2019

On The Compatibility Of Derived Structures On Critical Loci

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Abstract

We study the problem of compatibility of derived structures on a scheme which can be presented as a critical locus in more than one way. We consider the situation when a scheme can be presented as the critical locus of a function $w \in ?(S)$ and as the critical locus of the restriction $w|_X \in ?(X)$ for some smooth subscheme $X \subset S$. In the case when S is the total space of a vector bundle over X, we prove that, under natural assumptions, the two derived structures coincide. We generalize the result to the case when X is a quantized cycle in S and also give indications how to proceed when $X \subset S$ is a general closed embedding.

Degree Type

Dissertation

Degree Name

Doctor of Philosophy (PhD)

Graduate Group

Mathematics

First Advisor

Tony Pantev

Keywords

Batalin–Vilkovisky formalism, compatibility of derived structures, derived critical locus, shifted symplectic structure

Subject Categories

Mathematics

ON THE COMPATIBILITY OF DERIVED STRUCTURES ON CRITICAL LOCI

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A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2019

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Acknowledgments

First and foremost, I am most indebted and deeply grateful to my advisor Prof. Tony Pantev. So many times I had to resort both to the vast extent of his knowledge as well as to his abiding patience. In the end, I have learned much more from him than mere mathematics and for this I can only feel strangely fortunate, undeservingly so.

I would also like to thank department staff Mmes. Burns, Pallanti, Scarborough, and Tanna, who always managed to keep the bureaucracy at bay. How they do it is beyond me. I also cherish the company and instruction of my colleagues and friends in the department, especially Xiaofei, Joshua, Yixuan, Tong, and Márton. I'm also grateful for the guidance of my previous mentors, Prof. Kraljević and Prof. Škoda.

I wish to thank the Bock family for providing a haven I could call home, my old friends Zvonimir, Slaven, Marina, and Frano who inspire me to be beyond who I am today, and my family for all the necessary, and at times unappreciated, sacrifices.

I thank God, without whom none of this would have any meaning, especially for the people I am blessed with, the ones mentioned, as well as many others who fulfill my life.

ABSTRACT

ON THE COMPATIBILITY OF DERIVED STRUCTURES ON CRITICAL LOCI

Antonijo Mrcela

Tony Pantev

We study the problem of compatibility of derived structures on a scheme which can be presented as a critical locus in more than one way. We consider the situation when a scheme can be presented as the critical locus of a function $w \in \mathcal{O}(S)$ and as the critical locus of the restriction $w|_X \in \mathcal{O}(X)$ for some smooth subscheme $X \subset S$. In the case when S is the total space of a vector bundle over X, we prove that, under natural assumptions, the two derived structures coincide. We generalize the result to the case when X is a quantized cycle in S and also give indications how to proceed when $X \subset S$ is a general closed embedding.

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Chapter 1

Introduction

Field theories, both classical and quantum, provide an interesting and important motivation for derived geometry (*cf.* [9]). Here we briefly recall the main notions of field theories. For a more extensive treatment see [17].

A classical field theory on a space X is prescribed by the following two pieces of data:

- a space of fields;
- an action functional $\mathcal{S}$, which to every field φ assigns a number

$$\mathcal{S}[\varphi] = \int_X \mathcal{L}(\varphi, \partial\varphi, \partial^2\varphi, \ldots)$$

The functional $\mathcal{S}$ is determined by the density $\mathcal{L}$ on X, called Lagrangian density, which depends on the value of the field φ and its derivatives at the integration points $x \in X$. The space of fields is generally a space of sections of some sheaf over X. This can be $\mathcal{O}(X)$ in the case of scalar field theories, or the mapping space

$\mathrm{Map}(X, T)$ for some fixed target T in the case of σ-models. The space of fields characteristic of gauge theories (such as Chern–Simons or Yang–Mills theory) is the space of connections on a principal G-bundle over X. The problem of determining the stationary points of the action functional $\mathcal{S}$ leads to Euler–Lagrange equations.

Let G be a simple compact Lie group and X a closed oriented 3-manifold. Assume that $P \to X$ is the trivial principal G-bundle over X. Note that if G happens to be connected and simply connected, then every G-bundle on X is trivial. This is a consequence of the fact that $\pi_2(G)$ is trivial for any Lie group (see [15]), and therefore BG is 3-connected, which makes the space of classifying maps $[X, BG]$ homotopically equivalent to a point, whenever X is a real manifold of dimension at most 3. A typical choice for the group G is $SU(2)$ and this choice already exhibits all difficulties and essential features of the general theory.

Since P is trivial, we can use the trivializing section of P to pull back every principal connection from $\Omega^1(P, \mathfrak{g})$ to $\Omega^1(X, \mathfrak{g})$. Hence, the space of fields becomes $\Omega^1(X, \mathfrak{g})$. The action functional for the classical Chern–Simons theory is

$$\mathcal{S}(A) := \int_X \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right)$$

for $A \in \Omega^1(X, \mathfrak{g})$. The variation of this action is

$$\delta\mathcal{S}(A) = \int_X \mathrm{tr}\big(\delta A \wedge (dA + A \wedge A)\big)$$

and the Euler–Lagrange equation becomes $dA + A \wedge A = 0$, which is simply the flatness condition on the connection $d + A$.

Given a map $g\colon X \to G$, we can transform a connection $A \in \Omega^1(X, \mathfrak{g})$ by the gauge transfomation rule:

$$A \mapsto A^g := g^{-1}Ag + g^{-1}dg$$

In this way we obtain a right action of the gauge group $\mathrm{Map}(X, G)$ on the space of fields. This action can be interpreted as describing what happens with the fields when we change the chosen trivialization of the bundle $P \to X$. It can also be interpreted as the action on fields coming from the G-bundle automorphism $P \to P$ induced by g.

The Chern–Simons action is almost independent under the action of the gauge group. More precisely, we have

$$\frac{1}{4\pi^2}\big(\mathcal{S}(A^g) - \mathcal{S}(A)\big) = \int_X g^*\theta \ \in \mathbb{Z}$$

where θ is the Cartan 3-form on G. This is a closed, G-invariant form, found on any simple compact Lie group G, whose integral periods represent the generators of $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. In the case of $G = SU(2)$, the form θ is the normalized volume form on $SU(2)$ viewed as the 3-sphere. In this case the difference $\mathcal{S}(A^g) - \mathcal{S}(A)$ is just the multiple of the degree of the map $g\colon X \to S^3$. We see that the Chern–Simons functional is invariant under the action of the identity component of $\mathrm{Map}(X, G)$, hence it is invariant under infinitesimal gauge transformations.

The quantization procedure corresponds to the construction of the partition function for X, which is heuristically supposed to be of the form

$$Z := \int_{\{\text{all fields } \varphi\}} e^{\frac{i}{\hbar}\mathcal{S}(\varphi)}$$

Then, for any observable of the system, that is, a function f on the space of fields, we could compute its expectation value as

$$\langle f \rangle = \frac{1}{Z} \int_{\{\text{all fields } \varphi\}} f(\varphi) e^{\frac{i}{\hbar}\mathcal{S}(\varphi)}$$

These integrals are difficult to define directly as integrals coming from some kind of measure on the space of fields. However, they can be defined as an asymptotic series depending on $\hbar$, when $\hbar \to 0$. To illustrate how this could be done, we take a look at oscillating integrals in finite dimensions. If X is a compact manifold with a fixed volume form, and w a smooth function on X, then the asymptotics of the integral $\int_X e^{\frac{i}{\hbar}w(x)}$ as $\hbar \to 0$ is described by the stationary phase formula:

$$\sum_{x \in \mathrm{Crit}(w)} e^{\frac{i}{\hbar}w(x)} |\det \mathrm{Hess}_w(x)|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4} \operatorname{sign} \mathrm{Hess}_w(x)} \cdot (2\pi\hbar)^{\frac{\dim X}{2}} + o\Big(\hbar^{\frac{\dim X}{2}}\Big)$$

This formula can be improved to include higher order terms. For each critical point $x \in \mathrm{Crit}(w)$ one needs to multiply the approximation

$$e^{\frac{i}{\hbar}w(x)} |\det \mathrm{Hess}_w(x)|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4} \operatorname{sign} \mathrm{Hess}_w(x)} \cdot (2\pi\hbar)^{\frac{\dim X}{2}}$$

by the sum of the expressions of the form $\hbar^{-\chi(\Gamma)}\Phi_\Gamma(x)$, where the summation is over all graphs Γ (called the Feynman diagrams) with vertices of degree 3 or bigger, including disconnected graphs. The weight $\Phi_\Gamma(x)$ depends on the partial derivatives of w at the critical point x and the order of the group $\mathrm{Aut}(\Gamma)$. The number $\chi(\Gamma) \leq 0$ is the Euler characteristic of the graph.

We see that the stationary phase formula replaces the integral $\int_X e^{\frac{i}{\hbar}w(x)}$ as $\hbar \to 0$ with an algebraic expression which involves only the values of derivatives of w at the critical points. We would like to apply the analogous expansion to define the path integral $\int_{\{\text{all fields } \varphi\}} e^{\frac{i}{\hbar}\mathcal{S}(\varphi)}$ as $\hbar \to 0$. The problem is that the stationary phase formula requires the Hessian of w at the points $x \in \mathrm{Crit}(w)$ to be non-degenerate, which forces the critical points of w to be isolated. However, in our case, the gauge invariance of $\mathcal{S}$ makes the critical locus of $\mathcal{S}$ a union of orbits of the gauge group, so the critical points are far from being isolated. A remedy for this problem lies in the Batalin–Vilkovisky construction, first described in [3] and [4].

In the Batalin–Vilkovisky formalism, the space of BV fields $\mathcal{F}$ has the structure of a $\mathbb{Z}$-graded manifold equipped with a symplectic structure ω of internal degree -1. Additionally, there is a function $\mathcal{S}$ on the space of BV fields, called the master action, which satisfies the master equation $\{\mathcal{S}, \mathcal{S}\} = 0$. In this framework, the integral $\int_{\{\text{all fields } \varphi\}} e^{\frac{i}{\hbar}\mathcal{S}(\varphi)}$ gets replaced by

$$\int_{\{\varphi \in \mathcal{L}\}} e^{\frac{i}{\hbar}\mathcal{S}(\varphi)}$$

which is well defined via a stationary phase formula for an appropriate choice of a Lagrangian (with respect to ω) submanifold $\mathcal{L} \subset \mathcal{F}$. The space of BV fields is roughly constructed (see [20]) by starting with the space of fields of a classical field theory, such as Chern–Simons theory, and resolving its algebra of functions in two directions, by adding certain new fields. The Koszul–Tate resolution captures the functions on the space of solutions of Euler–Lagrange equations. It includes a corresponding anti-field for every normal field. The Chevalley–Eilenberg resolution encodes the gauge invariant functions. It includes a new field called ghost for every infinitesimal generator of the gauge group. The introduction of anti-fields and ghosts gives rise to additional fields such as ghosts for ghosts, anti-ghosts, i.e., anti-fields for ghosts, etc.

From the perspective of derived geometry (see [1], [23], [14]), the two previously mentioned resolutions introduce the derived and stacky structure on the space of fields, making it a derived stack. The reason for adding anti-fields and ghosts is to deal with the pathologies which arise when passing to a subspace such as the critical locus, which is often a non-transversal intersection of two spaces, or taking the quotient by a non-free action, respectively. The degenerate spaces get replaced by homotopically equivalent models which behave better and exhibit structures (for example, a natural symplectic structure, see [18]) necessary to carry out the procedures prescribed by field theories.

In the second chapter we will see how the critical locus of a function defined on a finite-dimensional space can be naturally equipped with a derived structure. However, there is a problem with this construction in terms of its applicability to our previous considerations because the spaces of fields are usually infinite-dimensional. Nevertheless, the reduced critical locus which is obtained by taking the quotient of the solution space of the Euler–Lagrange equation by the gauge group turns out to be of finite dimension (see [2] for the case of Yang–Mills theory). In light of this we can ask if the reduced space can be equipped with a derived structure by exhibiting it as the critical locus of a function acting on some finite-dimensional space. Note that it is often possible to exhibit the reduced space in the appropriate way locally. One example is a holomorphic version of the Chern–Simons theory (see [12], [16], [13]). It can be shown (*cf.* [12, Proposition 9.12]) that the reduced space of flat $(0, \bullet)$ G-connections on a Calabi–Yau 3-fold is locally the critical locus of the restriction of the holomorphic Chern–Simons functional on a certain finite dimensional space. This space is obtained by imposing norm and Laplacian eigenvalue constraints on the connections.

If it happens to be possible to exhibit the reduced space as the critical locus of the restriction of the action functional on some finite-dimensional space, then there might be many choices for such a finite-dimensional space, which prompts the question of compatibility of the obtained derived and (-1)-shifted symplectic structures on the reduced space.

In this thesis we study the problem of compatibility of derived structures on a scheme which can be presented as a critical locus in more than one way. More precisely, we consider the situation when a scheme can be presented as the critical locus of a function $w \in \mathcal{O}(S)$ and as the critical locus of its restriction $w|_X \in \mathcal{O}(X)$ for some smooth subscheme $X \subset S$. In the third chapter we consider the case when S is the total space of a vector bundle over X. We prove that, under natural assumptions on the relation between w and X, the two derived structures and the associated (-1)-shifted symplectic structures coincide. In the fourth chapter we generalize the result to the case of quantized cycles, and give indications how to deal with more general closed embeddings $X \subset S$.

Chapter 2

Derived critical loci

In this chapter we collect several notions related to the construction and properties of derived critical loci. In this manner we also fix the relevant notation. We follow closely the expositions in [24] and [19].

2.1 Derived zero loci of sections of vector bundles

First we will look into the local situation, so let the scheme X be a smooth affine scheme $\operatorname{Spec} R$, where R is a commutative algebra over a field $\Bbbk$ of characteristic 0. Let E be a projective R-module of finite type. The reader can have the example of the cotangent bundle $E = T^\vee X$ in mind. Let $\operatorname{Sym}_R(E^\vee)$ be the symmetric algebra on the R-dual $E^\vee$ of E. We're only interested in the structure of $\operatorname{Sym}_R(E^\vee)$ as the algebra of functions on the space $\operatorname{tot}(E)$ and we don't take into consideration its grading.

Since R might not be cofibrant as a $\mathrm{Sym}_R(E^\vee)$ algebra, we need to resolve it. Let $\bigwedge^{-\bullet} E^\vee$ be the non-positively graded exterior algebra of $E^\vee$ considered as an R-module. Its $(-m)$-th piece is equal to $\bigwedge^m E^\vee$. The non-positively graded $\mathrm{Sym}_R(E^\vee)$-module $\mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^{-\bullet} E^\vee$ in which the $(-m)$-th part equals to

$$\mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^m E^\vee$$

is degreewise projective over $\mathrm{Sym}_R(E^\vee)$. We have the augmentation map

$$\bigwedge^{-\bullet} E^\vee \to \left(\bigwedge^{-\bullet} E^\vee\right)_0 = \mathrm{Sym}_R(E^\vee) \to R$$

which comes from the canonical zero section $0 \in E$. Note that since $\bigwedge^{-\bullet} E^\vee$ is a graded commutative R-algebra, $\mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^{-\bullet} E^\vee$ is also a commutative graded algebra over $\mathrm{Sym}_R(E^\vee)$. The differential on this graded algebra is induced by the contraction and the canonical map $R \to \mathrm{End}_R(E) \simeq E^\vee \otimes_R E$ which acts by $1 \mapsto \mathrm{id}_E$. More explicitly, if we express id_E as $\sum_{i=1}^n \varepsilon_i \otimes e_i$, then the differential $d : \mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^m E^\vee \to \mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^{m-1} E^\vee$ acts as

$$d(p \cdot \sigma_1 \wedge \cdots \wedge \sigma_m) = \sum_{i=1}^n \varepsilon_i \cdot p \sum_{j=1}^m (-1)^{j+1} \sigma_j(e_i) \sigma_1 \wedge \cdots \wedge \widehat{\sigma_j} \wedge \cdots \wedge \sigma_m$$

This differential is clearly $\mathrm{Sym}_R(E^\vee)$-linear. Hence,

$$\mathrm{K}^\bullet(R, E) := \left(\mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^{-\bullet} E^\vee, d\right)$$

is a CDGA over the algebra $\mathrm{Sym}_R(E^\vee)$. It is well known (see for example [5]) that the cohomology of $\mathrm{K}^\bullet(R, E)$ is zero in negative degrees and is R in degree zero. This means that $\mathrm{K}^\bullet(R, E)$ resolves R.

Proposition 2.1.1. *The augmentation map* $\mathrm{K}^\bullet(R,E) \to R$ *is a cofibrant resolution of* R *in the model category of CDGAs over* $\mathrm{Sym}_R(E^\vee)$.

Any choice of a section $s \in E$ induces a map $\mathrm{Sym}_R(E^\vee) \xrightarrow{\mathrm{ev}_s} R$ of commutative algebras corresponding to the evaluation at s map $E^\vee \xrightarrow{\mathrm{ev}_s} R$. Let R_s denote R with the corresponding $\mathrm{Sym}_R(E^\vee)$-algebra structure. Note that R_s can be regarded as a CDGA over $\mathrm{Sym}_R(E^\vee)$ concentrated in degree 0, so we can take the tensor product of graded algebras R_s and $\mathrm{K}^\bullet(R,E)$. We get

$$R_s \otimes_{\mathrm{Sym}_R(E^\vee)} \mathrm{K}^\bullet(R,E) = R_s \otimes_{\mathrm{Sym}_R(E^\vee)} \left(\mathrm{Sym}_R(E^\vee) \otimes_R \bigwedge^{-\bullet} E^\vee\right) \simeq \bigwedge^{-\bullet} E^\vee$$

where $\bigwedge^{-\bullet} E^\vee$ on the right side is a $\mathrm{Sym}_R(E^\vee)$-module via the composite map $\mathrm{Sym}_R(E^\vee) \xrightarrow{\mathrm{ev}_s} R \hookrightarrow \bigwedge^{-\bullet} E^\vee$. Transferring $\mathrm{id}_{R_s} \otimes_{\mathrm{Sym}_R(E^\vee)} d$ by the isomorphism above, we obtain the differential d_s on the graded algebra $\bigwedge^{-\bullet} E^\vee$ which is just the standard Koszul differential on $\bigwedge^{-\bullet} E^\vee$ induced by the contraction along s (see [7]):

$$d_s(\varepsilon_1 \wedge \cdots \wedge \varepsilon_m) = \sum_{j=1}^{m} (-1)^{j+1} \varepsilon_j(s) \varepsilon_1 \wedge \cdots \wedge \widehat{\varepsilon_j} \wedge \cdots \wedge \varepsilon_m$$

where $\mathrm{id}_E = \sum_{i=1}^n \varepsilon_i \otimes e_i$. The differential d_s can be characterized as the unique antiderivation on the R-algebra $\bigwedge^{-\bullet} E^\vee$ which extends the evaluation $E^\vee \xrightarrow{\mathrm{ev}_s} R$.

We now give a geometric interpretation of the above algebraic constructions. Let $\mathrm{tot}(E) := \mathrm{Spec}(\mathrm{Sym}_R(E^\vee))$ be the total space of the projective R-module E. The canonical map of $\Bbbk$-algebras $R \hookrightarrow \mathrm{Sym}_R(E^\vee)$ corresponds to the projection $\mathrm{tot}(E) \twoheadrightarrow X$ and makes the space $\mathrm{tot}(E)$ a vector bundle over X with the module of sections E. The zero section $0 : X \hookrightarrow \mathrm{tot}(E)$ corresponds to the natural augmentation $\mathrm{Sym}_R(E^\vee) \to R$, while the section $s \in E$ induces another augmentation $\mathrm{Sym}_R(E^\vee) \xrightarrow{\mathrm{ev}_s} R$. We consider the homotopy fiber product

$$\begin{array}{ccc} \mathbb{R}Z(s) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{s} & \mathrm{tot}(E) \end{array}$$

in the category of étale derived stacks over $\Bbbk$ (see [23]). The derived affine stack $\mathbb{R}Z(s)$ is called the *derived zero locus* of the section s. To compute the homotopy fiber product, we may choose any cofibrant replacement of the natural augmentation $\mathrm{Sym}_R(E^\vee) \to R$ in the category of $\mathrm{Sym}_R(E^\vee)$-CDGAs, such as $\mathrm{K}^\bullet(R, E) \to R$, and then calculate the tensor product:

$$\begin{aligned} \mathbb{R}Z(s) \simeq \mathbb{R}\mathrm{Spec}\Big(R_s \otimes^{\mathbb{L}}_{\mathrm{Sym}_R(E^\vee)} R\Big) &\simeq \mathbb{R}\mathrm{Spec}\big(R_s \otimes_{\mathrm{Sym}_R(E^\vee)} \mathrm{K}^\bullet(R, E)\big) \\ &\simeq \mathbb{R}\mathrm{Spec}\big(\textstyle\bigwedge^{-\bullet} E^\vee, d_s\big) \end{aligned}$$

This translates to:

Proposition 2.1.2. *The standard Koszul complex* $\big(\bigwedge^{-\bullet} E^\vee, d_s\big)$ *associated to the section* $s \in E$ *is the algebra of functions on the derived zero locus* $\mathbb{R}Z(s)$ *of the section* s.

Generalization to the global case is straightforward. If X is a scheme over the field $\Bbbk$, $\mathcal{E}$ a locally free sheaf of finite rank, and $s \in H^0(X, \mathcal{E})$ a section of $\mathcal{E}$, then the derived zero locus is the homotopy fiber product

$$\begin{array}{ccc} \mathbb{R}Z(s) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{s} & \mathrm{tot}(\mathcal{E}) \end{array}$$

in the category of étale derived stacks over $\Bbbk$. It has a presentation in the form of a derived scheme with the usual zero locus $Z(s)$ as its topological space, and $\big(\bigwedge^{-\bullet}\mathcal{E}^\vee, d_s\big)|_{Z(s)}$ as its sheaf of functions. Note that the operation of taking the restriction to $Z(s)$ corresponds to taking the sheaf theoretic inverse image (not the $\mathcal{O}$-module pullback) by the natural inclusion $i : Z(s) \hookrightarrow \operatorname{tot}(\mathcal{E})$:

$$\big(\textstyle\bigwedge^{-\bullet}\mathcal{E}^\vee, d_s\big)|_{Z(s)} := i^{-1}\big(\textstyle\bigwedge^{-\bullet}\mathcal{E}^\vee, d_s\big)$$

2.1.1 Differential calculus on derived zero loci

Generally, if L' and L'' are two smooth subvarieties of a smooth variety S over $\Bbbk$, then the derived intersection of L' and L'', i.e., the homotopy pullback

$$\begin{tikzcd}
\mathbb{R}(L' \cap L'') \arrow[r] \arrow[d] & L'' \arrow[d, hook] \\
L' \arrow[r, hook] & S
\end{tikzcd}$$

in the category of étale derived stacks over $\Bbbk$, can be presented as a derived scheme having $L' \cap L''$ as its topological space and $\mathcal{O}_{L'} \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathcal{O}_{L''}$ as its sheaf of functions. The tangent complex $\mathbb{T}_{\mathbb{R}(L'\cap L'')}$ of $\mathbb{R}(L' \cap L'')$ is a complex concentrated in degrees 0 and 1, explicitly given by

$$\cdots \to 0 \to \Big[\underbrace{(i_{L'\cap L''\hookrightarrow L'})^* T_{L'} \oplus (i_{L'\cap L''\hookrightarrow L''})^* T_{L''}}_{0} \to \underbrace{(i_{L'\cap L''\hookrightarrow S})^* T_S}_{1}\Big] \to 0 \to \cdots$$

The only nonzero differential operates via derivatives of the inclusions:

$$d(i_{L'\cap L''\hookrightarrow L'})_s \oplus d(i_{L'\cap L''\hookrightarrow L''})_s : T_sL' \oplus T_sL'' \to T_sS$$

for $s \in L' \cap L''$. The zeroth cohomology of $\mathbb{T}_{\mathbb{R}(L'\cap L'')}$ encodes the tangent sheaf of the intersection of L' and L'', while the first cohomology measures the failure of transversality of the intersection of L' and L''.

In the case of derived zero locus of a section $s \in E$ of a vector bundle E over X we get

$$\mathbb{T}_{\mathbb{R}Z(s)} = \Big[\underbrace{(i_{Z(s)\hookrightarrow X})^* T_X \oplus (i_{Z(s)\hookrightarrow X})^* T_X}_{0} \to \underbrace{(i_{Z(s)\hookrightarrow \mathrm{tot}(E)})^* T_{\mathrm{tot}(E)}}_{1}\Big]$$

If $\nabla \colon E \to E \otimes \Omega^1_X$ is an algebraic connection on E, then there exists a natural quasiisomorphism between $\mathbb{T}_{\mathbb{R}Z(s)}$ and the complex

$$\Big[\underbrace{(i_{Z(s)\hookrightarrow X})^* T_X}_{0} \xrightarrow{\nabla s} \underbrace{(i_{Z(s)\hookrightarrow X})^* E}_{1}\Big] = (i_{Z(s)\hookrightarrow X})^* \Big[T_X \xrightarrow{\nabla s} E\Big]$$

The connection ∇ may exist only locally on X, and when it does exist, it is not unique. However, the differential in the complex above, which uses the $i_{Z(s)\hookrightarrow X}$ pullback of ∇s, is well defined globally and independent of the choice of ∇.

Dualizing the previous complex we get:

$$\Omega^1_{\mathbb{R}Z(s)} = (i_{Z(s)\hookrightarrow X})^* \Big[E^\vee \xrightarrow{(\nabla s)^\vee} \Omega^1_X\Big]$$

Zariski locally it is always possible to choose a connection ∇ which is flat. Using such a connection we can explicitly present $\Omega^1_{\mathbb{R}Z(s)}$ as a module over $\left(\bigwedge^{-\bullet} E^\vee, d_s\right)$, i.e., over the algebra of functions on $\mathbb{R}Z(s)$:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \bigwedge^3 E^\vee \otimes \Omega^1_X & \longrightarrow & \bigwedge^2 E^\vee \otimes \Omega^1_X & \longrightarrow & E^\vee \otimes \Omega^1_X & \longrightarrow & \Omega^1_X & & 0 \\
 & & \uparrow{\scriptstyle [\nabla,s]} & & \uparrow{\scriptstyle [\nabla,s]} & & \uparrow{\scriptstyle [\nabla,s]} & & \uparrow{\scriptstyle [\nabla,s]} & & \\
\cdots & \longrightarrow & \bigwedge^3 E^\vee \otimes E^\vee & \longrightarrow & \bigwedge^2 E^\vee \otimes E^\vee & \longrightarrow & E^\vee \otimes E^\vee & \longrightarrow & E^\vee & & -1
\end{array}$$

where the horizontal maps are induced by the Koszul differential, and the vertical maps are induced by ∇ and s. This double complex of vector bundles on X is a resolution of $\Omega^1_{\mathbb{R}Z(s)}$ which has an advantage of explicitly being a module over $\mathcal{O}(\mathbb{R}Z(s))$.

We can continue in a similar manner to describe $\Omega^2_{\mathbb{R}Z(s)}$ as a module over $\mathcal{O}(\mathbb{R}Z(s))$:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigwedge^2 E^\vee \otimes \Omega^2_X \arrow[r] & \boxed{E^\vee \otimes \Omega^2_X} \arrow[r] & \Omega^2_X & 0 \\
\cdots \arrow[r] & \bigwedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_X \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_X \arrow[r] \arrow[u] & \boxed{E^\vee \otimes \Omega^1_X} \arrow[u] & -1 \\
\cdots \arrow[r] & \bigwedge^2 E^\vee \otimes \mathrm{Sym}^2(E^\vee) \arrow[r] \arrow[u] & E^\vee \otimes \mathrm{Sym}^2(E^\vee) \arrow[r] \arrow[u] & \mathrm{Sym}^2(E^\vee) \arrow[u] & -2
\end{tikzcd}$$

The boxed pieces form the habitat of the (-1)-shifted 2-forms on the derived zero locus of $s \in E$. Hence, any such form on $\mathbb{R}Z(s)$ can be presented by a pair of sections $\alpha \in E^\vee \otimes \Omega^2_X$ and $\varphi \in E^\vee \otimes \Omega^1_X$ such that $[\nabla, s](\varphi) = i_s\alpha$.

The de Rham differential $d_{DR}\colon \Omega^1_{\mathbb{R}Z(s)} \to \Omega^2_{\mathbb{R}Z(s)}$ is given on each term of the double complex presenting $\Omega^1_{\mathbb{R}Z(s)}$ by the sum of ∇ and the Koszul type of map $\bigwedge^m E^\vee \otimes \mathrm{Sym}^n(E^\vee) \to \bigwedge^{m-1} E^\vee \otimes \mathrm{Sym}^{n+1}(E^\vee)$ which acts by contraction with $\mathrm{id}_E \in E^\vee \otimes E$, followed by the multiplication $E^\vee \otimes \mathrm{Sym}^n(E^\vee) \to \mathrm{Sym}^{n+1}(E^\vee)$.

2.2 The case of derived critical loci

Let X be a smooth algebraic variety over $\Bbbk$ and $w \in \mathcal{O}(X)$ a regular function on X. A critical point of w is a point $x \in X$ at which the differential dw vanishes. Critical points of w form a closed subvariety of X called the critical locus of w which we denote by $\mathrm{Crit}(w)$. This variety has a natural derived structure since $\mathrm{Crit}(w)$ can be understood as the zero locus of the section dw of the cotangent bundle of X:

$$\begin{tikzcd} \mathbb{R}\mathrm{Crit}(w) \arrow[r] \arrow[d] & X \arrow[d, "0"] \\ X \arrow[r, "dw"] & \mathrm{tot}(\Omega^1_X) \end{tikzcd}$$

Therefore, the structure of $\mathbb{R}\mathrm{Crit}(w)$ as a derived scheme is given by $\mathrm{Crit}(w)$ as its topological space and the Koszul complex $\left(\bigwedge^{-\bullet} T_X, d_w\right)|_{\mathrm{Crit}(w)}$ as its sheaf of functions. In this context we often use the term *potential* for the function w and the notation d_w for the differential in the Koszul complex of functions on $\mathbb{R}\mathrm{Crit}(w)$.

Remark 2.2.1. A part of the Batalin–Vilkovisky construction is a certain function, called the master action, which is defined on the space of BV fields. If we denote the space of BV fields by X, then the master action is a function $w \in \mathcal{O}(X)$ which has to satisfy the master equation $\{w, w\} = 0$. The action of the operator $\{w, -\}$ on the tangent bundle T_X (located in degree -1 of the Koszul complex) coincides, up to sign, with the action of the differential d_w on the algebra of functions on $\mathbb{R}\mathrm{Crit}(w)$. Indeed, using the properties of the Schouten–Nijenhuis bracket we have:

$$\{w, v\} = -\mathcal{L}_v w = -v(w) = -dw(v) = -d_w(v)$$

In general, the master equation makes the Hamiltonian vector field $\{\mathcal{S}, -\}$ square to zero, and hence it endows the algebra of functions on the space of BV fields with the structure of a cochain complex. This makes the space of BV fields a space with a derived structure.

Derived symplectic structure

Let S be a smooth variety (or more generally, smooth Deligne–Mumford stack) over $\mathbb{C}$ which has a symplectic form ω. We say that a smooth closed subvariety $L \subset S$ is *Lagrangian* if $\dim L = \frac{1}{2}\dim S$ and the form ω vanishes on L. It was shown in [18] (see Theorem 2.9 and Corollary 2.10 there) that if L' and L'' are two smooth closed Lagrangian subvarieties of S, then the derived fiber product $L' \times^h_S L''$ carries a canonical (-1)-shifted symplectic structure. Since the derived critical locus of $w \in \mathcal{O}(X)$ is the derived fiber product of two Lagrangian subvarieties of Ω^1_X, namely $X \xhookrightarrow{0} \mathrm{tot}(\Omega^1_X)$ and $X \xhookrightarrow{dw} \mathrm{tot}(\Omega^1_X)$, we conclude that it carries a canonical (-1)-shifted symplectic structure. Note that such a symplectic structure is a part of the Batalin–Vilkovisky construction.

In terms of the explicit description of the space of forms on the derived zero locus of the section $dw \in \Omega^1_X$ given in section 2.1.1, the canonical (-1)-shifted symplectic form on $\mathbb{R}\mathrm{Crit}(w)$ corresponds to the pair $dw = \mathrm{id}_{\Omega^1_X} \otimes dw \in T_X \otimes \Omega^2_X$ and $\mathrm{id}_{\Omega^1_X} \in T_X \otimes \Omega^1_X$. All the higher forms which make up the canonical key which closes this symplectic form are 0.

Remark 2.2.2. A Darboux's type of theorem, describing the local structure of shifted symplectic derived schemes and stacks, was proven in [8]. A derived scheme equipped with a shifted symplectic structure is étale locally (and sometimes Zariski locally) equivalent to the derived critical locus of a certain, explicitly described, shifted function, which depends on the shift of the symplectic structure. If X is a (-1)-shifted symplectic derived scheme, then X is Zariski locally equivalent to the derived critical locus $\mathbb{R}\mathrm{Crit}(w)$ of a regular function $w \in \mathcal{O}(U)$ defined on a smooth classical scheme U.

Chapter 3

The case of vector bundles

Let S be a smooth algebraic variety over $\mathbb{C}$. A critical point of a regular function $w \in \mathcal{O}(S)$ is a point $s \in S$ at which dw vanishes. Critical points of w form a closed subvariety of S called the critical locus of w which we denote by $\mathrm{Crit}(w)$. In this context we often use the term *potential* for the function w. Assume additionally that we have a smooth subvariety X of S. If $x \in X$ is a critical point of $w \in \mathcal{O}(S)$, then the Hessian of w at the point x is a well defined quadratic form on the tangent space T_xS.

Now take S to be the total space of a vector bundle E over a smooth variety X. Denote by $\pi\colon S = \mathrm{tot}(E) \to X$ the projection and use the zero section of E to view X as a subvariety of S. For every $x \in X$ define the subspace of fiber directions $F_x \subset T_xS$ to be the tangent space of the fiber $\pi^{-1}(x)$. Let $w \in \mathcal{O}(\mathrm{tot}(E))$ be a potential which satisfies the following assumptions:

(1) the critical locus $\mathrm{Crit}(w) \subset \mathrm{tot}(E)$ is contained inside X

(2) the differential $dw|_X$ vanishes in the fiber directions F_x

(3) at every point $x \in X \cap \mathrm{Crit}(w)$, the Hessian of w restricts to a non-degenerate bilinear form on F_x

The first two conditions imply that the closed subvariety $\mathrm{Crit}(w) \subset \mathrm{tot}(E)$ coincides with $\mathrm{Crit}(w|_X) \subset X \subset \mathrm{tot}(E)$, the critical locus of the restriction $w|_X \in \mathcal{O}(X)$. The third condition parallels the definition of nondegenerate critical manifolds in Morse–Bott theory, *cf.* [6], [2].

Let Y be the closed subvariety $\mathrm{Crit}(w)$, which is, under our assumptions, the same as the closed subvariety $\mathrm{Crit}(w|_X)$. Y can be understood as a derived scheme in two ways. On the one hand, Y is the critical locus of $w \in \mathcal{O}(\mathrm{tot}(E))$, and so it should have $\left(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\right)|_Y$ as its sheaf of functions. Note that here we define the operation of taking the restriction of a complex of sheaves to the subvariety Y as taking the sheaf theoretic inverse image (not the $\mathcal{O}$-module pullback[1]) by the natural inclusion $i : Y \hookrightarrow \mathrm{tot}(E)$:

$$\left(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\right)|_Y := i^{-1}\left(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\right)$$

On the other hand, Y is the critical locus of $w|_X \in \mathcal{O}(X)$ and therefore it should have $\left(\bigwedge^{-\bullet} T_X, d_{w|_X}\right)|_Y$ as its sheaf of functions. In this chapter we will compare these two derived structures on Y, together with the corresponding (-1)-shifted symplectic structures, and show that they are the same.

[1]see Remark 3.1.8 at the end of the next section

3.1 Line bundles

We begin by investigating the local situation around a point of the critical locus of $w \in \mathcal{O}(\mathrm{tot}(E))$. After choosing a local trivialization of the bundle, we can assume $X = \mathrm{Spec}(R)$ and $S = \mathrm{Spec}(R[t])$. Because of the global assumptions we have on our potential, we know that $w \in \mathcal{O}(S)$ has the form

$$w = r_0 + r_2 t^2 + \sum_{k \geq 3} r_k t^k \in R[t]$$

where $r_2 \in R$ is invertible. The tangent bundle splits in the following manner:

$$T_S = (R[t] \otimes_R T_X) \oplus \left(R[t] \left\langle \frac{\partial}{\partial t} \right\rangle \right)$$

Hence, the $(-m)$-th term of the Koszul complex $\left(\bigwedge^{-\bullet} T_S, d_w\right)$ is isomorphic to

$$R[t] \otimes_R \left(\bigwedge^m T_X \oplus \bigwedge^{m-1} T_X\right)$$

Take an arbitrary element of the $(-m)$-th term:

$$\varphi = (\alpha_0, \beta_0) + (\alpha_1, \beta_1)t + (\alpha_2, \beta_2)t^2 + (\alpha_3, \beta_3)t^3 + \cdots$$

where $\alpha_n \in \bigwedge^m T_X$ and $\beta_n \in \bigwedge^{m-1} T_X$. Applying the differential we get $d_w\varphi$ which can be written in the form

$$d_w\varphi = (\tilde{\alpha}_0, \tilde{\beta}_0) + (\tilde{\alpha}_1, \tilde{\beta}_1)t + (\tilde{\alpha}_2, \tilde{\beta}_2)t^2 + (\tilde{\alpha}_3, \tilde{\beta}_3)t^3 + \cdots$$

for some $\tilde{\alpha}_n \in \bigwedge^{m-1} T_X$ and $\tilde{\beta}_n \in \bigwedge^{m-2} T_X$.

We now calculate what happens when we apply d_w to an element of the form $\alpha_n t^n$. If α_n is representable as $f\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_m}}$, then we have

$$\begin{aligned}
d_w \alpha_n t^n &= \left(\sum_{j=1}^{m} (-1)^{j+1} \frac{\partial w}{\partial x_{i_j}} f \tfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\tfrac{\partial}{\partial x_{i_j}}} \wedge \cdots \wedge \tfrac{\partial}{\partial x_{i_m}} \right) t^n \\
&= \sum_{k \geq 0} \left(\sum_{j=1}^{m} (-1)^{j+1} \frac{\partial r_k}{\partial x_{i_j}} f \tfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\tfrac{\partial}{\partial x_{i_j}}} \wedge \cdots \wedge \tfrac{\partial}{\partial x_{i_m}} \right) t^{k+n} \\
&= \sum_{k \geq 0} d_k \alpha_n t^{k+n}
\end{aligned}$$

where d_k is the Koszul differential for the critical locus of the function $r_k \in \mathcal{O}(X)$. We see that $d_w \alpha_n t^n$ contributes $d_k \alpha_n$ to $\tilde{\alpha}_{k+n}$ and makes no contribution to any $\tilde{\beta}_n$. Now, if β_n corresponds to $f\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{m-1}}} \wedge \frac{\partial}{\partial t}$, we get:

$$\begin{aligned}
d_w \beta_n t^n &= \left(\sum_{j=1}^{m-1} (-1)^{j+1} \frac{\partial w}{\partial x_{i_j}} f \tfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\tfrac{\partial}{\partial x_j}} \wedge \cdots \wedge \tfrac{\partial}{\partial x_{i_{m-1}}} \wedge \tfrac{\partial}{\partial t} \right) t^n \\
&\quad + (-1)^{m+1} \frac{\partial w}{\partial t} f \tfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \tfrac{\partial}{\partial x_{i_{m-1}}} \ t^n \\
&= \sum_{k \geq 0} \left(d_k \beta_n \wedge \tfrac{\partial}{\partial t} \right) t^{k+n} + \sum_{k \geq 2} (-1)^{m+1} k r_k \beta_n t^{k-1+n}
\end{aligned}$$

Therefore, $d_w \beta_n t^n$ contributes $d_k \beta_n$ to $\tilde{\beta}_{k+n}$ and $(-1)^{m+1}(k+1) r_{k+1} \beta_n$ to $\tilde{\alpha}_{k+n}$.

Collecting the terms we get:

$$\tilde{\alpha}_0 = d_0\alpha_0$$

$$\tilde{\alpha}_1 = d_1\alpha_0 + d_0\alpha_1 + (-1)^{m+1}2\underline{r_2\beta_0}$$

$$\tilde{\alpha}_2 = d_2\alpha_0 + d_1\alpha_1 + d_0\alpha_2 + (-1)^{m+1}(3r_3\beta_0 + 2\underline{r_2\beta_1})$$

$$\tilde{\alpha}_3 = d_3\alpha_0 + d_2\alpha_1 + d_1\alpha_2 + d_0\alpha_3 + (-1)^{m+1}(4r_4\beta_0 + 3r_3\beta_1 + 2\underline{r_2\beta_2})$$

$$\vdots$$

$$\tilde{\alpha}_n = \sum_{i+j=n} d_i\alpha_j + (-1)^{m+1}\Big((n+1)r_{n+1}\beta_0 + nr_n\beta_1 + \cdots + 3r_3\beta_{n-2} + 2\underline{r_2\beta_{n-1}}\Big)$$

$$\vdots$$

If we assume $d_w\varphi = 0$, then every $\tilde{\alpha}_n$ in the sequence of equations above equals to 0. Since the function r_2 is invertible, the equations $\tilde{\alpha}_n = 0$ force the sequence $\beta_0, \beta_1, \beta_2, \ldots$ to be completely determined by the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ as we can express β_n in the following way:

$$2r_2\beta_n = (-1)^m\left(\sum_{i+j=n+1} d_i\alpha_j\right) - ((n+2)r_{n+2}\beta_0 + (n+1)r_{n+1}\beta_1 + \cdots + 3r_3\beta_{n-1})$$

Hence, we've proven the following lemma.

Lemma 3.1.1. *Let $\varphi \in \bigwedge^m T_S$ and write $\varphi = \sum_{n\geq 0}(\alpha_n, \beta_n)t^n$. If $d_w\varphi = 0$, then φ is completely determined by the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$*

Now pick $\varphi \in \ker d_w$ and let it be represented by $\sum_{n\geq 0}\alpha_n t^n$. Are there any constraints on α_n? Certainly we must have $d_0\alpha_0 = 0$. Additionally, if $\varphi \in \ker d_w$ is going to be in the image of d_w, than it must satisfy $\alpha_0 \in \operatorname{im} d_0$. We will now show that there are no other constraints on the image of d_w.

Proposition 3.1.2. *Let $\bar{\varphi} \in \bigwedge^m T_S$. Assume that $\bar{\varphi}$ is in the kernel of d_w and that it is represented by $\sum_{n\geq 0} \bar{\alpha}_n t^n$. Then $\bar{\varphi} \in \operatorname{im} d_w \Leftrightarrow \bar{\alpha}_0 \in \operatorname{im} d_0$.*

Proof. Let $\bar{\alpha}_0 = d_0\alpha_0$ and define

$$\varphi := (\alpha_0, \beta_0) + (0, \beta_1)t + (0, \beta_2)t^2 + (0, \beta_3)t^3 + \cdots$$

for some $\beta_n \in \bigwedge^m T_X$ which are to be determined. We write $d_w\varphi = \sum_{n\geq 0}(\tilde{\alpha}_n, \tilde{\beta}_n)t^n$. The desired equality $\bar{\varphi} = d_w\varphi$ implies the following sequence of equations:

$$\begin{aligned}
\bar{\alpha}_0 &= d_0\alpha_0 \\
\bar{\alpha}_1 &= d_1\alpha_0 + (-1)^m 2r_2\beta_0 \\
\bar{\alpha}_2 &= d_2\alpha_0 + (-1)^m(3r_3\beta_0 + 2r_2\beta_1) \\
\bar{\alpha}_3 &= d_3\alpha_0 + (-1)^m(4r_4\beta_0 + 3r_3\beta_1 + 2r_2\beta_2) \\
&\vdots
\end{aligned}$$

Since r_2 is invertible, we can solve inductively these equations and find $\beta_0, \beta_1, \beta_2, \ldots$ such that $\bar{\alpha}_n = \tilde{\alpha}_n$ for every n. But then we also have $\bar{\beta}_n = \tilde{\beta}_n$ because both $\bar{\varphi}$ and $d_w\varphi$ are in the kernel of d_w, and therefore, by Lemma 3.1.1, they are completely determined by $\sum_{n\geq 0} \bar{\alpha}_n t^n$ and $\sum_{n\geq 0} \tilde{\alpha}_n t^n$ respectively. $\square$

Now we continue investigating the kernel.

Lemma 3.1.3. *Let $\alpha_0 \in \bigwedge^m T_X$. If $\alpha_0 \in \ker d_0$, then there exists $\varphi \in \bigwedge^m T_S$ such that $\varphi \in \ker d_w$ and that φ is represented by the sequence $\alpha_0, 0, 0, 0, \ldots$.*

Proof. Let

$$\varphi := (\alpha_0, \beta_0) + (0, \beta_1)t + (0, \beta_2)t^2 + (0, \beta_3)t^3 + \cdots$$

and denote $d_w\varphi = \sum_{n\geq 0}(\tilde{\alpha}_n, \tilde{\beta}_n)t^n$. The requirement $d_w\varphi = 0$ implies a sequence of equations $\tilde{\alpha}_n = 0$, first of which, $d_0\alpha_0 = 0$, is satisfied by the assumption. Other equations $\tilde{\alpha}_n = 0$ for $n > 0$ form a system $L\beta = (-1)^m d\alpha_0$ with

$$L = \begin{pmatrix} 2r_2 & 0 & 0 & 0 & 0 & \cdots \\ 3r_3 & 2r_2 & 0 & 0 & 0 & \cdots \\ 4r_4 & 3r_3 & 2r_2 & 0 & 0 & \cdots \\ 5r_5 & 4r_4 & 3r_3 & 2r_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix}, \qquad d\alpha_0 = \begin{pmatrix} d_1\alpha_0 \\ d_2\alpha_0 \\ d_3\alpha_0 \\ d_4\alpha_0 \\ \vdots \end{pmatrix}$$

Solving the system in the usual manner we get $\beta_0, \beta_1, \ldots$ such that all the equations $\tilde{\alpha}_n = 0$ are satisfied. We need to show that the equations $\tilde{\beta}_n = 0$ are also satisfied. As we've seen on page 22, each $d_w\beta_n t^n$ contributes the factor $d_k\beta_n$ to $\tilde{\beta}_{k+n}$. Hence, the system of equations $\tilde{\beta}_n = 0$ is $D\beta = 0$, where

$$D = \begin{pmatrix} d_0 & 0 & 0 & 0 & 0 & \cdots \\ d_1 & d_0 & 0 & 0 & 0 & \cdots \\ d_2 & d_1 & d_0 & 0 & 0 & \cdots \\ d_3 & d_2 & d_1 & d_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

Note that D and L are lower triangular Toeplitz matrices. Such matrices commute, assuming that their entries commute, which is true in our case since the differentials d_k are R-linear. Since D and L commute, then D and L^{-1} also commute and therefore:

$$D\beta = 0 \Leftrightarrow DL^{-1}d\alpha_0 = 0 \Leftrightarrow L^{-1}Dd\alpha_0 = 0 \Leftrightarrow Dd\alpha_0 = 0$$

It remains to check the equality $Dd\alpha_0 = 0$. The n-th element of $Dd\alpha_0$ is

$$\sum_{k=0}^{n} d_{n-k}d_{k+1}\alpha_0 = d_0 d_{n+1}\alpha_0 + (d_n d_1 \alpha_0 + \cdots + d_1 d_n \alpha_0)$$

The terms in the parentheses cancel each other because the equality $d_i d_j = -d_j d_i$ holds in general. The first term is 0 since $d_0\alpha_0 = 0$. $\square$

Proposition 3.1.4. *The mapping $\bigwedge^m T_S \to R[t] \otimes_R \bigwedge^m T_X$ given by the formula*

$$\sum_{n\geq 0}(\alpha_n, \beta_n)t^n \mapsto \sum_{n\geq 0} \alpha_n t^n$$

restricts to an isomorphism $\Phi : \ker d_w \to \left\{\sum_{n\geq 0} \alpha_n t^n : \alpha_0 \in \ker d_0\right\}$.

Proof. By Lemma 3.1.1, the map Φ is an injection. To show that it is surjection, first we deal with the case $\alpha_0 = 0$. Pick any sequence $\alpha_1, \alpha_2, \ldots$. Similarly as in the proof of Proposition 3.1.2, we solve the system

$$\begin{pmatrix} 2r_2 & 0 & 0 & 0 & 0 & \cdots \\ 3r_3 & 2r_2 & 0 & 0 & 0 & \cdots \\ 4r_4 & 3r_3 & 2r_2 & 0 & 0 & \cdots \\ 5r_5 & 4r_4 & 3r_3 & 2r_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix} = (-1)^m \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \end{pmatrix}$$

to obtain β_n which we arrange in $\varphi := \sum_{n\geq 0}(0, \beta_n)t^n$. Now note that $d_w\varphi \in \ker d_w$ and $\Phi(d_w\varphi) = \sum_{n\geq 1} \alpha_n t^n$. Hence, the image of Φ contains $\left\{\sum_{n\geq 0} \alpha_n t^n : \alpha_0 = 0\right\}$. Since Φ is a vector space homomorphism, it only remains to see that $\alpha_0 \in \operatorname{im}\Phi$ whenever $\alpha_0 \in \ker d_0$, which is the content of Lemma 3.1.3. $\square$

Notice that the differential d_0 is precisely the differential $d_{w|X}$ on $\bigwedge^{-\bullet} T_X$. Now, combining the Propositions 3.1.4 and 3.1.2, we get the following description of the cohomology of the complex $\big(\bigwedge^{-\bullet} T_S, d_w\big)$.

Proposition 3.1.5. *Let R be a finitely generated smooth algebra over $\mathbb{C}$, and set $X := \operatorname{Spec}(R)$ and $S := \operatorname{Spec}(R[t])$. For any regular function $w \in \mathcal{O}(S)$ of the form*

$$w = r_0 + r_2 t^2 + \sum_{k \geq 3} r_k t^k \in R[t]$$

with $r_2 \in R$ invertible, the morphism of complexes $\big(\bigwedge^{-\bullet} T_S, d_w\big) \to \big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$ given by

$$\sum_{n \geq 0} (\alpha_n, \beta_n) t^n \mapsto \alpha_0$$

is a quasiisomorphism.

Remark 3.1.6. Note that our proof works without modification for formal neighborhoods of X, i.e., the previous proposition remains true if we replace $\operatorname{Spec}(R[t])$ with $\operatorname{Spec}(R[[t]])$.

Now we deal with the case of non-trivial line bundles. First we need to construct a morphism between complexes $\big(\bigwedge^{-\bullet} T_{\operatorname{tot}(E)}, d_w\big)$ and $\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$. The projection map $\pi\colon \operatorname{tot}(E) \to X$ induces, for every point $p \in \operatorname{tot}(E)$, the pushforward map $d\pi_p\colon T_p \operatorname{tot}(E) \to T_p X$. We associate to a vector field $\xi \in T_{\operatorname{tot}(E)}(U)$ a vector field $\vartheta_\pi \xi \in T_X(U \cap X)$ by restricting it on $X \cap U$ and pushing it forward:

$$(\vartheta_\pi \xi)_x := (d\pi_x)(\xi_x), \qquad \text{for } x \in U \cap X$$

This induces a map of sheaves $\vartheta_\pi\colon T_{\operatorname{tot}(E)} \to i_* T_X$ where $i\colon X \hookrightarrow \operatorname{tot}(E)$ is the inclusion, and then the map of complexes $\vartheta_\pi : \big(\bigwedge^{-\bullet} T_{\operatorname{tot}(E)}, d_w\big) \to i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$.

Theorem 3.1.7. *Let X be a smooth variety and let E be a line bundle over X. Let $w \in \mathcal{O}(\mathrm{tot}(E))$ be a potential which satisfies the following assumptions:*

(1) the critical locus $\mathrm{Crit}(w) \subset \mathrm{tot}(E)$ is contained inside X

(2) the differential $dw|_X$ vanishes in the fiber directions F_x

(3) at every point $x \in X \cap \mathrm{Crit}(w)$, the Hessian of w restricts to a non-degenerate bilinear form on F_x

Then the morphism of complexes $\vartheta_\pi : \big(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\big) \to i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$ is a quasiisomorphism. Hence, if we denote by Y the critical locus of w considered as a closed subvariety, then the derived structures that Y carries as the critical locus of w, and as the critical locus of $w|_X$, coincide.

Proof. We only need to work locally to prove that the morphism of complexes of sheaves ϑ_π is a quasiisomorphism. Note that the cohomology sheaves of both $\big(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\big)$ and $i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$ are supported on Y. Therefore, we only need to check points $p \in \mathrm{tot}(E)$ which lie inside Y, and this is the situation where the Proposition 3.1.5 applies.

The two derived structures on Y are obtained by taking the inverse images of the complexes of sheaves $\big(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\big)$ and $\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$. Using the adjoint pair (i^{-1}, i_*) corresponding to the natural inclusion $i\colon Y \hookrightarrow \mathrm{tot}(E)$, we conclude that the derived structures are quasiisomorphic. $\square$

Remark 3.1.8. The inclusion map $i\colon X \hookrightarrow S$ induces a map of tangent spaces $T_x i\colon T_x X \to T_x S$ which in turn induces a map of complexes

$$i\colon \left(\bigwedge\nolimits^{-\bullet} T_X, d_{w|_X}\right) \to i^*\left(\bigwedge\nolimits^{-\bullet} T_S, d_w\right)$$

However, this map is not necessarily a quasiisomorphism. For example, assume that $S = \mathrm{Spec}(R)$ is an affine smooth scheme and let $X = \mathrm{Spec}(R/(f))$ be a smooth hypersurface in S with a free normal bundle. Since both S and X are smooth and affine, the restriction of the tangent bundle of S to X splits. Choose a splitting, and then choose a vector field ∇f along X transverse to X, normalized so that $(df)|_X(\nabla f) = 1$. Then the complex $i^*\left(\bigwedge^{-\bullet} T_S, d_w\right) = R/(f) \otimes_R \left(\bigwedge^{-\bullet} T_S, d_w\right)$ is the cone of the morphism

$$\left(\bigwedge\nolimits^{-\bullet} T_X, d_{w|_X}\right) \to \left(\bigwedge\nolimits^{-\bullet} T_X, d_{w|_X}\right) \tag{3.1.1}$$

which acts as multiplication by the function $\frac{\partial w}{\partial n} = (dw)|_X(\nabla f)$ in each degree. Now assume that the map $i\colon \left(\bigwedge^{-\bullet} T_X, d_{w|_X}\right) \to i^*\left(\bigwedge^{-\bullet} T_S, d_w\right)$ is a quasiisomorphism. This map corresponds to the inclusion of $\left(\bigwedge^{-\bullet} T_X, d_{w|_X}\right)$ into the cone of (3.1.1). But then the long exact sequence of the triangle corresponding to (3.1.1) implies that all the cohomology groups of $\left(\bigwedge^{-\bullet} T_X, d_{w|_X}\right)$ vanish, i.e., that $\mathbb{R}\mathrm{Crit}(w|_X)$ is empty.

3.2 Rank n bundles

As in the previous section, we begin by analyzing the local picture. In this case, however, we will deal with formal neighborhoods. Let X be a smooth variety, and choose a point $x \in X$. Denote by R the local ring $\mathcal{O}_{X,x}$ and by $\widehat{R}$ the formal neighborhood $\widehat{\mathcal{O}_{X,x}}$ of the point x in X. The formal neighborhood of $(x,0)$ in the total space $S := \mathrm{tot}(E)$ of the trivial rank n bundle E over X corresponds to the ring $\widehat{R}[[t_1,\ldots,t_n]]$. Assume that the potential $w \in \mathcal{O}(S)$ can be expressed in this formal neighborhood as

$$w = r_0 + \sum_{i,j=1}^{n} r_{ij}(t_1,\ldots,t_n)t_it_j \in \widehat{R}[[t_1,\ldots,t_n]]$$

where $r_{ij} = r_{ji}$ and the matrix $(r_{ij}(0))$ is invertible. We would like to change the coordinates around $(x,0) \in S$ so that the quadratic part of w has a simpler form. In order to do this, we just follow the usual diagonalization procedure for quadratic forms as in the proof of Morse–Bott lemma. Suppose by induction that there exist coordinates $u_1,\ldots,u_n$ such that

$$w = r_0 + u_1^2 + \cdots + u_{k-1}^2 + \sum_{i,j\geq k}^{n} r'_{ij}(u_1,\ldots,u_n)u_iu_j$$

After a linear change in the last $n-k+1$ coordinates, we may assume $r'_{kk}(x,0) \neq 0$. Now introduce new coordinates $v_1,\ldots,v_n$ by setting $v_i := u_i$ for $i \neq k$ and

$$v_k := \sqrt{r'_{kk}(u_1,\ldots,u_n)}\left(u_k + \sum_{i>k} 2u_i \frac{r'_{ik}(u_1,\ldots,u_n)}{r'_{kk}(u_1,\ldots,u_n)}\right)$$

for some choice of the square root of $r'_{kk}(t_1, \ldots, t_n)$. In the new coordinates w has the form

$$w = r_0 + v_1^2 + \cdots + v_k^2 + \sum_{i,j>k}^{n} r''_{ij}(v_1, \ldots, v_n) v_i v_j$$

Therefore, after a suitable change of coordinates, our potential will look like

$$w = r_0 + t_1^2 + \cdots + t_n^2$$

The projection map $\vartheta_\pi : T_{\mathrm{tot}(E)} \to i_* T_X$ defined on page 27, induces a map of stalks on the formal neighborhood of $(x, 0)$ in S:

$$\widehat{T_{\mathrm{tot}(E)}} \to \widehat{i_* T_X}$$

and then a map of complexes

$$\widehat{\vartheta_\pi} \colon \left(\bigwedge\nolimits^{-\bullet} \widehat{T_{\mathrm{tot}(E)}}, d_w \right) \to i_* \left(\bigwedge\nolimits^{-\bullet} \widehat{T_X}, d_{w|X} \right)$$

We would like to show that this map is a quasiisomorphism. Transformation of coordinates described above only changes the coordinates which point in directions normal to X. Therefore, when transferred into the new coordinate system, the projection map $\widehat{\vartheta_\pi}$ will still act as the projection in the new coordinates. This means that we have reduced the proof of the fact that $\widehat{\vartheta_\pi}$ is a quasiisomorphism to the case when the potential on S has the form $w = r_0 + t_1^2 + \cdots + t_n^2$. This special case follows by induction on n using the Proposition 3.1.5.

Now that the local case has been addressed, we can immediately generalize Theorem 3.1.7.

Theorem 3.2.1. *Let X be a smooth variety and let E be a vector bundle over X. Let $w \in \mathcal{O}(\mathrm{tot}(E))$ be a potential which satisfies the following assumptions:*

(1) the critical locus $\mathrm{Crit}(w) \subset \mathrm{tot}(E)$ is contained inside X

(2) the differential $dw|_X$ vanishes in the fiber directions F_x

(3) at every point $x \in X \cap \mathrm{Crit}(w)$, the Hessian of w restricts to a non-degenerate bilinear form on F_x

Then the morphism of complexes $\vartheta_\pi \colon \left(\bigwedge^{-\bullet} T_{\mathrm{tot}(E)}, d_w\right) \to i_*\left(\bigwedge^{-\bullet} T_X, d_{w|_X}\right)$ is a quasiisomorphism. Hence, if we denote by Y the critical locus of w considered as a closed subvariety, then the derived structures that Y carries as the critical locus of w, and as the critical locus of $w|_X$, coincide.

Shifted symplectic structures

We conclude this chapter by comparing the two natural (-1)-shifted symplectic structures on the critical locus Y of the potential $w \in \mathcal{O}(\mathrm{tot}(E))$ which Y carries as the critical locus of both w and $w|_X$.

Recall from section 2.2 that the pair consisting of $dw \in T_{\mathrm{tot}(E)} \otimes \Omega^2_{\mathrm{tot}(E)}$ and $\mathrm{id}_{\Omega^1_{\mathrm{tot}(E)}} \in T_{\mathrm{tot}(E)} \otimes \Omega^1_{\mathrm{tot}(E)}$ locally represents the canonical (-1)-shifted symplectic form on the derived critical locus $\mathbb{R}\mathrm{Crit}(w)$. The canonical (-1)-shifted symplectic form on $\mathbb{R}\mathrm{Crit}(w|_X)$ is locally represented by $d(w|_X) \in T_X \otimes \Omega^2_X$ and $\mathrm{id}_{\Omega^1_X} \in T_X \otimes \Omega^1_X$. Since the differential $dw|_X$ vanishes in the fiber directions F_x, the pullback of $d(w|_X)$ by $\pi \colon \mathrm{tot}(E) \to X$ will equal to $dw|_X$ in a neighborhood of X. As for the identity maps, we only need to ensure that our choices of flat connections of Ω^1_X and $\Omega^1_{\mathrm{tot}(E)}$ in a neighborhood of $x \in X$ are compatible. Hence, we have the following:

Corollary 3.2.2. *Let X be a smooth variety and let E be a vector bundle over X. Let $w \in \mathcal{O}(\mathrm{tot}(E))$ be a potential which satisfies the assumptions (1), (2), and (3) from the previous theorem. Then the pullback of the natural (-1)-shifted symplectic form on the derived critical locus $\mathbb{R}\mathrm{Crit}(w|_X)$ by the quasiisomorphism ϑ_π equals the natural (-1)-shifted symplectic form on $\mathbb{R}\mathrm{Crit}(w)$. Furthermore, the canonical key closing the form on $\mathbb{R}\mathrm{Crit}(w|_X)$ gets mapped to the canonical key closing the form on $\mathbb{R}\mathrm{Crit}(w)$.*

Chapter 4

More general cases

Let S be a smooth algebraic variety over $\mathbb{C}$. A critical point of a regular function $w \in \mathcal{O}(S)$ is a point $s \in S$ at which dw vanishes. Critical points of w form a closed subvariety of S called the critical locus of w and denoted by $\mathrm{Crit}(w)$.

Let X be a smooth closed subvariety of S. If the closed subvarieties $\mathrm{Crit}(w) \subset S$ and $\mathrm{Crit}(w|_X) \subset X$ happen to coincide, we can pose the question whether their natural derived structures, which they acquire as critical loci of $w \in \mathcal{O}(S)$ and $w|_X \in \mathcal{O}(X)$, respectively, coincide. Without some assumptions on the relationship between X and w the two derived structures will differ.

In the previous chapter we considered the case when S was the total space of a vector bundle on X. In this chapter, we generalize the results to Kashiwara's quantized cycles, and then investigate some ways to deal with more general closed embeddings $X \hookrightarrow S$. We also comment on a possible generalization to critical loci of shifted potentials.

4.1 Quantized cycles

Let S be a smooth algebraic variety over $\mathbb{C}$ and X a smooth closed subvariety of S. We will assume that the conormal sequence corresponding to the embedding $X \subset S$ splits, so that the map from X to its first infinitesimal neighborhood in S admits a global retraction. The data of a subvariety $X \subset S$ together with a choice of global retraction σ is called a quantized cycle (see [11]).

Now assume that we have a potential $w \in \mathcal{O}(S)$ and a quantized cycle (X, σ). For every point $x \in X$ we have the projection map $\sigma_x : T_xS \to T_xX$ induced by the retraction of the quantized cycle. We can associate to a vector field $\xi \in T_S(U)$ a vector field $\vartheta_\sigma\xi \in T_X(U \cap X)$ by restricting it on $X \cap U$ and projecting:

$$(\vartheta_\sigma\xi)_x := \sigma_x(\xi_x), \qquad \text{for } x \in U \cap X$$

This induces a map of sheaves $\vartheta_\sigma \colon T_S \to i_*T_X$ where $i\colon X \hookrightarrow S$ is the inclusion, and then the map of complexes $\vartheta_\sigma : \big(\bigwedge^{-\bullet} T_S, d_w\big) \to i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$.

Now that we have constructed a map between the complexes which represent the algebras of functions on $\mathbb{R}\mathrm{Crit}(w)$ and $\mathbb{R}\mathrm{Crit}(w|_X)$, we need to impose conditions which would guarantee that this map is a quasiisomorphism. To do so we introduce the following notation. For every $x \in X$, we define the subspace $N_x \subset T_xS$ of directions normal to X as the kernel of the retraction σ:

$$N_x := \ker\Big(T_xS \xrightarrow{\sigma_x} T_xX\Big)$$

Now we have the following result.

Theorem 4.1.1. *Let S be a smooth variety and let (X, σ) be a quantized cycle consisting of a smooth closed subvariety $X \subset S$ and a retraction σ of the map from X to its first infinitesimal neighborhood in S. Let $w \in \mathcal{O}(S)$ be a potential which satisfies the following assumptions:*

(1) *the critical locus $\mathrm{Crit}(w) \subset S$ is contained inside X*

(2) *the differential $dw|_X$ vanishes in the normal directions N_x*

(3) *at every point $x \in X \cap \mathrm{Crit}(w)$, the Hessian of w restricts to a non-degenerate bilinear form on N_x*

Then the morphism $\vartheta_\sigma : \big(\bigwedge^{-\bullet} T_S, d_w\big) \to i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$ is a quasiisomorphism. Therefore, if we denote by Y the critical locus of w considered as a closed subvariety, then the derived structures that Y carries as the critical locus of w, and as the critical locus of $w|_X$, coincide.

Proof. The proof is similar to the one for the vector bundles. Since the cohomology sheaves of both $\big(\bigwedge^{-\bullet} T_S, d_w\big)$ and $i_*\big(\bigwedge^{-\bullet} T_X, d_{w|_X}\big)$ are supported on Y, we only need to work in neighborhoods of points $y \in Y$. After choosing a suitable coordinates for the formal neighborhood of y in S, we can apply the generalization of the Proposition 3.1.5 laid out in section 3.2, taking into consideration Remark 3.1.6. $\square$

We can compare the two (-1)-shifted symplectic structures on the critical locus Y in the same way as we did in the previous chapter.

Corollary 4.1.2. *Let S be a smooth variety and let (X, σ) be a quantized cycle consisting of a smooth closed subvariety $X \subset S$ and a retraction σ of the map from X to its first infinitesimal neighborhood in S. Let $w \in \mathcal{O}(S)$ be a potential satisfying the assumptions (1), (2), and (3) from the previous theorem. Then the pullback of the natural (-1)-shifted symplectic form on $\mathbb{R}\mathrm{Crit}(w|_X)$ by the quasiisomorphism ϑ_σ equals the natural (-1)-shifted symplectic form on $\mathbb{R}\mathrm{Crit}(w)$. Furthermore, the canonical key closing the form on $\mathbb{R}\mathrm{Crit}(w|_X)$ gets mapped to the canonical key closing the form on $\mathbb{R}\mathrm{Crit}(w)$.*

4.2 Closed embeddings

In this section we comment on the case of a general closed embeddings $X \hookrightarrow S$. One of the issues we encounter is how to impose a global condition on the relationship between the subvariety X and the potential $w \in \mathcal{O}(S)$ which would imply that $\mathbb{R}\mathrm{Crit}(w)$ and $\mathbb{R}\mathrm{Crit}(w|_X)$ have the same derived structure. This condition should revolve around the non-degeneracy of the Hessian of the function w in the directions normal to X. One way to express this is to require that the critical loci $\mathrm{Crit}(w)$ and $\mathrm{Crit}(w|_X)$ coincide not only as closed subvarieties, but as closed subschemes of S. Another issue is that without some extra structure, for example, the retraction in the case of quantum cycle, it becomes necessary to construct a morphism relating the derived structures on $\mathbb{R}\mathrm{Crit}(w)$ and $\mathbb{R}\mathrm{Crit}(w|_X)$ by some gluing procedure, perhaps using the ideas from [21] or [11].

Let S be a smooth variety and $X \subset S$ a smooth subvariety. We now assume that the potential $w \in \mathcal{O}(S)$ has the property that the critical loci $\mathrm{Crit}(w) \subset S$ and $\mathrm{Crit}(w|_X) \subset X \subset S$ coincide not only as closed subvarieties of S but also as closed subschemes of S. This assumptions implies that, for every closed point $x \in \mathrm{Crit}(w)$, it is possible to find coordinates in the formal neighborhood of $x \in S$ such that the potential w has the form

$$w = r + \sum_{|\alpha| \geq 2} r_\alpha t^\alpha \in R[[t_1, \dots, t_n]]$$

where R corresponds to a neighborhood of the point x in X, and $t_1, \dots, t_n$ describe directions normal to X in S (see [13, Lemma 1.14]). Under this choice of coordinates we have a retraction σ_x which we can use to impose the non-degeneracy condition on the Hessian of w at x, as in the case of quantized cycles. However, the assumption that $\mathrm{Crit}(w)$ and $\mathrm{Crit}(w|_X)$ coincide as closed subschemes automatically implies the non-degeneracy condition on $(r_{ij}(0))$. Hence, we can simply proceed to conclude, as in Theorem 4.1.1, that the morphism of complexes

$$\widehat{\vartheta_\sigma} \colon \left(\bigwedge\nolimits^{-\bullet} \widehat{T_S}, d_w \right) \to i_* \left(\bigwedge\nolimits^{-\bullet} \widehat{T_X}, d_{w|_X} \right)$$

constructed from σ_x is a quasiisomorphism.

One way to deal with the issue of gluing these quasiisomorphisms is to consider the formal neighborhood of X in S as a deformation of the formal neighborhood of X in the total space of the normal bundle of X in S. This is controlled by the Hochschild complex and the corresponding terms in the Hochschild complex can be used to modify the potential w to a non-commutative function for which the compatibility will hold on the nose. Additionally, it is likely that the arguments in [21] applied to appropriate Dolbeault cocycles will give a setup for which the quantized cycle proof will work directly.

4.3 Shifted potentials

In this section we briefly comment on the situation when the potential determining the critical locus is not an ordinary regular function, but is a shifted function. This generalization is a necessary one from the perspective of the Batalin–Vilkovisky formalism.

Let X be a smooth variety. An n-shifted function w is a section of the shifted structure sheaf $\mathcal{O}_X[n]$ of the variety X. We can interpret it as a map $w : X \to \mathbb{A}^1[n]$ or as a cohomology class $w \in H^n(X, \mathcal{O}_X)$. The differential dw of the n-shifted potential w is a section of the shifted cotangent bundle $\Omega^1_X[n]$ and the derived structure of the critical locus of w is given by the complex $\left(\bigwedge^{-\bullet} T_X[-n], d_w\right)$ where the differential is induced by the contraction with dw.

The space $\mathrm{tot}(\Omega^1_X[n])$ is defined as $\mathbb{R}\mathrm{Spec}_X(\mathrm{Sym}(T_X[-n]))$, hence testing it with $\mathbb{R}\mathrm{Spec}(A) \xrightarrow{x} X$ yields

$$\begin{aligned}\mathrm{Map}_{A-\mathrm{mod}}(A, x^*\Omega^1_X[n]) &\cong \mathrm{Map}_{A-\mathrm{mod}}(x^*T_X[-n], A)\\ &\cong \mathrm{Map}_{A-\mathrm{alg}}(\mathrm{Sym}_{\mathrm{A}}(x^*T_X[-n]), A)\end{aligned}$$

It is shown in [10], in a more general setting when X is an Artin stack, that the space $\mathrm{tot}(\Omega^1_X[n])$ is n-shifted symplectic. Theorem 2.9 in [18] then implies that $\mathbb{R}\mathrm{Crit}(w)$ carries a natural $(n-1)$-shifted symplectic structure.

Remark. If $w \in H^n(X, \mathcal{O}_X)$ is a shifted potential which is in the image of $H^n(X, \mathbb{C})$, then the derived critical locus $\mathbb{R}\mathrm{Crit}(w)$ is the $(n-1)$-shifted cotangent stack with its canonical symplectic structure. The case $n = 0$ corresponds to a regular function $w \in \mathcal{O}(X)$ which is locally constant, so that $dw = 0$. If X is smooth and projective over $\mathbb{C}$, then the Hodge theorem implies that the map $H^n(X, \mathbb{C}) \to H^n(X, \mathcal{O}_X)$ is surjective for all n and therefore the derived critical locus of any shifted function on X is the corresponding shifted cotangent bundle.

We expect that the derived and shifted symplectic structures on critical loci of shifted potentials are, under suitable assumptions, compatible with the restriction of the potential $H^n(S, \mathcal{O}_S) \to H^n(X, \mathcal{O}_X)$ induced by the inclusion $X \hookrightarrow S$.

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