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Ricci Flow On Cohomogeneity One Manifolds

Anusha Mangala Krishnan
University of Pennsylvania, anusha.march@gmail.com

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Abstract
In the first part of this thesis, in joint work with Renato Bettiol, we show that the geometric property of nonnegative sectional curvature is not preserved under the Ricci flow on closed manifolds of dimension greater than or equal to 4. This is in contrast to the situation for 3 dimensional manifolds. The main strategy is to study the Ricci flow equation on certain 4 dimensional manifolds that admit an isometric group action of cohomogeneity one.

Along the way we need to show that a certain canonical form for an invariant metric on a cohomogeneity one manifold, is preserved under the Ricci flow. In the particular situation of the above mentioned result, we prove the preservation of that canonical form using an ad hoc method. It is an interesting question whether this canonical form for a cohomogeneity one metric is preserved in general. In the second

part of the thesis we present a strategy to tackle this problem, explain its geometric consequences, and also explain the challenges in carrying out the strategy, along with some partial results.

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Anusha Mangala Krishnan

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Supervisor of Dissertation

Wolfgang Ziller, Professor of Mathematics

Graduate Group Chairperson

Julia Hartmann, Professor of Mathematics

Dissertation Committee:
Wolfgang Ziller, Professor of Mathematics
Davi Maximo, Assistant Professor of Mathematics
Robert Strain, Professor of Mathematics

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ABSTRACT

RICCI FLOW ON COHOMOGENEITY ONE MANIFOLDS

Anusha Mangala Krishnan

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In the first part of this thesis, in joint work with Renato Bettiol, we show that the geometric property of nonnegative sectional curvature is not preserved under the Ricci flow on closed manifolds of dimension greater than or equal to 4. This is in contrast to the situation for 3 dimensional manifolds. The main strategy is to study the Ricci flow equation on certain 4 dimensional manifolds that admit an isometric group action of cohomogeneity one.

Along the way we need to show that a certain canonical form for an invariant metric on a cohomogeneity one manifold, is preserved under the Ricci flow. In the particular situation of the above mentioned result, we prove the preservation of that canonical form using an ad hoc method. It is an interesting question whether this canonical form for a cohomogeneity one metric is preserved in general. In the second part of the thesis we present a strategy to tackle this problem, explain its geometric consequences, and also explain the challenges in carrying out the strategy, along with some partial results.

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Chapter 1

Introduction

The Ricci flow is the geometric PDE

$$\begin{aligned} \frac{\mathrm{dg}}{\mathrm{d}t} &= -2\,\mathrm{Ric}_{\mathrm{g}} \\ \mathrm{g}(0) &= \mathrm{g}_0 \end{aligned} \tag{1.0.1}$$

for evolving a metric g on a Riemannian manifold M with time. Here $\mathrm{Ric}_{\mathrm{g}}$ denotes the Ricci tensor associated to the metric g. This is a symmetric 2-tensor on the manifold that carries information about the curvature of g.

Heuristically the Ricci flow is like a heat equation for the metric. Thus, similar to the heat equation and temperature, the Ricci flow is expected to have regularizing properties for the metric. The underlying thread in using the Ricci flow to solve problems in geometry and topology, is to evolve the given metric on the manifold, to a *nice* metric (e.g. one of constant curvature) using the Ricci flow. Then other theorems from geometry allow one to draw topological conclusions.

In reality the Ricci flow is a *nonlinear* and *degenerate parabolic* PDE, which makes its analysis very complicated. Nevertheless, ever since it was first introduced by Hamilton in 1982, the Ricci flow has been used to prove a number of remarkable theorems in geometry and topology. Notably, Hamilton's [19] theorem that 3-manifolds with positive Ricci curvature are diffeomorphic to spherical space-forms; Perelman's [32] resolution of geometrization and the Poincare conjecture; Böhm-Wilking's [8] theorem that manifolds with positive curvature operator are diffeomorphic to spherical space forms; and the differentiable sphere theorem of Brendle-Schoen [9].

1.1 Ricci flow and nonnegative curvature

In applications of the Ricci flow to solve problems in geometry and topology, it is important to understand how geometric properties, in particular the curvature, behave under the flow. In particular, it is useful and important to know whether various positive or nonnegative curvature conditions are preserved under the flow. For example, Hamilton's theorem [19] made use of the facts proved by him in the same paper, using a tensor maximum principle, that the conditions of nonnegative sectional curvature ($\sec \geq 0$) and nonnegative Ricci curvature ($\mathrm{Ric} \geq 0$) are preserved on closed 3-dimensional manifolds under the Ricci flow. Maximum principle arguments also yield that the conditions of nonnegative curvature operator ($\mathcal{R} \geq 0$)

and of nonnegative scalar curvature (scal ≥ 0) are preserved on closed manifolds under the Ricci flow in all dimensions.

On the other hand, when one considers the conditions sec ≥ 0 and Ric ≥ 0 on manifolds of dimension 4 and greater, the situation is different. Böhm and Wilking [7] provided examples of homogeneous metrics with sec > 0 on the manifolds $M^{12} = \mathsf{Sp}(3)/\mathsf{Sp}(1)^3$ and $M^6 = \mathsf{SU}(3)/T^2$ that under the Ricci flow, evolve to metrics with mixed Ricci curvature and mixed sectional curvature respectively. In [29], Ni demonstrated examples of complete noncompact manifolds of all dimensions $n \geq 4$ with the property of sec ≥ 0 which are evolved by the Ricci flow to metrics of mixed sectional curvature. Maximo [27, 28] showed that the manifold $M^4 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ admits Kähler metrics with Ric ≥ 0 or Ric > 0 (but without sec ≥ 0) that under the Ricci flow evolve to metrics with mixed Ricci curvature. However until recently the status of sec ≥ 0 on closed manifolds of dimension 4 and 5 was unknown.

In joint work with Bettiol [4] we answer this question by exhibiting the first examples of closed 4-manifolds where the property of nonnegative sectional curvature fails to be preserved under the Ricci flow.

Theorem A. [Bettiol–Krishnan] There exist metrics with sec ≥ 0 on S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ that immediately lose the property of sec ≥ 0 when evolved by the Ricci flow.

By taking products of the above manifolds with round (i.e. constant curvature) spheres, one concludes:

Corollary B. The Ricci flow does not preserve sec ≥ 0 on closed manifolds of any dimension ≥ 4.

The proof of this theorem involves studying the Ricci flow on manifolds of cohomogeneity one, which are Riemannian manifolds with a large isometry group in a specific sense which we will describe below.

1.2 Ricci flow and symmetries

An important feature of the Ricci flow (arising from the diffeomorphism invariance of the Ricci tensor) is the fact that isometries are preserved along the flow. In fact, by work of Kotschwar [23], no new isometries are produced along the flow, so the isometry group of the evolving metric remains unchanged. A metric with a large isometry group can be described using a smaller number of variables, which can considerably simplify the analysis. The catch is that one would like to have a *time-independent* coordinate frame to study the evolving metric. As we will see, such a frame is not always available. However in certain situations, as in the metrics considered in Theorem A, such a frame does exist, and in those situations one can effectively use the presence of symmetries to prove results that shed light on the Ricci flow.

For example, if the initial metric g_0 is homogeneous (i.e. the isometry group G acts transitively, $M/\mathsf{G} = \{p\}$) then the evolving metrics will be homogeneous as well. Thus in the homogeneous case, each metric $\mathrm{g}(t)$ is completely determined by the inner product on *one* tangent space T_pM. This removes the spatial dependence and the Ricci flow equation reduces to an ODE in the time variable, which is much more tractable to study. Homogeneous Ricci flows have been studied extensively by several authors including Lauret, Böhm and Lafuente, see [25, 6] among others. It is important to note that the reduction of the Ricci flow PDE to an ODE in the homogeneous setting relies on standard existence and uniqueness theory for ODEs given an initial condition.

In terms of weakening the isometry assumption, the natural next step is to consider the Ricci flow on cohomogeneity one manifolds. A cohomogeneity one manifold consists of a Riemannian manifold (M, g) along with an isometric action by a Lie group G, such that the principal (generic) orbits are codimension one hypersurfaces in M. This is equivalent to the orbit space M/G being 1-dimensional. By symmetry, any invariant metric on a cohomogeneity one manifold can be described using one variable. Cohomogeneity one manifolds have been widely studied in other contexts and have been an important source of examples of interesting geometric structures, such as Einstein metrics [5], Ricci solitons [10], metrics of positive and nonnegative sectional curvature, and more recently, metrics of special holonomy [12].

The simplest examples of cohomogeneity one metrics are rotationally symmetric metrics. Angenent-Knopf [1] and Angenent-Isenberg-Knopf [2] studied the Ricci flow evolution of certain rotationally symmetric metrics on spheres and provided the first explicit descriptions of Type I and II singularity formation under the Ricci flow. In [21] and [22], Isenberg-Knopf-Sesum implicitly use a cohomogeneity one structure to provide evidence for conjectured stable/ attracting behaviour under the Ricci flow of rotationally symmetric metrics and Kähler metrics respectively. All of these papers contribute towards understanding singularity formation under the Ricci flow in higher dimensions ($n > 3$) where the absence of Hamilton-Ivey type pinching estimates makes a classification of singularity models a distant goal.

1.3 Ricci flow on cohomogeneity one manifolds

As indicated by the various results above, the systematic study of the cohomogeneity one Ricci flow is of natural interest and has several potential applications.

A useful step in gainfully studying the Ricci flow on cohomogeneity one manifolds is considering a special form known as a *diagonal metric*, and showing that this form of the metric is preserved under the flow. A diagonal metric is one that has the following *multiply warped product* structure along a curve $\gamma(r)$ that is orthogonal to all orbits:

$$\mathrm{g}(r) = h(r)^2 \mathrm{d}r^2 + \sum_{i=1}^{m} f_i(r)^2 \omega_i^2 \tag{1.3.1}$$

where r is a coordinate parameterizing the orbit space, and ω_i are G-invariant 1-forms on a fixed homogeneous space G/H which is the underlying manifold that each principal orbit is diffeomorphic to. The above formula describes the metric along a curve in M, and extends to all of M using the action of G.

It is a subtle and important point that this form of the metric is not forced upon you merely by the assumption of G-invariance. In fact all of the above cited works studying Ricci flow in the cohomogeneity one setting either explicitly or implicitly assume a larger isometry group of the initial metrics, which forces any invariant metric to be diagonal.

Now, consider the Ricci flow on a manifold where the initial metric is a diagonal metric as above. We would like to show that at time t the evolving metric $\mathrm{g}(t)$ has the form

$$\mathrm{g}(r,t) = h(r,t)^2 \mathrm{d}r^2 + \sum_{i=1}^{m} f_i(r,t)^2 \omega_i^2 \tag{1.3.2}$$

That is, we want to be able to study the evolving metric in a *time-independent* frame field. In settings where the initial metric has an isometry group large enough to force any invariant metric to be diagonal, the diagonal form of the metric will be preserved under the flow. (This is what we use in the proof of Theorem A.) On the other hand, this assumption of extra isometries significantly restricts the class of metrics that can be studied.

An obvious necessary condition for the diagonal form of the metric to be preserved under the flow is that the Ricci tensor of a diagonal metric also be diagonal. The Ricci tensor at a point on the geodesic γ is the sum of $\mathrm{Ric}_{\mathsf{G}/\mathsf{H}}$ and a contribution from the second fundamental form, where G/H is an orbit of the group action with the induced homogeneous metric. Since the second fundamental form contribution of a diagonal metric is diagonal (see e.g. Proposition 1.14 in [17]), the Ricci tensor of M is diagonal if and only if $\mathrm{Ric}_{\mathsf{G}/\mathsf{H}}$ is diagonal in the induced metric. This is an algebraic condition on G (alternately on its Lie algebra $\mathfrak{g}$) and we formally define it as follows:

Definition 1.3.1. A basis B for $\mathfrak{g}$ is said to be *stably Ricci-diagonal* if Ric(g) is diagonal in the basis $\mathcal{B} = B \cup \{\frac{\partial}{\partial r}\}$ whenever the metric g is diagonal in the basis $\mathcal{B}$.

(See 2.3.3 for an example where the basis is *not* stably Ricci diagonal.)

The concept of a stably Ricci-diagonal basis for a Lie algebra is introduced by Payne in [31] in order to study the Ricci flow on nilmanifolds. We have used this terminology to include diagonal metrics on cohomogeneity one manifolds.

We will now describe an algebraic condition on the Lie algebra of G that is sufficient to guarantee that Ric(g) is diagonal in the setting of compact semisimple Lie groups. This condition holds in a wide class of examples, including in the manifolds considered in Theorem A. Before writing the definition, we briefly provide some more information about cohomogeneity one manifolds that will be needed. More details can be found in Chapter 2.

We consider cohomogeneity one actions where M/G is isometric to a closed interval $[0, L]$. If γ is a minimal geodesic in M that parametrizes this orbit space, then the isotropy group at points $\gamma(r)$ for $0 < r < L$ are all the same, and this group is denoted by H, with Lie algebra $\mathfrak{h}$. The isotropy groups at points $\gamma(0)$ and $\gamma(L)$ are denoted by K_- and K_+ respectively, and their Lie algebras are denoted by

$\mathfrak{k}_-$ and $\mathfrak{k}_+$. Clearly $\mathsf{H} \subset \mathsf{K}_\pm \subset \mathsf{G}$ and hence $\mathfrak{h} \subset \mathfrak{k}_\pm \subset \mathfrak{g}$.

Definition 1.3.2. A basis B for $\mathfrak{g}$ is said to be *nice* for the cohomogeneity one manifold (M, G) if

1. it respects the inclusions $\mathfrak{h} \subset \mathfrak{k}_\pm \subset \mathfrak{g}$
2. the bracket of any two basis elements is a multiple of another basis element: for each i, j, $[X_i, X_j] = aX_l$ for some a, l.
3. if $[X_i, X_j]$ and $[X_r, X_s]$ are nonzero multiples of the same basis element X_k then $\{i, j\} \cap \{r, s\} = \emptyset$

The following proposition is the reason for making the above definition.

Proposition 1.3.3. *Assume* G, H *are compact semisimple Lie groups, then for a homogeneous metric on* G/H*, if* B *is a nice basis then* B *is stably Ricci diagonal.*

The concept of a nice basis for a Lie algebra was introduced by Lauret and Will and in [26] they show that the notions of nice basis and stably Ricci diagonal basis are equivalent for nilpotent Lie groups. For compact semisimple Lie groups it is not clear whether the reverse implication is true, i.e. if B is stably Ricci diagonal, one does not know if B is necessarily nice. Also note that for ease of studying the metric and Ricci tensor on cohomogeneity one manifolds, we have added the requirement that the basis respects the inclusions $\mathfrak{h} \subset \mathfrak{k}_\pm \subset \mathfrak{g}$.

Coming back to the question of Ricci flow on cohomogeneity one manifolds, it is not clear that the *stably Ricci diagonal* condition alone is sufficient to guarantee that the evolving metric remains diagonal under the Ricci flow. The issue is that off-diagonal terms could be appearing at a slower rate in time.

For example, one necessary property one has to prove is that a curve that is a geodesic orthogonal to the orbits in the initial metric g_0, stays orthogonal to the orbits in the evolving metric, and thus remains a geodesic up to reparametrization. This is not guaranteed by the fact that G acts by isometries. Another important fact is that the Killing vector fields X_i^* dual to the one forms ω_i are orthogonal only along γ and not at all points of M. Thus the resulting intial value problem is not a priori global in nature. However, we make the following conjecture:

Conjecture C. Let (M, G) be a cohomogeneity one manifold. Suppose that there exist Killing fields $\{X_i^*\}_{i=1}^m$ that are action fields on M coming from a nice basis of (M, G). Let g_0 be a cohomogeneity one metric on M that is diagonal in the basis $\mathcal{B} = \{\frac{\partial}{\partial r}, X_1^* \cdots, X_m^*\}$. If $\mathrm{g}(t)$ is a solution to the Ricci flow with $\mathrm{g}(0) = \mathrm{g}_0$ then $\mathrm{g}(t)$ is diagonal in the basis $\mathcal{B}$ as well for all t for which the flow exists.

If this conjecture holds one would have the following geometric implications:

- Under the assumptions of Conjecture C above, a geodesic orthogonal to all the orbits remains (up to reparametrization by arc length) a geodesic for as long as the flow exists.

- Under the assumptions of Conjecture C, the Killing vector fields $X_1^*, \cdots, X_m^*$ remain mutually orthogonal along the geodesic γ.

In the second part of this thesis we will present a strategy to solve this problem, and also state some partial results in this direction.

It is also an interesting question whether the same is true for all polar actions (where we allow $dim(M/\mathsf{G}) > 1$). That is, if S is a submanifold orthogonal to all orbits, does S remain orthogonal to the orbits when the metric is evolved by the Ricci flow?

This thesis is organized as follows. In Chapter 2, we review the basics of cohomogeneity one actions. In Chapter 3 we describe the *smoothness conditions* for a cohomogeneity one manifold with two singular orbits. In Chapter 4 we derive the system of PDEs satisfied by a cohomogeneity one metric evolving by the Ricci flow, assuming that the evolving metrics are also diagonal. In Chapter 5 we prove Theorem A. In Chapter 6 we will present a strategy for addressing Conjecture C along with some partial results. The contents of Chapter 5 are based on joint work with Renato G. Bettiol.

Chapter 2

Cohomogeneity one manifolds

In this Chapter, we recall some basic facts about cohomogeneity one group actions and describe the structure of invariant metrics on a cohomogeneity one manifold.

2.1 Cohomogeneity one structure

A Lie group G is said to act on a manifold M with *cohomogeneity one* if the orbit space M/G is 1-dimensional (equivalently, if the generic orbits of the group action are codimension one hypersurfaces). If M is compact, this implies that M/G is isometric to either an interval $[0, L]$ or a circle S^1. The former is guaranteed when the manifold is simply connected. We will assume from now on that $M/\mathsf{G} = [0, L]$.

Let π be the quotient map $M \to M/\mathsf{G}$. For each $r \in [0, L]$, the set $\pi^{-1}(r)$ is a G-orbit inside M. The preimages of values $0 < r < L$ are codimension one hypersurfaces in M, and are called *principal orbits*. The sets $B_- = \pi^{-1}(0)$ and $B_+ =$

$\pi^{-1}(L)$ are the nonprincipal orbits, which are called *exceptional* if their codimension is one, and *singular* if their codimension is ≥ 2. If M is simply connected then nonprincipal orbits are always singular.

Choose a point $x_- \in B_-$ and let $\gamma : [0, L] \to M$ be a minimal geodesic from B_- to B_+, such that $\gamma(0) = x_-$. Then γ is a horizontal lift of $[0, L]$ to M, and meets all orbits orthogonally. Let $x_+ = \gamma(L)$. Let $\mathsf{K}_\pm$ be the isotropy groups at $x_\pm$. The isotropy group at $\gamma(r)$ is the same group H for each $0 < r < L$, is called the *principal isotropy group* and is a subgroup of $\mathsf{K}_\pm$. Thus M decomposes as a union of homogeneous spaces, $B_\pm = \mathsf{G}/\mathsf{K}_\pm$ at the ends of the interval and $\mathsf{G} \cdot \gamma(r) = \mathsf{G}/\mathsf{H}$ for each $0 < r < L$.

By the Slice Theorem, the tubular neighborhoods $D(B_-) = \pi^{-1}([0, \frac{L}{2}])$ and $D(B_+) = \pi^{-1}([\frac{L}{2}, L])$ are disk bundles over the nonprincipal orbits B_- and B_+. If $D^{l_\pm+1}$ are disks of radius $\frac{L}{2}$ normal to $B_\pm$ inside $T_{x_\pm}M$, then $\mathsf{K}_\pm$ acts transitively on $S^{l_\pm} = \partial D^{l_\pm+1}$. This implies that $S^{l_\pm} = \mathsf{K}_\pm/\mathsf{H}$. (Thus this requirement of the quotients being diffeomorphic to spheres, puts a constraint on what combination of isotropy groups can occur.) We also have

$$D(B_\pm) = \mathsf{G} \times_{\mathsf{K}_\pm} D^{l_\pm+1}.$$

The manifold M is the union of these two disk bundles glued along their common boundary $\mathsf{G}/\mathsf{H} = \pi^{-1}(\frac{L}{2})$. The identification of G/H with $\partial D(B_\pm)$ is via the maps $g \cdot \mathsf{H} \mapsto [g, \gamma'(0)]$ and $g \cdot \mathsf{H} \mapsto [g, -\gamma'(L)]$ respectively. The data $\mathsf{H} \subset \mathsf{K}_\pm \subset \mathsf{G}$ is called a *group diagram* for the cohomogeneity one action.

Conversely, if we are given groups $\mathsf{H} \subset \mathsf{K}_\pm \subset \mathsf{G}$ where G is a compact Lie group and $\mathsf{K}_\pm/\mathsf{H} = S^{l_\pm}$ are spheres then we can construct a cohomogeneity one manifold as the union of disk bundles as above.

2.2 Invariant metrics

The minimal geodesic $\gamma(r)$ for $r \in [0, L]$ parametrizes the orbit space and any invariant metric is determined by specifying it along γ and then extending it to all of M by the G-action. A cohomogeneity one metric on the principal part of M has the following form along γ:

$$\mathrm{g}(r) = \mathrm{d}r^2 + \mathrm{g}_r, \ \ r \in (0, L) \tag{2.2.1}$$

where g_r is a one parameter family of homogeneous metrics on a fixed homogeneous space G/H. This metric extends across the singular orbits to yield a smooth metric on all of M if and only if the metric and its derivatives satisfy certain conditions at the endpoints $r = 0$ and $r = L$. These conditions, which are referred to as *smoothness conditions*, will be explained in more detail in Chapter 3.

We will now explain more carefully the description of our metrics. Let $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be the group diagram at a particular singular orbit, and let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie algebras. Let Q be a biinvariant metric on $\mathfrak{g}$ and $\mathfrak{m} = \mathfrak{k}^\perp$,

$\mathfrak{p} = \mathfrak{h}^\perp \cap \mathfrak{k}$ with respect to this metric. Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \oplus \mathfrak{m}$.

Let $\{X_i\}_{i=1}^m$ be a Q-orthogonal basis for $\mathfrak{h}^\perp$ that respects the decomposition $\mathfrak{h}^\perp = \mathfrak{p} \oplus \mathfrak{m}$. That is, there exists an index l such that

$$\mathfrak{p} = \text{span}\{X_1 \cdots , X_l\}$$

$$\mathfrak{m} = \text{span}\{X_{l+1}, \cdots , X_m\}$$

Assume also that the basis elements in $\mathfrak{m}$ are Q-orthonormal. The vector space $\mathfrak{h}^\perp$ can be identified with the tangent space to G/H at $[\mathsf{H}]$ in the following way. Let $\{X_i^*(r)\}_{i=1}^m$ be Killing vector fields along the curve γ, defined by

$$X_i^*(r) = \frac{\mathrm{d}}{\mathrm{d}s} \exp(s\, X_i) \cdot \gamma(r)\big|_{s=0}$$

Then $\{X_i^*(r)\}_{i=1}^m$ is a basis for $T_{[\mathsf{H}]}\mathsf{G}/\mathsf{H}$ at $\gamma(r) = [\mathsf{H}]$. Also, for $i = 1, \cdots , k$, let ω_i be the 1-form dual to the vector field X_i^*. We further assume that the vectors X_i can be chosen such that they also respect the decomposition $\mathfrak{h}^\perp = \mathfrak{m}' \oplus \mathfrak{p}'$ at the other singular orbit.

Now, we restrict our attention to so-called *diagonal metrics*, that is, metrics which on the principal part of M are of the form

$$\mathrm{g}(r) = h(r)^2 \mathrm{d}r^2 + \sum_{i=1}^m f_i(r)^2 \omega_i^2, \;\; r \in (0, L) \tag{2.2.2}$$

where ω_i is a 1-form on G/H dual to X_i^*. Thus $f_i(r)$ denotes the length of the Killing field $X_i^*(r)$ at the point $\gamma(r) \in M$. Also here $h(r)$ is the length of the vector $\frac{\partial}{\partial r} = \gamma'(r)$, and needs to be included when the parametrization of γ is not by arclength.

The expression 2.2.2 defines an invariant metric on the principal part of M, and extends to a smooth metric on all of M if and only if the functions $f_i(r)$ satisfy *smoothness conditions* at the endpoints $r = 0$ and $r = L$. The reader may refer to Chapter 3 and also the reference [37] for more details about smoothness conditions and how to compute them.

Remark 2.2.1. The metric is not necessarily diagonal at points not on the geodesic γ. The value of $\mathrm{g}(X_i^*, X_j^*)$ at an arbitrary point of M is determined by its value along γ, with the help of the group action. In particular for any $g \in \mathsf{G}$, the inner product at the point $g\mathsf{H} \in \mathsf{G}/\mathsf{H}$ can be determined from that at the point $\mathsf{H} \in \mathsf{G}/\mathsf{H}$ in the following way:

$$\mathrm{g}(X_i^*, X_j^*)|_{g\mathsf{H}} = \mathrm{g}(Ad_{g^{-1}}X_i^*, Ad_{g^{-1}}X_j^*)|_{\mathsf{H}}$$

Since the metric on the homogeneous space G/H is left-invariant but not necessarily biinvariant, the map $Ad_g : T_{[\mathsf{H}]}\mathsf{G}/\mathsf{H} \to T_{[\mathsf{H}]}\mathsf{G}/\mathsf{H}$ need not be an isometry. Thus the Killing vector fields X_i^* and X_j^* for $i \neq j$ will in general not be orthogonal at points not on γ.

We also recall some notation used while making computations for a diagonal cohomogeneity one metric. Let $P_r : \mathfrak{h}^\perp \to \mathfrak{h}^\perp$ be defined by $\mathrm{g}(X,Y)\big|_{\gamma(r)} = Q(P_r X, Y)$. Then $P = \operatorname{diag}(f_1^2, \cdots, f_m^2)$, and the shape operator S_r is given by $S = -\operatorname{diag}(f_1'/f_1, \cdots, f_m'/f_m)$. These will be used in certain computations in Chapter 6. For more details see [17].

2.3 Examples

In this section we will provide some examples of cohomogeneity one manifolds and group diagrams emphasizing the concepts of nice basis and stably Ricci-diagonal that were defined in the introduction. Certain 4-dimensional cohomogeneity one manifolds will be described in detail in Chapter 5, where they will be used in the proof of Theorem A. For more examples the reader may refer to [20], [18], [39].

2.3.1 T^2 action on S^3

Consider S^3 as the unit sphere in $\mathbb{C}^2$, $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. The torus $T^2 = \{(e^{i\theta}, e^{i\psi}) : \theta, \psi \in [0, 2\pi)\}$ acts on S^3 by multiplication in each complex factor. The two singular orbits have codimension two in S^3, in fact they are the unit circles in each factor of $\mathbb{C}$. This action has the following group diagram: $\mathsf{H} \subset \mathsf{K}_\pm \subset \mathsf{G}$: $\{(1,1)\} \subset \{(e^{i\theta}, 1)\}, \{(1, e^{i\psi})\} \subset T^2$. We select the natural basis $X_1 = (I, 0)$, $X_2 = (0, I)$ where I spans the Lie algebra of S^1. The group T^2 is abelian and all Lie brackets are zero, so this is trivially a nice basis.

2.3.2 A group diagram with a nice basis

The standard basis of $\mathfrak{so}(n)$ is $\{E_{ij} : 1 \leq i < j \leq n\}$ where E_{ij} is the skew-symmetric matrix with a $+1$ in the (i,j) entry, a -1 in the (j,i) entry, and zeros elsewhere. It satisfies

$$[E_{ij}, E_{jk}] = E_{ik} \text{ if } i \neq k$$

$$[E_{ij}, E_{kl}] = 0 \text{ if } \{i,j\} \cap \{k,l\} = \emptyset$$

so it is a nice basis for the Lie algebra $\mathfrak{so}(n)$. As a result, if we build up a cohomogeneity one manifold M from its group diagram, where we choose the groups to be $\mathsf{G} = \mathsf{SO}(n)$ and $\mathsf{K}_- = \mathsf{SO}(l) \times \mathsf{SO}(1)$, $\mathsf{K}_+ = \mathsf{SO}(l) \times \mathsf{SO}(1)$, $\mathsf{H} = \mathsf{SO}(l)$ with standard embeddings into $\mathsf{SO}(n)$, then $(M, \mathsf{SO}(n))$ will naturally have a nice basis. Here the $\mathsf{SO}(1)$ factors in $\mathsf{K}_\pm$ can be any circle in $\mathsf{SO}(n-l)$ whose Lie algebra is spanned by one of the standard basis vectors (it need not be the same circle in K_- and in K_+). Note that $\mathsf{K}_\pm/\mathsf{H}$ so defined are each a sphere (S^1 in this case, so the singular orbits have codimension two), so this group diagram does indeed yield a cohomogeneity one manifold.

Note that we can generalize this construction to other group diagrams that have similar block embeddings and nice bases: $\mathsf{G} = \mathsf{SO}(n)$, $\mathsf{K}_\pm = \mathsf{SO}(l+1)$, $\mathsf{H} = \mathsf{SO}(l)$.

2.3.3 Kervaire sphere S^5.

It has a cohomogeneity one action (see [14]) with the following group diagram $\mathsf{H} \subset \mathsf{K}_{\pm} \subset \mathsf{G}$:

$$\mathsf{G} = \mathsf{SO}(2) \times \mathsf{SO}(3),$$

$$\mathsf{K}_{-} = \mathsf{SO}(2) = (e^{-i\theta}, \mathrm{diag}(R(d\theta), 1)),$$

$$\mathsf{K}_{+} = \mathsf{O}(2) = (\det B, \mathrm{diag}(\det B, B)),$$

$$\mathsf{H} = \mathbb{Z}_2 = \langle -1, \mathrm{diag}(-1, -1, 1) \rangle,$$

where d is an odd integer. We select the following basis for $\mathfrak{g}$, which respects the inclusions $\mathfrak{h} \subset \mathfrak{k}_{\pm} \subset \mathfrak{g}$ and is orthonormal in the natural biinvariant metric on G:

$$X_1 = \frac{1}{d^2+1}(-I, dE_{12}), X_2 = \frac{1}{d^2+1}(dI, E_{12}), X_3 = (0, E_{13}), X_4 = (0, E_{23})$$

The order 2 element h in H acts on $\mathfrak{h}^{\perp}$ by the adjoint action, sending X_1 to X_1, X_2 to X_2, X_3 to $-X_3$, and X_4 to $-X_4$. Thus each of the basis elements spans a 1 dimensional Ad_{H} module. Additionally, we record the brackets in $\mathfrak{g}$:

$$
\begin{aligned}
[X_1, X_2] &= 0, \\
[X_1, X_3] &= -\frac{d}{d^2+1} X_4 \\
[X_1, X_4] &= \frac{d}{d^2+1} X_3 \\
[X_2, X_3] &= -\frac{1}{d^2+1} X_4 \\
[X_2, X_4] &= \frac{1}{d^2+1} X_3 \\
[X_3, X_4] &= -\frac{d}{d^2+1} X_1 - \frac{1}{d^2+1} X_2
\end{aligned}
$$

Therefore this is not a nice basis. We will now show that this basis is also not stably Ricci diagonal. We refer to Proposition 1.14 in [17] for the formulae for Ricci curvature of a diagonal metric on a cohomogeneity one manifold:

$$
\mathrm{Ric}(X_1, X_2) = \sum_{r,s} \frac{f_1^2 f_2^2 - 2f_r^4 + 2f_r^2 f_s^2}{4 f_r^2 f_s^2} \sum_{e_\alpha \in \mathfrak{n}_r} Q([X_1, e_\alpha]_{\mathfrak{n}_s}, [X_2, e_\alpha]_{\mathfrak{n}_s})
$$

It is easy to see that one can choose the metric in such a way that at some points, $\mathrm{Ric}(X_1, X_2) \neq 0$. Indeed, we can choose the metric such that at some point in the interior of the geodesic γ, the functions f_i all have the same value. Notice that the above formula is purely algebraic and does not involve any spatial derivatives (second fundamental form terms). Therefore at such a point, the expression simplifies to

$$
\begin{aligned}
\mathrm{Ric}(X_1, X_2) &= \frac{1}{4} \sum_{r,s} \sum_{e_\alpha \in \mathfrak{n}_r} Q([X_1, e_\alpha]_{\mathfrak{n}_s}, [X_2, e_\alpha]_{\mathfrak{n}_s}) \\
&= \frac{1}{4}(Q([X_1, X_3]_{\mathfrak{n}_4}, [X_2, X_3]_{\mathfrak{n}_4}) + Q([X_1, X_4]_{\mathfrak{n}_3}, [X_2, X_4]_{\mathfrak{n}_3})) \\
&= \frac{1}{4}\left(\frac{d}{(d^2+1)^2} + \frac{d}{(d^2+1)^2}\right) \\
&= \frac{1}{2}\frac{d}{(d^2+1)^2} > 0
\end{aligned}
$$

where we have only written the non-zero terms in the above sum. Thus we see that this basis is *not* stably Ricci diagonal.

Chapter 3

Smoothness conditions

In this Chapter we will describe the differential conditions which guarantee that a metric defined on the principal part of a cohomogeneity one manifold closes up smoothly at the singular orbits. These *smoothness conditions* at a singular orbit give constraints on the Taylor series of the coefficients of the metric along the geodesic $\gamma(r)$. They are determined by the group diagram.

We follow the discussion in [37]. For the most part we restate their results for the case of diagonal metrics. We also extract from their discussion the conditions needed for an invariant metric to be merely C^2 at a singular orbit.

3.1 Smooth metrics

Fix a singular orbit $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$. For simplicity we will only treat the case where the singular orbit has codimension 2, that is, $\mathfrak{p}$ is 1-dimensional. We will also make the assumption that we are working with a nice basis B for the cohomogeneity one manifold (M, G). By reordering the indices if needed, we can assume that $\mathfrak{p}$ is spanned by X_1, so that the function f_1 vanishes at $r = 0$. The slice V is a normal disk to the singular orbit G/K at the point $\gamma(0)$. In this case, V is a 2-disk. The metric defined on the principal part of the cohomogeneity one manifold is smooth at a singular point ($r = 0$ or $r = L$) if and only if it is smooth when restricted to points in the slice V.

We will now collect the conditions needed for the restriction $\mathrm{g}|_V$ to be smooth. Let L be the circle which is the identity component of K, and let $X_1 \in \mathfrak{k}$ be such that $\exp(2\pi X_1) = e$, then $\mathsf{L} = \{\exp(s\, X_1) : s \in [0, 2\pi]\}$. The group L acts on the slice V by rotation, however this action need not be effective. The ineffective kernel of the action is given by $\mathsf{L} \cap \mathsf{H}$, which is a finite cyclic group. Let $\#(\mathsf{L} \cap \mathsf{H}) = a$. Thus L acts on the slice V as rotation at speed a for some positive integer a.

It will be convenient for us to write the smoothness conditions in terms of the arclength parameter s along the curve γ. (Recall that for a diagonal metric, γ is a geodesic up to reparametrization.) We use prime ($'$) to denote derivative with respect to s. Note that $\frac{\partial}{\partial s} = \frac{1}{h}\frac{\partial}{\partial r}$.

The metric restricted to the slice V is smooth if and only if (see [15], Lemma 6.2 and also [37] Section 3.1)

$$f_1(s)^2 = \mathrm{g}_{\gamma(s)}(X_1^*, X_1^*) = a^2 s^2 + s^4 \phi(s^2)$$

Under the isotropy action of the circle L on $T_{\gamma(0)}\mathsf{G}/\mathsf{K} \sim \mathfrak{m}$, we see that $\mathfrak{m}$ splits as a sum of trivial and 2-dimensional modules as follows:

$$\mathfrak{m} = \ell_0 \oplus \ell_1 \oplus \cdots \oplus \ell_s \tag{3.1.1}$$

with $\mathsf{L}|_{\ell_0} = \mathrm{Id}$, and for $i > 0$, $\mathsf{L}|_{\ell_i} = R(d_i\theta)$, i.e. a rotation at speed d_i in the 2-plane ℓ_i for some integers d_i. In general this decomposition under the L action may be different from the decomposition under the action of H, and may not be compatible with the basis $\mathcal{B}$. However,

Proposition 3.1.1. *If $\mathcal{B}$ is a nice basis then one can choose the decomposition 3.1.1 such that for each i, ℓ_i is spanned by a basis consisting of elements of $\mathcal{B}$.*

Proof. Let $X_j \in \mathcal{B}$. Then by the property of nice basis, $[X_1, X_j] = \gamma_{1j}^k X_k$ for some index k. If $\gamma_{1j}^k = 0$ then X_j spans a trivial module for the action of L on $\mathfrak{m}$. In that case $X_j \in \ell_0$.

On the other hand if $\gamma_{1j}^k \neq 0$ then by skew-symmetry of the Lie bracket we have $\gamma_{1k}^j = -\gamma_{1j}^k \neq 0$. Since the basis is nice, we have $[X_1, X_k] = \gamma_{1k}^j X_j \neq 0$. This proves that $\mathrm{span}\{X_j, X_k\}$ is a 2-dimensional module for the action of L on $\mathfrak{m}$, thus we let it be one of the $\ell_i's$.

In this manner we run through the elements of $\mathcal{B}$ and see that each element X_j must either span a trivial module and thus belong in ℓ_0, or alternately belong to a 2-dimensional module spanned by X_j and another basis element X_k. This completes the proof. □

For a diagonal metric, with $\{X_l\}$ a nice basis, Lemma 3.3 and Lemma 3.5 (a) of [37] imply:

Lemma 3.1.2. *Let $\ell_i = span\{X_j, X_k\}$ be an irreducible L-module in $\mathfrak{m}$ on which L acts via a rotation $R(d_i\theta)$. Then $\mathrm{g}|_{\ell_i}$ is smooth if and only if there exist smooth functions ϕ_i such that*

$$f_j(s)^2 + f_k(s)^2 = \phi_1(s^2),\ \ f_j(s)^2 - f_k(s)^2 = s^{\frac{2d_i}{a}}\phi_2(s^2)$$

Lemma 3.1.3. *If $X_i \in \ell_0$, then $f_i(s)^2$ is an even function of s.*

3.2 C^2 metrics

For studying the Ricci flow, C^2 regularity of the metric is sufficient. In this section we will use the discussion in [37] to derive the conditions needed for an invariant metric to be C^2. One may suspect that this is equivalent to the assumption that the even functions ϕ_i are C^2. This is actually not the case, so we derive the conditions directly, using the strategy in [37]. The following is the main result of this chapter:

Theorem 3.2.1. *For a diagonal cohomogeneity one metric with codimension two singular orbits, the condition $\mathrm{g} \in C^2$ is characterized at a singular orbit by the following conditions on the components of* g*:*

- $f_1(0) = 0,\ {f_1}'(0) = a,\ {f_1}''(0) = 0$

- ${f_i}'(0) = 0$ *for each* $X_i \in \ell_0$

- *Appropriate conditions from Table 3.1 for each 2-dimensional module* ℓ_i, $i > 0$.

Proof. By isometries, it is enough to understand when the metric components are C^2 as we restrict ourselves to move around within the slice. Since the metric is diagonal, we only need consider inner products within $\mathfrak{p}$ and within $\mathfrak{m}$. The claim of the theorem then follows from Proposition 3.2.2, Corollary 3.2.5 and Table 3.1. □

Now we will prove the results needed to obtain the above theorem. To begin with, we characterize regularity of restriction of metric to the slice V. This gives constraints on the function f_1 which is the length of the vector in $\mathfrak{p}$.

Proposition 3.2.2. *Let* $f(s) := f_1(s)/a$.

1. $\mathrm{g}|_V \in C^0$ *if and only if* $f(0) = 0$ *and* $f'(0) = 1$

2. $\mathrm{g}|_V \in C^2$ *if and only if* $f''(0) = 0$

Proof. The restriction of the metric to the slice V is given by $g|_V = \mathrm{d}s^2 + f(s)^2 \mathrm{d}\theta^2$. Converting to Cartesian coordinates, we have

$$
\begin{aligned}
g_{xx} &= \cos^2\theta + \frac{f(s)^2}{s^2}\sin^2\theta = 1 + \left(\frac{f(s)^2}{s^2} - 1\right)\sin^2\theta \\
g_{yy} &= \sin^2\theta + \frac{f(s)^2}{s^2}\cos^2\theta = 1 + \left(\frac{f(s)^2}{s^2} - 1\right)\cos^2\theta \\
g_{xy} &= \sin\theta\cos\theta\left(1 - \frac{f(s)^2}{s^2}\right)
\end{aligned}
$$

If $g|_V$ is continuous at 0 then by continuity of g_{xx} at the origin, the limit $\lim_{s\to 0} \frac{f(s)^2}{s^2}$ must exist, which implies that $f(0) = 0$. Since the limit must be independent of θ, it is necessary that $\lim_{s\to 0} \frac{f(s)^2}{s^2} = 1$, and hence that $f'(0) = \lim_{s\to 0} \frac{f(s)}{s} = 1$. From the second expression for g_{xx} it is easy to see that $f(0) = 0$ and $f'(0) = 1$ are also sufficient conditions for g_V to be continuous.

We have

$$\frac{\partial^2}{\partial s^2}\left(\frac{f(s)^2}{s^2} - 1\right) = \frac{2}{s^4}\left[s^2 f'^2 + s^2 f f'' - 4 s f f' + 3 f^2\right]$$

For g to be twice continuously differentiable at the origin, the above expression must have a finite limit as s approaches 0. Evaluating by L'Hôpital's rule, we see that $f''(0) = 0$ is a necessary condition. Computing $\frac{\partial^2}{\partial x^2} g_{xx}$ and noting that $f''(0) = 0$ implies $\lim_{s\to 0} \frac{\partial^2 g_{xx}}{\partial x^2}$ is independent of θ, we see that $f''(0) = 0$ is also a sufficient condition. □

Next we turn our attention to the metric on $\mathfrak{m}$. Following the notation in [37], let the restriction of the metric to the 2-dimensional module $\ell_i = \text{span}\{X_j, X_k\}$ be given by functions $g_{11} = g(X_j^*, X_j^*)$, $g_{12} = g(X_j^*, X_k^*)$ and $g_{22} = g(X_k^*, X_k^*)$. (In our case of a diagonal metric, $g_{12}(s) = 0$.) As in the proof of Lemma 3.3 in [37], L invariance of the metric implies that the functions $\omega(z) = (g_{11} - g_{22}) + i g_{12}$ and

$\eta(z) = (g_{11} + g_{22})(z)$ satisfy:

$$\omega(se^{i\theta}) = e^{iq\theta}\omega(s), \; \eta(se^{ia\theta}) = \eta(s)$$

where $q = \frac{2d_i}{a}$ and $\omega(s)$ denotes the restriction of ω to the geodesic $\gamma(s)$. In particular, $(g_{11} + g_{22})$ must be invariant under rotations. Note that the metric on ℓ_i is C^k if and only if both $\omega(z)$ and $\eta(z)$ are C^k. In the following two propositions we derive the conditions needed for $\omega(z)$ and $\eta(z)$ to be C^2 functions on V.

Lemma 3.2.3. *Regularity of restriction of* ω*.*

1. $\omega(z) \in C^0$ *if and only if* $\omega(s) \in C^0$ *and* $\omega(0) = 0$ *when* $q \neq 0$*.*

2. *Suppose* $\omega(z) \in C^0$*. Then* $\omega(z) \in C^1$ *if and only if* $\omega(s) \in C^1$ *and* $\omega'(0) = 0$ *when* $q \neq 1$*.*

3. *Suppose* $\omega(z) \in C^1$*. Then* $\omega(z) \in C^2$ *if and only if* $\omega(s) \in C^2$ *and* $\omega''(0) = 0$ *when* $q \neq 2$*.*

Proof. 1. If $\omega(z) \in C^0$, then clearly the function $\omega(s)$ must be a C^0 function of s. Further, $\lim_{s\to 0} \omega(se^{i\theta}) = \lim_{s\to 0} e^{iq\theta}\omega(s) = e^{iq\theta}\omega(0)$, but for continuity this limit should be independent of θ, hence $\omega(0) = 0$. Conversely, if $\omega(s) \in C^0$ and $\omega(0) = 0$ then clearly $\omega(z) \in C^0$.

2. Next, suppose that $\omega(z) \in C^1$. Then $\omega(s)$ must be a C^1 function of s and

$$\begin{aligned}\frac{\partial\omega(z)}{\partial x} &= \frac{\partial}{\partial\theta}(e^{iq\theta}\omega(s))\frac{d\theta}{dx} + \frac{\partial}{\partial r}(e^{iq\theta}\omega(s))\frac{dr}{dx} \\ &= iqe^{iq\theta}\omega(s)\frac{(-sin\theta)}{s} + e^{iq\theta}\omega'(s)\frac{x}{s} = e^{iq\theta}\left(\cos\theta\omega'(s) - iq\sin\theta\frac{\omega(s)}{s}\right). \\ \implies \lim_{s\to 0}\frac{\partial\omega(z)}{\partial x} &= e^{iq\theta}\left(\cos\theta\omega'(0) - iq\sin\theta\omega'(0)\right) = e^{iq\theta}\left(\cos\theta - iq\sin\theta\right)\omega'(0).\end{aligned}$$

If $q \neq 1$ then the fact that this limit should be independent of θ yields that $\omega'(0) = 0$. If $q = 1$ then $(\cos\theta - iq\sin\theta) = e^{-i\theta}$ so the limit is automatically independent of θ and we do not require that $\omega'(0) = 0$. (The computation for $\frac{\partial\omega(z)}{\partial y}$ is similar and does not yield any new conditions.) Conversely, $\omega(s) \in C^1$ along with $\omega'(0) = 0$ whenever $q \neq 1$, imply that $\omega(z)$ is differentiable and its derivative is continuous everywhere (including at the origin).

3. Suppose that $\omega(z) \in C^2$. Then

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\omega(z) &= \frac{\partial}{\partial x}\left[e^{iq\theta}\left(\cos\theta\omega'(s) - iq\sin\theta\frac{\omega(s)}{s}\right)\right] \\ &= e^{iq\theta}\Bigg[(-2iq\cos\theta\sin\theta + \sin^2\theta)\frac{\omega'(s)}{s} + \cos^2\theta\omega''(s) \\ &\qquad + (-q^2\sin^2\theta + 2iq\sin\theta\cos\theta)\frac{\omega(s)}{s^2}\Bigg]\end{aligned}$$

Therefore at the origin,

$$\lim_{s\to 0}\frac{\partial^2\omega(z)}{\partial x^2} = e^{iq\theta}\left[\left(\sin^2\theta + \cos^2\theta - \frac{q^2\sin^2\theta}{2}\right) - iq\cos\theta\sin\theta\right]\omega''(0)$$

If $q \neq 2$ then θ-independence of this limit implies that $\omega''(0) = 0$. If $q = 2$ then we have that

$$\lim_{s\to 0} \frac{\partial^2 \omega(z)}{\partial x^2} = e^{i2\theta} \left[\left(\sin^2\theta + \cos^2\theta - 2\sin^2\theta \right) - i2\cos\theta \sin\theta \right] \omega''(0)$$
$$= e^{i2\theta} \left[\cos 2\theta - i \sin 2\theta \right] \omega''(0) = \omega''(0)$$

So when $q = 2$, there is no condition on $\omega''(0)$. And conversely, $\omega(s) \in C^2$ along with $\omega''(0) = 0$ when $q \neq 2$, is enough to guarantee $\omega(z) \in C^2$.

□

Lemma 3.2.4. *Let $F(x, y)$ be a rotationally symmetric function. Then $F \in C^2$ if and only if $F(s) \in C^2$ as a function of one variable and $F'(0) = 0$.*

Proof. First, suppose that $F(x, y)$ is a C^1 function. Then evidently we must have $F(s) \in C^1$ as a function of one variable. In addition, L-invariance implies that $F(s) = F(-s)$, so we can write

$$F'(0) = \lim_{s\to 0} \frac{F(s) - F(0)}{s} = \lim_{s\to 0} \frac{F(-s) - F(0)}{s} = -\lim_{s\to 0} \frac{F(-s) - F(0)}{-s} = -F'(0)$$

So $F'(0) = 0$. If $F(x, y)$ is in fact a C^2 function of (x, y) then clearly $F(s)$ when considered as a function of one variable, must be a C^2 function as well.

For the converse, suppose that $F(s) \in C^2$ and that $F'(0) = 0$. First notice that for $s \neq 0$, the coordinates (s, θ) are smoothly equivalent to (x, y) so it is clear that $F(x, y) \in C^2(V \setminus 0)$ and $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ exist at each point in $V \setminus 0$. At $s = 0$ we make the following computation (and the analogous one for y) to show that the partial derivatives exist at the origin as well (in fact they are equal to 0)

$$\lim_{h\to 0}\frac{F(0+h,0)-F(0,0)}{h}=\lim_{h\to 0}\frac{F(h)-F(0)}{h}=F'(0)=0$$

Then, using the chain rule at points $(x,y)\neq(0,0)$, it is easy to see that $\frac{\partial F}{\partial y}=F'(s)\cos\theta$ and $\frac{\partial F}{\partial x}=F'(s)\sin\theta$. Since $F'(0)=0$, we see that the limits of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ as $s\to 0$ are each 0, and in particular are independent of θ, thus proving that $F(x,y)\in C^1$. In a similar fashion we can use $F(s)\in C^2$ to show the continuity of second partial derivatives of F at $(0,0)$. $\square$

Corollary 3.2.5. *If $X_i\in\ell_0$ (i.e. the trivial module under the L-action on $\mathfrak{m}$) then the metric in the direction of X_i is in C^2 if and only if ${f_i}'(0)=0$.*

Corollary 3.2.6. *The function $\eta(z)$ is in C^2 if and only if $\eta(s)\in C^2$ and $\eta'(0)=0$.*

The content of Lemma 3.2.3 and Corollary 3.2.6 can be summarized in the following table. Recall that $q=\frac{2d_i}{a}$, where a is the speed at which L rotates the slice V, and d_i is the speed at which L rotates the 2-dimensional module ℓ_i.

	C^0	C^1	C^2
$q=1$	$f_j(0)-f_k(0)=0$	$f_j'(0)+f_k'(0)=0$	$f_j''(0)-f_k''(0)=0$
$q=2$	$f_j(0)-f_k(0)=0$	$f_j'(0)+f_k'(0)=0$ $f_j'(0)-f_k'(0)=0$	
$q\neq 1,2$	$f_j(0)-f_k(0)=0$	$f_j'(0)+f_k'(0)=0$ $f_j'(0)-f_k'(0)=0$	$f_j''(0)-f_k''(0)=0$

Table 3.1: Necessary and sufficient conditions for metric on ℓ_i to be C^2

Chapter 4

The Ricci flow equation for a cohomogeneity one metric

In this Chapter we will derive the coupled system of PDEs that are satisfied by a diagonal cohomogeneity one metric evolving by the Ricci flow. Consider a diagonal cohomogeneity one metric g

$$\mathrm{g}(r) = h(r)^2 \mathrm{d}r^2 + \sum_{i=1}^{m} f_i(r)^2 \omega_i^2$$

We use $K(\cdot,\cdot)$ to denote the Killing form of $\mathfrak{g}$. The structure constants γ_{ij}^k for $\mathfrak{g}$ in terms of a Q-orthonormal basis $\{X_i\}$ for $\mathfrak{g}$, are defined via:

$$[X_i, X_j] = \sum_k \gamma_{ij}^k X_k$$

Proposition 4.0.1. *The Ricci tensor of the metric* g *satisfies*

$$\begin{aligned}
\mathrm{Ric}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) &= -\sum_{j=1}^{m}\left(\frac{f_{j_{rr}}}{f_j}-\frac{f_{j_r}h_r}{hf_j}\right)\\
\mathrm{Ric}(X_i^*,X_i^*) &= -\frac{b_i}{2}+\sum_{j,k=1}^{m}\frac{f_i^4-2f_k^4}{4f_j^2f_k^2}(\gamma_{jk}^i)^2\\
&\quad+\left\{-\frac{f_{i_r}}{hf_i}\sum_{j=1}^{m}\frac{f_{j_r}}{hf_j}+\frac{f_{i_r}^{\,2}}{h^2f_i^2}-\frac{f_{i_{rr}}}{h^2f_i}+\frac{f_{i_r}h_r}{h^3f_i}\right\}f_i^2
\end{aligned}$$

where $b_i=K(X_i,X_i)$*,* $r\in(0,L)$ *and* $i=1,\cdots m$*.*

Proof. The unit tangent vector along the curve γ is given by $T=\frac{1}{h}\frac{\partial}{\partial r}$. Therefore by Proposition 1.14 and Remark 1.16 in [17], the Ricci tensor of a cohomogeneity one manifold (M,g) is given by:

$$\begin{aligned}
\mathrm{Ric}\left(\frac{1}{h}\frac{\partial}{\partial r},\frac{1}{h}\frac{\partial}{\partial r}\right) &= -\sum_{j}\frac{f_j''}{f_j}\\
\mathrm{Ric}(X_i^*,X_i^*) &= -\frac{b_i}{2}+\sum_{j,k}\frac{f_i^4-2f_k^4}{4f_j^2f_k^2}(\gamma_{jk}^i)^2\\
&\quad+\left\{-\frac{f_i'}{f_i}\sum_{j}\frac{f_j'}{f_j}+\frac{f_i'^2}{f_i^2}-\frac{f_i''}{f_i}\right\}f_i^2\|X_i\|_Q^2
\end{aligned}$$

As in previous chapters, we use $'$ to refer to derivative with respect to the arclength parameter s along the geodesic γ, defined by $\mathrm{d}s=h(r)\mathrm{d}r$. Then accounting for the reparametrization of γ by arclength, we substitute $\frac{1}{h}\frac{\partial}{\partial r}$ in place of $'$ in the above formulae. This completes the proof. □

We are now ready to write the Ricci flow equations for a diagonal cohomogeneity one metric. That is, assuming that the flow is through diagonal metrics, we write down the coupled PDEs that need to be satisfied by the components of the metric.

Proposition 4.0.2. *Let* $\mathrm{g}(t)$ *be a time-dependent diagonal metric evolving by the Ricci flow. Then the functions* $h, f_1, \cdots, f_m$ *satisfy the following system of PDEs:*

$$
\begin{aligned}
h_t &= \sum_{j=1}^{m} \left(\frac{f_{j_{rr}}}{h f_j} - \frac{f_{j_r} h_r}{h^2 f_j} \right) \\
f_{i_t} &= \frac{f_{i_{rr}}}{h^2} - \frac{f_{i_r} h_r}{h^3} + \frac{f_{i_r}}{h} \sum_{j=1}^{m} \frac{f_{j_r}}{h f_j} - \frac{f_{i_r}^{\,2}}{h^2 f_i} - \sum_{j,k=1}^{m} \frac{f_i^4 - 2 f_k^4}{4 f_i f_j^2 f_k^2} {\gamma_{jk}^{i}}^{2} + \frac{b_i}{2 f_i} \qquad (4.0.1) \\
& t \in (0, T),\ r \in (0, L),\ i = 1, \cdots m
\end{aligned}
$$

Proof. A time-dependent diagonal metric g and diagonal Ricci tensor can be written as:

$$
\begin{aligned}
\mathrm{g}(r,t) &= h(r,t)^2 \, \mathrm{d}r^2 + \sum_{i=1}^{m} f_i(r,t)^2 \, \omega_i^2 \\
\mathrm{Ric}_{\mathrm{g}}(r,t) &= \mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \mathrm{d}r^2 + \sum_{i=1}^{m} \mathrm{Ric}(X_i^*, X_i^*) \, \omega_i^2
\end{aligned}
$$

Differentiating the metric term by term with respect to t yields

$$
\frac{\mathrm{dg}}{\mathrm{d}t} = 2 h h_t \, \mathrm{d}r^2 + \sum_{i=1}^{m} 2 f_i f_{i_t} \, \omega_i^2
$$

Substituting these in the Ricci flow equation and comparing coefficients, along with Proposition 4.0.1, then yields the result. □

Chapter 5

Proof of Theorem A

In this Chapter we will prove Theorem A which was stated in the Introduction:

THEOREM (Bettiol–Krishnan [4]). There exist metrics with $\sec \geq 0$ on S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ that immediately lose the property of $\sec \geq 0$ when evolved by the Ricci flow.

The proof proceeds via studying the Ricci flow evolution of invariant cohomogeneity one metrics on these 4-manifolds. In fact these four are the only closed simply-connected 4-manifolds that admit cohomogeneity one structures, see [30]. We now list the group diagrams, corresponding to the cohomogeneity one actions that we use in order to describe invariant metrics on these manifolds. The table below does not list *all* cohomogeneity one actions existing on these manifolds, but only the ones that will be considered in the proof of Theorem A. As we will see, the group diagrams listed below share some common features, which will allow us to treat all four of the manifolds simultaneously while proving the theorem.

M	$\mathsf{H} \subset \{\mathsf{K}_-, \mathsf{K}_+\} \subset \mathsf{G}$
S^4	$\mathsf{S(O(1)O(1)O(1))} \subset \{\mathsf{S(O(2)O(1))}, \mathsf{S(O(1)O(2))}\} \subset \mathsf{SO}(3)$
$\mathbb{C}P^2$	$\mathbb{Z}_2 = \langle \mathrm{diag}(-1,-1,1)\rangle \subset \{\mathsf{S(O(1)O(2))}, \mathsf{SO}(2)_{1,2}\} \subset \mathsf{SO}(3)$
$S^2 \times S^2$	$\mathbb{Z}_n = \left\langle e^{2\pi i/n}\right\rangle \subset \left\{\{e^{i\theta}\}, \{e^{i\theta}\}\right\} \subset \mathsf{Sp}(1), \quad n$ even
$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$	$\mathbb{Z}_n = \left\langle e^{2\pi i/n}\right\rangle \subset \left\{\{e^{i\theta}\}, \{e^{i\theta}\}\right\} \subset \mathsf{Sp}(1), \quad n$ odd

Table 5.1: Group diagrams for 1-connected cohomogeneity one 4-manifolds

In the above table, $\mathsf{SO}(2)_{1,2}$ is the upper block diagonal embedding of $\mathsf{SO}(2)$ in $\mathsf{SO}(3)$; $\mathsf{S(O(1)O(2))}$ is the collection of elements in $\mathsf{O}(1) \times \mathsf{O}(2) \subset \mathsf{O}(3)$ which have determinant 1; $\mathsf{S(O(1)O(1)O(1))}$ is the finite group consisting of diagonal matrices in $SO(3)$; and $\mathsf{Sp}(1) \cong S^3 \subset \mathbb{H}$ is identified with the group of unit quaternions. More information can be found in the references [18], [20].

In each case, G is either $\mathsf{SO}(3)$ or $\mathsf{Sp}(1)$, so in each case the Lie algebra $\mathfrak{g}$ is isomorphic to the three-dimensional Lie algebra $\mathfrak{su}(2)$. The isotropy groups $\mathsf{K}_\pm$ at the singular orbits are unions of finitely many circles S^1, and the principal isotropy group H is always a finite group, so its Lie algebra $\mathfrak{h}$ is trivial. In particular, on the regular part $M \setminus B_\pm$, there are 3 linearly independent Killing vector fields X_1^*, X_2^*, and X_3^*, which are action fields corresponding to a basis of $\mathfrak{g}$.

More precisely, $X_i^*(p) = \frac{\mathrm{d}}{\mathrm{d}s} \exp(s\, v_i) \cdot p\big|_{s=0}$, where $\{v_i\}$ is the basis $\{I, J, K\}$ in the case of $\mathsf{Sp}(1)$, and $\{E_{23}, E_{31}, E_{12}\}$ in the case of $\mathsf{SO}(3)$, where E_{jk} is the skew-symmetric matrix with a $+1$ in the (j,k) entry, a -1 in the (k,j) entry, and zeros elsewhere.

Thus, along a minimal gedesic γ between the singular orbits, a diagonal metric can be written as

$$g = dr^2 + f_1(r)^2\omega_1^2 + f_2(r)^2\omega_2^2 + f_3(r)^2\omega_3^2, \qquad 0 < r < L, \tag{5.0.1}$$

5.1 The $\sec \geq 0$ metrics

Geometrically, the 4-manifold M in each case above, is foliated by a 1-parameter family of 3-manifolds that are finite quotients of S^3, collapsing at the endpoints of the interval to 2-dimensional (hence codimension two) singular orbits $B_\pm = \mathsf{G}/\mathsf{K}_\pm$. (This means that at each of the endpoints $r = 0$ and $r = L$ only one of the functions f_1, f_2, and f_3, vanishes.) For cohomogeneity one manifolds whose singular orbits have codimension two, one has the following theorem.

Theorem 5.1.1 (Grove–Ziller, [16])**.** *Any cohomogeneity one manifold with codimension two singular orbits admits a nonnegatively curved invariant metric.*

By this theorem, each of the 4-manifolds S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ admit metrics g_{GZ} with $\sec \geq 0$, and which are invariant with respect to the actions whose group diagrams are listed above. These are the non-negatively curved metrics used to prove Theorem A.

We will now discuss some details about these metrics. Some features common to all of them (originating from the gluing in the Grove-Ziller construction), are that they are diagonal metrics which have flat planes at all points, including planes along

$\gamma(r)$ that contain the tangent direction $\gamma'(r)$. Moreover, the two functions among f_1, f_2, and f_3 that do not vanish at the particular endpoint corresponding to a singular orbit B_- or B_+, are equal and constant in a neighborhood of that endpoint. The remaining function vanishes at that endpoint with nonvanishing first derivative. (This last fact is true for any invariant metric, not just g_{GZ}, and can be seen from the discussion on smoothness conditions in Chapter 3, for example, see Theorem 3.2.1.) We will prove that in each case there are sufficiently many isometries to ensure that *the metric remains diagonal along the Ricci flow.* These features are key in the proof of Theorem A.

We will now describe the group actions (and in some cases the additional discrete isometries present) in the case of each of the above 4-manifolds.

5.1.1 S^4

The $\mathsf{SO}(3)$-action on S^4 can be described as the restriction to the unit sphere of the action by conjugation on the space $V \cong \mathbb{R}^5$ of symmetric traceless 3×3 real matrices. The singular orbits $B_\pm$ are Veronese embeddings of $\mathbb{R}P^2$ formed by matrices with 2 equal eigenvalues of the same sign; while principal orbits are diffeomorphic to the real flag manifold $W^3 = S^3/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and formed by generic matrices in V. In the round metric, the following curve is a horizontal geodesic joining $x_- = \frac{1}{\sqrt{6}}\operatorname{diag}(1,1,-2) \in B_-$ to $x_+ = \frac{1}{\sqrt{6}}\operatorname{diag}(2,-1,-1) \in B_+$.

$$\gamma(r) = \operatorname{diag}\left(\tfrac{\cos r}{\sqrt{6}} + \tfrac{\sin r}{\sqrt{2}}, \tfrac{\cos r}{\sqrt{6}} - \tfrac{\sin r}{\sqrt{2}}, -\tfrac{2\cos r}{\sqrt{6}}\right) \in V, \qquad 0 < r < \tfrac{\pi}{3}.$$

In general, if we are considering a different metric on S^4, γ is merely a curve that is transverse to the orbits, and which parametrizes the orbit space. In this description, the round metric on S^4 takes the form (5.0.1) where

$$f_1(r) = 2\sin r, \quad f_2(r) = \sqrt{3}\cos r + \sin r, \quad f_3(r) = \sqrt{3}\cos r - \sin r. \tag{5.1.1}$$

Proposition 5.1.2. *Any* $\mathsf{SO}(3)$*-invariant metric* g *on* S^4 *is of the form 5.0.1.*

Proof. Given any $\mathsf{SO}(3)$-invariant metric g on S^4, there are isometries given by the elements $h_i \in \mathsf{H}$,

$$h_1 = \operatorname{diag}(1,-1,-1),$$

$$h_2 = \operatorname{diag}(-1,1,-1),$$

$$h_3 = \operatorname{diag}(-1,-1,1),$$

that fix each point $\gamma(r)$ and $\mathrm{d}h_i(\gamma(r))\colon T_{\gamma(r)}S^4 \to T_{\gamma(r)}S^4$ act as

$$\begin{aligned} \mathrm{d}h_1(\gamma(r)) &= \operatorname{diag}(1,1,-1,-1), \\ \mathrm{d}h_2(\gamma(r)) &= \operatorname{diag}(1,-1,1,-1), \\ \mathrm{d}h_3(\gamma(r)) &= \operatorname{diag}(1,-1,-1,1), \end{aligned} \tag{5.1.2}$$

with respect to the frame $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$ at $\gamma(r)$. Thus under the isotropy action of H, $T_{\gamma(r)}S^4$ splits as the direct sum of 4 inequivalent 1-dimensional representations spanned by the X_i^* and $\frac{\partial}{\partial r}$. Since the metric g at $\gamma(r)$ must be an $\mathrm{Ad}(\mathsf{H})$-invariant tensor on $T_{\gamma(r)}M$, hence g must be diagonal (i.e. of the form 5.0.1), i.e., $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$ is a g-*orthogonal frame* along $\gamma(r)$. □

Remark 5.1.3. The fact that for any invariant metric $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$ is a g-orthogonal frame along $\gamma(r)$ can also be seen by a simple calculation. Indeed, for $i \neq j$,

$$\begin{aligned} \mathrm{g}(X_i^*, X_j^*) &= \mathrm{g}\big(\mathrm{d}h_i(X_i^*), \mathrm{d}h_i(X_j^*)\big) = -\mathrm{g}(X_i^*, X_j^*) \\ \mathrm{g}\big(\tfrac{\partial}{\partial r}, X_j^*\big) &= \mathrm{g}\big(\mathrm{d}h_i\big(\tfrac{\partial}{\partial r}\big), \mathrm{d}h_i\big(X_j^*\big)\big) = -\mathrm{g}\big(\tfrac{\partial}{\partial r}, X_j^*\big), \end{aligned} \tag{5.1.3}$$

which implies $\mathrm{g}(X_i^*, X_j^*) = 0$ and $\mathrm{g}\big(\frac{\partial}{\partial r}, X_j^*\big) = 0$.

5.1.2 $\mathbb{C}P^2$

The $\mathsf{SO}(3)$-action on $\mathbb{C}P^2$ is obtained as the subaction of the transitive $\mathsf{SU}(3)$-action. The singular orbit B_- is the totally real $\mathbb{R}P^2 \subset \mathbb{C}P^2$, and $B_+ \cong S^2$ is the quadric $\left\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 : \sum_j z_j^2 = 0\right\}$. In the Fubini-Study metric, the following curve is a horizontal geodesic joining $x_- = [1:0:0] \in B_-$ to $x_+ = \left[\frac{1}{\sqrt{2}} : \frac{i}{\sqrt{2}} : 0\right] \in B_+$.

$$\gamma(r) = [\cos r : i \sin r : 0], \qquad 0 < r < \tfrac{\pi}{4}.$$

In this description, the Fubini-Study metric on $\mathbb{C}P^2$ takes the form (5.0.1) where

$$f_1(r) = \sin r, \quad f_2(r) = \cos 2r, \quad f_3(r) = \cos r. \tag{5.1.4}$$

Now we will describe an additional diffeomorphism of $\mathbb{C}P^2$, not coming from $SO(3)$. Consider the complex conjugation map

$$c\colon \mathbb{C}P^2 \to \mathbb{C}P^2, \qquad c\big([z_0 : z_1 : z_2]\big) = [\overline{z_0} : \overline{z_1} : \overline{z_2}], \tag{5.1.5}$$

which clearly commutes with the $\mathsf{SO}(3)$-action and is an involution with fixed point set B_-. Define $\phi = g \circ c$, where $g = \operatorname{diag}(1, -1, -1) \in \mathsf{SO}(3)$. It is easy to show that

Proposition 5.1.4. *The map ϕ is a diffeomorphism that fixes the above curve $\gamma(r)$ pointwise. Its linearization at any such point $\gamma(r)$ is the linear transformation on $T_{\gamma(r)}\mathbb{C}P^2$ with matrix $\phi_* = \operatorname{diag}(1, 1, -1, -1)$ with respect to the frame $\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\}$.*

Proof. The map ϕ is a composition of diffeomorphisms, hence a diffeomorphism itself. Also observe that ϕ fixes γ pointwise:

$$\begin{aligned}\phi(\gamma(r)) = g \circ c([\cos r : i\,\sin r : 0]) &= \operatorname{diag}(1, -1, -1)([\cos r : -i\,\sin r : 0]) \\ &= [\cos r : i\,\sin r : 0] = \gamma(r)\end{aligned}$$

Since ϕ fixes γ pointwise, clearly $\phi_*\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r}$. Also note that

$$\begin{aligned}
X_1^* &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\exp(s\,E_{23}) \cdot [\cos r : i\ \sin r : 0] \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}[\cos r : i\ \sin r \cos s : -i \sin r \sin s]) \\
&= [0 : 0 : -\sin r]
\end{aligned}$$

where the last expression should be understood to mean the projection to $\mathbb{C}P^2$ of a tangent vector to $S^5 \subset \mathbb{C}^3$. On the other hand,

$$\begin{aligned}
\phi_* X_1^* &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\mathrm{diag}(1, -1, -1) \cdot c \cdot \exp(s\,E_{23}) \cdot [\cos r : i\ \sin r : 0]) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\mathrm{diag}(1, -1, -1) \cdot c \cdot [\cos r : i\ \sin r \cos s : -i \sin r \sin s]) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\mathrm{diag}(1, -1, -1) \cdot [\cos r : -i\ \sin r \cos s : i \sin r \sin s]) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}[\cos r : i\ \sin r \cos s : -i \sin r \sin s] \\
&= [0 : 0 : -\sin r] \\
&= X_1^*
\end{aligned}$$

We also have

$$\begin{aligned}
X_3^* &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\exp(s\,E_{12}) \cdot [\cos r : i\ \sin r : 0]) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}[\cos s \cos r + i \sin s \sin r : -\sin s \cos r + i \cos s \sin r : 0] \\
&= [i \sin r : -\cos r : 0]
\end{aligned}$$

which implies that

$$\phi_* X_3^* = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (\mathrm{diag}(1,-1,-1)\cdot c \cdot \exp(s\,E_{12}) \cdot [\cos r : i\,\sin r : 0])$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\mathrm{diag}(1,-1,-1)\cdot[\cos s\cos r - i\sin s\sin r : -\sin s\cos r - i\cos s\sin r : 0])$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}[\cos s\cos r - i\sin s\sin r : \sin s\cos r + i\cos s\sin r : 0]$$

$$= [-i\sin r : \cos r : 0]$$

$$= -X_3^*$$

A similar computation shows that $\phi_* X_2^* = -X_2^*$, thus completing the proof. $\square$

Corollary 5.1.5. *For any metric of the form 5.0.1, ϕ is an isometry of the metric that fixes γ pointwise.*

Proof. Given any $p \in \mathbb{C}P^2$, there exists $g_p \in \mathsf{SO}(3)$ such that $g_p \cdot p$ lies in γ, and hence one may write $c(p) = (gg_p)^{-1}gg_p \cdot c(p) = (gg_p)^{-1}g \cdot c(g_p \cdot p)$ as a composition of diffeomorphisms whose linearization is isometric. It thus follows that c, and hence $\phi = g \circ c$, are isometries of $(\mathbb{C}P^2, \mathrm{g}_{\mathrm{GZ}})$. $\square$

In particular, ϕ is an isometry of $(\mathbb{C}P^2, \mathrm{g}_{\mathrm{GZ}})$ that fixes γ pointwise.

Proposition 5.1.6. *Any metric g on $\mathbb{C}P^2$ that is invariant under both $\mathsf{SO}(3)$ and ϕ is of the form 5.0.1 along γ.*

Proof. We claim that if g is any $\mathsf{SO}(3)$-invariant Riemannian metric on $\mathbb{C}P^2$ such that ϕ is an isometry, then $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$ is g-orthogonal and hence g must also be of the form (5.0.1). Indeed, using ϕ in conjunction with $\mathrm{diag}(-1,-1,1) \in \mathsf{H}$, one can produce sufficiently many isometries of $(\mathbb{C}P^2, \mathrm{g})$ that fix each point $\gamma(r)$ and act on $T_{\gamma(r)}\mathbb{C}P^2$ just as (5.1.2), so that an argument analogous to (5.1.3) may be carried out. □

5.1.3 $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

The $\mathsf{Sp}(1)$-actions on $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are induced by quaternionic left-multiplication on the first factor of $S^3 \times S^2 \subset \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ after taking the quotient by the diagonal circle action $e^{i\theta} \cdot (q, z, x) = \left(q\, e^{i\theta}, z\, e^{in\theta}, x\right)$. The orbit space $M_n = (S^3 \times S^2)/S^1$ of this circle action is diffeomorphic to $S^2 \times S^2$ if n is even, and to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ if n is odd. The singular orbits $B_\pm$ are both diffeomorphic to S^2, and lift to $S^3 \times \{\pm N\} \subset S^3 \times S^2$ where $N = \left(0, \frac{1}{2}\right) \in S^2\left(\frac{1}{2}\right) \subset \mathbb{C} \oplus \mathbb{R}$ is the North Pole, while principal orbits are diffeomorphic to the Lens space $S^3/\mathbb{Z}_n$. The following curve γ joining $x_- = \left[1, 0, -\frac{1}{2}\right]$ to $x_+ = \left[1, 0, \frac{1}{2}\right]$ is transverse to all orbits and parametrizes the orbit space.

$$\gamma(r) = \left[1, \tfrac{1}{2}\sin 2r, -\tfrac{1}{2}\cos 2r\right] \in M_n, \qquad 0 < r < \tfrac{\pi}{2},$$

where brackets indicate the coordinates induced by $\mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ in the quotient space. Similar to the previous examples, in this description, the metric g_{GZ} on M_n is of the diagonal form (5.0.1) with f_1, f_2, and f_3 satisfying analogous properties.

Consider the involutions given by conjugation by $j, k \in \mathsf{Sp}(1)$,

$$\phi_j, \phi_k \colon M_n \to M_n, \quad \phi_j\left([q,z,x]\right) = [-j\,q\,j, z, x], \quad \phi_k\left([q,z,x]\right) = [-k\,q\,k, z, x]. \tag{5.1.6}$$

Since $\mathsf{K}_- = \mathsf{K}_+ = \{e^{i\theta}\}$, therefore $j, k \in N(\mathsf{K}_\pm)$, so we see that the above maps are well-defined diffeomorphisms that leave invariant the $\mathsf{Sp}(1)$-orbits and act on them via conjugation, that is, the restrictions of ϕ_j and ϕ_k to $\mathsf{G}(\gamma(r)) \cong \mathsf{G}/\mathsf{H} = \mathsf{Sp}(1)/\mathbb{Z}_n$ are given by $\phi_j(g\mathsf{H}) = -jgj\mathsf{H}$ and $\phi_k(g\mathsf{H}) = -kgk\mathsf{H}$; recall that $j, k \in \mathsf{N}(\mathsf{H})$.

Proposition 5.1.7. *The maps ϕ_j and ϕ_k fix the geodesic $\gamma(r)$ pointwise and their linearizations at any such point are the linear transformations on $T_{\gamma(r)}M_n$ with matrices $(\phi_j)_* = \mathrm{diag}(1,-1,1,-1)$ and $(\phi_k)_* = \mathrm{diag}(1,-1,-1,1)$ with respect to the frame $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$.*

Proof. Similar to the proof of Proposition 5.1.4. □

Corollary 5.1.8. *For any metric of the form 5.0.1, each of ϕ_j and ϕ_k is an isometry of the metric that fixes γ pointwise.*

Proof. Given any $p \in M_n$, there exist $g_p, g' \in \mathsf{Sp}(1)$ such that $g_p \cdot p$ lies in γ and $\phi_j(g_p \cdot p) = (g')^{-1}\phi_j(p)$, so one may write $\phi_j(p) = g' \cdot \phi_j(g_p \cdot p)$ as a composition of diffeomorphisms whose linearizations are isometric, and analogously for ϕ_k. □

It thus follows that ϕ_j and ϕ_k are isometries of $(M_n, \mathrm{g}_{\mathrm{GZ}})$.

Proposition 5.1.9. *Any metric g on M_n that is invariant under $\mathsf{Sp}(1)$ and ϕ_j and ϕ_k is of the form 5.0.1 along γ.*

Proof. Using ϕ_j and ϕ_k, one can produce sufficiently many isometries of (M_n, g) so that an argument analogous to (5.1.3) may be carried out. □

Remark 5.1.10. When we make reference to *the Grove-Ziller metric* on $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we mean a Grove-Ziller metric g_{GZ} on any of the (infinitely many) cohomogeneity manifolds M_n where n has the appropriate parity.

5.2 Evolution under Ricci flow

In this section, we analyze the Ricci flow evolution of the cohomogeneity one 4-manifolds with $\sec \geq 0$ discussed above, showing that they remain diagonal under the flow (Proposition 5.2.1), and proving Theorem A.

5.2.1 Flow behavior

As mentioned in the introduction, the isometry group of (M, g_t) remains constant. In particular, cohomogeneity one metrics evolve via Ricci flow through other metrics invariant under the same cohomogeneity one action. Nevertheless, the horizontal geodesic γ joining the singular orbits, and hence the description (2.2.1) of the cohomogeneity one metric, may in general change with time. We now show that this is not the case for the Grove-Ziller metrics in the 4-dimensional examples discussed above, using their additional isometries.

Proposition 5.2.1. *The Ricci flow evolution* $\mathrm{g}(t)$ *of the metric* $\mathrm{g}_{\mathrm{GZ}} = \mathrm{g}(0)$ *on each* *of S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, is through other diagonal metrics*

$$\mathrm{g}(t) = h(r,t)^2\mathrm{d}r^2 + f_1(r,t)^2\omega_1^2 + f_2(r,t)^2\omega_2^2 + f_3(r,t)^2\omega_3^2, \quad 0 < r < L, \tag{5.2.1}$$

along the g_{GZ}-geodesic $\gamma(r)$, where h, f_1, f_2, and f_3, are smooth functions of r and t.

Proof. The metric g_{GZ} is a diagonal metric of the form (5.0.1), and $\gamma(r)$ is a g_{GZ}-geodesic parametrized by arclength. Since isometries are preserved, the Ricci flow evolution of g_{GZ} is through metrics $\mathrm{g}(t)$ which are invariant under the G-action as well as under (5.1.5) on $\mathbb{C}P^2$ and (5.1.6) on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, since these are isometries of the initial metric g_{GZ}. As discussed in Subsections 5.1.1, 5.1.2, and 5.1.3, by means of these isometries, the frame $\left\{\frac{\partial}{\partial r}, X_1^*, X_2^*, X_3^*\right\}$ along $\gamma(r)$ must be $\mathrm{g}(t)$-orthogonal. In particular, $\mathrm{g}(t)$ are diagonal cohomogeneity one metrics of the form (5.2.1) along $\gamma(r)$, which is $\mathrm{g}(t)$-orthogonal to the G-orbits and hence a horizontal $\mathrm{g}(t)$-geodesic (up to reparametrization). □

Remark 5.2.2. The Grove-Ziller metric g_{GZ} is smooth but not real-analytic, as there are points where all derivatives of f_1, f_2, and f_3 vanish, but these functions are not globally constant. However, the metrics $\mathrm{g}(t)$, $t > 0$, are real-analytic by Bando [3]. Moreover, since real-analyticity is preserved under Ricci flow, there does not exist a solution to the *backwards Ricci flow* with g_{GZ} as terminal condition.

Proposition 5.2.3. *Let (M, g) be a 4-manifold with a cohomogeneity one action* *of a Lie group* G *whose Lie algebra is isomorphic to* $\mathfrak{su}(2)$*. Assume that* g *is a diagonal metric of the form* (5.0.1) *and that its Ricci flow evolution* $\mathrm{g}(t)$ *is through other diagonal metrics, as in* (5.2.1)*. Then the functions* $h(r,t)$*,* $f_1(r,t)$*,* $f_2(r,t)$*, and* $f_3(r,t)$ *satisfy the degenerate parabolic system of partial differential equations*

$$\begin{aligned}
h_t &= -\left(\frac{f_{1r}}{f_1}+\frac{f_{2r}}{f_2}+\frac{f_{3r}}{f_3}\right)\frac{h_r}{h^2}+\left(\frac{f_{1rr}}{f_1}+\frac{f_{2rr}}{f_2}+\frac{f_{3rr}}{f_3}\right)\frac{1}{h}\\
f_{1t} &= \frac{1}{h^2}\,f_{1rr}+\frac{1}{hf_2f_3}\left(\frac{f_2f_3}{h}\right)_r f_{1r}-\frac{2}{f_2^2f_3^2}f_1^3+\frac{2(f_2^2-f_3^2)^2}{f_2^2f_3^2}\frac{1}{f_1}\\
f_{2t} &= \frac{1}{h^2}\,f_{2rr}+\frac{1}{hf_1f_3}\left(\frac{f_1f_3}{h}\right)_r f_{2r}-\frac{2}{f_1^2f_3^2}f_2^3+\frac{2(f_1^2-f_3^2)^2}{f_1^2f_3^2}\frac{1}{f_2}\\
f_{3t} &= \frac{1}{h^2}\,f_{3rr}+\frac{1}{hf_1f_2}\left(\frac{f_1f_2}{h}\right)_r f_{3r}-\frac{2}{f_1^2f_2^2}f_3^3+\frac{2(f_1^2-f_2^2)^2}{f_1^2f_2^2}\frac{1}{f_3}
\end{aligned} \tag{5.2.2}$$

where subscripts denote derivative with respect to that variable.

Proof. Follows directly from Proposition 4.0.2, using the structure constants of $\mathfrak{su}(2)$. □

5.2.2 Curvature evolution

We are now ready to analyze the evolution of sectional curvatures of g_{GZ} under Ricci flow, proving Theorem A.

Proof of Theorem A. Let M be any of the cohomogeneity one 4-manifolds discussed above, and equip it with the Grove-Ziller metric g_{GZ}. By Propositions 5.2.1 and 5.2.3, the Ricci flow evolution of $\mathrm{g}(0) = \mathrm{g}_{\mathrm{GZ}}$ is through other diagonal metrics of the form (5.2.1), satisfying (5.2.2).

The initial metric g(0) is such that, near each singular orbit $B_{\pm}$, the two functions among f_1, f_2, and f_3 corresponding to the two noncollapsing directions among X_1^*, X_2^*, and X_3^* are equal and constant. Up to relabelling, assume these are X_2^* and X_3^* near B_-, so that

$$f_2(r,0) = f_3(r,0) = const. > 0, \quad \text{for all} \quad 0 < r < \varepsilon, \tag{5.2.3}$$

while $f_1(0,t) = 0$ for all $t \geq 0$. Fix $0 < r_0 < \varepsilon$ and let $\sigma \subset T_{\gamma(r_0)}M$ be the tangent plane spanned by $\frac{\partial}{\partial r}$ and X_3^*. The sectional curvature of σ is given by

$$\sec_{g(t)}(\sigma) = -\frac{f_3''}{f_3} = -\frac{1}{f_3 h}\left(\frac{f_{3r}}{h}\right)_r = \frac{f_{3r}h_r}{f_3 h^3} - \frac{f_{3rr}}{f_3 h^2}$$

computed at $r = r_0$. As a consequence of (5.2.3), this plane σ is flat (i.e. $\sec_{g(t)}(\sigma) = 0$) at time $t = 0$. Moreover, as $h(r,0) \equiv 1$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sec_{g(t)}(\sigma)\Big|_{t=0} = -\frac{f_{3rrt}}{f_3}\Big|_{r=r_0,t=0}. \tag{5.2.4}$$

The right hand side of the equation for f_3 in (5.2.2) simplifies due to (5.2.3), yielding

$$f_{3t}\big|_{t=0} = \frac{2(f_2^2 - f_1^2)^2 - 2f_3^4}{f_3 f_2^2 f_1^2}, \qquad 0 < r < \varepsilon.$$

Differentiating the above expression in r twice and using (5.2.3) once more, we have

$$f_{3rrt}\big|_{r=r_0,t=0} = \frac{4({f_1}_r^2 + f_{1rr}f_1)}{f_3^3}\Big|_{r=r_0,t=0}.$$

By the smoothness conditions for a cohomogeneity one metric (in particular, see 3.2.1), $f_{1r} = ah$ for some positive integer a at $r = 0$ and f_1 must be an odd function of r; in particular, $f_{1rr}(0,t) = 0$. Also since $h(r,0) \equiv 1$, for small enough times $t > 0$, $h(r,t) > 0$, bounded away from 0. Therefore, up to choosing an even smaller $0 < r_0 < \varepsilon$, we have ${f_1}_r^2(r_0,0) > 0$, while both $f_1(r_0,0)$ and $f_{1rr}(r_0,0)$ are arbitrarily close to 0. It hence follows that (5.2.4) is strictly negative, so $\sec_{g(t)}(\sigma) < 0$ for all $t > 0$ sufficiently small, concluding the proof. □

Chapter 6

A strategy to prove Conjecture C

In the proof of Theorem A it was crucial to ascertain that the evolving metric does indeed remain diagonal in the same basis as the initial metric. It was possible to prove it in that situation with the help of additional symmetries, which may not be available in general. In this Chapter we present a strategy to prove Conjecture C (which addresses the question of whether a diagonal metric remains diagonal under the Ricci flow) along with partial results in support of the conjecture and strategy.

Let g_0 be an invariant metric on the cohomogeneity one manifold (M, G), that is diagonal in the basis $\mathcal{B} = \{\frac{\partial}{\partial r}, X_1^*, \cdots, X_m^*\}$ along a minimal geodesic $\gamma(r)$. Let $h^0(r), f_1^0(r), \cdots, f_m^0(r)$ be the components of g_0 in the basis $\mathcal{B}$. Assume that the basis $\mathcal{B}$ is stably Ricci diagonal.

Proposition 6.0.1. *The following two statements are equivalent:*

1. *There exists a solution $h(r,t), f_1(r,t), \cdots, f_m(r,t)$ to the coupled degenerate*

parabolic PDE system 4.0.1 subject to the boundary conditions at $r = 0$ *and* $r = L$ *that are determined by Theorem 3.2.1 and initial condition* $h(r,0) = h^0(r)$ *and* $f_i(r,0) = f_i^0(r)$ *for* $i = 1, \cdots, m$.

2. *There is a* C^2 *metric* $\mathrm{g}(t)$ *that evolves by the Ricci flow equation 1.0.1 and is diagonal in the basis* $\mathcal{B}$ *for each* t.

Proof. Assume Statement 1 holds, that is, there exists a $T > 0$ and functions $h, f_1, \cdots, f_m : (0,L) \times [0,T) \to \mathbb{R}$ that satisfy the PDE system 4.0.1 as well as the boundary conditions determined by Theorem 3.2.1. Define a diagonal metric $\mathrm{g}(t)$ on $M \setminus B_\pm$ via the formula

$$\mathrm{g}(r,t) = h(r,t)^2 \mathrm{d}r^2 + \sum_{i=1}^{m} f_i(r,t)^2 \omega_i^2, \ r \in (0,L) \tag{6.0.1}$$

Since $h(r,t)$, $f_i(r,t)$ satisfy the correct boundary conditions at $r = 0$ and $r = L$, hence $\mathrm{g}(t)$ extends to a C^2 metric on M. Therefore $\mathrm{g}(t)$ is a C^2 metric on M that satisfies the Ricci flow equation. By uniqueness of solutions to the Ricci flow, $\mathrm{g}(t)$ must be *the* solution to the Ricci flow, that is, Statement 2 holds.

Conversely, suppose that Statement 2 holds. Since g is C^2, hence it must satisfy the conditions determined by Theorem 3.2.1. Additionally, by Proposition 4.0.2, the components of g will satisfy the PDE system 4.0.1. Therefore Statement 1 holds. □

Thus the question of whether a diagonal metric flows through diagonal metrics (assuming the necessary condition that the basis is stably Ricci diagonal) is rephrased as a question of existence of solutions to an initial boundary value problem (IBVP) for a coupled PDE system. However, the boundary conditions specified by Theorem 3.2.1 are more than the number of functions appearing in the coupled PDEs. In other words, the IBVP is *overdetermined.* Additionally, the PDE system is only *degenerate* parabolic (notice that the equation $h_t = \cdots$ contains no h_{rr} term).

A possible strategy to overcome these difficulties is the following. Instead of looking for existence of solutions to the Ricci flow PDE system 4.0.1, study a related strictly parabolic PDE system, coming from the *Ricci-DeTurck flow* (see Section 6.2 below). Prove existence of solutions to this strictly parabolic PDE system subject to a *restricted set of boundary conditions* that makes the problem well-posed. Use this solution to the Ricci-DeTurck PDE system to obtain solutions to the Ricci flow PDE system under a restricted set of boundary conditions. Finally, prove that the remaining boundary conditions for a C^2 metric can be recovered with the help of the PDE. In the remainder of this chapter we will assume that we are in the setting of codimension two singular orbits and a nice basis $\mathcal{B}$.

6.1 Recovery of boundary conditions from a subset

In this section we will first describe a choice of subset of boundary conditions at a singular orbit. We will show that from this subset of boundary conditions, the remaining conditions for the metric to be C^2, can be recovered with the help of the PDEs 4.0.1. At the singular orbit corresponding to $r = 0$ we select the following mixture of C^0 and C^1 conditions, coming from invariance under the K_--action (see Chapter 3):

$$\begin{aligned}
&f_1(0,t) = 0 \\
&{f_i}'(0,t) = 0 \text{ for each trivial module } \ell_i = \text{span}\{X_i\} \text{ in } \mathfrak{m} \\
&\left.\begin{aligned} &f_j(0,t) - f_k(0,t) = 0 \\ &{f_j}'(0,t) + {f_k}'(0,t) = 0 \end{aligned}\right\} \text{ for each 2-dimensional module } \ell_{i'} = span\{X_j, X_k\}
\end{aligned} \tag{6.1.1}$$

The subset of boundary conditions at $r = L$ will be chosen analogously, using the smoothness conditions corresponding to K_+.

Observe that this yields a total of dim(M) -1 boundary conditions at each boundary point. It is important to note that this does not cover the full list of conditions required for the metric to be C^2. However we will show below that if there exists a solution to the PDE system 4.0.1 which at a singular orbit satisfies conditions of the form 6.1.1, then in fact that solution must actually satisfy *all* of the conditions required to be a C^2 cohomogeneity one metric on the manifold. The main result of this section is the following:

Theorem 6.1.1. *Let (M, G) be a cohomogeneity one manifold with codimension two singular orbits, and $\mathcal{B}$ a nice basis along a transverse curve γ in M that parametrizes the orbit space. Let* g *be a metric on $M \setminus B_{\pm}$ defined by 6.0.1 and satisfying the PDE system 4.0.1 and the boundary conditions 6.1.1. Assume also that the functions $h(r,t), f_i(r,t) \in C^{3,1}([0,L] \times [0,T))$. Then* g *defines a C^2 diagonal metric on M evolving by the Ricci flow.*

Proof. By Theorem 3.2.1 of Chapter 3, in order that 6.0.1 define a C^2 cohomogeneity one metric, the following additional boundary conditions need to be satisfied.

$$f_1''(0,t) = 0$$

$$(f_1')_t(0,t) = 0$$

$$f_j'(0,t) - f_k'(0,t) = 0 \text{ when } q \neq 1$$

$$f_j''(0,t) - f_k''(0,t) = 0 \text{ when } q \neq 2$$

Recall that prime ($'$) means derivative with respect to arclength along γ. By Lemma 6.1.2 and Corollary 6.1.3 below, these conditions can indeed be recovered from the boundary conditions 6.1.1 and the PDE system 4.0.1. □

Now we will prove the technical result needed in the proof of the above theorem.

Lemma 6.1.2. *Let* g(t) *be a diagonal cohomogeneity one metric satisfying the system* 4.0.1 *subject to the boundary conditions* 6.1.1*. Then the following are true at a singular orbit* $(r = 0)$*:*

1. $f_1''(0) = 0$.

2. $f_j'(0)(f_1'(0)^2 - 4{\gamma_{jk}^1}^2) = 0$ *for each pair* j, k *such that* $\{X_j, X_k\}$ *span a 2D module* l_i *for the adjoint action of* L *on* $\mathfrak{m}$*.*

3. $f_{1t}'(0) = 0$.

4. $f_j''(0) - f_k''(0) = 0$ *for each pair* j, k *such that* $\{X_j, X_k\}$ *span a 2D module* l_i *for the adjoint action of* L *on* $\mathfrak{m}$ *and* $q_i \neq 2$*.*

Proof. From the Ricci flow coupled PDEs, and regularity of f_1 up to the boundary, it follows that the right hand side of each equation in 4.0.1 has a well-defined limit as r approaches 0.

1. We have

$$\begin{aligned} f_{1t} &= f_1'' + f_1' \sum_{l \neq 1} d_l \frac{f_l'}{f_l} - \frac{f_1^3}{4} \sum_{j,k \neq 1} \frac{(\gamma_{jk}^1)^2}{f_j^2 f_k^2} + \frac{1}{f_1} \sum_{j,k \neq 1} \frac{(f_j^2 - f_k^2)^2}{f_j^2 f_k^2} (\gamma_{jk}^1)^2 \\ &= f_1'' + f_1' \sum_{i \in I_1} d_l \frac{f_i'}{f_i} + f_1' \sum_{\{j,k\} \in I_2} \frac{(f_j' f_k + f_j f_k')}{f_j f_k} - \frac{f_1^3}{4} \sum_{j,k \neq 1} \frac{(\gamma_{jk}^1)^2}{f_j^2 f_k^2} \\ &\quad + \frac{1}{f_1} \sum_{\{j,k\} \in I_2} \frac{(f_j^2 - f_k^2)^2}{f_j^2 f_k^2} (\gamma_{jk}^1)^2 \end{aligned}$$

where I_1 is the set of all indices i such that X_i spans a 1-dimensional module and I_2 denotes the set of all pairs of indices $\{j,k\}$ such that $\{X_j, X_k\}$ span a 2-dimensional module for the action of L on $\mathfrak{m}$.

At $r = 0$, $f_1(t) = 0$ for all t, and so $f_{1t}(r = 0) = 0$ for all t. Substituting the known boundary conditions ($f_1 = 0, f_i' = 0\,\forall i \in I_1, f_j = f_k, f_j' = -f_k'\,\forall\{j,k\} \in I_2$) at $r = 0$ in the above equation, we see that at $r = 0$,

$$0 = f_1'' + f_1' \cdot 0 + f_1' \cdot 0 - 0 + \sum_{\{j,k\}\in I_2} \frac{(f_j^2 - f_k^2)^2}{f_1 f_j^2 f_k^2}(\gamma_{jk}^1)^2$$

In each term in the last sum, the denominator vanishes to first order at $r = 0$ whereas the numerator vanishes to second order, implying that the last term is zero as well. We conclude that $f_1''(r = 0) = 0$.

2. In the PDE $f_{j_t} = \cdots$, the left hand side is well-defined at $r = 0$ by regularity of the solution to the PDE system. Therefore the right hand side must be well-defined as well. The right hand side of the equation has the following terms that have denominators that vanish at $r = 0$:

$$-\frac{1}{2}\sum_l \frac{(f_j^4 - f_l^4)}{f_j f_l^2 f_1^2}{\gamma_{jl}^1}^2 + \frac{f_j' f_1'}{f_1}$$

In fact the only non-zero term in the above sum is the one where $l = k$. Incorporating this observation and then rewriting the terms to have a common denominator results in the expression

$$\frac{-(f_j^4 - f_k^4){\gamma_{jk}^1}^2 + 2f_j' f_1' f_1 f_j f_k^2}{2f_1^2 f_j f_k^2}$$

Since the denominator vanishes to second order at $r = 0$ but the term must nonetheless have a well-defined limit at $r = 0$, we must have that the numerator also vanishes to (at least) second order. Therefore, if we compute the derivative of the numerator and evaluate it at $r = 0$ using the known boundary conditions, we obtain:

$$2f_j^3 f_j'(-4{\gamma_{jk}^1}^2 + f_1'^2) = 0$$

Then the claim follows since $f_j(0) \neq 0$.

3. By definition, $(f_1')_t = ({f_1}_s)_t = \frac{{f_1}_{rt}h - {f_1}_r h_t}{h^2}$. We will compute the right hand side of this equation by using the Ricci flow coupled PDEs. Firstly, using the equation $h_t = \cdots$, we see that

$$\begin{aligned}\frac{{f_1}_r h_t}{h^2} &= \frac{{f_1}_r}{h^2} \sum_j \left(\frac{{f_j}_{rr}}{hf_j} - \frac{{f_j}_r h_r}{h^2 f_j} \right) \\ &= f_1' \sum_{j=1}^m \frac{f_j''}{f_j}\end{aligned}$$

Evaluating this at $r = 0$ using the known boundary conditions (including $f_1''(r = 0) = 0$ which we have proved in Part 1 of this proposition) and

L'Hôpital's rule, we see that

$$\frac{f_{1r}h_t}{h^2}(r=0) = f_1''' + f_1' \sum_{j\neq 1} \frac{f_j''}{f_j}$$

On the other hand,

$$\frac{(f_{1t})_r}{h} = (f_{1t})' = f_1''' + f_1'' \sum_{j\neq 1} \frac{f_j'}{f_j} + f_1' \sum_{j\neq 1} \frac{f_j''}{f_j} - f_1' \sum_{j\neq 1} \frac{f_j'^2}{f_j^2} - \left(\sum_{j,k=1}^{m} \frac{f_1^4 - 2f_k^4}{4f_1 f_j^2 f_k^2} {\gamma_{jk}^1}^2 \right)'$$

From which we see that at $r = 0$,

$$\begin{aligned}(f_{1s})_t =& \frac{1}{2} \sum_{\{j,k\}} {\gamma_{jk}^1}^2 \left\{ \frac{4(f_j^2 - f_k^2)(f_j f_j' - f_k f_k')}{f_1 f_j^2 f_k^2} - \frac{(f_j^2 - f_k^2) f_1'}{f_1^2 f_j^2 f_k^2} \right\} - f_1' \sum_{j\neq 1} \frac{f_j'^2}{f_j^2} \\ =& \frac{1}{2} \sum_{\{j,k\}} {\gamma_{jk}^1}^2 \left\{ \frac{4(f_j^2 - f_k^2)(f_j f_j' - f_k f_k')}{f_1 f_j^2 f_k^2} - \frac{(f_j^2 - f_k^2) f_1'}{f_1^2 f_j^2 f_k^2} \right\} \\ & - f_1' \sum_{\{j,k\}} \left\{ \frac{f_j'^2}{f_j^2} + \frac{f_k'^2}{f_k^2} \right\}\end{aligned}$$

Evaluating at $r = 0$ using the known boundary conditions and L'Hôpital's rule yields:

$$(f_{1s})_t = 2 \sum_{\{j,k\}} \frac{[4{\gamma_{jk}^1}^2 - f_1'^2] f_j'^2}{f_1' f_j^2}$$

By part 2 of this proposition, each term in the sum has numerator zero, which completes the proof.

4. From the Ricci flow equations 4.0.1 we have

$$
\begin{aligned}
(f_j - f_k)_t = f_j'' - f_k'' + (f_j' - f_k')\sum_l \frac{f_l'}{f_l} - \frac{f_j^3}{4}\sum_{\alpha,\beta}\frac{(\gamma^j_{\alpha\beta})^2}{f_\alpha^2 f_\beta^2} + \frac{f_k^3}{4}\sum_{\alpha,\beta}\frac{(\gamma^k_{\alpha\beta})^2}{f_\alpha^2 f_\beta^2} \\
+ \frac{1}{2f_j}\sum_{\alpha,\beta}\frac{(f_\alpha^2 - f_\beta^2)^2}{f_\alpha^2 f_\beta^2}(\gamma^j_{\alpha\beta})^2 - \frac{1}{2f_k}\sum_{\alpha,\beta}\frac{(f_\alpha^2 - f_\beta^2)^2}{f_\alpha^2 f_\beta^2}(\gamma^k_{\alpha\beta})^2
\end{aligned}
$$

Using the boundary conditions to evaluate both at $r = 0$ and simplifying this expression with the help of L'Hôpital's rule ultimately yields $f_j''(0) - f_k''(0) = 0$.

□

Corollary 6.1.3. *When $a \neq 2\gamma^1_{jk}$, i.e. when $q \neq 1$, $f_j'(0) = 0$ and hence $f_k'(0) = 0$ so trivially $f_j'(0) - f_k'(0) = 0$.*

Proof. Follows easily by Parts 2 and 3 of the above Lemma, since at $t = 0$, $f_1'(r = 0) = a$. □

6.2 The Ricci-DeTurck flow for a cohomogeneity one manifold

In this Section we describe a strictly parabolic PDE system such that existence of solutions with sufficient regularity will imply existence of solutions for the Ricci flow PDE system. First, let us briefly recall the *DeTurck trick* in the setting of existence for the Ricci flow on closed manifolds. This is the technique used by DeTurck in [11] to provide a much simpler proof of short term existence for the Ricci flow on closed manifolds, as compared with Hamilton's original proof in [19] which relied on the Nash-Moser inverse function theorem. The Ricci-DeTurck flow is the geometric PDE for an evolving metric $\bar{g}(t)$

$$\begin{aligned} \frac{\partial \bar{g}}{\partial t} &= -2\,\mathrm{Ric}(\bar{g}) + \mathcal{L}_W \bar{g} \\ \bar{g}|_{t=0} &= g_0 \end{aligned} \tag{6.2.1}$$

Here $W = W^{\bar{g},\hat{g}}(p)$ is a time-dependent vector field on M given in coordinates by

$$W^k = \bar{g}^{pq}(\overline{\Gamma}^k_{pq} - \hat{\Gamma}^k_{pq}) \tag{6.2.2}$$

where $\hat{g}$ is a fixed background metric on the manifold. Denote by $\Phi(p,t)$ the flow of the vector field $-W$,

$$\begin{aligned} \frac{\partial \Phi(p,\tau)}{\partial \tau}\Big|_{\tau=0} &= -W(p) \\ \Phi(p,0) &= p \end{aligned} \tag{6.2.3}$$

and define a metric g on M by

$$g = \Phi^* \bar{g}. \tag{6.2.4}$$

Then the metric g satisfies the Ricci flow equation 1.0.1. On the other hand, the flow equation 6.2.1 is strictly parabolic, so standard theorems for existence of solutions to parabolic PDEs on *closed manifolds* yield existence for 6.2.1, while Equation 6.2.3

is an ODE. Thus existence for 1.0.1 is equivalent to existence for the PDE-ODE system 6.2.1, 6.2.3.

We will implement this strategy of converting to a PDE-ODE system in our context, i.e. a degenerate parabolic PDE system with boundary conditions. See [13], [34], where this is done in the context of Ricci flow on manifolds with boundary.

Now we will derive the (strictly parabolic) coupled PDE system that describes the Ricci-DeTurck flow on the cohomogeneity one manifold M, assuming the flow to be through diagonal metrics. That is, we assume that the evolving metric is of the form:

$$\bar{\mathrm{g}}(r,t) = \bar{h}(r,t)^2 \mathrm{d}r^2 + \sum_i \bar{f}_i(r,t)^2 \omega_i^2$$

We will also select the fixed background metric $\hat{\mathrm{g}}$ to be diagonal:

$$\hat{\mathrm{g}}(r) = \hat{h}(r)^2 \mathrm{d}r^2 + \sum_{i=1}^{m} \hat{f}_i(r)^2 \omega_i^2$$

We will assume that in the metric $\hat{\mathrm{g}}$, r is an arclength parametrization of γ, i.e. $\hat{h}(r) = 1$ for each $r \in [0, L]$.

Proposition 6.2.1. *In the setting of diagonal cohomogeneity one metrics, the vector field W has the following expression at points of γ:*

$$W = \left[\frac{1}{\bar{h}^2} \left(\frac{\bar{h}_r}{\bar{h}} - \frac{\hat{h}_r}{\hat{h}} \right) + \sum_i \frac{1}{\bar{f}_i^{\,2}} \left(-\frac{\bar{f}_i \bar{f}_{ir}}{\bar{h}^2} + \frac{\hat{f}_i \hat{f}_{ir}}{\hat{h}^2} \right) \right] \frac{\partial}{\partial r}$$

Proof. We need to compute the Christoffel symbols of the metric at points on γ. Note that the expression $\bar{g}^{pq}(\overline{\Gamma}^k_{pq} - \hat{\Gamma}^k_{pq})$ is tensorial in its coordinates so we can use the basis $\{\frac{\partial}{\partial r}, X_1^*, \cdots, X_m^*\}$ for computing it, even though it is not a frame of coordinate vector fields. The symbols $\overline{\Gamma}^k_{pq}$ and $\hat{\Gamma}^k_{pq}$ appearing below should be understood to be with respect to the above basis, not a coordinate frame. The index 0 corresponds to $\frac{\partial}{\partial r}$.

Implicit in the assumption that $\bar{g}$ is diagonal, is the assumption that under the flow, γ remains orthogonal to the orbits. Hence upto reparametrizing, γ is a geodesic. Denote its unit tangent vector by T. We have:

$$
\begin{aligned}
0 = \nabla_T T &= \nabla_{\frac{1}{\bar{h}}\frac{\partial}{\partial r}}\left(\frac{1}{\bar{h}}\frac{\partial}{\partial r}\right) \\
\implies 0 &= \nabla_{\frac{\partial}{\partial r}}\left(\frac{1}{\bar{h}}\frac{\partial}{\partial r}\right) = \frac{1}{\bar{h}}\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} + \left(-\frac{\bar{h}_r}{\bar{h}^2}\right)\frac{\partial}{\partial r} \\
\implies \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} &= \left(\frac{\bar{h}_r}{\bar{h}}\right)\frac{\partial}{\partial r}
\end{aligned}
$$

From this we can read off the Christoffel symbols $\bar{\Gamma}^0_{00} = \frac{\bar{h}_r}{\bar{h}}$ and $\bar{\Gamma}^i_{00} = 0$ for i $\neq 0$. Similarly, $\hat{\Gamma}^0_{00} = \frac{\hat{h}_r}{\hat{h}}$ and $\hat{\Gamma}^i_{00} = 0$ for $i \neq 0$.

Observe that

$$\nabla_{X_i} X_i = \nabla^r_{X_i} X_i + g_r(S_r X_i, X_i)T$$

For the first term, we have:

$$\nabla^r_{X_i} X_i = -\frac{1}{2}[X_i, X_i]_{\mathfrak{n}} + U(X_i, X_i) = 0 + U(X_i, X_i) = P_r^{-1} B_+(X_i, X_i)$$
$$= P_r^{-1}\frac{1}{2}([X_i, P_r X_i] - [P_r X_i, X_i]) = P_r^{-1}[X_i, P_r X_i] = P_r^{-1} 0 = 0$$

For the second term, first recall that $T = \frac{1}{\bar{h}}\frac{\partial}{\partial r}$. Next, recall that

$$S_r X_i = -\frac{1}{2} P_r^{-1} P_r' X_i = -\frac{\bar{f}_i{}'}{\bar{f}_i} X_i$$

Here prime ($'$) means derivative with respect to the arclength parameter along γ, so we obtain

$$S_r X_i = -\frac{\bar{f}_{ir}}{\bar{h}\bar{f}_i} X_i$$
$$\implies g_r(S_r X_i, X_i) = -\frac{\bar{f}_{ir}}{\bar{h}\bar{f}_i}\bar{f}_i{}^2 = -\frac{\bar{f}_i \bar{f}_{ir}}{\bar{h}}$$

As a result,

$$\nabla_{X_i} X_i = -\frac{\bar{f}_i \bar{f}_{ir}}{\bar{h}^2}\frac{\partial}{\partial r}$$

and therefore $\bar{\Gamma}^0_{ii} = -\frac{\bar{f}_i \bar{f}_{ir}}{\bar{h}^2}$ and $\bar{\Gamma}^j_{ii} = 0$ for $j > 0$. Similarly $\hat{\Gamma}^0_{ii} = -\frac{\hat{f}_i \bar{f}_{ir}}{\hat{h}^2}$ and $\hat{\Gamma}^j_{ii} = 0$ for $j > 0$. Substituting the expressions for the metric and the Christoffel symbols into the formula 6.2.2 completes the proof. □

The vector field derived above is of the form $W(r) = F(r)\frac{\partial}{\partial r} = F^{\bar{g},\hat{g}}(r)\frac{\partial}{\partial r}$. Note also that the vector field W derived above differs slightly from the one used in [34]. In writing the definitions we have emphasized the dependence of F and W on the metrics $\bar{g}$ and $\hat{g}$, but in the interest of compactness of notation we will often supress the superscript $\{\bar{g}, \hat{g}\}$ except where needed. In terms of geometry, since W is a multiple of the vector field $\frac{\partial}{\partial r}$, hence the flow Φ takes G-orbits to G-orbits. Hence on the principal part $M \setminus B_{\pm}$, Φ is given by

$$\Phi(t) : \mathsf{G}/\mathsf{H} \times (0, L) \to \mathsf{G}/\mathsf{H} \times (0, L)$$

$$(g\mathsf{H},\ \ r) \longmapsto (g\mathsf{H},\ \ \phi(r,t)).$$

In other words, Φ is essentially a (time-dependent) reparametrization of the interval $(0, L)$ by the function ϕ. By 6.2.3, we want ϕ to satisfy the ODE $\phi_t(r,t) = -F(\rho,t)\big|_{\rho=\phi(r,t)}$. That is,

$$\begin{aligned}\phi_t(r,t) = -\Bigg(&\frac{1}{\bar{h}(\rho,t)^2}\left(\frac{\bar{h}_\rho(\rho,t)}{\bar{h}(\rho,t)} - \frac{\hat{h}_\rho(\rho)}{\hat{h}(\rho)}\right)\\ &+\sum_i \frac{1}{\bar{f}_i(\rho,t)^2}\left(-\frac{\bar{f}_i(\rho,t)\bar{f}_{i\rho}(\rho,t)}{\bar{h}(\rho,t)^2} + \frac{\hat{f}_i(\rho)\hat{f}_{i\rho}(\rho)}{\hat{h}(\rho)^2}\right)\Bigg)\Big|_{\rho=\phi(r,t)} \qquad (6.2.5)\end{aligned}$$

with the initial condition $\phi(r,0) = r$. When we set up the Ricci-DeTurck flow PDE system below, it makes sense geometrically to have boundary conditions that ensure

$$\phi(0,t) = 0 \text{ and } \phi(L,t) = L \tag{6.2.6}$$

so that the flow Φ will keep the singular orbits fixed. In other words, the boundary conditions should yield $\phi_t|_{r=0} = 0$, $\phi_t|_{r=L} = 0$.

Proposition 6.2.2. *Let* $\bar{\mathrm{g}}(r,t)$ *be a family of diagonal cohomogeneity one metrics evolving via the Ricci-DeTurck flow. Then* $\bar{\mathrm{g}}$ *satisfies the equations:*

$$\begin{aligned}
\bar{h}_t &= \frac{-1}{\bar{h}} \operatorname{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + \bar{h}\left(F_r + F\frac{\bar{h}_r}{\bar{h}}\right) \\
\bar{f}_{it} &= \frac{-1}{\bar{f}_i} \operatorname{Ric}\left(X_i^*, X_i^*\right) + F\bar{f}_{ir}
\end{aligned}$$

Proof. We just need to compute $\mathcal{L}_W \bar{\mathrm{g}}$ in terms of F. Using $W = F\frac{\partial}{\partial r}$, we see that

$$\mathcal{L}_W \bar{\mathrm{g}}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 2\bar{\mathrm{g}}\left(\nabla_{\frac{\partial}{\partial r}} W, \frac{\partial}{\partial r}\right) = 2\bar{h}^2\left(F_r + F\frac{\bar{h}_r}{\bar{h}}\right)$$

For the directions tangent to the orbits, since $[W, X_i^*] = 0$, we obtain

$$\begin{aligned}
\mathcal{L}_W \bar{\mathrm{g}}\left(X_i^*, X_i^*\right) &= W(\bar{\mathrm{g}}(X_i^*, X_i^*)) - 2\bar{\mathrm{g}}([W, X_i^*], X_i^*) = W(\bar{\mathrm{g}}(X_i^*, X_i^*)) \\
&= F\frac{\partial}{\partial r}\left(\bar{f}_i^{\,2}\right) = 2F\bar{f}_i\bar{f}_{ir}
\end{aligned}$$

Thus we have

$$
\begin{aligned}
\bar{\mathrm{g}}_t &= 2\bar{h}\bar{h}_t \mathrm{d}r^2 + \sum_{i=1}^{m} 2\bar{f}_i \bar{f}_{it} \omega_i^2, \\
\mathrm{Ric}_{\bar{\mathrm{g}}} &= \mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \mathrm{d}r^2 + \sum_{i=1}^{m} \mathrm{Ric}\left(X_i^*, X_i^*\right) \omega_i^2, \\
\mathcal{L}_W \bar{\mathrm{g}} &= 2\bar{h}^2 \left(F_r + F\frac{\bar{h}_r}{\bar{h}}\right) \mathrm{d}r^2 + \sum_{i=1}^{m} 2F\bar{f}_i \bar{f}_{ir} \omega_i^2
\end{aligned}
$$

Substituting the above equations into equation 6.2.1 and comparing coefficients yields the result. □

Then Proposition 6.2.2 and Proposition 4.0.1 together imply the following explicit form for the Ricci-DeTurck equations, assuming the evolution to be through diagonal metrics, see also [34].

Proposition 6.2.3. *Suppose* $\bar{\mathrm{g}}(t)$ *is a time dependent diagonal cohomogeneity one metric on* M*, evolving by the Ricci-DeTurck flow. Then the components of* $\bar{\mathrm{g}}$ *satisfy the following (strictly parabolic) system of PDEs:*

$$
\begin{aligned}
\bar{h}_t &= \frac{\bar{h}_{rr}}{\bar{h}^2} - \frac{2\bar{h}_r^2}{\bar{h}^3} + \sum_{j=1}^{m} \frac{\bar{f}_{jr}^{\,2}}{\bar{h}\bar{f}_j^{\,2}} + \bar{h}_r \left(\sum_{j=1}^{m} \frac{\hat{f}_j \hat{f}_{jr}}{\bar{f}_j^{\,2}\hat{h}^2} + \frac{\hat{h}_r}{\bar{h}\hat{h}}\right) - 2\sum_{j=1}^{m} \frac{\hat{f}_j \hat{f}_{jr} \bar{h}}{\bar{f}_j^{\,3}\hat{h}^2} \bar{f}_{jr} \\
&\quad + \bar{h}\sum_{j=1}^{m} \frac{1}{\bar{f}_j^{\,2}} \left(\frac{\hat{f}_{jr}^{\,2}}{\hat{h}^2} + \frac{\hat{f}_j \hat{f}_{jrr}}{\bar{f}_j^{\,2}\hat{h}^2} - \frac{2\hat{f}_j \hat{f}_{jr} \hat{h}_r}{\bar{f}_j^{\,2}\hat{h}^3}\right) - \frac{\hat{h}_{rr}}{\bar{h}\hat{h}} + \frac{\hat{h}_r^2}{\bar{h}^2\hat{h}^2} \\
\bar{f}_{it} &= \frac{\bar{f}_{irr}}{\bar{h}^2} - \frac{\bar{f}_{ir}^{\,2}}{\bar{h}^2\bar{f}_i} - \frac{\bar{f}_{ir}\hat{h}_r}{\bar{h}^2\hat{h}} + \frac{\bar{f}_{ir}}{\hat{h}^2} \sum_{j=1}^{m} \frac{\hat{f}_j \hat{f}_{jr}}{\bar{f}_j^{\,2}} - \sum_{j,k=1}^{m} \frac{\bar{f}_i^{\,4} - 2\bar{f}_k^{\,4}}{4\bar{f}_i \bar{f}_j^{\,2} \bar{f}_k^{\,2}} {\gamma_{jk}^{i}}^{2} + \frac{b_i}{2\bar{f}_i}
\end{aligned}
\tag{6.2.7}
$$

$r \in (0,1)$, $i = 1, \cdots, m$

Thus, the PDE-ODE system 6.2.1, 6.2.3 is rephrased as 6.2.7, 6.2.5 in the cohomogeneity one setting. Since the vector field W is radial, one sees that the pullback equation 6.2.4 yields:

$$
\begin{aligned}
h(r,t) &= \phi_r(r,t)\bar{h}(\phi(r,t),t) \\
f_i(r,t) &= \bar{f}_i(\phi(r,t),t)
\end{aligned}
\tag{6.2.8}
$$

In other words, if $\bar{h}, \bar{f}_i$ satisfy the PDE system 6.2.7 then h, f_i defined by 6.2.8 will satisfy the Ricci flow coupled PDE system 4.0.1. Further, Theorem 6.1.1 coupled with 6.2.8 suggests that we should augment the PDE system 6.2.7 with boundary conditions at a singular orbit given by

$$
\begin{aligned}
&\bar{f}_1(0,t) = 0 \\
&\bar{f}_{i_r}(0,t) = 0 \text{ for each trivial module } \ell_i = \text{span}\{X_i\} \text{ in } \mathfrak{m} \\
&\left.\begin{aligned}
&\bar{f}_j(0,t) - \bar{f}_k(0,t) = 0 \\
&\bar{f}_{j_r}(0,t) + \bar{f}_{k_r}(0,t) = 0
\end{aligned}\right\} \text{ for each 2-dimensional module } \ell_{i'} = span\{X_j, X_k\}
\end{aligned}
\tag{6.2.9}
$$

The reason for this choice of boundary conditions is as follows:

Proposition 6.2.4. *Suppose that $\bar{h}, \bar{f}_i$ satisfy the Ricci-DeTurck flow PDE system 6.2.7 and the conditions defined by 6.2.9 at the boundary points $r = 0$, $r = L$. Also assume that 6.2.6 holds. Then the functions h, f_i defined by 6.2.8 satisfy the Ricci flow coupled PDE system 4.0.1 and the boundary conditions defined by 6.1.1 at $r = 0$ and $r = L$.*

Proof. By 6.2.8 and 6.2.6 it is clear that $\bar{f}_1(0,t) = 0$ if and only if $f_1(0,t) = 0$. Further, it is a simple computation to see that

$$f_{i_r}(r,t) = \phi_r(r,t) \cdot \bar{f}_{i_r}(\phi(r,t),t)$$

This (along with 6.2.6) implies that if $\bar{f}_{i_r}(0,t) = 0$ then $f_{i_r}(0,t) = 0$. In addition, we also obtain

$$f_{j_r}(0,t) + f_{k_r}(0,t) = \phi_r(0,t)(\bar{f}_{j_r}(0,t) + \bar{f}_{k_r}(0,t))$$

so that $\bar{f}_{j_r}(0,t) + \bar{f}_{k_r}(0,t) = 0$ implies $f_{j_r}(0,t) + f_{k_r}(0,t) = 0$. The argument at $r = L$ is the same, up to the relevant reordering of indices according to the K_+ action. □

Thus the problem has now been reduced to the following. Firstly, solving the strictly parabolic PDE system 6.2.7 subject to the linearly independent set of boundary conditions defined at $r = 0$ (and analogously at $r = L$) through 6.2.9, and obtaining a solution that is sufficiently differentiable up to the boundary of $[0,L] \times [0,T)$. In order to make this problem well-posed, we need also to select a boundary condition corresponding to the function $\bar{h}(r,t)$. One also needs to show that a solution to this problem will satisfy 6.2.6.

Thereafter, Theorem 6.1.1 implies that the metric defined through 6.2.8 is a C^2 metric on M. By Proposition 6.0.1 this will complete the proof that the Ricci flow is through diagonal metrics.

6.3 Solving the Ricci-DeTurck flow IBVP

The Ricci-DeTurck flow PDE system 6.2.7 is a non-linear parabolic PDE system. In the papers [13] and [34], the authors address the existence question for the flow on *manifolds with boundary.* The strategy suggested through those papers (see also [38]) is to use the nonlinear problem to create a linear problem L_v for each candidate metric $v(r,t)$ in a suitable function space. The existence theory for linear parabolic PDEs and systems with boundary conditions developed in the references [36], [24] and [33] then yield a solution $u(r,t)$ to the linear problem. Then, it is proved that the map $v \mapsto u$ has a fixed point in a suitable subset of the function space. This fixed point $u(r,t)$ must therefore be a solution to the original nonlinear problem. Theorems from [24] are also used to obtain higher regularity of solutions up to the boundary.

The major additional difficulty in our setting (arising out of the smoothness conditions for a *closed* manifold) is that some of the lower order terms in 6.2.7 can blow up at the boundary of the spatial domain $[0,L]$. Specifically, those terms that contain $\hat{f}_1$ or $\bar{f}_1$ in the denominator. As a result, when one sets up the linear problem, the constant terms in the equation do not lie in the usual L^p spaces, so one cannot proceed as in the above mentioned references. Tackling this question is currently beyond the scope of this thesis, but appears to be a promising future direction, which could yield a method of completing the proof of Conjecture C in the setting of a nice basis and codimension two singular orbits.

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