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Subtracted Geometry

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Abstract
In this thesis we study a special class of black hole geometries called subtracted geometries. Subtracted geometry black holes are obtained when one omits certain terms from the warp factor of the metric of general charged rotating black holes. The omission of these terms allows one to write the wave equation of the black hole in a completely separable way and one can explicitly see that the wave equation of a massless scalar field in this slightly altered background of a general multi-charged rotating black hole acquires an $SL(2,\mathbb{R}) \times \times SL(2,\mathbb{R}) \times SO(3)$ symmetry. The ”subtracted limit” is considered an appropriate limit for studying the internal structure of the non-subtracted black holes because

new 'subtracted' black holes have the same horizon area and periodicity of the angular and time coordinates in the near horizon regions as the original black hole geometry it was constructed from. The new geometry is asymptotically conical and is physically similar to that of a black hole in

an asymptotically confining box. We use the different nice properties of these geometries to understand various classically and quantum mechanically important features of general charged rotating black holes.

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SUBTRACTED GEOMETRY

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ABSTRACT

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In this thesis we study a special class of black hole geometries called subtracted geometries. Subtracted geometry black holes are obtained when one omits certain terms from the warp factor of the metric of general charged rotating black holes. The omission of these terms allows one to write the wave equation of the black hole in a completely separable way and one can explicitly see that the wave equation of a massless scalar field in this slightly altered background of a general multi-charged rotating black hole acquires an $SL(2,\mathbb{R}) \times \times SL(2,\mathbb{R}) \times SO(3)$ symmetry. The subtracted limit is considered an appropriate limit for studying the internal structure of the non-subtracted black holes because new 'subtracted' black holes have the same horizon area and periodicity of the angular and time coordinates in the near horizon regions as the original black hole geometry it was constructed from. The new geometry is asymptotically conical and is physically similar to that of a black hole in an asymptotically confining box. We use the different nice properties of these geometries to understand various classically and quantum mechanically important features of general charged rotating black holes.

Contents

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1. M. Cvetič, M. Guica and Z. H. Saleem, "General black holes, untwisted," JHEP **1309**, 017 (2013) doi:10.1007/JHEP09(2013)017 [arXiv:1302.7032 [hep-th]].

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3. M. Cvetič, G. W. Gibbons and Z. H. Saleem, "Quasinormal modes for subtracted rotating and magnetized geometries," Phys. Rev. D **90**, no. 12, 124046 (2014) doi:10.1103/PhysRevD.90.124046 [arXiv:1401.0544 [hep-th]].

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Chapter 1

Introduction

The discovery that black holes behave as thermodynamic objects has been one of the most important developments in fundamental physics. In the early 1970s Bekenstein and Hawking showed that black holes radiate as black bodies with a characteristic entropy that depends on the area of the black hole horizon $S = A/4\hbar G$. The quantity is naturally quantum gravitational since the planck constant and gravitational constant appear in the same equation. This is the reason why physicists have been searching for the quantum microscopic interpretation of the black hole entropy.

One of the most important hurdles in explaining the black hole entropy in terms of quantum microstates is the limited understanding of quantum gravity. However string theory, as the best available theory of quantum gravity has been instrumental in making breakthroughs in this direction. A very important step forward was achieved with Vafa-Strominger [1] calculation of the counting of the supersymmetric black hole microstates employing string theory and D-branes. Vafa and Strominger's results however worked for only certain type of black holes, called extremal black holes. These black holes have the property that they carry the maximum amount of electric and magnetic charge that is possible without making the black hole unstable.

Maldacena's [2] AdS/Cft correspondence was also very useful in understanding black hole thermodynamics using quantum field theory. Maldacena in his seminal paper conjectured that quantum gravity in higher dimensional AdS space is dual to a conformal gauge field theory in one less dimensional flat minkowski space. Soon after the AdS/Cft conjecture was proposed, Witten [3] showed how the thermodynamics of AdS black holes can be understood in terms of the dual gauge theory. Another success of the Maldacena conjecture was how it illuminated on the calculation of the entropy of BTZ black holes, raising the Brown-Hanneux[4] result from an anology to an actual counting of the states in a dual conformal field theory. This was really important for the black holes that arise in string theory since most of the times these have near horizon geometries of the form $BTZ \times S$. These results were also mostly obtained for the case of extremal or near extremal black holes.

The entropy matching of extremal black holes was also one of the main motivations for the proposed Kerr/Cft correspondence [5]. In the near-horizon extremal Kerr (NHEK) geometry with enhanced isometry group $SL(2;\mathbb{R}) \times U(1)$, one can find a set of boundary conditions for metric fluctuations, whose asymptotic symmetry group (ASG) enhances the U(1) to a single copy of the Virasoro algebra with central charge $c_L = 12J$. Such symmetry of the geometry however does not exist for non extremal black holes.

However even in the non-extremal case, the thermodynamic features, such as the entropy formula [10] and the first law of thermodynamics [7–9], are very strongly suggestive of a possible microscopic interpretation in terms of a two-dimensional conformal field theory of the general multi-charged rotating black holes in four [10] and five [11] dimensions. Furthermore, the wave equation for massless scalars in non-extremal black hole backgrounds exhibits an approximate $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ conformal symmetry at low energies, which is spontaneously broken by the temperatures [8, 9, 12]. Thus, one may expect that at least the low-energy dynamics of general black holes are described by a Cft.

Recently, [13, 14] advanced a concrete proposal - deemed “subtraction” - for how to relate general black holes to Cfts. The subtraction procedure consists of removing certain terms in the warp factor of the black hole geometry, in such a way that the scalar wave equation acquires a manifest local $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ conformal symmetry. The horizon area and the periodicities of the angular and time coordinates remain fixed. For this reason, the subtraction process is expected to preserve the internal structure of the black hole. Given that the geometry becomes asymptotically conical [15], rather than asymptotically flat, the physical interpretation of subtraction is the removal of the ambient asymptotically

Minkowski space-time in a way that extracts the "intrinsic" $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry of the black hole.

The subtraction procedure has been explicitly implemented both for five-dimensional three-charge rotating black holes [13] and four-dimensional four-charge ones [14]. It works particularly well in the context of the four-dimensional STU model [16] and its five-dimensional uplift, since in these cases the non-trivial matter field configurations that support the subtracted geometries are still solutions of the same Lagrangian as the original black holes. Moreover, one can use the extra dimensions available in the string-theory embedding of these models to show that the four-dimensional subtracted geometries uplift to $AdS_3 \times S^2$ [14] and the five-dimensional ones, to $AdS_3 \times S^3$ [13], thus making the connection with two-dimensional Cfts entirely explicit.

In chapter 2 we will introduce the framework in which subtracted geometries are introduced. We will also discuss the solution generating methods used to generate these geometries. Using these solution generating methods we will show how the entropy matching is achieved. The uplift of these geometries in five and six dimensions to $AdS_3 \times S_2$ and $AdS_3 \times S_3$ respectively will also be discussed. In chapter 3 we will study the electrodynamic properties of these geometries. We will again be employing solution generating methods to obtain these electrically and magnetically complex and interesting black hole geometries. In chapter 4 we will study the quasinormal modes for the rotating and magnetic cases and will obtain analytical results that were not obtained before for rotating and magnetic black holes. In chapter 5 we will obtain results for the entanglement entropy across the horizon of these subtracted black holes and show in what ways the logarithmic corrections differ from the original black holes. In chapter 6 we use the simple hypergeometric property of the wave equation solutions to obtain an analytical formula for the vacuum polarization of a minimally coupled scalar field. This has also not been obtained before. In chapter 7 we will explain the asymptotically conical behaviour of these geometries and try to understand the thermodynamic properties such as mass, charge and angular momentum. We will also derive the first law of thermodynamics and Smarr's law for these asymptotically conical geometries. We will present our conclusions and future directions in the last chapter.

Chapter 2

Solution Generation

In this chapter we use solution-generating techniques to construct interpolating geometries between general asymptotically flat, charged, rotating, non-extremal black holes in four and five dimensions and their subtracted geometries. In the four-dimensional case, this is achieved by the use of Harrison transformations, whereas in the five-dimensional case we use STU transformations. We also give the interpretation of these solution-generating transformations in terms of string (pseudo)-dualities, showing that they correspond to combinations of T-dualities and Melvin twists. Upon uplift to one dimension higher, these dualities allow us to untwist general black holes to AdS_3 times a sphere.

2.1 Un-twisting general $4d$ black holes

2.1.1 STU black holes and subtracted geometries

In this section, we will be working in the context of the four-dimensional STU model [16] - an $\mathcal{N} = 2$ supergravity theory coupled to three vector multiplets, characterized by the prepotential[1]

$$\mathcal{F} = -\frac{X^1 X^2 X^3}{X^0} \tag{2.1.1}$$

As usual, the bosonic content of this theory consists of the metric, four gauge fields A^Λ, $\Lambda = \{0, \ldots, 3\}$ and three complex scalars

$$z^I = \frac{X^I}{X^0}\,, \qquad I = \{1, 2, 3\} \tag{2.1.2}$$

[1]Throughout this article, we will be using the conventions and definitions of [27].

All the couplings of the theory, as well as the relationship between the various fields are entirely determined by the above $\mathcal{N} = 2$ prepotential.

We consider non-extremal rotating black hole solutions of this theory that are magnetically charged under three of the field strengths, with charges p^I, and electrically charged under the fourth field strength, with charge q_0. The metric of these solutions can be parametrized as

$$ds^2 = -e^{2U}(dt + \omega_3)^2 + e^{-2U} ds_3^2 \tag{2.1.3}$$

The three-dimensional base metric only depends on the rotation (a) and mass (m) parameters of the solutions, and takes the form

$$ds_3^2 = \frac{G}{X} dr^2 + G d\theta^2 + X \sin^2\theta d\phi^2 \tag{2.1.4}$$

$$X = r^2 - 2mr + a^2 \ , \qquad G = r^2 - 2mr + a^2 \cos^2\theta \tag{2.1.5}$$

The dependence on the charges is encoded in the conformal factor U and the angular velocity ω_3, as well as in the gauge fields and scalars that support the geometry. Parameterizing the charges as

$$q_0 = m \sinh 2\delta_0 \ , \qquad p^I = m \sinh 2\delta_I \tag{2.1.6}$$

and introducing the shorthands $c_i = \cosh\delta_i$, $s_i = \sinh\delta_i$, one finds that

$$\omega_3 = \frac{2ma\sin^2\theta}{G}[(\Pi_c - \Pi_s)r + 2m\Pi_s]d\phi \tag{2.1.7}$$

where

$$\Pi_c = c_0 c_1 c_2 c_3 \ , \qquad \Pi_s = s_0 s_1 s_2 s_3 \tag{2.1.8}$$

The conformal factor U is traded for a new quantity Δ

$$\Delta \equiv G^2 e^{-4U} \tag{2.1.9}$$

which has the nice property that it is polynomial in r. For the above, asymptotically flat, solutions

$$\begin{aligned} \Delta \;\; = \;\; & \big(a^2 \cos^2\theta + (r + 2ms_0^2)(r + 2ms_1^2)\big)\ \big(a^2 \cos^2\theta + (r + 2ms_2^2)(r + 2ms_3^2)\big) - \\ & -4a^2 m^2 (s_0 s_1 c_2 c_3 - c_0 c_1 s_2 s_3)^2 \cos^2\theta \end{aligned} \tag{2.1.10}$$

The black hole solutions are also supported by non-trivial gauge fields and scalars[2].

An interesting property of general black holes is that the wave equation for massless scalar perturbations is separable, and moreover it has a low-energy approximate $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. In order to render this $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry exact, [14] have introduced the so-called "subtracted" geometries, which differ from the original black hole metrics only by a change in the conformal factor Δ

$$\Delta \to \Delta_{sub} = (2m)^3 r(\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2\theta \tag{2.1.11}$$

The rotation parameter ω_3 in (2.1.7) is kept fixed. Since the asymptotic behaviour of Δ_{sub} is linear in r - as opposed to quartic - the new solutions are no longer asymptotically flat. Rather, they are asymptotically Lifshitz with dynamical exponent $z = 2$ and hyperscaling

[2]The scalar and vector sources are related by a subset of U-duality transformations to the original four-charge solution [11, 35]

violating exponent $\theta = -2$ (for definition and applications, see e.g. [36]). The physical picture that lies behind this replacement is that the subtraction procedure corresponds to enclosing the black hole into an "asymptotically conical box", which isolates its intrinsic dynamics from that of the ambient spacetime, while preserving its thermodynamic properties.

In [14] it was shown that in the static case the matter fields A^Λ, z^I supporting the subtracted geometry are still solutions of the STU model, albeit with unusual asymptotics. Furthermore, the explicit sources for the subtracting geometry of multi-charged rotating black holes were obtained in [15] as a scaling limit of certain STU black holes. Uplifting the subtracted geometries to five dimensions, one finds $AdS_3 \times S^2$ [14], which realizes the conformal symmetry of the four-dimensional wave equation in a linear fashion. In the following we will try to better understand the relationship between the original, asymptotically flat black holes, their subtracted geometries, and their five-dimensional uplift.

2.1.2 Solution-generating transformations

A powerful tool that we will be using extensively are solution-generating transformations that relate backgrounds of the four-dimensional STU model with a timelike isometry. These solution generating techniques can be used to both generate all the charged black holes of the previous subsection from the non-extremal Kerr solution[3], and to relate these general asymptotically flat black holes to their subtracted geometries.

The procedure is as follows. The four-dimensional STU Lagrangian itself has an $O(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ T-duality symmetry, which is enlarged at the level of the equations of motion to include a third $SL(2,\mathbb{R})$ electric/magnetic S-duality symmetry. Upon reduction to three dimensions, it is well known [37] that the "naïve" $O(3,3)$ three-dimensional global symmetry is enhanced to $O(4,4)$, since in three dimensions all one-form potentials can be dualized to scalars. When reducing along time the scalar Lagrangian becomes a non-linear

[3] They were employed in [11, 35] to generate four charge rotating black holes in four dimension with two magnetic and two electric charges. Here we are interested in the solution with one electric and three magnetic charges.

sigma model whose target space is an $SO(4,4)/(SO(2,2)\times SO(2,2))$ coset.

The four-dimensional origin of the sixteen scalars that parametrize the above coset is:

- four scalars, ζ^Λ, correspond to the electric potentials associated to the vector fields A^Λ
- four scalars, $\tilde{\zeta}_\Lambda$, are Hodge dual to the magnetic potentials associated to A^Λ
- six scalars, x^I and y^I, correspond to the real and respectively imaginary parts of the moduli fields z^I
- the scalar U corresponds to g_{tt} in the dimensional reduction (2.1.3)
- one scalar, σ, is Hodge dual to the Kaluza-Klein one-form ω_3

The symmetric coset space can be parametrized by the following coset element [38]

$$\mathcal{V} = e^{-U\,H_0}\cdot\left(\prod_{I=1,2,3} e^{-\frac{1}{2}(\log y^I)H_I}\cdot e^{-x^I E_I}\right)\cdot e^{-\zeta^\Lambda E_{q_\Lambda}-\tilde{\zeta}_\Lambda E_{p^\Lambda}}\cdot e^{-\frac{1}{2}\sigma E_0} \tag{2.1.12}$$

where the E_{p^Λ}, E_{q_Λ} etc. are generators of the $so(4,4)$ Lie algebra. An explicit parametrization of these generators is given in [27]. Thus, to any four-dimensional solution of the STU model one can associate a coset element $\mathcal{V}$ via the above procedure.

The $SO(4,4)$ symmetries act simply on the matrix $\mathcal{M}$, defined as

$$\mathcal{M} = \mathcal{V}^\sharp\mathcal{V}\,, \qquad \mathcal{V}^\sharp = \eta\mathcal{V}^T\eta \tag{2.1.13}$$

where η is the quadratic form preserved by $SO(2,2)\times SO(2,2)$. Namely, if $g\in SO(4,4)$, then the matrix $\mathcal{M}$ transforms as

$$\mathcal{M}\to g^\sharp\mathcal{M}g \tag{2.1.14}$$

We will be interested in several specific types of $SO(4,4)$ transformations.

Charging transformations

To each type of electric or magnetic charge that the four-dimensional back hole can have, there is an associated $so(4,4)$ Lie algebra element that generates it, while leaving the asymptotics of the solution flat

$$q_\Lambda \to E_{q_\Lambda} + F_{q_\Lambda}\ , \qquad p^\Lambda \to E_{p^\Lambda} + F_{p^\Lambda} \tag{2.1.15}$$

The expression for the $so(4,4)$ generators F_{q_Λ} and F_{p^Λ} is again given in [27]. Then, the charged black hole discussed in the previous section can be generated from the uncharged Kerr black hole by acting with the following group elements

$$g_{ch}(q_0, p^I) = e^{-\delta_0(E_{q_0}+F_{q_0})+\sum_I \delta_I(E_{p^I}+F_{p^I})} \tag{2.1.16}$$

where the various[4] δ_A have been defined in (2.1.6). Thus,

$$\mathcal{M}_{4-charge} = g_{ch}^\sharp\, \mathcal{M}_{Kerr}\, g_{ch} \tag{2.1.17}$$

In order to obtain the four-dimensional solution, one naturally has to re-dualize the three-dimensional scalars into vectors using (2.4.2) and then uplift.

Rescalings

One can also consider the action of the $so(4,4)$ Cartan generators H_I, H_0. Letting

$$g_S = e^{-c_0 H_0+\sum_I c_I H_I} \tag{2.1.18}$$

[4]Our notation is as follows. The index $I \in \{1,2,3\}$, the *symplectic* index $\Lambda \in \{0,\ldots,3\}$, while the *non-symplectic* index $A \in \{0,\ldots,3\}$.

one finds that they simply rescale the target space scalars as

$$U \to U + c_0 \,, \qquad \sigma \to e^{2c_0}\sigma \,, \qquad x^I \to e^{-2c_I} x^I \,, \qquad y^I \to e^{-2c_I} y^I \tag{2.1.19}$$

$$\zeta^0 \to e^{A}\zeta^0 \,, \qquad \zeta^I \to e^{A-2c_I}\zeta^I \,, \qquad \tilde{\zeta}_0 \to e^{B}\tilde{\zeta}_0 \,, \qquad \tilde{\zeta}_I \to e^{B+2c_I}\tilde{\zeta}_I \tag{2.1.20}$$

where we have let

$$A = c_0 + \sum_I c_I \,, \qquad B = c_0 - \sum_I c_I \tag{2.1.21}$$

Harrison transformations

Harrison transformations are generated by Lie group elements $e^{\alpha^\Lambda F_{p^\Lambda}}$ or $e^{\tilde{\alpha}_\Lambda F_{q_\Lambda}}$. In this paper, we will only be interested in the following Harrison transformations[5]

$$h_0 = e^{-\alpha_0 F_{q_0}} \,, \qquad h_I = e^{\alpha_I F_{p^I}} \tag{2.1.22}$$

The h_I transformations, eventually accompanied by certain rescalings, have been shown to relate non-rotating black holes to their subtracted geometries in [15, 27]. In this paper we would like to study the effect of all four Harrison transformation on a given four-dimensional asymptotically flat black hole, carrying arbitrary charge parameters δ_0, δ_I. Letting

$$g_H(\alpha_0, \alpha_I) = e^{-\alpha_0 F_{q_0} + \sum_I \alpha_I F_{p^I}} \tag{2.1.23}$$

we compute

$$\mathcal{M}_H(\alpha_0, \alpha_I) = g_H^\sharp \, \mathcal{M}_{4-charge} \, g_H \tag{2.1.24}$$

[5]Note that we dropped the tilde on α_0.

The effect of the Harrison transformations on the conformal factor Δ defined in (2.1.9) is to multiply the powers of r by various combinations of $(1-\alpha_A^2)$, where $A \in \{0, \dots, 3\}$

$$\Delta_H = (1-\alpha_0^2)(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)\, r^4 + \dots \tag{2.1.25}$$

in such a way that the coefficient of the r^4 term vanishes when any of the α_A equals one, the coefficient of r^3 is zero when any two of the α_A equal one, and so on. We give an explicit example of such a Δ_H in (2.6.18). It is thus clear that by performing at least three Harrison "infinite boosts" ($\alpha = 1$), we will obtain the same degree of divergence of Δ with r as the subtracted geometry has.

The subtracted geometry

To obtain the subtracted geometry, to the $\mathcal{M}_H$ defined in (2.1.24), we need to further apply a scaling transformation of the type (2.1.18). We find that when

$$\alpha_I = 1\,, \qquad \alpha_0 = \frac{\Pi_s \cosh\delta_0 - \Pi_c \sinh\delta_0}{\Pi_c \cosh\delta_0 - \Pi_s \sinh\delta_0} \tag{2.1.26}$$

$$e^{2c_0} = \frac{e^{\delta_1+\delta_2+\delta_3}}{\Pi_c \cosh\delta_0 - \Pi_s \sinh\delta_0}\,, \qquad e^{2c_I} = \frac{e^{2\delta_I}}{2m}\, e^{2(c_0-\delta_1-\delta_2-\delta_3)} \tag{2.1.27}$$

we recover precisely the subtracted geometries of [14] in the general rotating charged case. This result is very similar to that of [27], who showed that the subtracted geometry of a general charged rotating black hole can be obtained by applying the h_I Harrison transformations followed by a particular charging transformation and rescalings. We will further comment on the relationship with the result of [27] at the end of the next subsection.

The set of solutions to the STU model encoded in the matrix $\mathcal{M}_H(\alpha_0, \alpha_I)$ represent a four-parameter family of interpolating solutions between the original black hole and its subtracted geometry. In the non-rotating case, these interpolating solutions precisely coincide with those of [17], which we review in appendix 2.6.2. We also present the solution interpolating from the Kerr black hole to its subtracted geometry in appendix 2.6.3.

2.1.3 Discussion of the five-dimensional uplift

The microscopic interpretation of the subtracted geometry is clearest in the five-dimensional picture, since its uplift is $AdS_3 \times S^2$, which is holographically described by a CFT$_2$. In this subsection we will consider the five-dimensional uplift of a slightly generalized version of the subtracted geometries, namely the Harrison-transformed black holes with $\alpha_I = 1$ and α_0 arbitrary. Interestingly, all these backgrounds uplift to $AdS_3 \times S^2$, irrespective of the value of α_0.

The uplift Ansatz is given by

$$ds_5^2 = f^2(dz + A^0)^2 + f^{-1} ds_4^2 \,, \qquad f = (y^1 y^2 y^3)^{\frac{1}{3}} \tag{2.1.28}$$

where ds_4^2 is given in terms of the three-dimensional fields by (2.1.3) and A^0 by (2.4.1). Plugging in the solution discussed above we obtain

$$ds_5^2 = ds_3^2 + \ell^2 \left[d\theta^2 + \sin^2\theta \left(d\phi - \frac{a e^{-\delta_0 - \delta_1 - \delta_2 - \delta_3}}{4m^2} (dt + (\alpha_0 - 1)dz) \right)^2 \right] \tag{2.1.29}$$

where

$$\ell = 2m\, e^{\frac{2}{3}(\delta_1 + \delta_2 + \delta_3)} \tag{2.1.30}$$

is the radius of the S^2. The three-dimensional part of the metric, ds_3^2, is AdS_3 of radius 2ℓ in an unusual coordinate system

$$ds_3^2 = \frac{\ell^2 dr^2}{r^2 - 2mr + a^2} + \frac{e^{-\frac{2}{3}(\delta_1+\delta_2+\delta_3)}}{4m^2}\Big[-(a^2 e^{-2\delta_0} + 2mr - 4m^2 c_0^2)\, d\tilde{t}^2 +$$

$$+ \quad 2(a^2 e^{-2\delta_0} + 4m^2 c_0 s_0)\, d\tilde{t}dz + (2mr - a^2 e^{-2\delta_0} + 4m^2 s_0^2)\, dz^2\Big] \qquad (2.1.31)$$

The entire α_0 dependence is encoded in the new coordinate $\tilde{t}$

$$\tilde{t} = t + \alpha_0 z \qquad (2.1.32)$$

Thus, the effect of the h_0 Harrison transformation, which is non-trivial from a four-dimensional perspective, corresponds to a simple coordinate transformation in five dimensions[6]. In appendix 2.5.1 we show that the effect of the α_0 Harrison transformation on the five-dimensional uplift of *any* four-dimensional STU geometry with a timelike isometry is that of the coordinate transformation $t \to t + \alpha_0 z$.

The above five-dimensional geometry is supported by magnetic flux through the S^2, given by

$$A^I = e^{2\delta_I}\left(\frac{a}{2m}(dt + (\alpha_0 - 1)dz)\, e^{-\delta_0-\delta_1-\delta_2-\delta_3} - 2m d\phi\right)\cos\theta \qquad (2.1.33)$$

The associated magnetic charges are

$$p^I = 2m e^{2\delta_I} \qquad (2.1.34)$$

Note that they are different from the original charges (2.1.6). The Brown-Henneaux asymp-

[6]When $\alpha_0 = 1$, the AdS_3 factor can be written as a $U(1)$ Hopf fibre over AdS_2, where the Hopf fibre coordinate is the fifth dimension z. Also, the z component of the Kaluza-Klein gauge field in (2.1.29) vanishes. Thus, for $\alpha_0 = 1$, the four-dimensional geometry itself becomes $AdS_2 \times S^2$. This is in agreement with the well-known result that when all α_A are equal, the resulting Harrison transformation acts within Einstein-Maxwell gravity only, and that in the infinite boost limit it transforms the Schwarzschild metric to the Robinson-Bertotti one. This type of transformation was recently employed in [39].

totic symmetry group analysis [4] applied to the AdS_3 factor (2.1.31) yields a central charge

$$c = \frac{3(2\ell)}{2G_3} = \frac{12\pi\ell^3}{G_5} = \frac{48m^3}{G_4}\, e^{2(\delta_1+\delta_2+\delta_3)} \tag{2.1.35}$$

It is easy to check that $c = 6\, p^1 p^2 p^3$, as expected.

Let us now understand the five-dimensional uplift of the subtracted geometry itself. As explained, in order to get precisely the subtracted geometry one needs to perform the additional rescaling transformations H_0, H_I, with coefficients given by (2.1.27). From a five-dimensional point of view, these transformations simply multiply the metric by an overall factor

$$ds_5'^2 = e^{\frac{2}{3}(c_1+c_2+c_3)-2c_0} ds_5^2 \tag{2.1.36}$$

provided that we replace t and z by the rescaled coordinates

$$t' = e^{2c_0} t\ , \qquad z' = e^{c_0-c_1-c_2-c_3} z \tag{2.1.37}$$

Under the above rescaling, the radius of the AdS_3 becomes $\ell_{AdS_3} = 2\sqrt{2m}$. The associated Brown-Henneaux central charge is then

$$c = \frac{6(2m)^{\frac{3}{2}}}{G_4} \tag{2.1.38}$$

which only depends on the mass parameter. The action of the rescalings on the magnetic fields is

$$A^I \to e^{-c_0+c_1+c_2+c_3-2c_I} A^I \tag{2.1.39}$$

which implies that all magnetic charges are now equal $p^1 = p^2 = p^3 = \sqrt{2m}$. One can easily perform a coordinate transformation to put the metric (2.1.31) into BTZ form[7]

$$\frac{ds^2}{\ell^2} = T_-^2 du^2 + T_+^2 dv^2 + 2\rho\, du dv + \frac{d\rho^2}{4(\rho^2 - T_+^2 T_-^2)} \tag{2.1.40}$$

where we have defined

$$u = \frac{\sqrt{m^2 - a^2}}{8m^2 T_-}(-t' + (1+\alpha_0)z')\, e^{-\sum_I \delta_I + \delta_0}\ , \qquad v = \frac{1}{8mT_+}(t' + (1-\alpha_0)z')\, e^{-\sum_I \delta_I - \delta_0} \tag{2.1.41}$$

$$r = m + \frac{\sqrt{m^2 - a^2}}{T_+ T_-}\,\rho \tag{2.1.42}$$

Requiring that u, v be identified mod 2π as $z \to z + 2\pi$ and plugging in the values (2.1.26), (2.1.27) for α_0, c_A fixes the temperatures to

$$T_+ = \frac{(\Pi_c - \Pi_s)\sqrt{m}}{2\sqrt{2}}\ , \qquad T_- = \frac{(\Pi_c + \Pi_s)\sqrt{m^2 - a^2}}{2\sqrt{2m}} \tag{2.1.43}$$

It is then trivial to check that the Cardy formula in the dual CFT

$$S_{Cardy} = \frac{\pi}{3}\, c\, (T_+ + T_-) \tag{2.1.44}$$

with c given by (2.1.38), reproduces the Bekenstein-Hawking entropy of the general rotating black hole

$$S_{BH} = \frac{2\pi m}{G_4}[(\Pi_c - \Pi_s)(m + \sqrt{m^2 - a^2}) + 2m\Pi_s] \tag{2.1.45}$$

The central charge (2.1.38) does not agree with the Kerr/CFT central charge $c = 12J$ in

[7] The parameters $T_\pm$ are related to the dimensionless left/right moving temperatures in the dual CFT as $T_+ = \pi T_L$, $T_- = \pi T_R$. This redefinition slightly changes the form of Cardy's entropy formula (2.1.44).

the extremal limit. This could be explained by the fact that we are using different "frames" for computing the entropy. Nevertheless, we can bring the central charge to any desired value while keeping the entropy invariant by performing any rescaling transformation with $c_0 = 0$. Under it, the central charge transforms as

$$c \to c\, e^{c_1+c_2+c_3} \tag{2.1.46}$$

while the temperatures transform in the opposite way, thus leaving (2.1.44) unchanged. We further discuss these rescalings in the next subsection.

Finally, let us comment on the relationship with [27]. In that paper, the author applies the three maximal h_I Harrison transformations (followed by certain rescalings) to a black hole with arbitrary magnetic charges δ_I, but with electric charge given by $\tilde{\delta}_0$, where

$$\sinh \tilde{\delta}_0 = \frac{\Pi_s}{\sqrt{\Pi_c^2 - \Pi_s^2}} \tag{2.1.47}$$

rather than δ_0. Also, he does not use the h_0 Harrison transformation at all to reach the subtracted geometry.

Of course, one can reinterpret this procedure as starting with a general black hole with charge parameters δ_0, δ_I, to which one applies the h_I Harrison transformations with $\alpha_I = 1$, and then performs a charging transformation with parameter $\tilde{\delta}_0 - \delta_0$, followed by certain rescalings. It is not hard to check that the q_0 charging transformation simply corresponds to a boost in five dimensions. Thus, Virmani's procedure to obtain the subtracted geometry and ours simply differ by a five-dimensional coordinate transformation and some rescalings. Note that in both cases, the parameters of the transformations only depend on α_0, Π_s and Π_c.

2.1.4 Duality interpretation

The uplift of the subtracted geometry is $AdS_3 \times S^2$, supported by magnetic fluxes. This is the near-horizon geometry of three intersecting M5-branes in M-theory [40, 41], each of which wraps a different four-cycle on a six-torus T^6

	w^1	w^2	w^3	w^4	w^5	w^6	z
M5			-	-	-	-	-
M5	-	-			-	-	-
M5	-	-	-	-			-
p							-

Here z denotes the M-theory direction. The number of branes of each type is given by the flux of the corresponding gauge field through the S^2. Before the rescalings, we have

$$p^1 = 2me^{2\delta_1} \ , \qquad p^2 = 2me^{2\delta_2} \ , \qquad p^3 = 2me^{2\delta_3} \tag{2.1.48}$$

whereas after the scaling transformations we have $p^1 = p^2 = p^3 = \sqrt{2m}$. The dual CFT (known as the MSW CFT), whose central charge $c = 6p^1p^2p^3$ has been microscopically derived in [42], describes the low-energy excitations of the M5-brane worldvolume theory.

The above $AdS_3 \times S^2$ geometry has been obtained by applying to the five-dimensional uplift of a general non-extremal rotating four-dimensional black hole a set of h_I Harrison transformations with $\alpha_I = 1$, followed by an α_0 Harrison with parameter (2.1.26) and a rescaling. We will analyze the string/M-theory duality interpretation of each of these transformations, in reverse order.

The rescalings

The action of the rescalings (2.1.18) on the five-dimensional geometry is given by (2.1.36), (2.1.37) and (2.1.39). Since they change the radius of AdS_3, the M5 magnetic fluxes and the periodicity of the M-theory circle parametrized by z, these transformations do not act within the same theory. Rather, they take us from a given MSW CFT to another, of different central charge and defined on a circle of a different radius.

These transformations also do not generally leave the entropy invariant. On the AdS_3 length and temperatures they act as

$$\ell \to e^{\frac{1}{3}(c_1+c_2+c_3)-c_0}\,\ell\,, \qquad T_\pm \to e^{c_0-(c_1+c_2+c_3)}\,T_\pm \tag{2.1.49}$$

Since the central charge $c \propto \ell^3$, they leave invariant Cardy's formula (2.1.44) only if $c_0 = 0$. It is interesting to note that the only case in which the rescalings are not needed in order to match the entropy is that of the neutral Kerr black hole, which is also the one of most phenomenological interest.

The α_0 transformation

In appendix 2.5.1, we show that the α_0 Harrison transformation always corresponds to a coordinate transformation in M-theory, mixing the AdS_3 boundary coordinates as

$$\begin{pmatrix} z \\ t \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ \alpha_0 & 1 \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} \tag{2.1.50}$$

Note that the above diffeomorphism is not part of the Brown-Henneaux asymptotic symmetry group, because it mixes the left- and right-moving coordinates $u = z - t$ and $v = z + t$. Thus, this transformation changes the metric on the AdS_3 boundary, and therefore it corresponds to turning on a source for the dual stress tensor. For α_0 infinitesimal, we have

$$S_{CFT} \to S_{CFT} - \alpha_0 \int dt dz\, T^{zt} \tag{2.1.51}$$

This would be the entire story if the theory was defined on the plane. Nevertheless, in our case the M-theory circle is identified as $z \sim z + 2\pi$, so the theory is defined on the cylinder. The transformation (2.1.50) *does not* preserve the cylinder, and thus it is not a symmetry of the theory. In particular, it changes the entropy of the black holes. It would be interesting to precisely understand the holographic dual of this coordinate reidentification.

The α_I transformation

As we have discussed, the formula for the conformal factor Δ is completely symmetric under the interchange of the α_A. In the above, we have shown that the h_0 Harrison transformation corresponds to uplifting to M-theory and performing a specific coordinate transformation. It is then natural to ask whether the remaining α_I can also be interpreted as coordinate transformations in the appropriate frame.

That the answer should be yes is rather clear from the work of [31]. Those authors showed that a general *static* black hole can be "untwisted" to AdS_3 by going to the duality frame in which each of its charges becomes momentum and then performing an $SL(2,\mathbb{R})$ transformation in the (t, z) directions.

The black holes that we are considering carry D0 and D4 charges associated to various four-cycles in the compactification T^6. We have already observed that the h_0 Harrison transformation corresponds to uplifting to M-theory (which turns the D0 charge into momentum) and then performing the "shift" $SL(2,\mathbb{R})$ transformation (2.1.50). The remaining three Harrison transformations should then be identified with combinations of four T-dualities (which turn a given D4 into D0, and thus M-theory momentum), the shift transformation, reduction to type IIA, and four T-dualities back. In appendix 2.5.2 we show that, indeed, these combination of T-dualities and coordinate transformations has the same effect on certain scalars as the corresponding Harrison transformation.

Thus, we have succeeded in extending the results of [31] to general rotating black holes. While we only considered Harrison transformations represented by matrices of the form

(2.1.50), [31] also considered more general $SL(2,\mathbb{R})$ transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad ad - bc = 1 \tag{2.1.52}$$

The entries were further constrained by a condition essentially equivalent to reducing the degree of divergence of Δ. It was found that for the specific choice

$$a = \cosh^{-1}\delta\ , \quad b = 0\ , \quad c = -e^{-\delta}\ , \quad d = \cosh\delta \tag{2.1.53}$$

the entropy of the black hole is also preserved. As we have already discussed, all transformations with $c \neq 0$ do not preserve the cylinder that the theory is defined on, so they generically change the entropy, as we saw explicitly in the previous section. It is thus very interesting that - at least in the static case - there exists a choice of $SL(2,\mathbb{R})$ transformations that leave the entropy invariant. It would be instructive to check whether this choice persists in the general rotating case.

2.2 Un-twisting $5d$ black holes

2.2.1 Setup

Let us now turn to the analysis of five-dimensional black holes. We consider the non-extremal rotating generalization of the D1-D5-p black hole, first presented in [11]. These black holes are solutions of $\mathcal{N} = 2$ $5d$ supergravity coupled to two vector multiplets. The metric can again be written as a timelike fibre over a four-dimensional base space[8]

$$ds_5^2 = -\Delta^{-\frac{2}{3}}\tilde{G}(dt + \mathcal{A}^t)^2 + \Delta^{\frac{1}{3}} d\hat{s}_4^2 \tag{2.2.1}$$

[8] In this section we completely reset the notation of the previous one. Thus, the quantities $\Delta, G, \Pi_s, \Pi_c, \ell$ etc. have different interpretation from before. There is no simple relationship between the four-dimensional black holes studied in the previous section and the five-dimensional ones we study now.

The four-dimensional base space is spanned by the spatial coordinates $\{r, \theta, \phi, \psi\}$, and its metric is given by (2.7.3). As before, the base metric does not depend on the charges, but only on the mass and rotation parameters[9]. The remaining quantities are

$$\Delta = f^3 H_0 H_1 H_5 \ , \qquad \tilde{G} = f(f - 2m) \tag{2.2.2}$$

where

$$H_i = 1 + \frac{2m \sinh^2 \delta_i}{f} \ , \qquad f = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \tag{2.2.3}$$

As before, the parameters δ_i encode the electric charges of the black hole. As $r \to \infty$, $\Delta \propto r^6$, and the solutions are asymptotically flat. The main observation of [13] was that if one changes the conformal factor Δ as

$$\Delta \to \Delta_{sub} = (2m)^2 f(\Pi_c^2 - \Pi_s^2) + (2m)^3 \Pi_s^2 \tag{2.2.4}$$

while keeping $\mathcal{A}^t$ and $d\hat{s}_4^2$ fixed, the wave equation of a massless scalar propagating in the black hole geometry has exact local $SL(2R) \times SL(2, \mathbb{R})$ symmetry and the black hole thermodynamics is unchanged. In the five-dimensional case, the definition of Π_c and Π_s has changed to

$$\Pi_c = c_0 c_1 c_5 \ , \qquad \Pi_s = s_0 s_1 s_5 \tag{2.2.5}$$

Moreover, [13] showed that the five-dimensional subtracted geometry uplifts to $AdS_3 \times S^3$, thus geometrically realizing the hidden conformal symmetry visible in five dimensions.

In this section we will show that the "subtraction" procedure can again be implemented using combinations of string dualities and coordinate transformations. As before, these

[9] This suggests that one should be able to generate the general solutions of [11] by only reducing to four dimensions along time, rather than along both time and ψ as it was originally done.

transformations act naturally in one dimension higher, in this case six dimensions. Thus, we uplift the metric to a six-dimensional black string [43] using

$$ds_6^2 = G_{yy}(dy + A^3)^2 + G_{yy}^{-\frac{1}{3}} ds_5^2 \,, \qquad G_{yy} = \frac{H_0}{\sqrt{H_1 H_5}} \tag{2.2.6}$$

where the Kaluza-Klein gauge field A^3 can be found e.g. in [15]. This black string is a solution of a very simple six-dimensional theory, namely

$$S = \int d^6x \sqrt{g} \left(R + (\partial\phi)^2 - \frac{1}{12} F_{(3)}^2 \right) \tag{2.2.7}$$

which contains a three-form gauge field and a dilaton in addition to the metric. This theory is a consistent truncation of the type IIB supergravity action on T^4 with only Ramond-Ramond three-form field.

Given that the uplifts of both the original and the subtracted geometry are solutions of the theory (2.2.7) that share the same base metric $d\hat{s}_4^2$, it is natural that they be related by a symmetry that the six-dimensional action acquires upon reduction to four dimensions along $\{y, t\}$. The symmetries of the resulting four-dimensional action are nothing but the STU $SL(2, \mathbb{R})^3$ symmetries. The action of STU transformations directly on the six-dimensional geometry has been worked out in [34]. In the following subsection we will briefly review these transformations and show that they indeed connect the uplifts of the original and subtracted five-dimensional geometries.

2.2.2 Subtraction via STU

STU transformations are the symmetries of the $\mathcal{N} = 2$ four-dimensional supergravity theory with prepotential (2.1.1). This theory can be understood as the dimensional reduction of the six-dimensional action (2.2.7) on a two-torus. From the six-dimensional perspective, the STU transformations relate solutions of (2.2.7) which can be written as T^2 fibrations over the same four-dimensional base. We parametrize the metric as

$$ds_6^2 = ds_4^2 + G_{\alpha\beta}(dy^\alpha + \mathcal{A}^\alpha)(dy^\beta + \mathcal{A}^\beta) \ , \qquad y^\alpha = \{y, t\} \tag{2.2.8}$$

The six-dimensional $C^{(2)}$ field can be similarly decomposed as

$$C^{(2)}_{\alpha\beta} = \zeta\hat{\epsilon}_{\alpha\beta} \ , \qquad C^{(2)}_{\mu\alpha} = \mathcal{B}_{\mu\alpha} - C_{\alpha\beta}\mathcal{A}^\beta$$

$$C^{(2)}_{\mu\nu} = \mathcal{C}_{\mu\nu} - \mathcal{A}^\alpha_{[\mu}\mathcal{B}_{\nu]\alpha} + \mathcal{A}^\alpha_\mu C_{\alpha\beta}\mathcal{A}^\beta_\nu \tag{2.2.9}$$

and there is additionally the dilaton ϕ. We will be interested in the general rotating black string solution of [43]. We give the expressions for the four-dimensional fields $\mathcal{A}^\alpha$, $\mathcal{B}_\alpha$, $G_{\alpha\beta}$, ds_4^2, ζ, ϕ that characterize this solution in appendix 2.7.1.

Let us now briefly review the interpretation of the STU transformations in the type IIB frame, which is discussed at length in [34]. The last one, U, simply corresponds to a coordinate transformation in six dimensions, of the type

$$\mathcal{U} : \begin{pmatrix} y \\ t \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ t \end{pmatrix} \tag{2.2.10}$$

where $ad - bc = 1$.

The T transformation corresponds to a type IIB S-duality, followed by a T-duality along y, then by a coordinate transformation as above, a T-duality back on the new y coordinate, and finally an S-duality back. At least when $a = d = 1$ and $b = 0$, it was shown in [34] that it can alternatively be interpreted as

- a T-duality along y
- a timelike Melvin twist with $t \to t + c\, x^{11}$

- a T-duality back.

The S transformation is the same as the T transformation, both preceded and followed by four T-dualities on the internal T^4, whose role is to implement $6d$ electromagnetic duality on the initial and final geometries.

The T transformation

The first transformation that we will apply to the black string solution (2.7.2) - (2.7.5) is a T-type transformation, given by the $SL(2,\mathbb{R})$ matrix

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ \lambda_1 & 1 \end{pmatrix} \tag{2.2.11}$$

This transformation acts on (2.2.8) as

$$ds_6^2 \to \sqrt{\Sigma_1}\, ds_4^2 + \frac{G_{\alpha\beta}}{\sqrt{\Sigma_1}}\,(dy^\alpha + \mathcal{A}^\alpha + \lambda_1 \hat{\epsilon}^{\alpha\gamma}\mathcal{B}_\gamma)(dy^\beta + \mathcal{A}^\beta + \lambda_1 \hat{\epsilon}^{\beta\gamma}\mathcal{B}_\gamma) \tag{2.2.12}$$

where $\hat{\epsilon}_{\alpha\beta}$ is the ϵ symbol ($\hat{\epsilon}_{yt} = 1$) and Σ_1 is given by

$$\Sigma_1 = (1 + \lambda_1\zeta)^2 + \lambda_1^2 e^{-2\phi} \det G_{\alpha\beta} \tag{2.2.13}$$

The scalars ζ, ϕ and the determinant $\det G_{\alpha\beta}$ are inputs of the original geometry, which read

$$\det G_{\alpha\beta} = -\frac{1 - 2mf^{-1}}{H_1 H_5}\,, \qquad e^{2\phi} = \frac{H_1}{H_5}\,, \qquad \zeta = \frac{2ms_1c_1}{fH_1} \tag{2.2.14}$$

Plugging in, we find that Σ_1 takes the form

$$\Sigma_1 = \frac{4m^2 s_1^2 (s_1 + c_1\lambda_1)^2 + f^2\left(1 - \lambda_1^2\right) + 2fm\left(2s_1^2 + 2c_1s_1\lambda_1 + \lambda_1^2\right)}{\left(f + 2ms_1^2\right)^2} \tag{2.2.15}$$

where the function f is given in (2.2.3). Whenever $\lambda_1 \neq \pm 1$, the quantity Σ_1 asymptotically approaches a constant. Nevertheless, when $\lambda_1 = \pm 1$, then $\Sigma_1 \sim \mathcal{O}(r^{-2})$. This fact has a direct consequence on the asymptotic behaviour of the conformal factor Δ, which under T transforms as

$$\Delta \to \Delta_1 = \Sigma_1 \Delta \tag{2.2.16}$$

Thus, for $\lambda_1 = 1$, we can reduce the degree of divergence of Δ from r^6 to r^4. For precisely this value, Σ_1 is

$$\Sigma_1|_{\lambda_1=1} = \frac{2me^{2\delta_1}}{fH_1} \quad \Rightarrow \quad \Delta_1 = 2me^{2\delta_1} f^2 H_0 H_5 \tag{2.2.17}$$

and Δ_1 remains polynomial in r. The details of the above manipulations are given in appendix 2.7.2.

The S transformation

We can also act with the S transformation, whose action on the metric is very similar to (2.2.12)

$$ds_6^2 \to \sqrt{\Sigma_2}\, ds_4^2 + \frac{G_{\alpha\beta}}{\sqrt{\Sigma_2}} (dy^\alpha + \mathcal{A}^\alpha + \lambda_2 \hat{\epsilon}^{\alpha\gamma} \mathcal{B}'_\gamma)(dy^\beta + \mathcal{A}^\beta + \lambda_2 \hat{\epsilon}^{\beta\gamma} \mathcal{B}'_\gamma) \tag{2.2.18}$$

The four-dimensional gauge field $\mathcal{B}'_\alpha$ is - roughly speaking - the Hodge dual of $\mathcal{B}_\alpha$. The quantity Σ_2 is given by

$$\Sigma_2 = (1 + \lambda_2 \zeta')^2 + \lambda_2^2\, e^{2\phi} \det G_{\alpha\beta} \tag{2.2.19}$$

where the scalar ζ' is (roughly) the four-dimensional Hodge dual of the two form $\mathcal{C}_{\mu\nu}$. On the original black string background,

$$\zeta' = \frac{2ms_5c_5}{fH_5} \tag{2.2.20}$$

With these, we can compute Σ_2 explicitly. It is given by

$$\Sigma_2 = \frac{4m^2s_5^2(s_5 + c_5\lambda_2)^2 + f^2\left(1 - \lambda_2^2\right) + 2fm\left(2s_5^2 + 2c_5s_5\lambda_2 + \lambda_2^2\right)}{\left(f + 2ms_5^2\right)^2} \tag{2.2.21}$$

Note that, again, for $\lambda_2 = 1$ the asymptotics of Σ_2 change from $\mathcal{O}(1)$ to $\mathcal{O}(r^{-2})$. The intermediate steps of this calculation can be found in appendix 2.7.3.

To summarize, the combined effect of the S and T transformations on Δ is

$$\Delta \to \Sigma_1\Sigma_2\Delta \tag{2.2.22}$$

When $\lambda_1 = \lambda_2 = 1$, the final value of Δ is

$$\Delta_{fin} = 4m^2fH_0\, e^{2\delta_1+2\delta_5} \tag{2.2.23}$$

This has the same large r asymptotics as Δ_{sub} in (2.2.4), but it is not equal to it. Just like it is true of the subtracted geometries of the previous section, the uplifted black hole metric after an S and a T transformation becomes locally $AdS_3 \times S^3$. Consequently, there exists a coordinate transformation and a rescaling that takes it into the uplift of the subtracted geometry. We describe this transformation in the next subsection.

2.2.3 The final geometry

Setting $\lambda_1 = \lambda_2 = 1$, we find that the final metric is locally $AdS_3 \times S^3$

$$ds_6^2 = ds_3^2 + \ell^2\left[d\theta^2 + \sin^2\theta(d\phi + A^\phi)^2 + \cos^2\theta(d\psi + A^\psi)^2\right] \tag{2.2.24}$$

where

$$\ell^2 = 2me^{\delta_1+\delta_5} \tag{2.2.25}$$

The three-dimensional Kaluza-Klein gauge fields are constant and read

$$A^\phi = -(ad\tilde{t} + bd\tilde{y}) \ , \qquad A^\psi = -(ad\tilde{y} + bd\tilde{t}) \tag{2.2.26}$$

and the three-dimensional metric is

$$ds_3^2 = \ell^2 \left[r^2(d\tilde{y}^2 - d\tilde{t}^2) - (a^2 + b^2 - 2m)d\tilde{t}^2 - 2abd\tilde{t}d\tilde{y} + \frac{r^2dr^2}{(r^2+a^2)(r^2+b^2) - 2mr^2} \right] \tag{2.2.27}$$

The new coordinates $\tilde{t}$ and $\tilde{y}$ are related to t, y via

$$\tilde{t} = \ell^{-2}(c_0 t - s_0 y) \ , \qquad \tilde{y} = \ell^{-2}(c_0 y - s_0 t) \tag{2.2.28}$$

Thus, the δ_0 dependence of the six-dimensional metric can be trivially undone via the above coordinate transformation. The geometry (2.2.24) differs from the uplift of the subtracted geometry in two aspects: one needs to replace δ_0 by a new $\tilde{\delta}_0$ and ℓ by $\tilde{\ell}$, given by

$$\sinh \tilde{\delta}_0 = \frac{\Pi_s}{\sqrt{\Pi_c^2 - \Pi_s^2}} \ , \qquad \tilde{\ell}^2 = 2m\sqrt{\Pi_c^2 - \Pi_s^2} \tag{2.2.29}$$

This replacement amounts thus to a coordinate transformation and an overall rescaling. The metric can again be put in the form (2.1.40), by defining

$$\rho = r^2 - m + \frac{1}{2}(a^2 + b^2) \ , \qquad u = y - t \ , \qquad v = y + t \tag{2.2.30}$$

The temperatures that we can read off are

$$T_\pm = \frac{\sqrt{2m-(a\pm b)^2}}{2\hat{\ell}^2}\, e^{\mp\tilde{\delta}_0} \tag{2.2.31}$$

Pugging into Cardy's formula (2.1.44), we again get perfect match with the Bekenstein-Hawking entropy of the five-dimensional black hole which, in units of $G_5 = \pi/4$, reads

$$S = 2\pi m\sqrt{2m-(b-a)^2}\,(\Pi_c+\Pi_s) + 2\pi m\sqrt{2m-(b+a)^2}\,(\Pi_c-\Pi_s) \tag{2.2.32}$$

Note that the coordinate transformation $\delta_0 \to \tilde{\delta}_0$ and the rescaling $\ell \to \tilde{\ell}$ were absolutely necessary in order to match the entropy in general. The only case in which these transformations are not needed is the neutral case $\delta_i = 0$, for which just the S and T transformations are enough to produce the subtracted geometry.

2.3 Conclusion

In this chapter, we have shown that all non-extremal four- and five-dimensional black holes with general rotation and charges can be "untwisted" to AdS_3 times a sphere, thus generalizing the work of [31]. While it is possible that the untwisting may be done in several different ways [31] - i.e. by using different choices of $SL(2,\mathbb{R})$ matrices - our particular choice is universal (it does not depend on any of the black hole parameters) and has a very simple duality interpretation. Moreover, the powerful solution generating techniques that we use allow us to easily construct solutions that interpolate between the original black holes and their subtracted geometries, generalizing the work of [17].

2.4 Appendix A: Useful formulae

The* $3d \to 4d \to 5d$ *lift

Here we describe the relationship of the four-dimensional fields that appear in the STU Lagrangian to the three-dimensional fields and dualized scalars, as well as their uplift to five dimensions.

The four-dimensional gauge fields can be reduced to three dimensions via

$$A^{\Lambda}_{4d} = \zeta^{\Lambda}(dt + \omega_3) + A^{\Lambda}_3 \tag{2.4.1}$$

Next, the three-dimensional gauge fields are dualized into scalars via

$$-\,d\tilde{\zeta}_{\Lambda} = e^{2U}(\mathrm{Im}N)_{\Lambda\Sigma} \star_3 (dA_3{}^{\Sigma} + \zeta^{\Sigma} d\omega_3) + (\mathrm{Re}N)_{\Lambda\Sigma} d\zeta^{\Sigma}$$

$$-\,d\sigma = 2e^{4U} \star_3 d\omega_3 - \zeta^{\Lambda} d\tilde{\zeta}_{\Lambda} + \tilde{\zeta}_{\Lambda} d\zeta^{\Lambda} \tag{2.4.2}$$

The relationship between the five-dimensional gauge fields and the four-dimensional ones is

$$A^I_{5d} = -x^I(dz + A^0_{4d}) + A^I_{4d} \tag{2.4.3}$$

The real scalars in the five-dimensional $\mathcal{N} = 2$ Lagrangian are given by

$$h^I = f^{-1} y^I \,, \qquad f^3 = y^1 y^2 y^3 \tag{2.4.4}$$

and the uplift of the metric is given in (4.6.2).

The* $5d \to 6d$ *lift

Here we describe the relationship between the five-dimensional black hole geometries and the six-dimensional black string ones that we use in section 2.2. The reduction from six-dimensional Einstein frame to five dimensions is

$$ds_6^2 = G_{yy}(dy + A_{5d}^3)^2 + G_{yy}^{-\frac{1}{3}} ds_5^2 \,, \quad G_{yy} = h_3^{-\frac{3}{2}} \tag{2.4.5}$$

In terms of the four-dimensional fields that we have introduced in (2.2.8), we have

$$A_{5d}^3 = \mathcal{A}^y + \frac{G_{yt}}{G_{yy}}(dt + \mathcal{A}^t) \tag{2.4.6}$$

and

$$ds_{5E}^2 = G_{yy}^{\frac{1}{3}} ds_4^2 + \frac{\det G_{\alpha\beta}}{G_{yy}^{\frac{2}{3}}}(dt + \mathcal{A}^t)^2 \tag{2.4.7}$$

Comparing this expression with (2.2.1), we find that

$$\Delta = G_{yy} \left(\frac{f(f-2m)}{|\det G|} \right)^{\frac{3}{2}} \tag{2.4.8}$$

which is the equation we used to derive (2.2.22).

2.5 Appendix B: The Harrison transformations as dualities

2.5.1 The α_0 transformation

The action of the α_0 Harrison transformation on the various three-dimensional fields in the theory can be read off from the transformation of the matrix $\mathcal{M}$ and reads

$$e^{4U} \to \Xi_0^{-1} e^{4U} \,, \quad y^I \to \Xi_0^{\frac{1}{2}} y^I \,, \quad \Xi_0 = (1 + \alpha_0 \zeta^0)^2 - \alpha_0^2 f^{-3} e^{2U}$$

$$\zeta^\Lambda \to \frac{\zeta^\Lambda (1 + \alpha_0 \zeta^0) - \alpha_0 \, x^\Lambda f^{-3} e^{2U}}{\Xi_0} \,, \quad x^I \to x^I (1 + \alpha_0 \zeta^0) - \alpha_0 \zeta^I \tag{2.5.1}$$

where we have introduced $x^0 = 1$. The transformation rules for $\tilde{\zeta}_\Lambda$ and σ are rather cumbersome; instead, we can use (2.4.2) to compute the transformation of the Hodge dual fields ω_3, A_3^Λ, which behave simply as

$$\omega_3 \to \omega_3 - \alpha_0 A_3^0 \,, \qquad A_3^\Lambda \to A_3^\Lambda \tag{2.5.2}$$

We would like to understand the effect of the α_0 Harrison on the five-dimensional uplifted geometry. In terms of three-dimensional fields, the five-dimensional metric reads

$$ds^2 = f^2(dz + \zeta_0(dt + \omega_3) + A_3{}^0)^2 - f^{-1}e^{2U}(dt + \omega_3)^2 + e^{-2U}f^{-1}ds_3{}^2 \tag{2.5.3}$$

and the accompanying supporting gauge fields are

$$A^I = -x^I(dz + \zeta^0(dt + \omega_3) + A_3^0) + \zeta^I(dt + \omega_3) + A_3^I \tag{2.5.4}$$

Upon re-completing the squares in the required order, it is rather easy to see that the above transformations are induced by a simple change of coordinates

$$t \to t + \alpha_0 z \tag{2.5.5}$$

in the five-dimensional background (2.5.3).

2.5.2 The α_I transformations

We will concentrate for concreteness on α_1, which acts as

$$e^{4U} \to \Xi_1^{-1} e^{4U} \,, \qquad y^1 \to \Xi_1^{\frac{1}{2}}\, y^1 \,, \qquad x^1 \to x^1(1 - \alpha_1 \tilde{\zeta}_1) - \alpha_1 \tilde{\zeta}_0$$

$$\Xi_1 = (1 - \alpha_1 \tilde{\zeta}_1)^2 - \alpha_1^2 f^{-3} e^{2U} (x_2^2 + y_2^2)(x_3^2 + y_3^2) \tag{2.5.6}$$

Other fields that transform simply are

$$\begin{aligned}
\tilde{\zeta}_0 &\rightarrow \frac{\tilde{\zeta}_0(1 - \alpha_1 \tilde{\zeta}_1) - \alpha_1 x^1 e^{2U} f^{-3} (x_2^2 + y_2^2)(x_3^2 + y_3^2)}{\Xi_1} \\
\tilde{\zeta}_1 &\rightarrow \frac{\tilde{\zeta}_1(1 - \alpha_1 \tilde{\zeta}_1) + \alpha_1 e^{2U} f^{-3} (x_2^2 + y_2^2)(x_3^2 + y_3^2)}{\Xi_1}
\end{aligned}$$

$$\begin{aligned}
\zeta^2 &\rightarrow \frac{\zeta^2(1 - \alpha_1 \tilde{\zeta}_1) + \alpha_1 x^3 e^{2U} f^{-3} (x_2^2 + y_2^2)}{\Xi_1} \\
\zeta^3 &\rightarrow \frac{\zeta^3(1 - \alpha_1 \tilde{\zeta}_1) + \alpha_1 x^2 e^{2U} f^{-3} (x_3^2 + y_3^2)}{\Xi_1}
\end{aligned} \tag{2.5.7}$$

The transformation rules for the remaining fields are rather complicated, and we will not include them here. The claim is that the above transformations are equivalent to four T-dualities along the $w^{3,4,5,6}$ directions, a coordinate transformation as in (2.5.5), followed by four T-dualities back.

Uplifting to ten dimensions, the type IIA string frame metric is

$$ds_{10}^2 = ds_4^2 + y^1(dw_1^2 + dw_2^2) + y^2(dw_3^2 + dw_4^2) + y^3(dw_5^2 + dw_6^2) \tag{2.5.8}$$

and the NS-NS B-field reads

$$B^{(2)} = -x^1 dw^1 \wedge dw^2 - x^2 dw^3 \wedge dw^4 - x^3 dw^5 \wedge dw^6 \tag{2.5.9}$$

Under four T-dualities along $w^{3,4,5,6}$, the fields transform as

$$y^2 \to \frac{y^2}{y_2^2 + x_2^2} , \qquad x^2 \to \frac{x^2}{y_2^2 + x_2^2} \tag{2.5.10}$$

and similarly for x^3, y^3. The action of the four T-dualities on the Ramond-Ramond fields is roughly to interchange A^0_{4d} with (minus) the Hodge dual of A^1, and A^2 with $-A^3$. At the level of the three-dimensional scalars, we expect these exchanges to act as

$$\zeta^0 \to -\tilde{\zeta}_1 , \quad \zeta^1 \to \tilde{\zeta}_0 , \quad \zeta^2 \to -\zeta^3 \tag{2.5.11}$$

while U and ω_3 stay invariant.

Back to the general formulae, it is easy to check that combining the replacements (2.5.10)-(2.5.11) with the coordinate transformation (2.5.1)-(2.5.2), we obtain precisely the α_1 Harrison transformation formulae (2.5.6)-(2.5.7). Thus, for the subset of fields that we checked explicitly, this interpretation is correct.

2.6 Appendix C: Explicit "four-dimensional" examples

In this appendix we present explicit formulae for various four-dimensional black holes and interpolating geometries. The quotes above are due to the fact that we present the four-dimensional geometries either in terms of the three-dimensional scalar data, or in the form of the five-dimensional uplift.

In appendix 2.6.1 we present the scalar fields that yield the geometry of the general four-charge rotating black holes with three magnetic and one electric charge. To our best knowledge, the complete solution for all fields has not been published in the literature[10]. In appendix 2.6.2 we rederive the solution presented in [17] and relate our notation to theirs. Finally, in appendix 2.6.3 we present the five-dimensional uplift of the interpolating solution from the Kerr asymptotically flat black hole to its subtracted geometry. Since the formulae are rather cumbersome to write down, we present only the special cases $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$

[10] The general solution with three magnetic charges can be found in [27], whereas the explicit solution with two electric and two magnetic charges has been written down in [35], minus the gauge potentials.

and $\alpha_2 = \alpha_3 = 1$, both with α_0 set to its subtracted value $\alpha_0 = 0$.

2.6.1 The general four-charge black hole

We write herein the general four-dimensional asymptotically flat solution. The scalar field U is given implicitly in (2.1.10). Note that, despite the way it is presented, the expression for Δ is completely symmetric under interchanging the charges. The other scalar fields are given by

$$y^1 = \frac{\sqrt{\Delta}}{a^2\cos^2\theta + (r+2ms_2^2)(r+2ms_3^2)} \tag{2.6.1}$$

$$x^1 = \frac{2am\cos\theta(c_0c_1s_2s_3 - s_0s_1c_2c_3)}{a^2\cos^2\theta + (r+2ms_2^2)(r+2ms_3^2)} \tag{2.6.2}$$

The formulae for the remaining x^i, y^i are obtained by permutations of the above. The next simplest scalar is

$$\tilde{\zeta}_0 = \frac{2am\cos\theta}{\Delta}\left[s_0c_1c_2c_3(a^2\cos^2\theta + r(r+2ms_0^2)) - c_0s_1s_2s_3(a^2\cos^2\theta + (r-2m)(r+2ms_0^2))\right] \tag{2.6.3}$$

The formulae for the ζ^i are simply obtained from the above by replacing $\delta_0 \leftrightarrow \delta_i$. Next, we have

$$\begin{aligned}\zeta^0 \quad = \quad & \frac{1}{\Delta}\big[4m^2a^2\cos^2\theta\left((c_0^2+s_0^2)s_1c_1s_2c_2s_3c_3 - s_0c_0(2s_1^2s_2^2s_3^2 + s_1^2s_2^2 + s_2^2s_3^2 + s_3^2s_1^2)\right) + \\ & +2ms_0c_0\left(ra^2\cos^2\theta + \Pi_{i=1}^3(r+2ms_i^2)\right)\big]\end{aligned} \tag{2.6.4}$$

The expressions for the $\tilde{\zeta}_i$ are given by *minus* the above expression, after replacing $\delta_0 \leftrightarrow \delta_i$. It may be useful to also note that $\tilde{\zeta}_{2,3}$ can also be written as

$$\tilde{\zeta}_2 = \tilde{\zeta}_2^\star + x^1\zeta^3\,, \qquad \tilde{\zeta}_2^\star = -\frac{2ms_2c_2(r+2ms_3^2)}{a^2\cos^2\theta + (r+2ms_2^2)(r+2ms_3^2)} \tag{2.6.5}$$

and similarly for $\tilde{\zeta}_3$, with the obvious replacements. Finally, the expression for σ is given by

$$\sigma = \frac{4am\cos\theta(\Pi_c - \Pi_s)}{a^2\cos^2\theta + (r+2ms_2^2)(r+2ms_3^2)} - (\zeta^0\tilde{\zeta}_0 - \zeta^1\tilde{\zeta}_1 + 2x^1\zeta^0\tilde{\zeta}_1 + \zeta^2\tilde{\zeta}_2^\star + \zeta^3\tilde{\zeta}_3^\star) \tag{2.6.6}$$

The fields σ and $\tilde{\zeta}_\Lambda$ should be dualized to the one-forms ω_3 and A_3^Λ, which upon uplift yield the four-dimensional matter fields. It should be possible to check that ω_3 has the simple expression (2.1.7).

2.6.2 The static charged interpolating solution

The solution in the non-rotating charged case has been already given in [17]. We include a re-derivation of it in our notation. After four Harrison transformations with parameters α_A, the scale factor Δ takes the form

$$\Delta = \xi_0\,\xi_1\,\xi_2\,\xi_3 \tag{2.6.7}$$

where

$$\xi_A = (1-\alpha_A^2)\,r + \frac{1}{2}m\,e^{2\delta_A}\Big(1+\alpha_A + e^{-2\delta_A}(\alpha_A - 1)\Big)^2\,, \quad A\in\{0,\dots,3\} \tag{2.6.8}$$

The relationship between the parameters α_A and the ones - called a_A - used to parametrize the interpolating solutions in [17] is

$$a_A = \frac{\sqrt{1-\alpha_A^2}}{\sinh\delta_A + \alpha_A\cosh\delta_A} \tag{2.6.9}$$

Note that the values of the parameters a_A which correspond to the subtracted geometry in [17] match precisely with the values quoted in (2.1.26). One small advantage of our parametrization is that - unlike that of [17] - it is not singular when one of the charges vanishes.

The uplifted five-dimensional geometry takes the form

$$ds_5^2 = (\xi_1\xi_2\xi_3)^{\frac{2}{3}} \left(d\Omega_2^2 + \frac{dr^2}{G} \right) + (\xi_1\xi_2\xi_3)^{-\frac{1}{3}} ds_2^2 \tag{2.6.10}$$

where

$$ds_2^2 = \xi_0 (dz + A^0)^2 - \frac{G}{\xi_0} dt^2 \tag{2.6.11}$$

and the Kaluza-Klein gauge field reads

$$A^0 = \xi_0^{-1} \left(-\alpha_0 r + \frac{1}{2} m (1 + e^{2\delta_0}) \left(1 + \alpha_0 + e^{-2\delta_0}(\alpha_0 - 1) \right) \right) dt \tag{2.6.12}$$

As before, the α_0 dependence of the above metric, which is present only in the last parenthesis, ds_2^2, can be completely gauged away via the coordinate transformation (2.5.5). The $3d$ Einstein metric reads

$$ds_3^2 = (\xi_1\xi_2\xi_3)^2 \left(\frac{dr^2}{r(r-2m)} + \frac{ds_2^2}{\xi_1\xi_2\xi_3} \right) \tag{2.6.13}$$

When $\alpha_I = 0$, this spacetime is AdS_3 of radius $\ell = 4me^{\frac{2}{3}(\delta_1+\delta_2+\delta_3)}$. When at least two α_I are non-zero, including the asymptotically flat case, it is asymptotically conformal to AdS_3. It would be interesting if holography could be understood for this spacetime.

We also list the remaining five-dimensional fields, for completeness. We have

$$h^I = \frac{\xi_I}{(\xi_1\xi_2\xi_3)^{\frac{1}{3}}} \ , \qquad A^I_{(5d)} = -\frac{1}{2} m \left[(1+\alpha_I)^2 e^{2\delta_I} - (1-\alpha_I)^2 e^{-2\delta_I}\right] \cos\theta d\phi \tag{2.6.14}$$

2.6.3 The neutral rotating interpolating solution

While it is straightforward to generate the geometries that interpolate between the general charged rotating black holes and their subtracted geometry, the resulting formulae are rather uninspiring. Thus, we will limit ourselves to presenting only the simplest such rotating solution, for the neutral Kerr black hole. Introducing the notation

$$\epsilon_A = 1 - \alpha_A^2 \tag{2.6.15}$$

and

$$\Pi_4 = \epsilon_0\epsilon_1\epsilon_2\epsilon_3 \ , \qquad \Pi_3 = \epsilon_0\epsilon_1\epsilon_2 + \text{perms} \tag{2.6.16}$$

$$\Pi_2 = \epsilon_0\epsilon_1 + \text{ perms} \ , \qquad \Pi_1 = \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 \tag{2.6.17}$$

we find that the resulting warp factor is

$$\begin{aligned}
\Delta \ = \ & \Pi_4\, r^4 + (2m\Pi_3 - 8m\Pi_4)\, r^3 + [4m^2\Pi_2 - 12m^2\Pi_3 + (24m^2 + 2a^2\cos^2\theta)\Pi_4]\, r^2 + \\
+ \ & [8m^3\Pi_1 - 16m^3\Pi_2 + (24m^3 + 2a^2 m\cos^2\theta)\Pi_3 - (32m^3 + 8a^2 m\cos^2\theta)\Pi_4] r + \\
+ \ & 4m^2(a^2\cos^2\theta - 4m^2)\Pi_1 + 16m^4\Pi_2 - 4m^2(4m^2 + a^2\cos^2\theta)\Pi_3 + (a^2\cos^2\theta + 4m^2)^2\Pi_4 + \\
+ \ & 16m^4 + 8a^2m^2\alpha_0\alpha_1\alpha_2\alpha_3\cos^2\theta
\end{aligned} \tag{2.6.18}$$

Note that it has all the properties that we have mentioned in section 2.1.2. Turning on $\epsilon_{1,2,3}$ corresponds to turning on certain irrelevant deformations of the subtracted geometry.

Since the solution for the remaining fields is still rather cumbersome, we will be focusing on two special cases:

- equal deformations: $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1 - \alpha^2$
- one nonzero deformation: $\epsilon_1 = 1 - \alpha_1^2$ and $\epsilon_2 = \epsilon_3 = 0$

Since for neutral black holes we do not need to perform a h_0 Harrison transformation in order to reach the subtracted geometry, in both cases we will set $\alpha_0 = 0$.

Equal deformations

In this subsection we present the uplifted five-dimensional geometry after a deformation with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and $\alpha_0 = 0$. As a useful intermediate step, we write the three-dimensional one-forms

$$\omega_3 = \frac{2amr\sin^2\theta}{G}\,d\phi\,, \qquad A_3^0 = \frac{2am(r-2m)\alpha^3\sin^2\theta}{G}\,d\phi \tag{2.6.19}$$

$$A_3^1 = A_3^2 = A_3^3 = -\frac{2mX\alpha\cos\theta}{G}\,d\phi \tag{2.6.20}$$

To write down the final interpolating solution, it is useful to introduce some shortcuts. Thus, we let

$$\rho = (1-\alpha^2)\,r + 2m\alpha^2\,, \qquad \epsilon = 1-\alpha^2\,, \qquad Y = \rho^2 + a^2\epsilon^2\cos^2\theta \tag{2.6.21}$$

but we still don't replace α by ϵ when it appears with an odd power. The five-dimensional gauge fields then read

$$A_{5d}^1 = -\frac{2m\alpha}{Y}[a(\alpha dz - dt) + (\rho^2 + a^2\epsilon^2)d\phi]\,\cos\theta \tag{2.6.22}$$

We were unable to find much structure in the above solution, but it would be interesting if it existed. To reduce to three dimensions, we write the metric as

$$ds_5^2 = e^{-2U-2V} ds_3^2 + 4m^2 e^{2U} d\theta^2 + 4m^2 e^{2V} \sin^2\theta (d\phi + \hat{A})^2 \tag{2.6.23}$$

and find that, to first order in ϵ,

$$e^{2U} = e^{2V} = 1 + \frac{(r-2m)\epsilon}{m} + \dots \tag{2.6.24}$$

$$\hat{A}_t = -\frac{a}{4m^2} + \frac{3a(r-2m)\epsilon}{8m^3} + \dots\,, \qquad \hat{A}_z = \frac{a}{4m^2} + \frac{3a(m-r)\epsilon}{8m^3} + \dots \tag{2.6.25}$$

The r-dependence of the three-dimensional vector field indicates the presence of a $(1,2)$ operator (in addition to the $(2,2)$ ones found in [17]), whose coupling is proportional to the rotation parameter a. Note that unlike in the static case, beyond the leading order in ϵ, U and V will no longer be equal. The deformation of the three-dimensional Einstein metric reads

$$g_{tt} = -\frac{2mr - 4m^2 + a^2}{4m^2} - \frac{3(r-2m)^2\epsilon}{4m^2}\,, \qquad g_{zz} = \frac{2mr - a^2}{4m^2} + \frac{3(r^2 - 2mr + a^2)\epsilon}{4m^2}$$

$$g_{tz} = \frac{a^2}{4m^2} - \frac{3a^2\epsilon}{8m^2}\,, \qquad g_{rr} = \frac{4m^2}{X} + \frac{12m(r-2m)\epsilon}{X} \tag{2.6.26}$$

There are also additional massive vector fields coming from the dimensional reduction of the five-dimensional gauge field (2.6.22).

Single deformation

To study the effect of a single deformation, we set $\alpha_2 = \alpha_3 = 1$. The angular velocity ω_3 stays the same, whereas the gauge fields change to

$$A^0_3 = \frac{2am(r-2m)\alpha_1 \sin^2\theta}{G} \,, \qquad A^1_3 = -\frac{2mX\alpha_1\cos\theta}{G} \,, \qquad A^2_3 = A^3_3 = -\frac{2mX\cos\theta}{G} \tag{2.6.27}$$

The five-dimensional gauge fields read

$$A^1_{5d} = -\left(2m\alpha_1 d\phi + \frac{a(dz - \alpha_1 dt)}{2m}\right)\cos\theta \tag{2.6.28}$$

$$A^2_{5d} = A^3_{5d} = \left(-2md\phi + \frac{a(dt - \alpha_1 dz)}{\rho_1}\right)\cos\theta \tag{2.6.29}$$

where we have again introduced the shorthand

$$\rho_1 = (1-\alpha_1^2)\, r + 2m\alpha_1^2 \tag{2.6.30}$$

Note that the magnetic flux through the sphere is decreased. The metric takes the form (2.6.23) with $U = V$, where

$$e^{2U} = \left(\frac{\rho_1}{2m}\right)^{\frac{2}{3}} \,, \qquad \hat{A} = \frac{a(\alpha_1 dz - dt)}{2m\rho_1} \tag{2.6.31}$$

$$g_{tt} = -\frac{r^2(1-\alpha_1^2) + 2mr(2\alpha_1^2 - 1) + a^2 - 4m^2\alpha_1^2}{4m^2} \,, \qquad g_{tz} = \frac{a^2\alpha_1}{4m^2}$$

$$g_{zz} = \frac{r^2(1-\alpha_1^2) + 2mr\alpha_1^2 - a^2\alpha_1^2}{4m^2} \,, \qquad g_{rr} = \frac{\rho_1^2}{X} \tag{2.6.32}$$

The scalars that support the geometry are

$$h^1 = \left(\frac{\rho_1}{2m}\right)^{\frac{2}{3}} \quad , \qquad h^2 = h^3 = \left(\frac{2m}{\rho_1}\right)^{\frac{1}{3}} \tag{2.6.33}$$

It would be interesting if one could construct a consistent truncation of the five-dimensional action to three-dimensions, that contains this solution and then perform a detailed holographic analysis.

2.7 Appendix D: Details of the spectral flows

2.7.1 The general black string solution

The Einstein-frame metric of the general six-dimensional black string solution [43] is given by

$$ds_6^2 = ds_4^2 + G_{\alpha\beta}(dy^\alpha + \mathcal{A}^\alpha)(dy^\beta + \mathcal{A}^b) \tag{2.7.1}$$

where $y^\alpha = \{y, t\}$ and

$$\begin{aligned} ds_4^2 \;=\; & \sqrt{H_1 H_5}\left[\left(a^2 + r^2 + \frac{2a^2 m \sin^2\theta}{f-2m}\right)\sin^2\theta d\phi^2 + \left(b^2 + r^2 + \frac{2b^2 m\cos^2\theta}{f-2m}\right)\cos^2\theta d\psi^2 + \right. \\ & \left. + \frac{2abm\sin^2\theta\cos^2\theta}{f-2m} 2d\phi d\psi + \frac{fr^2 dr^2}{(r^2+a^2)(r^2+b^2) - 2mr^2} + f d\theta^2\right] \end{aligned} \tag{2.7.2}$$

The four-dimensional base metric is related to $d\hat{s}_4^2$ that appears in (2.2.1) by the rescaling

$$ds_4^2 = f\sqrt{H_1 H_5}\, d\hat{s}_4^2 \tag{2.7.3}$$

The Kaluza-Klein gauge fields read

$$\mathcal{A}^y = 2m\left(a\frac{s_0c_1c_5}{f-2m} - b\frac{c_0s_1s_5}{f}\right)\sin^2\theta d\phi + 2m\left(b\frac{s_0c_1c_5}{f-2m} - a\frac{c_0s_1s_5}{f}\right)\cos^2\theta d\psi$$

$$\mathcal{A}^t = 2m\left(a\frac{c_0c_1c_5}{f-2m} - b\frac{s_0s_1s_5}{f}\right)\sin^2\theta d\phi + 2m\left(b\frac{c_0c_1c_5}{f-2m} - a\frac{s_0s_1s_5}{f}\right)\cos^2\theta d\psi \tag{2.7.4}$$

$$G_{\alpha\beta} = \frac{1}{\sqrt{H_1H_5}}\begin{pmatrix} H_0 & -\frac{m\sinh 2\delta_0}{f} \\ -\frac{m\sinh 2\delta_0}{f} & \frac{2m\cosh^2\delta_0}{f} - 1\end{pmatrix}, \qquad \det G_{\alpha\beta} = -\frac{1-2mf^{-1}}{H_1H_5} \tag{2.7.5}$$

We have defined

$$H_i = 1 + \frac{2m\sinh^2\delta_i}{f}, \qquad f = r^2 + a^2\cos^2\theta + b^2\sin^2\theta \tag{2.7.6}$$

The solution is also supported by the ten-dimensional dilaton, which in RR frame reads

$$e^{2\phi} = \frac{H_1}{H_5} \tag{2.7.7}$$

and by the Ramond-Ramond two-form field, which can be found in [48]. Using (2.2.9), we also decompose the six-dimensional $C^{(2)}$ field to four dimensions, obtaining

$$\zeta = \frac{2ms_1c_1}{fH_1}, \qquad \mathcal{C} = ms_5c_5\cos^2\theta\left(\frac{a^2+r^2}{f} + \frac{a^2+r^2-2m}{f-2m}\right)d\phi\wedge d\psi \tag{2.7.8}$$

$$\mathcal{B}_y = 2m\left(a\frac{c_0s_1c_5}{f-2m} - b\frac{s_0c_1s_5}{f}\right)\sin^2\theta d\phi + 2m\left(b\frac{c_0s_1c_5}{f-2m} - a\frac{s_0c_1s_5}{f}\right)\cos^2\theta d\psi$$

$$\mathcal{B}_t = 2m\left(-a\frac{s_0 s_1 c_5}{f-2m} + b\frac{c_0 c_1 s_5}{f}\right)\sin^2\theta d\phi + 2m\left(-b\frac{s_0 s_1 c_5}{f-2m} + a\frac{c_0 c_1 s_5}{f}\right)\cos^2\theta d\psi \quad (2.7.9)$$

For our future manipulations, it is useful to introduce the scalar ζ', defined as

$$d\zeta' = v^2\sqrt{|\det G|}\star_4 H^{(3)}\ , \qquad H^{(3)} = d\,\mathcal{C} - \frac{1}{2}\mathcal{A}^\alpha\wedge d\mathcal{B}_\alpha - \frac{1}{2}\mathcal{B}_\alpha\wedge d\mathcal{A}^\alpha \qquad (2.7.10)$$

where $v^2 = e^{2\phi}$ is the volume of the internal four-torus. We simply find

$$\zeta' = \frac{2m s_5 c_5}{f H_5} \qquad (2.7.11)$$

2.7.2 The T transformation

The action of the T transformation on the four-dimensional fields is

$$e^{2\phi_1} = e^{2\phi}\Sigma_1\ , \qquad \zeta_1 = \frac{\zeta + \lambda_1(\zeta^2 + e^{-2\phi}\det G)}{\Sigma_1}\ , \qquad \mathcal{A}_1^\alpha = \mathcal{A}^\alpha + \lambda_1\hat{\epsilon}^{\alpha\beta}\mathcal{B}_\beta \qquad (2.7.12)$$

where Σ_1 is given by

$$\Sigma_1 = (1+\lambda_1\zeta)^2 + \lambda_1^2 e^{-2\phi}\det G \qquad (2.7.13)$$

Note that in Lorentzian signature, the transformation of $\mathcal{A}^\alpha$ differs by a sign from its spacelike counterpart, due to the different definition of $\hat{\epsilon}^{\alpha\beta}$. The fields $\mathcal{B}_\alpha$ and ζ' are unchanged, and

$$G^1_{\alpha\beta} = \frac{G_{\alpha\beta}}{\sqrt{\Sigma_1}}\ , \qquad ds_4^{1\,2} = \sqrt{\Sigma_1}\,ds_4^2 \qquad (2.7.14)$$

2.7.3 The S transformation

Let us now perform the S transformation. Its action on the four-dimensional fields is

$$e^{2\phi_2} = \Sigma_2^{-1} e^{2\phi_1} = \frac{\Sigma_1}{\Sigma_2}\, e^{2\phi} \ , \qquad \zeta_2 = \zeta_1 \ , \qquad \mathcal{A}_2^{\alpha} = \mathcal{A}_1^{\alpha} + \lambda_2 \hat{\epsilon}^{\alpha\beta} \mathcal{B}'_{\beta} \tag{2.7.15}$$

The factor Σ_2 is given by

$$\Sigma_2 = (1+\lambda_2 \zeta')^2 + \lambda_2^2\, e^{2\phi_1} \det G_1 = (1+\lambda_2\zeta')^2 + \lambda_2^2\, e^{2\phi} \det G \tag{2.7.16}$$

The four-dimensional gauge field $\mathcal{B}'_{\alpha}$ is determined by

$$d\mathcal{B}'_{\alpha} = e^{2\phi_1} \epsilon_{\alpha}{}^{\beta} \star_4 d\mathcal{B}_{\beta} + \zeta' \hat{\epsilon}_{\alpha\beta}\, d\mathcal{A}_1^{\beta} - \frac{e^{2\phi_1}\zeta_1}{\sqrt{\det G}} G_{\alpha\beta} \star_4 d\mathcal{A}_1^{\beta} \tag{2.7.17}$$

Note the sign difference with respect to the Euclidean signature formulae in [34]. To solve for $\mathcal{B}'_{\alpha}$, it is useful to rewrite the above expression in terms of the fields before the T transformation

$$\begin{aligned} d\mathcal{B}'_{\alpha} \;&=\; e^{2\phi}\, \frac{G_{\alpha\beta}}{\sqrt{\det G}} \left(\hat{\epsilon}^{\beta\gamma} \star_4 d\mathcal{B}_{\gamma} - \zeta \star_4 d\mathcal{A}^{\beta} \right) + \zeta'\, \hat{\epsilon}_{\alpha\beta}\, d\mathcal{A}^{b} + \\ &+\; \lambda_1 \left[e^{2\phi}\, \frac{G_{\alpha\beta}}{\sqrt{\det G}} \left(\zeta\, \hat{\epsilon}^{\beta\gamma} \star_4 d\mathcal{B}_{\gamma} - (\zeta^2 + e^{-2\phi} \det G) \star_4 d\mathcal{A}^{\beta} \right) + \zeta'\, d\mathcal{B}_{\alpha} \right] \end{aligned} \tag{2.7.18}$$

Writing

$$\mathcal{B}'_{\alpha} = \mathcal{B}'^{(0)}_{\alpha} + \lambda_1 \mathcal{B}'^{(1)}_{\alpha} \tag{2.7.19}$$

and integrating the above equation, we find

$$
\begin{aligned}
B_y^{\prime(0)} &= 2m\left(\frac{ac_0c_1s_5}{f-2m}-\frac{bs_0s_1c_5}{f}\right)\sin^2\theta d\phi - 2m\left(\frac{as_0s_1c_5}{f}-\frac{bc_0c_1s_5}{f-2m}\right)\cos^2\theta d\psi \\
B_t^{\prime(0)} &= -2m\left(\frac{as_0c_1s_5}{f-2m}-\frac{bc_0s_1c_5}{f}\right)\sin^2\theta d\phi + 2m\left(\frac{ac_0s_1c_5}{f}-\frac{bs_0c_1s_5}{f-2m}\right)\cos^2\theta d\psi
\end{aligned}
\tag{2.7.20}
$$

and

$$
\begin{aligned}
B_y^{\prime(1)} &= 2m\left(\frac{ac_0s_1s_5}{f-2m}-\frac{bs_0c_1c_5}{f}\right)\sin^2\theta d\phi - 2m\left(\frac{as_0c_1c_5}{f}-\frac{bc_0s_1s_5}{f-2m}\right)\cos^2\theta d\psi \\
B_t^{\prime(1)} &= -2m\left(\frac{as_0s_1s_5}{f-2m}-\frac{bc_0c_1c_5}{f}\right)\sin^2\theta d\phi + 2m\left(\frac{ac_0c_1c_5}{f}-\frac{bs_0s_1s_5}{f-2m}\right)\cos^2\theta d\psi
\end{aligned}
\tag{2.7.21}
$$

Finally, the metric becomes

$$
G^2_{\alpha\beta} = \frac{G^1_{\alpha\beta}}{\sqrt{\Sigma_2}} = \frac{G_{\alpha\beta}}{\sqrt{\Sigma_1\Sigma_2}}\ , \qquad ds^{2\,2}_4 = \sqrt{\Sigma_2}\, ds^{1\,2}_4 = \sqrt{\Sigma_1\Sigma_2}\, ds^2_4 \tag{2.7.22}
$$

Chapter 3

Electrodynamics: Melvin solutions

External magnetic fields can probe the composite structure of black holes in string theory. With this motivation we study magnetised four-charge black holes in the STU model, a consistent truncation of maximally supersymmetric supergravity with four types of electromagnetic fields. We employ solution generating techniques to obtain Melvin backgrounds,and black holes in these backgrounds. For an initially electrically charged static black hole immersed in magnetic fields, we calculate the resultant angular momenta and analyse their global structure. Examples are given for which the ergoregion does not extend to infinity. We calculate magnetic moments and gyromagnetic ratios via L armors formula. Our results are consistent with earlier special cases. A scaling limit and associated subtracted geometry in a single surviving magnetic field is shown to lift to $AdS_3 \times S_2$. Magnetizing magnetically charged black holes give static solutions with conical singularities representing strings or struts holding the black holes against magnetic forces. In some cases it is possible to balance these magnetic forces.

3.1 The STU Model and its Black Holes

The Lagrangian for the bosonic sector of the STU model, in the notation of [35], is

$$\begin{aligned} \mathcal{L}_4 \quad = \quad & R\,{*}\mathbb{1} - 12{*}d\varphi_i \wedge d\varphi_i - 12e^{2\varphi_i}\,{*}d\chi_i \wedge d\chi_i - 12e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3}\,{*}F_{(2)1} \wedge F_{(2)1} \\ & + e^{\varphi_2+\varphi_3}\,{*}F_{(2)2} \wedge F_{(2)2} + e^{-\varphi_2+\varphi_3}\,{*}\mathcal{F}^1_{(2)} \wedge \mathcal{F}^1_{(2)} + e^{-\varphi_2-\varphi_3}\,{*}\mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)}\Big) \\ & + \chi_1\left(F_{(2)1} \wedge \mathcal{F}^1_{(2)} + F_{(2)2} \wedge \mathcal{F}^2_{(2)}\right), \end{aligned} \tag{3.1.1}$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \le i \le 3$. The four field strengths can be written in terms of potentials as

$$\begin{aligned}
F_{(2)1} &= dA_{(1)1} - \chi_2\, d\mathcal{A}^2_{(1)}\,, \\
F_{(2)2} &= dA_{(1)2} + \chi_2\, d\mathcal{A}^1_{(1)} - \chi_3\, dA_{(1)1} + \chi_2\,\chi_3\, d\mathcal{A}^2_{(1)}\,, \\
\mathcal{F}^1_{(2)} &= d\mathcal{A}^1_{(1)} + \chi_3\, d\mathcal{A}^2_{(1)}\,, \\
\mathcal{F}^2_{(2)} &= d\mathcal{A}^2_{(1)}\,.
\end{aligned} \tag{3.1.2}$$

Note that (4.4.1) could be obtained by reducing the six-dimensional bosonic string action on $S^1 \times S^1$, and then dualising the 2-form potential $A_{(2)}$ to the axion that is called χ_1 here.

Four-charge rotating black hole solutions in the STU theory were constructed in [10]. We shall use the conventions and notation of [35], in which the metric for the four-charge black holes is given by

$$ds_4^2 = -\frac{\rho^2 - 2mr}{W}\,(dt + \mathcal{B}_{(1)})^2 + W\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\,\sin^2\theta\,d\phi^2}{\rho^2 - 2mr}\right). \tag{3.1.3}$$

where

$$\begin{aligned}
\Delta &= r^2 - 2mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta\,, \\
\mathcal{B}_{(1)} &= \frac{2ma\sin^2\theta(r\Pi_c - (r-2m)\Pi_s)}{(\rho^2 - 2mr)}\,d\phi\,, \\
W^2 &= r_1\, r_2\, r_3\, r_4 + a^4\cos^4\theta \\
&\quad + a^2\cos^2\theta\,[2r^2 + 2mr\sum_{i=1}^4 s_i^2 + 8m^2\Pi_s\,\Pi_c - 4m^2(2\Pi_s^2 + \sum_{i=1}^4 \Pi_s^i)]\,,
\end{aligned} \tag{3.1.4}$$

$r_i = r + 2ms_i^2$, $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$, and $\Pi_c = c_1c_2c_3c_4$ and $\Pi_s = s_1s_2s_3s_4$. We also define

$$\Pi_s^i = s_i^{-1}\,\Pi_s\,, \qquad \Pi_c^i = c_i^{-1}\,\Pi_c\,. \tag{3.1.5}$$

The expressions for the gauge potentials, axions and dilatons can be found [35].

The mass physical M, angular momentum J, charges Q_i and dipole moments μ_i were calculated in [10]. In the notation and conventions of [35] that we are using here, they are given by

$$
\begin{aligned}
M &= 14m\sum_{i=1}^{4}(c_i^2+s_i^2)\,, \qquad J = ma\,(\Pi_c-\Pi_s)\,, \\
Q_i &= 2ms_i\,c_i\,, \qquad \mu_i = 2ma\,(s_i\,\Pi_c^i - c_i\,\Pi_s^i)\,.
\end{aligned} \tag{3.1.6}
$$

In standard Maxwell electrodynamics, the magnetic moment of a particle of mass M and angular momentum J carrying a charge Q is given by $\mu = \mathbf{g}JQ/(2M)$, where $\mathbf{g}$ is the gyromagnetic ratio. Generically, for the four-charge black holes in the STU model, we can expect a relation of the form

$$
\mu_i = \frac{J}{2M}\sum_{j=1}^{4}\mathbf{g}_{ij}\,Q_j\,. \tag{3.1.7}
$$

From the quantities (7.0.14) given above it is not possible, in the absence of additional criteria, to derive a unique form for the "gyromagnetic matrix" $\mathbf{g}_{ij}$. However, if we impose the additional requirements that it be a symmetric matrix, and furthermore that it exhibit the same symmetries as the metric under permutation of the four charge parameters δ_i, then we are led to the following result:

$$
\begin{aligned}
i=j: \qquad & \mathbf{g}_{ii} = \frac{1}{2c_i^2}\,\sum_{k=1}^{4}(c_k^2+s_k^2)\,, \\
i\neq j: \qquad & \mathbf{g}_{ij} = -\frac{\Pi_s}{6c_ic_js_is_j}\,\frac{\sum_{k=1}^{4}(c_k^2+s_k^2)}{\Pi_c-\Pi_s}\,.
\end{aligned} \tag{3.1.8}
$$

In special cases the expression (3.1.8) for the gyromagnetic ratio reduces to previously-known results. For example, if we consider the single-charge case where $\delta_2=\delta_3=\delta_4=0$ then we obtain the "Kaluza-Klein" result [58, 59]

$$
\mathbf{g} = \mathbf{g}_{11} = 2-\tanh^2\delta_1\,. \tag{3.1.9}
$$

If two or more of the charges are non-zero, the gyromagnetic matrix has off-diagonal components. If we take all four charges to be equal, then

$$\mathbf{g}_{ij} = \frac{2(c^2+s^2)}{c^2}\,,\ i=j\,, \qquad \mathbf{g}_{ij} = -\frac{2s^2}{3c^2}\,,\ i\neq j\,, \tag{3.1.10}$$

and so with $Q_i = Q$ we have $\mathbf{g}_{ij}Q_j = 2Q$, implying the standard result [60] that $\mathbf{g} = 2$ for the Kerr-Newman black hole.

In the case of two non-zero equal charges, say, $Q_1 = Q_2 = Q$ and $Q_3 = Q_4 = 0$, we obtain the following nonzero gyromagnetic matrix coefficients:

$$\mathbf{g}_{11} = \mathbf{g}_{22} = 2\,,\ \mathbf{g}_{33} = \mathbf{g}_{44} = 2\,c^2\,,\ \mathbf{g}_{34} = \mathbf{g}_{43} = -\frac{2}{3}s^2\,. \tag{3.1.11}$$

Thus, $\mathbf{g}_{1j}Q_j = \mathbf{g}_{2j}Q_j = 2Q$ which implies $\mathbf{g} = 2$ for Q.

In the case of three non-zero equal charges, say, $Q_1 = Q_2 = Q_3 = Q$ and $Q_4 = 0$, we get the following nonzero gyromagnetic matrix coefficients:

$$\begin{aligned} &\mathbf{g}_{11} = \mathbf{g}_{22} = \mathbf{g}_{33} = 2+\tanh^2\delta\,,\ \mathbf{g}_{44} = 3c^2-1\,, \\ &\mathbf{g}_{i4} = \mathbf{g}_{4i} = -\frac{1}{3}\tanh^2\delta(2+\tanh^2\delta)\,,\ i=1,2,3\,. \end{aligned} \tag{3.1.12}$$

In this case $\mathbf{g}_{ij}Q_j = (2+\tanh^2\delta)Q$ for $i = 1,2,3$, and thus $\mathbf{g} = 2+\tanh^2\delta$. Furthermore, even though $Q_4 = 0$, a nonzero μ_4 is induced, since $\mathbf{g}_{4j}Q_j = -\tanh^2\delta(2+\tanh^2\delta)$ and thus $\mathbf{g}_4 = -\tanh^2\delta(2+\tanh^2\delta)$.

Another explicit example can be obtained with pair-wise equal charges, say, $Q_1 = Q_3$ and $Q_2 = Q_3$. In this case the pair-wise equal magnetic moments $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$ are related to the pair-wise equal charges as:

$$\mu_I = \frac{J}{2M}\sum_{J=1}^{2}\mathbf{G}_{IJ}Q_J\,, \quad I = 1,2\,, \tag{3.1.13}$$

where the coefficients of the gyromagnetic matrix $\mathbf{G}$ are

$$\mathbf{G}_{11} = \frac{2(3c_1^2 - 2 + 2c_2^2)}{3c_1^2}\,, \;\mathbf{G}_{22} = \frac{2(3c_1^2 - 2 + 2c_2^2)}{3c_2^2}\,, \;\mathbf{G}_{12} = \mathbf{G}_{21} = -\frac{4}{3}\frac{s_1 s_2}{c_1 c_2} \tag{3.1.14}$$

The matrix $\mathbf{G}$ has eigenvalues 2 and $2 + \frac{4}{3}(\tanh^2\delta_1 + \tanh^2\delta_2)$.

3.2 Pure Melvin-type Solution in the STU Model

Later in the chapter, we shall be constructing solutions in the STU model describing four-charge black holes immersed in external magnetic fields. These solutions will, under appropriate circumstances, be asymptotic to the STU model generalisations of the Melvin universe of Einstein-Maxwell theory. It is useful, therefore, first to consider the simpler case of these pure Melvin-type solutions, where there is no black hole but just the external magnetic fields. (To be precise, as explained in the introduction, when we use the expression "external magnetic fields" we mean that the fields numbered 1 and 3 carry external electric fields, while those numbered 2 and 4 carry external magnetic fields.) The STU model in the conventions we are using is given in appendix A. Melvin-type solutions can be found using the results presented in appendix A, starting from a purely Minkowski seed solution. They can also be read off from the expressions for magnetised black holes presented in section 3, by setting the black hole mass and charges to zero. Thus the metric is given by (4.5.1) and (4.5.5) with $\omega = 0$ and

$$\Delta = \prod_{i=1}^{4}\Delta_i\,, \qquad \Delta_i = 1 + \beta_i^2\, r^2\, \sin^2\theta\,, \tag{3.2.1}$$

and so

$$ds_4^2 = \sqrt{\Delta}\,(-dt^2 + dr^2 + r^2\, d\theta^2) + \frac{1}{\sqrt{\Delta}}\, r^2\, \sin^2\theta\, d\phi^2\,. \tag{3.2.2}$$

Note that here, and throughout the rest of the chapter, we use the notation that

$$\beta_i = 12 B_i\,, \tag{3.2.3}$$

where B_i is the physical asymptotic strength of the i'th field on the symmetry axis at large distance. This is done in order to avoid many cumbersome factors of 12 and powers of 12 in subsequent formulae. In the pure Melvin case under discussion here, where there is no black hole, the field strengths are in fact constant along the axis.

The scalar fields are given by

$$e^{2\varphi_1} = \frac{\Delta_1\Delta_3}{\Delta_2\Delta_4}\,, \qquad e^{2\varphi_2} = \frac{\Delta_2\Delta_3}{\Delta_1\Delta_4}\,, \qquad e^{2\varphi_3} = \frac{\Delta_1\Delta_2}{\Delta_3\Delta_4}\,, \tag{3.2.4}$$

with the axions all vanishing. The four electromagnetic potentials $\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$ are given by

$$\begin{aligned} A_{(1)1} &= -2\beta_1\, r\, \cos\theta\, dt\,, \qquad \mathcal{A}^1_{(1)} = -2\beta_3\, r\, \cos\theta\, dt\,, \\ A_{(1)2} &= \frac{\beta_2\, r^2\, \sin^2\theta}{\Delta_2} d\phi\,, \qquad \mathcal{A}^2_{(1)} = \frac{\beta_4\, r^2\, \sin^2\theta}{\Delta_4} d\phi\,. \end{aligned} \tag{3.2.5}$$

In terms of cylindrical coordinates (ρ, z) defined by $\rho = r\sin\theta$ and $z = r\cos\theta$, we have

$$ds_4^2 = \sqrt{\Delta}\,(-dt^2 + d\rho^2 + dz^2) + \frac{\rho^2}{\sqrt{\Delta}}\, d\phi^2 \tag{3.2.6}$$

with Δ_i in (3.2.1) now given by $\Delta_i = 1 + \beta_i^2\, \rho^2$. Making the further coordinate transformations to $x = \rho\cos\phi$ and $y = \rho\sin\phi$, the metric near the axis approaches Minkowski spacetime $ds_4^2 \to -dt^2 + dx^2 + dy^2 + dz^2$, and near the axis the field strengths approach

$$F_{(2)1} \to B_1\, dt\wedge dz\,, \quad F_{(2)2} \to B_2\, dx\wedge dy\,, \quad \mathcal{F}^1_{(2)} \to B_3\, dt\wedge dz\,, \quad \mathcal{F}^2_{(2)} \to B_4\, dx\wedge dy\,. \tag{3.2.7}$$

Thus, as mentioned above, the electric and magnetic field strengths have magnitude B_i on the axis for all values of z, in this pure Melvin case.

It is interesting to note that the 4-field Melvin solution can be obtained instead by means of a limiting procedure and analytic continuation from the four-charge static black hole solution in the STU model, generalising the procedure described in [56] for the Melvin solution in the Einstein-Maxwell theory. The four-charge black hole metric, which can be read off from the magnetised black holes in section 3 by sending the magnetic fields B_i to zero, is given by

$$ds^2 = -\frac{r(r-2m)}{\sqrt{r_1 r_2 r_3 r_4}}\, dt^2 + \sqrt{r_1 r_2 r_3 r_4}\Big[\frac{dr^2}{r(r-2m)} + d\theta^2 + \sin^2\theta\, d\phi^2\Big]\,, \tag{3.2.8}$$

where $r_i = r + 2ms_i^2$. We then write the 2-sphere metric in the form $d\theta^2 + \sin^2\theta\, d\phi^2 = 4(1+|\zeta|^2)^{-2}\, d\zeta d\bar{\zeta}$, where $\zeta = \tan 12\theta\, e^{\mathrm{i}\phi}$, and perform the scalings

$$r = \tilde{r}\,\lambda^{-1}\,,\qquad t = \tilde{t}\,\lambda\,,\qquad m = \tilde{m}\,\lambda^{-3}\,,\qquad s_i = \tilde{s}_i\,\lambda\,,\qquad \zeta = \tilde{\zeta}\,\lambda\,. \tag{3.2.9}$$

Sending $\lambda \to 0$ gives the metric

$$ds^2 = \frac{2\tilde{m}\tilde{r}}{\sqrt{\tilde{r}_1\tilde{r}_2\tilde{r}_3\tilde{r}_4}}\, d\tilde{t}^2 + \sqrt{\tilde{r}_1\tilde{r}_2\tilde{r}_3\tilde{r}_4}\Big(-\frac{d\tilde{r}^2}{\tilde{m}\tilde{r}} + 4d\tilde{\zeta}d\bar{\tilde{\zeta}}\Big)\,. \tag{3.2.10}$$

Defining

$$\tilde{r} = -12\tilde{m}\,\rho^2\,,\qquad \tilde{\zeta} = x + \mathrm{i}y\,, \tag{3.2.11}$$

and taking

$$x = 12\mathrm{i}\,\hat{t}\,,\qquad y = 12z\,,\qquad \tilde{t} = \frac{\mathrm{i}}{\tilde{m}}\,\tilde{\phi}\,,\qquad \tilde{s}_i = \frac{\mathrm{i}}{2\beta_i}\,,\qquad \tilde{m} = 2\sqrt{\beta_1\beta_2\beta_3\beta_4}\,, \tag{3.2.12}$$

we obtain the 4-field Melvin metric

$$ds^2 = \sqrt{\Delta}\,(-d\hat{t}^2 + d\rho^2 + dz^2) + \frac{\rho^2}{\sqrt{\Delta}}\, d\tilde{\phi}^2\,, \tag{3.2.13}$$

where $\Delta = \prod_i \Delta_i$ with $\Delta_i = 1 + \beta_i^2\,\rho^2$. We see that this metric coincides with (3.2.6), after a minor change of notation. Applying the same scalings and analytic continuations to the scalar fields and gauge fields in the four-charge black hole solutions, one reproduces the results given in (3.2.4) and (3.2.5).

3.3 Magnetised Electrically Charged Black Holes

Here, we consider the magnetisation of the four-charge solution of the STU model that reduces, when the charges are set equal, to the magnetisation of the electrically-charged Reissner-Nordström solution. Using the notation and conventions of [35], this is achieved when the field strengths numbered 1 and 3 carry magnetic charges, while the field strengths numbered 2 and 4 carry electric charges. In order to be able to present the magnetised solution in the most compact way, we shall denote the four charge parameters by (q_1, q_2, q_3, q_4).

Applying the procedure described in appendix A, we find that the metric is given by

$$ds_4^2 = H\,[-r(r-2m)dt^2 + \frac{r_1 r_2 r_3 r_4}{r(r-2m)}\,dr^2 + r_1 r_2 r_3 r_4 d\theta^2\,] + H^{-1}\,\sin^2\theta\,(d\phi - \omega dt)^2\,, \quad (3.3.1)$$

where

$$r_i = r + 2m s_i^2\,, \quad (3.3.2)$$

and we shall use the notation $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$. The function ω is given by

$$\omega = \sum_{i=1}^{4}\Big[-\frac{q_i\,\beta_i}{r_i} + \frac{q_i\,\Xi_i\,[r_i + (r-2m)\cos^2\theta]r}{r_i}\Big]\,, \quad (3.3.3)$$

where

$$q_i = 2m s_i c_i\,, \qquad \Xi_i = \frac{\beta_1\beta_2\beta_3\beta_4}{\beta_i}\,, \qquad \beta_i = 12 B_i\,, \quad (3.3.4)$$

and B_i denotes the external magnetic field strengths for each of the four gauge fields. Finally, the function H is given in this case by

$$H = \frac{\sqrt{\Delta}}{\sqrt{r_1 r_2 r_3 r_4}}\,, \quad (3.3.5)$$

where

$$
\begin{aligned}
\Delta &= 1+\sum_i \frac{\beta_i^2 r_1 r_2 r_3 r_4}{r_i^2}\sin^2\theta + 2[\beta_3\beta_4 q_1 q_2+\cdots]\cos^2\theta + [\beta_3^2\,\beta_4^2\,R_1^2\,R_2^2+\cdots] \\
&\quad -2(\prod_j \beta_j r_j)\sum_i \frac{q_i^2}{r_i^2}\sin^2\theta\cos^2\theta + [2\beta_2\beta_3\beta_4^2 q_2 q_3\,R_1^2+\cdots]\cos^2\theta + \prod_i \beta_i^2\,R_i^2 \\
&\quad +r_1 r_2 r_3 r_4 \sum_i \frac{\Xi_i^2\,R_i^2}{r_i^2}\sin^2\theta + [2\beta_1\beta_2\beta_3^2\beta_4^2 q_3 q_4\,R_1^2\,R_2^2+\cdots]\cos^2\theta\,, \qquad (3.3.6)
\end{aligned}
$$

and we have defined

$$
R_i^2 = r_i^2\,\sin^2\theta + q_i^2\,\cos^2\theta\,. \qquad (3.3.7)
$$

Note that in each of the square-bracketed terms, the ellipses denote all the analogous terms that arise by taking all inequivalent permutations of the indices 1, 2, 3 and 4.

The periodicity $\Delta\phi$ of the azimuthal coordinate ϕ is determined by the requirement that there should be no conical singularity at the north and south poles of the sphere. Since Δ is an even function of $\cos\theta$, the requirements at the north and the south poles are identical, and they imply that ϕ should have period given by

$$
\Delta\phi = 2\pi\alpha\,, \qquad \alpha = \Big(1+[\beta_1\beta_2 q_3 q_4+\cdots]+\prod_i \beta_i q_i\Big)\,, \qquad (3.3.8)
$$

where the ellipses in the square brackets represent the five additional terms that follow from the indicated term by taking all inequivalent permutations of the labels 1, 2, 3 and 4.

The physical charges carried by the four gauge fields can be calculated easily using the expressions in appendix 3.7.3. The non-zero ones are (P_1, Q_2, P_3, Q_4). For the sake of uniformity we shall change the notation and call these $(\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3, \widetilde{Q}_4)$ respectively. They turn out to be given by

$$
\widetilde{Q}_i = \frac{(q_i - \beta_i^2 q_1 q_2 q_3 q_4/q_i)}{\alpha}\,\frac{\Delta\phi}{2\pi}\,, \qquad (3.3.9)
$$

where α is defined in (3.3.8). We therefore have

$$\widetilde{Q}_i = q_i - \frac{\beta_i^2\, q_1\, q_2\, q_3\, q_4}{q_i}\,. \tag{3.3.10}$$

The solutions for the gauge potentials are given by

$$\begin{aligned}
A_{(1)1} &= \beta_1\, r(r-2m)\cos\theta\,\Big[\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}-\frac{1}{r_4}\Big]\,dt+\sigma_1\,(d\phi-\omega dt)\,,\\
A_{(1)2} &= \Big[-\frac{q_2}{r_2}+\sum_{i=1,3,4}\frac{r\,q_i\,\beta_1\beta_3\beta_4\,[r_i+(r-2m)\cos^2\theta]}{\beta_i\,r_i}\Big]\,dt+\sigma_2\,(d\phi-\omega dt)\,,\\
\mathcal{A}^1_{(1)} &= \beta_3\, r(r-2m)\cos\theta\,\Big[\frac{1}{r_3}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_4}\Big]\,dt+\sigma_3\,(d\phi-\omega dt)\,,\\
\mathcal{A}^2_{(1)} &= \Big[-\frac{q_4}{r_4}+\sum_{i=1}^{3}\frac{r\,q_i\,\beta_1\beta_2\beta_3\,[r_i+(r-2m)\cos^2\theta]}{\beta_i\,r_i}\Big]\,dt+\sigma_4\,(d\phi-\omega dt)\,,
\end{aligned} \tag{3.3.11}$$

where $\sigma_i \equiv \tilde{\sigma}_i/\Delta$. When ellipses occur within a bracketed expression, they denote the two additional terms obtained by cycling the three index values taken from the set $\{1,2,3,4\}$ that are not equal to i.

The axions and dilatons are given by

$$\chi_i = \frac{Z_i\cos\theta}{Y_i}\,,\qquad e^{2\varphi_i} = \frac{Y_i^2}{\Delta\, r_1 r_2 r_3 r_4}\,,\qquad i=1,2,3\,, \tag{3.3.12}$$

where

$$\begin{aligned}
Z_1 &= r_2r_4[(\beta_1q_3+\beta_3q_1)+\beta_2\beta_4(\beta_1q_1R_3^2+\beta_3q_3R_1^2)]\\
&\qquad -r_1r_3[(\beta_2q_4+\beta_4q_2)+\beta_1\beta_3(\beta_2q_2R_4^2+\beta_4q_4R_2^2)]\,, && (3.3.13)\\
Y_1 &= r_1r_3(1+2\beta_1\beta_3q_2q_4\cos^2\theta+\beta_1^2\beta_3^2R_2^2R_4^2)\\
&\qquad +r_2r_4(\beta_1^2R_3^2+\beta_3^2R_1^2+2\beta_1\beta_3q_1q_3\cos^2\theta)\,, && (3.3.14)\\
(Z_2,Y_2) &= (-Z_1,Y_1)\ \text{with}\ 1\leftrightarrow 2\,, && (3.3.15)\\
(Z_3,Y_3) &= (Z_1,Y_1)\ \text{with}\ 2\leftrightarrow 3\,. && (3.3.16)
\end{aligned}$$

3.3.1 Angular momentum

The angular momentum can be calculated using the standard procedure developed by Wald. The details of this calculation, and, in particular, the evaluation of the angular momentum in terms of the quantities in the dimensionally-reduced three-dimensional theory, are given in [51]. A subtlety in the calculation concerns the different boundary conditions that arise depending upon whether a gauge field carries an electric charge or a magnetic charge. If the charges were all electric, then the conserved angular momentum corresponding to the Killing vector $\xi = \partial/\partial\tilde{\phi}$, where $\tilde{\phi} = \phi/\alpha$ is the rescaled azimuthal coordinate that has period 2π and α is defined in (3.3.8), would be [51]

$$J = \frac{\alpha}{16\pi} \int_{S^2} d(\chi_4 + \sigma_i\, \psi_i) \wedge d\phi = \frac{(\Delta\phi)^2}{32\pi^2} \Big[\chi_4 + \sigma_i\, \psi_i\Big]_{\theta=0}^{\theta=\pi} . \tag{3.3.17}$$

As discussed in [51], this expression is invariant under the $U(1)^4$ abelian gauge transformations of the four gauge potentials that preserve the condition that the Lie derivatives of the gauge potentials with respect to the azimuthal Killing vector $\partial/\partial\phi$ vanish.

In our case, however, the fields $A_{(1)1}$ and $\mathcal{A}^1_{(1)}$ carry magnetic, rather than electric, charges. A simple way to evaluate the angular momentum is to perform dualisations of these two fields. Although rather involved in the four-dimensional theory itself, the dualisations can be easily implemented in the reduced three-dimensional theory, since then they amount to exchanging the roles of the σ_i and ψ_i axions for the fields in question. As can be seen from (3.7.3), since the the Kaluza-Klein vector $\bar{\mathcal{B}}_{(1)}$ must be invariant under duality it follows that the required duality transformations require also sending

$$\chi_4 + \sigma_i\, \psi_i \longrightarrow \chi_4 + \sigma_i\, \psi_i - \sigma_1\, \psi_1 - \sigma_3\, \psi_3 \, . \tag{3.3.18}$$

The conserved angular momentum for the four-charge black holes is therefore given by

$$J = \frac{(\Delta\phi)^2}{32\pi^2} \Big[\chi_4 + \sigma_2\, \psi_2 + \sigma_4\, \psi_4\Big]_{\theta=0}^{\theta=\pi} . \tag{3.3.19}$$

Evaluating this, we find

$$J = 12[\beta_1\, q_2 q_3 q_4 + \cdots] + 12 q_1 q_2 q_3 q_4\, [q_1 \beta_2 \beta_3 \beta_4 + \cdots]\,, \tag{3.3.20}$$

where the ellipses in each case denote the additional three symmetry-related terms.

3.3.2 Pairwise equal charges

A considerable simplification arises in the function Δ if we set the fields pairwise equal, so that

$$B_3 = B_1\,, \qquad B_4 = B_2\,, \qquad \delta_3 = \delta_1\,, \qquad \delta_4 = \delta_2\,. \tag{3.3.21}$$

We then find that

$$\Delta = \Big[1 + \sum_{i=1}^{2} \beta_i^2 (r_i^2 \sin^2\theta + q_i^2 \cos^2\theta) + 4\beta_1\beta_2 q_1 q_2 \cos^2\theta + \prod_{i=1}^{2} \beta_i^2 (r_i^2 \sin^2\theta + q_i^2 \cos^2\theta)\Big]^2\,. \tag{3.3.22}$$

With the fields set pairwise equal, i.e. $q_3 = q_1$, $q_4 = q_2$ and $\beta_3 = \beta_1$ and $\beta_4 = \beta_2$. We then find

$$\begin{aligned}
\mathcal{A}^2_{(1)} &= \Big[-\frac{q_2}{r_2} + \beta_1^2 q_2\, r\,\Big(1 + \frac{(r-2m)}{r_2}\,\cos^2\theta\Big) + 2\beta_1\beta_2 q_1\, r\,\Big(1 + \frac{(r-2m)}{r_1}\,\cos^2\theta\Big)\Big]\, dt \\
&\quad + \sigma_4\,(d\phi - \omega dt)\,, \\
\mathcal{A}^1_{(1)} &= -\frac{2\beta_1 r (r-2m)}{r_2}\,\cos\theta\, dt + \sigma_3\,(d\phi - \omega dt)\,,
\end{aligned} \tag{3.3.23}$$

with analogous expressions for $A_{(1)1}$ and $\mathcal{A}^1_{(1)}$. The fields σ_3 and σ_4 are given by

$$\begin{aligned}
\sigma_3 &= -q_1 \cos\theta\,(1 - \beta_1^2 R_2^2)\, Y^{-1}\,, \\
\sigma_4 &= \Big[\beta_2 R_1^2 + 2\beta_1 q_1 q_2 \cos^2\theta + \beta_1^2 \beta_2 R_1^2 R_2^2\Big]\, Y^{-1}\,,
\end{aligned} \tag{3.3.24}$$

where

$$
\begin{aligned}
R_i^2 &= r_i^2 \sin^2\theta + q_i^2 \cos^2\theta\,, \\
Y &= 1 + \beta_1^2 R_2^2 + \beta_2^2 R_1^2 + 4\beta_1\beta_2 q_1 q_2 \cos^2\theta + \beta_1^2\beta_2^2 R_1^2 R_2^2\,.
\end{aligned}
\tag{3.3.25}
$$

A different specialisation arises if we instead reverse the sign of the fields B_3 and B_4 before the pairwise identification, in other words, if we set

$$
B_3 = -B_1\,, \qquad B_4 = -B_2\,, \qquad \delta_3 = \delta_1\,, \qquad \delta_4 = \delta_2\,. \tag{3.3.26}
$$

Now, the function Δ becomes instead

$$
\begin{aligned}
\Delta = {} & [1 + 2\beta_1 q_2 \cos\theta + \beta_1^2(r_1^2 \sin^2\theta + q_1^2\cos^2\theta)][1 - 2\beta_1 q_2 \cos\theta + \beta_1^2(r_1^2 \sin^2\theta + q_1^2\cos^2\theta)] \times \\
& [1 + 2\beta_2 q_1 \cos\theta + \beta_2^2(r_2^2 \sin^2\theta + q_2^2\cos^2\theta)][1 - 2\beta_2 q_1 \cos\theta + \beta_2^2(r_2^2 \sin^2\theta + q_2^2\cos^2\theta)]\,.
\end{aligned}
\tag{3.3.27}
$$

Note that in this case the function ω now vanishes, and so the metric is purely static. In fact it is not hard to show that all the possible ways of making ω vanish involve making one or another of the following choices

$$
\begin{array}{lllll}
(1) & q_i = q_j\,, & q_k = q_\ell\,, & B_i = -B_j\,, & B_k = -B_\ell \\
(2) & q_i = q_j\,, & q_k = -q_\ell\,, & B_i = -B_j\,, & B_k = B_\ell \\
(3) & q_i = -q_j\,, & q_k = q_\ell\,, & B_i = B_j\,, & B_k = -B_\ell \\
(4) & q_i = -q_j\,, & q_k = -q_\ell\,, & B_i = B_j\,, & B_k = B_\ell\,,
\end{array}
\tag{3.3.28}
$$

where i, j, k and ℓ are all different and are chosen from 1, 2, 3 and 4. It can easily be seen that, as one would expect, the angular momentum (3.3.20) vanishes in all of these cases.

3.3.3 Asymptotic structure and ergoregions

It was observed in [50] that the metric component g_{tt} in the magnetised electrically charged Reissner-Nordström solution becomes arbitrarily large and positive at large distances near to the z axis, thus indicating the presence of an ergoregion extending to infinity. Not surprisingly, the same is in general true in the STU model generalisations of this solution that we are considering here. Specifically, if we introduce cylindrical coordinates $\rho = r\,\sin\theta$ and $z = r\,\cos\theta$, then it is easily seen from (4.5.1), (4.5.3) and (4.5.5) that to leading order in large z and small ρ we shall in general have

$$g_{tt} \sim +z^2\,\rho^2\,\Big(\sum_i \beta_i\,\Xi_i\Big)^2\,, \tag{3.3.29}$$

and thus an ergoregion extending to infinity. The reason for this metric behaviour is that the function ω given in (4.5.3) has the large-z expansion

$$\omega = 2z\,\sum_{i=1}^4 q_i\,\Xi_i - 2m\sum_{i=1}^4 q_i\,\Xi_i\,(1+s_i^2) + \mathcal{O}(\frac{1}{z})\,. \tag{3.3.30}$$

The ergoregion is avoided if one imposes the condition $\sum_i q_i\,\Xi_i = 0$ on the charges and magnetic fields, i.e. if

$$\beta_1\beta_2\beta_3\beta_4\sum_{i=1}^4 \frac{q_i}{\beta_i} = 0\,. \tag{3.3.31}$$

One way to achieve this is if one (or more) of the four field strengths is set to zero; for example, by taking $q_4 = 0$ and $\beta_4 = 0$. Under these circumstances the metric is still stationary, as opposed to static, but is asymptotically non-rotating at infinity. It can be seen from (3.3.20) that the angular momentum also vanishes in such a case.

Clearly there are also more general ways to satisfy (3.3.31), where all four fields are non-vanishing. If we assume that (3.3.31) is satisfied then it follows from (3.3.30) that the asymptotic metric near the axis is rotating with an angular velocity

$$\Omega_\infty = 2m\sum_{i=1}^4 q_i\,\Xi_i\,s_i^2 = 4m^2\beta_1\beta_2\beta_3\beta_4\,\sum_{i=1}^4 \frac{\sinh^3\delta_i\,\cosh\delta_i}{\beta_i}\,. \tag{3.3.32}$$

It can also be seen from (4.5.3) that if (3.3.31) holds then on the black hole horizon at

$r = 2m$, the angular velocity will be

$$\Omega_H = \sum_{i=1}^{4} \frac{q_i\,\beta_i}{2mc_i^2} = \sum_{i=1}^{4} \beta_i\, \tanh\delta_i\,. \tag{3.3.33}$$

Note that in general, the angular momentum (3.3.20) is non-vanishing if (3.3.31) is satisfied.

Of course if any of the conditions enumerated in (3.3.28) holds, then not merely is (3.3.31) satisfied but the metric is non-rotating everywhere, and also $J = 0$.

3.4 Scaling Limit, and Lift to Five Dimensions

The scaling limits of our magnetised non-extremal black holes, which will be parameterised by $\tilde m$, $\widetilde\Pi_s$, $\widetilde\Pi_c$ and $\tilde\beta_i$ ($i = 1, \cdots, 4$), can be obtained by taking a specific scaling limit [15] of the magnetised electric black holes of section 3 parameterised by m, δ_i, β_i with $\delta_1 = \delta_2 = \delta_3$. After taking the limit, the solution can then be lifted to five dimensions, where it can be seen to be $\mathrm{AdS}_3 \times S^2$.

The limit can be implemented by setting $\delta_1 = \delta_2 = \delta_3$ and making the scaling [15]:

$$\begin{aligned} m &= \tilde m\,\epsilon\,, \qquad r = \tilde r\,\epsilon\,, \qquad t = \tilde t\,\epsilon^{-1}\,, \qquad \beta_i = \tilde\beta_i\,\epsilon\,, \quad i = 1,2,3,4\,,\\ \sinh^2\delta_4 &= \frac{\widetilde\Pi_s^2}{\widetilde\Pi_c^2 - \widetilde\Pi_s^2}\,, \qquad \sinh^2\delta_i = (\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{1/3}\,\epsilon^{-4/3}\,, \quad i = 1,2,3\,, \end{aligned} \tag{3.4.1}$$

where ϵ is then sent to zero.

The implementation of the scaling limit (7.2.2) gives

$$(d\phi - \omega dt) \longrightarrow d\phi - (\tilde\beta_1 + \tilde\beta_2 + \tilde\beta_3)d\tilde t - \frac{2\tilde m\tilde\beta_4\,\widetilde\Pi_c\widetilde\Pi_s}{(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)\tilde r + 2\tilde m\widetilde\Pi_s^2}\, d\tilde t\,, \tag{3.4.2}$$

and

$$\Delta \longrightarrow 1 + \frac{8\tilde m^3\tilde\beta_4^2(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^2\sin^2\theta}{(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)\tilde r + 2\tilde m\widetilde\Pi_s^2}\,. \tag{3.4.3}$$

The quantities $\tilde\beta_1$, $\tilde\beta_2$ and $\tilde\beta_3$ drop out completely in the scaling limit if we send $\phi \longrightarrow$

$\phi + (\tilde\beta_1 + \tilde\beta_2 + \tilde\beta_3)\tilde t$. We shall assume from now on that this redefinition has been performed. Therefore, the obtained scaling limit of magnetised non-extremal black holes depend only on *four* independent parameters: $\tilde m$, $\tilde\Pi_c$, $\tilde\Pi_s$ and $\tilde\beta_4$.

In the case of vanishing magnetic fields, $\beta_i = 0$ it was possible [15] to identify the scaling limits with the *subtracted geometry* of a non-extreme black hole parameterised by $\tilde m, \tilde\delta_i$. In that case we have $\tilde\Pi_s \equiv \Pi_{i=1}^4 \sinh\tilde\delta_i$ and $\tilde\Pi_c \equiv \Pi_{i=1}^4 \cosh\tilde\delta_i$, determined by (7.2.2). In our case we have no independent derivation of a subtracted geometry and so no unique identification of $\tilde\delta_i$ is possible.

The lifting of the subtracted geometry solution to five dimensions is given by

$$ds_5^2 = e^{\varphi_1}\, ds_4^2 + e^{-2\varphi_1}\,(dz + \mathcal{A}^2_{(1)})^2\,. \tag{3.4.4}$$

Applying the scaling limit (7.2.2) here, together with $z = \tilde z\,\epsilon^{-1}$, we find that the five-dimensional metric ds_5^2 scales as $\epsilon^{-2/3}$, and defining $ds_5^2 = \epsilon^{-2/3}\, d\tilde s_5^2$ we have

$$\begin{aligned} d\tilde s_5^2 \;=\;& 4\tilde m^2(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{2/3}\,[d\theta^2 + \sin^2\theta\,(d\phi + \tilde\beta_4 d\tilde z)^2] \\ &+\frac{(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)\tilde r + 2\tilde m\widetilde\Pi_s^2}{2\tilde m(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{4/3}}\, d\tilde z^2 - \frac{(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)\tilde r - 2\tilde m\widetilde\Pi_c^2}{2\tilde m(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{4/3}}\, d\tilde t^2 \\ &-\frac{2\widetilde\Pi_c\widetilde\Pi_s}{(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{4/3}}\, d\tilde t d\tilde z + \frac{4\tilde m^2(\widetilde\Pi_c^2 - \widetilde\Pi_s^2)^{2/3}}{\tilde r(\tilde r - 2\tilde m)}\, d\tilde r^2\,. \end{aligned} \tag{3.4.5}$$

It can be seen that $\tilde\beta_4$ disappears from the five-dimensional metric if we make the further coordinate redefinition

$$\phi = \tilde\phi - \tilde\beta_4\,\tilde z\,, \tag{3.4.6}$$

This is a reflection of the fact that the magnetisation of the four-dimensional gauge field associated with the Kaluza-Klein vector $\mathcal{A}^2_{(1)}$ of the five-dimensional reduction can be implemented (or, in the above calculation, undone) by performing a rotation in the $(\phi, \tilde z)$ plane [11]. This transformation is related to a spectral flow in a dual conformal field theory

[11]The role of the specific Melvin transformation as a coordinate transformation in the $(\phi, \tilde z)$ plane of the

interpretation of AdS_3 geometries.

Finally, if we define new coordinates ρ, σ and τ by

$$\tilde{r} = 2\tilde{m}\,\cosh^2\rho\,, \qquad \tilde{z} = 2\mathrm{i}\,(2\tilde{m})^{3/2}\,(\widetilde{\Pi}_c\,\tau + \widetilde{\Pi}_s\,\sigma)\,, \qquad \tilde{t} = 2\mathrm{i}\,(2\tilde{m})^{3/2}\,(\widetilde{\Pi}_c\,\sigma + \widetilde{\Pi}_s\,\tau)\,, \tag{3.4.7}$$

the five-dimensional metric can be seen to become

$$d\tilde{s}_5^2 = 16\tilde{m}^2(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{2/3}\Big[(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\sigma^2) + 14(d\theta^2 + \sin^2\theta d\tilde{\phi}^2)\Big]\,, \tag{3.4.8}$$

which is the metric on $\mathrm{AdS}_3 \times S^2$.

3.5 Magnetostatic Black Holes

3.5.1 Magnetised magnetically charged black holes

Here we exchange the roles of the electric and the magnetic charges in the original four-charge seed solution. That is, the charges numbered 1 and 3 are now electric, while those numbered 2 and 4 are magnetic, in the conventions of [35]. (Some of the properties of the resulting metrics were discussed previously in [61–64].) We shall denote the four charge parameters by (p_1, p_2, p_3, p_4) in this case. In the case that the charges are set equal, the solution reduces to the magnetised magnetically-charge Reissner-Nordström black hole. Concretely, in the original seed solution, reduced to three dimensions, we replace (3.7.9) and (3.7.10) by

$$e^{2\varphi_1} = \frac{r_2\,r_4}{r_1\,r_3}\,, \qquad e^{2\varphi_2} = \frac{r_1\,r_4}{r_2\,r_3}\,, \qquad e^{2\varphi_3} = \frac{r_3\,r_4}{r_1\,r_2}\,, \qquad e^{2\varphi_4} = r_1\,r_2\,r_3\,r_4\,\sin^4\theta\,, \tag{3.5.1}$$

lifted geometry was first observed for dilatonic black holes in [57].

and

$$\begin{aligned}
\chi_1 &= 0 \qquad \chi_2 = 0\,, \qquad \chi_3 = 0\,, \qquad \chi_4 = 0\,,\\
\sigma_1 &= 0\,, \qquad \sigma_2 = p_2\cos\theta\,, \qquad \sigma_3 = 0\,, \qquad \sigma_4 = p_4\cos\theta\,,\\
\psi_1 &= p_1\cos\theta\,, \qquad \psi_2 = 0\,, \qquad \psi_3 = p_3\cos\theta\,, \qquad \psi_4 = 0\,,
\end{aligned} \tag{3.5.2}$$

The metric of the magnetised solution will still be given by (4.5.1), but now we have $\omega = 0$ and the function Δ in (4.5.5) is given by

$$\Delta = \prod_{i=1}^{4} \Delta_i\,, \qquad \Delta_i = (1 + \beta_i p_i \cos\theta)^2 + \beta_i^2\, r_i^2\, \sin^2\theta\,. \tag{3.5.3}$$

Because Δ is not an even function of $\cos\theta$ in this case, the periodicity conditions on ϕ for the metric to be free of conical singularities are different at the north and south poles of the sphere. Specifically, we find that the required periodicities are

$$\begin{aligned}
\theta = 0: &\qquad \Delta\phi = 2\pi \prod_i (1 + \beta_i p_i)\,,\\
\theta = \pi: &\qquad \Delta\phi = 2\pi \prod_i (1 - \beta_i p_i)\,.
\end{aligned} \tag{3.5.4}$$

The metric can be rendered free of conical singularities if the charges and magnetic fields satisfy the "no-force condition"

$$\prod_i (1 + \beta_i p_i) = \prod_i (1 - \beta_i p_i)\,. \tag{3.5.5}$$

Using the expressions given in section 3.7.3, we can calculate the physical electric and magnetic charges carried by the four gauge fields. In this case, the non-vanishing ones are (Q_1, P_2, Q_3, P_4). For the sake of uniformity, we shall relabel these as $(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4)$

respectively. They turn out to be given by

$$\widetilde{P}_i = \frac{p_i}{(1-\beta_i^2\, p_i^2)}\,\frac{\Delta\phi}{2\pi}\,. \tag{3.5.6}$$

The electromagnetic potentials are given by

$$\begin{aligned}
\hat{A}_{(1)1} &= \Big[-\frac{p_1}{r_1} + \frac{2\beta_1 r(r-2m)\cos\theta}{r_1} - \frac{\beta_1^2\, p_1\,[r_1^2 + r(r-2m)\,\cos^2\theta]}{r_1}\Big]\, dt\,,\\
\hat{A}_{(1)2} &= \frac{p_2\cos\theta + \beta_2\, R_2^2}{\Delta_2}\, d\phi\,,\\
\hat{\mathcal{A}}^1_{(1)} &= \Big[-\frac{p_3}{r_3} + \frac{2\beta_3 r(r-2m)\cos\theta}{r_3} - \frac{\beta_3^2\, p_3\,[r_3^2 + r(r-2m)\,\cos^2\theta]}{r_3}\Big]\, dt\,,\\
\hat{\mathcal{A}}^2_{(1)} &= \frac{p_4\cos\theta + \beta_4\, R_4^2}{\Delta_4}\, d\phi\,,
\end{aligned} \tag{3.5.7}$$

where $R_i^2 = r_i^2\sin^2\theta + p_i^2\cos^2\theta$. The scalar fields are given by

$$\begin{aligned}
e^{2\varphi_1} &= \frac{r_2\, r_4\, \Delta_1\, \Delta_3}{r_1\, r_3\, \Delta_2\, \Delta_4}\,,\qquad e^{2\varphi_2} = \frac{r_1\, r_4\, \Delta_2\, \Delta_3}{r_2\, r_3\, \Delta_1\, \Delta_4}\,,\qquad e^{2\varphi_3} = \frac{r_3\, r_4\, \Delta_1\, \Delta_2}{r_1\, r_2\, \Delta_3\, \Delta_4}\,,\\
\chi_1 &= 0\,,\qquad \chi_2 = 0\,,\qquad \chi_3 = 0\,.
\end{aligned} \tag{3.5.8}$$

3.5.2 $SL(2,\mathbb{R})^4$ truncations of the sigma model

The three-dimensional scalar sigma model associated with the timelike or spacelike reduction of the four-dimensional STU model has an $O(4,4)$ global symmetry. The Lagrangian in the case of the timelike reduction can be found in section 2.1 of [35]. The sixteen scalars comprise the original three dilatons $(\varphi_1, \varphi_2, \varphi_3)$ and three axions (χ_1, χ_2, χ_3) of the STU model; the Kaluza-Klein scalar φ_4 and the axion χ_4 dual to the Kaluza-Klein vector; the four axions σ_i coming from the direct dimensional reductions of the four gauge potentials; and finally the four axions ψ_i coming from the dualisations of the four gauge potentials in the dimensionally-reduced theory.

If we restrict attention to purely static configurations then χ_4 will vanish. If we furthermore restrict to configurations where the axions (χ_1, χ_2, χ_3) of the STU model vanish, then it can be seen from the sigma-model Lagrangian in eqn (7) of [35] that there are two possible disjoint truncations of the remaining scalar fields for which the vanishing of the four χ_i axions is consistent with their equations of motion.[12] Specifically, we can have either

$$\sigma_1 = \psi_2 = \sigma_3 = \psi_4 = 0 \tag{3.5.9}$$

or

$$\psi_1 = \sigma_2 = \psi_3 = \sigma_4 = 0\,. \tag{3.5.10}$$

In the truncation (3.5.9), if we define

$$\begin{aligned}
&u_1 = 12(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\,, \qquad u_2 = 12(-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)\,,\\
&u_3 = 12(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\,, \quad u_4 = 12(-\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4)\,,\\
&\alpha_1 = \psi_1 \qquad \alpha_2 = \sigma_2\,, \qquad \alpha_3 = \psi_3\,, \qquad \alpha_4 = \sigma_4\,,
\end{aligned} \tag{3.5.11}$$

then the three-dimensional sigma-model Lagrangian in equation (7) of [35], after the appropriate sign-changes because we are making a spacelike reduction, becomes

$$\mathcal{L}_{\rm scal} = \sum_{i=1}^{4} \Big(- 12(\partial u_i)^2 - 12 e^{2u_i}\,(\partial \alpha_i)^2\Big)\,. \tag{3.5.12}$$

This can be recognised as describing the coset $[SL(2,\mathbb{R})/O(2)]^4$. Similarly, if we consider

[12] In [35] a *timelike* reduction to three dimensions was performed. Here, we are instead reducing on the spacelike azimuthal Killing vector $\partial/\partial\phi$ rather than the timelike Killing vector $\partial/\partial t$. The formulae in [35] can be repurposed to the spacelike reduction with very straightforward modifications. In particular, the three-dimensional sigma-model Lagrangian in eqn (7) of [35] will take the same form in the case of the spacelike reduction, except that the kinetic terms for all the scalar fields will now have the standard negative sign appropriate to a Minkowski-signature theory.

instead the truncations (3.5.10), then defining instead

$$
\begin{aligned}
&u_1 = 12(-\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\,, \quad u_2 = 12(\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4)\,, \\
&u_3 = 12(-\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\,, \quad u_4 = 12(\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)\,, \\
&\alpha_1 = \sigma_1 \qquad \alpha_2 = \psi_2\,, \qquad \alpha_3 = \sigma_3\,, \qquad \alpha_4 = \psi_4
\end{aligned} \tag{3.5.13}
$$

gives again an $[SL(2,\mathbb{R})/O(2)]^4$ sigma model with Lagrangian (3.5.11).

(Note that if we considered a timelike reduction on the coordinate t rather than a spacelike reduction on the coordinate ϕ, we would end up with a Lagrangian like (3.5.12) except with a minus sign in front of the exponential terms. The coset in this case would be $[SL(2,\mathbb{R})/O(1,1)]^4$.)

The truncation described by (3.5.11) corresponds to the case where the gauge fields numbered 1 and 3 are purely electric, while those numbered 2 and 4 are purely magnetic. Since in this chapter we always consider Melvin backgrounds where fields 1 and 3 carry external electric fields, while 2 and 4 carry external magnetic fields, this means that we can remain within the truncation if we additionally allow fields 1 and 3 to carry electric charges, and fields 2 and 4 to carry magnetic charges. This is precisely the situation we considered in section 4, namely the STU model generalisations of the magnetically-charged Reissner-Nordström black hole in an external magnetic field. It can indeed be seen from equations (3.5.7) and (3.5.8), together with the staticity of the metric, that the solutions fall within the class described by the truncation (3.5.9) and (3.5.11).

By contrast, although the *charges* carried by the gauge fields in the solutions in section 3 are compatible with the truncation described by (3.5.10) and (3.5.13), the external fields are still appropriate for the other truncation, (3.5.9) and (3.5.11), and so the solutions in section 3 are not described by either of the truncated theories. And indeed, the axions χ_i are non-zero and the metric is not static.

3.5.3 Multi-centre BPS black holes in external magnetic fields

Returning to the truncation (3.5.9) and (3.5.11), we can in fact use it to describe more general situations than the "magnetised magnetically charged" black holes obtained in section 4. In particular, we can consider the case of multi-centre BPS black holes that are then immersed in external fields, provided that we align them all along a line so that we can apply the "Melvinising" transformation. For these purposes, it is useful first to present the general expressions for the transformations of the scalar fields under the "Melvinising" transformations. If we start with a seed solution for which the fields are denoted by bars, then after the transformation we will have

$$e^{u_i} = e^{\bar{u}_i}\,[(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}]\,, \qquad \alpha_i = \frac{\bar{\alpha}_i\,(1+\beta_i\,\bar{\alpha}_i) + \beta_i\,e^{-2\bar{u}_i}}{(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}}\,. \tag{3.5.14}$$

In particular this means that the transformed function φ_4 that appears in the metric ansatz (3.7.1) is given by

$$e^{-2\varphi_4} = e^{-2\bar{\varphi}_4}\prod_{i=1}^{4}[(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}]\,. \tag{3.5.15}$$

The multi-centre black holes in the STU model have metrics given by

$$ds^2 = -\Big(\prod_{i=1}^{4} H_i\Big)^{-1/2} dt^2 + \Big(\prod_{i=1}^{4} H_i\Big)^{1/2} d\vec{y}^2\,, \tag{3.5.16}$$

where the functions H_i are harmonic in the 3-dimensional Euclidean space with metric $d\vec{y}^2$. For black holes aligned along an axis we can conveniently use cylindrical coordinates in which

$$d\vec{y}^2 = d\rho^2 + \rho^2\,d\phi^2 + dz^2\,. \tag{3.5.17}$$

We shall take the harmonic functions to be given by

$$H_i = 1 + \sum_a \frac{p_i^{(a)}}{\sqrt{\rho^2 + (z-z_a)^2}}\,, \tag{3.5.18}$$

where the charges $p_i^{(a)}$ are constants and the black holes are located at the point z_a on the z axis. The metric is free of conical singularities on the z axis provided that ϕ has period 2π.

A field strength carrying an electric charge is described by a potential of the form

$$A^i_{\text{elec}} = -H_i^{-1}\, dt\,, \tag{3.5.19}$$

while a field strength carrying a magnetic charge is described by a potential of the form

$$A^i_{\text{mag}} = \sum_a \frac{p_i^{(a)}\,(z-z_a)}{\sqrt{\rho^2+(z-z_a)^2}}\, d\phi\,. \tag{3.5.20}$$

In our case, therefore, the potentials for fields 1 and 3 are of the form (3.5.19), while those for fields 2 and 4 are of the form (3.5.20). In the dimensionally-reduced three-dimensional language this implies that the axionic scalars α_i defined in (3.5.11) are all given in this seed solution by

$$\bar{\alpha}_i = \sum_a \frac{p_i^{(a)}\,(z-z_a)}{\sqrt{\rho^2+(z-z_a)^2}}\,. \tag{3.5.21}$$

The dilatonic scalar fields $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ in this multi-centre seed solution are given by

$$\vec{\varphi} = 12 \sum_i \epsilon_i\, \vec{c}_i\, \log H_i\,, \tag{3.5.22}$$

where

$$\mathcal{L} = \sqrt{-g}(R - 12(\partial\vec{\varphi})^2 - 14 \sum_i e^{\vec{c}_i\cdot\vec{\varphi}}\,(F^i)^2) \tag{3.5.23}$$

and ϵ_i is $+1$ if field i carries an electric charge and -1 if it carries a magnetic charge (see, for example, section 2.2 of [65]). Comparing the multi-centre metric given by (3.5.16) and (3.5.17) with the reduction ansatz (3.7.1), we see that in the multi-centre seed solution we shall have

$$e^{\bar{\varphi}_4} = \rho^2 \Big(\prod_i H_i\Big)^{1/2}\,, \tag{3.5.24}$$

and hence, from (3.5.11),

$$e^{-\bar{u}_i} = \rho\, H_i\,. \tag{3.5.25}$$

Applying the Melvinising transformations (3.5.14), we obtain the "magnetised magnetic" multi-centre black holes with metrics

$$ds^2 = e^{-\varphi_4}\,\rho^2\Big[-dt^2 + \Big(\prod_i H_i\Big)(d\rho^2 + dz^2)\Big] + e^{\varphi_4}\,d\phi^2\,, \tag{3.5.26}$$

where φ_4 is given by (3.5.15). Thus the metric is given by

$$ds^2 = Z^{1/2}\Big[-\Big(\prod_{i=1}^4 H_i\Big)^{-1/2} dt^2 + \Big(\prod_{i=1}^4 H_i\Big)^{1/2}(d\rho^2 + dz^2 + Z^{-1}\,\rho^2\,d\phi^2)\Big]\,, \tag{3.5.27}$$

where

$$Z = \prod_{i=1}^4[(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}]\,. \tag{3.5.28}$$

There will in general now be conical singularities along the z axis. This can be seen by looking at the form of the metric in the $(\rho,\phi))$ plane as ρ tends to zero. From (3.5.21) and (3.5.25) we see that as ρ tends to zero we shall have

$$Z \to \prod_{i=1}^4(1+\beta_i\,\bar{\alpha}_i)^2\,, \qquad \bar{\alpha}_i \to \sum_a p_i^{(a)}\,\mathrm{sign}(z-z_a)\,. \tag{3.5.29}$$

In the case of a single-centre black hole, the periodicity conditions for ϕ in order to avoid a conical singularity can be seen to reduce to those in equation (3.5.5).

3.6 Conclusions

In string theory charged black holes may be regarded as having a composite structure arising from their microscopic description in terms of intersecting D-branes/M-branes. This composite structure is reflected in the interactions of the black holes. In this chapter we have demonstrated this by using as external probes the various types of magnetic fields capable of exciting each of these constituents. We have found that the behaviour of black holes is indeed rather sensitive to which type of magnetic field is applied. By far the simplest case is that of Kaluza-Klein black holes, which are made up of a single constituent. Somewhat counterintuitively it turns out that the Maxwell-Einstein case is the most complex, which may be ascribed to the fact that all the constituents and probes are turned on.

Utilising the composite structure of charges and magnetic fields allows for a balance of different forces and torques and the taming of the extent of ergoregions. This work samples only a restricted subset of static four-charge generating black hole solutions. We anticipate that further studies of rotating five-charge generating solutions will reveal an even richer structure.

3.7 Appendix A: The STU Model

3.7.1 Reduction of the STU model to $D = 3$

We can "magnetise" the black hole solutions by performing a spacelike reduction to three dimensions on the azimuthal Killing vector $\partial/\partial\phi$, and then acting with the appropriate $O(4,4)$ transformations. This is analogous to the discussion in [35], except that there the reduction was performed on the timelike Killing vector $\partial/\partial t$[13]. Thus we make a standard Kaluza-Klein reduction with

$$ds_4^2 = e^{-\varphi_4}\, d\bar{s}_3^2 + e^{\varphi_4}\,(d\phi + \bar{\mathcal{B}}_{(1)})^2\,, \tag{3.7.1}$$

and

$$\begin{aligned} A_{(1)1} &= \bar{A}_{(1)1} + \sigma_1\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \qquad A_{(1)2} = \bar{A}_{(1)2} + \sigma_2\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \\ \mathcal{A}^1_{(1)} &= \bar{\mathcal{A}}^1_{(1)} + \sigma_3\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \qquad \mathcal{A}^2_{(1)} = \bar{\mathcal{A}}^2_{(1)} + \sigma_4\,(d\phi + \bar{\mathcal{B}}_{(1)})\,. \end{aligned} \tag{3.7.2}$$

[13] One can also employ a seed solution with analytically continued coordinates: $t \to i\phi$ and $\phi \to it$, perform the reduction on the the timelike Killing vector of the analytically continued solution, act on it with the appropriate generators of $O(4,4)$ transformations defined in [35], and finally, analytically continue the obtained solution back to original coordinates (t,ϕ).

where, when necessary, we place bars on three-dimensional quantities in order to distinguish them from four-dimensional ones. Note that throughout, we use the ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$ for the potentials, with σ_i being the axionic scalar coming from the direct Kaluza-Klein reduction of the i'th potential, and so on.

In three dimensions we then dualise 1-form potentials to scalars, in a fashion that is precisely analogous to the one described for the timelike reduction in [35]. The upshot is that the Kaluza-Klein 1-form $\bar{\mathcal{B}}_{(1)}$, whose field strength is $\bar{\mathcal{G}}_{(2)} = d\bar{\mathcal{B}}_1$, is replaced by the axion χ_4 with

$$e^{2\varphi_4}\,\bar{*}\bar{\mathcal{G}}_{(2)} = d\chi_4 + \sigma_1\, d\psi_1 + \sigma_2\, d\psi_2 + \sigma_3\, d\psi_3 + \sigma_4\, d\psi_4\,, \tag{3.7.3}$$

and the 1-form potentials in three dimensions coming from the reduction of the four 1-form potentials in four dimensions are dualised to axions ψ_i where

$$\begin{aligned}
-e^{-\varphi_1+\varphi_2-\varphi_3+\varphi_4}\,\bar{*}\bar{F}_{(2)1} &= d\psi_1 + \chi_3\, d\psi_2 - \chi_1\, d\sigma_3 - \chi_1\,\chi_3\, d\sigma_4\,,\\
-e^{-\varphi_1+\varphi_2+\varphi_3+\varphi_4}\,\bar{*}\bar{F}_{(2)2} &= d\psi_2 - \chi_1\, d\sigma_4\,,\\
-e^{-\varphi_1-\varphi_2+\varphi_3+\varphi_4}\,\bar{*}\bar{\mathcal{F}}^1_{(2)} &= d\psi_3 - \chi_2\, d\psi_2 - \chi_1\, d\sigma_1 + \chi_1\,\chi_2\, d\sigma_4\,,\\
-e^{-\varphi_1-\varphi_2-\varphi_3+\varphi_4}\,\bar{*}\bar{\mathcal{F}}^2_{(2)} &= d\psi_4 + \chi_2\, d\psi_1 - \chi_3\, d\psi_3 - \chi_1\, d\sigma_2 + \chi_2\,\chi_3\, d\psi_2\\
&\quad -\chi_1\,\chi_2\, d\sigma_3 + \chi_1\,\chi_3\, d\sigma_1 - \chi_1\,\chi_2\,\chi_3\, d\sigma_4\,.
\end{aligned} \tag{3.7.4}$$

The three-dimensional Lagrangian in terms of the dualised fields is a non-linear sigma model coupled to gravity, and can be written as

$$\bar{\mathcal{L}}_3 = \sqrt{-\bar{g}}\,[\bar{R} - 12\,\mathrm{tr}(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})]\,, \tag{3.7.5}$$

where $\mathcal{M} = \mathcal{V}^T\mathcal{V}$ and

$$\mathcal{V} = e^{12\varphi_i H_i}\,\mathcal{U}_\chi\,\mathcal{U}_\sigma\,\mathcal{U}_\psi\,. \tag{3.7.6}$$

Here

$$
\begin{aligned}
\mathcal{U}_\chi &= e^{\chi_1 E_{\chi_1}} e^{\chi_2 E_{\chi_2}} e^{\chi_3 E_{\chi_3}} e^{\chi_4 E_{\chi_4}}\,, \\
\mathcal{U}_\sigma &= e^{\sigma_1 E_{\sigma_1}} e^{\sigma_2 E_{\sigma_2}} e^{\sigma_3 E_{\sigma_3}} e^{\sigma_4 E_{\sigma_4}}\,, \\
\mathcal{U}_\psi &= e^{\psi_1 E_{\psi_1}} e^{\psi_2 E_{\psi_2}} e^{\psi_3 E_{\psi_3}} e^{\psi_4 E_{\psi_4}}\,. \qquad (3.7.7)
\end{aligned}
$$

H_i are the Cartan generators of $O(4,4)$, whilst E_{χ_i}, E_{σ_i} and E_{ψ_i} are the positive-root generators. (See [35] for a detailed description of the notation we are using here.)

3.7.2 Magnetisation of the four-charge static black hole

The usual four-charge black hole carries electric charges (Q) and magnetic charges (P) in the order (P_1, Q_2, P_3, Q_4), where we use our standard ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$ for the gauge fields. The static four-charge solution corresponds, in three dimensions, to

$$
\begin{aligned}
d\bar{s}_3^2 &= [-r(r-2m)dt^2 + \frac{r_1 r_2 r_3 r_4}{r(r-2m)}\, dr^2 + r_1 r_2 r_3 r_4 d\theta^2\,]\,\sin^2\theta\,, \\
r_i &= r + 2m s_i^2\,, \qquad (3.7.8)
\end{aligned}
$$

with

$$
e^{2\varphi_1} = \frac{r_1\, r_3}{r_2\, r_4}\,, \qquad e^{2\varphi_2} = \frac{r_2\, r_3}{r_1\, r_4}\,, \qquad e^{2\varphi_3} = \frac{r_1\, r_2}{r_3\, r_4}\,, \qquad e^{2\varphi_4} = r_1\, r_2\, r_3\, r_4\, \sin^4\theta\,, \qquad (3.7.9)
$$

and

$$
\begin{aligned}
\chi_1 &= 0 \qquad \chi_2 = 0\,, \qquad \chi_3 = 0\,, \qquad \chi_4 = 0\,, \\
\sigma_1 &= -q_1\cos\theta\,, \qquad \sigma_2 = 0\,, \qquad \sigma_3 = -q_3\cos\theta\,, \qquad \sigma_4 = 0\,, \\
\psi_1 &= 0\,, \qquad \psi_2 = q_2\cos\theta\,, \qquad \psi_3 = 0\,, \qquad \psi_4 = q_4\cos\theta\,, \qquad (3.7.10)
\end{aligned}
$$

The magnetisation of the four-charge solution can be implemented by transforming the coset representative $\mathcal{M}$ defined above according to

$$\mathcal{M} \longrightarrow S\mathcal{M}S^T \,, \tag{3.7.11}$$

where S is the $O(4,4)$ matrix

$$S = \exp(12B_1\, E_{\psi_1} + 12B_2\, E_{\sigma_2} + 12B_3\, E_{\psi_3} + 12B_4\, E_{\sigma_4})\,, \tag{3.7.12}$$

with (constant) parameters B_i being the asymptotic values of the magnetic fields of the four field strengths. One then retraces the steps of dualisation and lifts the transformed solution back to four dimensions to obtain the magnetised black hole.[14] The results are presented in section 3.

3.7.3 Magnetic and electric charges

The physical charges can be calculated very easily using the dimensionally-reduced quantities in three dimensions. Using the standard ordering of the $U(1)$ gauge fields, namely $\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$, the magnetic charges are given by

$$\begin{aligned}
P_1 &= \frac{1}{4\pi}\int_{S^2} dA_{(1)1} = \frac{1}{4\pi}\int_{S^2} d\sigma_1\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_1\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_2 &= \frac{1}{4\pi}\int_{S^2} dA_{(1)2} = \frac{1}{4\pi}\int_{S^2} d\sigma_2\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_2\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_3 &= \frac{1}{4\pi}\int_{S^2} d\mathcal{A}^1_{(1)} = \frac{1}{4\pi}\int_{S^2} d\sigma_3\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_3\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_4 &= \frac{1}{4\pi}\int_{S^2} d\mathcal{A}^2_{(1)} = \frac{1}{4\pi}\int_{S^2} d\sigma_4\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_4\Big]_{\theta=0}^{\theta=\pi}\,,
\end{aligned} \tag{3.7.13}$$

[14] We remind the reader that, as discussed in the introduction, when we speak, for the sake of brevity, of the "magnetised electrically-charged black hole" in the STU model we mean the one for which the field strengths numbered 1 and 3 carry magnetic charges and external electric fields, while those numbered 2 and 4 carry electric charges and external magnetic fields.

where $\Delta\phi$ is the period of the azimuthal coordinate ϕ.

The electric charges are given by integrating the equations of motion of the four fields $\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$. These give

$$
\begin{aligned}
Q_1 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1+\varphi_2-\varphi_3} *F_{(2)1} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_1\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_1\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_2 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1+\varphi_2+\varphi_3} *F_{(2)2} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_2\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_2\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_3 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1-\varphi_2+\varphi_3} *\mathcal{F}^1_{(2)} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_3\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_3\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_4 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1-\varphi_2-\varphi_3} *\mathcal{F}^2_{(2)} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_4\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_4\Big]_{\theta=0}^{\theta=\pi}\,.
\end{aligned}
\tag{3.7.14}
$$

(The ellipses here denote the additional terms in the equations of motion. In each case, the full set of terms conspire to give just the simple expressions presented here in terms of the fields ψ_i.)

3.8 Appendix B: STU Model in Other Duality Complexions

As we discussed before, in the formulation [35] that we are using in this chapter for the STU model, the usual four-charge black hole carries electric charges (Q) and magnetic charges (P) in the order (P_1, Q_2, P_3, Q_4), where we use our standard ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_1)$ for the gauge fields. To convert into the parameterisation used, for example, in [27], we need to dualise the potential $A_{(1)2}$ to $B_{(1)}$, whose field strength is the dual of $F_{(2)2}$. To do this, we start from the Lagrangian (4.4.1) and then add a Lagrange multiplier

$$
\mathcal{L}_{LM} = 4dB_{(1)}\wedge(F_{(2)2} - \chi_2\, d\mathcal{A}^1_{(1)} + \chi_3\, dA_{(1)1} - \chi_2\chi_3\, d\mathcal{A}^2_{(1)})\,, \tag{3.8.1}
$$

treating $F_{(2)2}$ now as an independent field that we solve for algebraically and substitute back into the total Lagrangian. This leads to the dualised Lagrangian

$$
\begin{aligned}
\widetilde{\mathcal{L}}_4 \;=\;& R\,{*\mathbb{1}} - 12{*d\varphi_i} \wedge d\varphi_i - 12e^{2\varphi_i}\,{*d\chi_i} \wedge d\chi_i - 2e^{\varphi_1-\varphi_2-\varphi_3}\,{*G_{(2)}} \wedge G_{(2)} \\
& -2e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3}\,{*F_{(2)1}} \wedge F_{(2)1} + e^{-\varphi_2+\varphi_3}\,{*\mathcal{F}^1_{(2)}} \wedge \mathcal{F}^1_{(2)} + e^{-\varphi_2-\varphi_3}\,{*\mathcal{F}^2_{(2)}} \wedge \mathcal{F}^2_{(2)}\Big) \\
& -4\chi_1\, F_{(2)1} \wedge \mathcal{F}^1_{(2)} + 4dB_{(1)} \wedge (\chi_3\, dA_{(1)1} - \chi_2\, d\mathcal{A}^1_{(1)} - \chi_2\chi_3\, d\mathcal{A}^2_{(1)})\,,
\end{aligned}
\tag{3.8.2}
$$

where $G_{(2)} = e^{-\varphi_1+\varphi_2+\varphi_3}\,{*F_{(2)2}}$, which is written in terms of the potential $B_{(1)}$ as

$$
G_{(2)} = dB_{(1)} - \chi_1\, d\mathcal{A}^2_{(1)}\,. \tag{3.8.3}
$$

If we now define

$$
\begin{aligned}
\widetilde{\chi}^1 \;=\;& -\chi_1\,, \qquad \widetilde{\chi}^2 = -\chi_2\,, \qquad \widetilde{\chi}^3 = \chi_3\,, \\
h^I \;=\;& f^{-1}\, e^{-\varphi_i}\,, \qquad f^3 = e^{-\varphi_1-\varphi_2-\varphi_3}\,, \qquad G_{IJ} = \mathrm{diag}\{(h^1)^{-2}, (h^2)^{-2}, (h^3)^{-2}\}\,, \\
A^{[0]}_{(1)} \;=\;& \mathcal{A}^2_{(1)}\,, \qquad A^{[1]}_{(1)} = B_{(1)}\,, \qquad A^{[2]}_{(1)} = A_{(1)1}\,, \qquad A^{[3]}_{(1)} = \mathcal{A}^1_{(1)}\,,
\end{aligned}
\tag{3.8.4}
$$

then (3.8.2) can be written,

$$
\begin{aligned}
\widetilde{\mathcal{L}} \;=\;& R{*\mathbb{1}} - 12G_{IJ}\,{*dh^I} \wedge dh^J - 32f^{-2}\,{*df} \wedge df - 12f^3\,{*F^{[0]}_{(2)}} \wedge F^{[0]}_{(2)} \\
& -12f^{-2}\,G_{IJ}\,{*d\widetilde{\chi}^I} \wedge d\widetilde{\chi}^J - 12f\,G_{IJ}\,({*F^{[I]}_{(2)}} + \widetilde{\chi}^I\,{*F^{[0]}_{(2)}}) \wedge (F^{[J]}_{(2)} + \widetilde{\chi}^J\,F^{[0]}_{(2)}) \\
& +12C_{IJK}\Big[\widetilde{\chi}^I\,F^{[J]}_{(2)} \wedge F^{[K]}_{(2)} + \widetilde{\chi}^I\,\widetilde{\chi}^J\,F^{[0]}_{(2)} \wedge F^{[K]}_{(2)} + 13\widetilde{\chi}^I\,\widetilde{\chi}^J\,\widetilde{\chi}^K\,F^{[0]}_{(2)} \wedge F^{[0]}_{(2)}\Big]\,,
\end{aligned}
\tag{3.8.5}
$$

where $F^{[\Lambda]}_{(2)} = dA^{[\Lambda]}_{(1)}$ and $C_{IJK} = |\epsilon_{IJK}|$. The charges carried by the four-charge black hole in [35] will now be of the form (Q, P, P, P), where the fields are ordered ($A^{[0]}_{(1)}$, $A^{[1]}_{(1)}$, $A^{[2]}_{(1)}$, $A^{[3]}_{(1)}$).

Note that we can in principle perform a further transformation on the Lagrangian (3.8.2), and dualise the gauge potential $\mathcal{A}^2_{(1)}$ also. This would result in a formulation where the standard four-charge black hole in [35] would be supported by four gauge fields that all carried magnetic charge. This dualisation can be achieved by adding a Lagrange multiplier $4d\widetilde{B}_{(1)}\wedge\mathcal{F}^2_{(2)}$ to (3.8.2), and then solving algebraically for $\mathcal{F}^2_{(2)}$ and substituting back into the total Lagrangian. The equation for $\mathcal{F}^2_{(2)}$ is quite complicated, taking the form

$$\alpha\,{*\mathcal{F}^2_{(2)}} = H_{(2)} + \beta\,\mathcal{F}^2_{(2)}\,, \tag{3.8.6}$$

where

$$\begin{aligned} \alpha &= e^{-\varphi_1-\varphi_2-\varphi_3} + \chi_1^2\,e^{\varphi_1-\varphi_2-\varphi_3} + \chi_2^2\,e^{-\varphi_1+\varphi_2+\varphi_3} + \chi_3^2\,e^{-\varphi_1-\varphi_2+\varphi_3}\,, \qquad \beta = 2\chi_1\,\chi_2\,\chi_3\,,\\ H_{(2)} &= d\widetilde{B}_{(1)} - \chi_2\,\chi_3\,dB_{(1)} - \chi_1\,\chi_3\,dA_{(1)1} + \chi_1\,\chi_2\,d\mathcal{A}^1_{(1)} + \chi_1\,e^{\varphi_1-\varphi_2-\varphi_3}\,{*dB_{(1)}}\\ &\quad +\chi_2\,e^{-\varphi_1+\varphi_2+\varphi_3}\,{*dA_{(1)1}} - \chi_3\,e^{-\varphi_1-\varphi_2+\varphi_3}\,{*d\mathcal{A}^1_{(1)}}\,. \end{aligned} \tag{3.8.7}$$

Equation (3.8.6) can be solved for $\mathcal{F}^2_{(2)}$, giving

$$\mathcal{F}^2_{(2)} = -\,\frac{\alpha\,{*H_{(2)}} + \beta\,H_{(2)}}{\alpha^2+\beta^2}\,, \tag{3.8.8}$$

but the result seems to be rather too complicated to be useful.

Chapter 4

Quasi-normal modes: Static and Rotating

The aim of this chapter is to analyse the quasi-normal solutions of the scalar wave equation in the background of the above mentioned subtracted rotating geometry and the subtracted magnetised geometry, by employing their hidden $SL(2,\mathbb{R}) \times SO(2,\mathbb{R}) \times SO(3)$ symmetry. We do so by first explicitly solving the wave equation for a massless scalar field in four dimensions, which due to the very special structure of the metric is separable and solvable in terms of hypergeometric functions and spherical harmonics both for subtracted rotating and subtracted magnetised geometries. In each case we obtain two branches of quasi-normal modes, with remarkably simple values of complex eigenfrequencies, one over-damped and one under-damped. Specifically, in the case of magnetised geometries the effect of the magnetic field turns out to be an additive shift of the real part of the eigenfrequency of the quasi-normal modes. The regularity of these solutions near the outer horizon is analysed in terms of Kruskal-Szekeres coordinates. These results are presented for subtracted rotating geometries in Section 2 and for subtracted magnetised geometries in Section 3.

The analysis is further extended by studying the wave equation for a minimally coupled massive scalar field in the five-dimensional lift of these subtracted geometries. For both rotating and magnetised cases, the lift on a circle S^1 results in a geometry that is locally BTZ $\times$ S^2, a product of the BTZ black hole and a two-sphere. As a consequence, the wave equation for a massive minimally coupled scalar field is separable and may be solved again in terms of the hypergeometric functions, spherical harmonics and a plane wave along the S^1 circle direction. Remarkably simple, explicit expressions for the frequencies of the two branches of the quasi-normal modes are obtained , where the quantised wave number along the S^1 circle shifts the real part of he eigenfrequencies. For the special case of the zero wave number and zero five-dimensional mass, one reproduces the results of Sections 2 and 3 as expected. Solutions for the non-zero wave numbers can be interpreted as quasi-normal modes for the massive four-dimensional Kaluza-Klein modes whose electric charge is proportional to the wave number. The regularity of these modes near the outer horizon is manifest after performing a Kaluza-Klein U(1) gauge transformation on the wave function. All of these results are presented in Section 4.

4.1 Subtracted Rotating Geometry

The metric for the four-charge rotating black hole solution of the STU model can be written in the form [68, 123]:

$$ds_4^2 = -\Delta_0^{-\frac{1}{2}} G(dt + \mathcal{A})^2 + \Delta_0^{\frac{1}{2}} (\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta d\phi^2), \tag{4.1.1}$$

with

$$\begin{aligned} X &= r^2 - 2mr + a^2 \ , \\ G &= r^2 - 2mr + a^2 \cos^2\theta \ , \\ \mathcal{A} &\equiv \frac{a \sin^2\theta A_{red}}{G} = \frac{2ma\sin^2\theta}{G} \left[(\Pi_c - \Pi_s) r + 2m\Pi_s \right] d\phi \ , \end{aligned} \tag{4.1.2}$$

and the warp factor Δ_0 given by

$$\begin{aligned} \Delta_0 = \quad & \prod_{i=1}^{4} (r + 2m\sinh^2\delta_i) + 2a^2\cos^2\theta [r^2 + mr \sum_{i=1}^{4} \sinh^2\delta_i + \ 4m^2(\Pi_c - \Pi_s)\Pi_s \\ & -2m^2 \sum_{i<j<k} \sinh^2\delta_i \sinh^2\delta_j \sinh^2\delta_k] + a^4\cos^4\theta \, . \end{aligned} \tag{4.1.3}$$

The mass, four charges and the angular momentum are parameterised as

$$\begin{aligned} G_4 M &= \frac{1}{4} m \sum_{i=1}^{4} \cosh 2\delta_i \, , \\ G_4 Q_i &= \frac{1}{4} m \sinh 2\delta_i \, , \quad i = 1, 2, 3, 4 \, , \\ G_4 J &= ma(\Pi_c - \Pi_s) \, , \end{aligned} \tag{4.1.4}$$

with G_4 the four-dimensional Netwon's constant and we employ the abbreviations

$$\Pi_c \equiv \prod_{i=1}^{4} \cosh \delta_i \ , \quad \Pi_s \equiv \prod_{i=1}^{4} \sinh \delta_i \ . \tag{4.1.5}$$

The two horizons, given by $X = 0$, are at

$$r_\pm = m \pm \sqrt{m^2 - a^2} \, . \tag{4.1.6}$$

It was shown in [68] that the replacement

$$\Delta_0 \to \Delta = (2m)^3 r(\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2\theta \, , \tag{4.1.7}$$

in the metric (6.0.5) reduces the highest power of r in Δ_0 and renders in the radial part of the massless scalar wave equation the irregular singular point at infinity regular, allowing for solutions in terms of hypergeometric functions. Moreover, the massless scalar wave equation is separable in terms of ordinary spherical harmonics, rather than the complicated spheroidal functions needed for the full four-charge black hole solution. This new metric has been dubbed a "subtracted geometry" and the massless scalar wave equation in this background exhibits a hidden SL(2,R)×SL(2,R)×SO(3) symmetry. Furthermore, at the outer and inner horizons the entropies

$$S_\pm = \frac{2\pi m}{G_4} \left[(\Pi_c + \Pi_s) m \pm (\Pi_c - \Pi_s) \sqrt{m^2 - a^2} \right] \, , \tag{4.1.8}$$

the inverse surface gravities

$$\frac{1}{\kappa_\pm} = 2m \left[\frac{m}{\sqrt{m^2 - a^2}} (\Pi_c + \Pi_s) \pm (\Pi_c - \Pi_s) \right] \, , \tag{4.1.9}$$

and the angular velocities

$$\Omega_\pm = \kappa_\pm \frac{a}{\sqrt{m^2 - a^2}} \, , \tag{4.1.10}$$

remain unchanged by this replacement, thus preserving the local geometry and thermodynamic properties of the metric. The expressions simplify significantly in the static case when $a = 0$.

It is straightforward to see that these black hole solutions and their subtracted geometry encompasses the following special cases:

$$\begin{aligned}
\text{Kerr-Newman:} \qquad & \delta_1 = \delta_2 = \delta_3 = \delta_4\,, \\
\text{Kerr:} \qquad & \delta_i = 0\,, \quad i = 1,2,3,4\,, \\
\text{Reissner-Nordström:} \qquad & \delta_1 = \delta_2 = \delta_3 = \delta_4\,, \quad a = 0\,, \\
\text{Schwarzschild:} \qquad & \delta_i = 0\,, \quad a = 0\,, \quad i = 1,2,3,4\,.
\end{aligned} \tag{4.1.11}$$

4.1.1 Kruskal-Szekeres Coordinates for Subtracted Rotating Geometry

In the following we construct Kruskal-Szekeres type coordinates to cover the outer horizon which allow us to identify suitable boundary conditions there[15]. At infinity the appropriate boundary condition is boundedness of the solution. The construction of Kruskal-Szekeres coordinates is in fact considerably simpler than that used for the Kerr solution [79, 80].

The subtracted metric (6.0.5), (4.1.2) with (4.1.7) can be cast in the following remarkably simple form[16]:

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\Big(-dt^2 + \frac{F^2 dr^2}{X^2}\Big) + \sqrt{\Delta}d\theta^2 + \frac{F^2 \sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2\,, \tag{4.1.12}$$

with

$$W = -\frac{aA_{red}}{F^2}\,, \quad F^2 = (2m)^2\left[2m(\Pi_c^2 - \Pi_s^2)r + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2\right]. \tag{4.1.13}$$

[15] One can analogously construct Kruskal-Szekeres type coordinates to cover the inner horizon region.

[16] This structure was also anticipated in [68] by evaluating the Laplacian of the subtracted rotating geometry.

X and A_{red} are defined in (4.1.2) and we display them again

$$X = r^2 - 2mr + a^2\,, \quad A_{red} = 2m(\Pi_c - \Pi_s)r + (2m)^2\Pi_s\,. \tag{4.1.14}$$

Importantly, X, F and W are only functions of r. We also note that the factor Δ (4.1.7) can be written in terms of F^2 as

$$\Delta = F^2 + (2m)^2 a^2 (\Pi_c - \Pi_s)^2 \sin^2\theta\,. \tag{4.1.15}$$

It is straightforward to show that

$$\frac{1}{\kappa_\pm} = \frac{2F(r_\pm)}{r_+ - r_-}\,, \tag{4.1.16}$$

and

$$\Omega_\pm = -W(r_\pm)\,. \tag{4.1.17}$$

This special property of the angular velocities and surface gravities leads to an asymmetry of two branches of the quasi-normal modes as analysed later in this Section.

We now construct Kruskal-Szekeres type coordinates to cover the horizon which allow us to identify suitable boundary conditions there. Due to the structure of the metric (4.1.12) the construction of Kruskal-Szekeres coordinates is straightforward.

The metric (4.1.12) allows for the introduction of retarded and advanced co-rotating Eddington-Finkelstein coordinates:

$$u = t - r*\,, \qquad v = t + r*\,, \qquad \phi_+ = \phi + W(r_+)t\,, \tag{4.1.18}$$

which satisfy

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 = g^{\alpha\beta}\partial_\alpha v \partial_\beta v\,. \tag{4.1.19}$$

The Hamilton-Jacobi equation is separable, yielding a solution

$$r* = \int^r \frac{F dr}{X}\,, \tag{4.1.20}$$

which is manifest for the metric (4.1.12).

The co-rotating Kiling vector

$$l^+ = \frac{\partial}{\partial t} - W(r_+)\frac{\partial}{\partial \phi}\,, \tag{4.1.21}$$

coincides with the null generator of the horizon. The angle ϕ_+ is constant along the orbits of the co-rotating Killing vector l^+:

$$l^+\phi_+ = (\partial_t - W(r_+)\partial_\phi)\phi_+ = 0\,. \tag{4.1.22}$$

We introduce Kruskal-Szekeres coordinates:

$$U = -e^{-\kappa_+ u}\,, \qquad V = e^{\kappa_+ v}\,, \tag{4.1.23}$$

and thus

$$\begin{aligned} \frac{dV}{V} + \frac{dU}{U} &= \frac{2\kappa_+ F dr}{X}\,, \\ \frac{dV}{V} - \frac{dU}{U} &= 2\kappa_+ dt\,. \end{aligned} \tag{4.1.24}$$

In terms of Kruskal-Szekeres coordinates the metric (4.1.12) takes the following form:

$$\begin{aligned} ds^2 &= \sqrt{\Delta}\,\frac{X}{F^2}\frac{dUdV}{\kappa_+^2 UV} + \sqrt{\Delta}d\theta^2 \\ &+ \frac{F^2\sin^2\theta}{\sqrt{\Delta}}\left[d\phi_+ + \frac{1}{2\kappa_+}(W(r) - W(r_+))(\frac{dV}{V} - \frac{dU}{U})\right]^2\,. \end{aligned} \tag{4.1.25}$$

In the vicinity of the outer horizon $r \sim r_+$ one has

$$r* = \int^r \frac{F(r)dr}{X} \sim \frac{F(r_+)}{r_+ - r_-} \ln(r - r_+) = \frac{1}{2\kappa_+} \ln(r - r_+)\,, \tag{4.1.26}$$

where we used (4.1.16) at the last step. This ensures

$$-UV = e^{2\kappa_+ r*} \sim (r - r_+)\,, \tag{4.1.27}$$

and the metric (4.1.25) is regular and analytic.

An argument given by Hawking and Reall [81] in the asymptotically AdS case may be adapted to show that if the co-rotating Killing vector l^+ (4.1.21) is timelike outside the horizon then there can be no super-radiance instability or a black hole bomb [82, 83].

The length squared of the co-rotating Killing vector l^+ (4.1.21) is

$$g^{\alpha\beta} l^+_\alpha l^+_\beta = -\frac{1}{\sqrt{\Delta}} \left[X + \frac{a^2 \sin^2\theta (\Pi_c - \Pi_s)^2 (r_+ - r_-)(r - r_+)}{[(\Pi_c - \Pi_s) r_+ + 2m\Pi_s]^2} \right] . \tag{4.1.28}$$

which is manifestly negative for $r > r_+$ and thus their is no super-radiance.

4.1.2 Massless Wave Equation and Quasi-Normal Modes

The massless scalar wave equation for the multi-charge black hole metric (6.0.5) is separable and the solutions expressible in terms of spheroidal functions of θ [66, 67]. The radial function may be expressed in terms of solutions of a confluent form of Heun's equation which has two regular singular points and an irregular singular point at infinity.

For the subtracted geometry metric (4.1.12) the massless scalar wave equation is also separable and of a specific form:

$$e^{-i\omega t} e^{in\phi} P^n_l(\theta) \chi(x)\,, \tag{4.1.29}$$

where $P_l^n(\theta)$ is an associated Legendre polynomial, the solution of the unit two-sphere S^2 Laplacian with eigenvalues $l(l+1)$, $l = 0, 1, \ldots$ and $n = \pm l, \pm(l-1), \ldots$

The radial equation takes the form [66, 67]:

$$\Big[\frac{\partial}{\partial x}(x^2 - \frac{1}{4})\frac{\partial}{\partial x} + \frac{1}{4(x-\frac{1}{2})}(\frac{\omega}{\kappa_+} - n\frac{\Omega_+}{\kappa_+})^2 - \frac{1}{4(x+\frac{1}{2})}(\frac{\omega}{\kappa_-} - n\frac{\Omega_-}{\kappa_-})^2 - l(l+1)\Big]\chi(x) = 0\,, \tag{4.1.30}$$

where

$$x = \frac{r - \frac{1}{2}(r_+ + r_-)}{r_+ - r_-}\ , \tag{4.1.31}$$

is designed so that the two horizons $r_\pm$ are at $x = \pm\frac{1}{2}$.

Due to (4.1.10) rotating solutions have the property:

$$\frac{\Omega_+}{\kappa_+} = \frac{\Omega_-}{\kappa_-}\,, \tag{4.1.32}$$

and thus the solutions to (4.1.30) depend only on one ratio $\Omega_+\kappa_+^{-1}$, only.

Solutions which are ingoing on the future horizon must be regular at $U = 0$ in Kruskal-Szekeres coordinates and this implies [66–68]

$$\begin{aligned}\chi(x) \;&=\; (x+\frac{1}{2})^{-(l+1)}(\frac{x-\frac{1}{2}}{x+\frac{1}{2}})^{-i(\omega-n\Omega_+)\frac{\beta_H}{4\pi}}\\ &\times\; F(l+1-i\frac{\beta_R\omega - 2n\beta_H\Omega_+}{4\pi}, l+1-i\frac{\beta_L\omega}{4\pi}, 1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi}; \frac{x-\frac{1}{2}}{x+\frac{1}{2}})\,,\end{aligned} \tag{4.1.33}$$

where

$$\frac{\beta_H}{2\pi} = \frac{1}{\kappa_+}\,,\quad \frac{\beta_R}{2\pi} = \frac{1}{\kappa_+} + \frac{1}{\kappa_-}\,,\quad \frac{\beta_L}{2\pi} = \frac{1}{\kappa_+} - \frac{1}{\kappa_-}\,. \tag{4.1.34}$$

Near the outer horizon $r^\star \to -\infty$, $(x-\frac{1}{2})(x+\frac{1}{2})^{-1} \to e^{2\kappa_+ r^\star}$ and so

$$\chi(x) \approx e^{-i(\omega-n\Omega_+)r^\star} F(l+1-i\frac{\beta_R\omega - 2n\beta_H\Omega_+}{4\pi}, l+1-i\frac{\beta_L\omega}{4\pi}, 1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi}; e^{2\kappa_+ r^\star})\,. \tag{4.1.35}$$

In Kruskal-Szekeres coordinates therefore

$$e^{-i\omega t}e^{in\phi}\chi(x)\approx e^{in\phi_+}V^{-i\frac{\omega-n\Omega_+}{\kappa_+}}(1+\dots)\,, \tag{4.1.36}$$

where the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon $U=0$.

At large x [66, 67]

$$\begin{aligned}\chi(x) &\approx x^{-(l+1)}\frac{\Gamma(1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(-2l-1)}{\Gamma(-l-i\frac{\beta_L\omega}{4\pi})\Gamma(-l-i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi})}\\ &+ x^l\frac{\Gamma(1-i\frac{\omega\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(2l+1)}{\Gamma(l+1-i\frac{\beta_L\omega}{4\pi})\Gamma(l+1-i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi})}\,.\end{aligned} \tag{4.1.37}$$

In order that χ be finite at spatial infinity, we must set

$$\begin{aligned} & i\omega\frac{\beta_L}{4\pi}=l+1+N_L\,,\\ \text{or}\qquad & i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi}=l+1+N_R\,,\end{aligned} \tag{4.1.38}$$

where $N_{L,R}=0,1,\dots$ This gives remarkably simple formulae for the frequencies of the quasi-normal modes

$$\begin{aligned} & \omega=-\frac{i}{2m(\Pi_c-\Pi_s)}(1+l+N_L)\,,\\ \text{or}\qquad & \omega=-\frac{i\sqrt{m^2-a^2}}{2m^2(\Pi_c+\Pi_s)}(1+l+N_R)+\frac{a}{2m^2(\Pi_c+\Pi_s)}n\,.\end{aligned} \tag{4.1.39}$$

Both frequencies result in damped modes, with the under-damped branch exhibiting oscillatory behaviour and the damping absent in the extremal limit $a\to m$. The specific asymmetry in frequencies of the two branches, resulting in the oscillatory behaviour of the under-damped branch only, is due to the special relationship between ratios (4.1.32). It is intriguing that the expressions are no more complex than those in the Kerr case [84]. In particular, eq. (4.1.39) agrees with eq. (0.28) of [84] which was obtained for the subtracted geometry of the neutral Kerr solution, i.e. the case with $\delta_i = 0$, and thus $\Pi_c = 1$ and $\Pi_s = 0$.

The subtracted geometry has a remarkable property that in the near-BPS limit ($m \to 0$, $a \to 0$, $\delta_i \to \infty$, with $me^{2\delta_i}$ and ma^{-1} finite) the near-horizon geometry of such black holes and their subtracted geometry are the same. As a consequence, the quasi-normal modes of the near-BPS black holes and those of their subtracted geometry are the same[17].

4.2 Subtracted Magnetised Geometry

The original subtracted Melvin metric was derived in [75] as a scaling limit of magnetised STU black holes. It describes a generalization of the (static) subtracted geometry, parameterised by an additional magnetic field parameter β_4 which is associated with the magnetic component of the Kaluza-Klein gauge field $\mathcal{A}_2$. The full solution is given in the Appendix 5.2.

Remarkably, one may cast this metric in the same form as the rotating subtracted metric (4.1.12), which we display again

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\Big(-dt^2 + \frac{F^2 dr^2}{X^2}\Big) + \sqrt{\Delta}d\theta^2 + \frac{F^2 \sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2\,, \tag{4.2.1}$$

[17] We are grateful for Shahar Hod for pointing out to us after the appearance of [84] that if one specialises to the near-BPS case of slowly rotating ($a \ll m$) Kerr-Newman black holes then $\beta_R \simeq 2\beta_H$ and the family of modes given by eq. (11) of [85] have identical frequencies to those of the second family of modes in eq. (0.28) of [84] and hence to the second family of (4.1.39) of this paper. The first family of (4.1.39) in this limit corresponds to negative imaginary frequencies whose absolute values are much larger than those of the second family, and thus this (ultra-damped) branch did not appear in [85].

where now

$$
\begin{aligned}
X &= r^2 - 2mr\,, \\
F^2 &= (2m)^3\left[(\Pi_c^2 - \Pi_s^2)r + (2m)\Pi_s^2\right]\,, \\
W &= -\frac{16m^4\Pi_s\Pi_c\beta_4}{F^2}\,, \\
\Delta &= F^2 + (2m)^6\beta_4^2(\Pi_c^2 - \Pi_s^2)^2\sin^2\theta\,.
\end{aligned}
\tag{4.2.2}
$$

This is effectively a generalization of the static subtracted geometry with the magnetic field parameter β_4 introducing a specific spatial rotation. The metric has two horizons

$$
r_+ = 2m\,, \quad r_- = 0\,. \tag{4.2.3}
$$

The inverse surface gravities of the inner and outer horizon are determined by

$$
\frac{1}{\kappa_+} = \frac{2F(r_+)}{r_+ - r_-} = 4m\Pi_c\,, \quad \frac{1}{\kappa_-} = \frac{2F(r_-)}{r_+ - r_-} = 4m\Pi_s\,, \tag{4.2.4}
$$

and are the same as the inverse surface gravities for the static subtracted geometry, i.e. (7.0.10) with $a = 0$. The angular velocities at the inner and outer horizon are are given by

$$
\Omega_+ = -W(r_+) = \beta_4\frac{\Pi_s}{\Pi_c}\,, \quad \Omega_- = -W(r_-) = \beta_4\frac{\Pi_c}{\Pi_s}\,. \tag{4.2.5}
$$

Note that in this case the ratios

$$
\frac{\Omega_+}{\kappa_+} = 4m\beta_4\Pi_s\,, \quad \frac{\Omega_-}{\kappa_-} = 4m\beta_4\Pi_c\,, \tag{4.2.6}
$$

are different, and now the radial part of the massless scalar wave equation (4.1.30) depends on both independent ratios.

4.2.1 Kruskal-Szekeres Coordinates for Subtracted Magnetised Geometry

The retarded and advanced co-rotating Eddington-Finkelstein coordinates are of the same form as in (4.1.18) and the Killing vector l^+ (4.1.21) again coincides with the null generator on the horizon.

We introduce the Kruskal-Szekeres coordinates (4.1.23) which yield (4.1.24) and the metric (4.2.1) takes the form (4.1.25). In the vicinity of the outer horizon $r \sim 2m$ one obtains $-UV \sim (r-2m)$, and thus the metric (4.1.25) is regular and analytic there.

We calculate the length squared of the co-rotating Killing vector l^+ (4.1.21)

$$\begin{aligned} g^{\alpha\beta} l^+_\alpha l^+_\beta &= -\frac{\sqrt{\Delta_s}}{F} \frac{(r-2m)}{(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2 + 8m^3\beta_4^2 \sin^2\theta(\Pi_c^2-\Pi_s^2)^2} \\ &\times \left[(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2\right] \times \left[r + 8m^3\beta_4^2\sin^2\theta \frac{1}{\Pi_c^2}(\Pi_c^2-\Pi_s^2)^2\right], \end{aligned} \tag{4.2.7}$$

which is negative outside the horizon, $r > 2m$. Thus, this geometry is stable with no super-radiance.

4.2.2 Massless Wave Equation and Quasi-Normal Modes

The massless wave equation is again separable with the same wave function Ansatz as (4.1.29). The radial wave equation can be cast in the same form as (4.1.30) with the inverse surface gravities (4.2.4) and angular velocities (4.2.5).

Solutions which are ingoing on the future horizon must be regular at $U = 0$ in Kruskal-Szekeres coordinates and this implies that [66–68]

$$\begin{aligned} &\chi(x) = (x+\frac{1}{2})^{-(l+1)}(\frac{x-\frac{1}{2}}{x+\frac{1}{2}})^{-i(\omega-n\Omega_+)\frac{\beta_H}{4\pi}} \\ &F(l+1-i\frac{\beta_R\omega - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi}, l+1-i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi}, 1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi}; \frac{x-\frac{1}{2}}{x+\frac{1}{2}}), \end{aligned}$$

where again

$$\frac{\beta_H}{2\pi} = \frac{1}{\kappa_+}\,, \quad \frac{\beta_-}{2\pi} = \frac{1}{\kappa_-} \quad \frac{\beta_R}{2\pi} = \frac{1}{\kappa_+} + \frac{1}{\kappa_-}\,, \quad \frac{\beta_L}{2\pi} = \frac{1}{\kappa_+} - \frac{1}{\kappa_-}\,. \tag{4.2.8}$$

Near the outer horizon $r^\star \to -\infty$, $(x-\frac{1}{2})(x+\frac{1}{2})^{-1} \to e^{2\kappa_+ r^\star}$ and so

$$\chi(x) \approx e^{-i(\omega - n\Omega_+)r^\star}$$
$$F(l+1-i\frac{\beta_R\omega - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi}, l+1-i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi}, 1-i\frac{\beta_H(\omega - n\Omega_+)}{2\pi}; e^{2\kappa_+ r^\star})\,.$$

In Kruskal-Szekeres coordinates therefore

$$e^{-i\omega t}e^{in\phi}\chi(x) \approx e^{in\phi_+}V^{-i\frac{\omega - n\Omega_+}{\kappa_+}}(1+\dots) \tag{4.2.9}$$

where the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon $U=0$.

At large x [66, 67]

$$\chi(x) \approx x^{-(l+1)}\frac{\Gamma(1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(-2l-1)}{\Gamma(-l-i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi})\Gamma(-l-i\frac{\omega\beta_R - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi})}$$
$$+x^l\frac{\Gamma(1-i\frac{\omega\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(2l+1)}{\Gamma(l+1-i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi})\Gamma(l+1-i\frac{\omega\beta_R - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi})}\,. \tag{4.2.10}$$

In order that χ be finite at spatial infinity, we must set

$$i\left(\frac{\omega\beta_L}{4\pi} - n\frac{\beta_H\Omega_+ - \beta_-\Omega_-}{4\pi}\right) = l+1+N_L\,,$$
$$\text{or} \qquad i\left(\frac{\omega\beta_R}{4\pi} - n\frac{\beta_H\Omega_+ + \beta_-\Omega_-}{4\pi}\right) = l+1+N_R\,, \tag{4.2.11}$$

where $N_{L,R} = 0, 1, \dots$ This gives remarkably simple and symmetric formulae for the frequencies of the quasi-normal modes

$$\begin{aligned} \omega &= -\frac{i}{2m(\Pi_c - \Pi_s)}(1 + l + N_L) - n\beta_4\,, \\ \text{or} \qquad \omega &= -\frac{i}{2m(\Pi_c + \Pi_s)}(1 + l + N_R) + n\beta_4\,. \end{aligned} \tag{4.2.12}$$

Both frequencies result in damped modes with a symmetric shift in advanced and retarded oscillatory behaviour due to the magnetic field parameter β_4.

An interesting observation can be made here about the magnetic field parameter in the above quasi-normal modes. According to the Bohr's correspondence principle, the frequency of oscillation of a classical system is equivalent to the frequency of transition of the corresponding quantum system. Guided by this principle, in [76], some observations were made which indicate that the real part of the quasi-normal modes is related to the quantized area spectrum of the quantum black hole. In our case the real part of the quasi-normal modes is related in a very simple way to the magnetic field parameter, thus making it easy to see how turning on the magnetic field affects the area spectrum of the quantum black hole.

4.3 Lifted Geometries and Quasi-Normal Modes

In Appendix E we derive the explicit lift of the subtracted geometries on a circle of size $2\pi R$ and parameterised by a coordinate z. The five-dimensional geometry is locally BTZ $\times\, S^2$ with the BTZ coordinates denoted by $\{t_3, r_3, \phi_3\}$ and the S^2 coordinates denoted by $\{\theta, \bar{\phi}\}$. The explicit transformation between $\{t, r, \theta, \phi, z\}$ coordinates, and the BTZ$\times S^2$ coordinates is given in the Appendix 5.3, too. The BTZ metric (4.6.5) can also be cast into local AdS_3 metric (4.7.7), parameterised by coordinates $\{T, \rho, \Phi\}$. The explicit transformation between the BTZ and the local AdS_3 coordinates is given in Appendix 5.4, following [77, 78]. The radius of AdS_3 is ℓ and the radius of S^2 is $\frac{\ell}{2}$. Specifically, $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$.

Since for this five-dimensional geometry the wave equation for the minimally coupled massive scalar field is separable and exactly solvable, this allows us to study explicitly the quasi-normal modes directly in five dimensions. Furthermore, the scalar field wave function can be expanded in terms of Kaluza-Klein modes, parameterised by a quantised wave number k along the circle direction z. We can therefore study the quasi-normal modes for each Kaluza-Klein mode by solving directly the wave equation in five dimensions for the complete tower of Kaluza-Klein states, i.e. we do not have to resort to solving a complicated equation for each Kaluza-Klein mode separately.

The wave equation for a massive, minimally coupled scalar field Φ in the local $\mathrm{AdS}_3 \times S^2$ background is separable and solved with the Ansatz

$$\mathbf{\Phi} = e^{-i\bar{\omega}T} e^{i\bar{k}\Phi} e^{in\bar{\phi}} \, P_l^n(\cos\theta)\, \chi(\rho)\,. \tag{4.3.1}$$

$P_l^n(\cos\theta)$, the associated Legendre function, is a solution for the Laplacian of the unit two-sphere S^2 with eigenvalues $l(l+1)$. Here $n = 0, \pm 1, \pm 2 ... \pm l$ and l is a non-negative integer. Again, $\{T, \Phi, \rho\}$ and $\{\theta, \bar{\phi}\}$ parameterise the local AdS_3 and S^2 coordinates, respectively. Furthermore, in our context the radius of AdS_3 is ℓ and that of S^2 is $\frac{\ell}{2}$ where we have $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$ (see Appendix 5.3).

The metric, describing a local AdS_3 (4.7.7)

$$ds^2_{AdS_3} = \ell^2 \,(-\sinh^2\rho\, dT^2 + d\rho^2 + \cosh^2\rho\, d\Phi^2)\,. \tag{4.3.2}$$

has the Laplacian

$$\Box_{AdS_3} = \partial_\rho^2 + \frac{2\cosh(2\rho)}{\sinh(2\rho)}\partial_\rho - \frac{1}{\sinh^2\rho}\partial_T^2 + \frac{1}{\cosh^2\rho}\partial_\Phi^2\,, \tag{4.3.3}$$

and enters the five-dimensional Klein-Gordon equation equation in the following form:

$$[\ell^2\,(\Box_{AdS_3} - 4l(l+1)) - M_5^2]\mathbf{\Phi} = 0 \tag{4.3.4}$$

Note again that $4\ell^2 l(l+1)$ is the eigenvalue of the two-sphere S^2 Laplacian with the two-sphere radius $\frac{\ell}{2}$. For the Ansatz (4.3.1) this equation becomes

$$\Big[\ell^2\big(\partial_\rho^2 + \frac{2\cosh(2\rho)}{\sinh(2\rho)}\partial_\rho + \frac{\bar{\omega}^2}{\sinh^2\rho} - \frac{\bar{k}^2}{\cosh^2\rho} - 4l(l+1)\big) - M_5^2\Big])\chi(\rho) = 0\,. \tag{4.3.5}$$

The solution, corresponding to the incoming wave at the outer horizon, is

$$\begin{aligned}\chi(\rho) &= (x+\frac{1}{2})^{-(\bar{l}+1)}(\frac{x-\frac{1}{2}}{x+\frac{1}{2}})^{-i\frac{\bar{\omega}}{2}} \\ & F(\bar{l}+1-i\frac{(\bar{\omega}+\bar{k})}{2}, \bar{l}+1-i\frac{(\bar{\omega}-\bar{k})}{2}, 1-i\bar{\omega}; \tanh^2\rho)\,.\end{aligned} \tag{4.3.6}$$

Here we have introduced

$$\bar{l}(\bar{l}+1) \equiv l(l+1) + \frac{M_5^2}{4\ell^2}\,. \tag{4.3.7}$$

While the analysis can be completed for massive minimally coupled five-dimensional scalars, in the following we will focus on massless ones, i.e. taking $M_5 = 0$ and thus $\bar{l} = l$. The only quantitative difference in the analysis for massive five-dimensional scalars is that the expressions below involve a change $l \to \bar{l} > l$, and thus a shift in the quasi-normal frequencies.

At this point we relate the respective local AdS$_3$ and S^2 coordinates $\{T, \Phi, \rho\}$ and $\{\theta, \bar{\phi}\}$ to $\{t, r, \theta, \phi, z\}$. This can be done by first employing Appendix C, where the explicit lift to the BTZ $\times\, S^2$ and the map to the BTZ and S^2 coordinates is given, and then employing Appendix D, where the transformation between the BTZ and local AdS$_3$ coordinates is provided. The result for the subtracted rotating geometry is

$$\begin{aligned} T &= \frac{4\sqrt{m^2-a^2}}{\ell^3}(\frac{t}{\kappa_+} - \frac{z}{\kappa_-})\,, \\ \Phi &= \frac{4\sqrt{m^2-a^2}}{\ell^3}(\frac{z}{\kappa_+} - \frac{t}{\kappa_-})\,, \end{aligned} \tag{4.3.8}$$

and

$$\cosh^2\rho = x + \frac{1}{2}\,, \quad \sinh^2\rho = x - \frac{1}{2}\,, \tag{4.3.9}$$

where x is defined in (4.1.31), i.e. $x = \left[r - \frac{1}{2}(r_+ + r_-)\right](r_+ - r_-)^{-1}$. Furthermore, for S^2 coordinates, θ is unchanged and the azimuthal angle $\bar{\phi}$ is related to ϕ as in (4.6.4):

$$\bar{\phi} = \phi - \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z + t)\,. \tag{4.3.10}$$

The 2π periodicity of $\bar{\phi}$ is ensured if $16ma(\Pi_c - \Pi_s)\ell^{-3} = a(2m)^{-2}(\Pi_c + \Pi_s)^{-1}$ is quantized in units of R^{-1}.

The radial equation (4.3.5) can be cast in the following form:

$$\left[\partial_x(x^2 - \frac{1}{4})\partial_x + \frac{\bar{\omega}^2}{4(x-\frac{1}{2})} - \frac{\bar{k}^2}{4(x+\frac{1}{2})} - l(l+1)\right]\chi(x) = 0\,. \tag{4.3.11}$$

The above coordinate transformations allow us to relate the quantum numbers in the Ansatz (4.3.1) to those of the standard Kaluza-Klein Ansatz:[18]

$$\mathbf{\Phi} = e^{-i\omega t}e^{ikz}e^{in\phi}\,P_l^n(\cos\theta)\,\chi(r)\,. \tag{4.3.12}$$

Namely, equating the two Ansätze (4.3.1) and (4.3.12), and employing the coordinate transformations (4.3.8) and (4.3.10) yields the following transformation between quantum numbers $\{\bar{\omega}, \bar{k}\}$ and $\{\omega, k\}$:

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{k}{\kappa_-} - n\frac{\Omega_+}{\kappa_+}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{k}{\kappa_+} + n\frac{\Omega_+}{\kappa_+}\,, \tag{4.3.13}$$

and n unchanged.

For the subtracted magnetised geometry the expressions for (4.3.8) are the same, but with $a = 0$ and static expressions for inverse surface gravities (4.2.4), i.e. $\kappa_+^{-1} = 4m\Pi_c$ and

[18] By abuse of notation we use above the same radial function notation.

$\kappa_-^{-1} = 4m\Pi_s$. The azimuthal angle is shifted due to the magnetic field β_4 as in (4.6.8):

$$\bar{\phi} = \phi - \beta_4 z\,. \tag{4.3.14}$$

Note that 2π periodicity of the S^2 azimuthal angle $\bar{\phi}$ is ensured if the magnetic field parameter β_4 is quantised in units of R^{-1}.

As a consequence, the transformation between the quantum numbers $\{\bar{\omega}, \bar{k}\}$ and $\{\omega, k\}$ is

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{k + n\beta_4}{\kappa_-}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{k + n\beta_4}{\kappa_+}\,, \tag{4.3.15}$$

and again, n unchanged.

These general expressions now allow us to recover results for the massless four-dimensional field with vanishing wave number $k = 0$. For the subtracted rotating geometry one obtains

$$\bar{\omega} = \frac{\omega}{\kappa_+} - n\frac{\Omega_+}{\kappa_+}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + n\frac{\Omega_+}{\kappa_+}\,, \tag{4.3.16}$$

just as in Section 2. Similarly for the magnetised subtracted geometry:

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{n\beta_4}{\kappa_-}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{n\beta_4}{\kappa_+}\,, \tag{4.3.17}$$

in agreement with Section 3.

We can also study massive Kaluza-Klein modes with the wave number $k \neq 0$, which is quantised in units of R^{-1}, where R is the radius of the circle S^1. Those are massive four-dimensional particles with mass $m_4 \propto k$, and they are charged under the Kaluza-Klein U(1) gauge symmetry with the charge $k = q$ (see Appendix 5.5). Their quasi-normal modes can be determined completely analogously to massless modes in Sections 2 and 3.

The solution (4.3.6), corresponding to the incoming wave at the outer horizon, is required to be finite at a large x, which is achieved for

$$\frac{\bar{\omega}+\bar{k}}{2} = -i(1+l+N_L)\,, \quad \text{or} \quad \frac{\bar{\omega}-\bar{k}}{2} = -i(1+l+N_R)\,, \tag{4.3.18}$$

where $l = 0, 1, \ldots$, and $N_L = 0, 1, \ldots$ or $N_R = 0, 1, \ldots$ This constrains a specific combination of ω and k. In the rotating case we have

$$\begin{aligned} & \omega = -\frac{i}{2m(\Pi_c - \Pi_s)}(1+l+N_L) + k\,, \\ \text{or} \qquad & \omega = -\frac{i\sqrt{m^2-a^2}}{2m^2(\Pi_c+\Pi_s)}(1+l+N_R) + \frac{a}{2m^2(\Pi_c+\Pi_s)}n - k\,. \end{aligned} \tag{4.3.19}$$

In the subtracted magnetised case we obtain

$$\begin{aligned} \omega &= -\frac{i}{2m(\Pi_c - \Pi_s)}(1+l+N_L) + n\beta_4 + k\,, \\ \text{or} \quad \omega &= -\frac{i}{2m(\Pi_c + \Pi_s)}(1+l+N_R) - n\beta_4 - k\,. \end{aligned} \tag{4.3.20}$$

Again, we obtained two branches of damped quasi-normal modes, both with oscillatory behaviour symmetrically advanced and retarded by $n\beta_4 + k$.

It is interesting to point out that the solution (4.3.6) for massive modes with $k \neq 0$ has a regular, analytic behaviour near the outer horizon, after one has made a gauge transformation $\chi(x) \to e^{ik\mathcal{A}_{2t+}t}\chi(x)$, where $\mathcal{A}_{2t+} = (2m)^4\Pi_c\Pi_s F^{-2}(r_+)$ is the time component of the Kaluza-Klein gauge potential $\mathcal{A}_2$ (4.4.4) or (4.5.18), evaluated at the outer horizon r_+. Namely, we obtain

$$\begin{aligned} e^{ik\mathcal{A}_{2t+}t}e^{-i\omega t}e^{in\phi}\chi(x) &\approx e^{ik\mathcal{A}_{2t+}t}e^{-i(\omega-n\Omega_+)t}e^{in\phi_+}e^{-i\bar{\omega}\kappa_+ r*}(1+\cdots) \\ &\approx e^{in\phi_+}V^{-i\frac{\omega-n\Omega_+}{\kappa_+}+i\frac{k}{\kappa_-}}(1+\ldots)\,, \end{aligned} \tag{4.3.21}$$

where we wrote the final expression in terms of Kruskal-Szekeres coordinates, and the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon

$U = 0$.

4.4 Appendix A: Subtracted Rotating Geometry with Sources

In [15] it was shown that the subtracted geometry (6.0.5), (4.1.2), (4.1.7) for four-charge rotating black hole is a solution of the equations of motion for the STU Lagrangian, describing the bosonic part of the N=2 supergravity Lagrangian coupled to three vector supermultiplets:

$$\begin{aligned}
\mathcal{L}_4 &= R\,{*\mathbf{1}} - \frac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}\,{*d\chi_i} \wedge d\chi_i - \frac{1}{2}e^{-\varphi_1}\,(e^{\varphi_2-\varphi_3}\,{*F_1} \wedge F_1 \\
&+ e^{\varphi_2+\varphi_3}\,{*F_2} \wedge F_2 + e^{-\varphi_2+\varphi_3}\,{*\mathcal{F}_1} \wedge \mathcal{F}_1 + e^{-\varphi_2-\varphi_3}\,{*\mathcal{F}_2} \wedge \mathcal{F}_2) \\
&- \chi_1\,(F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2)\,,
\end{aligned} \tag{4.4.1}$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \le i \le 3$. The four U(1) field strengths can be written in terms of potentials as

$$\begin{aligned}
F_1 &= dA_1 - \chi_2\, d\mathcal{A}_2\,, \\
F_2 &= dA_2 + \chi_2\, d\mathcal{A}_1 - \chi_3\, dA_1 + \chi_2\,\chi_3\, d\mathcal{A}_2\,, \\
\mathcal{F}_1 &= d\mathcal{A}_1 + \chi_3\, d\mathcal{A}_2\,, \\
\mathcal{F}_2 &= d\mathcal{A}_2\,.
\end{aligned}$$

The three axio-scalar fields and the four U(1) gauge potentials can be formally obtained as a scaling limit of a certain black hole solution (for details, see [15]), resulting in

$$\chi_1 = -\chi_2 = \chi_3 = -\frac{2ma(\Pi_c - \Pi_s)\cos\theta}{Q^2}, \quad e^{\varphi_1} = e^{\varphi_2} = e^{\varphi_3} = \frac{Q^2}{\sqrt{\Delta}}\,, \tag{4.4.2}$$

and the gauge potentials $A_1 = A_2 = A_3 \equiv A$ for gauge field strengths $*F_1 = F_2 = *\mathcal{F}_1 \equiv F$

and $\mathcal{A}_2$ for $\mathcal{F}_2$ are of the following form:

$$
\begin{aligned}
A = \quad & -\frac{r-m}{Q}\,dt - \frac{(2\,m)^2\,a^2\,(\Pi_c-\Pi_s)[\,r\,(\Pi_c-\Pi_s)+2m\Pi_s\,]\,\cos^2\theta}{Q\Delta}\,dt \\
& -\frac{(2m)^4\,a\,(\Pi_c-\Pi_s)[\,r\,(\Pi_c^2-\Pi_s^2)+2m\Pi_s^2\,]\,\sin^2\theta}{Q\Delta}\,d\phi\,, \qquad (4.4.3)\\
\mathcal{A}_2 = \quad & \frac{Q^3[(2m)^2\Pi_c\Pi_s+a^2(\Pi_c-\Pi_s)^2\cos^2\theta]}{2m(\Pi_c^2-\Pi_s^2)\Delta}\,dt + \frac{Q^3 2m\,a(\Pi_c-\Pi_s)\sin^2\theta}{\Delta}\,d\phi, \qquad (4.4.4)
\end{aligned}
$$

where

$$
Q = 2m(\Pi_c^2-\Pi_s^2)^{\frac{1}{3}}\epsilon^{-\frac{1}{3}} \equiv \tfrac{1}{2}\ell\epsilon^{-\frac{1}{3}}, \quad \text{as} \quad \epsilon\to 0\,. \qquad (4.4.5)
$$

and again, Δ defined as in (4.1.7):

$$
\Delta_0 \to \Delta = (2m)^3 r(\Pi_c^2-\Pi_s^2) + (2m)^4\Pi_s^2 - (2m)^2(\Pi_c-\Pi_s)^2 a^2\cos^2\theta\,. \qquad (4.4.6)
$$

The (formally infinite) factors of Q can in principle be removed from gauge potentials by removing corresponding factors from scalar fields. However, when lifting the scaling limit solution to five dimensions, it is useful to keep this scaling factor explicit; in the final five-dimensional metric an overall factor is not relevant.

4.5 Appendix B: Subtracted Magnetised Geometry with Sources

The magnetised solution of the static STU black hole was obtained in [75] and is of the form:

$$
ds_4^2 = H\,[-r(r-2m)dt^2 + \frac{r_1r_2r_3r_4}{r(r-2m)}\,dr^2 + r_1r_2r_3r_4 d\theta^2\,] + H^{-1}\,\sin^2\theta\,(d\phi-\tilde{\omega}dt)^2\,. \quad (4.5.1)
$$

Here

$$
r_i = r + 2ms_i^2\,, \qquad (4.5.2)
$$

and we shall use the notation $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$, with $i = 1, 2, 3, 4$. The function $\tilde{\omega}$ is given by

$$\tilde{\omega} = \sum_{i=1}^{4} \Big[-\frac{q_i\,\beta_i}{r_i} + \frac{q_i\,\Xi_i\,[r_i + (r-2m)\cos^2\theta]r}{r_i} \Big] , \tag{4.5.3}$$

where

$$q_i = 2ms_ic_i\,, \qquad \Xi_i = \frac{\beta_1\beta_2\beta_3\beta_4}{\beta_i}\,, \qquad \beta_i = 12B_i\,, \tag{4.5.4}$$

and B_i $(i = 1, 2, 3, 4)$ denote the external magnetic field strengths for each of the four gauge fields. Finally, the function H is given by

$$H = \frac{\sqrt{\bar{\Delta}}}{\sqrt{r_1r_2r_3r_4}}\,, \tag{4.5.5}$$

where

$$\begin{aligned} \bar{\Delta} \;=\; & 1 + \sum_i \frac{\beta_i^2 r_1r_2r_3r_4}{r_i^2}\,\sin^2\theta + 2[\beta_3\beta_4q_1q_2 + \cdots]\cos^2\theta + [\beta_3^2\,\beta_4^2\,R_1^2\,R_2^2 + \cdots] \\ & -2(\prod_j \beta_jr_j)\,\sum_i \frac{q_i^2}{r_i^2}\,\sin^2\theta\cos^2\theta + [2\beta_2\beta_3\beta_4^2q_2q_3\,R_1^2 + \cdots]\cos^2\theta + \prod_i \beta_i^2\,R_i^2 \\ & +r_1r_2r_3r_4\,\sum_i \frac{\Xi_i^2\,R_i^2}{r_i^2}\,\sin^2\theta + [2\beta_1\beta_2\beta_3^2\beta_4^2q_3q_4\,R_1^2\,R_2^2 + \cdots]\cos^2\theta\,, \end{aligned} \tag{4.5.6}$$

and we have defined

$$R_i^2 = r_i^2\,\sin^2\theta + q_i^2\,\cos^2\theta\,. \tag{4.5.7}$$

The Kaluza-Klein gauge field here is given by

$$\mathcal{A}_2 = \Big[\frac{q_4}{r_4} - \sum_{i=1}^{3} \frac{r\,q_i\,\beta_1\beta_2\beta_3\,[r_i + (r-2m)\cos^2\theta]}{\beta_i\,r_i}\Big]\,dt - \sigma_4\,(d\phi - \tilde{\omega}dt)\,, \tag{4.5.8}$$

where $\sigma_4 = \tilde{\sigma_4}\bar{\Delta}^{-1}$, and

$$
\begin{aligned}
\tilde{\sigma}_4 &= \frac{\beta_4 r_1 r_2 r_3}{r_4}\sin^2\theta + (\beta_1 q_2 q_3 + \cdots)\cos^2\theta + \beta_4(\beta_1^2 R_2^2 R_3^2 + \cdots) \\
&+2\beta_4(\beta_2\beta_3 q_2 q_3 R_1^2 + \cdots)\cos^2\theta + q_4[\beta_1^2(\beta_2 q_2 R_3^2 + \beta_3 q_3 R_2^2) + \cdots]\cos^2\theta \\
&+4\beta_1\beta_2\beta_3 q_1 q_2 q_3 q_4 \cos^4\theta \\
&-\frac{\beta_1\beta_2\beta_3 q_4^2 r_1 r_2 r_3}{r_4}\sin^2\theta\cos^2\theta - \beta_1\beta_2\beta_3\, r_4\left(\frac{q_1^2 r_2 r_3}{r_1} + \cdots\right)\sin^2\theta\cos^2\theta \\
&+\beta_1\beta_2\beta_3(\beta_2\beta_3 q_2 q_3 R_1^2 + \cdots)R_4^2\cos^2\theta + \beta_4 r_4\left[\frac{\beta_2^2\beta_3^2 r_2 r_3}{r_1}\, R_1^4 + \cdots\right]\sin^2\theta \\
&+2\beta_1\beta_2\beta_3\beta_4 q_4(\beta_1 q_1 R_2^2 R_3^2 + \cdots)\cos^2\theta + \beta_4\beta_1^2\beta_2^2\beta_3^2 R_1^2 R_2^2 R_3^2 R_4^2\,. \qquad (4.5.9)
\end{aligned}
$$

The dilaton field is given by

$$
e^{\varphi_1} = \frac{Y_1}{\sqrt{\bar{\Delta}}\, r_1 r_2 r_3 r_4}\,, \qquad (4.5.10)
$$

where

$$
\begin{aligned}
Y_1 &= r_1 r_3(1 + 2\beta_1\beta_3 q_2 q_4\cos^2\theta + \beta_1^2\beta_3^2 R_2^2 R_4^2) \\
&+ r_2 r_4(\beta_1^2 R_3^2 + \beta_3^2 R_1^2 + 2\beta_1\beta_3 q_1 q_3\cos^2\theta)\,. \qquad (4.5.11)
\end{aligned}
$$

For explicit expressions of all the fields see [75]. Note however in order to have the same sign for the gauge fields of the rotating and magnetised geometries, we have changed an overall sign for the gauge fields relative to [75].

The Scaling Limit

The subtracted geometry can be obtained by taking a scaling limit of the above magnetised electric black holes, analogously to the rotating case. The limit can be implemented by means of the scalings

$$
\begin{aligned}
&m \to m\,\epsilon\,, \qquad r = r\,\epsilon\,, \qquad t \to t\,\epsilon^{-1}\,, \qquad \beta_i \to \beta_i\,\epsilon\,, \quad i = 1,2,3,4\,, \\
&\sinh^2\delta_4 \to \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2}\,, \qquad \sinh^2\delta_i \to (\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}\,\epsilon^{-\frac{4}{3}}\,, \quad i = 1,2,3\,, \qquad (4.5.12)
\end{aligned}
$$

where ϵ is then sent to zero. In particular, this gives

$$(d\phi - \tilde{\omega} dt) \longrightarrow d\phi - (\beta_1 + \beta_2 + \beta_3)dt - \frac{2m\beta_4\,\Pi_c\Pi_s}{(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2}\,dt\,, \tag{4.5.13}$$

and

$$\bar{\Delta} \longrightarrow 1 + \frac{(2m)^3\beta_4^2(\Pi_c^2 - \Pi_s^2)^2\sin^2\theta}{(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2}\,, \qquad r_1r_2r_3r_4 \longrightarrow (2m)^3\left[(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2\right]\,. \tag{4.5.14}$$

The quantities β_1, β_2 and β_3 are removed by a gauge transformation $\phi \longrightarrow \phi + (\beta_1 + \beta_2 + \beta_3)t$. We shall assume from now on that this transformation has been performed. The final metric can be cast in the following form:

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\Big(-dt^2 + \frac{F^2dr^2}{X^2}\Big) + \sqrt{\Delta}d\theta^2 + \frac{F^2\sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2\,, \tag{4.5.15}$$

where

$$\begin{aligned} X &= r^2 - 2mr\,, \\ F^2 &= (2m)^3\left[(\Pi_c^2 - \Pi_s^2)r + (2m)\Pi_s^2\right]\,, \\ W &= -\frac{16m^4\Pi_s\Pi_c\beta_4}{F^2}\,, \\ \Delta &= F^2 + (2m)^6\beta_4^2(\Pi_c^2 - \Pi_s^2)^2\sin^2\theta\,. \end{aligned} \tag{4.5.16}$$

The dilation fields are of the form:

$$e^{\phi_1} = e^{\varphi_2} = e^{\varphi_3} = \frac{Q^2}{\sqrt{\Delta}}\,, \tag{4.5.17}$$

and the axion fields vanish. The Kaluza-Klein U(1) gauge field becomes

$$\mathcal{A}_2 = \frac{Q^3 2m\Pi_c\Pi_s}{(\Pi_c^2 - \Pi_s^2)F^2}dt - \frac{Q^3(2m)^3\beta_4(\Pi_c^2 - \Pi_s^2)\sin^2\theta}{\Delta}(d\phi + W dt)\,. \tag{4.5.18}$$

Note that at the horizon the combination $\phi + W(r_+)t = \phi_+$, and thus the second term in

(4.5.18) becomes the ϕ_+ component of the Kaluza-Klein gauge potential. The remaining three gauge potentials become identified and are of the form (4.4.3) by setting $a = 0$.

One can of course remove Q in the scalar and gauge fields via a gauge transformation. However, it is useful to keep it in the discussion of the lift and at the end remove the overall scaling parameter ϵ.

4.6 Appendix C: Subtracted Geometry Lifted to Five Dimensions

We now provide a lift of the subtracted rotating geometry to five-dimensions[19]. The five-dimensional metric for the scaling limit takes the form:

$$ds_5^2 = e^{\varphi_1} ds_4^2 \; + e^{-2\varphi_1}(dz + \mathcal{A}_2)^2 \,, \tag{4.6.1}$$

where we have to implement the scaling $z \to z\epsilon^{-1}$. This metric takes the form:

$$ds_5^2 = \epsilon^{-\frac{2}{3}}(ds_{S^2}^2 + ds_{BTZ}^2) \,, \tag{4.6.2}$$

where

$$ds_{S^2}^2 = \tfrac{1}{4}\ell^2 \left(d\theta^2 + \sin^2\theta d\bar{\phi}^2\right) \,, \tag{4.6.3}$$

with

$$\bar{\phi} = \phi - \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z + t) \,, \tag{4.6.4}$$

and

$$ds_{BTZ}^2 = -\frac{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}{\ell^2\, r_3^2} dt_3^2 + \frac{\ell^2 r_3^2}{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}\, dr_3^2 + r_3^2 (d\phi_3 + \frac{r_{3+} r_{3-}}{\ell r_3^2}\, dt_3)^2 \,, \tag{4.6.5}$$

[19] Partial results were provided in [15, 68]. Here we take particular care of the dimensions and of the periodicities of metric coordinates.

where

$$
\begin{aligned}
\phi_3 &= \frac{z}{R}\,, \\
t_3 &= \frac{\ell}{R}\,t\,, \\
r_3^2 &= \frac{16(2mR)^2}{\ell^4}\left[2m(\Pi_c^2-\Pi_s^2)r+(2m)^2\Pi_s^2-a^2(\Pi_c-\Pi_s)^2\right]\,.
\end{aligned}
\tag{4.6.6}
$$

Here, R is the radius of the circle S^1 and $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$ is the radius of the AdS$_3$. Furthermore

$$
r_{3\pm} = \frac{8mR}{\ell^2}\left[m(\Pi_c+\Pi_s) \pm \sqrt{m^2-a^2}(\Pi_c-\Pi_s)\right]\,. \tag{4.6.7}
$$

The periodicity of z coordinate is $2\pi R$, and thus the angular coordinate ϕ_3 has the correct periodicity of 2π. Note also that the 2π periodicity of $\bar{\phi}$ is ensured if $16ma(\Pi_c-\Pi_s)\ell^{-3} = a(2m)^{-2}(\Pi_c+\Pi_s)^{-1}$ is quantized in units of R^{-1}.

The lifted geometry is indeed locally AdS$_3 \times S^2$ with the radius of AdS$_3$ equal to ℓ and the radius of S^2 equal to $\frac{\ell}{2}$.

Subtracted Magnetised Geometry

This geometry also lifts to (4.6.2) where now $\bar{\phi}$ in (4.6.3) is defined as[20]

$$
\bar{\phi} = \phi - \beta_4\, z\,, \tag{4.6.8}
$$

and we set in all expressions above $a = 0$, i.e. the BTZ coordinates are related to $\{t, r, z\}$ as in (4.6.6) with $a = 0$. (Obviously, $\beta_4 = 0$ corresponds to the lift of the static subtracted geometry.) Note that the shift requires that β_4 be quantized in units of R^{-1}, in order for $\bar{\phi}$ to have the correct periodicity of 2π.

[20]It was observed in [87] that such a shift produces a magnetic field for the Kaluza-Klein U(1) gauge potential and thus a four-dimensional geometry in a Kaluza-Klein magnetic field.

4.7 Appendix D: Relation of the BTZ Black Hole Coordinates to the AdS$_3$ Coordinates

According to [77, 78] AdS$_3$ is the quadric

$$u^2 + v^2 - x^2 - y^2 = \ell^2 \,, \tag{4.7.1}$$

in $\mathbb{E}^{2,2}$ with the metric induced from

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \,. \tag{4.7.2}$$

In a local patch we have the embedding

$$u = \sqrt{A(r)}\cosh\Phi = \ell\cosh\rho\cosh\Phi \,, \tag{4.7.3}$$

$$x = \sqrt{A(r)}\sinh\Phi = \ell\cosh\rho\sinh\Phi \,, \tag{4.7.4}$$

$$y = \sqrt{B(r)}\cosh T = \ell\sinh\rho\cosh T \,, \tag{4.7.5}$$

$$v = \sqrt{B(r)}\sinh T = \ell\sinh\rho\sinh T \,. \tag{4.7.6}$$

The metric is of the form:

$$ds^2_{AdS_3} = \ell^2\,(-\sinh^2\rho\, dT^2 + d\rho^2 + \cosh^2\rho\, d\Phi^2) \,. \tag{4.7.7}$$

The relationship to the BTZ metric coordinates and parameters introduced in the Appendix 5.3 (eqs.(4.6.5,4.6.6)) is

$$A(r) = \ell^2\frac{r_3^2 - r_{3-}^2}{r_{3+}^2 - r_{3-}^2} \,, \qquad B(r) = \ell^2\frac{r_3^2 - r_{3+}^2}{r_{3+}^2 - r_{3-}^2} \,, \tag{4.7.8}$$

$$T = \frac{r_{3+}t_3 - r_{3-}\ell\phi_3}{\ell^2} \qquad \Phi = \frac{r_{3+}\ell\phi_3 - r_{3-}t_3}{\ell^2} \,, \tag{4.7.9}$$

where $r_{3\pm}$ is defined in (4.6.7).

Note that a shift in T is a boost in the Minkowski $v-y$ plane and a shift in Φ corresponds to a boost in the Minkowski $u-x$ plane. Since ϕ_3 of the BTZ metric (4.6.5) is periodic with period 2π, the coordinates $\{T,\Phi\}$ must be identified under the composition of two discrete boosts: $\big(T,\Phi\big) \to \big(T-\frac{2\pi r_{3-}}{\ell},\Phi+\frac{2\pi r_{3+}}{\ell}\big)$.

4.8 Appendix E: Kaluza-Klein Reduction of the Scalar Wave Equation

The five-dimenskonal Kaluza-Klein metric Ansatz

$$ds_5^2 = e^{\phi_1}\gamma_{\alpha\beta}dx^\alpha dx^\beta + e^{-2\phi_1}(dz+\mathcal{A}_{2\alpha}dx^\alpha)^2\,, \tag{4.8.1}$$

where $\{\alpha,\beta\}=0,1,2,3$, results in the five-dimensional wave equation given by

$$\nabla^\alpha\nabla_\alpha\mathbf{\Phi} - \nabla^\alpha\mathcal{A}_{2\alpha}\partial_z\mathbf{\Phi} - 2\mathcal{A}_2{}^\alpha\nabla_\alpha\partial_z\mathbf{\Phi} + (\mathcal{A}_2)^2\partial_z^2\mathbf{\Phi} = -e^{\phi_1}\partial_z^2\mathbf{\Phi}\,. \tag{4.8.2}$$

If we make the assumption that $\mathbf{\Phi}$ is separable in term of a four-dimensional wave function and a function of the fifth coordinate z:

$$\mathbf{\Phi}(x^\alpha,z) = \mathbf{\Phi}(x^\alpha)e^{if(z)}\,, \tag{4.8.3}$$

we can rewrite the above equation as

$$\gamma^{\alpha\beta}\left(\nabla_\alpha - i(\partial_z f)\mathcal{A}_{2\alpha}\right)\left(\nabla_\beta - i(\partial_z f)\mathcal{A}_{2\beta}\right)\mathbf{\Phi}(x^\alpha) = (\partial_z f)^2 e^{\phi_1}\mathbf{\Phi}(\mathbf{x}^\alpha)\,. \tag{4.8.4}$$

For the compactification on a circle S^1 with radius $2\pi R$, the above equation is solved with the Ansatz for $f(z)=kz$, where the wave number k is quantised in units of R^{-1}. The remaining effective four-dimensional wave equation can then be interpreted as the Klein-Gordon equation of the four-dimensional charged particle with a charge $q=k$ and an effective mass $\propto k$ which is modulated by the scalar field e^{ϕ_1}:

$$m_{eff}^2 = k^2 e^{\phi_1}\,. \tag{4.8.5}$$

Chapter 5

Entanglement Entropy

We compute the entanglement entropy of minimally coupled scalar fields on subtracted geometry black hole backgrounds, focusing on the logarithmic corrections. We notice that matching between the entanglement entropy of original black holes and their subtracted counterparts is only at the order of the area term. The logarithmic correction term is not only different but also, in general, changes sign in the subtracted case. We apply Harrison transformations to the original black holes and find out the choice of the Harrison parameters for which the logarithmic corrections vanish.

5.1 Entanglement entropy of original and subtracted black holes

5.1.1 Black hole entanglement entropy

Entanglement entropy of quantum fields, computed across the black hole event horizon Σ, gives a divergent expression of the form

$$S^{ent} \sim \frac{A_\Sigma}{\epsilon^2} + c_0 \ln\left(\frac{L}{\epsilon}\right) + S_{finite}\,, \tag{5.1.1}$$

where ϵ is a short-distance UV cutoff and L an IR cutoff. It is well known [142] that the divergences in this expression match the divergences in the one-loop effective action for quantum fields in the black hole background. This means that when we view the total black hole entropy as composed of a "bare gravitational" or "tree-level" entropy S^{tree}, plus the entanglement entropy as a "quantum correction" S^{loop}, then the total entropy $S^{(tot)}_{BH}$ takes the same general form as S^{tree} with the one-loop renormalized couplings replacing the bare couplings present in S^{tree}. These couplings which renormalize are the Newton constant G_4, and couplings $c_{1,2,3}$ for higher-order curvature terms R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho}$ added to the Lagrangian. (The assumption of a "bare" gravitational entropy can be disposed of in a specific model in which gravity is effective and wholly induced by quantum fields [143],[144].)

Both the tree-level entropy and the loop corrections are computed with the conical singularity method. A Euclidean manifold is obtained by Wick rotation of the Lorentzian black hole geometry. One creates a conical defect around the horizon (giving periodicity $2\pi\alpha$ to the Euclidean time coordinate, which loops around it). The tree-level entropy is then obtained from the expression:

$$S^{tree} = (\alpha\partial_\alpha - 1)\, S^B_\alpha|_{\alpha=1}\,, \tag{5.1.2}$$

where S^B_α is the bare gravitational action, including higher-order curvature terms, evaluated on the conical Euclidean manifold. The loop correction is given by the same equation but replacing the bare gravitational action by minus the log of the quantum partition function:

$$S^{loop} = -\,(\alpha\partial_\alpha - 1)\ln Z_\alpha\Big|_{\alpha=1} \tag{5.1.3}$$

These expressions have been computed for Kerr-Newman black holes by Solodukhin and Mann [145] and for arbitrary axisymmetric black holes by Jing and Yan [146]. They are respectively given by

$$S^{tree} = \frac{A_\Sigma}{4G_4^B} - 8\pi\int_\Sigma\left[\left(c_B^1 R + \frac{c_B^2}{2}\sum_{a=1}^{2} R_{\mu\nu}n_i^\mu n_i^\nu + c_B^3\sum_{a,b=1}^{2} R_{\mu\nu\alpha\beta}n_i^\mu n_j^\nu n_i^\alpha n_j^\beta\right)\right], \tag{5.1.4}$$

$$S^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} + \left\{ \frac{1}{144\pi}\int_\Sigma R - \frac{1}{45}\frac{1}{16\pi}\int_\Sigma \Big(\sum_{a=1}^{2} R_{\mu\nu} n_i^\mu n_i^\nu - 2\sum_{a,b=1}^{2} R_{\mu\nu\alpha\beta} n_i^\mu n_j^\nu n_i^\alpha n_j^\beta \Big) \right. \tag{5.1.5}$$

$$\left. - \frac{1}{90}\frac{1}{16\pi}\int_\Sigma (K^a K^a) + \frac{1}{24\pi}\Big(\lambda_1 - \frac{\lambda_2}{30}\Big)\int_\Sigma (K^a K^a - 2tr(K.K)) \right\} \ln\frac{L}{\epsilon}, \tag{5.1.6}$$

where $G_4^B, c_B^I, (I = 1, 2, 3)$ represent bare constants (tree-level), $K^a_{\mu\nu} = -\gamma^\alpha_\mu \gamma^\beta_\nu \nabla_\alpha n^a_\beta$ is the extrinsic curvature, $K^a = g^{\mu\nu} K^a_{\mu\nu}$ is the trace of the extrinsic curvature, and n_i^μ $(i = 1, 2)$ are unit vectors normal to Σ.

For the general axisymmetric black holes all the quantities dependent on the extrinsic curvature vanish, thus verifying that the tree-level and the loop formulas have the same general form and the entropy renormalizes. We quote from [146] a useful expression for the combination of Riemann tensor contractions:

$$R_{nn}(r_+, \theta) - 2R_{mnmn}(r_+, \theta) = \left\{ \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial \ln f}{\partial r} - \frac{1}{2}\frac{\partial g^{rr}}{\partial r}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r} + \frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right) - \frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2 \right\}_{r_+}. \tag{5.1.7}$$

Here the Boyer-Lindquist form of the Euclidean metric

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{t\varphi}dtd\varphi + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2, \tag{5.1.8}$$

is assumed, with g_{tt}, g_{rr}, $g_{t\varphi}$, $g_{\theta\theta}$ and $g_{\varphi\varphi}$ functions of the coordinates r and θ. The inverse metric component is $g^{rr} = 1/g_{rr}$, and $f = -g_{rr}\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)$.

Note that because R vanishes on the black hole metrics (6.0.5), the higher-order correction to the entropy is given essentially by the combination (5.1.7) integrated over the horizon.

Replacing the metric components of (5.1.7) and evaluating on the horizon we obtain:

$$
\begin{aligned}
S^{loop} &= \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{720\pi}\int_\Sigma \left(R_{nn}(r_H,\theta) - 2R_{mnmn}(r_H,\theta)\right) \\
&= \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{720\pi}\int_\Sigma \left(\frac{X''}{\Delta^{1/2}} + \frac{X'\,\Delta'}{2\Delta^{3/2}} - 2\frac{G'X'}{G\Delta^{1/2}} - 2\frac{G^2(\mathcal{A}')^2}{\Delta^{3/2}\sin^2\theta}\right)_{r_+}\,,
\end{aligned}
\tag{5.1.9}
$$

where a prime stands for a derivative with respect to r.

In the next subsections we give the results form this expression both for the original and the subtracted black hole geometries.

5.1.2 Results for original black holes

The evaluation of (5.1.9) for the original black hole geometry that has $\Delta = \Delta_0$, in the fully general case with four charges and rotation that is parametrized by (m, a, δ_I), is given by an expression of the form

$$
S^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{A_{red}(r_+)}{720}\int_{-1}^{1} \mathrm{d}u\,\frac{\kappa u^4 + \lambda u^2 + \mu}{(\alpha u^4 + \beta u + \gamma)^{3/2}}\,,
\tag{5.1.10}
$$

where the six parameters $(\alpha, \beta, \gamma, \kappa, \lambda, \mu)$ depend on the black hole parameters (m, a, δ_I) (as do, of course, the horizon radius r_+ and the function $A_{red}(r_+)$ defined above in (4.1.2)). The definitions of these six parameters are given in the Appendix. The expression is obtained through the change of variables $u = \cos\theta$. The integral it features is in general expressible as a lengthy combination of elliptic functions, which can take different forms in different regions of parameter space. For this reason we will provide here only the results in some particular cases of physical interest.

The expression derived in [145] for the entanglement entropy of the Kerr-Newman black hole (with rotation and a single charge parameter) is obtained in the limit $\delta_0 = \delta_1 = \delta_2 = \delta_3 \equiv \delta$.

It reads:

$$S_{KN}^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} + \frac{1}{45}\left[1 - \frac{3m^2\sinh^2(2\delta)}{4R_+^2}\left(1 + \frac{(a^2 + R_+^2)\arctan\frac{a}{R_+}}{a\,R_+}\right)\right]\log(\frac{r_+}{\epsilon})\,, \quad (5.1.11)$$

where

$$R_+ = r_+ + 2m\sinh^2\delta\,. \quad (5.1.12)$$

The correspondence between this expression and the result given by formula (4.12) in [145] is manifest if we translate suitably our notation to the one used in this reference. For ease of comparison we provide the following translation for the notations, where the left hand side correspond to the notations used in [145] and the right hand side to those used in the present work :

$$q \quad \longleftrightarrow \quad m\sinh(2\delta)\,, \quad (5.1.13)$$

$$m \quad \longleftrightarrow \quad m\cosh(2\delta)\,, \quad (5.1.14)$$

$$r_+ \quad \longleftrightarrow \quad r_+ + 2m\sinh^2\delta = R_{r_+}\,. \quad (5.1.15)$$

The results for the Reissner-Nordstrom and Schwarzschild black holes are obtained from the previous formula setting $a = 0$ and $a = 0 = \delta$ respectively. They read:

$$S_{RN}^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} + \frac{1}{90}\frac{(2 - \sinh^2\delta)}{\cosh^2\delta}\log\left(\frac{r_+}{\epsilon}\right)\,, \quad (5.1.16)$$

$$S_{Sch}^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} + \frac{1}{45}\log\left(\frac{r_+}{\epsilon}\right)\,. \quad (5.1.17)$$

On the other hand, the result for the static black hole with four different charges ($a = 0, \delta_{0,1,2,3} \neq 0$) reads

$$S^{loop}_{4q} = \frac{A_\Sigma}{48\pi\epsilon^2} \tag{5.1.18}$$
$$+ \frac{1}{360\,\Pi_c}\left(8(\Pi_c^2 - \Pi_s^2) - 3\sum s_I^2 - 6\sum_{I\neq J} s_I^2 s_J^2 - 9\sum_{I<J<K} s_I^2 s_J^2 s_K^2 - 4\prod_I s_I^2\right)\log\left(\frac{r_+}{\epsilon}\right),$$

where $s_I = \sinh\delta_I$ and $I = 0, 1, 2, 3$. This result does not feature previously in the literature. It reduces to (5.1.16) when we set $\delta_I = \delta$ for all I. Note that each of the results (5.1.16, 5.1.17, 5.1.18) has a log correction independent of the parameter m.

5.1.3 Results for subtracted black holes

We turn now to the evaluation of the entropy for the black holes with subtracted geometry. As remarked before, the black hole horizon area A_Σ is independent of Δ and therefore the leading order term of the entropy always matches the original one. We will show that this agreement is not preserved for the subleading order, i.e. the logarithmic correction involving the integral of (5.1.7).

The black hole entanglement entropy for subtracted geometry is computed by evaluating (5.1.9) with $\Delta = \Delta_{sub}$ as given by (2.2.4). The result for the fully general four-charge black hole with rotation is given by:

$$S^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} \tag{5.1.19}$$
$$-\frac{1}{180}\frac{m^3(\Pi_c+\Pi_s)^3 + \sqrt{(m^2-a^2)^3}(\Pi_c-\Pi_s)^3}{m[m\,(\Pi_c+\Pi_s) + \sqrt{m^2-a^2}(\Pi_c-\Pi_s)][m(\Pi_c^2+\Pi_s^2) + \sqrt{m^2-a^2}(\Pi_c^2-\Pi_s^2)]}\log\left(\frac{r_+}{\epsilon}\right).$$

The result at the subleading order is clearly different from the original black hole expression (5.1.10), which is much more complex and depends on all four charge parameters δ_I separately instead of through the combinations Π_c, Π_s. For completeness we include below the results for the subtracted versions of the Kerr-Newmann black hole, the Reissner-Nordstrom black hole, the Schwarzschild and the static four-charge black hole. These results are to be contrasted with the original expressions (5.1.11, 5.1.16, 5.1.17, 5.1.18).

$$
\begin{aligned}
S^{loop}_{KN-sub} &= \frac{A_\Sigma}{48\pi\epsilon^2} &(5.1.20)\\
&- \frac{1}{180}\frac{m^3(\cosh^4\delta+\sinh^4\delta)^3+\sqrt{(m^2-a^2)^3}(\cosh^4\delta-\sinh^4\delta)^3}{m(\tilde\gamma_1\tilde\gamma_2)} &(5.1.21)\\
S^{loop}_{RN-sub} &= \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{360}\frac{(\cosh^8\delta+3\sinh^8\delta)}{\cosh^8\delta}\log\left(\frac{r_+}{\epsilon}\right), &(5.1.22)\\
S^{loop}_{Sch-sub} &= \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{360}\log\left(\frac{r_+}{\epsilon}\right), &(5.1.23)\\
S^{loop}_{4q-sub} &= \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{360}\left(\frac{(\Pi_c^2+3\Pi_s^2}{\Pi_c^2}\right)\log\left(\frac{r_+}{\epsilon}\right). &(5.1.24)
\end{aligned}
$$

In (5.1.21), $\tilde\gamma_1$ stands for $m\,(\cosh^4\delta+\sinh^4\delta)+(m^2-a^2)^{1/2}(\cosh^4\delta-\sinh^4\delta)$ and $\tilde\gamma_2$ stands for $m\,(\cosh^8\delta+\sinh^8\delta)+(m^2-a^2)^{1/2}(\cosh^8\delta-\sinh^8\delta)$. These expressions are all obtained evaluating (5.1.19) in the appropriate limits. Just as before, the static results (5.1.22-5.1.24) have the log prefactor independent of m. It is seen, however, that they in every case the expression is different from the corresponding expression for the original black hole.

Nevertheless, there is a limit in which the expressions coincide. The subtracted geometry is designed to be a modification of the original black hole geometry that preserves the key features of extremal black holes even for non-extremal parameters. Therefore in the extremal limit the entropies of the original and the subtracted black holes should match exactly. In our parametrization, this limit is given by:

$$m \longrightarrow 0\,, \quad \delta_I \longrightarrow +\infty\,, \qquad \text{with } m\exp(2\delta_I) = 4G_4\,Q_I = \text{finite}\,. \tag{5.1.25}$$

When taking this limit, we find indeed agreement between the original and the subtracted entropies for extremal black holes:

$$S^{loop}_{ext} = S^{loop}_{ext-sub} = \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{90}\log\left(\frac{r_+}{\epsilon}\right). \tag{5.1.26}$$

5.2 Interpolating Schwarzschild geometry and vanishing log correction

If we compare the results (5.1.17) and (5.1.23), which express the entanglement entropy for the original and subtracted versions of the Schwarzschild black hole respectively, we notice an interesting feature: the sign of the logarithmic correction has changed from positive to negative. This raises the question of whether there exists an interpolating geometry for which this correction vanishes. One could interpret such a solution as a fixed point for the entropy, in the sense of the renormalization group, since the tree-level result is unaffected by the log correction.

As it happens, we can indeed construct solutions that interpolate between the original black hole geometry and the subtracted geometry. This is done through Harrison transforms, which are a four-parameter group of transformations acting on the black hole solution. It is shown in [71] that a four-parameter Harrison transform interpolates between the original geometry and a new black hole geometry, which after a rescaling yields the subtracted version of the original geometry. (A version of this construction requiring only two Harrison parameters had appeared previously in [15].) Therefore, one should expect to find a suitable combination of Harrison transformation parameters that corresponds to a geometry with a vanishing log term in the entropy.

Let us review briefly how the general Harrison transform is defined in [71]. We have four Harrison parameters (α_0, α_j), with $j = 1, 2, 3$. The effect of a Harrison transformation $g_H(\alpha_0, \alpha_j)$ on the black hole geometry is to modify the warp factor Δ (which is, in general, a polynomial of the fourth order in r) in the following way: The term with r^4 gets multiplied by $(1-\alpha_0^2)\prod_j(1-\alpha_j^2)$, so that it vanishes when any of the four parameters is set to 1. The term with r^3 is multiplied by a suitable permutation of terms combining three factors of the form $(1-\alpha^2)$, so that it vanishes when any two of the parameters are set to 1. The analogous operation happens with the second and first order terms. Detailed formulas, too long to quote here, are to be found in Appendix C of [71] for the particular cases of the Kerr solution and the general static solution.

To obtain the subtracted geometry from the original geometry, one needs to apply a Harrison transform with particular values of the four parameters, followed by a specific re-scaling of the metric. The values of the Harrison parameters that lead to the subtracted geometry are:

$$\alpha_j = 1\,, \qquad \alpha_0 = \frac{\Pi_s \cosh\delta_0 - \Pi_c \sinh\delta_0}{\Pi_c \cosh\delta_0 - \Pi_s \sinh\delta_0}\,, \tag{5.2.1}$$

and the subsequent re-scaling of the metric takes the form:

$$U \to U + c_0\,, \qquad \mathrm{e}^{2c_0} = \frac{\mathrm{e}^{\delta_1+\delta_2+\delta_3}}{\Pi_c \cosh\delta_0 - \Pi_s \sinh\delta_0}\,, \tag{5.2.2}$$

where $\exp(-4U) = \Delta/G^2$. The matter fields supporting the geometry get rescaled as well; we omit these details for briefness and refer the reader once more to [71] for the full formulas.

To move between the original and the subtracted versions of Schwarzschild, there is no metric re-scaling involved because all δ_I vanish and therefore so does c_0. Also, in this case both the initial and final values of the α_0 parameter are 0, so we may disregard it. The interpolating geometry we have is given by

$$ds^2 = -\Delta^{-1/2} G\, dt^2 + \Delta^{1/2}\left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G}\sin^2\theta\, d\phi^2\right)\,, \tag{5.2.3}$$

with

$$\Delta = \prod_I ((1-\alpha_I^2) r + 2m\alpha_I^2). \tag{5.2.4}$$

We can now compute the entropy directly for the interpolating geometries using (5.1.9). The lack of angular dependence makes the calculation trivial, and the result is

$$S^{loop}_{interpolating} = \left(\frac{3\alpha_0^2 + 3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2 - 8}{45}\right)\log\left(\frac{r_+}{\epsilon}\right)\,. \tag{5.2.5}$$

One can easily see that

$$
\begin{aligned}
S^{loop}_{interpolating} &= S^{loop}_{Sch} \qquad \text{for } \alpha_{0,1,2,3} = 0, \\
&= S^{loop}_{Sch-sub} \quad \text{for } \alpha_{1,2,3} = 1 \ , \ \alpha_0 = 0, \\
&= S^{loop}_{ext} \qquad \text{for } \alpha_{0,1,2,3} = 1, \\
&= 0 \qquad\quad \text{for } \sum_I \alpha_I^2 = \frac{8}{3} \ .
\end{aligned}
\tag{5.2.6}
$$

We conclude that a combination of Harrison parameters satisfying $\sum_I \alpha_I^2 = \frac{8}{3}$ takes us from the original Schwarzschild black hole to one with vanishing logarithmic corrections to the entropy.

5.3 Conclusion

We have studied the logarithmic corrections to the entanglement entropy of a minimally coupled scalar field in the subtracted geometry black hole background. Our main results are collected in formulas (5.1.19-5.1.24). They all differ from the corresponding results for non-subtracted black holes, indicating that the agreement of subtracted and non-subtracted entropies does not extend beyond the tree level. On the other hand, the subtracted results approach the original ones for the extremal BPS case in the appropriate limit. We noticed that the logarithmic correction term universally changes sign for all cases of subtracted black holes. For the schwarzchild case we found the interpolating solution which for certain choices of the Harrison parameters gives a vanishing logarithmic correction.

5.4 Appendix A: Parameters for the original general black hole expression

In this Appendix we provide the definitions of the parameters in the general expression for the entropy of original four-charge rotating black holes. We quote again formula (5.1.10):

$$
S^{loop} = \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{A_{red}(r_+)}{720} \int_{-1}^{1} \mathrm{d}u \, \frac{\kappa u^4 + \lambda u^2 + \mu}{(\alpha u^4 + \beta u + \gamma)^{3/2}} \, , \tag{5.4.1}
$$

The denominator is simply $\Delta_0^{3/2}$ with the change of variables $u = \cos\theta$. Hence we have:

$$\begin{aligned}
\alpha &= a^4, && (5.4.2)\\
\beta &= 2a^2(r_+^2 + C), && (5.4.3)\\
\gamma &= \prod_{I=0}^{4} (R_+^I)^2, && (5.4.4)
\end{aligned}$$

where $R_+^I = r_+ + 2m\sinh^2\delta_I$ and

$$C = mr_+ \sum_{I=0}^{3} \sinh^2\delta_I + \ 4m^2(\Pi_c - \Pi_s)\Pi_s - 2m^2 \sum_{I<J<K} \sinh^2\delta_I \sinh^2\delta_J \sinh^2\delta_K\,. \quad (5.4.5)$$

The remaining three parameters are given by:

$$\begin{aligned}
\kappa &= 4a^4\,, && (5.4.6)\\
\lambda &= 4\beta - 16a^2m^2\left(\Pi_c - \Pi_s\right)^2 + 2a^2(r_+ - r_-)(2r_+ - 2r_- - R_b)\,, && (5.4.7)
\end{aligned}$$

and

$$\begin{aligned}
\mu = {} & 4\gamma - 16a^2m^2\left(\Pi_c - \Pi_s\right)^2\\
& + \left[16mA_{red}(r_+)(\Pi_c - \Pi_s) + R_a + 4(r_+^2 + 2m\,r_+ + 2C)\right](r_+ - r_-)\,, \qquad (5.4.8)
\end{aligned}$$

where we use the abbreviations

$$R_a = R_+^0 R_+^1 R_+^2 + R_+^0 R_+^1 R_+^3 + R_+^0 R_+^2 R_+^3 + R_+^1 R_+^2 R_+^3, \quad (5.4.9)$$

and $R_b = 2r_+ + m\sum_I \sinh^2\delta_I$.

Chapter 6

Vacuum Polarization: Analytical Result

Quantum field theory in curved spacetime can be used to understand a lot of interesting features of black holes in a semiclassical approximation, most notably particle production near the black hole horizon [88]. The calculation of vacuum polarization or $\langle\phi^2\rangle$ (for a scalar field) is the simplest standard probe of quantum fluctuations in a black hole background, and can also be used to understand the symmetry breaking and Casimir effects near a black hole. Computation of $\langle\phi^2\rangle$ is also a preliminary step in evaluating the stress energy tensor $\langle T_{\mu\nu}\rangle$, which contributes to the backreaction through the semiclassical Einstein equation.

Candelas studied the vacuum polarization of a scalar field in the Schwarzschild black hole [89] and was able to obtain an analytical expression for $\langle\phi^2\rangle$ at the horizon. Candelas' methods extend easily to charged static black holes; there have also been numerical studies of vacuum polarization of scalar fields on general static black hole backgrounds beyond the event horizon (e.g. [90] for asymptotically flat solutions and [91] for the asymptotically anti-de Sitter case), and analytical computations at the horizon of a black hole threaded with a cosmic string [92]. The case of rotating black holes is much more challenging. Frolov [93] was able to calculate the analytical expression for $\langle\phi^2\rangle$ only at the pole ($\theta = 0$) of the event horizon, and Ottewill and Duffy [94] have provided a numerical evaluation throughout the black hole horizon. However so far no one has been able to give an analytical formula for $\langle\phi^2\rangle$ throughout the horizon of a four-dimensional rotating black hole. (An analytic approximation good for fields with large mass is available, however [95], and exact results are obtainable in $d = 3$ with AdS asymptotics [96, 97].)

The horizon vacuum polarization in the static subtracted metric was studied in [102]. In this chapter we shalll consider the subtracted geometry of the uncharged rotating Kerr black hole. We shall see that the special features of the subtracted rotating metric, in particular the well-defined nature of the thermal vacuum and the solvability of the wave equation, allow us to obtain analytical results that are unavailable for the standard Kerr black hole.

The subtracted Kerr metric is given by:

$$\begin{aligned} ds^2 = &-\Delta^{-1/2} G\, (dt + \mathcal{A}\, \mathrm{d}\tilde{\varphi})^2 \\ &+ \Delta^{1/2} \left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta\, d\tilde{\varphi}^2 \right) . \end{aligned} \tag{6.0.1}$$

with

$$\begin{aligned} X &= r^2 - 2Mr + a^2 \ , \quad G = r^2 - 2Mr + a^2\cos^2\theta \\ \mathcal{A} &= \frac{2Mar\sin^2\theta}{G}, \quad \Delta = 8M^3 r - 4M^2 a^2 \cos^2\theta \, . \end{aligned} \tag{6.0.2}$$

(The only difference between this metric and the standard Kerr metric is the form of the "warp factor" Δ. For the explicit form of gauge potentials and axio-dilatons of the STU model, supporting this geometry, see [15].) The horizons and their surface gravities and angular velocities are given by:

$$\begin{aligned} r_\pm &= M \pm \sqrt{M^2 - a^2}\,, \\ \kappa_\pm &= \frac{1}{2M}\left[\frac{M}{\sqrt{M^2-a^2}} \pm 1\right]^{-1} , \\ \Omega_\pm &= \kappa_\pm \frac{a}{\sqrt{M^2-a^2}}\,. \end{aligned} \tag{6.0.3}$$

We switch to co-rotating coordinates (t, r, θ, φ), with the new angular variable being defined by:

$$\varphi = \tilde{\varphi} - \Omega_+ t \, . \tag{6.0.4}$$

These are adapted to observers co-rotating with the black hole at the horizon. A noteworthy feature of subtracted geometry is that outside the horizon there is a globally defined timelike Killing vector, written as ∂_t in the co-rotating coordinates [103, 104]. This guarantees that there are no superradiant modes and ensures the existence of a Hartle-Hawking-like vacuum state adapted to the co-rotating observers. This is different from the case of ordinary Kerr black hole, where there is no such Killing vector [105, 106] and a physical co-rotating vacuum requires enclosing the black hole in a reflective box [94, 107]. The subtracted Kerr resembles more in this respect the Kerr/AdS black hole [96].

The general algorithm we follow for computing the horizon vacuum polarization in the Hartle-Hawking state starts by defining the Euclidean Green's function $G_H(x,x')$ (in a state regular at the horizon and infinity, and where the modes are adapted to co-rotating coordinates). Then we will evaluate $-iG_H$ with radial point splitting, perform the mode sum, and subtract the covariant divergent counterterms.

After writing the metric in coordinates (t,r,θ,φ) we perform the Wick rotation setting $t=-i\tau$. The metric becomes:

$$\begin{aligned} ds_E^2 &= -\frac{G}{\Delta^{1/2}}\left[\mathcal{A}\,\mathrm{d}\varphi - i(1+\mathcal{A}\Omega_+)d\tau\right]^2 \\ &+\Delta^{1/2}\left(\frac{dr^2}{X}+d\theta^2+\frac{X}{G}\sin^2\theta\,(d\varphi - i\Omega_+ d\tau)^2\right) . \end{aligned} \tag{6.0.5}$$

On writing the massless minimally coupled wave equation and proposing a solution of the form $\mathrm{e}^{in\kappa_+\tau}\mathrm{e}^{im\varphi}P_l^m(\cos\theta)\chi_{lmn}(r)$, we obtain straightforwardly a radial equation which, in the re-scaled variable $x=(r-\frac{1}{2}(r_++r_-))/(r_+-r_-)$, reads:

$$\begin{aligned} &\Big[\frac{\partial}{\partial x}\left(x^2-\frac{1}{4}\right)\frac{\partial}{\partial x}-\frac{n^2}{4\left(x-\frac{1}{2}\right)} \\ &+\frac{\beta_{mn}}{4\left(x+\frac{1}{2}\right)}-l(l+1)\Big]\,\chi_{lmn}(x)=0\,, \end{aligned} \tag{6.0.6}$$

where

$$\beta_{mn}=\frac{2Mn^2r_- - a^2(4m^2+n^2)-4iamnr_-}{r_+^2}\,. \tag{6.0.7}$$

Two independent solutions of the equation, respectively regular at the horizon and at infinity, are:

$$\chi_{lmn}^{(1,2)} = \frac{\left(x-\frac{1}{2}\right)^{\frac{n}{2}}}{\left(x+\frac{1}{2}\right)^{\frac{n}{2}+l+1}} F\left(a_{lmn}, b_{lmn}, c_{ln}^{(1,2)}; z^{(1,2)}\right) , \tag{6.0.8}$$

where

$$c_{ln}^{(1)} = n+1\,,\ c_{ln}^{(2)} = 2l+2\,,\ z^{(1)} = \frac{x-\frac{1}{2}}{x+\frac{1}{2}}\,,\ z^{(2)} = \frac{1}{x+\frac{1}{2}}\,,$$

$$(a_{lmn}, b_{lmn}) = l+1+\frac{|n|}{2} \pm \frac{\sqrt{\beta_{mn}}}{2}\,, \tag{6.0.9}$$

and the symmetry of the hypergeometric function makes irrelevant which branch of the square root is chosen.

The full Green's function is expanded as

$$G_H(-i\tau, x, \theta, \varphi\,; -i\tau', x', \theta', \varphi') = \frac{i\kappa}{2\pi\, r_0} \sum_{n=-\infty}^{\infty} e^{in\kappa(\tau-\tau')}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') G_{mln}(x,x')\,, \tag{6.0.10}$$

where $r_0 = r_+ - r_- = 2\sqrt{M^2-a^2}$, $\kappa \equiv \kappa_+$ as defined in (6.0.3), and

$$G_{mln}(x,x') = \frac{\Gamma(a_{mln})\Gamma(b_{mln})}{\Gamma(2l+2)\Gamma\left(1+|n|\right)} \chi_{mln}^{(1)}(x_<)\chi_{mln}^{(2)}(x_>)\,. \tag{6.0.11}$$

To evaluate the vacuum polarization at the horizon we set $x = 1/2, x' = \epsilon + \frac{1}{2}$ (note that this is a dimensionless regulator $\epsilon = (r'-r)/r_0$) and join the points in the other directions, calling the resulting Green's function $G_H(\epsilon, \theta)$. All the terms in the sum vanish except

$n = 0$, so we are reduced to:

$$
\begin{aligned}
-iG_H(\epsilon,\theta) = \frac{\kappa}{8\pi^2\, r_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} \left[P_l^m(\cos\theta)\right]^2 \\
\times \frac{\Gamma(l+1+i\alpha m)\Gamma(l+1-i\alpha m)}{\Gamma(2l+1)} (1+\epsilon)^{-(l+1)} \\
\times F\left(l+1+i\alpha m, l+1-i\alpha m, 2l+2, \frac{1}{1+\epsilon}\right) ,
\end{aligned}
\tag{6.0.12}
$$

where the parameter $\alpha \equiv a/r_+$ takes values between 0 and 1. We replace the hypergeometric by an integral expression using formula 9.111 of [108], leading to:

$$
\begin{aligned}
-iG_H(\epsilon,\theta) = \frac{\kappa}{8\pi^2\, r_0} \sum_{l=0}^{\infty} (2l+1) \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} \left[P_l^m(\cos\theta)\right]^2 \\
\times \int_0^1 \mathrm{d}t \left(\frac{t(1-t)}{1+\epsilon-t}\right)^l \frac{1}{1+\epsilon-t} \cos\left(m\alpha \ln\lambda\right) ,
\end{aligned}
\tag{6.0.13}
$$

where $\lambda = \left(\frac{(1+\epsilon)(1-t)}{t(1+\epsilon-t)}\right)$.

The addition theorem for the associated Legendre polynomials is used to compute the sum over m, and formula III.4 from [109] subsequently yields the sum over l. This leads, after a change of variables to $x = 1-t$, to the integral expression

$$
-\, iG_H(\epsilon,\theta) = \frac{\kappa}{8\pi^2\, r_0} \int_0^1 \mathrm{d}x\, f_\epsilon(x)\,; \tag{6.0.14}
$$

$$
f_\epsilon(x) = \frac{\frac{\epsilon^2+2\epsilon x+(2-x)x^3}{(x^2+\epsilon)^3}}{\left[1+\frac{4x(1-x)(x+\epsilon)}{(x^2+\epsilon)^2}\sin^2\theta\sin^2\left(\frac{\alpha}{2}\ln\lambda\right)\right]^{3/2}} , \tag{6.0.15}
$$

with $\lambda = \lambda(t(x))$. It is easy to see from numerical evaluation that the leading divergences in the integral as $\epsilon \to 0$ match those provided by the standard counterterms [110],

$$
G_{div} = \frac{1+\frac{1}{12}R_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu}}{8\pi^2\sigma} - \frac{1}{96\pi^2} R\ln(\mu^2\sigma)\,, \tag{6.0.16}
$$

where σ is the halved geodesic distance between the points and μ is an arbitrary mass scale.

It is more difficult, however, to obtain an explicit expression for the finite result of the subtraction. To make progress we perform the following sequence of changes of variables:

$$u = \frac{1}{2}\ln\left(\frac{x(1+\epsilon)}{(1-x)(x+\epsilon)}\right), \qquad w = \sinh u\,. \tag{6.0.17}$$

This leads to the more tractable expression for the integral $I_\epsilon \equiv \int_0^1 \mathrm{d}x\, f_\epsilon(x)$:

$$I_\epsilon = \int_0^\infty dw \frac{\sqrt{1+\epsilon}}{\left[\epsilon + (1+\epsilon)w^2 + v^2\sin^2(\alpha\sinh^{-1} w)\right]^{3/2}}, \tag{6.0.18}$$

where $v \equiv \sin\theta$. The intermediate u-integral expression is also obtainable directly from dimensional reduction from the Euclidean Green's function in $\mathrm{AdS}^3\times\mathrm{S}^2$, using the higher-dimensional embedding of subtracted geometry described in [68][21].

To analyze the small ϵ limit and subtract explicitly the counterterms, we set aside momentarily the $\sqrt{1+\epsilon}$ prefactor and split the integral in two subintervals, $I_\epsilon^<$ over $(0, \epsilon^{1/6})$ and $I_\epsilon^>$ over $(\epsilon^{1/6}, +\infty)$. In the second subinterval we can set ϵ to zero, at the expense of an error that vanishes as $\epsilon \to 0$. Then we can add and subtract terms compensating for the leading divergences at the lower limit, take $\epsilon \to 0$ safely in the subtraction, and integrate explicitly the added coutnerterms. This leads to:

$$\begin{aligned} I_\epsilon^> \sim \int_0^\infty \mathrm{d}w \Bigg[&\frac{1}{\left[w^2 + v^2\sin^2(\alpha\sinh^{-1} w)\right]^{3/2}} \\ &- \left(\frac{1}{w^3(1+\alpha^2v^2)^{3/2}} + \frac{v^2\alpha^2(1+\alpha^2)}{2w(1+w)(1+\alpha^2v^2)^{5/2}}\right)\Bigg] \\ &+ \frac{1}{2\epsilon^{1/3}(1+\alpha^2v^2)^{3/2}} - \frac{v^2\alpha^2(1+\alpha^2)\ln\epsilon}{12(1+\alpha^2v^2)^{5/2}}, \end{aligned} \tag{6.0.19}$$

where $\sim$ stands for equivalence as $\epsilon \to 0$. The second subintegral is thus reduced to a finite integral involving no regulator, that can be evaluated numerically, plus two explicit divergent terms.

[21] We thank Finn Larsen for bringing this point to our attention.

In the first subinterval, we can show that:

$$
\begin{aligned}
I_\epsilon^< &= \int_0^{\epsilon^{1/6}} \frac{\mathrm{d}w}{\left[\epsilon + (1+\epsilon)w^2 + v^2 \sin^2(\alpha \sinh^{-1} w)\right]^{3/2}} \\
&\sim \int_0^{\epsilon^{1/6}} \frac{\mathrm{d}w}{\left[\epsilon + (1+\epsilon)w^2 + v^2 \left(\alpha^2 w^2 - \frac{\alpha^2(\alpha^2+1)w^4}{3}\right)\right]^{3/2}} \,, \qquad (6.0.20)
\end{aligned}
$$

which is expressible (formula 3.163.3 of [108]) in terms of the incomplete elliptic integrals of first and second kind, $F(\gamma, k)$ and $E(\gamma, k)$. Here

$$
\gamma = \arcsin\left(\frac{\epsilon^{1/6}}{\sqrt{c_+}}\sqrt{\frac{c_- + c_+}{c_- + \epsilon^{1/3}}}\right) , \qquad k = \sqrt{\frac{c_+}{c_- + c_+}} \,, \qquad (6.0.21)
$$

and $c_\pm$ are the coefficients appearing in the denominator of the integrand when it is factored in a form proportional to $[(c_+^2 - w^2)(c_-^2 + w^2)]^{3/2}$. We need the expansions of the elliptic functions near $(\gamma, k) = (\frac{\pi}{2}, 1)$, which have been derived in [111]. In order to obtain all the divergent and finite contributions to $I_\epsilon^<$, we need F accurately to order 1 and E accurately to order ϵ. This in turns require obtaining the argument k accurately to order ϵ and γ to order $\epsilon^{4/3}$. The result of this expansion is the following expression for the divergent and finite pieces of $I_\epsilon^<$:

$$
\begin{aligned}
I_\epsilon^< &\sim -\frac{1}{2\epsilon^{1/3}(1+\alpha^2 v^2)^{3/2}} + \frac{1}{\epsilon\sqrt{1+\alpha^2 v^2}} \\
&+ \frac{1}{6(1+\alpha^2 v^2)^{5/2}} \times \Big(-3 - \alpha^2(7+4\alpha^2)v^2 \\
&+ \alpha^2(1+\alpha^2)v^2(\ln(8(1+\alpha^2 v^2)^{3/2}) - \ln\epsilon)\Big) \,. \qquad (6.0.22)
\end{aligned}
$$

There is an additional finite contribution coming from the prefactor $\sqrt{1+\epsilon}$ to the integral, which yields when expanded a 1/2 multiplied by the coefficient of the linear divergence of the integral. The complete result is thus expressed as:

$$
I_\epsilon = I_\epsilon^< + I_\epsilon^> + \frac{1}{2\sqrt{1+\alpha^2 v^2}} \,, \qquad (6.0.23)
$$

with the first to terms given by (7.1.16) and (6.0.19) respectively. We see that the $\epsilon^{-1/3}$ divergences cancel out, leaving only linear and logarithmic divergences that will match those of counterterms (6.0.16), leaving a finite renormalized result.

This concludes the computation of the explicit divergent and finite portions of the Green's function's coincidence limit. The counterterms (6.0.16) need now to be evaluated as a function of ϵ to the order $O(1)$. The form of σ can be computed from the formulas expressing σ in terms of coordinate separation:

$$\sigma = \frac{1}{2} g_{ab} \Delta x^a \Delta x^b + A_{abc} \Delta x^a \Delta x^b \Delta x^c + B_{abcd} \Delta x^a \Delta x^b \Delta x^c \Delta x^d + \cdots \tag{6.0.24}$$

where A, B are obtained from symmetrized derivatives of the metric tensor, as described in [112].

These expressions are valid in a coordinate system in which the metric is regular. We use the Kruskal coordinates for the subtracted geometry that have been derived in [104], which take the form (U, V, θ, φ) with $(-UV) \propto (r - r_+)$ near the horizon. Our radial coordinate separation is therefore written as $\Delta x^a = (-\delta, \delta, 0, 0)$ (with $\delta \propto \sqrt{\epsilon}$). After computing σ by this procedure (leading to an expression of the form $\sigma = \beta_1 \epsilon + \beta_2 \epsilon^2 + O(\epsilon^3)$) it is easy to obtain the Ricci counterterm in (6.0.16) because to the relevant order $O(\epsilon)$ we have $R_{\mu\nu} \sigma^{,\mu} \sigma^{,\nu} = R_{rr} \sigma^{,r} \sigma^{,r}$.

Once all the counterterms are computed by this procedure, when expressed in terms of the α parameter they take the relatively simple form:

$$G_{div} = \frac{1 + \alpha^2}{64\pi^2 M^2} \left[\frac{1}{\epsilon \sqrt{1 + \alpha^2 v^2}} - \frac{\alpha^2 v^2 (1 + \alpha^2) \ln \epsilon}{4(1 + \alpha^2 v^2)^{5/2}} + \frac{(-1 + \alpha^2(-4 + \alpha^2 + (7 + \alpha^2 + \alpha^4) v^2 + 3\alpha^2 v^4))}{12(1 + \alpha^2 v^2)^{5/2}} \right], \tag{6.0.25}$$

(plus a term of the form $R(r_+, \theta) \ln \mu^2$). Then, absorbing some R-proportional terms into the arbitrary constant μ, the final result for the vacuum polarization is:

$$
\begin{aligned}
\langle\phi^2\rangle_{r_+} = & R(r_+,\theta)\ln\mu^2 + \frac{1+\alpha^2}{64\pi^2 M^2}\left\{\frac{1}{12(1+\alpha^2 v^2)^{5/2}}\right. \\
& \times\Big[(1-\alpha^2(-4+\alpha^2(9+9\alpha^2+\alpha^4)v^2-3\alpha^2 v^4)) \\
& -3\alpha^2(1+\alpha^2)^2\ln(1+\alpha^2 v^2)\Big] \\
& + \int_0^\infty \mathrm{d}w\left[\frac{1}{\left[w^2+v^2\sin^2(\alpha\sinh^{-1}w)\right]^{3/2}}\right. \\
& \left.\left. -\left(\frac{1}{w^3(1+\alpha^2v^2)^{3/2}}+\frac{v^2\alpha^2(1+\alpha^2)}{2w(1+w)(1+\alpha^2v^2)^{5/2}}\right)\right]\right\}, \qquad (6.0.26)
\end{aligned}
$$

where

$$
R(r_+,\theta) = \frac{3\alpha^2(1+\alpha^2)^2 v^2}{8\,M^2(1+\alpha^2v^2)^{5/2}}\,. \qquad (6.0.27)
$$

Notice that in the absence of rotation spherical symmetry is recovered, with its value $\langle\phi^2\rangle_{r_+}^{Sch_{sub}} = (768\pi^2M^2)^{-1}$ matching the result obtained in [102] for the subtracted Schwarzschild black hole. In addition, the result at the pole takes the form $\langle\phi^2\rangle_{r_+,\theta=0} = (768\pi^2M^2)^{-1}(1+\alpha^2)(1+4\alpha^2-\alpha^4)$, agreeing with result found in [102] using a non-corotating vacuum state (at the pole, the distinction is irrelevant). The dot-dashed plot corresponds to the extremal case $a=M$.

It would be interesting to compare our results with numerical computations of the vacuum polarization in the standard Kerr metric (with a mirror in place to define the vacuum). Our calculation holds for the minimally coupled field, and the numerical results in [94] are for the conformal case, so a direct comparison is not yet available. We expect our calculations to be easily generalized to the case of fields with higher spins as well as to rotating charged black holes, including multi-charged solutions. We also expect our methods to be applicable to the computation of the stress-energy tensor, which would open the possibility of using the subtracted approximation to study analytically the backreaction for rotating four-dimensional black holes.

Chapter 7

Thermodynamics: Mass, Charge and Angular momentum

Black holes behave as thermodynamic objects. The thermodynamic properties of black holes are determined by the behavior of their geometry at the asymptotics due to the nature of spacetime curvature. The case of black holes in asymptotically flat spacetimes is very well understood [114] and is straightforward. On the other hand, the case of black holes in asymptotically non-zero negative cosmological constant (anti-deSitter (AdS) spacetime) possess new thermodynamic features, crucial in studies of gravity/field theory duality. In general, in fundamental theories where physical constants such as Yukawa couplings, gauge coupling constants or Newton constant as well as the cosmological constant arise as vacuum expectation values of scalar fields and hence can vary, the thermodynamic laws are changed to include these variations (see, e.g., [115]). These "constants" are thought of as the vacuum expectation values of fields at asymptotic infinity, and their variation can lead to new insights into thermodynamic behavior of gravitational systems, which can play an important role in the study of gravity/field theory duality.

In this chapter we focus on the study of thermodynamic properties of geometries which are asymptotically conical (AC). The fields supporting such geometries, instead of becoming constant at spatial infinity, vary as a function of radial distance at infinity. These geometries have very different asymptotic structure compared to the asymptotically flat and asymptotically AdS case. Their thermodynamics has not been explored in detail, and new insights there would provide a starting point for the study of gravity/field theory duality for AC spacetimes. The spacetime metrics have the asymptotic form:

$$ds^2 = -Y^{2p}dt^2 + B^2 dY^2 + Y^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,, \tag{7.0.1}$$

where p and B are constants.These AC metrics have the special properties that their radial distance BY is a non-trivial multiple of the area distance Y and their spatial metric restricted to the equatorial plane is that of a flat two-dimensional cone. The energy density of such metrics typically falls off as inverse squared of the radial distance and thus the geometry cannot have a finite total energy. Bisnovatyi-Kogan-Zeldovich's gas sphere [116, 117], Barriola-Vilenkin Global Monopole [118], the near horizon geometry of an extreme black hole in Einstein-Dilaton-Maxwell gravity, a black hole containing a global monopole and the cosmic string metric outside the string are all examples of asymptotically conical metrics. In this letter we study the thermodynamics of a special class of metrics of the asymptotic form (7.0.1), known as the "subtracted geometries" with $p = 3$, $B = 4$, and $Y = (8m^3r)^{\frac{1}{4}}$. The thermodynamics of these geometries is not known and we show that the subtleties lie in deriving the mass, asymptotic charges and gauge fields there. The conclusions here are generalizable to other cases of AC geometries and thus are of broader interest.

The Lagrangian density of this N=2 supergravity coupled to the three vector multiplets, also known as the STU-model is given by [123]:

$$\begin{aligned}
\mathcal{L}_4 \;\; &= \;\; R\,{*\mathbf{1}} - \frac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}\,{*d\chi_i} \wedge d\chi_i \\
&- \;\; \frac{1}{2}e^{-\varphi_1}\,(e^{\varphi_2-\varphi_3}\,{*F_1} \wedge F_1 + e^{\varphi_2+\varphi_3}\,{*F_2} \wedge F_2 \\
&+ \;\; e^{-\varphi_2+\varphi_3}\,{*\mathcal{F}_1} \wedge \mathcal{F}_1 + e^{-\varphi_2-\varphi_3}\,{*\mathcal{F}_2} \wedge \mathcal{F}_2) \\
&- \;\; \chi_1\,(F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2)\,,
\end{aligned} \tag{7.0.2}$$

where the index i ranges over $1 \leq i \leq 3$. The four field strengths in terms of potentials are given by:

$$
\begin{aligned}
F_1 &= dA_1 - \chi_2\, d\mathcal{A}_2\,,\ \mathcal{F}_1 = dA_1 + \chi_3\, d\mathcal{A}_2\,,\ \mathcal{F}_2 = d\mathcal{A}_2\,,\\
F_2 &= dA_2 + \chi_2\, d\mathcal{A}_1 - \chi_3\, dA_1 + \chi_2\,\chi_3\, d\mathcal{A}_2\,. \qquad (7.0.3)
\end{aligned}
$$

The four-charge rotating black hole metric is[22] :

$$
ds^2 = -\Delta_0^{-\frac{1}{2}} G(dt+\mathcal{A})^2 + \Delta_0^{\frac{1}{2}}(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G}\sin^2\theta d\phi^2), \qquad (7.0.4)
$$

where [23]:

$$
\begin{aligned}
X &= r^2 - 2mr + a^2\,,\\
G &= r^2 - 2mr + a^2 c_\theta^2\,,\\
\mathcal{A} &= \frac{2mas_\theta^2}{G}\,[(\Pi_c - \Pi_s)r + 2m\Pi_s]\,d\phi\,, \qquad (7.0.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_0 = &\prod_{I=1}^{4}(r+2ms_I^2) + 2a^2c_\theta^2[r^2 + mr\sum_{I=1}^{4} s_I^2 \qquad (7.0.6)\\
&+4m^2(\Pi_c-\Pi_s)\Pi_s - 2m^2\sum_{I<J<K} s_I^2 s_J^2 s_K^2]\ + a^4 c_\theta^4\,.
\end{aligned}
$$

The physical parameters (mass M, angular momentum J and charges Q_I) of the general four-charge black hole are parametrized in terms of auxiliary constants m, a, δ_I as:

$$
M = \frac{1}{4}m\sum_{I=0}^{3} c_{2I}\ , Q_I = \frac{1}{4}m\, s_{2I}\ , J = m\, a\,(\Pi_c - \Pi_s)\ , \qquad (7.0.7)
$$

[22]The four gauge potentials and three axio-dilaton fields are given in [123]. For the subtracted geometry analysis we can take the gauge potentials $A_1 = A_2 = A_3 \equiv A$ for gauge field strengths $*F_1 = F_2 = *\mathcal{F}_1 \equiv F$ and $A_4 \equiv \mathcal{A}$ for $\mathcal{F}_2 \equiv \mathcal{F}$. The gauge potential definitions of this letter differ from the ones used in [15, 123] by a factor of 1/2 to comply with standard literature convention.

[23]$s_\theta \equiv \sin\theta,\ c_\theta \equiv \cos\theta,\ s_I \equiv \sinh\delta_I,\ c_I \equiv \cosh\delta_I,\ s_{2I} \equiv \sinh 2\delta_I,\ c_{2I} \equiv \cosh 2\delta_I,\ \Pi_c \equiv \prod_{I=1}^4 c_I$ and $\Pi_s \equiv \prod_{I=1}^4 s_I$.

The subtraction procedure corresponds to replacing the "warp factor" Δ_0 with Δ, where:

$$\Delta = (2m)^3 r(\Pi_c^2 - \Pi_s^2) + (2m)^4\Pi_s^2 - (2ma)^2(\Pi_c - \Pi_s)^2 c_\theta^2\,, \tag{7.0.8}$$

while keeping everything else unchanged. Importantly, this leaves the global structure unchanged, with two horizons at $r_\pm = m \pm \sqrt{m^2 - a^2}$, with the same area and surface gravity there. (For the most general rotating black holes of the STU model with one more independent charge parameter [124], the subtracted geometry was obtained and analyzed in [125].)Therefore the entropy S, temperature T and angular potential Ω of the subtracted geometry remain the same as in the full geometry and are given by:

$$\begin{aligned} S =& \quad 2\pi m\left[(\Pi_c + \Pi_s)m + (\Pi_c - \Pi_s)\sqrt{m^2 - a^2}\right]\,, \\ T =& \quad \frac{\kappa_+}{2\pi}\,,\ \ \Omega = \kappa_+ \frac{a}{\sqrt{m^2 - a^2}}\,, \end{aligned} \tag{7.0.9}$$

where

$$\frac{1}{\kappa_+} = 2m\left[\frac{m}{\sqrt{m^2 - a^2}}(\Pi_c + \Pi_s) + (\Pi_c - \Pi_s)\right]\,. \tag{7.0.10}$$

The values of the fields sourcing the subtracted geometry are however changed. The gauge fields at $\theta = 0$ at the outer horizon r_+ are given by:

$$\begin{aligned} \mathcal{A}(r_+) =& \quad \frac{2m^2[(2m)^2\Pi_c\Pi_s + a^2(\Pi_c - \Pi_s)^2]}{R^4}\,dt\,, \\ A(r_+) =& \quad \frac{m - r_+}{4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}}\,dt \\ -& \quad \frac{m\,a^2\,(\Pi_c - \Pi_s)[\,r_+\,(\Pi_c - \Pi_s) + 2m\Pi_s\,]}{(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}R^4}\,dt\,, \end{aligned} \tag{7.0.11}$$

where at the outer horizon $r_+ = m + \sqrt{m^2 - a^2}$, $R^4 = (2m)^2[m(\Pi_c + \Pi_s) + \sqrt{m^2 - a^2}(\Pi_c - \Pi_s)]^2$. The gauge of these gauge fields is uniquely fixed by the scaling limit discussed below. (The gauge potentials at $\theta \neq 0$ can be found in [15].) The three dilatons and axions are given by,

$$e^{\varphi} \equiv e^{\varphi_{1,2,3}} = \frac{(2m)^2(\Pi_c^2 - \Pi_s^2)^{\frac{2}{3}}}{R^2}\,, \tag{7.0.12}$$

and

$$\chi \equiv \chi_1 = -\chi_2 = \chi_3 = -\frac{a(\Pi_c - \Pi_s)^{1/3} c_\theta}{2m}\,, \tag{7.0.13}$$

respectively. These three axio-dilaton fields are also fixed by the scaling limit. The asymptotic charges can then be easily obtained from the Gauss law,

$$\mathcal{Q} = \frac{m\Pi_c\Pi_s}{\Pi_c^2 - \Pi_s^2}\,, \qquad Q = m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}\,. \tag{7.0.14}$$

An important point to notice here is that the dilatons have a spatial dependence. This forces the gauge coupling constants to run logarithmically in the radial direction not even stabilizing at infinity. This is an important feature that the subtracted geometries share with Dilaton-Maxwell theory when a limit of vanishing Newtons constant is taken.

7.1 Thermodynamics

The definitions of mass and angular momentum are heavily dependent on the asymptotics of the curved geometry. Let us start by studying the mass of our subtracted geometry first. We can afford here to deal with the static case $a = 0$ since mass is defined independent of rotation. We can parameterize our static geometry by:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{7.1.1}$$

where $N = X^{\frac{1}{2}}\Delta^{-\frac{1}{4}}$ and $R = \Delta^{\frac{1}{4}}$. In the Hawking-Horowitz prescription [126] the mass is given by:

$$M_{HH} = -\frac{1}{8\pi}\int_{S_t\to\infty} N(K - K_0)\, d\Omega\,, \tag{7.1.2}$$

where $d\Omega = R^2 \sin\theta d\theta d\phi$, K is the extrinsic curvature of the boundary two sphere and K_0 in our case, will be the extrinsic curvature of the two-sphere *embedded in asymptotically conical geometry.* Up to $O(r^{-1})$ corrections we can show that $N \sim rR_0^{-1}$, $S_t \sim 4\pi R_0^2$, $R \sim R_0(1+m\Pi_s^2([2r(\Pi_c^2-\Pi_s^2)]^{-1})$ and $R_0 \equiv (2m)^{\frac{3}{4}} r^{\frac{1}{4}} (\Pi_c^2-\Pi_s^2)^{\frac{1}{4}}$. Calculating the Hawking-

Horowitz mass, we get:

$$M_{HH} = \frac{m}{4}\frac{\Pi_c^2 + \Pi_s^2}{\Pi_c^2 - \Pi_s^2}\,. \tag{7.1.3}$$

Next we would like to check that the Komar mass formula gives us the same results as the Hawking-Horowitz formalism. The Komar mass is defined as:

$$M_K = -\frac{1}{8\pi}\int_{S_t\to\infty} \star d\zeta_{(t)}\,, \tag{7.1.4}$$

where $\zeta_{(t)}$ is the time-like Killing vector. In the static subtracted geometry $\zeta_{(t)}$ is given by $-X\Delta^{-\frac{1}{2}}dt$. One can show that:

$$M_K = \frac{3}{4}r - \frac{1}{2}\frac{m\big(\Pi_c^2 - 2\Pi_s^2\big)}{\Pi_c^2 - \Pi_s^2} + O(\frac{1}{r^2})\,, \tag{7.1.5}$$

which diverges linearly with r. The appearance of this divergence is one of the most important features that separates the asymptotically conical case with the asymptotically flat case. We can however show that this divergence gets regulated once we take the asymptotic gauge fields and charges into account. Defining, $H^{\mu\nu} = \nabla^\mu\zeta^\nu_{(t)} - \nabla^\nu\zeta^\mu_{(t)}$ allows us to show that:

$$\nabla_\mu H^{\mu\nu} = -16\pi(T^\nu_\mu - \tfrac{1}{2}T\delta^\nu_\mu)\zeta^\mu_{(t)}\,. \tag{7.1.6}$$

Using the above relation for the case of static electrically charged subtracted geometry, we obtain:

$$\nabla_r\left(H^{rt} + 8\pi[3e^{\varphi}F^{rt}A_t(r) + e^{-3\varphi}\mathcal{F}^{rt}\mathcal{A}_t(r)]\right) = 0\,, \tag{7.1.7}$$

As $S_t\to\infty$, $\mathcal{A}_t(r)\to 0$ and thus only the term with $A_t(r)$ contributes. Furthermore we can identify that $R^2e^{\varphi}F^{rt} = Q$, and thus obtain:

$$M_{K_{reg}} = M_K + 3QA_t(r) = M_K(r) + \tfrac{3}{4}(m - r)\,, \tag{7.1.8}$$

where M_K is the unregulated Komar mass. Therefore the terms linear in r cancel and the regulated Komar mass is:

$$M_{K_{reg}} = M_{HH} \,, \tag{7.1.9}$$

i.e. the Komar formula gives the same result as the one obtained through the Hawking-Horowitz formalism.

We can also write the explicit expression of M_{HH} in terms of the reducible mass $M_{\text{irr}}^2 \equiv \frac{S}{4\pi}$, $\mathcal{Q}$ and Q:

$$M_{HH} = \frac{1}{4}\left(\frac{M_{\text{irr}}^4}{Q^3} + \frac{\mathcal{Q}^2 Q^3}{M_{\text{irr}}^4}\right) . \tag{7.1.10}$$

This formula can be used to give an analogue of the Christodoulou-Ruffini inequality (7.3.1) for our case, telling us the bound on how much mass of the black hole can be converted into energy.

Now we can move to define the angular momentum and study the $a \neq 0$ case. In the Hawking-Horowitz formalism the angular momentum is given by:

$$J_{HH} = -\frac{1}{8\pi}\int_{S_t\to\infty}(K_{ab} - Kh_{ab})N^a r^b d\Omega \,. \tag{7.1.11}$$

where a, b run over r, θ, ϕ. h_{ab} is the induced metric on the constant time hypersurface, N^a is the shift vector and r^a is the unit normal to the boundary two sphere. In the axially symmetric case of subtracted geometry the second term does not contribute because h_{ab} does not have a ϕr-component. On the other hand the Komar integral for the angular momentum is given by:

$$J_K = \frac{1}{16\pi}\int_{S_t\to\infty}\nabla^\mu \zeta^\nu_{(\phi)} dS_{\mu\nu} \,, \tag{7.1.12}$$

where $\zeta^\mu_{(\phi)}$ is the rotational Killing vector and the area element $dS_{\mu\nu} = -2n_{[\mu}r_{\nu]}d\Omega$ with $n^\mu = e^\mu_a n^a$ and $r^\mu = e^\mu_a r^a$ being the time-like and space-like normals to the surface S_t. We can show the equality of (7.1.11) and (7.1.12) by employing:

$$\nabla^\mu \zeta^\nu_{(\phi)} n_\mu r_\nu \quad = \quad K_{ab} N^a n^b \,. \tag{7.1.13}$$

It is a simple exercise to show:

$$J_{HH} = J_K = J \equiv am(\Pi_c - \Pi_s)\,. \tag{7.1.14}$$

Once we have defined mass and angular momentum we can easliy show that the Smarr law:

$$M_{HH} = 2TS + \mathcal{A}_t(r_+)\mathcal{Q} + 3A_t(r_+)Q + 2J\Omega\,, \tag{7.1.15}$$

and the first law of thermodynamics:

$$dM_{HH} = T\,dS + \mathcal{A}_t(r_+)\,d\mathcal{Q} + 3A_t(r_+)\,dQ + \Omega\,dJ\,, \tag{7.1.16}$$

continue to hold for our geometry. This gives us further confidence in our definitions of the mass and angular momentum. Another important point to notice is that in order for these laws to be obeyed, the gauge fixing of the fields was crucial and was uniquely fixed by the scaling limit.

7.2 Scaling Limit

There are two ways to obtain the subtracted geometries starting from the original ones. Firstly they can be obtained by applying the Harrison transformations [15] and secondly it can also be obtained via the scaling limit [15]. In this section we apply the scaling limit to the mass and angular momentum formulae obtained in the original black hole calculations and see how they agree with the subtracted geometry answers that we obtained above by direct calculations. The limit is implemented by means of the following scalings:

$$
\begin{aligned}
&m \to m\,\epsilon, \quad r \to r\,\epsilon, \quad t \to t\,\epsilon^{-1}, \quad a \to a\,\epsilon\,, \\
&\sinh^2\delta_{1,2,3} \to \frac{(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}}{\epsilon^{\frac{4}{3}}}, \sinh^2\delta_4 \to \frac{\Pi_s^2}{(\Pi_c^2 - \Pi_s^2)}\,.
\end{aligned}
\tag{7.2.1}
$$

The scaling limit ensures that the entropy, the surface gravity, the angular velocity *and* the angular momentum are the same as those of the asymptotically flat black hole, with the result for the angular momentum confirmed by the independent calculation above. The matching of the mass formula is however more involved. The mass of the original four-charge STU black hole is given by: $M = \frac{1}{4} m \sum_I \cosh 2\delta_I$. Its scaling limit is:

$$
\frac{M}{\ } = \frac{3m}{2} \frac{(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}}{\frac{4}{3}} + \frac{m}{4} \frac{\Pi_c^2 + \Pi_s^2}{\Pi_c^2 - \Pi_s^2} + \frac{3m}{4}\,. \tag{7.2.2}
$$

In the scaling limit the gauge potentials $A_{1,2,3}$ acquire an infinite constant term, not included in (7.0.12), which along with the large charges $Q_{1,2,3}$ ensure that in the Smarr relation, the product of original charges $Q_i = m \sinh 2\delta_i$ with A_i $(i = 1, 2, 3)$ contain divergent parts which cancel the divergent part in (7.2.2). Furthermore the constant term $\frac{3m}{4}$ in (7.2.2) is cancelled by a product of the sub-leading contribution in $Q_{1,2,3}$ and the divergent part of $A_{1,2,3}$. The remaining contributions are due to the precisely quoted charges (7.0.14) and gauge potentials (7.0.12), thus verifying that the mass of the subtracted geometry is indeed M_{HH}.

7.3 Conclusion and Discussion

It should be noted that our successful extension of a coherent black hole thermodynamic theory involving appropriately defined asymptotic charges to the case of subtracted geometries depends crucially on taking seriously their asymptotically conical nature. This differs both qualitatively and quantitatively from the the standard asymptotically flat and asymptotically AdS cases. Nevertheless the end result shares the universal features of those cases and gives further support to the idea that there are microscopic states or degrees of freedom (possibly stringy) counted by the entropy of black holes and the number of such states is insensitive to which environment they find themselves in.

Our analysis also resulted in the explicit expression (7.1.10) for M_{HH} in terms of $M_{\rm irr}$, $\mathcal{Q}$ and Q. This expression lends itself to propose an analog of the Christodulou-Ruffini inequality [127]:

$$M_{HH} \geq \frac{1}{4}\left(\frac{M_{\rm irr}^4}{Q^3} + \frac{\mathcal{Q}^2 Q^3}{M_{\rm irr}^4}\right) . \tag{7.3.1}$$

Such an inequality can be tested, at least in time symmetric data context [128] by taking the scaling limit of the initial data results for the STU model.

Furthermore for these initial data, the Einstein-Rosen Bridge structure is manifest from eq.(2.24) and eq.(2.25) of [104] where the reflection map of Kruskal-Szekeres coordinates $(U, V) \to (-U, -V)$ is an isometry that leaves the radial coordinate r invariant but fixes the $U = const.V$ surfaces, which in regions I and IV are constant time surfaces. Thus the initial data of the asymptotically conical 3-metrics has to be joined by an Einstein-Rosen throat. Further study of these properties of the subtracted geometry is of great interest.

Chapter 8

Conclusion

Subtracted geometries are an extremely interesting area of research in black hole physics because these geometries provide a new way of looking at the relationship of black holes geometries with the "Hidden Conformal Symmetry". The asymptotic conicality of these geometries is also a feature that has not been studied much in the past and may provide some interesting insights into asymptotic structure studies of black hole physics that are quite different from the quite well studied asymptotically flat and asymptotically AdS cases. Furthermore since the wave equation separability property of these geometries allow us to obtain an analytical solution for the minimally coupled scalar field, these geometries provide a very unique opportunity to understand many features of black holes, qualitatively that were previous only studied numerically.

An interesting application of our work interesting application of our work would be to find the detailed microscopic interpretation using string theory of general non-extremal black holes. Using the duality interpretation of the Harrison and S,T transformations that we gave, both the four- and five-dimensional problems can be reduced to understanding the effect of the time like Melvin twists, followed by several T-dualities, on the D0-D4 system. Another interesting direction would be to understand the contributions of other spin fields to the logarithmic corrections of the entanglement entropy of the subtracted geometry black holes. In the last few years a lot of work has been done on calculating and understanding the logarithmic corrections to black hole entropy using the euclidean gravity (or heat ker- nel) method. Subtracted geometries can be a helpful example to perform such calculations to further improve our understanding of these logarithmic corrections. The vacuum polarization calculation for other higher spin fields is another such avenue where subtracted geometry has the potential to provide new insights.

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