum_{2\leq j\leq s}x_j\cdot\theta_j(\lambda,y,\epsilon)}g^{x_1}(y)dx_2...dx_sdy\Big)dx_1,
\end{aligned}
$$

where $f_{1i}^{x_1}(x_2,...x_s,y)=f_{1i}(\pi_{1i}(x,0)+\pi_{1i}(0,y))=f_{1i}((x_1,...x_s,0)+\pi_{1i}(0,y))$, $1\leq i\leq k_1$, $g^{x_1}(y)=g(y)e^{ix_1\cdot\theta_1(\lambda,y,\epsilon)}$, $\eta_1^{x_1}(x_2...x_s,y)=\eta_1(x,y)$.

We will prove $\sum_{k_2}\sum_{k_1\geq 2} P_{1,k_1k_2}(x_2,...x_s,y)$ must be nondegenerate relative to $\{L_1\circ\pi_{1j}\}$, $1\leq j\leq k_1$, with some nondegeneracy bidegree (k_1,k_2), $k_1\geq 2$. Since

$$\ker(L_1\circ\pi_{1j})\cap T_1=\{(x_1,0..0,x_{s+1},...x_{m_1},0..0)|x_i\in\mathbb{R}, s+1\leq i\leq m_1\},$$

$$\ker(\pi_{1j})\cap T_2=\ker(L_1\circ\pi_{1j})\cap T_2,$$

we have $\ker(\pi_{1j})\subset\ker(L_1\circ\pi_{1j})$, so any polynomial in $L_1\circ\pi_{1j}$ can also be viewed as a polynomial in π_{1j}. So if $\sum_{k_2}\sum_{k_1\geq 2} P_{1,k_1k_2}(x_2,...x_s,y)$ is degenerate relative to $\{L_1\circ\pi_{1j}\}$, $1\leq j\leq k_1$, then P_1 must be degenerate relative to $\{\pi_{1j}\}$, $1\leq j\leq k_1$ or nondegenerate relative to $\{\pi_{1j}\}$ but with nondedegeneracy bidegree (k_1,k_2), $k_1\leq 1$, contradicting the assumption of P_1. So $\sum_{k_2}\sum_{k_1\geq 2} P_{1,k_1k_2}(x_2,...x_s,y)$ must be nondegenerate relative to $\{L_1\circ\pi_{1j}\}$, $1\leq j\leq k_1$, with some nondegeneracy bidegree (k_1,k_2), $k_1\geq 2$.

If we can show the following

$$\begin{aligned}|I_\lambda|=&|\int e^{i\lambda\sum_{k_2}\sum_{k_1\geq 2}P_{1,k_1k_2}(x_2...x_s,y)}\prod_{i=1}^{k_1} f_{1i}^{x_1}((0,x_2,...x_s,0)+\pi_{1i}(0,y))\\ &\eta_1^{x_1}(x_2...x_s,y)e^{i\sum_{2\leq j\leq s}x_j\cdot\theta_j(\lambda,y,\epsilon)}g^{x_1}(y)dx_2...dx_s dy|\\ \leq& C|\lambda|^{-\delta'}\prod_{i=1}^{k_1}||f_{1i}^{x_1}||_{L^\infty}||g^{x_1}||_{L^\infty}\end{aligned}$$

holds for some uniform $\delta'>0$ for any fixed x_1 with the constant independent of x_1, then (5.1.6) has power decay property.

Keep repeating the above argument, that is, checking the following conditions for every $1\leq r\leq s-1$. Here $L_r:(x_r,...x_s,y)\to(x_{r+1},...x_s,y)$.

(I) $\sum_{k_2}\sum_{k_1\geq 2} P_1(x_r, ...x_s, y)$ is nondegenerate with respect to

$\{L_r, \pi_{1j}\}$, $1 \leq j \leq k_1$.

(II) $\sum_{k_2}\sum_{k_1\geq 2} P_1(x_r, ...x_s, y)$ is degenerate with respect to $\{L_r, \pi_{1j}\}$, $1 \leq j \leq k_1$.

If it is case (I), we repeat the argument in the second and third step. If it is case (II), the problem is reduced to checking if $\{L_r \circ \pi_{ij}\}$ has uniform power decay property in the ambient space $\mathbb{R}^{s-r+m_2}$.

We end up with showing

$$
\begin{aligned}
&|\int\int e^{i\lambda \sum_{k_2}\sum_{k_1\geq 2} P_{1,k_1k_2}(x_s,y)} \prod_{i=1}^{k_1} f_{1i}^{\tilde{x}}((0,...0,x_s,0)+\pi_{1i}(0,y)) \\
&\eta_1^{\tilde{x}}(x_s,y) e^{ix_s\cdot\theta_s(\lambda,y,\epsilon)} g^{\tilde{x}}(y)dx_s dy| \qquad (5.1.7)\\
\leq& C|\lambda|^{-\delta}\prod_{i=1}^{k_1}||f_{1i}||_{L^\infty}\cdot||g^{\tilde{x}}||_{L^\infty}
\end{aligned}
$$

Here $\tilde{x} = (x_1, ...x_{s-1})$, $\eta_1^{\tilde{x}} = \eta_1(x_1, ...x_{s-1}, x_s, y)$,

$$f_{1i}^{\tilde{x}} = f_{1i}(\pi_{1i}(x_1, ...x_{s-1}, x_s, y)),$$

$$g^{\tilde{x}}(y) = g(y)e^{i\sum_{j=1}^{s-1} x_j\cdot\theta_j(\lambda,y,\epsilon)}$$

with fixed $x_1, ...x_{s-1}$.

If we denote

$$h^{\tilde{x}}(y) = \int e^{i\lambda P_1(x_s,y)}\prod_{i=1}^{k_1} f_{1i}^{\tilde{x}}((0,...0,x_s,0)+\pi_{1i}(0,y))e^{x_s\cdot\theta_s}\eta_1^s(x,y)dx_s,$$

by the same argument, we have

$$(5.1.7) \leq C\int(\int e^{i\lambda P_1^z(x_s,y)}\prod_{j=1}^{k_1} f_{1j}^{\tilde{x},z}(\pi_{1j}(x_s,y))\eta_1^{\tilde{x},z}(x_s,y)r^z(y)dx_s dy)dz,$$

where $P_1^z(x_s, y) = P_1(x_s, y) - P_1(x_s + z, y)$,

$$f_{1j}^{\tilde{x},z}(\pi_{1j}(x_s, y)) = f_{1j}^{\tilde{x}}(\pi_{1j}(x_s, y)) \cdot \bar{f}_{1j}^{\tilde{x}}(\pi_{1j}(x_s + z, y)),$$

for $1 \leq j \leq k_1$, $\eta_1^{\tilde{x},z}(x_s, y) = \eta_1^{\tilde{x}}(x_s, y)\bar{\eta}_1^{\tilde{x}}(x_s + z, y)$, $r^z(y) = e^{-i\lambda z \cdot \theta_s(\lambda, y, \epsilon)}$.

Notice now the ambient space is $\mathbb{R}^{m_2+1} = \{(x_s, y)|x_s \in \mathbb{R}, y \in \mathbb{R}^{m_2}\}$. We will still use the notation Λ_2 as the projection from $\mathbb{R}^{m_2+1} \to \mathbb{R}^{m_2} : (x_s, y) \to (y)$. By the same argument, to prove P_1^z has nondegeneracy norm greater than or equal to $|\lambda|^{\delta'}$ for some $\delta' > 0$, except some possible subset of z with small measure, we only need to show $\frac{\partial P_1(x_s,y)}{\partial x_s}$ is nondegenerate relative to $\{\pi_{1j}, \Lambda_2\}$, $1 \leq j \leq k_1$. If it is degenerate, we can check P_1 is nondegenerate relative to $\{\pi_{1j}\}$, $1 \leq j \leq k_1$ with only nondegneracy bidegree (k_1, k_2), $k_1 \leq 1$.

Now it is only left to show $\{\Lambda_2, L_{s-1} \circ \pi_{1j}\}$ has the uniform power decay property. Notice the range space of Λ_2 is a codimension 1 subspace of the ambient space. By lemma 2.3.1 and 2.4.2, $\{\ker(\pi_{1j}) \cap T_2\}$ has uniform power decay property can imply

$$\int\int e^{i\lambda P_1^z(x_s,y)} \prod_{j=1}^{k_1} f_{1j}^{\tilde{x},z}(\pi_{1j}(x_s, y))\eta_1^{\tilde{x},z}(x_s, y)r^z(y)dx_s dy$$

has uniform power decay for every fixed z, which is proved by condition (3).

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