Interfacial Wave Dynamics Of Core-Annular Flow Of Two Fluids

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Abstract
In core-annular flow of two different fluids, for a set of suitable flow conditions, various shapes of saturated waves such as bamboo, snake and corkscrew waves are observed. Some of the dominant parameters such as thickness ratio of the fluid, Reynolds number, viscosity ratio, density ratio, interfacial surface tension, and the direction of gravitational forces determine the final shape of the saturated wave and their ultimate stability in a nonlinear regime.

When the flow rate ratio is high, it is sometimes difficult to determine the differences between the final shape of the waves for up-flow and down-flow. For some combinations of thickness ratio, viscosity ratio, density ratio, Reynolds number and surface tension, waves tend to break and bubbles start to form. Interfacial surface tensions between these two fluids play a very important role in stabilizing the waves from breaking.

In this study, new sets of waves were discovered for core-annular flow, which modulate at certain flow parameter ranges. The critical parameter ranges are identified where the waves shift from saturated bamboo waves and bifurcate into modulated bamboo waves. A thorough analysis is performed for the first time to depict the windows of these critical parameters at which this transition takes place. A bifurcation diagram is constructed to capture the regime. A detailed wave shape analysis is performed to characterize these wave shapes and their periods of oscillation.

Due to challenges associated with large computational domain and enormous computational power requires to resolve the interfacial instability, a three-dimensional true non-axisymmetric model was never studied before. For the first time, effort is being undertaken to construct a viable 3-D Core-annular flow. A general purpose computational fluid dynamics package ANSYS Fluent is used for this analysis. Three dimensional models for both up-flow and down-flow were constructed and a novel explanation is presented to distinguish between the Bamboo waves, Cork-Screw waves, and Snake waves. The sensitivity of down-flow on initial conditions was also verified with 3-D models on some parameter space from selected publication.

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INTERFACIAL WAVE DYNAMICS OF CORE-ANNULAR FLOW OF TWO FLUIDS

Mohammed Asaduzzaman

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Mohammed Asaduzzaman
To my wife, Rummana, my daughter, Sabriya and son, Xavier.
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ABSTRACT

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Mohammed Asaduzzaman

Howard H. Hu

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CHAPTER 1 : Introduction

1.1 Background and motivation

When two immiscible fluids are forced to flow through a confined space simultaneously there is a natural tendency for the fluid with lower viscosity to migrate into the region of high shear. This natural tendency opens up lots of interesting technological applications where one fluid is used to lubricate another. One such application is transportation of crude viscous oil with another fluid with lower viscosity. The pumping energy required to push the viscous oil from its origin to a secondary destination is enormous, as it has to overcome the shear stress generated at the wall of the pipe. This lubricated mechanism of oil at the core and water at the annulus is called core-annular flow (CAF).

The ideal arrangement of this core-annular flow of oil and water has a perfectly cylindrical interface as shown in Figure 1(a). For ideal or prefect core-annular (PCAF) flow conditions, the interface between the two fluids are flat. But for a specific set of flow parameters, a wavy interface typically arises to levitate the core off the wall when the densities are different. General schematic of various modes is shown in Figure 1.
Figure 1: Schematic of possible Core Annular Flow Wave Shapes: (a) Flat Interface (b) Axisymmetric Bamboo Wave (c) Non-Axisymmetric Cork Screw Wave (d) Non-Axisymmetric Snake Wave

In Figure 1(b), wavy interface bamboo wave is illustrated. The bamboo waves are axisymmetric in nature and has pointed peak and wider trough and they are usually symmetric to peak. In Figure 1(c) and Figure 1(d), typical corkscrew and snake wave shapes are shown. Both corkscrew and snake waves are non-axisymmetric waves however corkscrew wave travels in both axial direction as well as azimuthal direction on the other hand snake waves only travel in axial direction.

Bai, Chen, and Joseph D. (1992) published their experimental work of the core-annular flow for a pipe and they observed both bamboo waves and corkscrew waves. A typical shape of bamboo wave is shown in Figure 2 (a) and a representative diagram of corkscrew wave is presented in Figure 2 (b).
Figure 2: (a) Bamboo waves observed in up flows of motor oil and water. The oil has a viscosity of 13.32 poise and a density of 0.881 g/cm³ at room temperature T = 22 °C. The volume flow rates are $Q_o = 0.11332$ gpm, $Q_w = 0.05284$ gpm (from BCJ 1990) (b) Corkscrew waves are observed in down-flows of motor oil and water. The oil has a viscosity of 13.32 poise and a density of 0.881 g/cm³ at room temperature T = 22°C and volume flow rate $Q_o = 0.8212$ gpm, $Q_w = 0.05284$ gpm

The inspiration of this work came from other prominent researcher in this field who studied the phenomenon of core-annular flow and corroborated their research with experimental work and theoretical observation.

In core-annular flow of two different fluids, for a set of suitable flow conditions, various shapes of saturated waves such as Bamboo waves, Snakes and Corkscrew waves are observed. Some of the dominant parameters such as thickness ratio of the fluid, Reynolds number, viscosity ratio, density ratio, interfacial surface tension, and the direction of gravitational forces determine the final shape of the saturated wave and their ultimate stability in a non-linear regime. Usually for up flow condition, bamboo waves are
generated which are axisymmetric in nature. On the other hand, for down flow, cork-screw and snake waves are observed which are non-axisymmetric and 3-dimensional in nature. Down flow wave shapes are more sensitive to initial flow conditions than that of up flow. When the flow rate ratio is high, it is sometimes difficult to determine the differences between the final shape of the waves for both up flow and down flow.

Interfacial surface tension between these two fluids play a very important role to stabilize the waves and prevent it from breaking at the interface. It is well known that interfacial surface tension stabilizes the short waves while it has lesser effect on longer wave length waves. In core-annular flow, numerous interesting phenomena take place while the two fluids try to reach a saturation condition while traveling together. Even though, there has been a large body of work conducted by many researchers, there is still room for new observations and analysis. There are wide ranges of publications to address the Bamboo wave with 2-D axisymmetric model. Due to challenges associated with large computational domain and the enormous computational power required to resolve the interfacial instability, a three-dimensional true non-axisymmetric model was never studied before.

1.2 Early research and experimental observation on core-annular flow

Isaacs and Speed (1904) filed a patent application where they first mentioned the lubrication of oil by water in a pipe. They created a concentric flow of two fluids where
one fluid is heavier than the other. The fluids were advanced through the pipe with a helical motion. The fluid with higher density separated from the lighter fluid as a result of the spiral motion. Lighter fluid eventually capsulated the higher density fluid and as a result reduction of the frictional resistance was possible.

The patent application by Clark and Shapiro (1949) [Clark, P.F & Shapiro, 1949 Method of Pumping Viscous Petroleum, U.S. Patent No. 2533878] explains the test of three miles length of 6-inch diameter pipe. This seminal work first addressed the problem of core-annular flow of very heavy viscous crude oil, petroleum. They emphasized their techniques on additives and surface-active agents in controlling the emulsification of water into oil.

An important series of experiments are carried out on water-lubricated pipelining by a group in Alberta, Canada, by Russell and Charles (1959), Russell, Hodgson and Govier (1959), Charles (1960) and Charles, Govier and Hodgson (1961). They observed various arrangements of oil and water from their experiments for flow through horizontal pipes. They are (a) stratified flow with heavy fluid below (b) oil bubbles and slugs in water (c) a concentric oil core and an annulus of water (core-annular flow) (d) various kinds of shear stabilized lubricated wavy flow, called wavy core flow and (e) water in oil (with or without) emulsions. When the densities are different, gravity destroys axis of symmetry in horizontal pipes, but for vertical pipeline flow, axisymmetry does not break.
Though the concept of water-lubricated pipeline is very fascinating, and the lubricated flows could be hydrodynamically stable, oil can easily foul the pipe wall [Joseph, Bai, Chen and Renardy (1997)]. Sometimes this fouling causes flow blockage in the pipe. Fouling could also occur due to accidental shutdown to the pipeline. Therefore, restart of the fouled pipe poses a practical challenge for the smooth operation of the water-lubricated pipe line. Some of these issues are addressed more elaborately in many literatures. The review of Oliemans and Ooms (1986) covered the early work prior to 1985 on the topic of water-lubricated pipeline and a detail source of historical reviews are presented in the book Fundamentals of Two-Fluid Dynamics, Part I and Part II of Joseph and Renardy (1993).

1.3 Perfect core-annular flow (PCF) or flat interface

Some of the early experimental work opened the door for scientists, mathematicians and engineers to analyze the observation with theoretical rigor. The ideal arrangement of water-lubricated pipeline is for the viscous oil in the core and surrounded by water in the annulus, with a perfect cylindrical interface. This concentric flat interface configuration is called perfect core-annular flow (PCAF). Russell and Charles (1959) solved the velocity distribution in PCAF to obtain relationships between the volumetric flow rates with the fluid viscosities and the applied pressure gradient. As expected, they found that in comparison to a pipeline flowing with oil only, the pressure gradient or the power requirement for such a pipeline can be theoretically reduced by a factor proportional to the viscosity ratio of oil to water when the oil is flowing in the lubricated mode.
However, PCAF can rarely be achieved in practice. For most of the practical systems, waves appear at the interface between the two fluids. These interfacial waves may reach saturated shapes and convect downstream with the flow. They may also finger into water and break into oil droplets. Studies on core-annular flows are generally focused on understanding the instabilities of the interface, the characteristics of the nonlinear interfacial waves, and their effects on the flow and flow patterns.

1.4 Linear stability analysis of core-annular flows

To understand the stability of a perfect core-annular flow, one performs a hydrodynamic linear stability analysis. In this analysis, on top of the PCAF one introduces disturbances to the flow field and to the interface. By limiting the disturbances to be infinitesimal, the resulting governing questions for the flow and the interfacial conditions can be linearized. The linearized system can be analyzed by considering disturbances with modes of certain shapes both in space and in time (normal modes). For core-annular flows, temporal stabilities of disturbances which take the form of periodic traveling waves are commonly considered. The resulting system is an eigenvalue problem that determines the exponential decay or growth of a particular mode of disturbance. If for a set of given flow parameters, all possible disturbances decay with time, then the corresponding PCAF is said to be stable. However, if there are certain disturbances that grow with time, the corresponding PCAF is unstable to this set of disturbances. One can also identify the
disturbance that grows the fastest, and this mode of disturbance is called the most
dangerous mode.

Hydrodynamics linear stability analysis for flows with interface has been developed
and used by many researchers over the years. It was accepted that the linear stability
analysis is able to determine the onset of instability for the perfect core-annular flows, and
to predict with reasonable accuracy the wavelength and wave speed of the resulting
interfacial waves even for situations when the waves are highly nonlinear.

Stability of two-layered viscosity stratified flow has been described by many
researchers over the years. Yih (1967) was the first to perform the linear stability of plane
Couette-Poiseuille flows in two fluid layers separated by an interface and bounded between
two walls. He suppressed the effect of gravity and density differences and focused his
attention on the viscosity difference and the volume ratio. Yih (1967) found that growth
rate is proportional to $\alpha^2 \mathcal{R}$, where $\alpha$ is the dimensionless wave number and $\mathcal{R}$ is the
Reynolds number. He also found that some of these flows are stable while others are
unstable. Flows with a small layer of less viscous fluid on the wall are stable. By
performing asymptotic long-wave analysis, he was able to show that such flows can be
linearly unstable to an interfacial mode for all non-zero Reynold’s number. This mode of
instability is attributed to the viscosity stratification of the two fluids.

Hooper and Boyd (1983) studied the stability of Couette flow of two fluids separated
by a plane layer in an infinite region, without boundaries. They found that the flow with a
flat free surface is always unstable to very short waves when the surface tension is neglected. However, when the surface tension is added it stabilizes the shortest waves.

Yiantsios and Higgins (1988) extended Yih’s (1967) study of two-layer viscosity-stratified plane poiseuille flow by adding interfacial surface tension and density differences, and by considering small and large wave numbers. Asymptotic analysis was performed and results were supplemented with numerical solutions of the Orr-Sommerfeld equations. Neutral stability curves were presented for various parameter ranges. In their study, the results are presented in temporal growth rate of the wave as a function of wave number (or wave length) disturbances. From these typical plots, one can identify the band of disturbances of stable (negative growth rate) and unstable (positive growth rate) regime and can also pinpoint the location of a specific wave number (or wave length) where the growth rate is maximum which corresponds to the most dominant or most dangerous mode. Additionally, from the growth rate versus wave number curve one can assemble the neutral stability diagram which corresponds to the contour lines of zero growth rate. Neutral stability curves distinguish the stable and unstable flow regimes for the given set of flow parameters for a specific problem.

The Joseph’s group at University of Minnesota was the first to analyze the stability of flows of two fluids arranged in a core-annular configuration. Joseph, Renardy and Renardy (1984) focused on the situation when the core is more viscous, considered the case with two fluids of the matched density, and neglected the effect of interfacial tension. They found that the lubricated transport was stable if the water fraction was not too great. This
was followed by a numerical study of Preziosi, Chen and Joseph (1989) in which all effects except gravity were included. The most unstable disturbance was found to be axisymmetric. They identified that inertia has a stabilizing effect, and the capillary instability can be completely stabilized by increasing Reynolds number. They also showed that their stability profile agrees with the experiments of Charles, Govier and Hodgson (1961). Hu and Joseph (1989) further explored the situation when the pipe wall is hydrophobic with an oil-water-oil (three-layer) configuration. The stability of the two coupled oil-water interfaces was analyzed and solved numerically by a finite element technique. They also evaluated various terms that arise in the global balance of energy of a small disturbance, which allowed them to identify three different mechanisms of instability: interfacial tension, interfacial friction, and Reynolds stresses. By direct comparison with the experiments, they showed that linear stability analysis could be used as a diagnostic tool in predicting flow regimes which arise in practice: stable core-annular flow; wavy core flows; bubbles and slugs of oil in water; bubbly mixtures of oil and water; and emulsions, mainly of water in oil. They showed that flow regimes, wavelength and wave speed were predicted with fair accuracy by their linear stability theory. In another study, Hu, Lundgren and Joseph (1990) solved the stability problem of core-annular flows in the singular limit of small ratio of viscosity of water to oil. Furthermore, Hu and Joseph (1989) considered the effects of the rotation on the stability of the core-annular flow of two fluids with different density and viscosity.

All the studies mentioned above were performed for core-annular flows in horizontal pipes by matching the density of the two fluids. As a result, the gravity effect is totally
neglected. The linear stability theory predicts that the most unstable disturbance is axisymmetric. For core-annular flow in the vertical configuration, the effects of the gravity and density difference of the two fluids can be incorporated into the analysis. Hickox (1971) studies the linear stability of Poiseuille flow of two fluids in a vertical pipe. He limited his attention to long waves and to the case where the fluid viscosity in the core is less than that in the annulus, which is of little practical interest of lubricated pipelining. He found that all such flows are always unstable to both axisymmetric and asymmetric disturbances. Furthermore, under some flow conditions, the growth rates of the asymmetric disturbances (with azimuthal wavenumber $n = 1$) could be larger than those of the axisymmetric disturbances.

Chen, Bai and Joseph (1990) explored the stability of a vertical core-annular flow in a circular pipe both numerically and experimentally. In their numerical computation, they restricted their attention to the axisymmetric mode. When the lubricating layer is thin and the density ratio is not too small, they found that it is possible to have stable perfect core-annular flows within a limited range of flow rates, and further verified the stable PCAF experimentally. For most of the flow parameters, they found that PCAF is unstable either in a form of capillary instability due to the interfacial tension, or in a form of ‘interfacial friction’ due to viscosity jump, or in a gravity mode due to the mismatch of the density. In their accompanying experiments, they recorded large-amplitude axisymmetric waves in the up-flow section of the pipeline, and non-axisymmetric waves in the down-flow section. Similar non-axisymmetric helical shaped waves were also reported in early experiments by Freeman and Tavlarides (1979) for concurrent jet flows. To understand the role of the
non-axisymmetric waves, Boomkamp and Miesen (1992) examined the linear stability of core-annular flow to non-axisymmetric disturbances in the limit of very viscous oil in water. They found that the growth rates of non-axisymmetric disturbances are approximately the same as those of the corresponding axisymmetric ones, thus inferred that the non-axisymmetric modes are important and should be taken into account in the description of finite amplitude interfacial waves in such core-annular flows.

A more extensive experimental study of the stability of vertical core-annular flow was performed by Bai, Chen and Joseph (1992). They observed large amplitude axisymmetric waves, which they termed bamboo waves, in the up-flow section of the pipe, and non-axisymmetric waves, which they termed corkscrew waves, in the down-flow section. Hu and Patankar (1995) explored the stability of core-annular flow in vertical pipe with respect to non-axisymmetric disturbances, and found that when the oil core is thin, the interface is most unstable to the non-axisymmetric sinuous mode of disturbance with azimuthal wave number $n = \pm 1$ and predicted that the core moves in the form of corkscrew waves as observed in experiments of Bai, Chen and Joseph. This sinuous mode of disturbance is the most dangerous mode for quite a wide range of material and flow parameters and persists in vertical pipes with both upward and downward flows.

1.5 Weakly nonlinear stability analysis of core-annular flows
Theoretically, the linear stability analysis is only valid for small disturbances. Surprisingly, its results turn out to be quite accurate in predicting wavelengths, wave speeds and flow types in flow regimes which are far from the perfect core-annular state. In the neighborhood after PCAF becomes unstable, weakly nonlinear stability theories in which some effects of nonlinearity are retained can be used to describe dynamics of the resulting flow.

Nonlinear stability analysis of plane Couette-Poiseuille flows in two fluid layers separated by an interface and bounded between two walls (the same system as in Yih (1967) were performed by Hooper and Grimshaw (1985), and by Renardy (1989). Hooper and Grimshaw (1985) conducted a long wave analysis using a technique of multiple scales, derived a nonlinear amplitude equation for a wave train in the frame of reference moving with its group velocity. The resulting amplitude equation takes the form of the Kuramoto-Sivashinsky equation. They showed that the interface between the two fluids can evolve into saturated waves of finite amplitude.

Similarly, Renardy (1989) used a center manifold theorem to derive the nonlinear amplitude equation which takes the form of the Ginzburg-Landau equation. She showed that the numerical value of the Landau coefficient depends on the specific flow conditions, such as whether the volume flux or the pressure gradient is fixed. She computed the values of the Landau coefficient for relevant flows and showed that steady travelling waves are supported at the interface.
Renardy (1997) also performed a weakly nonlinear stability analysis for vertical CAF in the down-flow section to examine the onset of non-axisymmetric disturbances. She identified the flow regimes where non-axisymmetric mode of disturbance with azimuthal wave number \( n = \pm 1 \) is the most unstable and examined the interaction between the \( n = 1 \) mode with \( n = -1 \) mode, leading to either the waves traveling the azimuthal direction, known as the corkscrew waves, or standing waves, known as snake waves. Both of them travel in the axial flow direction. As the names imply, the corkscrew waves travel with the flow in the helical motion, however, the snake waves are simply meandering side-to-side while translating with the flow. She identified a regime of Reynolds number and showed that a small change in Reynolds number upsets the stability of the waves and wave shapes change from corkscrew to snake and back to corkscrew wave. She also identified zones where neither corkscrew wave nor snake waves are observed. Renardy (1997) presented the results of down-flow and concluded that the corkscrew wave tends to be preferred when annulus is narrow, while snakes are more likely when the annulus is wide.

### 1.6 Direct numerical simulations of core-annular flows

Direct numerical simulation is a very powerful tool where the Navier-Stokes equations for the flow in the core and annulus coupled with interface shape are solved numerically. This approach is widely used by many researchers over the years as an effective tool to capture the incremental change of interface wave shape over time in both linear and non-linear regime.
Bai, Kelkar and Joseph (1996) was the first group to perform such simulations of axisymmetric core-annular flow of a density matching and very viscous oil core in water. In the limit of an infinitely viscous core, they assumed that the core moves with a uniform velocity as a rigid solid and is deformed by the pressure forces in the water and computed steady periodic solutions of the Navier-Stokes equations in the water annulus together with the shape of the interface. They were able to obtain periodic nonlinear waves with steep slopes at the front face of the wave crest and shallower slopes at the lee side of the crest, similar to the bamboo waves observed in the experiments of Bai, Chen and Joseph for the vertical up-flows. They further showed that the asymmetric wave front steepens further, and wave peak becomes more pointed as the flow speed increases, in agreement with experimental observations. However, it was observed that experimental waveforms have more pointed and symmetric peaks in comparison to the simulated ones. It is not clear whether these differences are due to the assumption of the matched densities, or as a matter of fact that in their simulations they used a much larger value of interfacial tension, 26 dyne/cm, versus the experimental value of 8.54 dyne/cm recorded in Bai, Chen and Joseph (1992). In their study, they also noticed that the critical wavelengths predicted from the maximum growth rate of small disturbances from the linear stability analysis tend to be smaller than those measured in experiments, thus proposed a new scheme for selecting wavelengths of the bamboo-waves by matching the numerically computed hold-up ratio with the value measured from the experiments. The hold-up ratio is defined as the ratio of the superficial velocities of oil to water, and its value was found to be fixed around 1.39 in the experiments of Bai, Chen and Joseph (1992). However, their computed wavelengths by matching the hold-up ratio seemed to be longer than those observed in the experiments.
Three years later, Li and Renardy (1999) computed unsteady axisymmetric vertical core-annular flows using a volume-of-fluid (VOF) scheme to fully resolve the flow both in the core and in the annulus and the shape of the interface. They targeted their calculations with the same parameters documented in the experiments of Bai, Chen and Joseph. Their computations were performed in a periodic domain in the flow direction with the periodic length selected as one wavelength determined from the maximum growth of the linear stability analysis. Their numerical models were validated both in the computed growth rates of small disturbances against the predictions from linear stability analysis, and in the saturation wave amplitude and waveforms against the predictions from weakly nonlinear results in Renardy. Their numerical scheme was able to recover nonlinear steady bamboo waves with pointed peaks and almost symmetric waveforms in better agreement with the measurements of Bai, Chen and Joseph (1992) for up-flows. Furthermore, as the flow speed is increased, the bamboo waves shorten, and peaks become more pointed. However, it was found that when the flow rates of oil and water were fixed as done in the experiments of Bai, Chen and Joseph (1992), their simulation failed to achieve a steady value for the hold-up ratio, and computed hold-up ratio was much larger than the experimentally measured value. They argued that since the oil and water flow rates are not constant quantities as perfect core-annular flow evolves into the nonlinear regime of the bamboo waves, it might be wise to specify a fixed pressure gradient. By fixing the pressure gradient at an appropriate value, they were able to find a steady value of the hold-up ratio as the interface evolves into nonlinear steady bamboo waves, and this hold-up ratio is much closer to the experimental value. In their simulations, they also noticed a new type bamboo wave
in which the flow (the velocity and pressure) field and the resulting hold-up ratio are time-dependent with a distinct time period.

More numerical simulations for the same axisymmetric vertical core-annular flows were executed by Kouris and Tsamopoulos (2001) using a pseudo-spectral method. In their simulations, even though the same periodic conditions were imposed in the flow direction, they employed a much longer periodic length to allow for multiple waves of different lengths to develop and interact. Oftentimes, periodic lengths of 5-9 times the wavelength predicted from the maximum growth of small disturbances from the linear stability analysis were used in their simulations. To directly compare with experiments of Bai, Chen and Joseph (1992), in their simulations they fixed the volume fraction of each fluid and the total flow rate of both fluids. They computed cases for both high and low values of the interfacial tension, 26 dyne/cm and 8.54 dyne/cm, respectively. They did obtain nonlinear bamboo waves with sharper crest (pointing towards the annular fluid) than troughs, a typical feature of the bamboo waves. Their computed bamboo waves also showed the co-existence of waves with different wavelengths. For all the simulated cases, these bamboo waves appear to be steady with a non-deforming waveform traveling in the axial flow direction. They were unable to detect the time-dependent bamboo waves observed by Li and Renardy (1999). However, for one of their computed cases with lower value of interfacial tension (case IV1), they did report a second slower temporal oscillation of core flow rate with a distinct time period.
1.7 Outline and scope of the thesis

In this study, our focus is to study the core-annular flow of a cylindrical pipe with some selected parameter ranges to study various nonlinear wave shape patterns. The goal is to understand and explain what occurs to the waves when it reaches nonlinear saturation regime. Most of the studies were performed with 2-D axisymmetric model to describe the nature of various wave shapes generated due to change in flow parameters. The formation of bamboo waves is described. Oscillation and modulation of the wave amplitudes were identified for the first time. A detailed bifurcation diagram is also presented for the first time to map out the onset of wave propagation from flat interface to traveling wave to oscillating and modulated waves. Special emphasis is given to identify a viable 3-D model to study the pattern selection problem and the sensitivity of the initial conditions on final shape of various nonlinear asymmetric waves. This type of 3-D modeling work is also conducted for the first time to identify the formation of non-axisymmetric model.

In Chapter 1, a brief introduction is illustrated about the core-annular flow with emphasis of the previous work done in this area.

In Chapter 2, mathematical formulation along with linear stability theory is presented to understand the stability of core-annular flow. Basic flow equations are derived, which is also called flat interface solution. The effect of gravity is taken into account in our analysis. Basic flow is then perturbed and linear stability analysis is performed to come up
with the simplified equations. A FORTRAN code is written to solve those equations numerically to study the stability of the wave.

In Chapter 3, ANSYS Fluent code is used to validate a benchmark case simulation results with published work of Li and Renardy (1999) and other experimental results and linear stability analysis. All of the flow parameters for the benchmark case study are taken from Li and Renardy (1999) and experimental set-up of Bai et al. (1992).

In Chapter 4 and 5, the bulk of our new findings are described in detail. Oscillation and modulations of the waves were described for a certain range of flow parameters. A detailed description of the bifurcation of the wave is described from flat interface solution to a travelling wave regime and the branch out to an oscillating and modulated regime. Only a certain range of surface tension parameters and Reynolds numbers were considered to map out the regime.

In Chapter 6, 3-D model of the benchmark up-flow is constructed to study the true nature of the saturated waves. This is done successfully for the first time. A novel approach is described to distinguish the nature of various wave shapes from 3-D analysis by tracing the trajectories of the centroidal coordinate of core fluid from an arbitrary cross section of the flow domain. Results from the 3-D analysis is compared with equivalent 2-D axisymmetric model.
In Chapter 7, 3-D models of the down-flow is described in detail. Results of the 3-D waves are compared with the pattern selection studies of Renardy (1997). Sensitivity of the change in wave shape were studied for a slight change in Reynolds number and compared with the theoretical study of Renardy (1997). Our ANSYS Fluent 3-D results show the true nature of asymmetric shape of the waves. A novel approach is presented to identify various non-symmetric wave shapes such as corkscrew and snake waves from the simulation results. This kind of full blown 3-D model to simulate non-axisymmetric wave shapes (corkscrew and snake waves) are presented for the first time.

In, Chapter 8, we summarized the new findings from our research.

In Chapter 9, a conclusion is drawn from our present work and some future extension of the work is proposed.
CHAPTER 2 : Mathematical Formulations

2.1 Governing equations and boundary conditions

In this chapter, the basics of the core-annular flow of two immiscible fluids passing through a pipeline of circular cross section is presented. The interface between the two fluids could be flat or wavy depending on flow conditions. In this section, a general mathematical formulation of core-annular flow is illustrated.

The schematic diagram of the core-annular flow is depicted in Figure 3. Let us consider cylindrical coordinate system with $r$, $\theta$, and $x$ as three ordinates. The radius of the pipe is at $r = R_2$. The axis of the pipe is located at $r = 0$. The interface between the two fluids is defined at $r = R_1 + \delta(\theta, x, t)$. For flat interface problem or basic flow problem, the interface location reduces to $r = R_1$. The density and viscosity of the core fluid (fluid 1) is denoted as $\rho_1$ and $\mu_1$, and the density and viscosity of the annulus fluid (fluid 2) is denoted as $\rho_2$ and $\mu_2$. Both density and viscosity of the core and annular fluids are considered to be constants. The gravity is acting in the negative $x$-direction as shown in Figure 3. Fluid could travel in the positive $x$-direction for an up-flow and negative $x$-direction for down flow.
The three components of the conservation of momentum equation in $r$, $\theta$, and $x$ directions for both core and annular flow are presented below,

$$\rho_l \left[ \frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla) u_r - \frac{u_r u_r}{r} \right] = -\frac{\partial p}{\partial r} + \mu_l \left[ \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] + \rho_l g_r, \quad (2.1)$$

$$\rho_l \left[ \frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} \right] = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu_l \left[ \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] + \rho_l g_\theta, \quad (2.2)$$

$$\rho_l \left[ \frac{\partial u_x}{\partial t} + (\mathbf{u} \cdot \nabla) u_x \right] = -\frac{\partial p}{\partial x} + \mu_l \nabla^2 u_x + \rho_l g_x. \quad (2.3)$$

Here, pressure is represented by $p$ and the component of the velocity in the $r$, $\theta$, and $x$ directions are represented by $u_r$, $u_\theta$, and $u_x$ respectively. Also, $\rho_l$ and $\mu_l$ are density and dynamic viscosity of fluid 1 (core, $l = 1$) and fluid 2 (annulus, $l = 2$). Components of the
gravitational acceleration are $g_r$, $g_\theta$ and $g_x$ respectively. For the configuration shown in
Figure 3, $g_r = 0$, $g_\theta = 0$, and $g_x = -g$. In addition, we define,

$$
\mathbf{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_x \frac{\partial}{\partial x},
$$

(2.4)

and

$$
\nabla^2 = \frac{1}{r \, \partial r} \left( r \, \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}.
$$

(2.5)

The momentum equation could also be expressed in a vector form as

$$
\rho_l \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \mu_l \nabla^2 \mathbf{u} + \rho_l \mathbf{g}.
$$

(2.6)

Similarly, the continuity equation in the cylindrical coordinate could be written as

$$
\frac{\partial \rho}{\partial t} + \frac{1}{r \, \partial r} (\rho r u_r) + \frac{1}{r \, \partial \theta} (\rho u_\theta) + \frac{\partial}{\partial x} (\rho u_x) = 0.
$$

(2.7)

Its vector form could be represented as

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.
$$

(2.8)

In order to present the conditions at the interface, it is convenient to introduce a scalar function $F(x(r, \theta, x), t)$. This scalar function describes the interface shape as a set of points that satisfy,

$$
F(x(r, \theta, x), t) \equiv 0.
$$

(2.9)

Essentially, a material particle at the interface will always remain at the interface. Since, $F$ is always zero at any point on the interface, its time derivative following any material point on the interface will also be zero, i.e.,
\[ \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0. \]  \hspace{1cm} (2.10)

From analytical geometry, the unit normal for the interface surface \( F(\mathbf{x}(r, \theta, x), t) = 0 \) could be defined as

\[ \mathbf{n} = \frac{\nabla F}{|\nabla F|}. \]  \hspace{1cm} (2.11)

Equation (2.10) is the most general form of the kinematic condition of the interface. In this particular problem shown in Figure 3, the interface could be defined as

\[ r = R_1 + \delta(\theta, x, t). \]  \hspace{1cm} (2.12)

Here, \( \delta(\theta, x, t) \) is the deviation of the interface from the flat one, at \( r = R_1 \). Therefore, in this case equation (2.9) takes the form,

\[ F = r(t) - R_1 - \delta(\theta(t), x(t), t) = 0. \]  \hspace{1cm} (2.13)

Using equation (2.13), the kinematic condition (2.10) reduces to,

\[ -\delta_t + u_r - u_\theta \delta_\theta - u_x \delta_x = 0. \]  \hspace{1cm} (2.14)

Similarly, equation (2.11) becomes,

\[ \mathbf{n} = \frac{\mathbf{e}_r - \delta_\theta \mathbf{e}_\theta - \delta_x \mathbf{e}_x}{\sqrt{1 + \delta_\theta^2 + \delta_x^2}}. \]  \hspace{1cm} (2.15)

Here, \( \mathbf{e}_r, \mathbf{e}_\theta, \) and \( \mathbf{e}_x \), are unit normals in \( r, \theta, \) and \( x \) direction respectively and \( \delta_t, \delta_\theta \) and \( \delta_x \) are partial derivatives of the deviation \( \delta(\theta, x, t), i.e., \delta_t = \frac{\partial \delta}{\partial t}, \delta_\theta = \frac{\partial \delta}{\partial \theta}, \) and \( \delta_x = \frac{\partial \delta}{\partial x} \) respectively.

At this point, it is important to introduce a notation of the jump. For any quantity \( F \) across the interface,
\[ \llbracket F \rrbracket = F|_{r=(R_1+\delta)_+} - F|_{r=(R_1+\delta)_-}. \] (2.16)

At any specified point on the interface, the velocity is continuous between the two fluids (no-slip). Therefore,

\[ \llbracket u \rrbracket = 0. \] (2.17)

The surface traction at the interface is balanced by surface forces, or

\[ \llbracket -pI + 2\mu E \rrbracket \cdot n = \nabla_2 \sigma + H\sigma n. \] (2.18)

Here, \( \sigma \) is surface tension and \( \nabla_2 \sigma \) is the surface gradient of surface tension \( \sigma \), which could be introduced by temperature or concentration gradient along the interface. \( H \) is the mean curvature, and \( E \) is the strain rate tensor,

\[ E = \frac{1}{2}(\nabla u + \nabla u^T). \] (2.19)

To complete the mathematical specification of core-annular flow, the boundary conditions at the center of the pipe and at the wall needed to be satisfied,

\[ u = \text{finite, at } r = 0, \] (2.20)

and

\[ u = 0, (i.e., u_r = 0; u_\theta = 0; u_x = 0), \text{ at } r = R_2. \] (2.21)

### 2.2 Base flow

In section 2.1, necessary governing equations and boundary conditions are stated for the general core-annular flow. To start the analysis for this problem, let us consider first the solution where the interface is flat at \( r = R_1 \) and surface tension \( \sigma \) is constant. For
this flat interface or perfect core-annular flow or basic flow problem, the solution of the velocity and pressure fields can be simplified to

\[
\begin{align*}
&u_r = 0, \\
&u_\theta = 0, \\
&u_x = w(r), \\
&p = P(x).
\end{align*}
\] (2.22)

The boundary conditions (2.20) and (2.21) reduce to

\[w \text{ is finite at } r = 0,\] (2.23)

and

\[w = 0 \text{ at } r = R_2,\] (2.24)

respectively.

For this basic flow, there is no fluctuation of the interface, therefore \( \delta = 0 \) and \( n = \hat{e}_r \). At the interface \( r = R_1 \), components of the velocities match between the two phases. Therefore, equation (2.17) reduces to

\[(w)_1 = (w)_2.\] (2.25)

Traction condition (2.18) at the interface could be written as

\[\left[-P + 2\mu E\right] \cdot \hat{e}_r = H\sigma \hat{e}_r.\] (2.26)

After simplification, equation (2.26) reduces to

\[\left[-P + 2\mu E_{rr}\right] \hat{e}_r + \left[2\mu E_{\theta r}\right] \hat{e}_{\theta} + \left[2\mu E_{xx}\right] \hat{e}_x = H\sigma \hat{e}_r.\] (2.27)

Since, \( u_r = 0 \), and \( u_\theta = 0 \), traction condition (2.27) in the axial direction reduces to

\[\left[\frac{\mu}{\frac{d}{d\theta}}\right] w = 0.\] (2.28)

Similarly, the traction condition in the radial direction reduces to
\[\left[-P + 2\mu E_{rr}\right] = H\sigma.\]  \hspace{1cm} (2.29)

Here, \(\sigma\) is surface tension and \(H\) is simplified curvature defined by \(H = \left(\frac{1}{R_i}\right)\). Therefore, equation (2.29) reduces to
\[P_2 - P_1 = \frac{\sigma}{R_1}.\]  \hspace{1cm} (2.30)

Thus, \(dP_1/dx = dP_2/dx = dP/dx\).

After substituting the components of velocity from equation (2.22) to momentum equation (2.2) to (2.3) we obtain the following result
\[0 = f + \mu_l \left[\frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr}\right] - \rho_1 g.\]  \hspace{1cm} (2.31)

Here,
\[f = -\frac{dP}{dx}.\]  \hspace{1cm} (2.32)

Rearranging equation (2.31), we obtain
\[\left[\frac{1}{r} \frac{d}{dr}\right] \left[r \frac{dw}{dr}\right] = -\left(\frac{f - \rho_1 g}{\mu_l}\right).\]  \hspace{1cm} (2.33)

The general solution of equation (2.33) in the core and annulus can be written as,
\[w(r) = \begin{cases} 
-\frac{(f - \rho_1 g)}{4\mu_1} r^2 + A_1 \ln r + B_1, & 0 \leq r \leq R_1, \\
-\frac{(f - \rho_2 g)}{4\mu_2} r^2 + A_2 \ln r + B_2, & R_1 \leq r \leq R_2.
\end{cases}\]  \hspace{1cm} (2.34)

The four conditions listed in equations (2.23), (2.24), (2.25) and (2.28) uniquely determine the four coefficients \(A_1, A_2, B_1\), and \(B_2\). The final solution can be expressed as:
\[ w(r) = \begin{cases} 
\frac{(f - \rho_1 g)}{4\mu_1} (R_1^2 - r^2) + \frac{(f - \rho_2 g)}{4\mu_2} (R_2^2 - R_1^2) + R_1^2 \frac{(\rho_1 - \rho_2) g}{2\mu_2} \ln \frac{R_2}{R_1} & 0 \leq r \leq R_1, \\
\frac{(f - \rho_2 g)}{4\mu_2} (R_2^2 - r^2) - R_1^2 \frac{(\rho_1 - \rho_2) g}{2\mu_2} \ln \frac{r}{R_2} & R_1 \leq r \leq R_2. 
\end{cases} 
\] (2.35)

In order to express the general solution of the base flow in the dimensionless form, we may scale the length with the radius of the interface \( R_1 \), and velocity with the center line velocity \( w_0 \). The dimensionless velocity profile for the core and annulus velocity could be expressed as,

\[ \bar{w} (\bar{r}) = \frac{w}{w_0} = \begin{cases} 1 - mK \frac{\bar{r}^2}{\Lambda} & 0 \leq \bar{r} < 1, \\
[a^2 - \bar{r}^2 - 2(K - 1) \ln(\bar{r}/a)]/\Lambda & 1 \leq \bar{r} \leq a. 
\end{cases} \] (2.36)

Here, the dimensionless parameters are,

\[ m = \frac{\mu_2}{\mu_1}, \]
\[ \zeta = \frac{\rho_2}{\rho_1}, \]
\[ a = \frac{R_2}{R_1}, \]
\[ K = \frac{(f - \rho_1 g)}{(f - \rho_2 g)}. \] (2.37)

Also, \( \Lambda \) is defined by,

\[ \Lambda = Km + a^2 - 1 + 2(K - 1) \ln a. \] (2.38)

The centerline velocity \( w_0 \), at \( r = 0 \) could be obtained from equations (2.35) and (2.38), and could be expressed as

\[ w_0 = \frac{(f - \rho_2 g)}{4\mu_2} \Lambda R_1^2. \] (2.39)
If the gravity is ignored, i.e., \( g = 0 \), then the general solution listed in equation (2.35) becomes

\[
w(r) = \begin{cases} 
\frac{f}{4\mu_1} (R_1^2 - r^2) + \frac{f}{4\mu_2} (R_2^2 - R_1^2) & 0 \leq r \leq R_1, \\
\frac{f}{4\mu_2} (R_2^2 - r^2) & R_1 \leq r \leq R_2.
\end{cases}
\]

The corresponding non-dimensional form of the velocity is

\[
\bar{w}(\bar{r}) = \begin{cases} 
1 - \frac{m \bar{r}^2}{(a^2 + m - 1)} & 0 \leq \bar{r} < 1, \\
\frac{a^2 - \bar{r}^2}{(a^2 + m - 1)} & 1 \leq \bar{r} \leq a.
\end{cases}
\]

### 2.3 Perturbed flow

The basic flow solution with a flat interface described in section 2.2 may not be stable. To determine the stability of the basic flow, it is necessary to perform a linear stability analysis. In order to do that, the basic flow solution is perturbed with a small disturbance such that

\[
\begin{align*}
    u_r &= 0 + u, \\
    u_\theta &= 0 + v, \\
    u_x &= w(r) + \omega, \\
    p &= P + \varrho,
\end{align*}
\]

and the interface is at,

\[
r = R_1 + \delta(\theta, x, t).
\]

Here, \( u, v, \omega, \varrho \) and \( \delta \) are of infinitesimal magnitude. Substituting the perturbed components of velocity from equation (2.42) into the governing equations (2.1) to (2.3),
dropping the terms related to the basic flow, and neglecting multiplication terms of two small quantities, we obtain the following linearized governing equations in terms of \( u, \nu, w, \) and \( p \).

\[
\rho_l \left( \frac{\partial u}{\partial t} + w \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial r} + \mu_l \left[ \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial \nu}{\partial \theta} \right],
\]  \hspace{0.5cm} (2.44)

\[
\rho_l \left( \frac{\partial \nu}{\partial t} + w \frac{\partial \nu}{\partial x} \right) = -\frac{1}{r} \frac{\partial \nu}{\partial \theta} + \mu_l \left[ \nabla^2 \nu - \frac{\nu}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right],
\]  \hspace{0.5cm} (2.45)

\[
\rho_l \left( \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} + w'u \right) = -\frac{\partial p}{\partial x} + \mu_l \nabla^2 w.
\]  \hspace{0.5cm} (2.46)

Similarly, the continuity equation (2.4) reduces to

\[
\frac{1}{r} \cdot \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{\partial w}{\partial x} = 0.
\]  \hspace{0.5cm} (2.47)

The boundary condition listed in equation (2.20) and equation (2.21) become

\[ u, \nu, \text{and } w \text{ are finite, at } r = 0, \]  \hspace{0.5cm} (2.48)

and

\[ u = \nu = w = 0 \text{ at } r = R_2, \]  \hspace{0.5cm} (2.49)

respectively.

For the perturbed flow, the interface is located at \( r = R_1 + \delta \). However, it is convenient to apply the interface conditions at the unperturbed location, \( r = R_1 \). This requires the use of Taylor series expansion for any quantity \( F \) near the interface

\[
F|_{r = R_1 + \delta} \approx F|_{r = R_1} + \delta \frac{\partial F}{\partial r}|_{r = R_1}.
\]  \hspace{0.5cm} (2.50)

Applying equation (2.50) to the kinematic condition (2.14), and dropping the higher order terms, we have, at \( r = R_1 \),
\[ u = \delta_t + \delta_x w. \]  

(2.51)

Similarly, velocity conditions at \( r = R_1 \),

\[
\begin{align*}
\llbracket u \rrbracket &= 0, \\
\llbracket \nu \rrbracket &= 0, \\
\llbracket w \rrbracket + \delta \llbracket w' \rrbracket &= 0.
\end{align*}
\]

(2.52)

Shear traction at the interface \( r = R_1 + \delta \) is obtained by \( \theta \) and \( x \) component of equation (2.27). We also need to use the equation (2.50) and equation (2.16). After neglecting the multiplication of two small terms and dropping \( \llbracket \mu w' \rrbracket = 0 \) at \( r = R_1 \), we obtain the following shear traction condition at the interface \( r = R_1 \),

\[
\llbracket \mu \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial r} \right) \rrbracket + \delta \llbracket \mu w'' \rrbracket = 0,
\]

(2.53)

and

\[
\llbracket \mu \left( \frac{\partial u}{\partial \theta} + R_1 \frac{\partial \nu}{\partial r} - \nu \right) \rrbracket = 0.
\]

(2.54)

Similarly, normal traction (\( r \) component of equation (2.27)) condition at the interface reduces to

\[
-\llbracket p \rrbracket + 2 \llbracket \mu \frac{\partial u}{\partial r} \rrbracket = \frac{\sigma}{R_1^2} \left( \delta + \delta_{\theta \theta} + R_1^2 \delta_{xx} \right).
\]

(2.55)

Here, the curvature of the interface is defined by \( H = \frac{(\delta + \delta_{\theta \theta} + R_1^2 \delta_{xx})}{R_1^2} \).
2.4 Dimensionless equations for the perturbed flow

From here on, we will discuss the equation and boundary conditions in the dimensionless form. We will use the same notation for the dimensionless variables in our discussion without much confusion.

Here, we introduce a few additional non-dimensional parameters such as Reynolds number and surface tension parameters. Reynolds number is defined as

\[ \mathbb{R}_l = \frac{\rho_l w_0 R_1}{\mu_l}, \quad l = 1, 2; \quad (\text{or } \mathbb{R}_1 = \mathbb{R}. ) \]  

(2.56)

The surface tension parameter is defined as

\[ S = \frac{\sigma}{\rho_1 w_0^2 R_1}. \]  

(2.57)

The surface tension parameter, \( S \), strongly depends upon center line velocity. An alternative surface tension parameter is defined by \( J \), which is independent of centerline velocity,

\[ J = \frac{\sigma R_1}{\rho_1 v_1^2} = \frac{\sigma R_1}{\mu_1^2 \rho_1}. \]  

(2.58)

\( J \) and \( S \) are related by \( S = J \mathbb{R}_1 \). In our analysis we used the surface tension parameter \( J \) which is shown in equation (2.58).

Hu and Patankar (1995) defined \( \mathbb{R}_g \), a non-dimensional Reynolds number for vertical core-annular flow

\[ \mathbb{R}_g = g R_1^3 \left( \frac{\rho_1}{\mu_1} \right)^2. \]  

(2.59)
\( R_g \) is co-related with non-dimensional quantity with the driving force \( K \) listed in equation (2.37) and with other dimensionless numbers such as \( R, m, a, \) and \( \zeta \) by the following equation,

\[
K = \frac{4m R - (\zeta - 1) R_g (a^2 - 1 - 2 \ln a)}{4m R + (\zeta - 1) R_g (m + 2 \ln a)}.
\] (2.60)

For upward flow, \( R_g \) is negative and for downward flow the value of \( R_g \) is positive. For horizontal pipe flow with equal densities, \( R_g = 0 \).

Now let us scale the velocities \((u, v, w)\) with \( w_o \), length \((x,r)\) with \( R_1 \), time with \( R_1 w_o \) and pressure with \( \rho_l w_o^2 \). In this way, the governing equations, (2.44) to (2.47) reduce to

\[
\left[ \frac{\partial u}{\partial t} + \bar{w} \frac{\partial u}{\partial x} \right] = - \frac{\partial p}{\partial r} + \frac{1}{R_1} \left[ \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right],
\] (2.61)

\[
\left[ \frac{\partial v}{\partial t} + \bar{w} \frac{\partial v}{\partial x} \right] = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{R_1} \left[ \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right],
\] (2.62)

\[
\left[ \frac{\partial w}{\partial t} + \bar{w} \frac{\partial w}{\partial x} + \bar{w}' u \right] = - \frac{\partial p}{\partial x} + \frac{1}{R_1} \nabla^2 w,
\] (2.63)

and

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial x} = 0.
\] (2.64)

It is observed that, the non-dimensional form of the equations are obtained by substituting \( \rho_l = 1 \) and \( \mu_l \) by \( 1/R_1 \).

The boundary conditions remain same for \( r = 0 \) and \( r = a \) which is shown in equation (2.48) and (2.49). At the interface, equations (2.51) to (2.52) change slightly.
when the dimensional velocity $w$ is replaced by dimensionless velocity $\bar{w}$. Therefore, the kinematic condition at the interface evaluated at $r = 1$ reduces to

$$u(1, \theta, x, t) = \delta_t + \bar{w}(1)\delta_x. \quad (2.65)$$

The velocity boundary condition (2.52) at the interface evaluated at $r = 1$ could be expressed in the non-dimensional form

$$[u] = 0, \quad [\nu] = 0, \quad [\omega] + \delta[\bar{w}'] = 0. \quad (2.66)$$

Similarly, non-dimensional form of shear stress conditions (2.53) and (2.54) at the interface evaluated at $r = 1$, become

$$\left[ \frac{\zeta_l}{\mathbb{R}_l} \left( \frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial r} \right) \right] + \delta \left[ \frac{\zeta_l}{\mathbb{R}_l} \bar{w}'' \right] = 0, \quad (2.67)$$

and

$$\left[ \frac{\zeta_l}{\mathbb{R}_l} \left( \frac{\partial u}{\partial \theta} + \frac{\partial \nu}{\partial r} - \nu \right) \right] = 0. \quad (2.68)$$

In the same fashion, non-dimensional form of normal stress conditions (2.55) at the interface evaluated at $r = 1$, becomes

$$-\left[ \frac{\zeta_l}{\mathbb{R}_l} \phi \right] + 2 \left[ \frac{\zeta_l}{\mathbb{R}_l} \frac{\partial u}{\partial r} \right] = S(\delta + \delta_{\theta\theta} + \delta_{xx}). \quad (2.69)$$

Here,

$$\zeta_l = (\zeta_1, \zeta_2) \text{ for } l = 1 \text{ and } l = 2, \text{ and } \zeta_1 = 1, \, \zeta_2 = \frac{\rho_2}{\rho_1} = \zeta \text{ and } \mathbb{R}_l = \mathbb{R}_1 \text{ or } \mathbb{R}_2$$

From all the governing equations and boundary conditions, it is revealed that the stability of the core-annular flow problem is associated with six non-dimensional
parameters $m, a, \zeta, J, R_1$ and $R_2$. Out of six parameters, only five of them are independent. Because $R_1$ and $R_2$ are related to viscosity and density ratios by the following relationship $R_1 / R_2 = m / \zeta$.

### 2.5 Normal modes

In section 2.4, the governing equations and boundary conditions are linear. Stability for this linear problem can be examined by means of normal mode decomposition. This procedure allows us to decompose the perturbations into their Fourier modes and analyze one mode at a time, i.e., by replacing $[u_t, v_t, w_t, p_t]$ and $\delta$ from the following relationships.

\[
[u, v, w, p](r, \theta, x, t) = [iu, v, w, p](r) \exp[in\theta + i\alpha(x - ct)], \quad (2.70)
\]

and

\[
\delta(\theta, x, t) = \delta \exp[in\theta + i\alpha(x - ct)]. \quad (2.71)
\]

Here, $u, v, w,$ and $p$ are the perturbed components of velocities and pressure. In addition, $u(r), v(r), w(r)$ and $p(r)$ are complex valued functions. Respectively, $n$ and $\alpha$ are the wave numbers of disturbances in the azimuthal and axial directions. Also, $n = 0$ represents axisymmetric (varicose) mode, and $n = 1$, represents asymmetric (sinuous or snake) mode. $\delta$ is a complex constant which represents the deviation of the interface from a perfect cylinder of radius one. $c$ is a complex number, $c = c_r + ic_i$, its real part represents the wave speed and its imaginary part represents the growth or decay rate of this mode of disturbances.
The goal of the linear stability analysis is to determine the growth or decay rate for a disturbance with given wave numbers \( n \) and \( \alpha \). Introduction of the normal mode into the linearized system of governing equations leads to a major simplification by transforming partial differential equations into ordinary differential equations in \( r \). Derivatives in time, axial and azimuthal directions are replaced by algebraic terms. The problem transforms into an eigenvalue problem and we need to find eigenvalue \( c \) such that the disturbances \((u, v, w, p, \text{and } \delta)\) are non-zero. By substituting equation (2.70) in equations (2.61) to (2.64), the following system of ordinary differential equations are obtained

\[
\alpha (\bar{w}_l - c)u = \rho' - \frac{i}{\Re l} \left[ uu'' + \frac{u'}{r} + \left( \alpha^2 + \frac{n^2 + 1}{r^2} \right) u - \frac{2n}{r^2} v \right], \quad (2.72)
\]

\[
\alpha (\bar{w}_l - c)v = -\frac{n\rho}{r} - \frac{i}{\Re l} \left[ vv'' + \frac{v'}{r} + \left( \alpha^2 + \frac{n^2 + 1}{r^2} \right) v - \frac{2n}{r^2} w \right], \quad (2.73)
\]

\[
\alpha (\bar{w}_l - c)w + \bar{w}_l u = -\alpha p - \frac{i}{\Re l} \left[ ww'' + \frac{w'}{r} + \left( \alpha^2 + \frac{n^2}{r^2} \right) w \right], \quad (2.74)
\]

and

\[
u' + \frac{u}{r} + \frac{n}{r} v + \alpha w = 0. \quad (2.75)
\]

The boundary conditions at \( r = 0 \) becomes

\[
u(0), v(0), w(0) \text{ and } p(0) \text{ are finite.} \quad (2.76)
\]

and at \( r = a \), the boundary condition at the wall becomes

\[
u(a) = v(a) = w(a) = 0. \quad (2.77)
\]
After substituting equations (2.70) and (2.71) in the kinematic condition (2.65), we obtain

\[ u(1) = \alpha [\bar{w}(1) - c] \delta. \]  

(2.78)

At the interface \( r = 1 \), the velocity jump condition of equation (2.66) reduces to the following form after substituting \( \delta \) from equation (2.78).

\[
\begin{align*}
\lbrack u \rbrack &= 0, \\
\lbrack \nu \rbrack &= 0, \\
\lbrack w \rbrack [\bar{w}(1) - c] \alpha + u(1) \lbrack w' \rbrack &= 0.
\end{align*}
\]  

(2.79)

Similarly at \( r = 1 \), the shear stress conditions from equations (2.67) to (2.68) transform into the following form after replacing \( \delta \) from equation (2.78).

\[
\begin{align*}
\left[ \frac{\zeta_l}{\mathbb{R}_l} (\nu' - \alpha u) \right] + \left[ \frac{\zeta_l}{\mathbb{R}_l} \bar{w}'' \right] u(1) &= 0, \\
\left[ \frac{\zeta_l}{\mathbb{R}_l} (\nu' - \nu - n u) \right] &= 0.
\end{align*}
\]  

(2.80) and  

(2.81)

Also, at \( r = 1 \), the normal stress condition of equation (2.69) becomes

\[
-\lbrack \zeta_l \rho \rbrack + 2i \left[ \frac{\zeta_l}{\mathbb{R}_l} u' \right] = \frac{J}{\mathbb{R}_l} (1 - \alpha^2 - n^2) \frac{u(1)}{\alpha [\bar{w}(1) - c]}
\]  

(2.82)

It is possible to express \( \rho \) in terms of \( u \) and \( \nu \) by using equation (2.82). This result could be used to eliminate the pressure term \( \rho \) from the remaining equations. Therefore, the stability analysis for the core-annular flow could be performed by solving the set of equations from (2.72) to (2.82).

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2.6 Numerical method and solution strategy

Before, going into the solution strategy, it is necessary to understand the conditions stated in equation (2.76), i.e., \( u(0), v(0) \) \( w(0) \) are finite at \( r = 0 \). Joseph (1976) showed that \( u(r,\theta,x,t) \) is single valued and therefore independent of \( \theta \) at \( r = 0 \). Preziosi, Chen and Joseph (1989) also addressed this condition by using the method of Frobenious. They summarized the boundary conditions at the center for various azimuthal wave numbers, i.e., for axisymmetric and non-axisymmetric mode. They are listed here,

\[
\begin{align*}
n = 0: & \quad u(0) = v(0) = \alpha w(0) + 2u'(0) = 0, \\
n = 1: & \quad u(0) + v(0) = w(0) = 0, \\
n \geq 2: & \quad u(0) = v(0) = w(0) = 0.
\end{align*}
\] (2.83)

Linear stability analysis of the core-annular flow and normal mode decomposition of the flow equations constitute an eigenvalue system. Hu and Joseph (1989) also showed that after discretization using finite element method, the perturbed system of equations ultimately reduces to a general complex-valued eigenvalue system of the form,

\[
A\mathbf{x} = cB\mathbf{x}.
\] (2.84)

Where, \( A \) and \( B \) are global matrices of \( 3N \times 3N \) and

\[
\mathbf{x} = [u_1, v_1, u_2, v_2, \ldots, u_N, v_N]^T \text{ is a column vector. Here, } N \text{ is the total number of nodes that span the domain } (0, \alpha) \text{ and } u' = \frac{du}{dr}. \] The solution of the eigenvalue system can be obtained with a FORTRAN program [Hu (1995)] which is listed in Appendix-F. For the given set of parameters, \( \alpha, m, \zeta, J, \Re_g, \Re, n, \) and \( \alpha \) this FORTRAN program calculates the complex eigenvalue \( c \).
CHAPTER 3 : Numerical Analysis and Code Verification

3.1 Approaches to solve core-annular flow

The motivation of this work is to study the core-annular flow, especially the interfacial wave shapes and their nature at nonlinear regime. There could be a few different approaches to do that.

**Analytical solution:** It is possible to have an analytical solution for a basic flow or a flat interface problem as described in Chapter 2.2. When the wave becomes unstable, and reaches a nonlinear regime, it is not possible to investigate the wave with an analytical approach as there is not a closed form solution.

**Linear stability analysis:** Linear stability analysis could be a useful tool to identify the stability of the flat interface and the onset of the waves in the linear regime. But this technique is unable to predict the nonlinear evolution of the wave, and the wave shape once the wave reaches nonlinear regime. Nonetheless, the linear stability analysis is a very powerful tool to understand the growth rate of the waves and could be used to predict the nature of the wave even in nonlinear regimes.

In a given system, for a given set of flow parameters, as the waves evolve over time, many waves are formed. Linear stability analysis is a powerful tool to predict the most dominant wave for the set of given flow parameters. Usually growth rate vs. wave number plot could be obtained from the linear stability analysis. The wave length corresponding to the maximum growth rate is the most dominant wave in the system. Linear stability
analysis is used in our work to determine the most dominant wave length to represent the length of the flow domain for different flow parameters.

**Direct numerical simulation:** It is possible to solve full blown Navier-Stokes equations for the core-annular flow to understand the flow field. Special caution is needed to resolve the interface and the assignment of correct initial and boundary condition to capture the evolution of the wave. The technique to solve the instability problem with direct numerical simulation is discussed in detail in the problem set-up section.

**Experimental evaluation:** Bai et al. (1992) and others already performed multiple experiments to characterize the core-annular flow. To confirm a specific case with experimental set-up is time consuming and an expensive endeavor. Therefore, there are only a few publications in experimental work.

In our study, direct numerical simulation is used to perform most of the analysis. ANSYS Fluent (v16 and v17), a general purpose computational fluid dynamics software is used to model the nonlinear behavior of the core-annular flow.

### 3.2 Code verification

The bulk of the modeling work in this thesis is performed with ANSYS Fluent general purpose CFD code. COSMOSOL™ Multiphysics software was also considered to evaluate a 2-D axisymmetric model. Due to its robustness in handling large scale problem and accessibility, the ANSYS Fluent code is being used for this thesis. Fluent’s multiphase
VOF (Volume of Fluid) method is used to track the interface. ANSYS Fluent supports structured quad or hexahedral mesh. It has a built-in superior interface tracking algorithm which captures accurate interface. In addition to all of the above advantages, ANSYS Fluent works extremely well for simulating large models with HPC (high performance computing) environment; this feature is absolutely needed for three-dimensional numerical studies.

In ANSYS Fluent Theory Guide, several topics such as treatment of governing equations, discretization, multiphase flow modeling techniques such as VOF method, and interface tracking techniques are described in great details. As mentioned earlier, in this study, VOF method is used to track the interface. The VOF model is a surface-tracking technique applied to a fixed Eulerian mesh. It is designed for two or more immiscible fluids where the position of the interface between the fluids is of interest. In the VOF model, a single set of momentum equations is shared by the fluids and the volume fraction of each of the fluids in each computational cell is tracked throughout the domain. ANSYS Fluent has a superior numerical scheme and interpolation method to resolve the interface between two immiscible fluids. ANSYS Fluent code offers both explicit and implicit schemes. The interface fluxes can be interpolated using interface tracking or capturing schemes such as

1. Geo-Reconstruct
2. Compressive Scheme

For the sake of clarity an elaborate explanation of each scheme is presented below from ANSYS Fluent Theory Guide.
Geo-reconstruct: The geometric reconstruction scheme represents the interface between fluids using a piecewise-linear approach. In ANSYS Fluent, this scheme is the most accurate and is applicable for general unstructured meshes. The geometric reconstruction scheme is generalized for unstructured meshes from the work of Youngs (1982). It assumes that the interface between two fluids has a linear slope within each cell and uses this linear shape for calculation of the advection of fluid through the cell faces. See Figure 4.

![Figure 4: Interface calculation and interpolation scheme of ANSYS Fluent (Fluent Theory Manual v18.2, page 561).](image)

(a) Actual interface shape (b) Interface shape represented by the geometric reconstruction (piecewise-linear) scheme.

Compressive scheme: The compressive scheme is a second order reconstruction scheme based on the slope limiter. The slope limiters are used in spatial discretization schemes to avoid the spurious oscillations or wiggles that would otherwise occur with high order spatial discretization schemes due to sharp changes in the solution domain. This scheme is used in our analysis when we used implicit scheme.

ANSYS Fluent’s geometrical reconstruction scheme is as good as sharp interface method which accurately captures the interface between two fluids. In Figure 4, a
comparison is drawn between actual interface vs. interface predicted by geometrical
reconstruction scheme. It is recommended to use geometrical reconstruction scheme for
explicit solver and compressive scheme for implicit solver. For this study, these two
schemes are used to track the interface over time.

Another very important parameter which influence the solution is the Courant
Number. Courant number is defined by

\[ C = \frac{v_{\text{fluid}} \cdot \Delta t}{\Delta x}. \]  

(3.1)

Here, \( C \) is the Courant Number, \( v_{\text{fluid}} \) is the fluid velocity, and \( \Delta x \) is the cell size and \( \Delta t \)
is time step. From equation (3.1) it is clear that if the fluid velocity is reduced Courant
Number also gets smaller. Therefore, if we limit the maximum courant number to a certain
value, a larger time step could be possible if the fluid velocity is reduced. It is
recommended to run the model with Courant Number less than 0.25 in explicit scheme for
better interface resolution.

3.3 Problem set-up to validate ANSYS Fluent code (benchmark case study)

A benchmark problem is selected from published work of Li and Renardy (1999).
Flow parameters are selected from their study of direct numerical simulation of an
axisymmetric core-annular flow (up-flow) which corresponds to Bai, Chen and Joseph’s
(1991) experimental work. We used this benchmark case to compare ANSYS Fluent’s
modeling results with those from Li and Renardy’s direct numerical simulation.
The non-dimensional values of the flow parameters are listed in Table 1. To construct a numerical simulation with ANSYS Fluent, we converted these non-dimensional flow properties into the dimensional values. In an effort to do that, we need to assign the diameter of the pipe, fluid properties and appropriate boundary condition. The geometry of the pipe and fluid properties are taken from Bai, Chen and Joseph’s (1991) experimental work. All the dimensional parameters considered for this benchmark study of the 2-D axisymmetric models are listed in Table 2.

### Table 1: Parameters for the benchmark problem.

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Non-Dimensional Number</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness Ratio</td>
<td>$a = \frac{R_2}{R_1}$</td>
<td>1.61</td>
</tr>
<tr>
<td>Viscosity Ratio</td>
<td>$m = \frac{\mu_2}{\mu_1}$</td>
<td>0.00166389</td>
</tr>
<tr>
<td>Density Ratio</td>
<td>$\zeta = \frac{\rho_2}{\rho_1}$</td>
<td>1.1</td>
</tr>
<tr>
<td>Surface Tension Parameter</td>
<td>$J = \frac{\sigma \rho_1 R_1}{\mu_1^2}$</td>
<td>0.063354</td>
</tr>
<tr>
<td>Driving Force Ratio</td>
<td>$K = \frac{(f - \rho_1 g)}{(f - \rho_2 g)}$</td>
<td>-2.067</td>
</tr>
<tr>
<td>Reynolds Number (Gravity)</td>
<td>$R_g = g R_1^3 \left(\frac{\rho_1}{\mu_1}\right)^2$</td>
<td>-0.576</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>$R = \frac{\rho_1 w R_1}{\mu_1}$</td>
<td>3.73754</td>
</tr>
<tr>
<td>Wave Numbers (Azimuthal)</td>
<td>$n$</td>
<td>0</td>
</tr>
<tr>
<td>Wave Numbers (Axial)</td>
<td>$\alpha$</td>
<td>2.4</td>
</tr>
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</table>
Table 2: Parameters considered for benchmark study

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Symbol</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pipe Radius</td>
<td>$R_2$</td>
<td>4.762E-3 m</td>
</tr>
<tr>
<td>Flat Interface Radius</td>
<td>$R_1$</td>
<td>2.958E-3 m</td>
</tr>
<tr>
<td>Oil Density</td>
<td>$\rho_1$</td>
<td>905 Kg/m$^3$</td>
</tr>
<tr>
<td>Water Density</td>
<td>$\rho_2$</td>
<td>995 Kg/m$^3$</td>
</tr>
<tr>
<td>Oil Viscosity</td>
<td>$\mu_1$</td>
<td>0.601 Pa-Sec</td>
</tr>
<tr>
<td>Water Viscosity</td>
<td>$\mu_2$</td>
<td>0.001 Pa-Sec</td>
</tr>
<tr>
<td>Pressure Gradient*</td>
<td>$f$</td>
<td>9.473E3 Pa/m</td>
</tr>
<tr>
<td>Wave Length</td>
<td>$\lambda$</td>
<td>7.744E-3 m</td>
</tr>
<tr>
<td>Surface Tension</td>
<td>$\sigma$</td>
<td>8.548E-3 N/m</td>
</tr>
<tr>
<td>Center Line Velocity</td>
<td>$w_0$</td>
<td>0.839 m/sec</td>
</tr>
</tbody>
</table>

* Pressure Gradient calculated from equation (2.37)

It is important to point out the assignment of the pressure gradient in ANSYS Fluent. In our analytical calculation, $K$ is the driving force. $K$ is related to density of $\rho_1$ and $\rho_2$ and pressure gradient $f$ by equation (2.37). Li and Renardy (1999) implemented driving force $K$ directly in their numerical analysis. To construct a model in ANSYS Fluent, it is necessary to impose periodic boundary condition at two ends of the domain with pressure gradient. In ANSYS Fluent, we cannot directly impose the value of $K$, instead we need to define a gradient of dynamic pressure at periodic boundary condition. Therefore, gradient of dynamic pressure in ANSYS Fluent is assigned by equation (3.2)
\( (\nabla P)_{\text{Fluent}} = -f + \rho_{\text{ref}} g \).

(3.2)

Here, \( \rho_{\text{ref}} \) is the reference density.

Therefore, care should be taken to assign a reference density as an operating density in ANSYS Fluent during the simulation set-up to correctly assign the pressure gradient term. The reference density could be either \( \rho_1 \) or \( \rho_2 \). If no reference density is assigned as an operating density in ANSYS Fluent simulation, the program calculates an operating density by taking the area weighted average of the two densities. In that case, care should be taken to assign the appropriate gradient of dynamic pressure in ANSYS Fluent’s periodic boundary condition panel. As long as proper care is taken to assign the pressure gradient during the simulation set-up, ANSYS Fluent simulation should produce correct results.

An example is illustrated here to clarify the pressure gradient calculation for ANSYS Fluent model. If density of oil \( \rho_{\text{ref}} = \rho_1 = 905 \text{ Kg/m}^3 \) is assigned as an operating density in ANSYS Fluent, then the calculated pressure gradient for the benchmark case \( (\mathbb{R} = 3.737) \) would be \( (\nabla P)_{\text{Fluent}} = -595.05 \text{ Pa/m} \). If no reference density is assigned as an operating density in ANSYS Fluent, then the program calculates a reference density by taking the area weighted average of both oil and water density. For the benchmark case with initial wavy interface, the calculated reference density is \( \rho_{\text{ref}} = 959.5 \text{ kg/m}^3 \). The corresponding pressure gradient for this case would be \( (\nabla P)_{\text{Fluent}} = -60 \text{ Pa/m} \).

Another very important parameter is in the selection process of the dimensional length of the pipe for numerical simulation. To compare the experimental observation of the
bamboo wave, one option is to start with a long pipe and assign the velocity of the core and annulus fluid and let the wave evolve over time. But it is almost impossible to run the simulation to replicate the experimental set-up of the long pipe in Bai et al. (1996) due to enormous computational cost.

Bai, Chen and Joseph’s (1991) experimental study of core-annular flow showed that once the waves reached saturation, waves were spatially periodic with a distinct wavelength. Li and Renardy (1996) demonstrated that the wave with this distinct wavelength corresponds to maximum growth rate predicted by linear stability analysis. They used this wavelength to perform their direct numerical simulation. A similar approach is also taken by R.R. Nourgaliev, M. S. Liou, and T. G. Theofanous (2007). They (Theofanous, et al. (2007)) also considered one wavelength of spatially periodic wave and assigned periodic boundary condition with applied pressure gradient to study the evolution of the wave.

In our study, we also considered the wave as spatially periodic. The wavelength is obtained from the theory of linear stability analysis. As mentioned in Chapter 2, from a given set of non-dimensional flow parameters $a, m, \zeta, J, \Re_g, \Re,$ and $n,$ the growth rate $(c)$ vs. wave number $(\alpha)$ plot could be generated by executing the FORTRAN program (listed in Appendix-F). For the given set of flow parameters for the benchmark case, the growth rate vs. wave number results are plotted in Figure 5.
Figure 5: Plot of growth rate vs. wave number for the benchmark case.

From Figure 5, it is observed that maximum growth takes place at wave number $\alpha = 2.4$. This wave number represents the exponential growth of the waves and describes the most dominant wave in the system. Therefore, this wave length would represent the characteristics of the wave formation. The wave length $\lambda$, is simply related to the wave number by

$$\lambda = \frac{2\pi R_1}{\alpha}. \quad (3.3)$$

For the benchmark case, the maximum growth rate takes place at the wave number $\alpha = 2.4$. Calculated dimensional wavelength associated with the maximum growth rate is $\lambda = 7.744$ mm. Therefore, the length of the flow domain for the benchmark case would be 7.744 mm.

To model the benchmark case with all the parameters listed in Table 1, it is necessary to assign the initial and boundary conditions to the model. To assign the velocity field in dimensional form, it is needed to determine the centerline velocity from equation (2.39).
The magnitude of centerline velocity is found to be $w_0 = 0.839$ m/sec. The velocity profile at the core and annulus are calculated from equation (2.36) which is shown in Figure 6.

![Figure 6: Velocity profile at the core and annulus for the basic flow in dimensional form.](image)

To study the evolution of the wave, initially a perturbed wave of sinusoidal form with very small initial amplitude is introduced at the interface. The equation of initial disturbed wave is given as,

$$Y = A \cos \left( \frac{2\pi}{\lambda} x \right) + R_1. \quad (3.4)$$

Here, $Y$ is the radial position of the wave from the axis. $A$ is the initial amplitude.

There are two ways to set-up and initialize the problem.

1. **Fixed Wall**: A schematic diagram of the 2-D axisymmetric model and the corresponding boundary conditions are shown in Figure 7 for the fixed wall configuration. Axial velocity profile from equation (2.36) of the basic flow is introduced in the domain and the wall is not moving. Notice that the velocity at the interface is high for this configuration. This
high interface velocity affects the local Courant number. For a limiting maximum Courant Number, higher velocity of the fluid will demand smaller time steps. Therefore, to resolve the interface accurately, smaller time steps are needed for this configuration. This translates into longer computational time.

![Diagram showing wave propagation in a pipe with fixed and periodic boundary conditions.](image)

**Figure 7:** Schematic diagram of 2-D axisymmetric problem set-up and boundary conditions for fixed wall configuration.

2. **Moving Wall:** To speed up the simulation, it is recommended to run the simulation in a moving reference frame where the wall of the pipe could be assigned a reference velocity $w_f$ (such as the centerline velocity). To make the model work in a moving frame of reference, this reference velocity is subtracted from the initial velocity field at the domain obtained from equation (2.36). The idea is similar to looking at the wave from a boat moving in the river. The resultant velocity field is shown in Figure 8.
Both approaches produce the same results but the moving frame of reference method runs significantly faster because of smaller velocity near the interface allows larger time step. It helps in resolving the interface faster than the fixed wall model.

The initial assignment of velocity, disturbed interface shape and the initialization of the two fluid domains are assigned with a user defined function or subroutine (UDF) written in “C++” programming language and is compiled in ANSYS Fluent. UDF for both 2-D and 3-D models are presented in Appendix-E. Pressure gradient is the driving force and is assigned as a periodic boundary condition as shown in Figure 7 and Figure 8. Gravitational acceleration direction also has to be assigned in ANSYS Fluent.

To calculate the growth rate from the ANSYS Fluent simulation, we need to capture infinitesimal disturbances at the interface. This requires fine mesh near the interface to capture small change. A schematic of significantly refined mesh construction is shown in
Figure 9. For the growth rate study, we can start with a very small initial amplitude \( A = 1 \times 10^{-4} \) m of disturbances [as shown in equation (3.4)] and let it grow over time to obtain the growth rate of the initial interface wave. Also, the time step selected to run this model is 1E-5 seconds. A model with denser mesh and smaller time steps are computationally expensive. On the other hand, to capture the saturated wave shape, we can assign a larger initial amplitude of the disturbed wave along with more coarse mesh to capture the change of interface wave. Therefore, we explored different mesh configurations to reduce the cell count so that the model would run in a reasonable amount of time.

<table>
<thead>
<tr>
<th>Model</th>
<th>( X )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_3 )</th>
<th>( Y_4 )</th>
<th>( \Delta X ) (m)</th>
<th>( \Delta Y ) (m)</th>
<th>Aspect Ratio</th>
<th>Initial Amp</th>
</tr>
</thead>
<tbody>
<tr>
<td>(300×660) 198 K Cells</td>
<td>300</td>
<td>150</td>
<td>200</td>
<td>260</td>
<td>50</td>
<td>2.5813E-5</td>
<td>5E-6</td>
<td>5.16</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Figure 9: Mesh layout for growth rate calculation. Total computational cell size is (300×560) = 198K.

The detail of the mesh layout is shown in Figure 10. Total cell count for this model is 52,500. This mesh configuration is used to study the saturated wave.
Figure 10: Mesh layout for the benchmark case. Total computation mesh size is \( \sim 52K \).

This mesh configuration (\( \sim 52 \) K) is significantly smaller than 198 K cell mesh configuration. Also notice that mesh is primarily refined near the flat interface radius \( R_1 \) so that changes in wave shape at the interface could be captured more accurately. Mesh is biased to grow in vertical direction on either side of the interface to reduce the mesh count. It is observed that the smallest mesh size in the vertical direction for this configuration is \( \Delta Y = 1.2E-5 \) m. That is significantly smaller than 198 K mesh size. Therefore, the model could be run with smaller time steps. This mesh configuration is used as a benchmark case study to compare the ANSYS Fluent simulation results with the published results of Li and Renardy (1999). For benchmark case study, it is also beneficial to start with larger initial wave amplitude of \( A = 3 \times 10^{-4} \) m and let it grow faster to saturation wave in order to
reduce the computational cost. In order to validate the model, few other mesh configurations are also studied and the results are presented at the end of this chapter.

A typical axisymmetric model consists of a disturbed initial sinusoidal wave with amplitude $A = 3 \times 10^{-4}$ m. The time step of the simulation is $1 \times 10^{-4}$ seconds or smaller. During this progression, the wave grows over time and shape of the wave changes from sinusoidal shape to a slightly deviated shape as it reaches saturation. The evolution of the waves is shown in Figure 11 at different times. The simulation results suggest that wave shape and amplitude do not change in any significant way after one second of simulation time. Therefore, we can conclude that the wave reached saturation around one second of simulation time.
Figure 11: Propagation of wave shapes at different times. (a) Wave shapes at time 0.0108 seconds, 0.5008 seconds, 1.008 seconds and 2.008 seconds (b) Comparison of different wave shapes from initial stage at time zero second to saturation.
Similar conclusions could be drawn from the velocity vector plots shown in Figure 12. At the beginning of the simulation, an initial velocity is assigned in the domain from the analytical solution of the core-annular flow by executing a user defined subroutine (UDF) which is shown in Appendix-C. Figure 12 shows the change of axial velocity over time. After approximately one second of simulation time, the magnitude of the velocity vector does not change in any appreciable amount, which also suggests that the waves reached to a saturation stage.

Figure 12: Axial velocity distribution at different simulation time. (a) Velocity vector at simulation time 0.0108 seconds (b) Velocity at 0.508 seconds (c) Velocity at 1.0008 seconds (d) Velocity at 2.0008 seconds.
3.4 Comparison of ANSYS Fluent (v16.1) simulation results with published results (Jie Li & Renardy’s (1999))

To evaluate the efficacy of ANSYS Fluent (v16.1) model, the simulation results must be compared with the published results. The following results were reported by Li & Renardy (1999) for the given set of parameters \((a = 1.61, \ m = 0.00166389, \ \zeta = 1.1, \ J = 0.063354, \ \Re = 3.73754)\).

(a) Wave Amplitude
(b) Wave Shape
(c) Wave Speed and
(d) Hold-up Ratio

Before comparing the published results, let us discuss different ways of defining wave amplitude. In Figure 13, schematic of a wave shape at the interface is presented with the maximum and minimum height of the wave.

![Figure 13: Schematic of wave shape with maximum and minimum wave height.](image-url)
A typical definition of the amplitude of the wave could be,

\[ A_m = \frac{Y_{\text{Max}} - Y_{\text{Min}}}{2}. \]  \hspace{1cm} (3.5)

To calculate the growth rate, this definition of amplitude is used. Also, most of the ANSYS Fluent simulation results are presented with this amplitude.

In addition to the definition of \( A_m \), an additional amplitude i.e., the maximum amplitude, is defined by subtracting the average height \( (R_1) \) of the wave from the maximum height of the wave

\[ A_{\text{max}} = Y_{\text{Max}} - R_1. \]  \hspace{1cm} (3.6)

This definition of amplitude \( (A_{\text{max}}) \) is used to compare the ANSYS Fluent Simulation results with published results.

### 3.5 Comparison of growth rate

To calculate the growth rate from ANSYS Fluent simulation results first, amplitudes of the waves were saved as a function of time. The amplitude \( (A_m) \) is calculated from equation \( (3.5) \). A typical amplitude vs. time plot from the simulation results is shown in Figure 14 in dimensional form. Notice that the amplitude shows oscillation and around one second of simulation time it starts to reach saturation (with periodic oscillation).
In order to calculate the growth rate, it is necessary to convert the amplitude vs. time data from the simulation results into a non-dimensional form. The amplitude $A_m$ and time $t$ are non-dimensionalized by equation (3.7).

$$A_m^* = \frac{A_m}{R_1},$$

$$t^* = \frac{tw_0}{R_1}. \tag{3.7}$$

Here $A_m^*$ and $t^*$ are non-dimensional amplitude and non-dimensional time, respectively.

From the ANSYS Fluent simulation results, non-dimensional amplitudes vs. non-dimensional times are plotted on a log-linear plot shown in Figure 15. Growth rate is calculated by fitting an exponential curve fit shown in equation (3.8) with the few initial data marked with a cross symbol.

$$f(x, a, b) = ae^{bt^*}. \tag{3.8}$$
Here $b$ is the linear growth rate of the amplitude, $t^*$ is non-dimensional time and $a$ is a constant. The calculated magnitude of the growth rate is $b = 0.06286$. Growth rate obtained from ANSYS Fluent simulation results and the linear stability analysis, matches fairly well which is shown in Table 3.

![Figure 15](image-url)  

Figure 15: Curve fit to calculate the growth rate of the wave from the growth of the wave amplitude as a function of time.

<table>
<thead>
<tr>
<th>Growth rate comparison of ANSYS Fluent simulation results vs. Growth Rate from linear stability analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ANSYS Fluent Simulation</strong></td>
</tr>
<tr>
<td>Growth Rate</td>
</tr>
</tbody>
</table>

Growth rate is sensitive to mesh size, time step selection, and Courant number. Of course, the solution has to be a converged solution.
### 3.6 Comparison of wave amplitude

From ANSYS Fluent simulation results, maximum amplitude \( A_{max} \) vs. simulation time is plotted in Figure 16. Notice that the data presented here is in dimensional form.

![Figure 16: Maximum saturated wave amplitude \( A_{max} \) vs. simulation time in dimensional form.](image)

Li and Renardy (1999), reported non-dimensional maximum amplitude \( A_{max} \) vs. non-dimensional time plot in a log-linear scale which is shown in Figure 17. The amplitude considered in their work is the maximum amplitude shown in equation (3.6).

In an effort to compare ANSYS Fluent™ simulation results with published results, dimensional maximum amplitude \( A_{max} \) vs. time plot has to be converted in non-dimensional plot with a log-linear scale. The results are shown in Figure 17. An excellent agreement is found between the saturated maximum wave amplitudes. The % variation of the saturated maximum amplitudes lies within the 1% range. Therefore, an excellent agreement is observed between the published results and ANSYS Fluent simulation results.
Figure 17: Comparison of maximum saturated wave amplitudes \( (A_{\text{max}}) \) between published results and ANSYS Fluent simulation results in Log10-Linear scale. (Non-Dimensional Amplitude Vs. Non-Dimensional Time).

### 3.7 Wave shape comparison

A typical saturated wave shape of ANSYS Fluent simulation result is shown in Figure 18. Usually, after one second of simulation time (dimensional), wave amplitude reaches saturation. Here the volumetric fraction of the oil is represented by blue color and the volumetric fraction of water is represented by red color. The wave shapes resemble bamboo waves as described by experimental and published results of Li and Renardy (1999). The simulation could run longer to ensure that, the wave generated is indeed a saturated wave. For the present study, the simulation is run for three to five seconds.
Figure 18: Saturated wave shape after one second of simulation time.

In Figure 19, wave shapes generated by ANSYS Fluent simulation results are compared with published results (Li & Renardy, 1999) in non-dimensional wave height.

![Graph showing wave height comparison]

Figure 19: Saturated wave shape comparison between published (Li and Renardy, 1999) results and ANSYS Fluent simulation results.

The plus (+) sign represents the shape of wave from Li and Renardy’s (1999) published results. Solid and dotted lines represent the saturated wave shapes corresponding to the peak and valley locations of the oscillating amplitudes shown in Figure 14.
Fluent’s simulation results shows a striking agreement with published results of Li and Renardy (1999).

Li and Renardy (1999) are the first researchers who noticed a temporal periodicity in their modeling work (in hold-up ratio oscillation) but did not explicitly describe the evolution of the wave shape. They only reported one wave shape. In fact, our study suggests that for the given set of the flow parameters (benchmark case), wave amplitude oscillate and modulate. This means that wave shapes are also changing as the wave travels forward and there is a period associated with this change of wave pattern. A further detailed discussion is presented in Chapter 4 and Chapter 5.

3.8 Wave speed calculation and comparison

To calculate wave speed, wave shapes at two consecutive time instances with an interval of 0.001 seconds are plotted in Figure 20. The traveling distance is measured by calculating Peak to Peak wave distance. Wave speed at saturation is defined by total distance travelled by the wave at two instances of time divided by the magnitude of time interval. From the simulation results, the calculated wave speed is 0.669 m/sec. For the given flow parameters, the centerline velocity could be calculated from the basic flow equation (2.39). The center line velocity \( w_0 \) is 0.839 m/sec. Therefore, the non-dimensional wave speed is \( 0.669 / 0.839 = 0.797 \).
The calculated saturated non-dimensional wave speed published by Li and Renardy (1999) is 0.806. Excellent agreement in wave speed is observed between ANSYS Fluent simulation result and published results. It is also worth noting that for the given set of flow parameters linear stability analysis predicts the wave speed of 0.9431. The results are summarized in Table 4.

Table 4: Comparison of wave speed between ANSYS Fluent simulation results vs Renardy’s 2-D axisymmetric model and linear stability analysis.

<table>
<thead>
<tr>
<th></th>
<th>Fluent™ Simulation</th>
<th>Published Results (Li &amp; Renardy)</th>
<th>Linear Stability Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave Speed</td>
<td>0.797 (Saturated)</td>
<td>0.8068 (Saturated)</td>
<td>0.9431 (in linear regime)</td>
</tr>
</tbody>
</table>
3.9 Hold-up ratio

Bai, Kelker & Joseph (1996) extensively discussed the hold-up ratio. Some of their findings are discussed here. The hold-up ratio is the ratio of ratios, the ratio of volume flow rates to the ratio of volumes. In other words, the hold-up ratio $h$ is the ratio of volume flow rates of the oil and water $\left(\frac{Q_o}{Q_w}\right)$ to the ratio of volume of the oil to volume of the water $\left(\frac{V_o}{V_w}\right)$ in the pipe, or

$$h = \frac{\left(\frac{Q_o}{Q_w}\right)}{\left(\frac{V_o}{V_w}\right)} = \frac{\left(\frac{Q_o}{Q_w}\right)}{\frac{R_1^2}{R_2^2 - R_1^2}}.$$  

(3.9)

Hold-up ratio represented in equation (3.9) could also be defined as the ratio of superficial velocities by simple manipulation,

$$h = \frac{\left(\frac{Q_o}{\pi R_1^2}\right)}{\frac{Q_w}{\pi (R_2^2 - R_1^2)}} = \frac{c_o}{c_w}.$$  

(3.10)

Here, $c_o$ and $c_w$ are superficial velocities of oil and water. They are defined by

$$c_o = \frac{Q_o}{\pi R_1^2}, \text{ and } c_w = \frac{Q_w}{\pi (R_2^2 - R_1^2)}.$$  

(3.11)

In perfectly mixed flow, for example, a well-emulsified solution of water in oil, the hold-up ratio $h$ is one. In lubricated pipelining, the two fluids are not well-mixed and the hold-up ratio is not likely to be one. Thus, the hold-up will tend to be greater than unity when the water is the component in contact with the pipe wall and to be less than unity when oil is in contact with the pipe wall. This idea is not correct in vertical flow where the effects
of buoyancy are important. A sample hold-up ratio calculation is shown below. Here the density ratio is assumed to be one and gravitational acceleration is ignored. The hold-up ratio certainly depends on fluid properties and flow parameters but is also strongly influenced by flow types.

For perfect core-annular flow, the interface is purely cylindrical with uniform radius of the core. Therefore, from the basic flow equation described in Chapter 2, the velocities of the core and annulus section are given by equation (2.40). Therefore, the flow rates for the core \( Q_o \) and annulus \( Q_w \) would be evaluated as

\[
Q_o = 2\pi \int_0^{R_1} rw(r)dr = 2\pi \left[ \frac{f}{16\mu_1} R_1^4 + \frac{f}{8\mu_2} (R_2^2 - R_1^2) \right],
\]

\[
Q_w = 2\pi \int_{R_1}^{R_2} rw(r)dr = 2\pi \left[ \frac{f}{16\mu_2} (R_2^2 - R_1^2)^2 \right].
\]

After simplification, flow rate for oil and water could be written in dimensionless form as shown below

\[
Q_o = \frac{f\pi R_1^4}{8\mu_2} [m + 2(a^2 - 1)],
\]

\[
Q_w = \frac{f\pi R_1^4}{8\mu_2} (a^2 - 1)^2.
\]

The flow rate ratio could also be written in the non-dimensional form as

\[
\gamma = \frac{Q_o}{Q_w} = \frac{[m + 2(a^2 - 1)]}{(a^2 - 1)^2}.
\]

From equation (3.14), we can correlate thickness ratio \((a)\) with volume flow ratio \((\gamma)\) and viscosity ratio \((m)\) by
\[ a = \sqrt{1 + \frac{1}{\gamma} \left( 1 + \sqrt{1 + m\gamma} \right)} = \frac{R_2}{R_1}. \] 

(3.15)

Now the oil volume fraction \( (\eta^2) \) could be conveniently defined by volume of the oil to total volume of oil and water inside the pipe

\[ \eta^2 = \left( \frac{R_1}{R_2} \right)^2 = \frac{1}{a^2} = \frac{1}{1 + \frac{1}{\gamma} \left( 1 + \sqrt{1 + m\gamma} \right)}. \] 

(3.16)

Since the oil fraction is known, water volume fraction would be determined by the following relationship,

\[ 1 - \eta^2 = 1 - \frac{1}{1 + \frac{1}{\gamma} \left( 1 + \sqrt{1 + m\gamma} \right)} = \frac{1 + \sqrt{1 + m\gamma}}{\gamma + 1 + \sqrt{1 + m\gamma}}. \] 

(3.17)

Volume ratio could also be determined from the oil and water fraction definition,

\[ \frac{V_0}{V_w} = \frac{\eta^2}{1 - \eta^2} = \frac{\gamma}{1 + \sqrt{1 + m\gamma}}. \] 

(3.18)

Therefore, hold-up ratio could be determined by,

\[ h = \frac{\frac{Q_o}{V_o}}{\frac{Q_w}{V_w}} = 1 + \sqrt{1 + m\gamma}. \] 

(3.19)

If the oil viscosity is very high, i.e., as \( m \to 0 \), the hold-up ratio \( h \to 2 \).

It is observed that for perfect core-annular flow, the hold-up ratio is approximately 2.

In this analysis, the gravity is not taken into account, and the density for both fluids match.
Therefore, for up-flow and down-flow in a circular pipe, the hold-up ratio will differ from two. Hold-up ratio is also influenced by flow rate and surface tension.

For benchmark case of flow parameters of \( a = 1.6, m = 0.00166, \zeta = 1.1, K = -2.067, \zeta = 1.1, R = 3.737, R_2 = 4.762 \text{ mm and } R_1 = 2.958 \text{ mm}\), we can calculate the hold-up ratio for up-flow flat interface or PCAF flow condition by using the velocity profile listed in equation (2.36). The volumetric flow rate for oil and water would be as follows

\[
Q_o = 2\pi \int_0^{R_1} rw(r)dr = 2\pi \int_0^1 \bar{r} \left(1 - mK \frac{r^2}{\Lambda}\right) d\bar{r} = 3.138, \tag{3.20}
\]

\[
Q_w = 2\pi \int_{R_1}^{R_2} rw(r)dr
\]

\[
= 2\pi \int_1^{1.61} \bar{r} \left([a^2 - \bar{r}^2 - 2(K - 1) \ln(\bar{r}/a)]/\Lambda\right) d\bar{r}
\]

\[
= 1.637 .
\]

Once the flow rate of the oil and water is known for up-flow benchmark case, we can calculate the hold-up ratio from equation (3.9). Which comes out to be a value of 3.05. In determining the hold-up ratio gravitational acceleration is taken into account.

Therefore, it is interesting to see that hold-up ratio for the horizontal pipe where the viscosity ratio is very high and gravity is ignored the hold-up ratio approaches a value of 2 . On the other hand, for our benchmark vertical up-flow condition the hold-up ratio is around 3 for flat interface or PCAF flow condition. It is important to note that the hold-up ratio for horizontal pipe is around 2 and that is independent of pressure gradient. However, the hold-up ratio (3.05) calculated from the velocity profile of the up-flow depends on the pressure gradient and only applicable for the benchmark flow parameters.
It is relatively easy to calculate the volume of core (oil) and annulus (water). For 2-D axisymmetric model, the surface could be rotated 360 degrees to obtain core and annulus volumes. Once the volumes are obtained, it is relatively easy to calculate the volumetric flow rate by computing the volume integral of the axial velocity over the volumes of the core and annulus. All of these calculations are performed by Ensight™ post processing tool.

For the benchmark case, a comparison of hold-up ratio between the published results and ANSYS Fluent simulation results are presented in Figure 21.

![Figure 21: Comparison of hold-up ratio between ANSYS Fluent Simulation results vs. published results of Li and Renardy (1999).](image)

From the numerical simulation, it is observed that once the wave reaches saturation, the magnitude of hold-up ratio also oscillates around 2.15. A comparison between the ANSYS
Fluent simulation results and published results are presented in Table 1. It is evident that ANSYS Fluent simulation result agrees with published results of Li and Renardy (1999).

Table 5: Comparison of hold-up ratio between ANSYS Fluent results versus Renardy’s published results.

<table>
<thead>
<tr>
<th>Hold Up Ratio</th>
<th>ANSYS Fluent Simulation</th>
<th>Published Results (Li &amp; Renardy)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>~2.12</td>
<td>~2.15</td>
</tr>
</tbody>
</table>

Hold-up ratio as a function of time shown in Figure 21 shows oscillation. This oscillation of hold-up ratio also indicates temporal periodicity. Li and Renardy (1999) reported temporal periodicity in hold-up ratio for the first time, but they did not explain what happens to the wave shape and wave amplitudes due to this occurrence. They also did not mention the evolution of the wave shape and how it modulates over time due to this temporal periodicity. They tried to establish this temporal periodicity of the hold-up ratio by showing repeating vortex pattern after a specified period. A detailed explanation on how different flow parameters affect this kind of temporal periodicity did not exist. How does range of flow parameters affect the temporal periodicity. In their study they did not perform a detail and systematic study (not the main interest of their work) to observe the bifurcation of the saturated traveling with constant wave amplitude into modulated wave with distinct temporal periodicity for different flow parameters. Our study specifically addresses some of these questions in great details.
3.10 Comparison of benchmark model with various mesh configurations

Most of the ANSYS Fluent simulation results presented in the earlier section, we used a mesh size of 52,500 (~52K) cells. Few other models with different mesh configurations and time steps were studied to make sure that the converged results obtained from the ANSYS Fluent simulation are mesh independent. Description of the few mesh configurations from other models are depicted here. The selected models are:

1. 24 K Cell Model (Smallest)
2. 52K Cell Model (Benchmark Case)
3. 72K Cell Model (Medium)
4. 198K Cell Model (Large)

In Figure 9, a very fine mesh configuration of 198K is presented and in Figure 10, a benchmark case study mesh of 52 K is shown. Details of the two other mesh configurations of 24K, and 72K cell models are shown in Figure 22. For 52K cell size model, only a very small area near the flat interface line was meshed with denser cells. But when the parametric study is performed for lower surface tension case, it is observed that the growth of the wave is significant and the interface of the waves surpassed the initial dense mesh regime. To capture the wave growth and instability at lower surface tension parameters, the finer mesh area is extended which is shown in Figure 22. The new extended cell count is 72 K.
At the beginning of our research work, we mostly concentrated on 2-D axisymmetric analysis of the core-annular flow. As we learned more about the strength of the ANSYS Fluent software and parallel computing, we wanted to take an ambitious step to simulate 3-dimensional core-annular flow. It would have been extremely time consuming to run a simulation in full 3-dimensional model of the benchmark case with 52K or 72K cell
densities (2-D-slice). Therefore, mesh densities were reduced even further to 24 K cells. For 24 K cell model, mesh densities were reduced in both \( X \) direction and \( Y \) direction. But critical area near the flat interface where the interface will grow in \( Y \) direction, mesh densities are kept constant. We performed multiple models with lower mesh density near the interface and identified that \( \Delta Y = 1.2 \times 10^{-5} \) m or smaller element length capture the growth of the initial wave and produces better convergence and smaller variations in saturated wave amplitudes. We learned from our earlier discussions that a denser mesh configuration of 198 K cells is significantly larger than other models and the computational cost is also enormous. Most of the 2-D axisymmetric models were run with 32 core machines and one second of simulation time takes around 4 to 5 hours (for 52 K mesh) of clock time.

Saturated wave amplitudes from ANSYS Fluent simulation are plotted in Figure 23 for four different mesh configurations of 24K, 52 K, 72 K and 198 K. From the plot, it is observed that percent variation of the wave amplitudes of the saturation waves lie within 5% range when compared with the case of 198 K model. The fluctuations of the amplitudes also lie within 3% range. Therefore, all four mesh models could be considered for further analysis. Notice that for 52 K and 198 K model, the initial starting amplitude was assigned to \( A = 1 \times 10^{-4} \) m and for 24K and 70K models, the initial starting amplitude was assigned to \( A = 3 \times 10^{-4} \) m. All the models converge to a very small window of saturation amplitude. Therefore, a mesh independent converged solution could be obtained with the proposed mesh configurations.
Figure 23: Comparison of maximum amplitudes of saturated waves at different mesh configurations.

Before ending this section, it is noteworthy to state that right after initialization, at the beginning of the simulation, a very fine time step is required for the model to converge in first few iterations. In Fluent, variable time stepping tool is an excellent feature which could be used. A range of fine time step such as $1 \times 10^{-8}$ seconds to $1 \times 10^{-4}$ seconds could be assigned. The Courant number also has to be assigned less than 1 (our recommendation is 0.25). This variable time stepping feature will alleviate any initial convergence issues. Once the model stabilizes to a desirable time step, it is recommended to run the model with fixed time stepping features of ANSYS Fluent.
CHAPTER 4 : 2-D-Axisymmetric Core-annular Flow: Effect of Surface Tension

4.1 ANSYS Fluent simulation results: Wave amplitudes at various surface tension parameters

In Chapter 3, the results for a benchmark case study \((a = 1.61, m = 0.00166, \zeta = 1.1, J = 0.063354, \Re = 3.73754)\) were presented, where we observed the formation of saturation waves. We also noticed periodic oscillation of the wave amplitudes. Li and Renardy (1999) observed time periodic behavior, but details of wave propagation and modulation of the waves were not described in great detail. Also, the full range of systematic parametric study was not presented in their paper. In this chapter, we will use surface tension parameter as the controlling parameter to explore evolution of the saturated wave in the nonlinear regime.

In Figure 24 changes in wave amplitudes over time are presented for various values of surface tension parameter \(J\). The benchmark or base case is designated by \(“J_b”\) for surface tension parameter \(J_b = 0.063354\), presented in Chapter 3. That represents a surface tension magnitude of \(\sigma = 0.085 \text{ N/m}\). In this plot, surface tension parameters vary from \((1/8)\) times the benchmark case to \(10\) times of benchmark surface tension parameter. For simplicity wave length is kept constant and only interfacial surface tension is changed for this study. Figure 24 summarizes the evolution of the wave propagation from perfect core-annular flow, to saturated bamboo wave to modulated bamboo wave as the surface tension parameter decreases. For the set of flow parameters, surface tension restrains the
growth of the wave. In fact, surface tension prevents the wave from breaking and forming into bubbles. In this particular analysis, wave length is kept constant.

From Figure 24 we observed four distinct flow regimes. They are

Figure 24: Wave amplitudes vs. simulation time for various surface tension parameters. ($J = 1 \times J_b = 0.063354, a = 1.61, m = 0.00166389, \zeta = 1.1, R = 3.73754$).

1. Perfect core-annular flow regime (PCAF) or flat interface regime ($J \geq 9.5J_b$):

From Figure 24 it is clear that for the base case or benchmark case (surface tension parameter $J = J_b = 0.063354$), wave amplitude shows oscillation and modulation. But if the surface tension parameter is increased, wave amplitude starts to reduce and eventually dies down. For this study, when the surface tension parameter is higher than or equal to 9.5 times the bench mark surface tension parameter (i.e. $J \geq 9.5J_b$), initial disturbed
interface eventually becomes flat interface and the problem becomes perfect core annular flow (PCAF).

2. *Saturated bamboo wave regime* \((3J_b \leq J < 9.5J_b)\) (SBW): In this regime wave reaches a stable equilibrium and travels with constant amplitude. Wave shape and wave speed remain the same over time.

3. *Modulated bamboo wave regime* \(((1/8)J_b < J < (3J_b))\) (MBW): In this regime, both wave shape and wave amplitude oscillate periodically with time. That means that the wave is not just a traveling wave of constant amplitude rather the amplitude of the wave changes periodically. From this study, the range of modulated regime is observed for surface tension parameters smaller than \(3J_b\) and larger than \((1/8)J_b\).

4. *Unstable wave break regime* \((J < (1/8)J_b)\) (UWB): When the surface tension is less than \((1/8)J_b\), lower viscosity fluid drawn out as fingers that penetrate into the higher viscosity layer and make the wave unstable and eventually cause the wave to break. If the surface tension parameters get any lower, the waves become very unstable and the interface starts to break.

4.2 Perfect core annular flow regime (PCAF) or flat interface regime: \((J \geq 9.5J_b)\)

At higher value of surface tension, initial wave damps out over time. The PCAF is stable. In this study, it is observed that initially imposed wave dies down when the surface
tension parameter is larger than nine times the benchmark model surface tension parameter.

To prove this point, the evolution of the wave shapes at different times are shown in Figure 25 and in Figure 26 for surface tension parameter \( J = 9.5J_b \).

![Figure 25: Evolution of wave shape at different simulation times (seconds) for surface tension parameter \( J = 9.5J_b \).](image-url)
Figure 26: Enlarged magnitude of wave shape at different times for surface tension parameter of $J = 9.5J_b$.

From Figure 26, it is observed that wave shapes at different times eventually reach flat interface in around two seconds of simulation time. From this discussion, it is clear that the wave reaches flat interface when the surface tension parameter is $J \geq 9.5J_b$. At this surface tension value, the wave stabilizes and form a flat interface.

To compare ANSYS Fluent simulation results with linear stability analysis, growth rate vs. surface tension parameters results from linear stability analysis are presented in Figure 27. It is observed that the growth rate is negative or the solution is stable when the surface tension parameter is slightly more than $8J_b$. ANSYS Fluent simulation results predict a very small amplitude saturated bamboo wave at $8.5J_b$ which could be a stable solution and a flat interface or perfect core annular flow solution at $9.5J_b$ of surface tension parameter. This variation of simulation results would be attributed to fixed wavelength
used in ANSYS Fluent simulation. Linear study suggests that as the surface tension changes wave length also changes for a given set of flow parameter.

Therefore, the prediction of flat interface or stable solution from ANSYS Fluent simulation results are in a fairly good agreement with linear stability analysis.

4.3 **Saturated bamboo wave regime**: \((3J_b \leq J < 9.5J_b)\)

If the surface tension gets smaller than \(9.5J_b\) and greater than \(3J_b\), initial disturbed wave eventually developed into a saturated traveling wave with constant wave speed and fixed wave shape. This regime is called saturated bamboo wave regime. In this regime, the amplitudes of the waves are constant over time. As shown in Figure 28, the amplitudes of the waves are constant over time and the values of the amplitude keep increasing as the surface tension parameter \(J\) reduces.
Figure 28: Amplitudes of the waves at saturated (non-modulated) bamboo wave regime with respect to surface tension parameters.

Constant wave amplitude also indicates that all waves are travelling waves and therefore, all waves will be of same shape irrespective of time. To illustrate this point, wave shapes are drawn for surface tension parameter, \( J = 3J_b \) at every 0.05 seconds time interval. The amplitude and times are shown in Figure 29. Waves at every 0.05 seconds are shifted in the horizontal direction to match up with the first reference wave starting at an arbitrary time \( t \) which is shown in Figure 30. From Figure 30, it is observed that wave shapes at every 0.005 seconds interval perfectly match with each other. Therefore, they are the exact same wave and it travels at a constant speed. Wave shapes for surface tension parameters \( J = 4J_b, J = 5J_b \) and other cases of saturated bamboo waves are shown in Appendix-A. The wave shapes remain constant over time for a selected surface tension parameter along with a constant wave speed.
Figure 29: Temporal location of wave shape for every 0.05 seconds time interval for surface tension parameter $J = 3J_b$.

Figure 30: Wave shape at every 0.05 seconds time interface for surface tension parameter $J = 3J_b$. Waves were shifted in the horizontal direction to match with reference wave at time $t$.

In Figure 31, waves shapes for all of the surface tension parameters in the saturated bamboo wave regime are presented in actual dimension and in Figure 32 in non-dimensional form. From Figure 31, it is clear that amplitude of the wave reduces with the increase of the surface tension. It is also evident that for saturated bamboo wave regime, the shape of the wave gets pointed near the peak and widens near the trough as the surface tension reduces. The change in shape of the wave is the direct attribution of the surface
Figure 31: Comparison of wave shapes for saturated bamboo wave regime (in scale) at different surface tension parameters.

Figure 32: Comparison of wave shapes in non-dimensional form at saturated bamboo wave regime at different surface tension parameters.

tension; with the reduction of surface tension force, the wave tends to contract inside and a pointed peak is formed. Also note that the wave shapes are not perfectly symmetric and as the surface tension reduces, the slight asymmetry becomes more visible.
In Figure 33, saturated wave amplitudes are plotted as a function of surface tension parameters for saturated bamboo wave regime along with a curve fit function. Notice that at surface tension parameter $J = 9.5J_b$ wave has reached to a perfect core annular flow (PCAF) or flat interface and the amplitude of the waves reduces to zero. As the surface tension reduces, wave amplitudes keep on rising and the relationship is determined by the nonlinear curve-fit shown in equation (4.1)

$$y(x) = \frac{1}{16000} (x_c - x)^{3/4}. \quad (4.1)$$

Here $x_c = 9.5J_b$.

![Amplitude Data (Saturated Bamboo Wave Regime) Curve Fit [y(x)]](image)

**Figure 33:** Amplitude vs surface tension parameters for saturated bamboo wave regime. Curve fit of the amplitudes with respect to surface tension shows nonlinear relationship.

It could be concluded that in this saturated bamboo wave regime, surface tension is large enough to contain the wave shape fixed at all times, but it is not too large to dampen the initial disturbance to flat interface. On the other hand, surface tension is not too small to initiate unstable wave propagation. In this regime, surface tension tends to stabilize the
wave and prevent it from getting unstable. It is just the right amount of surface tension which makes the wave to travel at constant wave speed with a constant wave shape at all times once the saturation is reached.

4.4 **Modulated bamboo wave regime: \((1/8)J_b \leq J < 3J_b\)**

In Figure 34, an enlarged version of Figure 24 is shown. It is observed that if the surface tension is less than \(3J_b\) and more than or equal to \((1/8)J_b\), wave amplitude oscillates. It is also observed that wave amplitude reaches to a saturation magnitude within one second of simulation time for surface tension parameters of \(J_b\), \(2J_b\), \((1/2)J_b\) and it takes a little longer time (3 to 5 seconds of simulation time) for smaller values of surface tension parameters such as \((1/4)J_b\) and \((1/8)J_b\).

![Figure 34: Saturated and modulated bamboo wave regime with variable amplitudes.](image)

In fact, surface tension tries to stabilize the wave. At lower value of surface tension, fingering takes place; i.e., fluid with lower viscosity penetrates the regime of higher viscosity and eventually the wave breaks if the surface tension parameter is too small. At
higher surface tension parameters, the wave amplitude reaches a saturation magnitude within one second of simulation time and traveling wave propagates with constant amplitude without any modulation which is shown in Figure 28. In order to understand why the wave amplitudes oscillate and modulate in this regime, we need to study the evolution of wave shapes more closely. Since the benchmark case $J = J_b$ falls into the modulated bamboo wave regime, in this section a detail analysis of the evolution of the wave shapes at this surface tension parameter is presented. An enlarged view of the amplitude vs. time is plotted in Figure 35. Notice that initially data were saved on larger time interval (0.01 sec) for 0 to 3 seconds of simulation time. Since our focus is nonlinear saturated waves, more data points were saved with a smaller time interval (0.001 seconds) for 3 to 4 seconds of simulation time. Our goal is to examine and explain the nature of the wave shapes and patterns once the wave reaches saturation (with periodic oscillation). This kind of detailed study of the evolution of the waves are described for the first time.

![Figure 35: Base Case Amplitude Vs. Time for surface tension parameters $J = J_b = 0.063354$.](image-url)
In Figure 36, amplitude vs. time curve is plotted for saturated (with periodic oscillation) waves from 3 seconds to 4 seconds of simulation time for the benchmark case. Uniform oscillation of the amplitude is observed. In this figure, three vertical lines are the simulation times of three Peak to Peak amplitudes. The time interval for Peak to Peak oscillation is 0.034 seconds of simulation time.

Figure 36: Amplitude Vs. Time for the saturated wave for the benchmark case with surface tension parameter \( J = J_b \). Wave shape corresponding to each peak wave is drawn in Figure 37. To compare the wave shape, three waves are shifted in the horizontal direction only (maximum height of the waves are shifted to maximum height of the first reference wave). The dotted lines are shifted waves in \( x \)-direction. It is clear from Figure 37 that three waves overlap. This indicates that the waves at each peak are identical. They perfectly match with each other. Therefore, we can confirm that the wave shape repeats itself at every peak and the period of the repetition of the wave is 0.034 seconds of simulation time.
Figure 37: Wave shape at three Peak to Peak locations for benchmark case of surface tension $J = J_b$.

Similarly, Figure 38 shows the amplitude versus time curve with solid lines representing three distinct valleys. The interval between the valley waves are also 0.034 seconds.

Figure 38: Amplitude vs. Time curve for the benchmark case of surface tension $J = J_b$. Three vertical lines are showing the Valley to Valley amplitude as a function of time.

To compare the valley waves, three waves are selected and shifted in horizontal direction, which is shown in Figure 39. All three valley waves perfectly overlap with each other and are shown with dotted lines. Therefore, all of the valley waves are identical in their shape and has the same period of 0.034 seconds.
From the above discussion, it is confirmed that for the base case, wave shape for the peak and valley repeat itself and with a period of 0.034 seconds of simulation time. All of the peak waves are identical to each other and all of the valley waves are identical with each other.

Figure 39: Valley to Valley three waves are identical in shape for benchmark case of surface tension $J = J_b$. Waves are shifted only in $X$-direction.

Figure 40 shows the comparison of the peak and valley wave shapes when they are shifted in the $x$-direction to match at maximum height. It is observed that peak and valley waves vary in their shapes. Even though all of the peak waves are identical with each other and same is true for all of the valley waves, it is not true when Peak to Peak wave shapes are compared with Valley to Valley wave shapes. Therefore, peak and valley waves are not the same waves. They are different and unique. This is also another critical finding of this research work.
Figure 40: Comparison between the peak and valley wave shape for $J = J_b$.

Since the wave shapes are unique between the peak and valley of the oscillating amplitudes, there must be waves which will differ in shapes in-between the peaks and valleys. To describe the evolution of the waves as time progresses, wave shapes are drawn in every 0.003 second time interval. The temporal locations of each wave amplitudes are shown in Figure 41 and the corresponding wave shapes are shown in Figure 42. It is observed that wave shape changes as the time progresses. Waves were shifted to the horizontal direction to match the maximum height of the first reference wave. The shifted waves are shown in Figure 43. It is evident that the wave shapes do not coincide with each other, rather they are different from each other and they only repeat itself after a certain time interval (0.034 seconds).

Figure 41: Temporal location of waves and its amplitude at every 0.003 seconds of simulation time. Covering the waves between peak and valley. Vertical lines are the identification of temporal location and corresponding wave amplitudes.
Figure 42: Wave shape at every 0.003 seconds time interval. \( J = 1 \times J_b = 0.063, a = 1.61, m = 0.0016, \zeta = 1.1, \Re = 3.737 \).

Figure 43: Wave shapes at 0.003 second interval. Waves are shifted in horizontal direction to match the maximum height of the first reference wave at time \( t \).

In order to further investigate the wave shape at different times in-between the peak and valley of the wave amplitudes, wave height is non-dimensionalized by subtracting the
minimum height of the wave from the wave shape and then dividing the result with the difference between the maximum and minimum height of the wave. The non-dimensional wave shapes are shown in Figure 44. It is evident that wave shape does not collapse into a single master curve. It is rather interesting to see that wave peak remains same but the waves stretch near the trough.

![Figure 44](image)

Figure 44: Non-dimensional wave shape at every 0.003 seconds time interval capturing the wave shape from peak to valley. Wave shapes were shifted in the horizontal direction to match with the maximum height of the first wave at time t.

From the discussion above, it is very clear that the wave shapes follow a pattern. It repeats itself after a period of 0.034 seconds. This proves that waves not only oscillate with a distinct period but also modulate with time; which is one of the vital findings of this research.
All of the above discussions are based on the results of our benchmark studies for the case of surface tension parameter $J = 0.063354$. Similar results were obtained when the wave shapes were compared for all surface tension parameters within $(1/8)J_b \leq J < 3J_b$; i.e. for modulated bamboo wave regime.

To examine the wave shape for a different surface tension parameter within the modulated bamboo wave regime, the results of surface tension parameter $J = 2J_b = 0.127$ are discussed here. Figure 46 shows the temporal location of Peak to Peak waves and Figure 48 shows the corresponding Peak to Peak waves shapes. For the sake of comparison, the peak waves are shifted in the horizontal direction and they perfectly overlap. Therefore, all the peak waves are identical. The period between the Peak to Peak wave is 0.032 seconds.
Figure 45: Temporal location of Peak to Peak wave for the surface tension parameter $J = 2J_b = 0.127$.

Figure 46: Solid lines are Peak to Peak wave shapes for surface tension parameter $J = 2J_b$. Dotted lines are the shifted waves and they perfectly coincide with each other. Peak to Peak waves repeat itself after every 0.034 seconds.

Similarly, temporal location of Valley to Valley waves for surface tension parameter $J = 2J_b$ is shown in Figure 47. The waves were also shifted in the horizontal direction to match with each other in Figure 48. They perfectly overlap. Therefore, all the valley waves are the same wave and the period between them is also 0.032 seconds. Therefore, waves repeat itself with Peak to Peak interval and Valley to Valley interval. It is now proved that all the peak waves have the same shape and amplitude and all the valley waves have same shape and amplitude.
Figure 47: Temporal location of the Valley to Valley waves for the surface tension parameter $J = 2J_b = 0.127$.

Figure 48: Solid lines are Valley to Valley wave shape for surface tension parameter $J = 2J_b$. Dotted lines are the shifted waves and they perfectly coincide with each other. Valley to Valley waves repeat itself after every 0.034 seconds.

When peak and valley waves are compared with each other, they are not the same wave. In Figure 49 comparison between the peak and valley waves for surface tension parameter $J = 2J_b$ is presented. It is evident that they do not overlap. Therefore, peak and valley waves differ from each other. They are unique waves.
Figure 49: Peak to Peak and Valley to Valley wave shape comparison for surface tension parameter $J = 2J_b$.

Wave shape comparison for the surface tension parameter $J = (1/2)J_b$, $J = (1/4)J_b$ and $J = (1/8)J_b$ cases are shown in Appendix-A. The conclusions remain the same. For all modulated bamboo wave regime, wave amplitudes oscillate and modulate.

From the results presented above, it is clear that for the set of surface tension parameters $(1/8)J_b \leq J < 3J_b$, waves not only oscillate but also modulate. When the wave amplitude oscillates, wave shape also follows a distinct pattern. The wave shape changes with time and repeats itself with a unique period. Figure 50 shows the change in period as a function of surface tension parameters. It is observed that period of the wave decreases with the increase of surface tension parameter. This implies that surface tension seems to stabilize the flow and therefore increases its speed and makes the period slightly smaller.
Figure 50: Period of oscillation for different surface tension parameters.

Figure 51 compares all modulated Peak to Peak saturated wave shapes in true scale and Figure 52 shows them in non-dimensional form. Valley to Valley wave shapes follow a similar pattern. Wave height is non-dimensionalized by subtracting the minimum height of the wave from the wave shape and then dividing the result with the difference between the maximum and minimum height of the wave. It is observed that the wave shapes for both Peak to Peak waves and Valley to Valley waves become compressed and pointed at the crest and widen at the trough as the surface tension is reduced. It is also very prominent from those two figures that the waves are not perfectly symmetric and with the reduction of surface tension this asymmetry is more obvious.
Figure 51: Peak to Peak wave shape for modulated wave at different surface tension parameters.

Figure 52: Peak to Peak wave shape in non-dimensional form for all of the waves of modulated bamboo wave regime.
From all of the above discussions, it is revealed that, for modulated bamboo wave regime, wave is not only a traveling wave but it modulates as the time progresses. In Figure 34, a closer look on the saturated wave amplitude is depicted for oscillating amplitudes. Even though amplitude reaches saturation (with periodic oscillation) at one second of simulation time, for a guaranteed saturated wave, simulation is run longer to make sure that saturation wave still holds.

In Figure 53, saturated wave amplitudes are compared at simulation time of three to four seconds. It is observed that magnitude of the oscillation amplitude is slightly higher at $J_b$ and $(1/2)J_b$, but as the surface tension is increased, magnitude of the oscillation keeps on reducing. On the other hand, when the magnitudes of the surface tension parameters are decreased from the benchmark case, amplitude of oscillation tends to increase up to a certain maximum and then it starts to drop as the surface tension is reduced.

![Graph showing oscillation amplitude vs. simulation time](image)

**Figure 53:** Comparison of saturated and modulated wave amplitude oscillation.

The amplitudes of oscillation with respect to surface tension parameters are shown in Figure 54. The oscillation amplitude is defined by dividing the difference of maximum
amplitude and minimum amplitude by two. This is done for the fluctuating wave amplitudes for each surface tension cases of the modulated bamboo wave regime.

From Figure 51, it is observed that, at surface tension parameter \((1/8)J_b\), wave peak starts to get pinched (compressed) at the crest and starts to get unstable. Therefore, it reaches the lowest amplitude of oscillation at saturation. Surface tension smaller than \((1/8)J_b\) is unstable and waves start to break for surface tension parameter less than \((1/8)J_b\).

![Graph showing wave amplitude vs. surface tension parameter](image)

Figure 54: Amplitudes of oscillation of the modulated waves as a function of surface tension parameters \(J\).

In Figure 55, a further amplification of the modulated wave amplitude is drawn which shows the maximum (peak) and the minimum (valley) locations. The maximum and minimum values of the modulated wave amplitude is used to draw the branching of the modulation of the wave amplitudes which is shown in Figure 55.
Figure 55: Enlarged view of the modulated wave amplitude vs. time with distinct peak and valley of the oscillation for different surface tension parameters.

Figure 56: Maximum and minimum magnitude of oscillating amplitudes of waves at modulated bamboo wave regime.
Therefore, from Figure 56 it is clear that at saturated bamboo wave regime, wave travels with constant amplitude and no oscillation is observed in this regime but once the wave reaches in the modulated bamboo wave regime, the wave starts to modulate and as a result, the amplitude of the wave changes as the time progresses and repeats itself after a specified period. In Figure 56, only two magnitudes (maximum and minimum amplitudes) of modulated waves are presented to show the branching of the waves from saturated bamboo wave regime to modulated bamboo wave regime.

4.5 Bifurcation of saturated bamboo waves

From the above discussion, it is reasonable to draw a diagram of amplitude changes over the range of surface tension parameter changes. The results from Figure 56 and Figure 33 are plotted in Figure 57. Figure 57, shows clearly that at surface tension parameters greater than $9.5J_b$, the amplitude of the wave is zero. That is, the interface is flat or we consider this problem as perfect core annular flow (PCAF). As the surface tension decreases from $9.5J_b$, amplitude of the waves continue to increase until the surface tension parameter reaches $3J_b$. This regime is known as saturated bamboo wave regime. We noticed that the amplitude of the waves is constant, that means that the traveling waves move with a constant shape and with a constant wave speed. It is also evident that increase in wave amplitude is not linear in this saturated bamboo wave regime. Rather it follows a non-linear growth with the decrease in surface tension parameter. As the surface tension continues to decrease, the wave amplitudes start to oscillate and modulate, therefore, it
branches out and shows a distinct maximum and minimum amplitude. Therefore, Figure 57 is the bifurcation diagram of the saturated wave amplitude over the range of surface tension parameters. This is a new finding of our research work.

![Bifurcation diagram](image)

Figure 57: Bifurcation diagram: Amplitude of the oscillation of the wave at different surface tension parameters as a multiplication of benchmark surface tension $J_b$.

### 4.6 Unstable wave (break) regime: ($J < (1/8) J_b$)

In the earlier section, we showed that when the surface tension parameter gets smaller, wave crest starts to contract. If the surface tension parameter becomes lower than $(1/8) \times J_b$, waves show localized pinching near the crest of the wave. A secondary curvature develops near the crest of the wave and tries to break the interface. Once the
interface is broken, the waves get unstable and eventually reaches to saturation wave with lower magnitude. In Figure 58, wave shapes are shown for the surface tension parameter \((1/10)J_b\) case. If we examine Figure 58 more closely, we observe that near the secondary curvature close to the crest of the wave, a small bubble is formed. This means that the interface is broken at that time and any result which describes the wave after the interface is broken, does not correspond to saturation waves even though waves tend to reach a lower saturation amplitude after running the simulation for longer time. In other words, what is obtained after the interface is broken is not a stable result.

Figure 58: Evolution of the wave shape at different simulation times (seconds) for surface tension parameter \(J = (1/10)J_b\).

Figure 59 and Figure 60 show the evolution of the waves for surface tension parameter \((1/16)J_b\). Again, it is observed that a secondary curvature is developed near the crest and a bubble is formed, which breaks the interface.
Figure 59: (a) Evolution of the wave shape at different times for surface tension parameter $J = (1/16)J_b$. (b) Wave shape with enlarged view close to wave break.

Figure 60: Wave shape close to wave breaking for surface tension parameter $J = (1/16)J_b$.
From the above discussion, it is clear that, as the surface tension parameter gets smaller and smaller, the wave interface starts to break and wave reaches to a totally nonlinear unstable regime. That is why this regime is considered as nonlinear wave break regime.

4.7 Wave speed vs. surface tension

Wave speed is another very important parameter to investigate. Figure 61 shows the relationship between the Non-Dimensional wave speed and surface tension. From our analysis, we found a nonlinear relationship for the wave speed as a function of surface tension. As the surface tension increases wave speed increases, but eventually reaches a plateau as the wave tries to reach flat interface results.

![Wave Speed vs. Surface Tension](image)

Figure 61: Wave Speed as a function of surface tension.

As the surface tension increases the wave speed also increases. This could be attributed to surface tension which stabilizes the waves at higher surface tension parameters. Waves can propagate faster at higher value of surface tension.
Additionally, we can draw conclusions from the linear stability analysis. In our simulation we kept the wave number or wave length (pipe length) fixed. We also kept the Reynolds number fixed. We only changed the surface tension for the given length of pipe. Linear stability analysis suggests that when we keep the Reynolds number fixed and change the surface tension, it also changes the wave number or wave length. Since we kept the wave length fixed to keep the pipe length fixed for our analysis, the flow regime will correspond to a different Reynolds number or flow rate. That might also attribute to higher wave speeds.

### 4.8 Hold-up ratio vs. surface tension

Figure 62 shows hold-up ratios at various surface tension parameters. It is interesting to see that the hold-up ratio shows the oscillation for the oscillating and modulating wave regime and remains constant for non-modulated wave regime. This is clear proof that waves oscillate and modulate for some surface tension parameters and remains constant for others.

Interestingly enough, as the surface tension increases; hold-up ratio also increases. At first glance, this considerable increase in hold-up ratio seems counterintuitive, but a careful analysis confirms that hold-up ratio could be significantly affected by the surface tension. ANSYS Fluent simulation results indicate that at higher value of surface tension at around \( J = 9.5J_b \), a flat interface or PCAF condition will occur for the given set of flow parameters. The hold-up ratio predicted by our simulation is 3.07 for PCAF condition.
Figure 62: Hold-up ratio at different surface tension parameters.

Discussion in Chapter 3 on hold-up ratio (section 3.9) for the up-flow condition also determines a magnitude of hold-up ratio 3.05 for flat interface condition for vertical flow of the given set of benchmark flow conditions where gravity is being taken into account. The results from both ANSYS Fluent simulation results and analytical results are compared in Table 6.

Table 6: Hold-up ratio comparison between ANSYS Fluent simulation Results vs. Analytical solution PCAF of a vertical up-flow condition at benchmark flow parameters

<table>
<thead>
<tr>
<th>Surface Tension</th>
<th>ANSYS Fluent Simulation</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.05</td>
<td></td>
<td>3.05</td>
</tr>
</tbody>
</table>

Therefore, hold-up ratio obtained from ANSYS Fluent simulation matches extremely well with pure analytical calculation of hold-up ratio. Thus, this wide variation of hold-up ratio at different values of surface tension could be rationalized.
The final takeaway message is that surface tension affects the hold-up ratio as the wave shape alters with surface tension. The increase of the hold-up ratio with the increase of surface tension parameter is caused by the fact that the wave amplitude drops as the surface tension parameter increases, as indicated by the bifurcation diagram presented in Figure 57. For waves with larger amplitude, the mean velocity of the oil core reduces, thus the hold-up ratio is smaller for smaller values of surface tension.

At this point, it is also worthwhile to mention that in our analysis we used a fixed wavelength and only changed the values of interfacial surface tension to study the nature of the wave shape in non-linear regime. Our analysis predicts PCAF or flat interface at $J = 9.5J_b$, where wave amplitudes diminishes to zero. Linear stability analysis predicts PCAF at around $J = 8J_b$ for the given set of benchmark flow parameters and for a fixed value of wave number $\alpha = 2.4$, where growth rate starts to become negative which is presented in Figure 27. Therefore, ANSYS Fluent simulation results and results obtained from linear stability analysis match fairly well. A slight variation could also be attributed to fixed wavelength of $\lambda = 7.744$ mm which was used in our analysis.

Ideally, linear stability analysis could have been used as a guiding tool to determine the change in wavelength at various surface tension parameters instead of a fixed wavelength. In Figure 63, growth rates obtained from linear stability analysis are presented as a function of wave numbers at various magnitudes of surface tensions for our benchmark flow parameters. It is observed that the wave number corresponding to the maximum
growth rate changes as the surface tension parameters vary. The wavelength keeps on reducing with the reduction of surface tension parameters.

Figure 63: Growth rate versus wave number at different surface tension parameters at fixed Reynolds number of $\mathbb{R} = 3.737$

In Table 7 wave numbers and corresponding wavelengths are tabulated for maximum growth rate at various surface tension parameters.

Table 7: Wave numbers and corresponding wavelengths at the most dominant growth rates for different surface tension parameters at fixed Reynolds number of $\mathbb{R} = 3.737$.

<table>
<thead>
<tr>
<th>Surface Tension</th>
<th>$J = (1/4)J_b$</th>
<th>$J = (1/2)J_b$</th>
<th>$J = J_b$</th>
<th>$J = 2J_b$</th>
<th>$J = 5J_b$</th>
<th>$J = 6J_b$</th>
<th>$J = 8.5J_b$</th>
<th>$J = 9.5J_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave Number</td>
<td>2.6</td>
<td>2.5</td>
<td>2.4</td>
<td>2.0</td>
<td>1.6</td>
<td>1.5</td>
<td>0.35*/1.3**</td>
<td>0.35*/1.3**</td>
</tr>
</tbody>
</table>

*Wave number /Wavelength corresponding to the largest growth rate at smaller wave number.
** Wave number /Wavelength corresponding to the largest growth rate at larger wave number.
It is also noted that for surface tension parameter $J = 8.5J_b$, two localized maximum growth rates are observed. Importantly, the mode which influences the highest growth rate at smaller wave number is associated with capillary mode of disturbances and influence the wave with larger wavelengths. On the other hand, mode of disturbances associated with maximum growth rate at larger value of wave number is associated with shear mode of disturbances where surface tension influences the dynamics of short waves. Therefore, two competing modes of disturbances interact with each other for this particular flow parameter. In this study, our focus is to investigate the mode of disturbances associated with the later mode. Therefore, it is recommended to use the wavelength associated with the growth rate at the larger wave number. To avoid confusion, it is also recommended to exclude the cases where two localized maximum growth rates are observed for a given set of flow parameters from consideration because of unknown mode of disturbances influencing the growth rate.
CHAPTER 5 : 2-D-Axisymmetric Core-annular Flow: Effect of Reynolds Number

5.1 Linear stability analysis results at different Reynolds number

Thus far, we have only discussed the wave dynamics for a constant Reynolds number of \( \mathbb{R} = 3.737 \). For that study, we started with a two-dimensional axisymmetric model. To study the nonlinear behavior of the wave shape at saturation, we took advantage of linear stability theory. Linear stability analysis provided us with the growth rate vs. wave number curve as shown in Figure 5. As explained in Chapter 3, the wave length is calculated by considering the wave number corresponding to the most dangerous or dominant mode where the growth rate is maximum.

In Chapter 4, we observed the response of surface tension on wave dynamics for a fixed flow rate. We established the fact that surface tension is a key variable. For this case, surface tension prevents the wave from breaking. It is a stabilizing force. As the surface tension is reduced, the wave becomes unstable and eventually starts to break. Another interesting finding is that surface tension is a key parameter which enhances or diminishes modulation of the wave over a certain range and makes the wave more stable at higher surface tensions where the wave becomes a traveling wave. If the surface tension is increased even further, flow becomes a flat interface problem or perfect core annular flow.

In this section, our goal is to understand the response of various flow rate conditions or Reynolds numbers, by keeping the surface tension parameter \( J = 1 \times j_b = 0.063354 \)
fixed. To do this we needed to come up with the wave length corresponding to the most dominant modes for various Reynolds numbers. By applying linear stability analysis (FORTRAN PROGRAM), we plotted the growth rate vs. wave number curves for Reynolds number $\mathbb{R} = 0.5$ to $\mathbb{R} = 6$. Wave length could be determined from the most dominant wave number where the growth rate is maximum. In Figure 64, Growth rate vs. wave number curves for Reynolds number $\mathbb{R} = 0.5$ to $\mathbb{R} = 6.0$ are shown. Wave number corresponding to the maximum growth rate are identified with vertical lines for different Reynolds numbers. It is observed that as the Reynolds number increases, wave number corresponding to maximum growth rate gets larger. In other words, as the Reynolds number increases, the wave length for the most dominant mode gets smaller [equation (3.3)].

Figure 64: Growth Rate Vs. Wave Number for various Reynolds number with fixed flow parameters of $J = J_b = 0.063354$, $a = 1.61$, $m = 0.00166389$, $\zeta = 1.1$.

Also, notice that as the Reynolds number increases, a secondary peak develops near the smaller values of wave numbers which is obvious for Reynolds number $\mathbb{R} = 4.5$, $\mathbb{R} =$
5 and $\Re = 6.0$. If the flow rate is increased even further or the Reynolds number increases from $\Re = 6$ to $\Re = 7$, the maximum growth rate is observed at lower values of wave number which is shown in Figure 65. Linear stability analysis suggests that the maximum values of the growth rate for those peaks are associated with different modes of disturbances. As we learned earlier, two competing modes such as shear and capillary mode of disturbances compete with each other and eventually one of the mode of disturbances dominates the other mode and shows higher growth rate than the other. Therefore, maximum growth rate at a smaller wave number does not represent the mode of disturbances for a fixed interfacial surface tension parameter of $J = 1 \times J_b = 0.063354$, rather it represents a different mode of disturbance.

Figure 65: Growth rate vs. Wave number at various Reynolds numbers. $\Re = 7$ and $\Re = 10$ shows maximum growth at lower wave number.

In Figure 66, the wave numbers corresponding to the maximum growth rate for various Reynolds numbers are shown and corresponding wave lengths are presented in Table 8.
We observe that there is a sudden drop of wave number when the flow rate is slightly increased, for example when the Reynolds number changes from $\mathbb{R} = 6$ to $\mathbb{R} = 7$.

Figure 66: Wave Number corresponds to maximum growth rate as a function of Reynolds number. For fix flow parameters of $J = J_b = 0.063, a = 1.61, m = 0.00166, \zeta = 1.1$.

Table 8: Wave numbers at most dominant growth rate for different Reynolds numbers and corresponding wave lengths.

<table>
<thead>
<tr>
<th>Reynolds Number</th>
<th>$\mathbb{R} = 0.5$</th>
<th>$\mathbb{R} = 1.0$</th>
<th>$\mathbb{R} = 1.5$</th>
<th>$\mathbb{R} = 2.0$</th>
<th>$\mathbb{R} = 2.73$</th>
<th>$\mathbb{R} = 3.0$</th>
<th>$\mathbb{R} = 3.73$</th>
<th>$\mathbb{R} = 4.5$</th>
<th>$\mathbb{R} = 5$</th>
<th>$\mathbb{R} = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave Number</td>
<td>0.9</td>
<td>1.18</td>
<td>1.45</td>
<td>1.7</td>
<td>2.4</td>
<td>2.625</td>
<td>2.825</td>
<td>3.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wave Length (mm)</td>
<td>20.651</td>
<td>15.751</td>
<td>12.818</td>
<td>10.933</td>
<td>7.744</td>
<td>7.08</td>
<td>6.579</td>
<td>5.808</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This shift in the wave number indicates that for the parameters we are studying ($a = 1.61, m = 0.00166, \zeta = 1.1, J = 0.063$), we should limit our case studies within Reynolds number $\mathbb{R} = 6$. Therefore, cases with Reynolds numbers higher than 6 (such as $\mathbb{R} = 7$, and $\mathbb{R} = 10$) are excluded from our analysis.

Once the wave length or length of the fluid domain is attained for different Reynolds numbers, it is important to assign an appropriate pressure gradient along the length of the pipe with periodic boundary conditions in ANSYS Fluent. If the operating density is assigned as the density of oil $\rho_1$, then the following pressure gradient needs to be assigned for the listed cases of Reynolds numbers in ANSYS Fluent.
Table 9: Pressure gradient assigned in Fluent™ model for various cases of Reynolds Number.

<table>
<thead>
<tr>
<th>Reynolds Number</th>
<th>1</th>
<th>1.5</th>
<th>1.75</th>
<th>2</th>
<th>3</th>
<th>3.5</th>
<th>3.737</th>
<th>4.5</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluent Pressure Gradient (Pa/m)</td>
<td>-805.88</td>
<td>-767.37</td>
<td>-748.12</td>
<td>-728.87</td>
<td>-651.85</td>
<td>-613.35</td>
<td>-595.05</td>
<td>-536.33</td>
<td>-497.82</td>
<td>-420.8</td>
</tr>
</tbody>
</table>

* Operating density assigned in ANSYS Fluent: \( \rho_1 = \text{Oil Density} = 905 \text{ kg/m}^3 \)

Another very important observation is to assign the correct direction of gravitational acceleration for up-flow or down-flow condition. For the up-flow model, gravity is acting opposite to the flow direction in the negative \( x \)-direction.

In Chapter 4, we studied the variations of the wave form in nonlinear saturated regime. We observed that when the wave reached saturation, it oscillates and modulates for a certain range of surface tension parameters. Similar oscillation and modulation is also observed when the Reynolds number changes. Our study only highlights a small range of Reynolds number variation as listed above.

5.2 Distinction between the modulated and non-modulated wave

To study the effect of Reynolds number on saturated wave, mesh densities for \( X \) and \( Y \) directions are kept in same densities. That is, both \( X \) and \( Y \)-direction mesh sizes are kept at an equivalent of 72 K mesh configurations as discussed in Chapter 3. \( X \)-direction mesh seeds were adjusted to take into account different wave lengths with the change of Reynolds number. To ensure that wave amplitudes have reached to saturation wave, simulations are run for up to five seconds.
Results of saturated wave amplitudes are compared in Figure 67 for different cases of Reynolds numbers by keeping the surface tension parameter $J = J_b = 0.063354$ constant. The wave amplitude is defined in equation (3.5). Again, two distinct regimes were observed. For the first regime, wave amplitudes show oscillation and modulation for the range of Reynolds number $1.5 < R < 5.75$ and for the second regime wave amplitudes do not show any oscillation for the range of Reynolds number $5.75 \leq R \leq 1.5$.

It is very surprising that even at a very small Reynolds number $(R = 1)$, wave amplitudes tend to fluctuate and take a longer time to reach saturation. However, wave amplitudes do not show any oscillation and modulation when the Reynolds number is less than 1.5.

By examining Figure 67, we noticed two distinct regimes. They are

(a) **Modulated bamboo wave regime** $(1.5 < R < 5.75)$ (**MBWR**): In this regime, wave oscillates and modulates as it travels. As a result, wave amplitude also changes with time. Usually, wave shape changes as the time progresses and repeats itself with in a fixed period.

(b) **Saturated bamboo wave regime** $(1.5 \geq R \geq 5.75)$ (**SBWR**): In this regime, wave travels with a fixed wave shape and with a constant wave amplitude.
Figure 67: Wave amplitudes at different ranges of Reynolds numbers for a fixed surface tension parameter, $J = J_b = 0.0633$ and other fixed parameters $a = 1.61$, $m = 0.00166$, $\zeta = 1.1$.

5.3 **Modulated bamboo wave regime: ($1.5 < \Re < 5.75$)**

Figure 67 depicts the oscillation of amplitude for the range of Reynolds numbers $1.5 < \Re < 5.75$. If we closely examine the wave amplitudes over time, we notice that waves not only oscillate but also modulate as it is observed in Chapter 4 for the case of surface tension parameters. To prove this fact, let us analyze the waves for the flow parameter of Reynolds number $\Re = 2$.

Three vertical lines in Figure 68 show the amplitude and corresponding time for the Peak to Peak wave shapes which are plotted in Figure 69. To compare the waves, they are
shifted in the horizontal direction to match with maximum wave height. It is evident that Peak to Peak wave shapes match perfectly with each other.

![Amplitude of saturated wave](image)

**Figure 68:** Amplitude of saturated wave for Reynolds number $\mathbb{R} = 2$. Three vertical lines are showing the location of the Peak to Peak wave amplitude and the corresponding simulation time.

![Comparison of Peak to Peak saturated waves](image)

**Figure 69:** Comparison of Peak to Peak saturated waves for Reynolds number $\mathbb{R} = 2$.

Similarly, Valley to Valley wave amplitudes and corresponding simulation time are shown with three vertical lines in Figure 70. Wave shapes are compared in Figure 71. Again, it is observed that they perfectly match with each other.
Figure 70: Amplitude of saturated wave for Reynolds number $\mathbb{R} = 2$. Three vertical lines are showing the location of the Valley to Valley wave amplitudes and the corresponding simulation time.

Figure 71: Comparison of Valley to Valley saturated waves for Reynolds number $\mathbb{R} = 2$ for a fixed surface tension parameter of $(J = J_b = 0.0633)$.

From the above discussions, it is clear that wave amplitudes oscillate and all of the peak waves repeat itself after a simulation time interval of 0.04 seconds. Similarly, all of the valley waves are the same waves and the frequency of occurrences is 0.04 seconds. The comparison between the peak and the valley wave shapes are shown in Figure 72. They do not match with each other. They are totally different and unique waves.
Figure 72: Comparison between the Peak to Peak wave shapes and Valley to Valley wave shapes for the case of Reynolds number $R = 2$.

Therefore, it is safe to assume that in between the peak and the valley location, wave shape changes over time. That is why the amplitude of the wave shows modulation. In fact, our analysis suggests that wave shapes indeed change between the peak and valley locations.

In Figure 73, wave amplitudes and corresponding times are shown with vertical lines for every 0.003 seconds of simulation time. The wave shapes are plotted in Figure 74.

Waves were translated in the horizontal direction to match with the maximum height of the initial wave as shown in Figure 75.

Figure 73: Wave amplitude vs. time curve showing temporal location of every 0.003 seconds.
Figure 74: Wave shapes at every 0.003 seconds time interval for the case of Reynolds number $R = 2$.

Figure 75: Comparison of wave shapes at every 0.003 seconds for the case of Reynolds number $R = 2$. Waves are translated only in the horizontal direction to match with the maximum height of the initial reference wave at time $t$.

From Figure 75, it is clear that wave shape changes with time as it propagates and repeats itself after a period of 0.04 seconds of simulation time. This underscores the
argument that wave not only oscillates but also modulates. A similar trend is true for all of the other cases of Reynolds numbers within the range of modulated Bamboo wave regime and they are plotted in Appendix-B.

In Chapter 4, we observed that at modulated wave regime, wave travels with a distinct period, similar trend is also observed for the cases of Reynolds number. The change in period associated with the change in Reynolds number is shown in Figure 76. Our analysis suggests that the period decreases as the Reynolds number increases. It could be attributed to the nature of wave propagation at different Reynolds numbers and how the wave speed is affected by the Reynolds number. Our analysis suggests that with the increase of Reynolds number, wave speed also increases and as a result periods of the wave decrease. Some of these findings are discussed in a later section.

![Figure 76: Period of repetition of the wave shapes at various Reynolds number with fixed flow parameters of $J = J_b = 0.0633$, $a = 1.61$, $m = 0.00166$ and $\zeta = 1.1$.](image)

In the earlier section, the nature of the waves was described with the change in Reynolds number. Wave shapes for Reynolds numbers $\Re = 2$ and $\Re = 3.73$ were described in great detail. Both cases of Reynolds numbers fall into the modulated Bamboo wave regime where the wave shape changes with time. In order to compare all of the wave
shapes at different Reynolds number within the modulated Bamboo wave regime, waves from each peak location are plotted in Figure 77. It is observed that the wave length varies as the Reynolds number increases. For the sake of comparison, in Figure 78, waves were shifted in the horizontal direction to match with the peak of first reference wave of Reynolds number $\Re = 2$. All of the waves were drawn in multiple wave lengths with actual scale. In Figure 79, all of the waves were plotted again in non-dimensional length scale.

![Figure 77: Peak to Peak Wave shape (actual dimension) showing only one wave length at different Reynolds numbers for a fixed surface tension parameter of $J_b = 0.0063$ and other fixed parameters ($a = 1.61, m = 0.00166, J_b = 0.0633$).](image-url)
Figure 78: Peak to Peak wave shape (actual dimension) shifted in x-direction showing multiple wave lengths at different Reynolds numbers for fixed parameters $J = Jb = 0.063$, $a = 1.61$, $m = 0.0016$ and $\zeta = 1.1$.

From the above discussion, it is clear that at lower Reynolds numbers, the growth of the wave peak is higher than that of other larger Reynolds numbers. It is also evident that at lower Reynolds numbers, for example at $Re = 2$, crest of the wave gets narrower and pointed at the peak and the trough of the wave gets wider when compared with other waves at higher Reynolds numbers.
Figure 79: Peak to Peak wave shapes in non-dimensional form for modulated bamboo waves at different Reynolds numbers.

We noticed that, in modulated Bamboo wave regime, wave amplitudes oscillate and modulate. An enlarged version of the amplitude vs. simulation time is shown in Figure 80 for saturated waves at different Reynolds number. It is clear that as the Reynolds number increases, maximum and minimum amplitudes of oscillation of the modulated Bamboo wave decreases.
Figure 80: Wave amplitude of oscillated and modulated wave regime of Reynolds number for a fixed surface tension parameter ($J_b = 0.006354$) other fixed parameters ($a = 1.61$, $m = 0.0016$ and $\zeta = 1.1$).

Figure 81: Fluctuation of maximum and minimum wave amplitudes for saturated waves at various Reynolds number for modulated Bamboo wave regime.

The amplitude obtained from maximum and minimum values of oscillation are plotted in Figure 81 for each Reynolds number cases. It is observed that amplitude of the oscillation grows on both sides of Reynolds number three ($\Re = 3$) and then eventually
drops before reaching to saturated (fixed amplitude) bamboo waves at the lower end of the Reynolds number $\mathbb{R} = 1.5$ and at the higher end of the Reynolds number $\mathbb{R} = 5.75$. It could be attributed that for the given set of flow parameters, the wave undergoes some sensitive flow regime which triggers the wave to change from saturated Bamboo waves to modulated Bamboo waves and back to saturated Bamboo waves over the range of flow parameters.

![Fluctuating Amplitude of Modulated Bamboo Wave](image)

Figure 82: Variation of amplitude of the saturated waves at different Reynolds number within modulated Bamboo wave regime.

5.4 **Saturated bamboo wave regime: (5.75 ≤ $\mathbb{R}$ ≤ 1.5)**

We have observed that for the range of Reynolds number $1.5 < \mathbb{R} < 5.75$, wave amplitudes and wave shapes at every increment in-between the peak and valley are slightly different and the waves repeat itself within a distinct period. On the other hand, for the range of Reynolds numbers, $\mathbb{R} \geq 5.75$ and $\mathbb{R} \leq 1.5$, if we plot the wave amplitude over
time we observe that at saturation, wave amplitudes do not change with time. They remain constant over time for each Reynolds number. The change in saturated Bamboo wave amplitudes over time for different Reynolds numbers is shown in Figure 83. Again, from Figure 83 it is evident that for the range of Reynolds number \( 5.75 \leq \mathbb{R} \leq 6.0 \), saturated wave amplitudes show no oscillation or modulation. Waves are constant amplitude traveling waves with fixed wave shape and wave speed.

![Diagram](image)

**Figure 83:** Wave amplitude of non-oscillated wave regime of Reynolds number for a fixed surface tension parameter \( J = J_b = 0.063 \) other fixed parameters (\( a = 1.61, m = 0.0016 \) and \( \zeta = 1.1 \)).

To display this fact, wave shapes are taken at every 0.005 seconds of simulation time for Reynolds number \( \mathbb{R} = 6 \). The temporal locations of the waves are shown in Figure 84. The corresponding wave shapes are plotted in Figure 85. It is evident that amplitude of the wave remains constant over time and when the waves were shifted in the horizontal direction to coincide with each other, they perfectly match. Therefore, waves are indeed constant amplitude traveling waves and they have the same shapes at every time increment and also travel with a constant wave speed. Similar conclusion is true for \( \mathbb{R} = 1 \) and \( \mathbb{R} = 1.5 \). They are shown in Appendix-B.
Figure 84: Saturated wave amplitude for Reynolds number $\Re = 6$ with time. Vertical lines show the instances of time where the waves shapes are compared at every 0.005 seconds of simulation time.

Figure 85: Wave shapes are shifted in the x-direction to coincide with the initial reference wave at time $t$.

In Figure 86, wave shapes are drawn for saturated Bamboo wave regime for different Reynolds number. Notice that wave length is highest for the smallest Reynolds number. In Figure 87, wave shapes were shifted in horizontal direction with respect to the maximum reference height of Reynolds number $\Re = 1$. Wave shapes were compared in Figure 87 in dimensional form and in Figure 88 in non-dimensional form. From these two figures, it is clear that, as the Reynolds number decreases, wave peak tends to become narrower and the
growth of the wave height becomes larger. Waves become pointed at the crest and widen at the trough. Also notice that waves are not perfectly symmetric.

Figure 86: Wave shapes at saturated bamboo wave regime at different Reynolds numbers; showing dimensional wave length and wave height.

Figure 87: Saturated bamboo wave shape (actual dimension) at different Reynolds number for a fixed surface tension parameter of $J = J_b = 0.0633$. and other fixed parameters ($\alpha = 1.61$, $m = 0.0016$, $\zeta = 1.1$).
5.5 **Bifurcation of saturated bamboo wave:**

From the discussion above it is clear that for the set of flow parameters (Reynolds number), two distinct regimes have evolved. One of the regime consists of constant amplitude traveling bamboo wave and the second regime comprises of the modulated bamboo wave. Therefore, by taking maximum and minimum of the wave amplitude of oscillation from Figure 80 and taking the constant amplitude of the saturated bamboo wave from Figure 83, bifurcation diagram is constructed. The bifurcation diagram is shown in Figure 89. This diagram clearly depicts the initiation of oscillating and modulating regime.
when the Reynolds number is greater than $R = 1.5$. Branching of the amplitudes continues until Reynolds number smaller than $R = 5.75$. When the Reynolds number reaches at $R=5.75$ or larger, suddenly waves choose to travel with constant amplitude.

![Graph showing bifurcation diagram of wave amplitude at different Reynolds numbers for a fixed surface tension parameter of $J = J_b = 0.0633$ and other fixed parameters ($a = 1.61, m = 0.00166, \zeta = 1.1$).](image)

The argument behind this branching of the modulated regime could be attributed to the sensitivity of the flow parameters on the stability of the solution. At the onset of transition from modulated regime to saturated regime, a very small change in flow rate or Reynolds number variation can alter the formation of wave propagation.

Before leaving this section, it is important to address the combined effect of Reynolds numbers and surface tension parameters on the nature of the wave shape. Therefore, by
combining the bifurcation diagram presented for the range of surface tension parameters in Figure 57 and the bifurcation diagram for the change of Reynolds number listed in Figure 89, a qualitative phase diagram could be constructed which is presented in Figure 90. It is evident from Figure 90 that for a fixed value of Reynolds number $R = 3.737$, when the surface tension value is very small [smaller than $J = (1/8)J_b$], the wave starts to break. As the surface tension increases, wave shapes become more stable and time periodic forming modulated bamboo waves. The modulated bamboo waves persist up to surface tension parameter $J = 3J_b$. Surface tension parameter larger than $J = 3J_b$ makes the wave even more stable but it loses its time periodic behavior. In other words, the wave at this value of surface tension travels with a fixed wave shape and a fixed wave speed. These waves are saturated bamboo waves and when the surface tensions reach higher than $J = 9.5J_b$ wave amplitude diminishes and becomes flat interface or perfect core-annular flow. Similarly, when the surface tension is fixed at $J = J_b$, and Reynolds numbers vary from $R = 1$ to $R = 6$, we observe a window of modulated bamboo wave regime for $R = 1.5$ to $R = 5.75$ and saturate wave regime when the Reynolds number is smaller than $R = 1.5$ and larger than $R = 5.75$. By connecting the boundaries of modulated wave regime in both surface tension and Reynolds number, we construct the green dotted line which represents the boundaries between the modulated and saturated bamboo waves. Similarly, we can infer a red line to estimate the boundary between the wave break regime and modulated bamboo wave regime. Since we have very limited sets of data to determine the actual mapping of the phase diagram, it is rather challenging to extrapolate a boundary between the saturated bamboo wave and perfect core-annular flow. Therefore, a small
green vertical line is drawn to depict a boundary between the saturated bamboo wave regime and perfect core-annular flow.

Figure 90: Phase diagram describing the stability of the flow regime for the range of Reynolds numbers and surface tension parameters.

It is also important to note that if we use linear stability analysis to estimate the wavelength for different surface tension cases, then there may be slight shift in the phase diagram but qualitatively they will follow similar conclusions.
5.6 Comparison of wave speed at different Reynolds numbers

Wave speed at different Reynolds numbers are plotted in Figure 91. A slight nonlinear relationship exists between the wave speed and Reynolds number. As the Reynolds number increases, wave speed also increases. The rationale behind the increase in wave speed is that, with the increase in Reynolds number, waves move from higher amplitude growth to lower amplitude growth and waves become more stable at higher Reynolds number around $R = 5.75$.

![Wave speed graph](image)

Figure 91: Wave speed (non-dimensional) at different Reynolds numbers for a fixed surface tension parameter of $J = J_b = 0.063$. and other fixed parameters ($a = 1.61, m = 0.00166, \zeta = 1.1$).

5.7 Hold-up Ratio at different Reynolds numbers

From earlier studies of surface tension, we noticed that for modulated wave regime, hold-up ratio shows oscillation and modulation. Similar phenomenon is observed when the surface tension was kept constant but Reynolds number was varied. Figure 92 shows
the variation of hold-up ratio with respect to various Reynolds number at saturation. It is evident that hold-up ratio increases with the increase of Reynolds number which is consistent with our intuition. Oscillation and modulation of the hold-up ratio is also prominent for the Reynolds number of modulated bamboo wave regime. Figure 93 shows an enlarged view of the plot for the modulated wave regime, where oscillation of the hold-up ratio is prominent for the range of Reynolds number $\mathbb{R} = 2$ to $\mathbb{R} = 5$. Again, for the non-modulated wave regime, no oscillation and modulation of the hold-up ratio was observed. It is rather constant in magnitude once the waves reach saturation.

Figure 92: Hold-up ratio at different Reynolds numbers for fixed flow parameters of $J = J_b = 0.063$, $a = 1.61$, $m = 0.00166$, $\zeta = 1.1$. 
Figure 93: Enlarged view of Hold-up ratio at different Reynolds numbers for fixed flow parameters of $J = J_b = 0.0633$, $a = 1.61$, $m = 0.00166$, $\zeta = 1$. 
CHAPTER 6 : Numerical Analysis of Three-Dimensional (3-D) Core-annular Flow

In this chapter, we will discuss the challenges to perform the nonlinear analysis of the wave formation of core-annular flow in three dimensions (3-D). Due to enormous computational efforts required to resolve the interfacial behavior in three dimensions, it was put on hold for a prolonged amount of time by other researchers. The goal of this chapter is to verify our earlier results of the 2-D axisymmetric benchmark case study with a full blown 3-D model.

In Chapter 3 and 4, a 2-D axisymmetric model of the benchmark case of a core-annular flow is studied with the following parameters: $a = 1.61$, $m = 0.0016$, $\zeta = 1.1$, $J = 0.063$, $\mathbb{R} = 3.737$. It is observed that an axisymmetric modulated bamboo wave is generated for the above-mentioned flow parameters of the benchmark case [Bai et al. (1992), Li and Renardy (1999)]. The techniques learned from 2-D axisymmetric models were utilized to construct a full blown 3-dimensional model. For 3-D model we used cartesian coordinate system to construct 3-dimensional mesh. We used 2-D axisymmetric quad elements of 24K mesh configuration as described in Chapter 3 to represent the planner mesh. For 3-D mesh generation, we first created 2-D quad mesh on the circular face of the 3-D domain then it is extruded in the axial direction to generate 3-D hexahedral cells. Detail of mesh seeds densities are shown in Figure 94. The total cell count for 3-D mesh configuration is 5.37 million.
Figure 94: Details of mesh seed densities of 3-D model.

6.1 Problem set-up of 3-D model

In Chapter 3 (section 3.3), steps to set-up a numerical simulation for the benchmark case is described for 2-D axisymmetric model. Simulation of the three-dimensional model
follows the similar approach. That is, we need to assign the initial and boundary conditions appropriately. The 3-D initial interface is generated by rotating the 2-D disturbed interface curve (as described in Chapter 3) in azimuthal direction. We also need to assign three-dimensional domains for the core and annular fluids. Initial velocity inside the core and annulus domains are defined by user defined subroutine which is presented in Appendix-C. Along with the initial condition, we must apply the periodic boundary condition at two ends of the domains with appropriate axial (x-component) pressure gradient which is shown in Figure 95.

![Figure 95: Schematic of 3-D problem set-up and boundary conditions for the benchmark model.](image)

We also need to make sure that the direction of the gravitational acceleration is taken into account and the reference operating density is also assigned appropriately in ANSYS Fluent.
Since there is no constraint such as axis of symmetry in 3-D model, the waves are free to evolve with time. One big advantage of a full blown 3-D model is that it can predict both axisymmetric wave or non-axisymmetric wave depending on flow parameters. In this chapter, our goal is to compare the results of the full blown 3-D model for the 2-D axisymmetric model of the benchmark case as described in Chapter 3. It is interesting to observe that, the 3-D model is able to recover the results obtained from 2-D axisymmetric model with fair accuracy. The details of 3-D results are presented below.

6.2 Evolution of wave formation in 3-D model

We discussed earlier that an initial disturbed interface is introduced in 3-D model by a user defined subroutine in ANSYS Fluent. As the time progresses, interfacial waves grow and eventually reach saturation. The first six pictures shown in Figure 96 illustrate the evolution of the interfacial waves for the first 0.5 seconds. Waves are still evolving. As the time progresses, waves eventually reach saturation at around 1.5 seconds. The last three pictures of Figure 96 show the wave shape after 1.5 seconds of simulation time where changes in amplitudes of the wave are diminishing over time. More details on saturated wave and its natures are described in a later section of this chapter.
Figure 96: Evolution of the interfacial wave in three dimensions at different times.
Once the wave reaches saturation, the saturated wave shape resembles an axisymmetric wave. We must ensure that the resultant waves are indeed axisymmetric. In order to do this, volumetric fraction of oil and water for $X$-$Y$ and $Z$-$X$ plane are drawn in Figure 97 for an instant of time. The blue color represents oil and red color represents water. The interface is green where the volumetric fraction is 0.5. A quick examination of Figure 97 reveals that wave shape at the interface on $X$-$Y$ plane is symmetric to axis of the pipe. The interface waves are designated as upper wave and lower wave for $X$-$Y$ plane. Similarly, the interface line in $Z$-$X$ plane is also symmetric with respect to centerline and they are labeled as left and right waves.

Figure 97: Volume fraction of water showing blue as oil and red as water. Contour plot of volumetric fraction showing the saturated wave shape at the interface between the oil and water.

To examine whether the waves are truly axisymmetric, the upper and lower interfacial waves are plotted in $X$-$Y$ plane in Figure 98. Similarly, left and right waves are also plotted in $Z$-$X$ plane. Then the $X$-$Y$ plot and $Z$-$X$ plot are superimposed. Superposition of the upper wave with the left wave coincides exactly. On the other hand, superposition of lower and right waves perfectly matches with each other. Therefore, the saturated wave is indeed an
axisymmetric wave. The three-dimensional model of the core-annular flow confirms that the wave generated by the specific parameters for this benchmark case are truly axisymmetric.

Figure 98: Superposition of the wave shapes. Upper wave is superimposed with left wave and lower wave is superimposed with right wave. They perfectly match.

In addition to the superposition of the wave shapes, another approach is discussed here to ensure that the waves generated at each instance of time shown in Figure 96 are truly axisymmetric. In Figure 99, volume fraction contour of oil and water is drawn at an arbitrary cross section inside the domain. If we examine the cross section from the front view or Z-Y plane, we notice that cross sections are circular but the area changes with time. Therefore, if we plot the centroidal coordinates of the cross section, namely $Z_m$ and $Y_m$, 

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for axisymmetric wave, the centroidal coordinates \((Z_m, Y_m)\) at every instance of time will coincide with each other and the plot will appear as the point plot shown in Figure 100.

Therefore, we can conclude that for axisymmetric wave or Bamboo wave there is no movement of the centroidal coordinates from the center or axis of the pipe and as a result a point plot is obtained.

Figure 99: Cross section of the domain at different times showing the volume fraction of core with blue color and annulus with red color.

\[ Y_m = \text{Coordinate of } Y \text{ at Centroid of Core (Oil)} \]

\[ Z_m = \text{Coordinate of } Z \text{ at Centroid of Core (Water)} \]

Figure 100: The centroid of the core is plotted in Y-Z plane. It is a point plot.
From 3-D simulation, centroidal coordinates of the oil volumetric fractions are drawn as a function of time. They are shown in Figure 101. In Figure 101(a), centroidal coordinates of $Y_m$ and $Z_m$ constitute a point graph in $Y$-$Z$ plane. On the other hand, $Z_m$ and $Y_m$ vs. time plot, Figure 101(b), shows a straight line which suggests that as the wave travels, the cross section of the core fluid changes at the arbitrary location but the centroid position does not change with time. It remains in the center.

![Zm Vs. Ym Over Time](image)

Figure 101: Centroidal coordinate $Z_m$ vs. $Y_m$ at different simulation times of 0.5 to 1.5 seconds. (a) Coordinate $Z_m$ and $Y_m$ over time (b) Centroidal coordinate location ($Z_m$ and $Y_m$) over time in a 3-D plot.

From all of the discussions above, it is clear that the centroid of the core fluid at any cross section of the pipe flow remains at the center over time and the wave is axisymmetric. The trajectories of the centroidal coordinate ($Z_m$, $Y_m$) remains at a fixed point at the center of the pipe. Therefore, the wave will remain axisymmetric at all times.

Another very important finding of 3-D simulation is the confirmation of the wave shape. In Figure 102, saturation wave shapes are plotted at two different times. Notice
that the interface wave plotted here is for three wave lengths. The waves are pointed at the peak and wider at the trough and symmetric in both side of the peak and of course axisymmetric. Therefore, it is true that for the set of given parameters $a = 1.61, m = 0.00166, \zeta = 1.1, J = 0.0633, \text{and } \mathbb{R} = 3.737$, the saturated wave evolves into a bamboo wave and it is corroborated by full blown 3-D and 2-D axisymmetric model and by experimental work.

Figure 102: Wave shape from 3-D simulation. Interface waves of three consecutive domains (three wave lengths) show the wave shape for the benchmark flow parameters. They constitute the shape of a bamboo stem.
6.3 Comparison of wave amplitude between 2-D axisymmetric model and 3-D model

To analyze the results obtained from the 3-D simulation, results of volume fraction of oil and water is presented at the plane of $z = 0$. Since the wave is axisymmetric, any planer cut would represent the wave shape. Figure 103 shows three-dimensional saturated surface wave (green). From this figure maximum and minimum heights are determined at the interface between the oil and water. The blue color represents oil and the red color represents the water. Maximum and minimum height of the upper interface wave is tracked to calculate amplitude of the wave. Amplitude of the wave is defined in chapter 3 by equation (3.5).

![Figure 103: Volumetric fraction of the saturated wave. Blue represents oil, red represents water and green represents three-dimensional interfacial surface. A view at $z=0$ also shows the maximum and minimum interfacial wave height.](image)

In Figure 104, amplitude of the saturated waves from both 2-D axisymmetric model and 3-D model are compared for equivalent mesh configuration of 24K. The agreement between the wave amplitudes is fairly good when the wave reaches saturation around 1.5
In earlier time, growth of the wave varies between the 2-D and 3-D model. This is associated with the set-up of initial amplitude of the disturbances between the two models. For the 2-D model, the initial amplitude of disturbance is 0.0003 m and for the 3-D model initial amplitude of disturbance the wave is 0.0001 m. To better represent the saturated amplitude, more data points are plotted with smaller time intervals for saturated wave than that of the waves at earlier times.

![Graph comparison between 2D and 3D model](image)

Figure 104: Comparison between the amplitudes of waves of 2-D axisymmetric model and 3-D model.

It is also important to note that for this benchmark case, 3-D model also predicts oscillation of the wave amplitudes and a good agreement is found between the amplitudes of 2-D axisymmetric model and 3-D model. The small variation could be attributed to mesh configuration of 3-D model as the mesh density in azimuthal direction might influence the final results. Denser mesh in the azimuthal direction might improve the accuracy of the 3-D model.
D model. Again, it will be computationally very expensive to run 3-D model with very dense mesh.

At this time, it is noteworthy to point out that 3-D model is not only capable of recovering 2-D axisymmetric model results but it actually validates the 2-D model. In 2-D axisymmetric model, we impose axis of symmetry. On the other hand, 3-D model is free to predict the final shape of the wave without any imposed axis of symmetry. Therefore, the final shape of the wave could be either axisymmetric bamboo wave, or it could be non-axisymmetric corkscrew or snake wave or could be any other complex wave shape depending on the stability of the solution of a given set of flow parameters. For the given set of benchmark flow parameters, 3-D model prefers axisymmetric bamboo wave as a stable solution. Therefore, 3-D model validates that the solution obtained from 2-D axisymmetric model is stable and it is indeed bamboo wave.

6.4 Analysis of wave shapes for 3-D model

In Figure 104, we observed oscillation of the wave amplitudes. Oscillation in wave amplitudes implies that wave shape modulates with time. From the 2-D analysis, we observed that the wave pattern repeats itself after a defined period. A similar outcome is observed from the 3-D model. In Figure 105, oscillation of the wave amplitudes is shown with Peak to Peak wave location as a function of time. In Figure 106, Peak to Peak wave shapes are drawn from the 3-D model. Waves were shifted to match with the maximum wave height of the initial wave at time $t$. Surprisingly, all of the Peak to Peak waves
overlaps and the period of the Peak to Peak oscillation is 0.035 seconds (vs. 2-D, 0.034 seconds)

![Graph showing amplitude vs. time](image1)

Figure 105: Oscillation of the wave amplitudes of 3-D model. Vertical lines showing Peak to Peak wave locations.

![Graph showing height vs. distance](image2)

Figure 106: Comparison of Peak to Peak waves of the 3-D model. Waves are shifted in the \(x\)-direction to match with initial peak wave at time \(t\).

Similarly, oscillation of the wave amplitude over time is shown in Figure 107. Three vertical lines corresponds to time of the three valley waves. Wave shapes corresponding to three valley waves are plotted in Figure 108. The waves were shifted in horizontal direction to match with the maximum height of the reference wave at time \(t\). Again, Valley
to Valley wave shapes coincide. Also, the period of the Valley to Valley wave is 0.035 seconds (2-D, 0.034 seconds). Therefore, both Peak to Peak and Valley to Valley waves have the same period of 0.035 seconds.

Figure 107: Oscillation of the wave amplitudes of 3-D model. Vertical line showing Valley to Valley wave location.

Figure 108: Comparison of Valley to Valley waves of the 3-D model. Waves are shifted in the x-direction to match with initial valley wave at time t.

Peak to Peak and Valley to Valley wave shapes are compared in Figure 109. They are shifted in x-direction to match with the maximum height of each wave. It is clear from
Figure 109 that peak and valley waves slightly vary. They are not the same waves. Which agrees with 2-D axisymmetric model. Again, both Peak to Peak waves and Valley to Valley waves have the same period of 0.035 seconds.

Therefore, it is safe to consider that in core-annular flow, as the wave travels forward, wave shapes oscillates at a fixed frequency. Both 2-D axisymmetric mode and 3-D model predicts the oscillation and modulation of the waves. This is a new finding of this research work and this behavior is also presented for the first time with a true three-dimensional model.

![Figure 109: Comparison of Peak to Peak and Valley to Valley wave shapes. Valley wave is shifted in x-direction to match the maximum wave height of the Peak wave.](image)

6.5 **Comparison of 3-D wave shape with 2-D axisymmetric model and published results**

To compare the results obtained from ANSYS Fluent simulation, wave shape from both 3-D and 2-D axisymmetric models are compared with the published results of Li and
Renardy (1999). In Figure 110, peak wave shapes from both 2-D axisymmetric and 3-D models are plotted. Wave shapes are in good agreement except that the crest of the wave of the 3-D model is slightly lower than that of the 2-D axisymmetric model and published results. That might be attributed to the growth of the wave in the azimuthal direction due to reduced mesh size.

![Figure 110: Wave shape comparison of published 2-D results, 2-D axisymmetric Fluent™ model and 3-D Fluent™ model.](image)

### 6.6 Comparison of wave speed between 2-D-axisymmetric and 3-D model

A comparison of wave speed between 2-D-axisymmetric model and full-blown 3-D model is shown in Table 10. Wave speed obtained at saturation for 3-D is about 3% smaller when compared with published results of Li and Renardy (1999) and about 1.6% smaller...
when compared with other ANSYS Fluent 2-D-axisymmetric models. This deviation in 3-D results could be attributed to mesh configuration. With denser mesh in both x and θ direction, the potential to improve the accuracy of 3-D model is greater. Again, as the mesh size increases, computation time could be enormous with limited hardware and software availability. Typical time to run a 3-D model in 256 core machines is around two to three days for one second of simulation time.

Table 10: Comparison between wave speed obtained from 2-D-axisymmetric model and 3-D model

<table>
<thead>
<tr>
<th>Model</th>
<th>Fluent™ Simulation</th>
<th>Published Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-D Axisymmetric Model</td>
<td>0.797</td>
<td>0.8068</td>
</tr>
<tr>
<td>3-D Model</td>
<td>0.784</td>
<td></td>
</tr>
</tbody>
</table>

6.7 **Hold-up ratio comparison between 2-D and 3-D models**

Hold-up ratios from the 3-D and 2-D axisymmetric models were compared in Figure 111. Hold-up ratio for 3-D model and 2-D models are shifted in time scale to compare the results in non-dimensional time. A good agreement is found between the 2-D and 3-D simulation results. 3-D simulation shows slightly higher values than that of 2-D-axisymmetric model. Again, with the denser mesh in the azimuthal direction, it may produce more accurate results. Another reason which might attribute to higher magnitude
of the hold-up ratio for 3-D model would be truncated volume from the cartesian mesh.

![Graph showing comparison of hold-up ratio in non-dimensional time scale at saturation between 2-D axisymmetric and 3-D models of ANSYS Fluent results vs. Published 2-D results.]

Figure 111: Comparison of hold-up ratio in non-dimensional time scale at saturation between 2-D axisymmetric and 3-D models of ANSYS Fluent results vs. Published 2-D results.
CHAPTER 7 : Numerical Analysis of Non-Symmetric 3-D Waves: 
Corkscrew and Snake Waves

We learned that in a perfect core-annular flow of two fluids, the core fluid has a 
cylindrical interface of constant radius centered on the axis of the pipe. Annulus fluid 
forms an outer layer and surrounds the core fluid. In linear stability analysis, the response 
of the flow to perturbation of the form $exp[i n \theta + i \alpha (x - ct)]$ is examined. Where, $n$ is 
the wave number of disturbance in azimuthal direction, $\theta$ is azimuthal angle, $\alpha$ is the axial 
wave number and $x$ is the axial axis, $t$ is time. Once the perfect core-annular flow loses 
its stability at finite axial wave number $\alpha$ to an axisymmetric $n = 0$ disturbance, then the 
deformation of the interface can reach to a varicose shape. The deformed wave shape 
resembles Bamboo waves. This is explained by Preziosi, Chen and Joseph (1989), and by 

Renardy (1997) performed a weakly nonlinear stability analysis for vertical CAF in 
the down-flow section to examine the onset of non-axisymmetric disturbances. She 
identified the flow regimes where non-axisymmetric mode of disturbance with azimuthal 
wave number $n = \pm 1$ is the most unstable; where for both $n = 1$ and $n = -1$, wave 
evolves into corkscrew wave but the travel direction in azimuthal directions are just 
opposite to each other. That is, if the corkscrew wave for $n = 1$ travels in counter 
clockwise direction then, for $n = -1$, the wave will travel in clockwise direction. Renardy 
(1997) examined the interaction between the $n = 1$ mode with $n = -1$ mode, leading to 
either the waves traveling the azimuthal direction, known as the corkscrew waves, or
standing waves, known as snake waves. Both of them travel in the axial flow direction. As the names imply, the corkscrew waves travel with the flow in the helical motion, however, the snake waves are simply meandering side-to-side while translating with the flow.

For $n = 2$, interface shape could be even more complex. Some of the schematics of the different shapes of the interfacial wave shapes are presented by Hu and Patankar (1995) and shown in Figure 112.

![Figure 112: Geometric representation of various models of interfacial waves.](image)

Hu and Patankar (1995) also performed linear stability analysis on both axisymmetric ($n = 0$) and non-axisymmetric ($n \neq 0$, i.e., $n = 1$ and $n = 2$, etc.) mode of perturbation. They examined the stability of the flow under the experimental conditions documented in Table 1 of Bai et al. (1992). The parameters considered are for down flow condition and they are listed as follows: $(a, m, \zeta, J, \Re_g, \Re) = (1.7, 0.00166, 1.1, 0.063, 0.488, 1.2)$. The
analysis results show change of growth rate as a function of wave number. The results are plotted in Figure 113.

![Growth rate versus wave number for axisymmetric and asymmetric disturbances. The flow conditions are taken the same as experiment No. 6 in Bai et al. (1992) \( \alpha = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427 \) and \( \alpha = 0.531 \).](image)

Hu and Patankar (1995) showed that the largest growth rate occurs for asymmetric sinuous mode \( n = 1 \), and it is the most dangerous mode which gives rise to a corkscrew shape wave. From Figure 113 it is also clear that for symmetric mode, i.e., \( n = 0 \), a significantly lower growth rate is observed and it has two unstable wavelengths, one of them takes place at longer wave length (smaller wave number) due to interfacial surface tension and the other at larger wave number due to shear. Growth rates for the higher modes i.e., for \( n \geq 2 \) is much smaller than \( n = 1 \) and \( n = 0 \) mode and also notice that the magnitude of the growth rate is negative, which suggests that higher modes of disturbances are stable. Hu and Patankar (1995) also shows that most dangerous wave number \( \alpha = 0.531 \), where the wave shape turns out to be a corkscrew. The corresponding wave length

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[\lambda = \frac{2\pi R_1}{\alpha} = 3.31 \text{ cm}] \text{ is in agreement with the wave length observed by experimental work of Bai, et al. (1992). The wave speed obtained by Hu and Patankar (1995) is reported to be 40.5 cm/sec. This result also matches fairly well with experimental observation.}

In this chapter, we will discuss the evolution of the non-axisymmetric wave and attempt to explain the nature of corkscrew and snake wave at saturation with 3-D computational model. In addition, there is an attempt to prove some of the theoretical results of Yuriko Renardy (1997). She described a detailed mathematical model of the pattern selection problem. She predicted the bifurcation of the wave into travelling waves and standing waves. She also mapped out a detailed regime for the traveling wave also known as corkscrew wave and standing wave or snake wave. The sensitivity of the initial conditions for certain parameters are depicted in a regime where the traveling wave will change to standing waves and back to traveling waves. She showed that for certain change in initial parameters such as Reynolds number ($\text{\(\mathbb{R}\)}$), neither corkscrew nor snake waves would be observed while keeping all the other flow parameters unchanged. Pattern selection results for the following flow parameters $a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.06, K = -0.5427 (\text{\(\mathbb{R}_g\)} = 0.488)$ and $\alpha = 0.531$ at various Reynolds number is presented in Figure 9 of Renardy’s (1997) paper. The exact plot is reproduced here in Figure 114 in our study to show the regime of alteration of different wave shapes. Figure 114 is a very powerful figure as it clearly depicts the nature of sensitivity of waves patterns with the change in Reynolds number by keeping rest of the flow parameters constant.
Figure 114: Pattern selection results for \( (a = 1.7, \ m = 0.00166, \ \zeta = 1.1, \ j = 0.063354, \ K = -0.5427 \) and \( \alpha = 0.531 \). (a) The real parts of the Landau coefficients; a stability condition for corkscrew is that \( \text{Re}(\beta_2) < \text{Re}(\beta_1) \), and vice versa for snakes. (b) The second stability condition, that of supercriticality, for corkscrews is that \( \text{Re}(\beta_1) < 0 \), and for snakes that \( \text{Re}(\beta_1 + \beta_2) < 0 \). The results from both (a) and (b) are combined in the middle plot to show the intervals of Reynolds number \( R_1 \) in which corkscrews (C) are preferred, snakes (S) are preferred, or neither (N).

We can summarize the sensitivity of wave shape change with respect to the Reynolds number in the following way:

1. \( R > 1.2 \): Neither Corkscrew nor Snake wave is observed
2. \( 0.62 \leq R \leq 1.2 \): Corkscrew wave is observed
3. \( 0.417 \leq R < 0.62 \): Snake wave is observed
4. \( 0.2 \leq R < 0.417 \): Corkscrew wave is observed
This is indeed a very novel finding by Renardy (1997). So far there had not been any experimental verification or any numerical model to support her prediction for the entire regime of Reynolds number to validate the sensitivity of wave shape reported by her study. In our work attempt is being made to verify some of her claims. Three case studies are conducted with full blown 3-D model at three different Reynolds numbers, Case(I) $\mathbb{R} = 1.2$; Case (II) $\mathbb{R} = 0.9$; and Case (III) $\mathbb{R} = 0.525$ as shown in Figure 114.

Bai et al. (1992) performed experiment for the Case I of $\mathbb{R} = 1.2$ with the same parameters ($a = 1.7, \ m = 0.00166, \ \zeta = 1.1, \ J = 0.063354, \ K = -0.5427$ and $\alpha = 0.531$). They reported their results in Table 1(experiment number 6) of their paper. The inside diameter of the pipe was 0.952 cm. They found traveling waves or corkscrew waves under the listed flow conditions. They also reported the wave length of 3.3 cm and wave speed as 43.12 cm/sec. Hu and Patanker (1995) also performed linear stability analysis with the same parameter and showed that $n = 1$ is the most dangerous mode which gives rise to corkscrew wave shape. Their results match fairly well with Renardy’s (1997) prediction for this particular case.

7.1 3-D non-axisymmetric model set-up

3-D Non-axisymmetric model set-up is similar to the set-up of 3-D bamboo wave case. Here we have the new domain length or wave length of 3.31. Other parameters such as initial velocity field and pressure gradient has also changed due to change in Reynolds
number and other flow parameters. Also notice that the flow is down flow. The new mesh configuration is shown in Figure 115. Total cell count is 7.525 million.

Simulation is started with an initial disturbed interface of amplitude 0.0003 m. After running for a very long time, around three seconds of simulation time, waves start to show the true nature of non-axisymmetric corkscrew behavior. Another interesting finding from this simulation is that before reaching corkscrew wave, it initially shows some behavior of snake like wave and then eventually turns into stronger corkscrew wave at a later time. Now, the question is how can we prove this transformation of the wave shape from the
simulation results? In this section, we will try to explain the evolution and nature of the wave generated at three different cases selected from Figure 114. Before we discuss the results obtained from the 3-D model, we would like to emphasize further on the characteristic of corkscrew and snake waves.

We learned from the earlier section that corkscrew waves are non-axisymmetric waves and travel in both axial as well as azimuthal directions. Snake waves are also non-axisymmetric waves but only travel in the axial direction. It does not have any rotational movement. Since they are both non-axisymmetric waves, at times it is very difficult to distinguish the type of wave just by pure visual observation from the simulation results. Therefore, we need to produce a better way to distinguish them. In chapter 6, we described a method to explain bamboo waves by plotting the trajectories of the centroidal coordinates of the core fluid of an arbitrary cross-section over time. We ascertained that for axisymmetric waves such as bamboo waves, the centroidal coordinates never shifted from the axis of pipe and as a result, a point plot was obtained when the centroidal coordinates $Z_m$ and $Y_m$ were plotted over time. When similar techniques were applied, we notice that for corkscrew waves, the trajectories of the centroidal coordinates $(Z_m, Y_m)$ follow a circular path which is shown in Figure 116. For snake waves, the trajectories of the centroidal coordinates $(Z_m, Y_m)$ of the core fluid move back and forth in a line which is shown in Figure 117. This method is used in this chapter to identify and distinguish the nature of the wave shape.
Figure 116: A typical trajectories of the centroidal coordinates of core fluid of an arbitrary cross-section over time for a corkscrew wave. The movement of the centroidal coordinate follows a circular path.

Figure 117: A typical trajectories of the centroidal coordinates of core fluid of an arbitrary cross-section over time for a snake wave. The movement of the centroidal coordinate follows a straight line.
7.2 Analysis of case I (\(\mathbb{R} = 1.2\)) : Corkscrew wave shapes

Before, getting into the details, let us first try to understand the nonlinear evolution of the wave at longer time. Figure 118 shows the snapshot of the evolution of the non-axisymmetric wave at two different times (at 3.0475 and 3.070 seconds). The picture at the top shows the cross section of the core and annulus section at mid-plane and the picture at the bottom shows the interface in three dimensions. It is obvious that the waves are non-axisymmetric.

Figure 118: Evolution of 3-D non-axisymmetric corkscrew wave over time [\(a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063, K = -0.5427, \alpha = 0.531\) and \(\mathbb{R} = 1.2\)].

To calculate the wave speed, from the simulation results, waves were extracted from the top center cross section of the core-annular section as shown in Figure 119.
Figure 119: Slice of core-annular section at the center $X$-$Y$ plane of the pipe. [$\alpha = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063$, $K = -0.5427$, $\alpha = 0.531$ and $R=1.2$].

The waves are plotted in Figure 120 at two instances of time. The distance between the maximum height is the travel distance at the time interval. Therefore, the calculated wave speed is 0.381 cm/sec.

![Wave Shape at Time $t$](image1)

$t = 3.0475$ sec

$t+dt = 3.0550$ sec

d = 0.00285 m

Figure 120: Interface wave shape at two different times obtained from top $X$-$Y$ plane section for the following flow parameters $\alpha = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063$, $K = -0.5427$, $\alpha = 0.531$ and $R=1.2$.

The wave shapes are also plotted in Figure 121 with two wave lengths for better visualization. Obviously, they are distinctly different from the axisymmetric bamboo wave.
Figure 121: Nonaxisymmetric corkscrew wave shape taken at interface at the center cross section plane. For better visualization, wave shapes are drawn for two wave lengths. \( a = 1.7, \ m = 0.00166, \ \zeta = 1.1, \ j = 0.063, \ K = -0.5427, \ \alpha = 0.531 \) and \( \mathbb{R}=1.2 \).

From the above discussion, we can compare two important results. First, the wave generated by the flow parameters are indeed non-axisymmetric wave. Secondly, the computed wave speed from ANSYS Fluent simulation results matches fairly well with experimental observation and linear stability analysis. The variations in 3-D model could be improved by refining the mesh. Also, in the 3-D model, wave speed fluctuates when it is calculated by two consecutive waves. A representative speed is reported here to compare with published results. The comparison of wave speed from different sources are tabulated in Table 11.

Table 11: Comparison of wave speed of ANSYS Fluent 3-D-non-axisymmetric model with published results and experiment.

<table>
<thead>
<tr>
<th>Wave Speed (cm/sec)</th>
<th>Experiment (Bai, Joseph 1992)</th>
<th>Linear Stability Analysis (Hu and Patankar)</th>
<th>ANSYS Fluent Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>43.12</td>
<td>40.5</td>
<td>38.10</td>
</tr>
</tbody>
</table>
So far, we only showed that the waves are non-axisymmetric but still did not establish the fact that they are corkscrew waves. In Chapter 6, we took an approach to explain bamboo waves by monitoring the movement of centroidal coordinates of the core fluid with respect to time. Since wave shapes were axisymmetric in nature, the centroid of the core never shifted from the center. Therefore, the plot of the centroidal coordinates \((Z_m, Y_m)\) was a point plot over time. But corkscrew wave shows movement in both azimuthal direction as well as in axial direction. On the other hand, Snake wave only shows movement in the axial direction but not in azimuthal direction.

Let us now try to examine the movement of the centroidal coordinate to explain the nature of the wave shape. In Figure 122, evolution of the wave shapes is shown between 3 seconds to 3.43 seconds. Movement of the centroid of the core fluid is monitored over time at an arbitrary cross section around 2/3 of the domain at four instances of times (3.02 seconds, 3.035 seconds, 3.08 seconds, and 3.1475 seconds). It is observed from Figure 122 that the centroidal coordinates show a circular movement when plotted in \(Y_m\) and \(Z_m\) coordinates over time.
Figure 122: Evolution of corkscrew wave. Movement of the centroidal coordinates over time \([\alpha = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063, K = -0.5427, \alpha = 0.531 \text{ and } R=1.2]\). (a) At time = 3.08 seconds (b) time = 3.035 seconds (c) time = 3.08 seconds and (d) time = 3.1475 seconds.

Figure 123 shows the plot in \(Y_m\) and \(Z_m\) and Figure 124 shows the movement of the \(Y_m\) vs. time and \(Z_m\) vs. time of the core fluid for all instances of time between 3 seconds to 3.47 seconds of simulation time. A similar conclusion could be attained from Figure 125.
where the movement of the centroidal coordinates are drawn for two different orientations over time in a 3-D plot.

Figure 123: Movement of the coordinate $Y_m$ and $Z_m$ over time [$a = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063$, $K = -0.5427$, $\alpha = 0.531$ and $R=1.2$].

Figure 124: Movement of $Y_m$ vs. time and $Z_m$ Vs. time [$a = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063$, $K = -0.5427$, $\alpha = 0.531$ and $R=1.2$].
Again, it is evident from Figure 123, Figure 124, and Figure 125 that the centroidal coordinate trajectories follow a circular path as the core fluid moves over time. It confirms the movement of the waves in azimuthal direction and the waves generated are indeed corkscrew waves because the waves move in both azimuthal and axial direction.

In Figure 126, few representative interfacial wave shapes are plotted at four different instances of time. For better visualization, wave shapes are drawn in at two consecutive wavelengths. They are the 3-D representation of non-axisymmetric corkscrew wave.
Figure 126: Corkscrew wave shapes. Interface movement at different times (2-wave lengths are added) \(a = 1.7, m = 0.00166, \zeta = 0.11, J = 0.063, K = -0.5427, \alpha = 0.531\) and \(\mathbb{R}=1.2\).
From the discussion above, it is clear that the wave shapes are indeed corkscrew wave at saturation. It is also verified by experimental observation by Bai, et al. (1992).

Before, moving into the discussion for the next flow conditions to examine the type of waves, it is important to point out a few other observations at an earlier simulation time before the wave interface reaches to saturated corkscrew shape for this particular (Case-I) flow parameters \([a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063, K = -0.5427, \alpha = 0.531\) and \(R=1.2\)]. When the evolution of the interfaces is examined at an earlier simulation time such as 2.35 seconds, 2.45 seconds, and 2.55 seconds as shown in Figure 127, it is interesting to observe that the trajectories of the centroid of the core fluid \((Y_m, Z_m)\) is showing movement in a line instead of a circular motion. This plot indicates that the waves are not axisymmetric because the centroidal coordinate \((Y_m \text{ vs. } Z_m)\) plot is not a point plot rather a diagonal line plot (showing movement from the center or axis of the pipe). Therefore, the waves are non-axisymmetric. Since the centroidal coordinate does not show any circular motion at an earlier time of the simulation, consequently the wave only travels in the axial direction. It can be confirmed by the \(Y_m \text{ vs. time and } Z_m \text{ vs. time}\) plot. Therefore, these interface waves are indeed snake waves (travel only in axial direction and centroidal coordinate of the core fluid of an arbitrary cross-section move back and forth in line). Simulation results suggests that the waves are behaving more like a snake wave up to 2.6 seconds of simulation time as shown in Figure 127.
Figure 127: Evolution of the interface waves at earlier simulation time. Plots are showing the change in centroidal coordinate of the core fluid over time. Movement of the centroidal coordinate $Z_m$ and $Y_m$ over time is in a straight line. [$a = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $f = 0.063$, $K = -0.5427$, $\alpha = 0.531$ and $R = 1.2$].

As the simulation time keeps on increasing, the interfacial waves gradually shift from snake waves to corkscrew wave. Just by observation of the wave it is sometimes difficult to discern the actual shape of the wave as both corkscrew and snake wave looks similar in
nature. Only the movement of the centroidal coordinate in a circular path or in a line over time distinguishes the true nature of the wave shapes. This change in wave shape will be clear from the subsequent discussion. In Figure 128, evolution of the interface waves and the movement of the centroidal coordinates of the core fluid is plotted at a later time.

Figure 128: Evolution of interface waves at longer simulation time. $Z_m$ vs. $Y_m$ plot is the movement of centroidal coordinate of the core fluid at an arbitrary location. This plots shows a shift from line to a circular trajectory. $Y_m$ vs. Time and $Z_m$ vs. Time curve show the movement of the centroidal coordinate over time. [$a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063, K = -0.5427, \alpha = 0.531$ and $R=1.2$].
The $Y_m$ vs. $Z_m$ plot over time in Figure 128 shows the gradual shifting of the trajectories of the centroid from a diagonal line to a circular path. This behavior indicates that the snake wave is transforming gradually into a corkscrew wave as the simulation time progresses from 2.6 seconds to 3 seconds. The previous discussion (Figure 122 to Figure 125) confirms that at later simulation time (3 seconds to 3.4 seconds), the saturated wave is indeed strong corkscrew wave.

### 7.3 Analysis of Case II ($\mathbb{R} = 0.9$): Corkscrew Wave Shapes

In Figure 114, pattern selection chart for non-axisymmetric cases are shown. Flow parameters with $\mathbb{R} = 1.2$ case is discussed above. The rationale in selecting $\mathbb{R} = 1.2$ case was obvious because Joseph’s group had already studied this particular case and published their experimental observation. Hu and Patanker (1995) also performed linear stability analysis on this flow condition which is summarized in earlier section. By examining the pattern selection results of Li and Renardy (1997) shown in Figure 114, it is evident that the $\mathbb{R} = 1.2$ case falls in corkscrew regime. If the Reynolds number is slightly larger than $\mathbb{R} = 1.2$, then the wave falls at neither corkscrew nor snake wave regime. Therefore, the $\mathbb{R} = 1.2$ case is very close to the borderline or transition area where the wave changes abruptly from corkscrew to neither corkscrew nor snake wave regime. To avoid this sensitivity, a second case is selected at Reynolds number $\mathbb{R} = 0.9$, which lies at the middle of the corkscrew regime shown in Figure 114. The goal of selecting this particular
parameter is to prove Renardy’s prediction of corkscrew wave at saturation with a full blown 3-D non-axisymmetric model.

To analyze the shape of the wave for $\mathbb{R} = 0.9$, first evolution of the waves at different simulation times are shown in Figure 129. Again, the centroidal coordinates are plotted to examine the trajectories. The centroidal coordinate plot of $Z_m$ vs. $Y_m$ shows that the centroid of the core fluid follows a circular trajectory over time.
Figure 129: Evolution of the interfacial wave shape for Case II ($\mathbb{R} = 0.9$). The centroidal coordinates ($Z_m$ and $Y_m$) of the core fluid trajectories shows a circular path. Therefore, the interfacial wave travels in both azimuthal and axial direction and the waves are corkscrew waves. [$a = 1.7, \, m = 0.00166, \, \zeta = 1.1, \, J = 0.063354, \, K = -0.5427$ and $\alpha = 0.531$ and $\mathbb{R} = 0.9$]
Therefore, the interfacial wave is non-axisymmetric and travels in the azimuthal and axial direction. Hence, the interfacial saturated wave shapes are indeed corkscrew waves. In Figure 129, only four instances of wave evolution (at time 3.0325 seconds, 3.25 seconds, 3.3575 seconds and 3.4250 seconds) are shown. For better visualization, the trajectories of the centroid of the core fluid from 3 to 4 seconds of simulation time is presented in Figure 130, Figure 131, and Figure 132. These three figures confirm the movement of the centroidal coordinates of core fluid in both azimuthal and axial direction.

Figure 130: Trajectories of the centroidal coordinates ($Z_m$ vs. $Y_m$) of the core fluid at an arbitrary cross section over time for case II. [$a = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063354$, $K = -0.5427$ and $\alpha = 0.531$ and $R = 0.9$].
Figure 131: Trajectories of the centroidal coordinates ($Y_m$ vs. time and $Z_m$ vs. time) of the core fluid at an arbitrary cross section over time for case II. [$a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427$ and $\alpha = 0.531$ and $\mathbb{R} = 0.9$].

Figure 132, shows the 3-D plot of the centroidal coordinate trajectories at two different orientations. It is interesting to see that the movement of the centroidal coordinates almost resembles a corkscrew shape over time. Since the motion of the centroidal coordinate shows movement in circular path the waves are indeed corkscrew wave.

Figure 132: Trajectories of the centroidal coordinates ($Z_m$ and $Y_m$) of the core fluid at an arbitrary cross section over time for case II in 3-D plot [$a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427$ and $\alpha = 0.531$ and $\mathbb{R} = 0.9$].
In Figure 133 and Figure 134, a few representative interfacial wave shapes are shown for this case of $\mathbb{R}_1 = 0.9$. Again, the interfacial waves are corkscrew waves.

Figure 133: Few representative wave shapes at different simulation times. $[\alpha = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427$ and $\alpha = 0.531$ and $\mathbb{R}_1 = 0.9]$. 

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Wave Shape and wave speed

To calculate the wave speed for the Case II, two representative wave shapes are considered. They are shown in Figure 135. To get the wave, a 2-D cut plane is taken at the center and
only the upper portion of the waves are presented in Figure 136. Two waves are plotted in Figure 137. The distance between the maximum heights of those two waves are calculated and the time taken to travel the distance is attained to compute the wave speed. The calculated wave speed for Case II ($\Re_1 = 0.9$) is 26.7 cm/sec. Figure 138, shows the same waves in two wave lengths for better visualization. Wave shapes for non-axisymmetric corkscrew waves are completely different from axisymmetric waves and they are unique and have relatively a blunt peak.

Figure 135: Interfacial wave shape at time 3.0625 seconds and 3.07 seconds. (a) Interfacial wave surface in 3-D (b) Interfacial wave shape at a 2-D cross section at the center plane.

Figure 136: Interfacial wave shape at the cut plane taken at the center showing only the top wave shape.
Figure 137: Wave shapes at two different times obtained from top X-Y plane section. Only the top portion of the waves are shown. \[a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427 \text{ and } \alpha = 0.531 \text{ and } \mathbb{R} = 0.9\].

Figure 138: Non-axisymmetric corkscrew wave shapes taken at interface at the center cross section (X-Y) plane. Only the top portion of the waves are shown. For better visualization, wave shapes are drawn for two wave lengths. \[a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427 \text{ and } \alpha = 0.531 \text{ and } \mathbb{R} = 0.9\].
7.4 Analysis of Case III (Re = 0.525): Corkscrew Wave Shapes

In earlier discussion, wave shapes of two different cases were presented based on Renardy’s analytical prediction for Case I (Re = 1.2) and Case II (Re = 0.9). In those studies, all the other flow parameters remain constant except Reynolds number. Renardy’s prediction matches with ANSYS Fluent 3-D models for both Re = 1.2 and Re = 0.9. Corkscrew wave shapes are observed for both cases.

To validate the pattern selection and sensitivity of the wave shape on flow parameters, another case study is considered at Re = 0.525 by holding the rest of the flow parameters $a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427$ and $\alpha = 0.531$ constant. This is labeled as Case III. Renardy’s prediction on sensitivity of Reynolds number on the wave shapes is shown in Figure 114. At Re = 0.525, the wave shape predicted by Renardy’s analysis came out as a snake wave.

Analysis results from true 3-D model and evolution of the few wave shapes for Case III (Re = 0.525) is shown in Figure 139. The movement of the centroidal coordinates are also plotted in Figure 140 and Figure 141. A 3-D plot of the trajectories of the centroidal coordinate is presented over time in Figure 142 at two different views. The movement of the centroidal coordinates of the core fluid show linear movement in $Z_m$ and $Y_m$ plot over time. The linear movement is relatively small. Therefore, it is safe to conclude that a weak snake wave is observed from 3-D dimensional analysis for case III (Re = 0.525).
Figure 139: Evolution of the wave shapes for Case III ($\Re = 0.525$). [$a = 1.7, m = 0.00166, \zeta = 1.1, J = 0.063354, K = -0.5427$ and $\alpha = 0.531$]
Figure 140: Movement of the centroidal coordinate ($Z_m$ vs. $Y_m$) of the core fluid at an arbitrary location over time.

Figure 141: Movement of the centroidal coordinates ($Z_m$ vs. time and $Y_m$ vs. time) of the core fluid at an arbitrary location over time.
Figure 142: movement of the centroidal coordinates in 3-D plot with respect to time. Movement of the centroidal coordinate \((Z_m, Y_m)\) over time confirms that the wave shape is indeed a snake wave.

Figure 143, shows some of the interfacial wave shapes at four different times. The interface is drawn at two wave lengths. At first glance, the wave shapes resemble stretched Bamboo waves. A cross sectional view at X-Z plane is shown in Figure 144. This view also resembles bamboo wave but they are not really bamboo waves rather they are snake waves. Only the movement of the centroidal coordinates of an arbitrary cross section of the core fluid confirms that, the centroid of the core fluid moves back and forth in a line. The magnitude of the movement is small but it shows movement only in a specified line and not in any rotational direction. Therefore, the waves are non-axisymmetric and only move in axial direction not in the azimuthal direction. This is a pure indication of snake wave.
Figure 143: Figure: Re 0.525: Case III, combination of bamboo wave and weak snake wave. Interface wave shape (green).
Figure 144: Re 0.525: Case III, combination of bamboo wave and weak snake wave. Interface wave shape (green).

The magnitude of the movement of the centroidal coordinate is very small. As a result, the asymmetric nature of the wave is not very prominent for this case. A close examination of the interface wave reveals that the waves are not symmetric. In Figure 145 and in Figure
waves shapes of the $X$-$Z$ plane are compared. Upper interface wave shape is compared with lower interface wave at two instances of times (3.48 second and 3.7 seconds)

Figure 145: Comparison of the Top and Bottom wave shapes at simulation time 3.48 sec. Bottom wave height is adjusted to match with Top wave height.

Figure 146: Comparison of the Top and Bottom wave shapes at simulation time 3.7 sec. Bottom wave height is adjusted to match with top wave height.
Certainly, they are not axisymmetric. There is deviation near the trough of the wave. As the wave propagates over time, the difference in wave shape persists and therefore, the long-term response is non-axisymmetric wave form. It is noteworthy to mention that, even though the wave shape resembles stretched bamboo wave, we proved that the waves are indeed non-axisymmetric and the movement of the centroidal coordinate of the core fluid of an arbitrary cross section moves in a line, therefore, the waves are snake waves.

**Wave Shape and Wave Speed:**

To measure the wave speed, wave shapes from two different instances of times are drawn in Figure 147. In Figure 147(a), 3-D interfacial surface is shown. In Figure 147(b), a cross section in the mid-section showing the volume fraction of the fluids. Blue color represents core (oil) fluid and red color represents annulus (water) fluid. In Figure 147(c), only upper section of the volume fraction is shown. The interface wave shape is drawn in Figure 148 and in Figure 149.
Figure 147: (a) Interface surface (3-D) at simulation time 3.48 seconds (b) Volume fraction of the center plane showing the clear interface; blue represents core fluid and red represents annulus fluid (c) Volume fraction of the upper section of the wave with interface.

From Figure 148, the distance traveled in simulation time $t = 3.48$ second and $t + dt = 3.51$ second is 0.00314 m. The computed wave speed for Case III [$a = 1.7$, $m = 0.00166$, $\zeta = 1.1$, $J = 0.063354$, $K = -0.5427$ and $\alpha = 0.531$, $\mathbb{R} = 0.525$ ] is 10.5 cm/sec.

Figure 148: Upper interface wave of the mid-section showing the maximum at time 3.48 seconds and at time 3.51 seconds.
Figure 149: Non-axisymmetric snake wave shape taken at interface at the center cross section (X-Y) plane. Only the upper portion of the wave is shown. For better visualization, wave shapes are drawn for two wave lengths. \[ a = 1.7, m = 0.00166, \zeta = 1.1, f = 0.063354, K = -0.5427 \text{ and } \alpha = 0.531, R = 0.525 \].

In Figure 150, wave speed from all three cases are plotted. Clearly, Reynolds number plays a role to influence the wave speed. Wave speed increases as the Reynolds number increases.

Figure 150: Change in wave speed as a function of Reynolds number.
From all the above discussions, it is indeed possible to analyze true three-dimensional model with Ansys Fluent. The models are not only capable of predicting symmetric bamboo waves but is very capable of predicting the non-axisymmetric waves such as corkscrew and snake waves.

The following conclusions could be drawn by visualization of the wave shape and the discussion presented above for different wave shapes:

(a) Bamboo Waves: The centroid of the core fluid at any arbitrary cross section of the pipe flow remains at the center over time and the waves are axisymmetric. The trajectories of the centroidal coordinates \((Z_m \text{ and } Y_m)\) remains at a fixed point at the center of the pipe.

(b) Corkscrew Waves: The centroid of the core fluid at any arbitrary cross section moves in radial direction and also rotates along the azimuthal direction over time. Therefore, the trajectories of the centroidal coordinate of the core fluid rotate along a circular path.

(c) Snake Wave: The centroid of the core fluid at any arbitrary cross section moves only in radial direction and no rotation of the centroidal coordinate is observed over time. Therefore, the movement of the centroid is always limited to a fixed line. Another distinction is that the waves are not axisymmetric.

From all the results of the non-symmetric models, it is also evident that the prediction of Renardy’s theoretical work (1992) is indeed a seminal piece of work and the 3-D models
are capable enough to predict the pattern selection problem and was able to predict the sensitivity of the Reynolds number in determination of the final wave pattern.
CHAPTER 8 : What is New in Our Research?

A significant amount of work is done to understand the nature of core-annular flow in the past. Here are some of the findings which are new in our research.

1. A detailed characterization of the core-annular flow waves is performed in nonlinear regime using ANSYS Fluent.

2. In our work we presented details about the evolution, and propagation of the core-annular flow in nonlinear regime. For up-flow, it is well known that for a certain parameters of flow field, bamboo waves are generated. In this study, a new form of bamboo waves is observed where it is not just a traveling wave but it oscillates and modulates in a certain range of flow parameters.

3. In this study, special emphasis is given to characterize these waves in two parameter regimes (surface tension and Reynolds number). A detailed analysis is performed to show that the bamboo wave shape changes with time and at a certain time interval, wave shape repeats itself. That means that there is a period associated with the propagation of the waves. The amplitudes of these wave shape also show oscillation and modulation for a certain rage of flow parameters. On the other hand, a slight change in flow parameters alters the formation of the wave where the wave amplitudes do not show any oscillation and modulation. They are rather a constant amplitude traveling waves. Based on this characteristic of the wave form at various flow conditions a bifurcation diagram is constructed. Bifurcation diagram distinctly identifies the regime of modulated wave vs. non-modulated traveling waves. Bifurcation diagram is also presented to show the effect
of wave shape and its nature on Reynolds number and on surface tension. The depiction of the bifurcation diagrams of this kind are also shown for the first time.

4. For the first time, effort is being undertaken to perform full three-dimensional analysis of the core-annular flow. With the advent of advanced computing hardware and improved solver algorithm, it is feasible to perform full blown 3-D model of the core-annular flow. Our 3-D simulation results ensure that waves generated by benchmark flow parameters are indeed axisymmetric bamboo waves which is also verified by experimental work and two-dimensional axisymmetric model. Three-dimensional model also confirms the oscillation and modulation of the wave observed in 2-D axisymmetric model. Movement of the centroidal coordinate trajectories of core fluid of an arbitrary cross section of a pipe at different time is considered to distinguish various kinds of 3-D wave shapes.

5. Present work also addresses the modeling approach for non-axisymmetric core-annular flow in three dimensions for the first time. From our modeling results, an elegant approach is described to distinguish different wave shapes such as corkscrew and snake wave. Theoretical work of Renardy’s (1997) pattern selection is verified with 3-D models. The sensitivity of the flow parameters on wave pattern were studied and compared with experimental and theoretical work from linear stability analysis.
CHAPTER 9: Conclusions

Our work presented a methodology to model Core-annular flow with ANSYS Fluent. A very good agreement is found between our 2-D axisymmetric model versus other published results and experimental observation. For the first time, we were able to show that for certain sets of flow parameters, waves generated at nonlinear regime follow a certain characteristic. As the wave propagates, wave tends to modulate and follow a periodic pattern. Therefore, magnitudes of amplitudes of the waves show oscillation and modulation for a selected range of flow parameters. On the other hand, with the change of flow parameter such as surface tension and Reynolds number, this oscillation and modulation of the wave amplitudes diminishes and the wave travels at constant amplitude. In our analysis, we have depicted a bifurcation diagram to identify the regime of flow parameters where wave characteristic changes from flat interface solution to constant amplitude traveling wave to branching out to modulated waves. Wave speed, hold-up ratio and waves shapes were also compared for different flow regimes. It is interesting to see that hold up ratio also show oscillation in the modulated bamboo wave regime. For the first time a full blow 3-D model is built using ANSYS Fluent. Both symmetric Bamboo wave and non-symmetric corkscrew and snake waves were modeled successfully. By tracing the trajectories of the centroid of the core fluid at an arbitrary cross section of the flow domain, the nature of the wave shape could be identified from the simulation results. This novel approach distinguishes the difference between the Bamboo wave, corkscrew wave and snake wave. Three discreet models were run in ANSYS Fluent to verify
Renardy’s (1997) pattern selection theory. Our 3-D models were able to predict the non-axisymmetric corkscrew and snake wave for the flow regime described by Renardy (1997).

In our research we only explored two parameter spaces and were successful to obtain oscillating and modulating regime of the bamboo wave. Wave dependency with other parameters such as density ratio and viscosity ratio along with thickness ratio could be studied in the future. Instead of modeling with one wave length, simulation could be performed with multiple wave lengths to analyze the evolution of the waves in nonlinear regime. It would be interesting to investigate mathematically to learn why this kind of bifurcation takes place in order to map out a full range of regime.
APPENDICES:

Appendix-A: Effect of Surface Tension

Modulated Bamboo Waves: Peak to Peak and Valley to Valley wave shape for surface tension parameter $J = (1/2)*J_b$.

Figure 151: Peak to Peak wave shape comparison for the surface tension parameter $J = (1/2)J_b = 0.03167$. Solid lines are the waves and dotted lines are shifted waves to match the initial wave at time $t$. They perfectly overlap.
Figure 152: Valley to Valley wave shape comparison for the surface tension parameter $J = (1/2)J_b = 0.03167$. Solid lines are the waves and dotted lines are shifted waves to match the initial wave at time $t$. They perfectly overlap.

Figure 153: Peak to Peak and Valley to Valley wave shape comparison for surface tension parameter $J = (1/2)J_b = 0.03167$. 

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Modulated Bamboo Waves: Peak to Peak and Valley to Valley wave shape for surface tension parameter $J = (1/4)J_b$.

Figure 154: Peak to Peak wave shape comparison for $J = (1/4)J_b$. 

![Amplitude Vs. Time (Peak to Peak, $J = (1/4)J_b$)](image)

![Wave Shapes at Different Times: Peak to Peak: ($J=(1/4)J_b$)](image)
Figure 155: Valley to Valley wave shape comparison for $J = (1/4)J_b$.

Figure 156: Peak to Peak and Valley to Valley wave shape comparison for $J = (1/4)J_b$. 

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Non-modulated/Saturated Bamboo Waves: Wave Shape for surface tension parameter $J = 4J_b$.

Figure 157: Amplitude Vs. Time. Three lines showing the temporal location of the waves at every 0.005 sec from starting time $t$.

Figure 158: Wave shape at every 0.005 seconds from initial starting time $t$. 
Non-modulated/Saturated Bamboo Waves: Wave Shape Analysis: $J = 5J_b$ Case

Figure 159: Amplitude Vs. time curve. Three lines showing the temporal location of the waves at every 0.005 seconds interval.

Figure 160: Wave shape at very 0.005 seconds for the surface tension parameter of $J = 5J_b$. 
Appendix-B: Effect of Reynolds Number

Modulated Bamboo Waves: Peak to Peak and Valley to Valley wave shape for a fixed surface tension parameter $J=J_b$ and for Reynolds number $\Re=2.5$.

![Amplitude Vs. Time (Re = 2.5, J=0.063354)](image-url)

![Wave Shapes at Different Times: Peak to Peak : (Re = 2.5, J=0.063354)](image-url)

Figure 161: Peak to Peak wave shape for $\Re=2.5$. 

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Figure 162: Valley to Valley wave shape for $\Re = 2.5$.

Figure 163: Peak and Valley Wave shape comparison for $\Re = 2.5$. 
Modulated Bamboo Waves: Peak to Peak and Valley to Valley wave shape for a fixed surface tension parameter $J=J_b$ and for Reynolds number $\mathbb{R}=3.0$.

Figure 164: Peak to Peak wave shape for $\mathbb{R}=3.0$. 

![Amplitude Vs. Time](image1.png)

![Wave Shapes at Different Times: Peak to Peak](image2.png)
Figure 165: Valley to Valley wave shape for $\mathbb{R} = 3.0$.

Figure 166: Peak to Peak and Valley to Valley wave comparison for $\mathbb{R} = 3.0$. 

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Modulated Bamboo Waves: Peak to Peak and Valley to Valley wave shape for a fixed surface tension parameter \( J=J_b \) and for Reynolds number \( \Re=3.5 \).

Figure 167: Peak to Peak wave shape for \( \Re=3.5 \).
Figure 168: Valley to Valley wave shape for $\mathbb{R}=3.5$.

Figure 169: Comparison of Peak and Valley wave shape for $\mathbb{R}=3.5$. 

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Non-Modulated or Saturated Bamboo Waves: Wave shape for a fixed surface tension parameter $J=J_b$ and for Reynolds number $\Re=1$.

![Wave Shapes at Different Times: (Re =1.0 , J=0.063354)](image_url)

Figure 170: Wave shape at interval of every 0.02 seconds for $\Re=1.0$. 

![Amplitude Vs. Time (Re = 1.0)](image_url)

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Appendix-C: Subroutines for initialization for 2-D and 3-D models

Subroutine for 2-D model:
/*Initialization, specify in Define-User Defined-Function Hooks*/
/*Assume the wave equation is \( y_{\text{wave}} = \text{amp} \times \cos(2 \pi / \text{wave\_length} \times x + \theta) + y_0 \)*/
#include "udf.h"
#define PI 3.14159
#define amp 0.0003 /*amplitude: please customize*/
#define wave_length 0.007744 /*wave\_length: please customize*/
#define theta 0.0 /*phase angle: please customize*/
#define y0 2.958E-3 /*offset: please customize*/

#define domain_ID_secondary 3 /*domain ID for the secondary phase*/
#define domain_ID_mixture 1 /*domain ID for the mixture phase*/

DEFINE_INIT(init_velocities_wave_shape, domain)
{
  cell_t c;
  Thread *ct; /* Threads are pointers to a structure */
  real pos[ND_ND], y_wave, x, y;
  real a,m,Re,K,rhol,rho2,Vo,R1,R2,g,mu2, mu1,dr,Rg,uo,Vw;
  g=-9.81;
  R1=y0;
  R2=4.762E-3;
  a=R2/R1;
  Re=3.73754;
  mu2=0.001;
  mu1=0.601;
  m=mu2/mu1;
  rhol=905.0;
  rho2=995.0;
  dr=rho2/rhol;
  Rg=g*R1*R1*R1*(rhol/mu1)*(rhol/mu1);
  K=-2.067;
  Vo=(Re*mu1)/(rhol*R1);
  uo=Vo;
  Vw=0.0;
  /*Initialize velocities*/
domain = Get_Domain(domain_ID_mixture);

thread_loop_c(ct,domain)
{  /* Loop through the cells in each cell thread */
  begin_c_loop(c,ct)
  {
    C_CENTROID(pos,c,ct);
     /*Please customize use the equation*/
     x = pos[0]; /*x coordinate of the cell*/
     y = pos[1]; /*y coordinate of the cell*/
     y_wave = amp*cos((2.0*PI/wave_length)*x+theta) +y0;
     if(y < y_wave)
      C_U(c,ct) = (1-
      (m*pow((y/y_wave),2)*K)/(m*K+pow((R2/y_wave),2)
      -1 + 2*(K-1)*log((R2/y_wave))))*uo-Vw;
      else
      C_U(c,ct) = (pow((R2/y_wave),2) -pow((y/y_wave),2)-2*(K-1)*log((y/y_wave)/(R2/y_wave)))/(m*K+pow((R2/y_wave),2) -1 + 2*(K-1)*log((R2/y_wave)))*uo-Vw;
      
      C_V(c,ct)= 0;
  }
  end_c_loop(c,ct)
  }

/*Initialize the volume fraction*/
/* Loop through all the cell threads in the domain */
domain = Get_Domain(domain_ID_secondary);
thread_loop_c(ct,domain)
{  /* Loop through the cells in each cell thread */
  begin_c_loop(c,ct)
  {
    C_CENTROID(pos,c,ct); /*Get the coordinates of each cell and
store them in pos*/
    x = pos[0]; /*x coordinate of the cell*/
    y = pos[1]; /*y coordinate of the cell*/
    y_wave = amp*cos(2.0*PI/wave_length*x+theta) +y0;
    if(y < y_wave)
     C_VOF(c,ct) = 0;
     else
     C_VOF(c,ct) = 1;
  }
  end_c_loop(c,ct)
  }
}
Subroutine for 3-D Model:

/*Initialization, specify in Define-User Defined-Function
Hooks*/
/*Assume the wave equation is \( y_{wave} = \text{amp} \times \cos(2 \times \pi \times \text{wave_length} \times x + \text{theta}) + y0 \)*/
#include "udf.h"
#define PI 3.14159
#define amp 0.0003 /*amplitude: please customize*/
#define wave_length 0.00774 /*wave length: please customize*/
#define theta 0.0 /*phase angle: please customize*/
#define y0 2.958E-3 /*offset: please customize*/

#define domain_ID_secondary 3 /*domain ID for the secondary phase*/
#define domain_ID_mixture 1 /*domain ID for the mixture phase*/

DEFINE_INIT(init_velocities_wave_shape, domain)
{
    cell_t c;
    Thread *ct; /* Threads are pointers to a structure */
    real pos[ND_ND], y_wave, x, y, z, r;
    real a, m, Re, K, rho1, rho2, Vo, R1, R2, g, mu2, mu1, dr, Rg, uo, Vw;
    g = -9.81;
    R1 = y0;
    R2 = 4.762E-3;
    a = R2 / R1;
    Re = 3.7375;
    mu2 = 0.001;
    mu1 = 0.601;
    m = mu2 / mu1;
    rho1 = 905.0;
    rho2 = 995.0;
    dr = rho2 / rho1;
    Rg = g * R1 * R1 * R1 * (rho1 / mu1) * (rho1 / mu1);

    K = (4 * m * Re - (dr - 1) * Rg * (a * a - 1 - 2 * log(a))) / (4 * m * Re + (dr - 1) * Rg * (m + 2 * log(a)));
    Vo = (Re * mu1) / (rho1 * R1);
    uo = Vo;
    Vw = 0.0;
    /*Initialize velocities*/
    domain = Get_Domain(domain_ID_mixture);
    thread_loop_c(ct, domain)
begin_c_loop(c,ct) {
  C_CENTROID(pos,c,ct);
}

x = pos[0];
y = pos[1];
z = 0.0;
#if RP_3-D
  z = pos[2];
#endif
  r = sqrt(y*y + z*z);
  y_wave = amp*cos((2.0*PI/wave_length)*x+theta) +y0;
  if(r < y_wave) {
    /*
       C_U(c,ct) = (1.0-(m*pow((r/y_wave),2.0)*K)/(m*K+pow((R2/y_wave),2.0)-1.0 + 2.0*(K-1.0)*log((R2/y_wave))))*uo-Vw;
       */
    C_U(c,ct) = (1.0-(m*pow((r/y_wave),2.0)*K)/(m*K+pow((R2/y_wave),2.0)-1.0 + 2.0*(K-1.0)*log((R2/y_wave))))*uo;
  } else {
    /*
       C_U(c,ct) =pow((R2/y_wave),2.0) -pow((r/y_wave),2.0)-2.0*(K-1.0)*log((r/y_wave)/(R2/y_wave))/((m*K+pow((R2/y_wave),2.0)-1.0 + 2.0*(K-1.0)*log((R2/y_wave))))*uo-Vw;
       */
    C_U(c,ct) =pow((R2/y_wave),2.0) -pow((r/y_wave),2.0)-2.0*(K-1.0)*log((r/y_wave)/(R2/y_wave))/((m*K+pow((R2/y_wave),2.0)-1.0 + 2.0*(K-1.0)*log((R2/y_wave))))*uo;
  }
  C_V(c,ct)= 0;
/*Please customize use the equation*/
}
end_c_loop(c,ct)
}

/*Initialize the volume fraction*/
/* Loop through all the cell threads in the domain */
domain = Get_Domain(domain_ID_secondary);
thread_loop_c(ct,domain)
{ /* Loop through the cells in each cell thread */
begin_c_loop(c,ct)
{
C_CENTROID(pos,c,ct); /*Get the coordinates of each cell and store them in pos*/
x = pos[0];
y = pos[1];
z = 0.0;
#if RP_3-D
z = pos[2];
#endif
r = sqrt( y*y + z*z );
y_wave = amp*cos(2.0*PI/wave_length*x+theta) +y0;
if(r < y_wave)
  C_VOF(c,ct) = 0;
else
  C_VOF(c,ct) = 1;
}
end_c_loop(c,ct)
}
Appendix-D: Linear Stability Analysis: FORTRAN PROGRAM

C* LINEAR STABILITY OF CORE-ANNULAR FLOW BY FINITE ELEMENT METHOD
C* GENERAL CASE n=0, axisymmetric
C***************************************************************************
PROGRAM CAFGNRL
        IMPLICIT COMPLEX (C)
        DIMENSION CA(160,160), CB(160,160), P(6,3), DP(6,3), DDP(6,3),
        CZ(160,160), RR(53), II(2,50), C(160), CEA(160), CEB(160),
        CU(160), WW(3)
        complex*16 CA, CB, CEA, CEB, CZ, CWK
        real*8 P, DP, DDP, WW, RR
        real*8 RA, AM, RG, TJ, AD, ALP, RE, RK, WA
        complex*16 C, CU, CEIG
        real*8 D, A1
        COMMON /X/ CA, CB, CEA, CEB, CZ, CWK
        /X7/ P, DP, DDP, WW, RR
        /X2/ ALP, RE
        /X1/ RA
        /X10/ RR
        /L1/ NR1, NR2
        /L2/ II, IE
        /L3/ IP, NW
        OPEN(UNIT=6, FILE='hu')
        CALL SHAV

        INPUT PARAMETERS
        NR1=20
        NR2=20
        102 WRITE(*,1001)
        NC=0
        READ(*,*) RA, AM, RG, TJ, AD
        WRITE(6,1002) NR1, NR2, NC, TJ, RA, AM, AD, RG
        20 WRITE(*,1005)
        READ(*,*) ALP, RE

        AUTOMATICALLY GENERATE THE GRID FOR ELEMENTS
        101 CALL RGDV
        write(*,*) (RR(i), i=1, ip)

        GENERATE ELEMENTAL MATRICES AND ASSEMBLE INTO GLOBAL ONES
        RK=(4.*AM*RE+(1.-AD)*RG*(RA**2-1.-2.*DLOG(RA)))/
        & (4.*AM*RE-(1.-AD)*RG*(AM+2.*DLOG(RA)))
        WA=AM*RK+RA**2-1.0+2.0*(RK-1.0)*DLOG(RA)
        CALL EMG

        CALL IMSL/EIGZC TO COMPUTE ALL CEIGENVALUES FOR A*U=C*B*U
        IA=160
1  IB=160
2  N=NW-2
3  IJOB=1
4  IZ=160
5  CALL
6  deigzc(CA,IA,CB,IB,N,IJOB,CEA,CEB,CZ,IZ,CWK,INFER,IER)
7  IF(INFER.EQ.0) GO TO 105
8  WRITE(*,1006) INFER
9  105  DO 1 I=1,N
10    C(I)=(-1.d30,-1.d30)
11    DO 2 I=1,N
12      D=dabs(dreal(CEB(I)))+dabs(dimag(CEB(I)))
13      IF(D.LT.1.d-10) GO TO 2
14      C(I)=CEA(I)/CEB(I)*ALP
15    CONTINUE
16  CEIG=(-1.d30,-1.d30)
17  DO 3 I=1,N
18      A1=DIMAG(C(I))
19      IF((A1.GT.DIMAG(CEIG)).AND.(A1.LT.1.d2)) THEN
20        CEIG=C(I)
21        MP=I
22    ENDIF
23    CONTINUE
24  CEIG=CEIG/ALP
25  DO 6 K=1,N
26    CU(K)=CZ(K,MP)
27    CU(NW)=0.
28    CU(NW-1)=0.
29    WS=DREAL(CEIG)
30    GR=DIMAG(ALP*CEIG)
31    WRITE(*,1003) ALP,WS,GR
32    WRITE(6,1003) ALP,WS,GR
33  CONTINUE
34  PREPARE FOR NEXT VALUE OF COMPUTATION
35  IF(IM.EQ.0) GO TO 20
36  IF(IM.EQ.2) GO TO 102
37  IF(IM.EQ.5) STOP
38  WRITE(*,1007)
39  READ(*,*) IM
40  FORMAT(1X,'a=   m=    Rg=   J*=   Density Ratio=')
41  1001  FORMAT(/' N1=',I2,1X,'N2=',I2,1X,'N=',I2,1X,'J=',F8.2,1X,
42        'a=',F7.3,1X,'m=',F10.5,1X,'D=',F5.2,1X,'Rg=',F6.3)
43  1002  FORMAT(' Alpha=',1PE10.3,' Re='1PE10.3,' WaveSpeed=',1PE13.6,' GrowthRate=',1PE13.6)
44  1003  FORMAT(' Alpha= ',1PE10.3,' Re= ',1PE10.3,
45        ' WaveSpeed=' ,1PE13.6 , ' GrowthRate=' ,1PE13.6)
46  1004  FORMAT(' I= ',(0,ALP),'(1 RE),(2 RESTART),(5 STOP)')
47  1005  FORMAT(' Alpha= Re= ')

224
! THIS IS THE SUBROUTINE FOR INTERPOLATION FUNCTIONS AND THEIR DERIVATIVES
!
SUBROUTINE SHAP(Y,N)
    !... Code...
RETURN
END

! THIS THE SUBROUTINE TO CALCULATE VALUES OF INTERPOLATION FUNCTIONS AT EACH INTEGRATION POINT (3)
SUBROUTINE SHAV
    !... Code...
RETURN
END
WW(2)=4./9.
WW(3)=WW(1)
DO 14 I=1,3
    IN=I
    CALL SHAP(HCE(IN),IN)
14    CONTINUE
RETURN
END

C---------------------------------------------------------------
C!     THIS IS THE SUBROUTINE TO GENERATE NODE COORDINATE RR, !
C!     NODE NUMBER IP AND ELEMENTAL-GLOBAL NODE NUMBER         !
C!     CAYRESPONDING ARRAY II                                  !
C---------------------------------------------------------------
SUBROUTINE RGDV
DIMENSION RR(53),II(2,50)
real*8 RR, D,DT,E
real*8 RA,AM,RG,TJ,AD,ALP,RE,RK,WA
COMMON /X10/RR/L2/II,IE/L3/IP,NW/L1/NR1,NR2/X1/RA
1  /X2/ALP,RE,TJ,AM,AD,WA,RK,NC
RR(1)=0.
IP=1
C
D=1.d0/NR1
DO 1 I=1,NR1
    IP=IP+1
1     RR(IP)=RR(IP-1)+D
C
IP=IP+1
RR(IP)=RR(IP-1)
E=dsqrt(AM/RE/ALP)
D=(RA-1.d0)/(NR2+2)
M=DLOG(10.d0*D/E+1.d0)/DLOG(2.d0)+0.5
IF(M.LE.2) M=2
DT=D/(2**M-1)
DO 12 I=1,M
    IP=IP+1
    RR(IP)=RR(IP-1)+DT
12  DT=DT*2.d0
    DO 2 I=1,NR2
        IP=IP+1
    2     RR(IP)=RR(IP-1)+D
    DO 13 I=1,M
        IP=IP+1
    13   DT=DT/2.d0
NW=2*IP
C
IE=0
DO 3 I=1,NR1
3
IE=IE+1
II(1,IE)=I
3     II(2,IE)=I+1
N=NR1+1
DO 4 I=1,NR2+2*M
IE=IE+1
II(1,IE)=N+I
4     II(2,IE)=N+I+1
RETURN
END

C
----------------------------------------------------------------
C!   THIS IS THE SUBROUTINE TO CALCULATE RL, DL, W, DW/DR FOR A     !
C!   GIVEN VALUE OF R                                            !
C----------------------------------------------------------------
SUBROUTINE VALU(RL,DL,W,DW,R,RA,AM,AD,RK,WA)
real*8 RL,DL,W,DW,R,RA,AM,AD,RK,WA
IF(R.GT.1.0) GO TO 10
RL=1.
DL=1.
W=1.0-AM*RK*R*R/WA
DW=-2.*AM*RK*R/WA
RETURN
10     RL=AM
       DL=AD
       W=(RA*RA-R*R-2.0*(RK-1.0)*DLOG(R/RA))/WA
       DW=-2.0*(R+(RK-1.0)/R)/WA
20     RETURN
END

C
----------------------------------------------------------------
C!   THIS IS THE SUBROUTINE TO GENERATE ELEMENTAL METRICES AND     !
C!   PUT THEM INTO GLOBAL ONES: CA AND CB                      !
C----------------------------------------------------------------
SUBROUTINE EMG
DIMENSION
CA(160,160),CB(160,160),CEA(4,4),CEB(4,4),P(6,3),
1   DP(6,3),DDP(6,3),RR(53),II(2,50),ER(2),EP(6,3),EDP(6,3),
1   EDDP(6,3),WW(3)
c
complex*16 CA,CB,CEA,CEB,CI,C,C1,C2,C3
real*8 P,DP,DDP,WW,RR, EP,EDP,EDDP,ER,D,DR
real*8 R,R2,A1,S1,S2,S3, T1,T2,T3
real*8 RA,AM,RG,TJ,AD,ALP,RE, RK,WA, RL,DL,W,DW,DDW
c
COMMON
/X/CA,CB/X7/P,DP,DDP,WW/X2/ALP,RE,TJ,AM,AD,WA,RK,NC

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C
CI=(0.,1.)
C=CI/RE
DO 1 I=1,NW
DO 1 J=1,NW
CA(I,J)=0.
1     CB(I,J)=0.
C--- LOOP OVER EACH ELEMENT
DO 2 IEE=1,IE
DO 3 J=1,2
3     ER(J)=RR(II(J,IEE))
DO 4 I=1,4
DO 4 J=1,4
CEA(I,J)=0.
4     CEB(I,J)=0.
C
C--- INTERPOLATION FUNCTION FOR EACH ELEMENT
DR=ER(2)-ER(1)
DO 21 I=1,6
DO 21 N=1,3
EP(I,N)=P(I,N)
EDP(I,N)=DP(I,N)
21    EDDP(I,N)=DDP(I,N)
DO 22 N=1,3
DO 23 I=2,4,2
EP(I,N)=EP(I,N)*DR
EDP(I,N)=EDP(I,N)*DR
23     EDDP(I,N)=EDDP(I,N)*DR
DO 24 I=1,6
EDP(I,N)=EDP(I,N)/DR
24     EDDP(I,N)=EDDP(I,N)/(DR*DR)
22     CONTINUE
C
C--- LOOP OVER EACH INTEGRATE POINT
DO 5 N=1,3
R=ER(1)*EP(5,N)+ER(2)*EP(6,N)
R2=R*R
D=W(N)*DR*R
CALL VALU(RL,DL,W,DW,R,RA,AM,AD,RK,WA)
A1=ALP*ALP
C
DO 10 J=1,4
S1=EP(J,N)
S2=EDP(J,N)+S1/R
S3=EDDP(J,N)+S2/R-2.*S1/R2
DO 10 I=1,4
T1=EP(I,N)
T2=EDP(I,N)+T1/R
T3=EDDP(I,N)+T2/R-2.*T1/R2
CEA(J,I)=CEA(J,I)+D*RL*(T3*S3+2.*A1*T2*S2+A1*A1*T1*S1)*C-
1 D*DL*ALP*(W*(A1*T1*S1+T2*S2)-DW*T1*S2)
10 CEB(J,I)=CEB(J,I)-D*DL*ALP*(A1*T1*S1+T2*S2)
C
5 CONTINUE
C--- ASSEMBLE INTO GLOBAL ONES
DO 2 LI=1,2
  IL=II(LI,IEE)
  DO 15 L=1,2
    I=2*(LI-1)+L
    IK=2*(IL-1)+L
    DO 15 JJ=1,2
      JL=II(JJ,IEE)
      DO 15 K=1,2
        J=2*(JJ-1)+K
        JK=2*(JL-1)+K
        CA(IK,JK)=CA(IK,JK)+CEA(I,J)
        CB(IK,JK)=CB(IK,JK)+CEB(I,J)
  15 CONTINUE
2 CONTINUE
C--- ADD THE TERMS CAUSED BY INTERFACE
W=1.-AM/RK/WA
DW=2.0*(1.0-AM)*RK/WA
DDW=4.0*AM*(1.0-RK)/WA
C1=-TJ*(1.-ALP**2)*ALP/(RE*RE*DW)
C3=-CI*DDW/(RE*DW)
47 C2=ALP**2/RE*CI
I=2*(NR1+1)
CA(I-1,I)=CA(I-1,I)+C1+C3+C2
CA(I-1,I+2)=CA(I-1,I+2)-C1-C3-C2*AM
CA(I,I-1)=CA(I,I-1)+C2*(1.-AM)
CA(I,I)=CA(I,I)+C3
CA(I,I+2)=CA(I,I+2)-C3
C--- ADD THE EQUATIONS AT INTERFACE POINTS
DO 25 J=1,2
  L=I+J
  L1=I-2+J
  DO 25 K=1,NW
    CA(L1,K)=CA(L1,K)+CA(L,K)
    CB(L1,K)=CB(L1,K)+CB(L,K)
  25 CONTINUE
CA(I+1,I-1)=-1.
CA(I+1,I+1)=1.
CA(I+2,I-1)=DW
CA(I+2,I)=-W
CA(I+2,I+2)=W
CB(I+2,I)=-1.
CB(I+2,I+2)=1.
RETURN
END
For the complete FORTRAN program contact Dr. Howard Hu.
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Water-lubrication in a vertical pipe was studied in the experiments of Bai Chen and Joseph [1991].


