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Some \(l^p\)-Improving Bounds For Radon-Like Transforms

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ABSTRACT

SOME L^p-IMPROVING BOUNDS FOR RADON-LIKE TRANSFORMS

Dominick Villano

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We prove $L^p - L^q$ boundedness for a wide class of Radon-like transforms. The technique of proof leverages the existing one-dimensional theory to produce a non-trivial bounds in any dimension. For certain combinatorially simple transforms, this range is sharp up to endpoints. Additionally, we make observations connecting the L^p-improving properties of a Radon-like transform to the zero set of certain homogeneous polynomials.

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Chapter 1

Introduction

1.1 Background

Interesting Lesbesgue space mapping properties of the classical Radon transform can be traced back to the work of Oberlin and Stein in [OS82]. For a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, an element of the unit sphere $u \in S^{d-1}$, and a real number t, define the Radon transform of f as follows

$$R_d f\,(u,t) := \int_{x\cdot u=t} f\,(x)\, d\sigma_1\,(x)\,.$$

Here $d\sigma$ is the induced $d-1$ dimensional Lebesgue measure on each hyperplane.

It is is a simple consequence of Fubini's theorem that R_d is bounded from $L^1\,(\mathbb{R})$ to $L^1\left(S^{d-1}\times\mathbb{R}\right)$. What Oberlin and Stein observed is that R_d is in fact bounded from $L^{(d+1)/d}\left(\mathbb{R}^d\right)$ to $L^{d+1}\left(S^{d-1}\times\mathbb{R}\right)$. This can be interpreted as a statement that locally integrability becomes less wild after averaging in a suitable manner.

In fact, the local mapping properties of geometric averaging operators predates the work of Oberlin and Stein. Interest in the problem dates at least as far back as the 1970's to work of Littman [Lit73] and Strichartz [Str70]. They were interested in convolutions with singular measures supported on a hypersurface, in particular the sphere. Precisely, they show that if f is Borel measurable and $d\sigma_2$ is the surface measure on the $d-1$ dimensional unit sphere, then the operator $T_{S,d}$ defined via the formula

$$T_{S,d}f\left(x\right) := \int_{S^{d-1}} f\left(x-y\right) d\sigma\left(y\right)$$

is bounded from $L^p\left(\mathbb{R}^d\right)$ to $L^q\left(\mathbb{R}^d\right)$ if and only if $\left(\frac{1}{p},\frac{1}{q}\right)$ lies in the closed triangle with vertices $\left(0,0\right),\left(1,1\right),\left(\frac{d}{d+1},\frac{1}{d+1}\right)$

As in the case of the classical Radon transform, curvature plays a key role in the local L^p-improving properties of the operator. The notion of rotational curvature introduced by Phong and Stein in [PS86a, PS86b, PS91] unifies the classical Radon transform and the spherical averaging operator. They show that, for hypersurfaces, the best possible mapping properties of a geometric averaging operator are related to the nonvanishing of a Monge-Ampere determinant. Specifically, if $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and

$$\det \begin{bmatrix} 0 & \nabla_x \Phi \\ \left(\nabla_y \Phi\right)^T & \frac{\partial^2 \Phi}{\partial_{x_i} \partial_{y_j}} \end{bmatrix} \neq 0,$$

then the operator

$$T_\Phi f\left(x\right) := \int_{\Phi(x,y)=0} f\left(y\right) a\left(a\right) d\sigma_x\left(y\right)$$

is bounded from $L^2(\mathbb{R}^d)$ to $L^2_{(d-1)/2}(\mathbb{R}^d)$. This gain in Sobolev regularity then allows (for instance), the argument of Strichartz to be applied and recover exactly the same L^p mapping region as the spherical averaging operator. Although rotational curvature fills the role of a necessary and sufficient condition for the best possible L^p improving properties of a geometric averaging operator, no such condition correctly exists in less favorable conditions, even for hypersurfaces.

Eventually, a more general but less quantitative result was obtained in landmark work of Chirst, Nagel, Stein, and Wainger [CNSW99]. They introduce vector field techniques which demand a rephrasing of the problem in order to state. This is the formulation that will be used for the remainder of the thesis.

Let $U \subset \mathbb{R}^d$ be a small neighborhood of the origin. Suppose $\pi_1 : U \to \mathbb{R}^{d-k_1}$ and $\pi_2 : U \to \mathbb{R}^{d-k_2}$ are smooth submersions. The projections π_i generate a Radon-like transform R, which is defined via duality

$$\int_{\mathbb{R}^{d-k_2}} Rf(y)\, g(y)\, dy := \int_U f(\pi_1(x))\, g(\pi_2(x))\, a(x)\, dx. \qquad (1.1.1)$$

Here a is a smooth cutoff function supported in U. Note that this operator is now entirely local, and so is bounded from L^p to L^q for $1 \leq q \leq p$ by Holder's inequality. Additionally, since the L^p spaces are nested on a compact set, any nontrivial bound corresponds to a gain in integrability, earning the title L^p-improving.

What Chirst, Nagel, Stein, and Wainger demonstrated is that L^p-improving estimates exist if and only if the distributions $\ker D\pi_1$ and $\ker D\pi_2$ generate the entire tangent space under the Lie Bracket. In the case of hypersuface averages, we have the numerology $k_1 = k_2$ and $d = 2k_1 - 1$. Additionally, $\ker D\pi_1$ and $\ker D\pi_2$ intersect trivially and rotational curvature translates as follows: for any smooth nonzero vector field X defined on the support of a with $X(x) \in \ker D\pi_1(x)$, there is a smooth vector field $Y \in \ker D\pi_2$ such that $\ker D\pi_1$, $\ker D\pi_2$ and $[X, Y]$ span the tangent space.

As the higher dimensional theory was developing, a breakthrough for the case of averaging over curves occurred in Christ's 1998 paper "Convolution, Curvature, and Combinatorics: a case study" [Chr98]. The paper obtains essentially optimal restricted weak type bounds for the operator given by

$$T_\gamma f(x) := \int_{-1}^{1} f\left(x - \left(t, t^2, \ldots, t^{d-1}, t^d\right)\right) dt.$$

Chirst's main innovation is to introduce a singular coordinate system by iterating the above map. The singularities of this system are then carefully avoided using what has now become know as the method of refinements.

The power of this philosophy can be seen in the work of Tao and Wright [TW03], who synthesized the vector field techniques of Christ, Nagel, Stein, and Wainger and iteration techniques of Christ to obtain optimal-up-to-endpoint restricted weak type bounds of averages over any smooth family of curves. Of particular note is their notion of a set of central width, which allows the method of refinements to be executed even without explicit knowledge of the coordinate system's singularities. This concept has no obvious higher-dimensional analogue. Another important innovation is their theory of two-parameter Carnot-Carathéodory balls, which generalizes the one-parameter work of Nagel, Stein, and Wainger [NSW85].

Other signifigant and interesting results include the work of Bak [Bak00], Choi [Cho11, Cho03], Christ, Dendrinos, Stovall, and Street [CDSS17], Dendrinos, Laghi, and Wright [DLW09], Drury and Guo [DG91], Erdoğan and Oberlin [EO10], Gressman [Gre09, Gre15], Lee [Lee04], Secco [Sec99]. Sharp bounds were obtained by Iosevich and Sawyer [IS96] in the case that the geometry is given by homogeneous polynomials. This thesis provides an alternate proof Seeger's L^p-mapping theorem in [See98]. Finally, in [Gre16], Gressman provides a curvature conditions which fills the role of rotational curvature in certain situations: they ensure the largest possible range of boundedness up to endpoints.

Of particular signifigance to the present thesis is the work of Street [Str11, Str14] and Stovall [Sto11]. In [Str11, Str14], Streeet generalizes and refines the theory of two paramater balls to multiparameter balls. In [Sto11], Stovall generalizes the results of [TW03] to the multilinear setting.

The rotational curvature assumption of Phong and Stein, the one-dimensional work of Tao and Wright, and the generalized curvature conditions of Gressman in [Gre16] share two common features. First, up to endpoints, they produce the sharp range of $L^p - L^q$ boundedness. Second, they are all invariant under choice of vector field, meaning that relevant spanning sets remain spanning after picking a different basis of vector fields. This need not be the case in general.

The purpose of this thesis is to prove some bounds when such invariance is not present. The strategy is to try to understand higher dimensional smooth distributions as the ensemble of all the smooth curves they contain. This is fairly unwieldy and so some particularly useful families of vector fields are identified and used to prove necessary and sufficient conditions. Sometimes these conditions align, producing bounds that are sharp up to endpoints.

1.2 Definitions and Main Results

We will deal exclusively with Banach space exponents; all p_i mentioned will satisfy $1 \leq p_i \leq \infty$.

We will introduce all Radon-like transforms via the double fibration formulation (1.1.1). So, let $d \geq 4$ and let $U \subset \mathbb{R}^d$ be a small neighborhood of the origin. Suppose $\pi_1 : U \to \mathbb{R}^{d-k_1}$ and $\pi_2 : U \to \mathbb{R}^{d-k_2}$ are smooth submersions and that $\ker(D\pi_1) \cap \ker(D\pi_2)$ is trivial. Without loss of generality $\pi_i(0) = 0$. For notational ease, let $k_3 = d - k_1 - k_2$ and define

$$\Delta_i := \ker(D\pi_i).$$

The projections π_i generate a Radon-like transform R, which is defined via duality

$$\int_{\mathbb{R}^{d-k_2}} Rf(y)\, g(y)\, dy := \int_U f(\pi_1(x))\, g(\pi_2(x))\, a(x)\, dx.$$

Here a is a smooth cutoff function supported in U.

If there is a positive constant C such that for all measurable functions f_1, f_2, and all smooth cutoff functions a with sufficiently small support, we have the inequality

$$\int_U f(\pi_1(x))\, g(\pi_2(x))\, a(x)\, dx \le C \|f\|_{L^{p_1}\left(\mathbb{R}^{d-k_1}\right)} \|g\|_{L^{p_2}\left(\mathbb{R}^{d-k_2}\right)}, \tag{1.2.1}$$

we say R is of strong type (p_1, p_2'), where $1/p + 1/p' = 1$.

If there is a positive constant C' such that for all measurable subset $E \subset \mathbb{R}^{d-k_1}$, $F \subset \mathbb{R}^{d-k_2}$, and all smooth cutoff functions a with sufficiently small support, we have the inequality

$$\int_U \chi_E(\pi_1(x))\, \chi_F(\pi_2(x))\, a(x)\, dx \le C' |E|^{1/p_1} |F|^{1/p_2}, \tag{1.2.2}$$

then we say R is of restricted weak type (p_1, p_2'). By real interpolation, a proof of which may also be found [Gra14], if an operator T is restricted weak type $p_{1,1}, p_{2,1}$ and $p_{1,2}, p_{2,2}$, then T is strong-type $p_{1,\theta}, p_{2,\theta}$ where $0 < \theta < 1$ and

$$p_{i,\theta}^{-1} := \theta p_{i,1}^{-1} + (1-\theta)\, p_{i,2}^{-1}.$$

The constant implicit in the symbol $\lesssim$ can depend only on the p_i and π_i.

In order to state the main results of this thesis, a special class of vector fields associate to Δ_1 and Δ_2 must be identified.

Definition 1.2.1. A *basis* of (Δ_1, Δ_2) is a (labeled) collection $B := \{X_1, \dots, X_{k_1+k_2}\}$ of smooth vector fields defined on U that is point-wise linearly independent and satisfies $X_i \in \Delta_1$ for $i \le k_1$, $X_i \in \Delta_2$ for $i > k_1$.

We will be interested in spanning conditions of certain subsets of bases, and so need to introduce a bookkeeping system. The following definitions are slight variations of terminology appearing in [TW03] and [Sto11].

Definition 1.2.2. If B is a basis of (Δ_1, Δ_2) and $S \subseteq B$, let Δ_S be the distribution spanned by the Lie algebra generated by S. If $\dim \Delta_S / (\Delta_1 \oplus \Delta_2) = k_3$, S is called *spanning*.

A word associated to S is a j-tuple $w \in \{1, \ldots, |S|\}^j$ for some $j \geq 1$. The degree of a word is the ordered pair $\deg w := (\deg w_1, \deg w_2)$ where $\deg w_i$ counts the number of entries of w that belong to Δ_i. If I is any finite collection of words,

$$\deg I := \sum_{w \in I} \deg w.$$

Finally, to each word we assign a vector field, denoted X_w, defined via the recursive formula:

$$X_{(w,j)} := [X_w, X_j].$$

Let $W(S)$ denote the set of all words associated to S.

Bases that come equipped with a third, complementary distribution will be important to the theory.

Definition 1.2.3. A basis B of (Δ_1, Δ_2) is called *direct* if $[X_i, X_j] = 0$ whenever $1 \leq i, j \leq k_1$ or $k_1 + 1 \leq i, j \leq k_1 + k_2$ and there exists a k_3-dimensional distribution Δ_3 with the following properties

1. Both $\Delta_1 \cap \Delta_3$ and $\Delta_2 \cap \Delta_3$ are trivial.

2. If $w \in W(B)$ has length ≥ 2, $X_w \in \Delta_3$.

If $S \subseteq B$ for some direct basis B, and S is spanning, write $S \prec (\Delta_1, \Delta_2)$. For such S, if I is a finite collection of elements of $W(S)$, I spans S if $\{X_w(0) \mid w \in I\}$ spans

$\Delta_S(0)$. The mapping polytope of S, $P(S) \subset \mathbb{R}^+ \times \mathbb{R}^+$ is the interior of the convex hull of the set

$$\{x \mid \text{ There exists some } I \text{ that spans } S \text{ with } \deg I \leq x\}.$$

Here the inequality is taken coordinate-wise. Finally, the theorem:

Theorem 1.2.4. *Suppose $p_1 \geq 1$, $p_2 \geq 1$, and $p_1^{-1} + p_2^{-1} > 1$. Define*

$$(c_1, c_2) := \left(\frac{p_2}{p_1 + p_2 - p_1 p_2}, \frac{p_1}{p_1 + p_2 - p_1 p_2} \right).$$

Let $P(\Delta_1, \Delta_2)$ be the interior of the convex hull of the union

$$\bigcup_{S \prec (\Delta_1, \Delta_2)} P(S).$$

If

$$(c_1, c_2) \in P(\Delta_1, \Delta_2),$$

then R is of strong type (p_1, p_2'). In certain cases, this region is sharp up to end-points, meaning that if the distance between $P(\Delta_1, \Delta_2)$ and (b_1, b_2) is positive, then R is not of strong type (p_1, p_2').

The proof will proceed by proving the restricted weak type bound on the interior of $P(\Delta_1, \Delta_2)$. By real interpolation, the strong-type bound follows. There is a corresponding necessary condition, which is used to prove sharpness, when possible. This necessary condition involves a class of bases more general than direct bases. Roughly speaking, theses will be bases for which minimal spanning sets correspond to submanifolds. We delay the precise statements and proof.

Chapter 2

Direct Bases and Sufficiency

2.1 Existence of Direct Bases

It is not immediately clear from the definition that direct bases always exist. However, they do.

Lemma 2.1.1. *Direct bases exist.*

Proof. In a suitably small neighborhood of the origin, let $Y_1, \ldots, Y_d$ be pairwise commuting, linearly independent vector fields such that $Y_1, \ldots, Y_{k_1}$ span Δ_1. Let $Z_1, \ldots, Z_{k_2}$ be an arbitrary basis of Δ_2. Write $Z_i = \Sigma a_{i,j} Y_j$, where $a_{i,j}$ are smooth functions. Since Δ_1 and Δ_2 span a $k_1 + k_2$ dimensional subspace at the origin, after relabeling the $Y_j's$ if necessary, we may assume that the k_2 rightmost columns of the following matrix span a k_2 dimensional subspace

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k_1+k_3} & a_{1,k_1+k_3+1} & \dots & a_{1,d} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k_1+k_3} & a_{2,k_1+k_3+1} & \dots & a_{2,d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k_2,1} & a_{d,2} & \dots & a_{k_2,k_1+k_3} & a_{k_2,k_1+k_3+1} & \dots & a_{k_2,d} \end{bmatrix}.$$

Restricting to a possibly smaller neighborhood to avoid the zero set of certain functions and again relabeling the Y_j's if necessary, A may be row reduced to obtain

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \dots & \tilde{a}_{1,k_1+k_3} & 1 & 0 & \dots & 0 \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & \dots & \tilde{a}_{2,k_1+k_3} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{k_2,1} & \tilde{a}_{d,2} & \dots & \tilde{a}_{k_2,k_1+k_3} & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Define

$$\tilde{Z}_i := \Sigma \tilde{a}_{i,j} Y_j + Y_{k_1+k_3+i}.$$

Since each $\tilde{Z}_i$ is a linear combination of $Z_1, \dots, Z_{k_2}$ (with smooth functions for coefficients), $\left\{\tilde{Z}_1, \dots, \tilde{Z}_{k_2}\right\}$ spans Δ_2. Further, any nonzero vector field in Δ_2 must have nonzero Y_j coefficient for some $j \geq k_1+k_3+1$. Since Δ_2 is involutive $[\tilde{Z}_i, \tilde{Z}_\ell] = 0$ for all $1 \leq i, \ell \leq k_2$.

Finally, there is a a smooth $k_1 + k_3$ dimensional distribution Σ_1 that contains Δ_1 (namely the span of $Y_1, \dots, Y_{k_1+k_3}$) and has trivial intersection with Δ_2, such that for any $Y \in \Delta_1$ and any $1 \leq i \leq k_2$, $[Y, \tilde{Z}_i] \in \Sigma_1$.

Repeating this argument with the roles of Δ_1 and Δ_2 reversed yields a $k_2 + k_3$ dimension distribution Σ_2 containing Δ_2 such that $\Sigma_2 \cap \Delta_1$ is trivial and a pairwise-commuting collection of vector fields $\tilde{Y}_1, \dots, \tilde{Y}_{k_1}$ that span Δ_1 with the following property: for any $Z \in \Delta_2$ and any $1 \leq j \leq k_1$, $[Z, \tilde{Y}_j,] \in \Sigma_2$.

Set $B := \left\{\tilde{Y}_1, \dots, \tilde{Y}_{k_1}, \tilde{Z}_1, \dots, \tilde{Z}_{k_2}\right\}$. Then any $w \in W(B)$ of length ≥ 2 lies in $\Sigma_1 \cap \Sigma_2$, which is k_3 dimensional and has trivial intersection with both Δ_1 and Δ_2. $\square$

2.2 Trading Multilinearity for Dimensions

Before moving on to a proof of the Theorem 1.2.4, we first record a brief but useful method for translating multilinear, one-dimensional estimates to the higher dimensional bilinear setting. We prove a quantitative version the of Hörmander implies L^p-improving result of Christ, Nagel, Stein, and Wainger [CNSW99] as a consequence.

Suppose $V \subset \mathbb{R}^d$ is a suitably small neighborhood of the origin and that $\pi_1 : V \to \mathbb{R}^{d_1}$, $\pi_2 : V \to \mathbb{R}^{d_2}$ are smooth submersions with $\ker D\pi_1 \subset \ker D\pi_2$. Then for any measurable subset $\Omega \subset V$,

$$|\pi_1(\Omega)| \lesssim |\pi_2(\Omega)| \tag{2.2.1}$$

Here the implicit constant depends on V.

Proof. Since π_1 is a submersion, $\pi_1(\Omega)$ is open and bounded. There is a third smooth submersion $\pi_3 : \pi_1(\Omega) \to \mathbb{R}^{d_2}$ such that $\pi_3 \circ \pi_1 = \pi_2$. So

$$|\pi_1(\Omega)| = \int_{\pi_2(\Omega)} f(x)\, dx$$

where $f(x) \leq \sup_x \left|\pi_3^{-1}(x)\right| \lesssim 1$ $\square$

This simple fact links one-dimensional, multilinear estimates to bilinear, high dimensional estimates, which we state in the geometric, restricted weak-type formulation.

Lemma 2.2.1. *Suppose* $\{X_1, \dots, X_{k_1+k_2}\}$ *are smooth, non-vanishing vector fields on* U *satisfying* $X_i(x) \in \Delta_1(x)$ *for* $i \leq k_1$ *and* $X_i(x) \in \Delta_2(x)$ *for* $i > k_1$. *Let* π_{X_i} *be smooth submersions on* U *such that* X_i *spans* $\ker D\pi_{X_i}$. *If* $(b_1, \dots, b_{k_1+k_2})$ *are non-negative real numbers such that for all measurable subsets* $\Omega \subset U$

$$\prod_{i=1}^{k_1+k_2} \left(\frac{|\Omega|}{|\pi_{X_i}(\Omega)|}\right)^{b_i} \frac{1}{|\Omega|} \lesssim 1 \tag{2.2.2}$$

then

$$\left(\frac{|\Omega|}{|\pi_1(\Omega)|}\right)^{B_1} \left(\frac{|\Omega|}{|\pi_2(\Omega)|}\right)^{B_2} \frac{1}{|\Omega|} \lesssim 1$$

where $B_1 := \sum_{i=1}^{k_1} b_i$ *and* $B_2 := \sum_{i=k_1+1}^{k_1+k_2} b_i$.

Proof. Using (2.2.1), the proof is a simple calculation:

$$\begin{aligned}
&\left(\frac{|\Omega|}{|\pi_1(\Omega)|}\right)^{B_1} \left(\frac{|\Omega|}{|\pi_2(\Omega)|}\right)^{B_2} \frac{1}{|\Omega|} \\
&\quad = \prod_{i=1}^{k_1} \left(\frac{|\pi_{X_i}(\Omega)|}{|\pi_1(\Omega)|}\right)^{b_i} \prod_{i=k_1+1}^{k_1+k_2} \left(\frac{|\pi_{X_i}(\Omega)|}{|\pi_2(\Omega)|}\right)^{b_i} \prod_{i=1}^{k_1+k_2} \left(\frac{|\Omega|}{|\pi_{X_i}(\Omega)|}\right)^{b_i} \frac{1}{|\Omega|} \lesssim 1
\end{aligned}$$

$\square$

In [Sto11], Stovall classifies (up to endpoints), the tuples $(b_1, \dots, b_{k_1+k_2})$ for which (2.2.2) holds. We recall one piece of terminology from that paper, as it will be used more than once.

Definition 2.2.2. If $S := \{X_1, \ldots, X_{k_1+k_2}\}$ is a collection of smooth non-vanishing vector fields on U, not necessarily linearly independent, such that $X_i \in \Delta_1$ for $i \leq k_1$ and $X_i \in \Delta_2$ for $i > k_1$, write $S \leqq (\Delta_1, \Delta_2)$. For $w \in W(S)$, the *multilinear degree* of w is the $k_1 + k_2$-tuple

$$\deg_{ML} w := (\deg_{ML} w_1, \ldots, \deg_{ML} w_{k_1+k_2})$$

where $deg_{ML} w_i$ counts the number of occurrences of X_i in w. If $I \in (W(S))^d$,

$$\deg_{ML} I = \sum_{w \in I} \deg w.$$

We say I is spanning if $\{X_w(0) \mid w \in I\}$ is linearly independent. Then, the multilinear polytope of S, $P_{ML}(S) \subset \mathbb{R}^{k_1+k_2}$, is the the interior of the convex hull of the region

$$\{x \mid x \geq \deg_{ML} I \text{ for some spanning } I\}.$$

Corollary 2.2.3 (Quantitative Hömander implies L^p-improving)**.** *Suppose that*

$$(c_1, c_2) \in \bigcup_{S \leqq (\Delta_1, \Delta_2)} \bigcup_{(b_1, \ldots, b_n) \in P_{ML}(S)} \left(\sum_{i=1}^{k_1} b_i, \sum_{i=1}^{k_2} b_i \right).$$

Then R is of strong type (p_1, p_2').

Proof. Fix some S such that (c_1, c_2) lies in the above region. Let π_{X_i} be as in Lemma 2.2.1. Then (2.2.2) is satisfied by work of Stovall [Sto11], and the corollary follows by Lemma 2.2.1. □

In principle, it could be that Theorem 1.2.4 provides no new bounds beyond those provided by Corollary 2.2.3, which, evidently, has a very brief proof. However, we know of no such example at the moment. See the section concerning symmetry and tensor decomposition for a more through discussion.

2.3 Proof of Theorem 1.2.4

First, we use direct bases to show the existence of a particularly useful coordinate system.

Lemma 2.3.1. *Suppose $S \prec (\Delta_1, \Delta_2)$ and $S \subseteq B$ where B is a direct basis. Define $k := \dim \Delta_S$ and $\ell := d - k$. If $\ell \geq 1$, label the elements of $B \setminus S := \{X_1, \ldots, X_\ell\}$ so that for some $1 \leq r \leq \ell$, $1 \leq r$ implies $X_i \in \Delta_1$ and $i \geq r+1$ implies $X_i \in \Delta_2$. For $1 \leq i \leq \ell$, define $S_i := S \cup \{X_1, \ldots X_i\}$. Note $S_i \prec (\Delta_1, \Delta_2)$. Then there exists a coordinate system $(x_1, \ldots, x_d)$ defined on a neighborhood of the origin such that*

$$\Delta_S = \operatorname{span}\{\partial x_1, \ldots, \partial x_k\} \tag{2.3.1}$$

and

$$\Delta_{S_i} = \operatorname{span}\{\partial x_1, \ldots, \partial x_{i+k}\}$$

Proof. The distribution Δ_S is involutive and constant dimensional and so there exists a local coordinate $x_1, \ldots, x_d$ system satisfying (2.3.1). Write

$$X_1 = \sum_{i=1}^{d} c_i \partial x_i$$

where c_i are smooth functions. Note that $c_i(0) \neq 0$ for a least one $i \geq k+1$. Fix such an i and call it i_0. Shrinking the neighborhood further to avoid the zero set

c_{i_0}, define the vector field

$$\tilde{X}_1 = \sum_{i=k+1}^{d} \frac{1}{c_{i_0}} c_i \partial y_i$$

This vector field belongs to Δ_{S_1}. Since Δ_{S_1} is involutive $[\tilde{X}_1, \partial x_i] = 0$ for all $1 \leq i \leq k$. Extend $\partial x_1, \ldots, \partial x_k, \tilde{X}_1$ to coordinates and repeat the argument. □

Since the region described in Theorem 1.2.4 is the convex hull of sets of the form $(x, y) \geq (x_0, y_0)$ where x_0, y_0 are positive integers, the region has only finitely many vertices. So, there is a neighborhood of the origin $\bar{U}$ such that for every vertex of the region, a coordinate system satisfying the conclusions of Lemma 2.3.1 exists on $\bar{U}$.

We are now ready to prove the theorem. Fix p_1 and p_2 so that there is some $S_0 \prec (\Delta_1, \Delta_2)$ with $(c_1, c_2) \in P(S_0)$. Since we are working in the interior of the given region, strong type boundedness will follow from restricted weak type bounds by real interpolation. If $|S_0| = k_1 + k_2$, Theorem 1.2.4 is simply a version of quantitative Hörmander implies L^p improving. If $|S_0| < k_1 + k_2$, better bounds are implied, but more work has to be done.

Let $(x_1, \ldots, x_d)$ be coordinates furnished from Lemma 2.3.1, relative to some direct basis B_0 where $S_0 \subset B_0$. As before, let $k := \dim \Delta_{S_0}$ and $r = \dim (\Delta_{S_0} \oplus \Delta_1) - k$. Let (y_0, y_1, y_2) be the following convenient grouping of coordinates

$$y_0 := (x_1, \ldots, x_k), y_1 := (x_{k+1}, \ldots, x_{k+r}), y_2 := (x_{k+r+1}, \ldots, x_d).$$

For any measurable subset Ω, define the following slices:

$$\Omega_{(\bar{y_1},\bar{y_2})} := \Omega \cap \{(y_0, y_1, y_2) \mid (y_1, y_2) = (\bar{y_1}, \bar{y_2})\}$$

$$\Omega_{\bar{y_2}} := \Omega \cap \{(y_0, y_1, y_2) \mid y_2 = \bar{y_2}\}.$$

In particular, we have

$$U_{(\bar{y_1},\bar{y_2})} := U \cap \{(y_0, y_1, y_2) \mid (y_1, y_2) = (\bar{y_1}, \bar{y_2})\}$$

$$U_{\bar{y_2}} := U \cap \{(y_0, y_1, y_2) \mid y_2 = \bar{y_2}\}.$$

Define $s_i := \dim \Delta_{S_0} - |S_0 \cap \Delta_i|$. Note that $s_1 + d - (k + r) = d - k_1$ and $s_2 + r = d - k_2$. For the next few calculations, we will include some normally suppressed notation: if ρ is a positive integer and E is measurable $\left|E\right|_\rho$ indicates the ρ-dimensional (induced) measures of E. By Corollary 2.2.3 applied to each of the leaves of the foliation defined by Δ_S

$$\begin{aligned} |\Omega| = \iint \left|\Omega_{(y_1,y_2)}\right|_k dy_1 dy_2 \\ \lesssim_{(y_1,y_2)} \iint \left|\pi_1\left(\Omega_{(y_1,y_2)}\right)\right|_{s_1}^{1/p_1} \left|\pi_2\left(\Omega_{(y_1,y_2)}\right)\right|_{s_2}^{1/p_2} dy_1 dy_2. \end{aligned} \tag{2.3.2}$$

Then to prove the restricted weak-type analogue of (1.2.2), it suffices to show that dependence of the constant on (y_1, y_2) can be removed and that

$$\iint \left|\pi_1\left(\Omega_{(y_1,y_2)}\right)\right|_{s_1}^{1/p_1} \left|\pi_2\left(\Omega_{(y_1,y_2)}\right)\right|_{s_2}^{1/p_2} dy_1 dy_2 \lesssim \left|\pi_1(\Omega)\right|_{d-k_1}^{1/p_1} \left|\pi_2(\Omega)\right|_{d-k_2}^{1/p_2}. \tag{2.3.3}$$

We delay the proof of the former and focus just on (2.3.3). In general, bounding the size of its projection by the size of some arbitrary slice is not possible. But in this case, the interaction between the coordinates and the projections makes such a bound possible and trivial: For each (y_1, y_2), there is a map

$$\pi_1^{(y_1,y_2)} : U_{y_2} \to U_{(y_1,y_2)}$$

such that for all $x \in U_{y_2}$, $\pi_1(x) = \pi_1 \circ \pi_1^{(y_1,y_2)}(x)$ and for all measurable $\Omega \subset U$,

$$\Omega_{(y_1,y_2)} \subset \pi_1^{(y_1,y_2)}\left(\Omega_{y_2}\right).$$

This means that

$$\left|\pi_1\left(\Omega_{(y_1,y_2)}\right)\right|_{s_1} \leq \left|\pi_1\left(\pi_1^{(y_1,y_2)}\left(\Omega_{y_2}\right)\right)\right|_{s_1} = \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}. \tag{2.3.4}$$

By a similar argument,

$$\left|\pi_2^{y_2}\left(\Omega_{y_2}\right)\right|_{d-k_2} \leq \left|\pi_2\left(\Omega\right)\right|_{d-k_2} \tag{2.3.5}$$

The inequalities (2.3.4) and (2.3.5), along with Jensen's inequality, transfer the bound from a slice to the whole space:

$$\begin{aligned}
&\iint \left|\pi_1\left(\Omega_{(y_1,y_2)}\right)\right|_{s_1}^{1/p_1} \left|\pi_2\left(\Omega_{(y_1,y_2)}\right)\right|_{s_2}^{1/p_2} dy_1 dy_2 \\
&\qquad \leq \int \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}^{1/p_1} \int \left|\pi_2\left(\Omega_{(y_1,y_2)}\right)\right|_{s_2}^{1/p_2} dy_1 dy_2 \\
&\qquad \lesssim \int \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}^{1/p_1} \left(\int \left|\pi_2\left(\Omega_{(y_1,y_2)}\right)\right|_{s_2} dy_1\right)^{1/p_2} dy_2 \\
&\qquad = \int \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}^{1/p_1} \left|\pi_2\left(\Omega_{y_2}\right)\right|_{s_2+r}^{1/p_2} dy_2 \\
&\qquad = \int \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}^{1/p_1} \left|\pi_2\left(\Omega_{y_2}\right)\right|_{d-k_2}^{1/p_2} dy_2 \\
&\qquad \leq \int \left|\pi_1\left(\Omega_{y_2}\right)\right|_{s_1}^{1/p_1} dy_2 \left|\pi_2\left(\Omega\right)\right|_{d-k_2}^{1/p_2} \lesssim \left|\pi_1\left(\Omega\right)\right|_{d-k_1}^{1/p_1} \left|\pi_2\left(\Omega\right)\right|_{d-k_2}^{1/p_2}
\end{aligned}$$

Now we explain why the the implicit constant's dependence on (y_1, y_2) can be removed. This is because the implied constant in Stovall's multilinear theory remains bounded under small perturbation. Much of the work has already been done by Street [Str11, Str14]. Specifically, Street shows that the implicit constants present in all pertinent multi-parameter Carnot-Carétheodory ball estimates can be chosen to depend only on the C^M norms of the vector fields, provided that the parameter ε can be bounded below. For more details, see Theorem 5.5 of [Str11].

This requirement on ε is not a problem, as the constant produced in the exponent when performing the refinement process in [Sto11] will behave well under perturbation. This means that ε can be bounded from below, as the multilinear polytope can only get bigger under a small perturbation.

The key inequalities in [Sto11] are

$$\alpha^b \alpha_k^{C\varepsilon} \lesssim \sum_{I \in \mathbf{I}_0} \delta^{\deg_{ML} I} \lesssim B\left(x_0; \delta_1, \ldots, \delta_k\right) \sim B_{\mathbf{j}}\left(x_0; \delta_1, \ldots, \delta_k\right) \tag{2.3.6}$$

and

$$\alpha_k^{C\varepsilon} B_{\mathbf{j}}\left(x_0; \delta_1, \ldots, \delta_k\right) \lesssim \left|\Phi_{\mathbf{j}}^n\left(T_n\right)\right|. \tag{2.3.7}$$

Here b is the relevant multilinear exponent, B and $B_{\mathbf{j}}$ are two types of multiparamater Carnot-Carétheodory balls, and $\Phi_{\mathbf{j}}^n\left(T_n\right)$ is a subset of $B_{\mathbf{j}}$. See [Sto11] for precise definitions. All implied constants depend on ε

By Street's work [Str11], the implicit constant in the first two inequalities of (2.3.6) can be taken uniform over a perturbation. The constant in last comparability estimate of (2.3.6) can also be taken uniform, since the estimate follows from a compactness argument whose main ingredient is a map associated to $B(x_0; \delta_1, \ldots, \delta_k)$, whose implicit constant Street also controls.

There only remains the inequality in (2.3.7). This comes down to check the hypotheses of Theorem 7.1 in Christ's paper [Chr08]. Hypotheses (i) and (ii) are certainly satisfied and Stovall's method of checking of the third condition will also remain valid after a small perturbation.

2.4 Tensor Decomposition and Optimization

It follows from the proof that, roughly, the smaller the $S \prec (\Delta_1, \Delta_2)$, the better the bounds. More precisely, let $S_1, S_2 \prec (\Delta_1, \Delta_2)$ and suppose there are k_3-tuples of words of length at least two

$$(w_{1,1}, \ldots w_{1,k_3}) \in (W(S_1))^{k_3}$$

$$(w_{2,1}, \ldots w_{2,k_3}) \in (W(S_2))^{k_3}$$

such that $\left\{X_{w_{1,i}}(0)\right\}$ and $\left\{X_{w_{2,j}}(0)\right\}$ are linearly independent and

$$\sum_i \deg w_{1,i} = \sum_j \deg w_{2,j}.$$

If $|S_1| < |S_2|$, then the techniques used in the proof of Theorem 1.2.4 produce a strictly greater region of boundedness when using S_1 and $(w_{1,1}, \ldots w_{1,k_3})$ than when using S_2 and $(w_{2,1}, \ldots w_{2,k_3})$.

Additionally, if M_i are non-singular $k_i \times k_i$ matrices and $B := \{X_1, \ldots, X_{k_1+k_2}\}$

is a direct basis of (Δ_1, Δ_2), then

$$\tilde{B} := \{M_1(X_1), \dots, M_1(X_{k_1}), M_2(X_{k_1+1}) \dots, M_2(X_{k_1+k_2})\}$$

is also a direct basis of (Δ_1, Δ_2). Here we are identifying $\Delta_i(0)$ with $\mathbb{R}^{k_i}$ and X_i with elements of the appropriate standard basis. These two observations allow us to shift the question of R's boundedness to a question about the symmetry properties of high-rank tensors.

Let B be a direct basis of (Δ_1, Δ_2) and let $\tau := (w_1, \dots, w_{k_3})$ a k_3-tuple of words of length at least two. If $\deg \tau = (d_1, d_2)$, then τ produces a multilinear map

$$T_\tau : \left(\mathbb{R}^{k_1}\right)^{d_1} \times \left(\mathbb{R}^{k_2}\right)^{d_2} \to \mathbb{R}.$$

The procedure is straightforward but notationally lengthy. First, let $\boldsymbol{X}_1 := (X_1, \dots, X_{k_1})$ and $\boldsymbol{X}_2 := (X_{k_1+1}, \dots, X_{k_1+k_2}\}$. For any $u \in \mathbb{R}^{k_1}$ and $v \in \mathbb{R}^{k_2}$ define

$$u \cdot \boldsymbol{X}_1 := \sum_{i=1}^{k_1} u_i X_i, \qquad v \cdot \boldsymbol{X}_2 := \sum_{j=k_1+1}^{k_1+k_2} v_j X_j$$

Then, for $w_\ell \in \tau$ of degree $(d_{1,\ell}, d_{2,\ell})$, let $X_{w_\ell}\left(u_1, \dots, u_{d_{1,\ell}}, v_1, \dots v_{d_{2,\ell}}\right)$ be the vector field obtained by taking the iterated commutator formula for X_{w_ℓ} and replacing the i^{th} occurrence of an element in $\boldsymbol{X}_1$ with $u_i \cdot \boldsymbol{X}_1$ and the j^{th} occurrence of an element in $\boldsymbol{X}_2$ with $v_j \cdot \boldsymbol{X}_2$. The map

$$T_{w_\ell}\left(u_1, \dots, u_{d_{1,\ell}}, v_1, \dots v_{d_{2,\ell}}\right) = X_{w_\ell}\left(u_1, \dots, u_{d_{1,\ell}}, v_1, \dots v_{d_{2,\ell}}\right)(0)$$

is a multilinear map from $\left(\mathbb{R}^{k_1}\right)^{d_{1,\ell}} \times \left(\mathbb{R}^{k_2}\right)^{d_{2,\ell}}$ to $\Delta_3(0)$. Finally, define

$$\begin{aligned} T_\tau &\left(u_1^1, \ldots, u_{d_{1,1}}^1, v_1^1, \ldots v_{d_{2,1}}^1, \ldots, u_1^{k_3}, \ldots, u_{d_{1,k_3}}^{k_3}, v_1^{k_3}, \ldots v_{d_{2,k_3}}^{k_3}\right) \\ &\qquad := \det{}_{k_3}\left(T_{w_1}, \ldots, T_{w_{k_3}}\right)(0) \end{aligned} \tag{2.4.1}$$

Here $\det_{k_3}$ is the volume form on the leaf of Δ_3 passing through the origin. The symmetry properties of T_τ influence the analysis of R. However, the symmetry properties of an arbitrarily high-rank tensor are hard to understand. Here we state the general lemma, which may be useful in some applications, and a more specific lemma which can be used, for instance, in the case of translation invariant convolution with a submanifold.

Lemma 2.4.1. *Suppose that* $\deg \tau = (d_1, d_2)$, *and for* $1 \leq r_1 \leq k_1$, $1 \leq r_2 \leq k_2$ *there is a* r_1 *dimensional subspace* $U \subset \mathbb{R}^{k_1}$ *and an* r_2*- dimensional subspace* $V \subset \mathbb{R}^{k_2}$ *such that* T_τ *restricted to* $U \times V$ *is not identically zero. If*

$$(c_1, c_2) \subset \left\{x \in \mathbb{R}^2 \mid x > \deg \tau + (r_1, r_2)\right\},$$

then R *is strong-type* (p_1, p_2').

Proof. Let $e_1, \ldots, e_{r_1}$ be a basis of U and $f_1, \ldots, f_{r_2}$ a basis for V. There is a d_1-element sequence $e_{i_1}, \ldots e_{1_{d_1}}$ and d_2-element sequence $f_{j_1}, \ldots, f_{j_{d_2}}$ such that

$$T_\tau\left(e_{i_1}, \ldots e_{1_{d_1}}, f_{j_1}, \ldots, f_{j_{d_2}}\right) \neq 0$$

This corresponds to an $S' \prec (\Delta_1, \Delta_2)$ with $|S'| = r_1 + r_2$ and a τ' spanning S' with $\deg \tau' = \tau + r_1 + r_2$. The lemma follows by Theorem 1.2.4. □

The extreme case $\dim U = \dim V = 1$ is particularly relevant when $k_3 = 1$, which has the obvious but useful property that a spanning set has only one important element. This was exploited by Seeger in [See98]. See the examples section for an in depth discussion of this case.

On the other hand, if, for all direct bases B and all spanning sets τ, the only U and V satisfying the hypotheses of Lemma 2.4.1 have $\dim U = k_1$ and $\dim V = k_2$, then Theorem 1.2.4 is equivalent to Corollary 2.2.3. We know of no such example.

Another situation which arises in practice is that each T_{w_ℓ} is symmetric. This occurs, when R is given by translation invariant convolution. Again, see the examples section for a more in-depth discussion.

Lemma 2.4.2. *Suppose that* $T_\tau := \det_{k_3}\left(T_{w_1}, \ldots, T_{w_{k_3}}\right)(0)$ *is nonzero and, for* $1 \leq \ell \leq k_3$*, each* T_{w_ℓ} *is symmetric in the sense that permuting any of the entries from* $\mathbb{R}^{k_1}$ *with each other leaves* T_{w_ℓ} *unchanged and similarly for the entries from* $\mathbb{R}^{k_2}$*. If*

$$(c_1, c_2) \subset \left\{x \in \mathbb{R}^2 \mid x > \deg\tau + (\min\{k_1, k_3\}, \min\{k_2, k_3\})\right\}$$

then R *is strong-type* (p_1, p_2')*.*

Proof. By the symmetry hypothesis, for any T_{w_ℓ}, the image of its restriction to all subspaces of the form $U \times V$ with $\dim U = \dim V = 1$ is a homogeneous polynomial in $k_1 + k_2$ variable of degree $(\deg w_\ell)_1 + (\deg w_\ell)_2$. More precisely, the polynomial has degree $(\deg w_\ell)_1$ in the U variables and degree $(\deg w_\ell)_2$ in the V variables. The coefficients are various directions $X_w(0)$, and if T_{w_ℓ} is not the zero map, at least one of these is not zero.

Set all the V variables equal to one. Consider the (affine) Veronese map from $\mathbb{R}^{k_1} \to \mathbb{R}^N$ where N is the number of monomials of degree $(\deg w_\ell)_1$ in k_1 variables:

$$x = (x_1, \ldots, x_{k_1}) \to (x^\alpha)_{|\alpha| = (\deg w_\ell)_1}.$$

Here the α are multiindices. The image of this map is not contained in any hyperplane, and so the restriction of T_{w_ℓ} is nonzero, as a polynomial.

To see that the restriction of T_τ is nonzero, notice that

$$\det_{k_3}\left(T_{w_1}, \ldots, T_{w_{k_3}}\right)(0) \neq 0$$

implies the existence of words $w_1', \ldots, w_{k_3}'$ such that $X_{w_1'}, \ldots, X_{w_{k_3}'}$ are linearly independent. Using the multi-linearity of the determinant expand the polynomial T_τ yields a polynomial with at least one nonzero coefficient. Then the claim follows by Lemma 2.4.1. $\square$

Chapter 3

Split bases and Necessity

We now give a necessary condition for R to be of strong type (p_1, p_2'). Like the sufficient condition, it is phrased in terms of a special class of bases.

Definition 3.0.1. A basis B of (Δ_1, Δ_2) is called *split* if $[X_i, X_j] = 0$ when either $1 \le i, j \le k_1$ or $k_1 + 1 \le i, j \le k_1 + k_2$ and for every vertex v of $P_{ML}(B)$ there exists some spanning $I = \{w_1, \ldots, w_d\} \in (W(B))^d$ with the following properties. (Here we make the labeling choice that that $w_i = i$ for $1 \le i \le k_1 + k_2$.)

1. $\deg_{ML} I = v$

2. The distribution defined by

$$\operatorname{span}\{X_{w_i} \mid 1 \le i \le k_1, 1 + k_1 + k_2 \le i \le d\}$$

   is involutive on U.

3. The distribution defined by

$$\text{span}\left\{X_{w_i} \mid 1 + k_1 \leq i \leq d\right\}$$

is involutive on U.

Any $I \in W(B)^d$ satisfying these properties for some vertex v will be called split. If B is split, write $B \lhd (\Delta_1, \Delta_2)$.

Any direct basis is split, so Lemma 2.1.1 implies the existence of split bases as well. There are split bases that are not direct. See the next section for a simple example.

Split bases are bases for which minimal spanning sets correspond to submanifolds.

Theorem 3.0.2. *Let $\phi_{(k_1,k_2)} : \mathbb{R}^2 \to \mathbb{R}^{k_1+k_2}$ be the linear map defined by*

$$\phi_{(k_1,k_2)}\left(x_1, x_2\right) = \left(x_1, \ldots, x_1, x_2, \ldots, x_2\right)$$

where x_1 is repeated k_1 times and x_2 is repeated k_2 times. Suppose $p_1 \geq 1$, $p_2 \geq 1$, and $p_1^{-1} + p_2^{-1} > 1$. If

$$\phi_{(k_1,k_2)}\left(c_1, c_2\right) \notin \bigcap_{B \lhd (\Delta_1, \Delta_2)} P_{ML}\left(B\right),$$

then R is not of restricted weak type (p_1, p_2').

Proof. The strategy is to show that, if B is split basis of (Δ_1, Δ_2), and if $\phi\left(c_1, c_2\right) \notin P_{ML}\left(B\right)$, then the counterexamples in the one-dimensional multilinear setting are also counterexamples in the bilinear, high-dimensional situation.

If $\phi(c_1, c_2)$ does not lie in the region defined in the main theorem, there is a split basis B_0 such that $\phi(c_1, c_2) \notin P_{ML}(B_0)$. Let π_{X_j} be a smooth submersion which takes a U to $\mathbb{R}^{d-1}$ with X_j tangent to its level sets. By work of Stovall, for positive η small enough, there is a measurable set $\{\Omega_\eta\}$ contained in some small neighborhood of the origin such that as $\eta \to 0$

$$A_\eta := \prod_{j=1}^{k_1} \left(\frac{|\Omega_\eta|}{\left|\pi_{X_j}(\Omega_\eta)\right|} \right)^{c_1} \prod_{j=k_1+1}^{k_1+k_2} \left(\frac{|\Omega_\eta|}{\left|\pi_{X_j}(\Omega_\eta)\right|} \right)^{c_2} \frac{1}{|\Omega_\eta|} \to \infty.$$

Since

$$\left(\frac{|\Omega_\eta|}{|\pi_1(\Omega_\eta)|} \right)^{c_1} \left(\frac{|\Omega_\eta|}{|\pi_2(\Omega_\eta)|} \right)^{c_2} \frac{1}{|\Omega_\eta|} = \left(\frac{\prod_{j=1}^{k_1} \left|\pi_{X_j}(\Omega_\eta)\right|}{|\pi_1(\Omega_\eta)| \, |\Omega_\eta|^{k_1-1}} \right)^{c_1} \left(\frac{\prod_{j=k_1+1}^{k_1+k_2} \left|\pi_{X_j}(\Omega_\eta)\right|}{|\pi_2(\Omega_\eta)| \, |\Omega_\eta|^{k_2-1}} \right)^{c_2} A_\eta,$$

to prove Theorem 3.0.2, it suffices to show

$$\frac{\prod_{j=1}^{k_1} \left|\pi_{X_j}(\Omega_\eta)\right|}{|\pi_1(\Omega_\eta)| \, |\Omega_\eta|^{k_1-1}} \sim 1 \tag{3.0.1}$$

and

$$\frac{\prod_{j=k_1+1}^{k_1+k_2} \left|\pi_{X_j}(\Omega_\eta)\right|}{|\pi_2(\Omega_\eta)| \, |\Omega_\eta|^{k_2-1}} \sim 1. \tag{3.0.2}$$

Note that neither (3.0.1) and (3.0.2) involve any interaction between Δ_1 and Δ_2. The estimates are, in a sense, a statement about the rectangularity of Ω_η in two different coordinate systems.

To that end, here is a review of Stovall's construction, which is influenced by the work of Tao and Wright [TW03] and Chirst, Nagel, Stein, and Wainger [CNSW99].

Additionally, all of what follows can be placed in the more general work of Street concerning multiparameter Carnot-Carétheodory balls.

In what follows, ε is a small positive parameter, and K is a large positive parameter. All implicit constants in this section depend on ε. If $\delta := (\delta_1, \ldots, \delta_{k_1+k_2})$ is some $k_1 + k_2$-tuple of small positive real numbers and $I \in W(B_0)^d$, define

$$(K\delta)^{\deg_{ML} I} := \prod_{j=1}^{k_1+k_2} (K\delta_j)^{(\deg_{ML} I)_j}$$

Since (Δ_1, Δ_2) satisfy the Hörmander condition, there is some $I' = \{w_1, \ldots, w_d\} \in W(B_0)^d$ so that

$$\lambda_{I_0}(0) := \det(X_{w_1}, \ldots, X_{w_d})(0) \neq 0$$

Setting $D := \sum_{j=1}^{k_1+k_2} (\deg_{ML} I')_j$, define

$$\mathbf{I} := \left\{ I \in W(S_0) : (\deg_{ML} I)_j \leq \frac{D}{\varepsilon}, 1 \leq j \leq k_1 + k_2 \right\}.$$

We will always assume ε is small enough that $\mathbf{I}$ contains all the vertices of $P_{ML}(B_0)$. Since there are finitely many vertices of $P_{ML}(B_0)$, there are finitely many split $I \in W(B_0)^d$, and by shrinking the neighborhood, we may assume that $\lambda_I(x) \sim 1$ and

$$\det\left(X_1, \ldots, X_{k_1}, X_{w_{k_1+k_2+1}}, \ldots, X_{w_d}\right)(x) \sim 1 \tag{3.0.3}$$

$$\det\left(X_{k_1+1}, \ldots, X_{k_1+k_2}, X_{w_{k_1+k_2+1}}, \ldots, X_{w_d}\right)(x) \sim 1. \tag{3.0.4}$$

for all split I. Here det is the induced volume form on the appropriate submanifolds, which exist since I is split.

Then define the vector valued function

$$\Lambda(x) := \left((K\delta)^{\deg_{ML} I} \lambda_I(x)\right)_{I\in\mathbf{I}}$$

There is some $I_0 \in \mathbf{I}$ such that $(K\delta)^{\deg_{ML} I_0} \lambda_{I_0}(0) \sim |\Lambda(0)|$. In fact, if the entries of δ are small enough, there is a split I_0 satisfying the above. Fix this $I_0 := \{w_1, \ldots, w_d\}$ and define the mapping

$$\Phi_\delta(t_1, \ldots, t_d) := \exp\left(\sum_{j=1}^{d} K^{-1}(K\delta)^{\deg_{ML} w_j} t_j X_{w_j}\right)(0).$$

Since $D\Phi$ is nonsingular at the origin, the pullback vectors

$$Y_{w,\delta} := (D\Phi)^{-1}\left(K^{-1}(K\delta)^{\deg_{ML} w} X_w\right)$$

can be defined on a neighborhood on the origin, which, at first glance, may depend on δ. In fact, the following lemma from [Sto11, TW03] guarantees that it does not. It requires two more assumption on δ, which are usually called the smallness and non-degeneracy conditions respectively

$$\delta_j \leq c(\varepsilon, K), \qquad 1 \leq j \leq k_1 + k_2 \tag{3.0.5}$$

$$\delta_j \leq C\delta_i^{\varepsilon}, \qquad 1 \leq i, j \leq k_1 + k_2 \tag{3.0.6}$$

Here, $c(\varepsilon, K)$ is a constand depending on ε and K, and C is a constant depending on ε. Since the actual structure of Ω_η matters, some facts about Φ from [Sto11] will be important. The following lemma holds for all ε suitably small, K suitably large, and δ satisfying the smallness and non-degeneracy conditions. In fact, this lemma is true in a more general situation, (in particular, the basepoint of Φ can vary), but for present purposes, this will be enough.

Lemma 3.0.3 (Stovall, [Sto11], Tao, Wright [TW03]). *1. There exists a $C \sim 1$ so that Φ_δ is a diffeomorphism on $B_C(0)$, the ball of radius C centered at the origin.*

2. On $B_C(0)$, $Y_{w_j} = \partial_j + O\left(\frac{|t|}{K}\right)$.

3. If E is a measurable subset of $B_C(0)$, $|\Phi(E)| \sim K^{-d}(K\delta)^{\deg_{ML} I_0} |E|$.

All implicit constants depend on ε.

Stovall's Ω_η are the image under $\Phi_{\bar{\eta}}$ of some ball $B \subset B_C(0)$ of sufficiently small radius r, which depends on ε, where $\bar{\eta}$ is of the form $(\eta^{a_1}, \ldots, \eta^{a_{k_1+k_2}})$ and η is sufficiently small. We will abuse notation by writing η for $\bar{\eta}$ and Φ_η for $\Phi_{\bar{\eta}}$. From the above lemma we have

$$|\Omega_\eta| \sim K^{-d} r^d (K\eta)^{\deg_{ML} I_0}. \tag{3.0.7}$$

To prove, (3.0.1) and (3.0.2), there remains estimating $|\pi_i(\Omega)|$ and $\left|\pi_{X_j}(\Omega)\right|$. We prove only (3.0.1) and note that (3.0.2) is proved in exactly the same way.

Let M be the $d - k_1$ dimensional leaf of the foliation defined by

$$X_{k_1+1}, \ldots, X_{k_1+k_2}, X_{w_{k_1+k_2+1}}, \ldots, X_{w_d}$$

that passes through the origin. Note that the I_0 being split guarantees that this foliation is, in fact, $d - k_1$ dimensional. Then M is (uniformly) transverse to the fibers of π_1, and so π_1 restricted to M is a diffeomorphism. Then, for any measurable function f defined on a sufficiently small neighborhood $\bar{U}$ of $\mathbb{R}^{d-k_1}$,

$$\int_{\bar{U}} f(y)\, d\mu(y) \sim \int_M f \circ \pi_{\Delta_1}(x)\, d\nu(x). \tag{3.0.8}$$

Here $d\mu$ and $d\nu$ are the densities that come from (restriction of) the volume form. The $\sim$ comes from a determinant expression in the change of variables forumla, which depends only on the geometry of the vector fields. The submanfold M is well suited to Φ_{η_0}. Specifically, if V is the subspace of $\mathbb{R}^d$ with zeros in the first k_1 components, then $\Phi_\eta(V \cap B) = M \cap \Omega_\eta$. By change of variables and (3.0.4)

$$\begin{aligned}\int_M f \circ \pi_1(x) d\nu(x) &= \int_{V\cap B_C} f \circ \pi_1 \circ \Phi_\eta(t) \left|\det D\bar{\Phi}_\eta(t)\right| dt \\ &\sim K^{-(d-k_1)} \prod_{j=k_1+1}^{d} (K\eta)^{\deg_{ML} w_j} \int_{V\cap B_C} f \circ \pi_1 \circ \Phi_\eta(t)\, dt.\end{aligned} \tag{3.0.9}$$

Here $\bar{\Phi}_\eta$ is the restriction of Φ_η to V. A few different choices of f will be useful. For estimating $|\pi_1(\Omega_\eta)|$, define

$$f_0(y) := \begin{cases} 1 & \text{if } \pi_1^{-1}(y) \cap \Omega_\eta \neq \emptyset \\ 0 & \text{else} \end{cases}$$

Since $Y_{w_i} = \partial_i + O\left(\frac{|t|}{K}\right)$, the set of $t \in V$ such that $f_0 \circ \pi_1 \circ \Phi_\eta(t) \neq 0$ contains the set $V \cap B_r$ and is contained in the set $V \cap B_{2r}$ if K is sufficiently large. And so, by (3.0.8) and (3.0.9)

$$|\pi_1(\Omega_\eta)| \sim K^{-(d-k_1)} r^{d-k_1} \prod_{i=k_1+1}^{d} (K\eta)^{\deg_{ML} w_i}. \tag{3.0.10}$$

For estimating $\left|\pi_{X_j}\left(\Omega_\eta\right)\right|$, for $1 \le j \le k_1$ define

$$f_j\left(y\right) = \left|\pi_{X_j}\left(\pi_{\Delta_1}^{-1}\left(y\right) \cap \Omega_\eta\right)\right|.$$

The strategy for estimating $f_j\left(y\right)$ is extremely similar to the strategy for estimating $\left|\pi_1\left(\Omega_\eta\right)\right|$. The only difference is that we work fiber by fiber.

For $t \in V \cap B_{2r}$, let let $L\left(t\right)$ denote the leaf defined by $Y_1, \ldots, Y_{k_1}$ that passes through t. For $1 \le i \le k_1$ let $L_i\left(t\right)$ be the leaf defined by $Y_1, \ldots, \hat{Y}_i, \ldots, Y_{k_1}$ that passes through t. Here, $\hat{Y}_i$ means that Y_i is omitted from the list. If $|t| > r\left(1 + \frac{1}{K}\right)$, $L\left(t\right)$ is empty since K is big.

Let π_{t,Y_i} be the smooth submersion that maps $L\left(t\right)$ to $L_i\left(t\right)$, is the identity on $L_i\left(t\right)$, and has level sets whose tangent space is spanned by Y_i. If $\pi_1 \circ \Phi_\eta\left(t\right) = y$, then, by change of variables

$$\begin{aligned} f_j\left(y\right) &\sim K^{k_1-1}\prod_{\substack{i=1\\ i\neq j}}^{k_1}\left(K\eta\right)^{\deg_{ML} w_i}\int\limits_{\pi_{t,Y_i}\left(B_{2r}\cap L(t)\right)}\left|\det\left(X_1, \ldots, \hat{X}_j, \ldots, X_{k_1}\right)\right| ds \\ &\sim \left|\pi_{t,Y_i}\left(B_{2r} \cap L\left(t\right)\right)\right| K^{k_1-1}\prod_{\substack{i=1\\ i\neq j}}^{k_1}\left(K\eta\right)^{\deg_{ML} w_i}. \end{aligned} \tag{3.0.11}$$

Here, as before, $\hat{X}_i$ means leave X_i out of the list and det is the restriction of the volume form to the appropriate sub-manifold. The second squiggle follows from the fact that, on U,

$$\det\left(X_1, \ldots, \hat{X}_j, \ldots, X_{k_1}\right)\left(x\right) \sim 1.$$

The last step is to understand $|\pi_{t,Y_i}(B_{2r}\cap L(t))|$, which is done by leaning on the estimate $Y_{w_i}=\partial_i+O\left(\frac{|t|}{K}\right)$. Let $\tilde{\pi}$ be the map defined by

$$\tilde{\pi}(t_1,\ldots,t_d):=(t_1,\ldots,t_{k_1}).$$

For $t\in V\cap B$, define

$$g(t):=\max\left\{0,\sqrt{r^2-|t|^2}\right\}.$$

Let B' be the $d-k_1$ dimensional ball of radius r/K centered at the origin

$$h_1(t):=\sup_{b'\in B'}g(t-b'),\, h_2(t):=\inf_{b'\in B}g(t-b').$$

If $\bar{B}(\rho)$ denotes the k_1-dimensional ball centered at the origin of radius ρ, we have

$$\bar{B}(h_2(t))\subset\tilde{\pi}(L(t))\subset\bar{B}(h_1(t)) \tag{3.0.12}$$

once again, because $Y_{w_i}=\partial_i+O\left(\frac{|t|}{K}\right)$. The strategy is to show that

$$\int(h_2(t))^{k_1}\,dt\sim\int(h_1(t))^{k_1}\,dt \tag{3.0.13}$$

which means that $\tilde{\pi}(L(t))$ is more or less $\bar{B}(g(t))$ This is essentially automatic, but here is a double check that the estimate is independent of big K. Push everything to polar coordinates and suppose, for instance, $K>10$, then

$$\begin{aligned}\int(h_1(t))^{k_1}\,dt&\sim\left(\frac{r}{K}\right)^{d-k_1}r^{k_1}+\int_{r/K}^{r+r/K}t^{d-k_1-1}\left(r^2-\left(t-\frac{r}{K}\right)^2\right)^{k_1/2}dt\\&\lesssim r^{k_1}\left(\left(\frac{r}{K}\right)^{d-k_1}+\int_{r/K}^{r+r/K}t^{d-k_1-1}dt\right)\lesssim r^d,\end{aligned}$$

and

$$\int (h_2(t))^{k_1}\,dt \sim \int_0^{r-r/K} t^{d-k_1-1}\left(r^2-\left(t+\frac{r}{K}\right)^2\right)^{k_1/2} dt$$
$$\gtrsim \int_0^{r/2-r/K} t^{d-k_1-1}\left(r^2-\left(t+\frac{r}{K}\right)^2\right)^{k_1/2} dt$$
$$\gtrsim r^{k_1}\int_0^{r/2-r/K} t^{d-k_1-1}dt \sim r^d.$$

Then (3.0.12), (3.0.13) and the estimate $Y_{w_i} = \partial_i + O\left(\frac{|t|}{K}\right)$ imply

$$\int_{V\cap B_{2r}} \left|\pi_{t,Y_i}\left(B_{2r}\cap L(t)\right)\right| dt \sim \int_{V\cap B_r} \left(r^2-|t|^2\right)^{\frac{k_1-1}{2}} dt \sim r^{d-1}.$$

Combined with (3.0.9) and (3.0.11), this gives

$$\left|\pi_{X_j}\left(\Omega_\eta\right)\right| \sim K^{-(d-1)} r^{d-1} \prod_{\substack{i=1\\ i\neq j}}^{d} (K\eta)^{\deg_{ML} w_i}. \tag{3.0.14}$$

With that (3.0.7), (3.0.10), and (3.0.14) established, (3.0.1) follows. □

Chapter 4

Examples and Sharpness

4.1 Convolution with the k-flat corkscrew, a flat sharp example

There are examples where the regions described in Theorem 1.2.4 and Theorem 3.0.2 coincide, meaning that, up to endpoints, the region is sharp. This is easiest to see with examples that are flat in all but one direction. Specifically, if there is a direct basis $B := \{X_1, \ldots, X_{k_1+k_2}\}$ such that $i \leq k_1 - 1$ or $j \leq k_1 + k_2 - 1$ implies

$$[X_i, X_j] = 0,$$

then any word of length greater than two involved in a spanning set is a finite sequence of only k_1's and $k_1 + k_2$'s. This immediately implies that the two regions coincide, and that Theorem 1.2.4 is sharp up to endpoints. Here is an explicit example:

Let $1 < k < d$ be positive integers and define $\gamma_0 : \mathbb{R}^{k+1} \to \mathbb{R}^d$ via

$$\gamma_0 (t_1, \ldots, t_{k+1}) := \left(t_1, t_2, \ldots, t_k, t_{k+1}, t_{k+1}^2, t_{k+1}^3, \ldots, t_{k+1}^{d-k}\right)$$

and, if f is a function on $\mathbb{R}^d$ define R_{γ_0} as

$$R_{\gamma_0} (f) := \int_{[-1,1]^{k+1}} f\left(x - \gamma_) (t)\right) dt.$$

Define the following vector fields in $\mathbb{R}^{k+1+d} := (t, x)$

$$X_i := \partial t_i, 1 \leq i \leq k+1$$

$$Y_i := \partial t_i - \partial x_i, 1 \leq i \leq k$$

$$Y_{k+1} := \partial t_{k+1} - \sum_{j=1}^{d-k} j t_{k+1}^{j-1} \partial X_{k+j}$$

Then $\Delta_1 = \text{span}\{X_1, \ldots, X_{k+1}\}$ and $\Delta_2 = \text{span}\{Y_1, \ldots, Y_{k+1}\}$. Both

$$\{X_1, \ldots, X_k, Y_1, \ldots, Y_{k+1}\}$$

and

$$\{X_1, \ldots, X_{k+1}, Y_1, \ldots, Y_k\}$$

are pairwise commuting collections of vector fields. And so, except for the endpoints, R_{γ_0} maps L^p to L^q for $q \geq p$ if and only if (p^{-1}, q^{-1}) lies in the closed trapezoid with verticies

$$(0,0), (1,1), \left(\frac{2}{d-k+1}, \frac{2(d-k-1)}{(d-k+1)(d-k)}\right), \left(1 - \frac{2(d-k-1)}{(d-k+1)(d-k)}, 1 - \frac{2}{d-k+1}\right)$$

4.2 A sharp example which is not flat, combinatorial flatness

Such totally flat examples are not the only case in which the region given in Theorem 1.2.4 is sharp. What is really important is the existence of a split basis in which only one vector field from Δ_1 and one vector field from Δ_2 control the spanning sets. Such pairs distributions might be called combinatorially flat.

In $\mathbb{R}^4$ consider the pair of distributions $\Delta_1 := \operatorname{span}\{\partial_{x_1}, \partial_{x_2}\}$ and $\Delta_2 := \operatorname{span}\{\partial_{x_3} + (x_1x_3 + x_2)\,\partial_{x_4}\}$. Let R_{CF} be the associated Radon-like transform. The basis

$$B_0 = \{\partial_{x_1}, \partial_{x_2}, \partial_{x_3} + (x_1x_3 + x_2)\,\partial_{x_4}\}$$

is direct. Computation of brackets in this basis yields that R_{CF} is bounded, for $q \geq p$, when (p^{-1}, q^{-1}) lies in the interior of the triangle with vertices

$$(0,0)\,, (1,1)\,, \left(\frac{2}{3}, \frac{1}{3}\right).$$

Consider the basis

$$B_1 := \{X_1 := \partial_{x_1} - (x_3/2)\,\partial_{x_2}, X_2 := \partial_{x_2}, X_3 := \partial_{x_3} + (x_1x_3 + x_2)\,\partial_{x_4}\}\,,$$

which is split but not direct. The only brackets which do no vanish identically are

$$[X_1, X_3] = \frac{1}{2}\left(\partial_{x_2} + x_3\partial_{x_4}\right), [X_2, X_3] = \partial_{x_4}.$$

which, by Theorem 3.0.2 means that the region above is sharp. So, up to endpoints, R_{CF} has exactly the same mapping properties as the Radon-like transform associated to the pair $\bar{\Delta}_1 := \text{span}\{\partial_{x_1}, \partial_{x_2}\}$, $\bar{\Delta}_2 := \text{span}\{\partial_{x_3} + x_2\partial_{x_4}\}$.

Note that the multilinear polytope of B_1 is strictly contained in the multilinear polytope of any direct basis. Here is a proof. Define

$$Z_1 := a\partial_{x_1} + b\partial_{x_2}$$

$$Z_2 := c\partial_{x_1} + d\partial_{x_2}$$

$$Z_3 := e\left(\partial_{x_3} + (x_1x_3 + x_2)\,\partial_{x_4}\right)$$

here a, b, c, d, e are smooth functions and $\{Z_1, Z_2, Z_3\}$ is assumed to be a direct basis of (Δ_1, Δ_2). Without loss is generality, $b(0) \neq 0$ and so $[Z_1, Z_3]$ has nonzero ∂_{x_4} coefficient at the origin. If $d(0) \neq 0$, $[Z_1, Z_3]$ also has nonzero ∂_{x_4} coefficient at the origin, the multilinear polytope of $\{Z_1, Z_2, Z_3\}$ is the closed convex hull of the set

$$\{p \mid (p_1, p_2, p_3) \geq (1, 2, 2)\} \cup \{p \mid (p_1, p_2, p_3) \geq (2, 1, 2)\}$$

There remains the case $d(0) = 0$. The directness of $\{Z_1, Z_2, Z_3\}$ forces $[Z_2, Z_3] = \eta[Z_1, Z_3]$ where η is a smooth function and $\eta(0) = 0$. Since

$$[Z_2, Z_3] = e\,(cx_3 + d)\,\partial_{x_4} + Z_2(e)\,X_3 - Z_3(c)\,\partial_{x_1} - Z_3(d)\,\partial_{x_2},$$

this implies $Z_3(c)(0) = Z_3(d)(0) = Z_2(e)(0) = 0$. So

$$[Z_3, [Z_2, Z_3]] = [Z_3, e\,(cx_3 + d)\,\partial_{x_4} + Z_2(e)\,X_3 - Z_3(c)\,\partial_{x_1} - Z_3(d)\,\partial_{x_2}]$$

Since $Z_3(c)(0) = Z_3(d)(0) = 0$, the ∂_{x_4} coefficient at the origin will be the same as the ∂_{x_4} coefficient of

$$[Z_3, e\,(cx_3 + d)\,\partial_{x_4} + Z_2\,(e)\,X_3].$$

Since the ∂_{x_4} coefficient of X_3 is zero at the origin, we can consider only the ∂_{x_4} coefficient of

$$[Z_3, e\,(cx_3 + d)\,\partial_{x_4}],$$

which is nonzero. This implies that the multilinear polytope of $\{Z_1, Z_2, Z_3\}$ is the closed convex hull of the set

$$\{p \mid (p_1, p_2, p_3) \geq (1, 2, 2)\} \cup \{p \mid (p_1, p_2, p_3) \geq (2, 1, 3)\}.$$

This means that inclusion of split bases instead of only direct bases changes conclusion of Theorem 3.0.2 in a nontrivial way.

4.3 Translation Invariant averages

When R is traslation invariant convolution with a submanifold, the Lie algebra generated by (Δ_1, Δ_2) is somewhat simple, and so the optimization in Lemma 2.4.2 yields the following bounds

Let $1 < k < d$ be positive integers and suppose

$$\gamma_1\,(t_1, \ldots, t_k) := (t_1, t_2, \ldots, t_k, a_1\,(t_1, \ldots, t_k)\,, \ldots, a_{d-k}\,(t_1, \ldots, t_k))$$

parameterizes a small neighborhood of a k-dimensional submanifold. If f is a function on $\mathbb{R}^d$ define, for suitably small epsilon, R_{γ_2} as

$$R_{\gamma_2}(f) := \int_{[-\varepsilon,\varepsilon]^k} f(x - \gamma_2(t))\, dt.$$

Define the following vector fields in $\mathbb{R}^{k+d} := (t, x)$

$$X_i := \partial t_i, 1 \leq i \leq k+1$$

$$Y_i := \partial t_i - \nabla_{t_i} \cdot \gamma_2(t)\, 1 \leq i \leq k$$

Then $\Delta_1 = \operatorname{span}\{X_1, \ldots, X_k\}$ and $\Delta_2 = \operatorname{span}\{Y_1, \ldots, Y_k\}$, and

$$B := \{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$$

is a direct basis of (Δ_1, Δ_2). Moreover, the Lie algebra generated by B has a particularly convenient structure: if w is any word associated to (Δ_1, Δ_2) such that $X_w(0) \neq 0$, replacing all but the first instance of any Y_i in w with X_i leaves X_w unchanged. Since any word with only one entry belonging to Δ_2 corresponds to a symmetric multilinear map, Lemma 2.4.2 applies. Specifically, for any multiindex of α of $\mathbb{R}^k$, define

$$D^\alpha \gamma_2(t) := (D^\alpha a_1(t), \ldots, D^\alpha a_{d-k}(t))$$

Suppose the exits multilindices $\alpha_1, \ldots, \alpha_{d-k}$ such that

$$\det(D^{\alpha_1}\gamma_2, \ldots, D^{\alpha_{d-k}}\gamma_2)(0) \neq 0.$$

Set $|\alpha| := \sum |\alpha_i|$ and $k_0 := \min\{k, d-k\}$. Then, by Theorem 1.2.4 and Lemma 2.4.2, R_{γ_2} is bounded from L^p to L^q, for $q \geq p$, if (p^{-1}, q^{-1}) lies in the interior of the trapezoid with vertices

$$(0,0)\,,(1,1)\,,\left(\frac{|\alpha|-(d-k)+k_0}{|\alpha|-(d-k)+2k_0-1},\frac{k_0+1}{|\alpha|-(d-k)+2k_0-1}\right),$$
$$\left(1-\frac{k_0+1}{|\alpha|-(d-k)+2k_0-1},1-\frac{|\alpha|-(d-k)+k_0}{|\alpha|-(d-k)+2k_0-1}\right)$$

4.4 The case of Hypersurfaces, a result of Seeger

Finally, we point out that when $k_3 = 1$ the techniques of this paper may be used to recover a result of Seeger [See98].

Denote the associated Radon-like transform R_{CD1}. First, note that in this case, the distinction between direct bases and bases is immaterial. Given any collection of linearly independent vector fields $\left\{\bar{X}_1,\ldots,\bar{X}_{k_1+k_2}\right\}$, there is a direct basis $\{X_1,\ldots,X_{k_1+k_2}\}$ such that $\bar{X}_i(0) = X_i(0)$ for all $1 \le i \le k_1+k_2$. And so, for any minimal spanning word in basis $\left\{\bar{X}_1,\ldots,\bar{X}_{k_1+k_2}\right\}$ there is a spanning word of the same degree in the basis $\{X_1,\ldots,X_{k_1+k_2}\}$.

The next step is to prove the following lemma

Lemma 4.4.1 (Seeger, [See98])**.** *Suppose B is a direct basis and that $w \in W(B)$ with $X_w(0) \neq 0$. Suppose further that for all $w' \in W(B)$ with $\deg w' < \deg w$, $X_{w'}(0) = 0$. Then T_w is symmetric in the sense of Lemma 2.4.2.*

Proof. The claim is immediate if w has length two or three. Let w have length $n \ge 4$ and write

$$w := (i_1,\ldots,i_n)\,.$$

For $3 \leq k \leq n-1$, let w_k be the word $w_k := (i_1, \ldots, i_{k-1})$. Then, by the Jacobi identity

$$X_w = \left[\ldots[[X_{w_k}, X_{i_k}], X_{i_{k+1}}]\ldots], X_{i_n}\right] = \left[\ldots[[X_{w_k}, X_{i_{k+1}}], X_{i_k}]\ldots], X_{i_n}\right]$$
$$+ \left[\ldots[X_{w_k}, [X_{i_k}, X_{i_{k+1}}]]\ldots], X_{i_n}\right]$$

If we can show that $\left[\ldots[X_{w_k}, [X_{i_k}, X_{i_{k+1}}]]\ldots], X_{i_n}\right](0) = 0$, the claim follows, since it implies that the map is symmetric in all variables except the first two. To show that, we prove a slightly more general statement:

If $w_{k,1}$ and $w_{k,2}$ are two elements of $W(B)$ of length at least two such that

$$\deg_{ML} w_{k,1} + \deg_{ML} w_{k,2} = \deg_{ML}(w_k, i_{k+1}).$$

then

$$\left[[\ldots[[X_{w_{k,1}}, X_{w_{k,2}}], X_{i_{k+2}}]\ldots], X_{i_n}\right](0) = 0. \tag{4.4.1}$$

The proof proceeds by induction on $k-(n-1)$. In the case that $k = n-1$, the claim is immediate as (4.4.1) is a bracket of two vector fields that vanish at the origin. If $k < d-1$,

$$\left[[\ldots[[X_{w_{k,1}}, X_{w_{k,2}}]X_{i_{k+2}}]\ldots], X_{i_n}\right] = -\left[[\ldots[[X_{i_{k+2}}, X_{w_{k,1}}]X_{w_{k,2}}]\ldots], X_{i_n}\right]$$
$$- \left[[\ldots[[X_{w_{k,2}}, X_{i_{k+2}}], X_{w_{k,1}}]\ldots], X_{i_n}\right]$$

and (4.4.1) follows by the induction hypothesis. □

Then, by Lemma 2.4.2, if w is a word associated to any basis with $\deg w = (d_1, d_2)$ and

$$(c_1, c_2) \in \left\{x \in \mathbb{R}^2 \mid x > (d_1+1, d_2+1)\right\}$$

then R_{CD1} is strong type (p_1, p_2'), which is exactly Seeger's result.

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