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Essays on Monetary Theory and Policy

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Abstract
This dissertation consists of two essays concerning Monetary Theory and Policy. Essay 1, "Monetary Policy and Asset Prices", develops a dynamic model in which money and a financial asset compete as media of exchange. We show that money circulates if and only if real assets are scarce, in the sense that their supply is not sufficient to satisfy the demand for liquidity. Our model also generates a connection between asset prices and monetary policy. When money grows at a higher rate, inflation is higher and the return on money decreases. In equilibrium, no arbitrage amounts to equating the real return of both objects. Thus, the price of the asset increases in order to lower its real return. Essay 2, "Monetary Policy and Interest Rates with Collateralized Borrowing", proposes a model of money that analyzes collateral and its interaction with monetary policy and interest rates. The value of the asset that serves as collateral, which is specific to each agent, plays a crucial role. This asset is different from fiat money and a standard financial asset. Therefore, its valuation is not standard. I prove existence and uniqueness of equilibrium in an environment where the Mundell-Tobin effect is not present. Heterogeneity of borrowers has a relevant effect on the impact of changes in the interest rates.

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ESSAYS ON MONETARY THEORY
AND POLICY

José Suárez-Lledó

A DISSERTATION
in
ECONOMICS

Presented to the Faculties of the University of Pennsylvania in Partial
Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2007

Supervisor of Dissertation
Graduate Group Chairperson
To my family and Beatriz
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ABSTRACT

ESSAYS ON MONETARY THEORY

AND POLICY

José Suárez-Lledó

Randall Wright

This dissertation consists of two essays concerning Monetary Theory and Policy. Essay 1, “Monetary Policy and Asset Prices”, develops a dynamic model in which money and a financial asset compete as media of exchange. We show that money circulates if and only if real assets are scarce, in the sense that their supply is not sufficient to satisfy the demand for liquidity. Our model also generates a connection between asset prices and monetary policy. When money grows at a higher rate, inflation is higher and the return on money decreases. In equilibrium, no arbitrage amounts to equating the real return of both objects. Thus, the price of the asset increases in order to lower its real return. Essay 2, “Monetary Policy and Interest Rates with Collateralized Borrowing”, proposes a model of money that analyzes collateral and its interaction with monetary policy and interest rates. The value of the asset that serves as collateral, which is specific to each agent, plays a crucial role. This asset is different from fiat money and a standard financial asset. Therefore, its valuation is not standard. I prove existence and uniqueness of equilibrium in an environment where the Mundell-Tobin effect is not present. Heterogeneity of borrowers has a relevant effect on the impact of changes in the interest rates.
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Chapter 1

Introduction

This dissertation consists of two essays on monetary theory and policy: "Monetary Policy and Asset Prices" (Chapter 2), and "Monetary Policy and Interest Rates with Collateralized Borrowing" (Chapter 3).

Essay 1 develops a dynamic framework that generates a link between inflation (generated by expansionary monetary policy) and asset prices. We study the properties of a monetary model in which a real asset is valued for its rate of return and for its liquidity. We show that money is essential if and only if real assets are scarce, in the precise sense that their supply is not sufficient to satisfy the demand for liquidity. Our model generates a clear connection between asset prices and monetary policy. When money grows at a higher rate, inflation is higher and the return on money decreases. In equilibrium, no arbitrage amounts to equating the real return of both objects. Therefore, the price of the asset increases in order to lower its real return. This negative relationship between inflation and asset returns is in the spirit of research in finance initiated in the early 80's.

Essay 2 presents a model of money that introduces collateral and analyzes its interaction with monetary policy and interest rates. Borrowing capacity, and ultimately consumption, is linked to the value of the asset that serves as collateral, which is specific to each agent. This asset is different from fiat money and a standard financial asset and...
is still valued for its liquidity as a collateralizable asset in contingencies. Thus, the valuation of the asset will not follow conventional considerations. The Friedman rule is still optimal. Nominal interest rate targeting captures one of the driving forces in the model: the impact of borrowing constraints on real loan demand. Numerical simulations confirm both that the Mundell-Tobin effect is not present in this model, and that changes in the interest rates have a smaller impact in homogeneous economies populated by constrained agents than in economies with both types.
Chapter 2

Monetary Policy and Asset Prices

2.1 Introduction

Before describing the contents of the present chapter, it must be emphasized that most of the findings of this essay can also be found in a working paper by Geromichalos, Licari, and Suárez-Lledó (see (16)).

Monetary policy controls the money supply, which determines the rate of inflation, and hence the rate of return on (or the cost of holding) currency.\(^1\) The purpose of this paper is to study the effect of monetary policy on the price of and rate of return on other assets. We use a model in the tradition of modern monetary theory, extended to include real assets in fixed supply just like the standard “trees” in Lucas (30). However, these assets are not only stores of value in our model. They also compete with currency as a medium of exchange. We show that money is essential (i.e. monetary equilibria Pareto dominate non-monetary equilibria) if and only if real assets are scarce, in the precise sense that their supply is not sufficient to satisfy the demand for liquidity. In this case, real assets and money are concurrently used as means of payment, and an increase in

\(^1\) In simple models, in steady state the growth rate of the money supply pins down the inflation rate, and through the arbitrage condition known as the Fisher equation, this pins down the nominal interest rate; it does not matter if policy controls money, inflation or the interest rate, since any one determines the other two.
inflation causes agents to want to move out of cash and into other assets. In equilibrium, this increases the price of these assets and lowers their rates of return.

Hence, the model predicts clearly that inflation reduces the return on other assets, which is something that has been discussed extensively in the finance literature for some time. Examples of papers that report this negative relationship are Fama (14), Geske and Roll (17), and Marshall (32). Geske and Roll, for example, characterize the connection between asset returns and inflation as a puzzling empirical phenomenon that does not necessarily ascribe causality one way or the other. An early attempt to explain this finding in a general equilibrium framework was made by Danthine and Donaldson (13).

In this paper money is assumed to yield direct utility. As opposed to this reduced-form monetary model we choose a setting in which the frictions that make money essential are explicitly described. Building on Lagos and Wright (28) (henceforth LW), we provide a model based on micro foundations within which the effects of monetary policy on asset prices can be analyzed.

Several models based on LW have been created to study different questions related to finance and asset pricing. For example, Lagos and Rocheteau (26) allow capital to be traded in a decentralized market and focus on the issue of over-investment. Lagos (25) introduces stocks and bonds into the model and examines issues like the equity premium and the risk-free rate puzzles. Like these papers, we allow alternative assets to compete as media of exchange, but our focus is on the competition between fiat money and real assets, and the implications of monetary policy on the price of and the rate of return on these assets. Assets are valued for what they yield, which includes direct rate of returns, as is standard in finance, and potentially some liquidity services, as is standard in monetary theory. This essay provides a tractable model where money and other assets

\( ^2\)Applications of earlier monetary theory to finance include Bansal and Coleman (2) and Kiyotaki and Moore (21). In the former, the authors examine the joint distribution of asset returns, velocity, inflation, and money growth. In the latter, the analysis focuses on the interaction between liquidity, asset prices, and aggregate economic activity.
coexist, and where monetary policy affects the equilibrium prices and rates of return on these assets in a straightforward and empirically relevant way.

The rest of the chapter is structured as follows. In Section 2.2 we describe the baseline model. In Section 2.3 we consider an economy without money and study the equilibrium properties of a model where the financial asset serves as the only medium of exchange. Section 2.4 introduces money and allows us to study the link between monetary policy and asset prices. Section 2.5 concludes.

2.2 The Model

Time is discrete and each period consists of two subperiods. During the first subperiod trade occurs in a decentralized market, whereas in the second subperiod economic activity takes place in a centralized (Walrasian) market. There exists a $[0, 1]$ continuum of agents who live forever and discount future at rate $\beta \in (0, 1)$. Agents consume and supply labor in both subperiods. Preferences are given by $U(x, h, X, H)$ where $x$ and $h$ ($X$ and $H$) are consumption and labor in the decentralized (centralized) market, respectively. Following LW, we use the specific functional form

\[ U(x, h, X, H) = u(x) - c(h) + U(X) - AH. \]

We assume that $u$ and $U$ are twice continuously differentiable with $u(0) = 0$, $u' > 0$, $u'(0) = \infty$, $U' > 0$, $u'' < 0$, and $U'' \leq 0$. To simplify the presentation we assume $c(h) = h$, but this is not important for our substantive results. Let $q^*$ denote the efficient quantity, i.e. $u'(q^*) = 1$, and suppose there exists $X^* \in (0, \infty)$ such that $U'(X^*) = 1$ with $U(X^*) > X^*$.

During the first subperiod, economic activity is similar to the standard search model. Agents interact in a decentralized market and anonymous bilateral trade takes place.
The probability of meeting someone is $\alpha$. The first subperiod good, $x$, comes in many specialized varieties. Agents can transform labor into a special good on a one-to-one basis, and they do not consume the variety that they produce. When two agents, say $i$ and $j$, meet in the decentralized market, there are three possibilities. The probability that $i$ consumes what $j$ produces but not vice versa (single coincidence meeting) is $\sigma \in (0, \frac{1}{2})$. By symmetry, the probability that $j$ consumes what $i$ produces but not vice versa is also $\sigma$. Finally, the probability that neither of the agents likes what the other agent produces is $1 - 2\sigma$.\footnote{For simplicity we assume that the probability that both agents like each other’s good (double coincidence meeting) is zero.} In a single coincidence meeting, if $i$ consumes what $j$ produces, we will call $i$ the buyer and $j$ the seller.

During the second subperiod agents trade in a centralized market. With centralized trade it is irrelevant whether the second-subperiod good comes in one or many varieties. We assume that at the second subperiod all agents consume and produce the same general good. Agents in the centralized market can transform one unit of labor into $w$ units of the general good, where $w$ is a technological constant. We normalize that constant to $w = 1$ without loss of generality. It is also assumed that both the general and the special goods are non-storable.

There is another object, called (fiat) money, that is perfectly divisible and can be stored at any quantity $m \geq 0$. We let $\phi$ denote its value in the centralized market. The key new feature of our model is the introduction of a real asset. One can think of this asset as a Lucas tree. We assume that this tree lives forever, and that agents can buy its shares in the centralized market at price $\psi$. Let $T > 0$ denote the total stock of the real asset. We assume that $T$ is exogenously given and constant. As opposed to fiat money, the asset yields a real return in terms of the general consumption good. We let the gross return (dividend) of the asset be denoted by $R > 0$. An agent that owns shares of the tree can consume the fruit (which here is just the general good) and sell the shares in the
Walrasian market. She can also carry shares of the tree into the decentralized market in order to trade. Hence, in this economy money and the asset can potentially compete as a media of exchange.

2.3 Equilibrium in a Model without Money

We first consider an economy without money. Analyzing this simpler version of the model, provides intuition for the next section, where money and the asset can co-exist as media of exchange. We begin with the description of the value functions, treating the prices and the distribution of asset holdings as given. These objects will be endogenously determined in equilibrium. Agents are allowed to keep any positive quantity of assets at home before they visit the decentralized market.\(^4\) The state variables for a representative agent are the units of the asset that she brings into the decentralized market, \(b\), and the units of the asset that she keeps at home, \(a\). To reduce notation we define \(z \equiv (a, b)\).

First, consider the value function in the centralized market for an agent with portfolio \(z\). We normalize the price of the general good to 1. The Bellman’s equation is given by

\[
W(z) = \max_{x, H, z_{+1}} \{U(X) - H + \beta V(z_{+1})\}
\]

s.t. \(X + \psi(b_{+1} + a_{+1}) = H + (R + \psi)(b + a)\),

where \(b_{+1}, a_{+1}\) are the units of the asset carried into the next period’s decentralized market and left at home, respectively.\(^5\) It can be easily verified that, at the optimum,

\(^4\) As opposed to money, the asset here serves as a store of value. Moreover, the terms of trade for an agent may depend on the amount of the asset that she carries into the decentralized market. Therefore, it may be prudent to keep some assets at home. As we shall see in what follows, this is precisely what happens in equilibrium, whenever the supply of assets is sufficiently large to satisfy the demand for liquidity in the economy.

\(^5\) We impose non-negativity on all the control variables except \(H\), which is allowed to be negative. Once equilibrium has been found, one can introduce conditions that guarantee \(H \geq 0\) (see LW).

7
\( X = X^* \). Using this fact and replacing \( H \) from the budget constraint into \( W \) yields

\[
W(z) = \max_{z,1} \{U(X^*) - X^* - \psi(b_{z,1} + a_{z,1}) + (R + \psi)(b + a) + \beta V(z_{z,1})\}. \tag{2.1}
\]

This expression implies the following results. First, the choice of \( z_{z,1} \) does not depend on \( z \). In other words, there are no wealth effects. This result relies on the quasi-linearity of \( U \). Second, \( W \) is linear,

\[
W(z) = \Omega + (R + \psi)(b + a).
\]

We now turn to the terms of trade in the decentralized market. Consider a single-coincidence meeting between a buyer and a seller with state variables \( z \) and \( \tilde{z} \), respectively. We use the generalized Nash solution, where the bargaining power of the buyer is denoted by \( \theta \in (0,1] \). Let \( q \) represent the quantity of the special good and \( d_b \) the units of the asset that change hands during trade. The bargaining problem is

\[
\max_{q,d_b} \left[ u(q) + W(a, b - d_b) - W(z) \right]^{\theta} \left[ -q + W(\tilde{a}, \tilde{b} + d_b) - W(\tilde{z}) \right]^{1-\theta}
\]

s.t. \( d_b \leq b \). Using the linearity property of \( W \), the problem simplifies to

\[
\max_{q,d_b} \left[ u(q) - (R + \psi)d_b \right]^{\theta} \left[ -q + (R + \psi)d_b \right]^{1-\theta}
\]

s.t. \( d_b \leq b \). Clearly, the linearity of \( W \) implies that the solution to the bargaining problem does not depend on the variables \( a, \tilde{a}, \tilde{b} \). It depends on \( b \) only if the constraint binds. The solution to this problem is a variation of the standard bargaining result obtained in this type of models.
Lemma 2.3.1. The bargaining solution is the following:

If \( b < b^* \), then
\[
\begin{cases}
q(b) = q(b), \\
d_b(b) = b.
\end{cases}
\]

If \( b \geq b^* \), then
\[
\begin{cases}
q(b) = q^*, \\
d_b(b) = b^*.
\end{cases}
\] (2.2)

where \( b^* \) solves \( (R + \psi) b = z(q^*) + (1 - \theta)u(q^*) \), \( q(b) \) solves \( (R + \psi) b = z(q) \), and \( z(q) \) is defined by

\[
z(q) \equiv \frac{\theta u'(q)q + (1 - \theta)u(q)}{\theta u'(q) + (1 - \theta)}.
\] (2.3)

Proof. It is straightforward to verify that the suggested solution satisfies the necessary first-order conditions, which are also sufficient in this problem.

Lemma 2.3.2. For all \( b < b^* \), we have \( \dot{q}(b) > 0 \) and \( \dot{q}(b) < q^* \).

Proof. See Appendix I.

Next, consider the value function of an agent in the decentralized market. Let \( F(b) \) be the distribution of asset holdings in this market. We set \( \xi = \alpha \sigma \). The Bellman's equation is\(^6\)

\[
V(z) = \xi \{ u[q(b)] + W[a, b - d_b(b)] \} + \xi \int \left\{ -q(\tilde{b}) + W[a, b + d_b(\tilde{b})] \right\} dF(\tilde{b}) + (1 - 2\xi) W(z).
\]

\(^6\) The first term is the payoff from buying \( q(b) \) and going to the centralized market with state variables \( (a, b - d_b(b)) \). The second term is the expected payoff from selling \( q(\tilde{b}) \) and going to the centralized market with state variables \( (a, b + d_b(\tilde{b})) \). Both expressions reflect the fact that the only relevant variable for the determination of the terms of trade is the amount of assets that the buyer brings into the decentralized market. The last term is the payoff from going to the centralized market without trading in the decentralized market.
Using (2.1) and the linearity property of \( W \), we can re-write the decentralized market value function as

\[
V(z) = \kappa + v(z) + \max_{z_{t+1}} \left\{ \left[ -\psi + \beta(R + \psi_{t+1}) \right](b_{t+1} + a_{t+1}) + \beta \xi \left\{ u[q_{t+1}(b_{t+1})] - (R + \psi_{t+1})d_{b_{t+1}}(b_{t+1}) \right\} \right\},
\]

where \( \kappa = U(X^*) - X^* + \xi \int \left\{ -q(b) + (R + \psi)d_{b}(b) \right\} dF(b) \) is a constant and

\[
v(z) = \xi \left\{ u[q(b)] - (R + \psi)d_{b}(b) \right\} + (R + \psi)(b + a).
\]

In order to study the optimal behavior of the agent we focus on the term inside the maximum operator in (2.4). We define this term as \( J(z_{t+1}) \) and refer to it as the objective function. This function consists of two terms. The term \(-\psi + \beta(R + \psi_{t+1})\) is the net gain of carrying an additional unit of the asset from the centralized market of the current period into the centralized market of the next period. Sometimes we refer to the negative of this term as the cost of carrying the asset across periods. The expression \( u[q_{t+1}(b_{t+1})] - (R + \psi_{t+1})d_{b_{t+1}}(b_{t+1}) \) is the gain from trade, of carrying an additional unit of the asset into the next period's decentralized market. We refer to this term as the surplus of the buyer.

Next, we show that in any equilibrium the net gain of carrying the asset across periods is non-positive.

**Lemma 2.3.3.** In any equilibrium, \( \psi \geq \beta(R + \psi_{t+1}) \).

**Proof.** See Appendix I.

The following lemma establishes some important properties of the optimal choice of the agent. To ease the presentation, we assume that \( e(q) \equiv \xi u'(q)/z'(q) + 1 - \xi \) is a strictly decreasing function of \( q \) in the range \((0, q^*)\). This assumption insures that there exists a
unique choice of \( b_{+1} \) that maximizes the objective function.\(^7\) As in LW, conditions that guarantee the monotonicity of \( e(q) \) are \( \theta \approx 1 \) or \( u' \) is log-concave.

**Lemma 2.3.4.** Assume that \( e'(q) < 0 \) for all \( q \in (0,q^*) \). In every period, the optimal choice of \( b_{+1} \) is unique and satisfies \( b_{+1} \in (0,\bar{b}] \), where \( \bar{b} \equiv \{ b : (R + \psi)b = z(\bar{q}) \} \), and \( \bar{q} \) solves \( u'(\bar{q}) = z'(\bar{q}) \). The optimal choice of \( a_{+1} \) satisfies

\[
    a_{+1} = \begin{cases} 
        0, & \text{if } \psi > \beta(R + \psi_{+1}), \\
        \in \mathbb{R}_+, & \text{if } \psi = \beta(R + \psi_{+1}).
    \end{cases} \tag{2.5}
\]

**Proof.** See Appendix I.  \( \square \)

According to Lemma 2.3.4, the agent’s optimal choice of \( a_{+1} \) depends only on whether the cost of transferring assets across time is zero or positive. The optimal choice of \( b_{+1} \) never exceeds \( \bar{b} \), and it is equal to \( \bar{b} \) only if \( \psi - \beta(R + \psi_{+1}) = 0 \).\(^8\) In this case, the quantity of the special good purchased in the decentralized market is \( \bar{q} \). Sometimes, we refer to this value of \( q \) as the constrained efficient quantity. For every \( \theta < 1 \), we have \( \bar{q} < q^* \), and so (unconstrained) efficiency is never obtained in equilibrium.\(^9\) Another implication of Lemma 2.3.4 is that the first order condition with respect to \( b_{+1} \) is always satisfied with equality,

\[
    -\psi + \beta(R + \psi_{+1})e[q_{+1}(b_{+1})] = 0. \tag{2.6}
\]

---

\(^7\) If the set of maximizers of \( J(z_{+1}) \) with respect to \( b_{+1} \) is not a singleton, one can still draw conclusions about the model at the cost of a higher level of complexity. To avoid that, we assume that \( e'(q) < 0 \) for all \( q \in (0,q^*) \). Notice that this assumption is sufficient but not necessary for the existence of a unique optimal choice of \( b_{+1} \).

\(^8\) It is straightforward to verify that \( \bar{b} \) maximizes the surplus of the buyer.

\(^9\) This result is in the spirit of the Hosios’ condition for efficiency (see (18)). According to this, the bargaining solution is efficient if it splits the surplus so that each participant is compensated for her contribution to the match. In this model the whole surplus is due to the buyer. This implies that, if \( \theta < 1 \), the buyer does not get all the surplus from the match and consequently never carries the socially optimal quantity \( b^* \) into the decentralized market.
Finally, uniqueness of the optimal choice of $b_{t+1}$ implies that the distribution of asset holdings is degenerate (i.e. all the agents carry the same amount of the asset into the decentralized market).

**Definition 2.3.1.** An equilibrium for this economy is a value function $V(z)$ that satisfies Bellman’s equation, a solution to the bargaining problem given by $d_b(b) = b$ and $q(b) = \tilde{q}(b)$, and a bounded path of $\psi$ such that (2.5) and (2.6) hold at every date with $a + b = T$.

The key factor which determines whether $a_{t+1}$ is equal to or greater than zero in equilibrium is the stock of the asset. High $T$ leads to equilibria with $a_{t+1} > 0$, while low $T$ induces agents to carry their whole amount of assets into the decentralized market.

Consider first equilibria with $a_{t+1} = 0$. In this case, we can replace $b_{t+1}$ with $T$ in both (2.6) and the solution to the bargaining problem. The latter can be re-written as

$$ (R + \psi)T = z(q). \tag{2.7} $$

Since (2.7) holds at every date, we get

$$ z(q) = \beta z(q_{t+1})e(q_{t+1}) + RT, \tag{2.8} $$

where it is understood that $q = q(T)$ and $q_{t+1} = q_{t+1}(T)$. Equation (2.8) is a first order difference equation in $q$. Equilibrium can now be re-defined as a path of $q$ that stays in $(0,\bar{q})$ and satisfies (2.8).

We focus on steady state equilibria, which are defined to be solutions to (2.8) with the additional requirement that $q_{t+1} = q$.\(^{10}\) A steady state equilibrium satisfies

$$ G(q^S) = 0, \tag{2.9} $$

where $G(q) \equiv -z(q) + \beta z(q)e(q) + RT$. Since $G(0) > 0$, a sufficient condition for the

---

\(^{10}\) A detailed discussion on dynamics in this class of models is presented in Lagos and Wright (27).
existence of a steady state \( q^S \in (0, \bar{q}) \) is \( G(\bar{q}) \leq 0 \). This condition reduces to

\[
T \leq \frac{(1 - \beta)z(\bar{q})}{R} = \bar{T}.
\]

(2.10)

We now show that (2.10) is also necessary for the existence of a steady state equilibrium with \( q^S \leq \bar{q} \) and \( a_{+1} = 0 \). To see why this is true, suppose that an equilibrium of the class described above exists. Equation (2.7) implies that \((R + \psi)T = z(q^S)\) holds at every date. Leading this equation by one period, multiplying it by \( \beta \), and subtracting it from the original equation yields

\[
T [\psi - \beta(R + \psi_{+1})] = (1 - \beta)z(q^S) - RT.
\]

Lemma 2.3.3 implies that the left-hand side of the expression above is non-negative. Therefore, a necessary condition for the existence of a steady state equilibrium with \( q^S \leq \bar{q} \) and \( a_{+1} = 0 \) is \( T \leq \bar{T} \).

Next, we establish uniqueness of equilibrium under the additional assumption that \( e'(q) < 0 \) for all \( q \in (0, q^*) \). Equation (2.7) implies that the steady state equilibrium prices satisfy \( \psi = \psi_{+1} \) at every date. Using this result, the steady state quantity and the price of the asset can be determined by solving the following system of equations

\[
\psi = -R + \frac{z(q^S)}{T},
\]

(2.11)

\[
\psi = \frac{\beta}{1 - \beta e(q^S)} \beta e(q^S).
\]

(2.12)

Equation (2.11) defines \( \psi \) as a strictly increasing function of \( q^S \). In addition, (2.12) implies that \( \frac{d\psi}{dq^S} = \frac{\beta e^2(q^S)}{(1 - \beta e(q^S))^2} \), which is negative if \( e'(q) < 0 \). We conclude that, if \( T \leq \bar{T} \), there exists a unique steady state equilibrium with \( a_{+1} = 0 \) and \( q^S \leq \bar{q} \) (and \( q^S = \bar{q} \) only if \( T = \bar{T} \)). For \( T < \bar{T} \), it can be easily verified that \( q^S \) is strictly increasing in \( \beta, \xi \) and \( \theta \). The second-order condition of the maximization problem states that, at the optimum,
By Lemma 2.3.2, the signs of the terms $J_{bb}(z_{11}) < 0$ and $e'(q)$ coincide. Hence, we conclude that in any equilibrium $e'(q) < 0$. Finally, notice that $T$ appears only in (2.11). Therefore, a shift in $T$ results in a higher $q^S$ and a lower $\psi$.

Next consider $T > \tilde{T}$. We show that for every $T$ in this range there exists a unique equilibrium with $a_{+1} > 0$. Lemma 2.3.4 suggests that the agent chooses $a_{+1} > 0$ only if prices satisfy $\psi = \beta(R + \psi)$. This implies that the price of the asset in such an equilibrium evolves according to the simple difference equation $\psi_{+1} = -R + (1/\beta)\psi$. It can be easily verified that the only bounded solution to this equation satisfies $\psi = \bar{\psi} \equiv [\beta/(1 - \beta)] R$ at every date. For $T$ in the range under consideration we have $q^S = \bar{q}$, and so $e(q^S) = 1$. Hence, the price of the asset implied by (2.12) for $q^S = \bar{q}$ coincides with the formula given above. To conclude the argument, it suffices to show that, for $\psi = \bar{\psi}$, the real balances exceed $z(q)$. This condition is needed to justify the claim that in this equilibrium $a_{+1} > 0$. In fact, we have

$$(R + \bar{\psi})T > \left( R + \frac{\beta}{1 - \beta} R \right) \frac{(1 - \beta)z(\bar{q})}{R} = z(\bar{q}).$$

**Proposition 2.3.1.** Consider the model without money and assume that $e'(q) < 0$ for all $q \in (0, q^*)$. For every $T > 0$ there exists a unique steady state equilibrium where the real asset serves as a medium of exchange. If $T \geq \bar{T}$, constrained efficiency is achieved, i.e. $q^S = \bar{q}$. For this range of $T$ we have $\psi = \bar{\psi}$ and $a_{+1} = T - \bar{T}$. If $T < \bar{T}$, then $a_{+1} = 0$ and $q^S \in (0, \bar{q})$. In this case $q^S$ is strictly increasing in $T$ and $\psi$ is strictly decreasing.

### 2.4 Money and Real Assets

In this section we introduce a new object, called (fiat) money, which is perfectly divisible and can be stored at any positive quantity $m \geq 0$. Agents can bring any amount of this object into the decentralized market in order to trade. We assume that the supply of
money evolves according to $M_{t+1} = (1 + \mu)M_t$, where $\mu$ is a constant. The new money is injected into the economy as a lump-sum transfer after the agents leave the centralized market. The state variables for the agent are $m, a$ and $b$. To reduce notation we redefine $z \equiv (m, a, b)$. First, consider the value function in the centralized market. The Bellman's equation is

$$W(z) = \max_{X, H, z_{t+1}} \left\{ U(X) - H + \beta V(z_{t+1}) \right\}$$

s.t. $X + \phi m_{t+1} + \psi (b_{t+1} + a_{t+1}) = H + \phi m + (R + \psi)(b + a)$,

where, except from the new variables $m, m_{t+1}$ and the value of money $\phi$, everything is the same as in the previous section. Replacing $H$ from the budget constraint and using the fact that at the optimum $X = X^*$, we have

$$W(z) = \max_{z_{t+1}} \left\{ U(X^*) - X^* - \phi m_{t+1} - \psi (b_{t+1} + a_{t+1}) \right\} + \phi m + (R + \psi)(b + a) + \beta V(z_{t+1}) \right\}.$$  \hspace{1cm} (2.13)

As in Section 4.3, the quasi-linearity of $U$ rules out wealth effects, and $W$ is linear,

$$W(z) = \Omega + \phi m + (R + \psi)(b + a).$$

We now turn to the terms of trade in the decentralized market. Consider a single-coincidence meeting between a buyer and a seller with state variables $z$ and $\tilde{z}$, respectively. Let $d_m$ represent the amount of money spent in order to buy the special good.

\[\textit{Notice that while } a \text{ stands for the amount of the real asset that the agent keeps at home, there is no analogous variable for money. Unlike the asset, which serves as a store of value, fiat money is only valued for its liquidity services. Therefore, the agent never has an incentive to leave a positive amount of money at home before visiting the decentralized market.}\]
The generalized Nash bargaining problem is given by

\[
\max_{q,d_m,d_b} \left[ u(q) + W(m - d_m, a, b - d_b) - W(z) \right]^{\theta} \\
\times \left[ -q + W(\bar{m} + d_m, \bar{a}, \bar{b} + d_b) - W(\bar{z}) \right]^{1-\theta}
\]

s.t. \( d_m \leq m \) and \( d_b \leq b \). Using the linearity property of \( W \), this problem simplifies to

\[
\max_{q,d_m,d_b} \left[ u(q) - \phi d_m - (R + \psi)d_b \right]^{\theta} \left[ -q + \phi d_m + (R + \psi)d_b \right]^{1-\theta}
\]

s.t. \( d_m \leq m \) and \( d_b \leq b \). Clearly, the solution to the bargaining problem can only depend on the money and asset holdings of the buyer.

**Lemma 2.4.1.** Define total real balances as \( \pi \equiv \phi m + (R + \psi)b \). The bargaining solution is the following:

If \( \pi < \pi^* \), then

\[
q(m,b) = \hat{q}(\pi), \\
d_m(m,b) = m, \quad d_b(m,b) = b.
\]

If \( \pi \geq \pi^* \), then

\[
q(m,b) = q^*, \\
\phi d_m(m,b) + (R + \psi)d_b(m,b) = z(q^*),
\]

where \( z(q) \) is defined by (2.3), \( \pi^* \) is the set of pairs \((m,b)\) such that \( \phi m + (R + \psi)b = z(q^*) \), and \( \hat{q}(\pi) \) solves \( \pi = z(q) \).

**Proof.** It can be easily verified that the suggested solution satisfies the first-order conditions, which are also sufficient in this problem. \( \square \)

Lemma 2.4.1 reveals that all pairs \((m,b)\) that yield the same \( \pi \) buy equal quantity in the match. In other words, the seller is only interested in the term \( \phi m + (R + \psi)b \) that
the buyer carries and not in the composition of the portfolio.

Lemma 2.4.2. For all \( \pi < \pi^* \), we have \( q'(\pi) > 0 \) and \( q(\pi) < q^* \).

Proof. The proof is identical to the one of Lemma 2.3.2. \( \square \)

Next, consider the value function in the decentralized market. We redefine \( F(m, b) \) to be the joint distribution of money and asset holdings in this market. The Bellman's equation is\(^{12}\)

\[
V(z) = \xi \left\{ u[q(m, b)] + W[m - d_m(m, b), a, b - d_b(m, b)] \right\} \\
+ \xi \int \left\{ -q(\tilde{m}, \tilde{b}) + W[m + d_m(\tilde{m}, \tilde{b}), a, b + d_b(\tilde{m}, \tilde{b})] \right\} dF(\tilde{m}, \tilde{b}) \\
+ (1 - 2\xi) W(z).
\]

Using (2.13) and the linearity property of \( W \), we can re-write the value function in the decentralized market as

\[
V(z) = \lambda + \nu(z) + \max_{z_+} \left\{ \left( \frac{-\phi + \beta \phi_{+1}}{m_{+1}} + \left[ -\psi + \beta (R + \psi_{+1}) \right] \right) \left( b_{+1} + a_{+1} \right) \\
+ \beta \xi \left\{ \left( q_{+1} \right) - \phi_{+1} d_m - (R + \psi_{+1}) d_b \right\} \right\},
\]

where

\[
\lambda = U(X^*) - X^* + \beta \phi_{+1} \mu M \\
+ \xi \int \left\{ -q(\tilde{m}, \tilde{b}) + \phi d_m(\tilde{m}, \tilde{b}) + (R + \psi) d_b(\tilde{m}, \tilde{b}) \right\} dF(\tilde{m}, \tilde{b})
\]

\(^{12}\)The first term is the payoff from buying \( q(m, b) \) and going to the centralized market with state variables \( (m - d_m(m, b), a, b - d_b(m, b)) \). The second term is the expected payoff from selling \( q(\tilde{m}, \tilde{b}) \) and going to the centralized market with state variables \( (m + d_m(\tilde{m}, \tilde{b}), a, b + d_b(\tilde{m}, \tilde{b})) \). Both expressions reflect the fact that the only relevant variables for the determination of the terms of trade are the money and asset holdings of the buyer in the decentralized market. The last term is the payoff from going to the centralized market without trading in the decentralized market.
is a constant,

\[ v(z) = \xi \left\{ u[q(m, b)] - \phi d_m(m, b) - (R + \psi) d_b(m, b) \right\} + \phi m + (R + \psi)(b + a), \]

and it is understood that \( q_{+1}, d_{b+1}, \) and \( d_{m,+1} \) are all functions of \( m_{+1} + \mu M \) and \( b \).

As in the previous section, we define the term inside the maximum operator in (2.15) as \( J(z_{+1}) \), and refer to it as the objective function. This function consists of three terms. The first term is the cost of carrying \( m_{+1} \) units of money from period to period, and the second term is the cost of carrying the real asset across periods. The last term is the discounted, expected surplus of the buyer.

**Lemma 2.4.3.** In any equilibrium \( \phi \geq \beta \phi_{+1}, \) and \( \psi \geq \beta (R + \psi_{+1}). \)

**Proof.** The proof is identical to the one of Lemma 2.3.3. \( \square \)

We now study the optimal behavior of the agent. The choice variable \( a_{+1} \) enters the objective function linearly, and it is multiplied by the cost of holding the asset. Therefore, the optimal choice of \( a_{+1} \) is still given by (2.5). The next lemma characterizes the optimal choice of \( m_{+1} \) and \( b_{+1} \). To keep the analysis simple, we assume that the function \( e(q) = \xi u'(q)/z'(q) + 1 - \xi \) is strictly decreasing for all \( q \in (0, q^*) \).

**Lemma 2.4.4.** Assume that \( e'(q) < 0 \) for all \( q \in (0, q^*) \). The optimal choices of \( m_{+1} \) and \( b_{+1} \) satisfy \( \phi(m_{+1} + \mu M) + (R + \psi)b_{+1} = \pi_{+1} \in (0, \bar{\pi}] \). The optimal choice is unique in terms of \( \pi_{+1} \).

**Proof.** See Appendix I. \( \square \)

Lemma 2.4.4 indicates that all agents enter the decentralized market with the same total real balances \( \pi_{+1} \). In general, one cannot determine the choices of \( m_{+1} \) and \( b_{+1} \).
separately. However, the bargaining solution implies that \( z(\hat{q}) = \pi \). Therefore, the fact that \( m_{t+1} \) and \( b_{t+1} \) are not uniquely determined is not a problem, since the real balances are sufficient to determine \( q \) in equilibrium. Similarly, the joint distribution function \( F(m, b) \) is not degenerate in terms of \((m, b)\) but is degenerate if we express it in terms of real balances, i.e. \( \bar{F}(\pi) \). Finally, the first-order conditions with respect to \( m_{t+1} \) and \( b_{t+1} \) are given by

\[
-\phi_+ \geq \beta \phi_{+,1} e(\hat{q}_{+,1}) \leq 0, \quad = 0 \text{ if } m_{t+1} > 0, \quad (2.16)
\]
\[
-\psi + \beta (R + \psi_{+,1}) e(\hat{q}_{+,1}) = 0. \quad (2.17)
\]

We now provide a definition of equilibrium for the economy described in this section.\(^{15}\)

**Definition 2.4.1.** An equilibrium for the economy with money and the real asset is a value function \( V(z) \) that satisfies Bellman’s equation, a solution to the bargaining problem given by \( d_m(m, b) = m, \ d_b(m, b) = b \) and \( q(m, b) = \hat{q}(\pi) \), a path of \( \phi \), and a bounded path of \( \psi \) such that (2.5), (2.16) and (2.17) hold at every date with \( a + b = T \) and \( m = M \), and the sequence \( \{(\phi M)_t\}_{t=0}^{\infty} \) is bounded.

We now turn to the characterization of equilibria. The following lemma establishes a useful result that holds for equilibria with \( a_{t+1} = 0 \).

**Lemma 2.4.5.** In any steady state equilibrium with \( a_{t+1} = 0 \), \( \psi = \psi_{+,1} \) and

\[
\frac{\phi}{\phi_{+,1}} = 1 + \mu. \quad (2.18)
\]

**Proof.** See Appendix I. \( \square \)

\(^{15}\) As opposed to the supply of the real asset, the supply of money can grow over time, and so a bounded path of \( \phi \) is not enough to guarantee that the real money balances, \( \phi M \), are also bounded. Instead of that, we require a path of \( \phi \) such that the sequence \( \{(\phi M)_t\}_{t=0}^{\infty} \) is bounded.
Recall that in any steady state equilibrium with \( a_{t+1} = 0 \), \( \phi M + (R + \psi)T = z(q^S) \) holds at every date. Leading this equation by one period, multiplying it by \( \beta \), and subtracting it from the original equation implies

\[
M \left[ \phi - \beta \phi_{t+1}(1 + \mu) \right] + T \left[ \psi - \beta(R + \psi_{t+1}) \right] = (1 - \beta)z(q^S) - RT,
\]

which, utilizing (2.18), can be re-written as

\[
M\phi(1 - \beta) + T \left[ \psi - \beta(R + \psi_{t+1}) \right] = (1 - \beta)z(q^S) - RT.
\]

Both terms on the left-hand side of this expression are non-negative. Hence, a necessary condition for the existence of a steady state equilibrium with \( a_{t+1} = 0 \) is

\[
T \leq \left[ (1 - \beta)z(q^S) \right] / R \leq \bar{T},
\]

where \( \bar{T} \) is defined in (2.10) and the last inequality follows from the fact that \( z \) is strictly increasing in \( q \).

If \( T > \bar{T} \), the only possible equilibrium has \( a_{t+1} > 0 \). As in the previous section, we guess and verify that such an equilibrium exists for every \( T \) in the relevant region. Since the candidate equilibrium has \( a_{t+1} > 0 \), optimality requires \( \psi = \beta(R + \psi_{t+1}) \), and the sequence \( \{\psi_t\}_{t=0}^\infty \) is bounded only if \( \psi = \bar{\psi} \) in every period, where \( \bar{\psi} = [\beta/(1 - \beta)] R \) as in Section 3. It remains to show that, for \( \psi = \bar{\psi} \), the total real balances exceed \( z(\bar{q}) \). We have

\[
\phi M + (R + \bar{\psi})T \geq \left( R + \frac{\beta}{1 - \beta} R \right) T > \left( R + \frac{\beta}{1 - \beta} R \right) \frac{(1 - \beta)z(\bar{q})}{R} = z(\bar{q}),
\]

which concludes the argument.

According to the analysis above, if \( T \geq \bar{T} \), the agent carries just enough units of the
asset to purchase $\bar{q}$ and keeps the rest at home. One might suggest that the agent can keep some more assets at home (this is still optimal since $\psi = \beta(R + \psi_{+1})$) and carry some money into the decentralized market instead. This should be equivalent to bringing no money and $b_{+1} = \bar{T}$, as long as the new composition of the agent's portfolio, say $(\bar{\mu}, \bar{b})$, satisfies $\pi = \rho \bar{\mu} + (R + \psi)\bar{b} = z(\bar{q})$. The definition of the objective function reveals that this argument is not accurate. The pairs $(0, \bar{T})$ and $(\bar{\mu}, \bar{b})$ yield the same surplus in the match, but the cost terms associated with each pair might differ. In particular, since here $\psi = \beta(R + \psi_{+1})$, the cost of carrying the asset across periods is zero. Consequently, the agent would choose $m_{+1} > 0$ only if the cost of holding money is also zero, i.e. $\phi = \beta \phi_{+1}$.

Using (2.18) this condition reduces to $\mu = \beta - 1$, i.e. the monetary authority should follow the Friedman Rule. We conclude that for economies with positive nominal interest rate and $T \geq \bar{T}$ there do not exist monetary equilibria.

If we allow for economies with zero nominal interest rate, then the Friedman Rule can be followed, and there exists a monetary equilibrium. However, in this case money does not add anything to welfare. Since money and the real asset are equally costly, agents substitute the asset with money in the decentralized market. Regardless of the composition of the agents' portfolio, the equilibrium quantity always satisfies $q^S = \bar{q}$. Therefore, if $T \geq \bar{T}$, we say that there is no essential role for money.

Next, consider $T < \bar{T}$. In this case the real asset is not enough to satisfy the demand for liquidity in the economy, and so money has an essential role. Focussing on steady state equilibria and using Lemma 2.4.5, (2.16) and (2.17) become

\[
e(q^S) = \frac{1 + \mu}{\beta},
\]

\[
\psi = \frac{\beta}{1 - \beta e(q^S)} Re(q^S).
\]

Equation (2.19) implies that the steady state quantity is negatively related to the growth
rate of money supply. Moreover, (2.19) and (2.20) together imply

\[ \psi = \psi(\mu) = \frac{(1 + \mu)R}{-\mu}. \]  

(2.21)

Equation (2.21) provides an expression for the price of the real asset as a function of the policy rule. Since \( \psi \) cannot be negative we impose the restriction \( \mu < 0 \). The reason for this result is that in this model the return of the real asset is always positive. Hence, money is valued in this economy only if it also has a positive return. This requires that the monetary authority deflates the economy. For every \( \mu \in (\beta - 1, 0) \), we have \( \psi'(\mu) > 0 \) and \( \psi''(\mu) > 0 \). As \( \mu \to 0 \) (constant money supply), the price of the real asset goes to infinity, and as \( \mu \to \beta - 1, \psi \to \bar{\psi} \), which is equal to the price of the real asset when \( T \geq \bar{T} \).

The previous asset pricing equation can be interpreted as follows: As \( \mu \) increases, inflation is higher and the rate of return on money, \( \phi_{S_1}/\phi - 1 \), decreases. In equilibrium, no arbitrage implies that the rate of return on both objects has to be the same. Therefore, \( \psi \) increases in order to lower the rate of return on the asset, \( (\psi + R)/\psi - 1 = R/\psi \equiv \rho_a \).

It turns out that not all \( \mu \in (\beta - 1, 0) \) are consistent with a monetary equilibrium. For any given \( T < \bar{T} \), define \( \psi^T \) and \( q^{S,T} \) as the (unique) steady state solution for the economy without money. As we have already seen, \( q^{S,T} < q \) and \( \psi^T > \bar{\psi} \). Next, define \( \mu^T \equiv \left\{ \mu : \frac{1+\mu}{}R = \psi^T \right\} \). It follows from (2.21) that this value of \( \mu \) is uniquely defined, and that for all \( T < \bar{T}, \mu^T < 0 \) and \( d\mu^T/dT < 0 \). Then, consider any \( \mu_1 \in (\mu^T, 0) \), and let \( \psi_1 \) and \( q_1^{S} \) be the equilibrium values determined through (2.21) and (2.19), respectively. Since \( \mu_1 > \mu^T \), we have \( \psi_1 > \psi^T \) and \( q_1^{S} < q^{S,T} \). In any steady state equilibrium \( \phi M + (R + \psi)T = z(q^S) \). This implies that following the policy rule \( \mu_1 \) would lead to \( \phi M < 0 \), which is not consistent with equilibrium. We conclude that the range of equilibrium.

\[ ^{16} \text{Clearly, } \mu^T \text{ is a neutral monetary policy, in the sense that it yields the same equilibrium quantity and asset price as in the economy without money.} \]
monetary policies for which a monetary equilibrium can be supported is \((\beta - 1, \mu^T]\).\(^{17}\)

For every \(\mu \in (\beta - 1, \mu^T]\), (2.19) implies that \(q^S \geq q^{S,T}\), and so the introduction of money in the economy improves welfare. If, on the other hand, \(\mu > \mu^T\), then in any monetary equilibrium the steady state quantity would be lower than the quantity associated with the non-monetary equilibrium. Agents realize that and choose not to carry any amount of money. We summarize these results in the following proposition.

**Proposition 2.4.1.** Consider the model with money and the real asset. Assume that \(e'(q) < 0\) for all \(q \in (0, q^*)\). If \(T \geq T\), then \(q^S = \bar{q}, \psi = \bar{\psi}, \text{ and } a_{t+1} = T - \bar{T}\). For this range of \(T\), the equilibrium is always non-monetary, assuming that \(\mu > \beta - 1\). If \(T < \bar{T}\), money has an essential role and the range of policies consistent with a monetary equilibrium is \((\beta - 1, \mu^T]\), where \(\mu^T\) is strictly decreasing in \(T\). In every monetary equilibrium the policy rule determines \(\psi\) and \(q^S\). The former is increasing in \(\mu\), and the latter is decreasing. The model predicts a negative relationship between inflation and asset returns. For every \(\mu \in (\beta - 1, \mu^T]\), the introduction of money improves welfare. As \(\mu \to \beta - 1\), \(\psi \to \bar{\psi}\) and \(q^S \to \bar{q}\). Therefore, whenever the supply of the asset is not sufficiently large to satisfy the demand for liquidity, constrained efficiency is achieved only if the Friedman Rule is followed.

\(^{17}\) This restriction holds for all \(T\), not only for \(T < \bar{T}\). When \(T \geq \bar{T}\), we have \(\psi = \bar{\psi}\), and consistency requires \(\mu^T = \beta - 1\). Hence, for this specific region of \(T\), the range of admissible policies becomes the empty set. However, if we allow for cases with zero nominal interest rate, the lower bound of admissible policies includes \(\beta - 1\) and the relevant set is a singleton, namely the Friedman Rule.
Chapter 3

Monetary Policy and Interest Rates with Collateralized Borrowing

3.1 Introduction

Agents usually anticipate consumption by borrowing. In the last years the development of credit markets has walked hand by hand with a significant increase in consumer debt. Monacelli (2006) more specifically reports on the considerable degree of co-movement between total private consumption and household mortgage debt in the last fifteen years. On the other side of this remarkable credit extension, information and repayment issues have often arisen. Monacelli also reports on the significant increase in secured debt as a percentage of household debt.

In principle, agents may use assets that they own as collateral in order to alleviate those information and incentive problems. It is for good reason that secured debt has become a vastly common form of borrowing contract and collateral is widely required in most credit market transactions.¹ There is a variety of assets that can serve as collateral,

¹ According to Ferraris and Watanabe (2007) the value of all commercial and industrial loans intermediated by US banks and secured by collateral amount to 46.9% of the total value of loans. Iacoviello(2005) also reports on a large proportion of borrowing secured by real estate.
among which durable goods seem to be the predominant instrument. A reason for this could be that, in general, their price is supposed to be less volatile since it is subject to longer term consumption or investment decisions. Nevertheless, these and other assets may often present several features that could cause their price to be not so stable.

Bernanke and Gertler (1999, 2001) maintain that monetary policy should only respond to variability in asset prices insofar as they pose a threat to price stability or help forecast inflationary pressures. However, they reach that conclusion for a model in which the asset other than money is capital, as part of an investment portfolio and productive input. In this framework they analyze different degrees of responsiveness of monetary policy to asset price movements that come mainly from non-fundamental reasons, like asset price bubbles. I will try to argue that things may be different when, in line with Iacoviello (2005) and Kiyotaki and Moore (1997), these assets are used as collateral and borrowing constraints are tied to the value of the asset. Iacoviello (2005) is more related to Bernanke, Gertler, and Gilchrist (1996) or Bernanke et al. (1999) where they describe a financial accelerator mechanism by which a shock to a firm’s net worth leads to a change in the value of its collateralizable assets and this to an adjustment in production.

Instead, I here describe a world in which asset price movements may affect final consumption through collateral constraints. Ferraris and Watanabe (2007) present a model in which capital is used as collateral but in principle its price does not play a role and they concentrate on issues like optimal capital accumulation. My paper may be closer to Kiyotaki and Moore (1997). However, the asset that serves as collateral in their paper is also a factor of production, what may affect the relation between asset prices, credit limits, and borrowing constraints in a very different way.

In this paper I build a model that both provides explicit micro foundations that make money essential and allows agents the possibility to store assets than can be used as collateral in a credit market. Even though the latter issue has been treated to a certain extent in the literature, the introduction of collateral in search-theoretic models is a very
recent development. Furthermore, the value of the assets that serve as collateral plays a crucial role in this paper; for I model the value of collateral in a very natural way that allows me to analyze heterogeneity on the borrower's side. To the best of my knowledge, this is a feature in the paper that has not been explicitly considered before and will render interesting implications for monetary policy.

Nonetheless, one of the most relevant insights delivered by the model involves the asset pricing mechanism. In this paper asset prices will not be determined following conventional considerations. Agents in this model will value assets for not only their specific expected return, but also for the liquidity they can provide in some contingencies when used as collateral. Assets other than money are modeled in a way that they may yield less return than financial assets, but they possess better liquidity properties. On the other hand, these assets may be less liquid than fiat money but will bear an additional return, higher than that of mere cash. Therefore, I am able to support equilibria with money and a stored asset that may have different properties than fiat money or financial assets, and that will still be valued for its "indirect" liquidity as a collateralizable asset. In fact, money and the asset that serves as collateral are rather complements in my model.

In other models with collateral and borrowing constraints (Bernanke and Gertler (1999), Iacoviello (2005), Kiyotaki and Moore (1997), and Monacelli (2006)) heterogeneity is introduced by means of giving agents different impatience rates. Thus, the more patient agents become lenders and the other borrowers. However, borrowers are homogeneous in the sense that in equilibrium the borrowing constraints for all of them are always binding. As I mentioned before, a new element that I introduce is heterogeneity on the borrower's side.\(^2\) There will still be some agents who become lenders and other

\(^2\) In Berentsen and Waller (2005) agents are also homogeneous but their borrowing constraints are never binding in equilibrium, i.e. they are all unconstrained. Ferraris and Watanabe (2007) analyze homogeneous borrowers as well. However, they study separately the cases in which their borrowing constraints are binding from those in which they are not.
agents who become borrowers. However, within the group of borrowers there will be a fraction for which the value of the asset used as collateral is not high enough to get an optimal amount of credit. Their borrowing constraint will therefore be binding in equilibrium. On the other hand, the value of collateral for the remaining borrowers may guarantee such optimal loan amount. Bernanke and Gertler (1995, 1999, 2001) assert that the condition of balance-sheets is determinant of agent's ability to borrow and lend. In my model it is the value of the asset used as collateral what will determine how much an agent can borrow and whether an agent faces a borrowing limit or not. I believe that modeling heterogeneity on the borrower's side is not only innovative but will bear interesting implications for the analysis of monetary policy. More precisely, I argue that the presence of this kind of heterogeneity will have consequences on the evolution of interest rates.

My work is in this sense related to that by Berentsen and Waller (2005). They study optimal policy where the monetary authority seeks to improve welfare by stabilizing short-run aggregate shocks. As a result they obtain that, away from the Friedman rule, the optimal policy is to smooth interest rates in order to smooth consumption. However, since their paper focuses on different questions, they do not model the use of collateral and their borrowers are homogeneous. I perform several numerical experiments obtaining that changes in the interest rates have a smaller impact in homogeneous economies populated by constrained agents than in economies with both types. In other words, in an experiment where the monetary authority wishes to influence consumption after a shock to the value of collateral, interest rates evolve more smoothly in an economy with heterogeneous borrowers than in a situation in which all borrowers are constrained. Inversely, when we go from a scenario where all borrowers are unconstrained to a world with heterogeneous borrowers then interest rates will need to be changed more abruptly.

The rest of the paper is structured as follows. Section 3.2 describes the main aspects of the model. Section 3.3 discusses the features of the equilibrium. In section 3.4 numer-
ical simulations are performed to produce more sensible insights. Section 3.5 elaborates on relevant issues and other points that were not previously discussed in detail, and Section 3.6 concludes.

### 3.2 The Model

The structure of the environment is similar to that in Berentsen, Camera, and Waller (2007) which itself builds on Lagos and Wright (2005). Time is discrete and I consider a $[0,1]$ continuum of agents that live forever and discount future at rate $\beta \in (0,1)$. In every period there are three markets that open sequentially in the following order: first, a credit market; second, a decentralized trade market where agents meet bilaterally; and finally a centralized Walrasian market. The commodities traded in the second and third market are perfectly divisible and of different types.

At the beginning of every period, and before entering the first market, uncertainty is resolved and agents receive a preference shock so that they find out both whether they are going to be producers or consumers in the second market of that period, and whether they will match with someone in that market. Producers will only produce and sell the good in the decentralized market to agents who are consumers. Therefore, producers will not really need money in the second market as opposed to consumers, who need it to buy the good in the second market. Thus, at any period of time $t$, after uncertainty is resolved, a credit market opens in which agents can reallocate their money balances before going into trade.

In this credit market agents who are sellers and buyers who know they will not match may want to deposit their money in the bank and earn the interest at the nominal rate $i_t$. Buyers who know they will match want to borrow money to finance consumption in the next market. The net borrowing of a certain agent is denoted by $l_t$. The amount of loans that buyers can take could in principle be limited by the value of their stored
asset at that moment in time. Then the credit market closes and trade takes place in a
decentralized market where agents meet and bargain over the terms of trade in a bilateral
and anonymous manner. Finally, in the third market agents work, consume, and readjust
their portfolios for the next period.

In order to simplify the analysis and get a clear intuition of the functioning of the
model I assume without loss of generality\textsuperscript{3} that loan and deposit contracts cannot be
rolled over. As a consequence all financial contracts will have a duration of only one
period. While one-period debt contracts are very convenient, in equilibrium they turn
out to be sufficient to cover the needs of any trading situation. Thus, in the third market
agents repay their loans or redeem their deposits, and decide how much money and
collateral to take into next period.

Consumers in market 2 get utility $u(q)$ from consuming an amount $q > 0$ of the
good, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. Producers, on the
other hand, incur in a cost $c(q)$ for producing $q$ units of output in the second market,
where $c'(q) > 0$, $c''(q) \geq 0$, $c'(0) = 0$. Furthermore, for the sake of simplicity I assume
that $c(q) = q$. Agents in the centralized market get utility $U(x)$ from consumption with
$U'(x) > 0$, $U''(x) \leq 0$, $U'(0) = \infty$, and $U'(\infty) = 0$.\textsuperscript{4} A technology is available such
that allows agents to produce one unit of input with one unit of labor generating one
unit of disutility.

3.2.1 Money and Collateral

A most important object in the model is (fiat) money, that is perfectly divisible and can
be stored in any amount $m_t \geq 0$. Papers that model search frictions with money as a
medium of exchange, from the early work of Kiyotaki and Wright (1989, 1993) to more

\textsuperscript{3} With quasi-linear utility agents are happy to pay off all debt at the end of every period.

\textsuperscript{4} Lagos and Wright (2005) explain how these assumptions, along with quasi-linearity of preferences,
lead to a degenerate distribution of money and asset holdings at the beginning of every period. In
particular, they show how the assumptions regarding the utility functions allow to derive technical
conditions to ensure that in equilibrium all agents produce and consume in the centralized market.
modern models, provide a good description of those frictions that make money essential. The same basic assumptions are also established here. In particular, in the decentralized market there are some agents who want to consume but cannot produce, and others who can produce but do not desire to consume. Thus, trade in this market occurs in bilateral and anonymous meetings where trading histories are private information. These reasons generate the need to use fiat money as a medium of exchange money (Kocherlakota (1998), Wallace (2001)). Therefore, I am able to construct a model where money and credit coexist, which is the basic framework where collateral can be introduced and analyzed.

As it was reviewed in the introduction, although durable goods are the preferred instrument there are currently a variety of assets that may serve as collateral. In line with Berentsen and Monnet (2007), I assume that there exists a storage technology such that agents can store some amount of the good in the centralized market, \( a_t \), to be used as collateral in the next credit market. This technology will allow agents to recover \( \delta a_t \) in the next centralized market. The return \( \delta \) is basically an idiosyncratic shock. When the time comes to be on the credit market, each agent's collateral, \( a_t \), will have a different value, \( \delta a_t \). I assume that these shocks follow a certain distribution, \( F(\delta) \), about which only two assumptions are made: \( F(\delta) \) is continuous, and such that \( \delta \in [0, \widehat{\delta}] \), where \( \widehat{\delta} \) is the highest that the collateral of any given agent can be valued.\(^5\)

This asset is special in the following sense: its return is specific to the agent that owns it. The idiosyncratic shock can be interpreted as a shock that endows the stored amount of good of each agent with specific characteristics so that different values are assigned to the asset across agents. Thus, in a Kiyotaki and Wright (1989) fashion I assume that each agent likes the particular features of her asset but not those of the asset owned by any other agent. This rules out the possibility that this asset can be traded, which is

\(^5\) Berentsen, Camera, and Waller (2007) and Ferraris and Watanabe (2007) both model deterministic returns to the asset that is used as collateral.
another difference with a standard financial asset. The way I model this asset and its relation with collateral and the credit market holds a very close resemblance with the way in which mortgage contracts and the housing market works. People want to own houses, but they want to own the house that they like. One particular agent enjoys her own house, let us say because it is decorated in a certain way, but not necessarily her neighbor's. Thus, people do not want to trade their houses both because they like theirs and because they want to live in them. However, whenever people find themselves in need for liquidity they can collateralize their assets. That is, they can always borrow against the value of their houses. \(^6\)

In the exact same manner, I allow agents to store this asset that will not only be valued differently but that is also collateralizable. This modeling approach is very convenient to keep the model analytically tractable and allows me to properly analyze the use of collateral in credit markets. Moreover, one most important advantage of this approach is that a new asset is introduced that has characteristics different from those of financial assets and money and that is still valued both for its indirect liquidity as well as for its specific return. As a first approach I analyze the model taking as given a certain distribution function for the idiosyncratic shocks, \(F(\delta)\). Later on, I perform several policy experiments of relevance for some of the most recent concerns in monetary and stabilization policies. In particular, I will analyze how the decisions of agents and policy variables would change when this distribution changes.

### 3.2.2 Monetary Policy

The per capita money stock in market 3 at any given period in time \(t\) is denoted by \(M_t\). The gross growth rate of money supply is given by \(\gamma = M_t / M_{t-1}\), where \(\gamma > 0\). Normally most monetary authorities set their policy objectives in terms of the interest rate and

\(^6\) Ferraris and Watanabe (2007) report that 47.9% of US households had home-secured debt in 2004, where their house was the guarantee of payment.
usually they are able to exert a very precise control over it. Likewise, in this paper I consider a monetary authority that takes the interest rate, \( i \), as its policy variable. The money growth rate \( \gamma \) will then be determined so that it is compatible with the chosen level of \( i \). This approach makes it considerably easier to solve the model and facilitates comparisons with a majority of models where the interest rate is also the monetary policy variable. Changes in the monetary base take place through deterministic lump-sum injections or withdrawals of money, \( \tau M_t \), at the beginning of every period. Therefore, the net change in the aggregate money stock would be \( \tau M_{t-1} = (\gamma - 1)M_{t-1} \). Henceforth, in order to keep notation simple, those variables corresponding to the next period will be indexed by \(+1\), and those corresponding to the previous one will be indexed by \(-1\).

Let \( P \) denote the price of the good in the centralized market. Then, \( \phi = 1/P \) denotes the value of money, the real price of money in that market. Notice that in this model there are no aggregate shocks. Decisions may vary as a consequence of a shock to the distribution of the value of collateral \( F(\delta) \), or to the interest rate \( i \), but there is no other shock that affects the whole economy. I intend to focus on equilibria where real allocations are constant through time. These will be referred to as stationary equilibria. In particular,

\[
\phi M = \phi_{-1} M_{-1} = z. \tag{3.1}
\]

In these equilibria \( \phi_{-1}/\phi = P/P_{-1} = M/M_{-1} = \gamma \), which means that I am restricting my attention to equilibria where the growth rate of money supply \( \gamma \) is constant. Finally, in this paper I will only consider economies with positive interest rates, \( i > 0 \). Among other technical reasons, focusing on these economies facilitates considerably the task of solving and analyzing the model. In what follows, I will solve and study the model backwards to better track the decisions of the agents.

\footnote{This, however, is not a restrictive assumption. In other similar models that analyze optimal monetary policy only the limit case in which \( \gamma \to \beta \) is considered, which means that \( i \to 0 \).}
3.2.3 Third Market

Let $P$ be the price of the good and $\phi = 1/P$ the value of money in this centralized market. At this stage agents get the return from having stored the real asset in the last period, $\delta a$, and decide how much to work, $h$, how much to consume, $x$, and how to readjust their portfolios for next period. That is, they also choose how much money, $m_{t+1}$, and how much of the real asset, $a_{t+1}$, to carry into next period. Finally, in order to settle financial claims, agents who were consumers in market 2 and borrowed $l$ units of money have to pay back $(1 + i)l$ from their loan contracts. This amount is transferred to agents who were producers or consumers who did not match, and thus redeem their deposits.

Given a distribution $F(\delta)$ and a policy arrangement $i$, let $V_3(m, l, a)$ be the expected lifetime utility of entering the third market with $m$ units of money, $l$ net borrowing, and $a$ units of the stored asset in period $t$. The problem of a representative agent in this market is

$$V_3(m, l, a) = \max_{x, h, m_{t+1}, a_{t+1}} [U(x) - h + \beta V_1(m_{t+1}, a_{t+1})]$$

subject to

$$x + h + m_{t+1} + a_{t+1} = h + \delta m - \phi (1 + i)l + \delta a.$$

Substituting the budget constraint into (3.2) gives us

$$V_3(m, l, a) = \phi [m - (1 + i)l] + \delta a$$

$$+ \max_{x, m_{t+1}, a_{t+1}} \left[U(x) - x - \phi m_{t+1} - a_{t+1} + \beta V_1(m_{t+1}, a_{t+1})\right].$$
The first order conditions are

\begin{align}
U'(x) &= 1 \\
\phi_{-1} &= \beta V_1^m \\
1 &= \beta V_1^a
\end{align}

(3.3) (3.4)

The superscript indicates the variable with respect to which the function is being differentiated.\(^8\) Thus, \(V_1^m\) is the marginal value of taking an additional unit of money into the first market of period \(t\), and equation (3.4) is the asset pricing equation. The envelope conditions are

\begin{align}
V_3^m &= \phi; V_3^d = -\phi(1 + i); V_3^a = \delta.
\end{align}

(3.5)

As can be observed, the value function is linear. This, together with the assumptions on preferences and production costs listed in the description of the model, ensures that there are no wealth effects and therefore all agents enter the following period with same amount of money and collateral.

### 3.2.4 Second Market

At the point of entering the second market there are three types of agents: among those who match there are agents who want to consume but cannot produce, buyers (b), and those who want to produce but do not wish to consume, sellers (s). Agents belonging to the third type, others (o), are those who do not match. They can be both producers or consumers. In case a buyer meets a seller, they bargain bilaterally. The terms of trade are determined in a simple bargaining procedure of take-it-or-leave-it offers made by the

\(^8\) Note that equations (3.3) and (3.4) have been lagged one period for convenience of analysis.
buyer that fits well the depicted framework.\textsuperscript{9} It can be easily shown that the equilibrium equations and main results do not vary under other pricing mechanisms, like competitive pricing.\textsuperscript{10}

An agent that walks into the second market with \( m \) units of money, net borrowing \( l \), and \( a \) units of collateral has lifetime utility \( V_{2j}(m_j, l_j, a), j = b, s, o \). Agents who do not match in the second market will just walk through it and into the third market with their asset and money holdings and their loans untouched. Thus, their value function is simply

\[
V_{2o}(m_o, l_o, a) = V_3(m_o, l_o, a).
\]

The value functions for sellers and buyers who match are, respectively,

\[
V_{2s}(m_s, l_s, a) = -c(q) + V_3(m_s + d, l_s, a),
\]
\[
V_{2b}(m_b, l_b, a) = u(q) + V_3(m_b - d, l_b, a).
\]

where \( m_j = m + l_j, j = b, s \). The term \( d \) refers the amount of cash paid for the quantity consumed. Thus, an agent who has consumed \( q \) units of the good in market 2 enters market 3 with \( m_b - d \) units of money. Similarly, an agent who has produced and sold \( q \) units of the good in market 2 enters market 3 with \( m_s + d \) units of money. The problem

\textsuperscript{9}Rocheteau and Wright (2005) also analyze competitive pricing. They show that, at least for a model of credit without collateral, some results may vary according to different pricing procedures. The quantity traded under bargaining may be lower than under competitive pricing due to the classic holdup problem also pointed out in Lagos and Wright (2005). However, in my model the quantity traded does not need to be lower since I consider take-it-or-leave-it offers, which means that buyers have all the bargaining power and the holdup problem disappears.

\textsuperscript{10}For a comprehensive analysis of several relevant market structures and their different results please refer to Rocheteau and Wright (2005).
to be solved by the buyers looks like

$$\max_{q,d} \quad u(q) + V_3(m_b - d, l_b, a) - V_3(m_b, l_b, a) \quad \text{s.t.} \quad -c(q) + V_3(m_s + d, l_s, a) - V_3(m_s, l_s, a) \geq 0$$

$$d \leq m_b.$$  

(3.6)

Since I have assumed that $c(q) = q$, and the third market value functions are linear, this problem reduces to

$$\max_{q,d} \quad u(q) - \phi d \quad \text{s.t.} \quad -q + \phi d \geq 0$$

$$d \leq m_b.$$  

(3.7)

The solution to this problem is

$$\text{if} \quad q^* \leq \phi m_b, \quad \text{then} \quad \begin{cases} q = q^* \\ d = \frac{q^*}{\phi} \end{cases}$$

$$\text{if} \quad q^* > \phi m_b, \quad \text{then} \quad \begin{cases} q = \hat{q} \\ d = m_b \end{cases}.$$  

(3.8)

where $\hat{q}$ solves $c(q) = \phi m_b$. Given the maintained assumptions, this means that $\hat{q} = \phi m_b$.

In either case both the amount consumed, $q$, and the amount of money paid for it, $d$, depend only on $m_b$. Which itself depends on $z$ and $a$, given that the amount borrowed is related to the amount of the stored asset, as it is mentioned in the introduction and will be shown in the first market. Thus

$$q(z, a) = \begin{cases} q^* \quad \text{if} \quad q^* \leq \phi m_b \\ \hat{q} \quad \text{otherwise.} \end{cases}$$

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Agents in the centralized market will never choose to carry an amount of money, $m$, such that they can obtain the efficient quantity in the decentralized market, $q(z, a) = q^*$, unless $\gamma = \beta$ and the buyer has all the bargaining power. The later holds in this case, however I focus on economies where $i > 0$, that is, $\gamma > \beta$. Although it is true that in this model for $\gamma = \beta$ agents would carry $m = m^*$, nobody would need to take loans, $a = 0$, and $q(z, 0) = q^*$. Nevertheless, if $\gamma > \beta$, then $m < m^*$, and people want to borrow in order to get closer to $q^*$. This means that $a > 0$, and $q(z, a) = \hat{q} < q^*$. Notice that $q^*$ in principle does not depend on $F(\delta)$, since these are idiosyncratic shocks that do not affect the equation that defines $q^*$, namely, $u'(q^*) = c'(q^*) = 1$.

The first order condition for this problem can be expressed as follows,

$$\phi [u'(q) - 1] = \lambda_q$$

where $\lambda_q$ is the multiplier assigned to the constraint in problem (3.6). It is easy to show that in equilibrium, $d = m_b$ and $q = \phi m_b$ always for any $i > 0$. This implies that $u'(q) > 1$, i.e. $u'(q) > c'(q)$, the buyer spends all his money, and trades are not efficient.\footnote{See Appendix II.} It also means that since the budget constraint binds for any $i > 0$, then the interest rate basically acts as a tax on consumption. Finally, using the envelope theorem and equations (3.5) and (3.9) we get

$$V_{2j}^j = -\phi(1 + i), \ V_{2j}^a = \delta, \ j = b, s.$$  

$$V_{2s}^m = V_3^m = \phi$$

$$V_{2s}^m = V_3^m + \lambda_q = \phi u'(q).$$
3.2.5 First Market

Uncertainty is solved right before entering the first market. Then a credit market opens that allows the possibility to reallocate otherwise idle balances. In other words, agents who find out that are going to be sellers and match, and also agents who will not match, will deposit their money in banks with the expectation to earn a nominal interest $i$. On the other hand, agents that turn out to be buyers who will match may want to borrow that money at nominal rate $i$ in order to finance higher consumption. Financial intermediation is done by perfectly competitive banks that accept deposits from producers and extend loan contracts to consumers. Agents are anonymous and this is an environment with lack of enforcement and certain lack of record keeping. In this world, agents do not know anyone else's histories and banks impose that agents can only borrow up to a limit based on the value of collateral.

The probability of being a buyer and matching in the second market is $n$. The probability of being a seller and matching as well is $s$. Finally, the probability of being either a seller or a buyer but not matching would then be $1 - n - s$. Thus, an agent who enters the first market with $m$ units of money and $a$ units of the real asset stored from last period has expected lifetime utility

$$V_1(m, a) = \int [nV_{b2}(m_b, l_b, a) + sV_{s2}(m_s, l_s, a) + (1 - n - s)V_{o2}(m_o, l_o, a)] dF(\delta) \quad (3.11)$$

where $m_j = m + l_j$, $j = b, s, o$. Once trading types have realized, the problem for sellers and for buyers who do not match (lenders) is the same, and different from that of buyers who match (borrowers).

\footnote{This also implies that banks cannot punish agents who default on their loan contracts. However, they can impose borrowing limits based on collateral, as it is mentioned next.}
Lenders, types \( j = s, o \). These types solve the following problem

\[
\max_{l_j} \ V_{2j}(m_j, l_j, a) \tag{3.12}
\]

s.t. \( 0 \leq m + l_j \).

The first order condition for this problem is

\[
V^{m}_{2j} + \lambda_j + V^{l}_{2j} = 0
\]

where \( \lambda_j \) is the multiplier corresponding to the constraint. It is not hard to see that these agents will become net lenders, that is \( \lambda_j > 0, \ j = s, o \). Since they will earn a positive interest, it is always profitable for them to lend the money that they will not need. At \( i = 0 \) their optimal strategy is to deposit any amount in the interval \([0, m]\).

Ultimately, if \( i = 0 \) it could be assumed that these agents still lend out the money since they do not use it and there is no intra-period inflation. Using (3.10) the previous first order condition becomes

\[
\lambda_j = \phi i, \ j = s, o. \tag{3.13}
\]

Borrowers, type \( j = b \). One of the main elements in this model is introduced in the problem of the buyer: a borrowing constraint that depends on the value of collateral at any point in time. This constraint captures the idea that in an environment with limited enforcement a borrowing limit is established by the financial intermediaries that require collateral in order to issue loans: the nominal amount of loan plus interest must not be greater than the nominal value of available collateral, \( \phi(1+i)l \leq \delta a \). The difference here is that each agent will face a different constraint according to the particular value of its
collateral. The problem that each buyer of type \( b \) solves is

\[
\max_{l_b} V_{2b}(m_b, l_b, a) \quad \text{s.t.} \quad 0 \leq m + l_b \leq \frac{\delta a}{\phi(1+i)} ,
\]

where the second constraint is the borrowing constraint, with multiplier \( \lambda_b \). The first constraint has multiplier \( \lambda_b \). Again, one can easily see that buyers of type \( b \) will always become net borrowers, i.e. \( \lambda_b = 0 \), since money is required to buy consumption.\(^{13}\)

Assumptions on preferences, like the usual Inada conditions, guarantee that \( m + l_b > 0 \) always, since \( u'(0) = +\infty \). Therefore, the problem for these buyers would in fact read

\[
\max_{l_b} V_{2b}(m_b, l_b, a) \quad \text{s.t.} \quad l_b \leq \frac{\delta a}{\phi(1+i)} .
\]

The first order condition is

\[
V_{2b}' - \lambda_b + V_{2b}^m + \lambda_b = 0
\]

which, using (3.10) and \( \lambda_b = 0 \), can be written as follows

\[
\phi [u'(q) - (1 + i)] = \lambda_b .
\]

As mentioned in the introduction, within the group of buyers of type \( b \) (consumers) there will be two types: those who are borrowing constrained and those who are not. Thus,\(^{13}\)Berentsen, Camera and Waller (2007) prove that financial intermediation improves welfare (consumption) away from the Friedman rule. Not only \( m_b > 0 \) because \( m > 0 \), but also \( l_b > 0 \).
the constraint will be binding for the first ($\lambda_{lb} > 0$), but not so for the second ($\lambda_{lb} = 0$).\footnote{Monacelli (2006) makes technical assumptions to ensure that around the steady state both the budget and borrowing constraints of all the agents are satisfied with equality. As he argues in his paper, this was needed in order to be able to log-linearize the model. Iacoviello (2005) also builds a model such that borrowing constraints could be non-binding only in some uncertainty cases. However, he assumes that uncertainty is sufficiently small so that they will always be binding for all agents.}

In other words, the first order condition for the constrained agents reduces to

$$u'(q_c) > 1 + i,$$

whereas for the unconstrained

$$u'(q_u) = 1 + i$$

where $q_c$ and $q_u$ are the amounts of good in the decentralized market consumed by constrained and unconstrained consumers, respectively. Equation (3.20) tells us that unconstrained buyers take loans until the marginal benefit of doing so is equal to its marginal cost. Trades are efficient in this case. Constrained agents are not able to reach this efficiency and will therefore borrow as much as they are allowed to. Therefore, in equilibrium the marginal utility of constrained borrowers will be higher than that of unconstrained agents.

Notice, however, that the fact that for unconstrained borrowers equation (3.20) holds may wrongly lead to the conclusion that, since their loan contracts are efficient, they actually get the efficient consumption $q_u(z, a) = q^*$. On the contrary, for positive interest rates the equation $u'(q_u) = 1 + i$ actually means that $u'(q_u) > 1$. Therefore, $q_u(z, a) = \hat{q}_u < q^*$. Nonetheless, it is true that, since $q = \phi(m + l_b)$ and $u'$ is decreasing in $q$, constrained borrowers get $\hat{q}_c < \hat{q}_u < q^*$.
3.3 Equilibrium

As mentioned in a previous section, I focus on symmetric equilibria where all agents of a given type behave in the same way and where real money balances are constant through time. One of the most relevant aspects in the model refers to the way in which the number of agents that are constrained in equilibrium is determined. Recall that I have assumed $\delta \in [0, \hat{\delta}]$, with some distribution. From equation (3.20), since $u'$ is continuous and strictly decreasing, and $\delta$ is continuously distributed, the next lemma follows.

**Lemma 3.3.1.** There exists a critical value, $\delta_c$, that lies in the interior of the distribution $F(\delta)$, and is defined by

$$u'(z + \frac{\delta_c a}{1 + i}) = 1 + i.$$  \hfill (3.21)

Its expression can be easily derived from equation (3.21).\footnote{Given that $u'$ is continuous and monotonically decreasing, its inverse exists.} This critical delta will divide the group of borrowers in constrained and unconstrained. Thus, we can obtain the nominal loan demand of both constrained and unconstrained agents, $l_b^c$ and $l_b^u$ respectively:

$$l_b = \begin{cases} 
  l_b^c = \frac{\delta a}{\phi(1+i)}, & \text{if } \delta < \delta_c \\
  l_b^u = \bar{l}_u, & \text{if } \delta \geq \delta_c
\end{cases}$$

where $l_b^u$ is given by the equation $u'(z + \phi l_b^u) = 1 + i$. This in fact means that $l_b^u = \bar{l}_b = \frac{\delta a}{\phi(1+i)}$. Thus, for $\delta < \delta_c$, $l_b$ is increasing in $\delta$, and for $\delta \geq \delta_c$, $l_b = \bar{l}_b$, which is constant. The reason for the latter is that unconstrained borrowers take loans until the marginal benefit of doing so is equal to its marginal cost.

Also, both loan demands are naturally decreasing in the interest rate. Notice that
it is straightforward to see that \( l_b^c < l_b^u \). Therefore, since \( q \) is strictly increasing in \( l_b \), \( q = z + \phi l_b \), then \( q_c = z + \phi l_b^c < q_u = z + \phi l_b^u \). Figure 1 shows the dynamics of \( q \) as a function of \( \delta \).

On the other hand, market clearing in the credit market requires that the total amount of loans demanded by constrained and unconstrained borrowers cannot exceed the total amount of money deposited by lenders, \( s z + (1 - n - s)Z = (1 - n)Z \),

\[
\int_0^\delta \frac{\delta a}{1 + i} dF(\delta) + \int_{\delta}^{\delta_c} \frac{\delta_c a}{1 + i} dF(\delta) = \frac{1 - n}{n} z .
\]  \hspace{1cm} (3.22)

I will now turn to the equilibrium equations. Differentiating (3.11) and using the envelope theorem and equations (3.13) and (3.18), the marginal values of money and collateral can be expressed as \(^{16}\)

\[
V_1^m = \int \phi [u'(q) + (1 - n)(1 + i)] dF(\delta) ,
\]  \hspace{1cm} (3.23)

and

\[
V_1^a = \int \left[ n\delta \left( \frac{u'(q)}{1 + i} - 1 \right) + \delta \right] dF(\delta) .
\]  \hspace{1cm} (3.24)

The marginal value of money has the following interpretation. If the agent is a buyer who matches in the second market, an additional unit of money yields \( u'(q) \) units of marginal utility in that market. In any other case the agent can deposit the additional unit of money which will yield the nominal return \((1 + i)\). Notice that the credit market increases the marginal value of money because agents can earn interest on idle cash.

Remember that equation (3.4) is the pricing equation for the asset that is used as collateral. Now that we have derived the expression for \( V_1^a \), we know what the asset pricing equation looks like. The price of this asset is always that of the good in the centralized

\(^{16}\)It is straightforward to check that the value function is concave in both \( m \) and \( a \).
market: 1. However, in accordance with such price, the pricing equation will also reflect both the specific expected return and the liquidity properties of the collateralizable asset. Then, using (3.4) and the first order condition (3.24), and rearranging them, we can derive the expression for the liquidity premium of the asset used as collateral

$$1 - \beta \delta = \int \left[ n \left( \beta \delta \frac{u'(q)}{1 + \delta} - \beta \delta \right) \right] dF(\delta).$$

(3.25)

where $\delta$ is just the mean value of the distribution. The cost of storing an additional unit of collateral is 1, and the expected return of doing so is $\beta \int_0^{\delta} \delta \ dF(\delta) = \beta \delta$. Thus, the right-hand side of (3.25) is the collateral’s liquidity premium. However, as I will make clear below, this premium is positive, which means that $\beta \delta < 1$. Therefore, agents need an incentive to hold collateral other than its investment return. The liquidity properties of collateral provide this incentive.

Plugging (3.23) and (3.24) into (3.3) and (3.4) respectively, and rearranging we obtain the following equilibrium conditions

$$\frac{\gamma - \beta (1 + i)}{\beta (1 + i)} = \int n \left( \frac{u'(q)}{1 + \delta} - 1 \right) dF(\delta),$$

(3.26)

and

$$\frac{1 - \beta \delta}{\beta} = \int \delta n \left( \frac{u'(q)}{1 + \delta} - 1 \right) dF(\delta).$$

(3.27)

We know from (3.20) that the right-hand side of (3.26) and (3.27) is zero for any $\delta \geq \delta_c$, that is, for unconstrained borrowers, while it is strictly positive for $\delta < \delta_c$. Finally, if we use equation (3.21) to obtain an expression for $\delta_c$, the last two equilibrium equations
can be rewritten as

\[
\frac{\gamma - \beta(1 + i)}{\beta(1 + i)} = \int_0^{\delta_s(z,a)} \delta n \left( \frac{u'(z + \frac{\delta a}{1+i})}{1+i} - 1 \right) dF(\delta), \tag{3.28}
\]

\[
\frac{1 - \beta \delta}{\beta} = \int_0^{\delta_s(z,a)} \delta n \left( \frac{u'(z + \frac{\delta a}{1+i})}{1+i} - 1 \right) dF(\delta). \tag{3.29}
\]

The terms in parenthesis on the right-hand sides of equations (3.28) and (3.29) reflect the excess of marginal utility with respect to the marginal cost of borrowing in the credit market. Thus, the right-hand side of equation (3.28) can be interpreted as the average excess of marginal utility over the marginal cost of borrowing among constrained borrowers.

**Definition 3.3.1.** Given a distribution function \( F(\delta) \) and an interest rate \( i \), a symmetric stationary monetary equilibrium is a choice of real balances, \( z \), and collateral holdings, \( a \), that satisfy (3.22) and (3.29), and a growth rate of money, \( \gamma \), that is consistent with \( i \) and satisfies (3.28).

As I mention in previous sections, I contemplate the frequent case in which the monetary authority tries to exert control over interest rates. Consequently, the nominal interest rate on loans in this model is set exogenously. Once \( i \) is fixed we need a rate of growth of money \( \gamma \) that is consistent with that level of the interest rate. That \( \gamma \) is pinned down by (3.28), and is defined as a function of \( z, a \). Notice, on the other hand, that (3.29) defines \( a \) as a function of \( z \). And \( z \) is also endogenously determined by (3.22) as a function of \( a \).

**Proposition 3.3.1.** Assume that \( \delta \) is continuously distributed and that \( u' \) is continuous and monotonically decreasing. For positive interest rates there exists a unique symmetric monetary equilibrium with credit in which another asset is valued for its liquidity as collateralizable in contingencies as well as for its specific return. Borrowers are dif-
differentiated by a critical value $S_c$ between constrained and unconstrained. Equilibrium consumption for both types is decreasing in $i$ and $\gamma$, and increasing in the value of collateral, $F(\delta)$. Finally, in equilibrium consumption by constrained agents is less than that of unconstrained ones, $q_c < q_u < q^*$, with $(q_c, q_u) \rightarrow q^*$ as $\gamma \rightarrow \beta$.

The proof of everything that is stated in this proposition has already been provided in previous sections, except that of existence and uniqueness, which I relegate to the appendix. In their paper on channel systems for monetary policy, Berentsen and Monnet (2007) defined a limit to the deterministic returns on collateral, $1 = \beta R$, where $R$ was such return. In the model I analyze in this paper there is also an upper bound to the return of collateral. Remember that the right-hand side of (3.28) and (3.29) is strictly positive. If we look at equation (3.29), this in fact means that $1 - \beta \delta > 0$, which implicitly defines the upper bound

$$\delta < \frac{1}{\beta}.$$  \hfill (3.30)

It makes sense at this point to turn our attention to some relevant questions: How would an equilibrium be affected if the interest rate was modified? How would decisions change if the distribution of collateral values changed? How do equilibrium variables interact with each other? Because of the serious difficulties due, among other reasons, to the nonlinearities in the model, only some analytical answers to these questions will be provided later. However, a more sensible understanding of the functioning of the model can be achieved with numerical simulations.

### 3.4 Quantitative Analysis

At the end of this section I present some experiments that I estimate of relevance for recent developments in monetary and stabilization policy. In these experiments het-
erogeneity on the borrower’s side will play a crucial role. I parameterize the model as follows and in order to make it comparable to other existing literature. Assume first that \( u(q) = q^\alpha \), \( \alpha = 1/2 \), and \( c(q) = q \). Also, \( \delta \sim \text{U}[0,\delta^\ast] \). This means that

\[
\delta_c = \frac{1 - 4z(1 + i)^2}{4a(1 + i)}.
\]

Also assume that \( \beta = 0.96 \), which implies that \( \delta^\ast < 2/\beta = 2.0833 \), and for simplicity, \( n = 0.5 \). First, I analyze the equilibria when the central bank modifies its policy variable \( i \). I consider a range from 0 to 0.1, for a given distribution \( F(\delta) \). I obtain that as \( i \) increases \( \gamma \) also has to increase. This is what we normally see central banks do: when inflation is high or inflationary pressures exist, interest rates are also higher in order to bring inflation under control. However, \( z \) and \( a \) decrease. Why? If \( i \) is increased borrowing becomes more expensive, borrowing constraints tighten, and real demand for loans falls. Therefore \( a \), which is what allows agents to borrow, decreases. This also happens because, since \( \gamma \) is higher, money is now less valued and \( a \) would then be used to get an asset that loses value. Holdings of money balances are reduced for the same reason.

All that is happening is that we are considering a change in \( i \) with no change in \( F(\delta) \), that is, agents are facing more expensive borrowing but do not see better returns to their stored assets. It is true though that with higher interest rates sellers would have an incentive to carry more money that they can lend. However, when agents are to decide on money balances for next period, they do not know whether they will be sellers. Another important observation is that consumption, both constrained and unconstrained, falls with a higher \( \gamma \), quite as in Berentsen et al. (2007), Bernanke, Gertler, and Gilchrist (1996), Kiyotaki and Moore (1997), and Iacoviello (2005), just to name a few. The critical value of \( \delta \) remains unchanged in relation to any movement of \( i \) or \( \gamma \).

If we now assume that the central authority is running a moderate policy \( i = 0.02 \), we can study the effects of a shock to the distribution of collateral values on the decisions.
of agents. I consider different values of the upper bound, ranging from 1.96 to 2.08. In other words, the decisions of agents will be modified resulting from situations where the average value of collateral has fallen below its real price in the centralized market, or resulting from more favorable conditions where that average value is above its real price. The results are exactly the opposite to those in the case with $i$ increasing: when $\hat{\delta}$ increases, $\delta_c$ decreases: the more favorable the shock to the value of collateral the less people will be borrowing constrained. Also, $z$ and $a$ go up, and $\gamma$ falls.

When people's collateral is more valued borrowing constraints are less tight and real demand for loans rises. Thus, for a given level of the interest rate, borrowers demand more money in loans and lenders are willing to provide that money in view of higher expected earnings. In this scenario it is necessary that money is more valued in order to meet a higher demand for money and for agents to be willing to hold it. Consequently, $\gamma$ goes down. Also, since the value of collateral is now higher people know that they can get more loans and therefore bring more of $a$ to the credit market.

An interesting result, in contrast to the related literature, is that while average unconstrained consumption $q_u$ is increasing the average constrained consumption $q_c$ is decreasing. This is due to the fact that as $\hat{\delta}$ increases $\delta_c$ decreases making the number of constrained people ever smaller, and this reduces the average constrained consumption. Nevertheless, the total effect on consumption should be stronger. Average constrained consumption falls but unconstrained agents are growing in number and their consumption is higher than that of constrained agents. Indeed, total consumption increases as the average value of collateral increases, just like in other models.

In what follows I try to assess the impact of changes in the interest rate induced by the central bank. One possible way to think about this is the following. As we have seen, shocks to $F(\delta)$ and $i$ have opposite effects on $z$, $a$, and $\gamma$. This means that monetary

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17 The real price of collateral in the centralized market is 1. This can be seen from the budget constraint. Thus, if $\hat{\delta} = 1.96$ then $\hat{\delta} = \frac{\delta}{\delta} = 1.96 < 1$. And if $\hat{\delta} = 2.083$ then $\hat{\delta} = \frac{\delta}{\delta} = 2.083 > 1$.

18 Bernanke et al. (1996) and Bernanke and Gertler (1999) present similar mechanisms.
policy can try to use \( i \) to offset changes in the value of collateral.\(^{19}\) In the last years we have witnessed a boom in the housing market in Europe, and to a certain extent in the US. Prices have risen considerably bringing households with mortgage contracts to the edge. However, in the last months the market is beginning to slow down, certainly in Europe, and house prices are growing ever more slowly, if not decreasing already. If we translate this situation into our model, a progressive fall in house prices would mean a drop in \( \hat{\delta} \). As I explained before, this means that both \( z \) and \( a \) would decrease, with \( \gamma \) and \( \delta_c \) increasing. The latter meaning that the lower the value of collateral the more people will be constrained. This could in fact cause a drop in consumption. However, the monetary authority could reduce \( i \) in order to induce an increase in \( z \) and \( a \), and ultimately bring consumption up back to its level previous to the drop in the value of collateral. In this sense I believe that the type of heterogeneity featured in my model will bear interesting insights. How does monetary policy differ from the case in which borrowers are homogeneous? Will changes in the interest rates have a greater or lower impact in heterogeneous economies than in homogeneous ones?

The intuition comes from a simple analysis of the model. In equilibrium, consumption by the unconstrained agents is \( q_u = z + l_u^R \), while consumption by the constrained is \( q_c = z + l_c^R \), where \( l_u^R \) and \( l_c^R \) are their real demands for loans, which have been defined before. Now, what is the difference between the change in constrained consumption, \( q_c \), and the change in unconstrained consumption, \( q_u \), when monetary policy, \( i \), changes? Which one varies more?

**Proposition 3.4.1.** For a given distribution, \( F(\delta) \), consider a change in the interest rate, \( i \). The reactions in consumption by constrained and unconstrained agents are such

\(^{19}\) Bernanke and Gertler (1999) describe policies for aggregate demand stabilization that work in the same direction.
This proposition basically states that \( |d_l^u/di| > |d_l^c/di| \), and therefore \( |dq_u/di| > |dq_c/di| \). Thus, changes in the interest rate imply larger movements in loan demand by unconstrained agents, which also lead to larger adjustments in consumption of unconstrained agents than in consumption of constrained agents.

I now report the results of simulations for two different economies: one mostly populated by constrained borrowers (85.07\% = \( \delta_c/\delta = 1.68437/1.98 \)), the other populated of mostly unconstrained borrowers (82.97\% = \( 1 - \delta_c/\delta = 1 - 0.35407/2.08 \)). In these simulations the economies adapt to an increasing interest rate. As we saw before, total consumption \( q_T \) goes down as \( i \) increases. However, what matters here is the degree of adjustment in total consumption. Comparison of simulations for both economies tell us that consumption in a mostly unconstrained economy reacts approximately 31\% more than in a mostly constrained economy.

What are the implications of these results? Let us consider a shock to the values of collateral that ends up affecting total consumption, \( q_T = q_u + q_c \). Assume for a moment that the monetary authority wished to offset these effects on consumption. Intuitively, if all borrowers are constrained, the monetary authority would need to move the interest rate by, say, \( |y|\% \) to achieve a readjustment of, say, \( |x|\% \) in \( q \). However, if borrowers are heterogeneous, then by proposition 3.4.1 we know that the monetary authority has to change the interest rate by only \( |z|\% < |y|\% \) in order to achieve the same adjustment in \( q \). The reason is that now there is a fraction of agents who adjust their \( q \) by more than the others in response to a change in the interest rate. Therefore, interest rates have a greater impact in economies where borrowers are heterogeneous than in economies...
where all of them are homogeneously constrained. The argument works inversely when we go from a situation with all borrowers unconstrained to a world with heterogeneous borrowers. Interest rates in this case would have a lower impact. I test this conjecture in one last experiment.

Imagine we are in the following scenario. A certain economy is in good health and the value of their assets is high, \( \hat{\delta} = 2.08 \). Then, \( \delta_c = 0.35407 \), which means that this economy is mostly populated with unconstrained borrowers (82.97%). Total consumption in this world is \( q_r = 0.1104 \) and agents have to pay a 4% nominal interest rate on loans. Suppose that market conditions are worsened and the value of collateral, \( \delta \), drops such that it leads to a decrease in total consumption of 1%. In this new scenario \( q_r = 0.1093 \) and 80% of the agents are unconstrained. If the central bank wants to increase the consumption level back to \( q_r = 0.1104 \), it must reduce \( i \) by 12.5% and set it at a nominal 3.5%.

Let us turn now to the case of an economy that initially has 85.06% of constrained borrowers, and total consumption is \( q_r = 0.0843 \) at the same nominal interest rate of 4%. After the new conditions affect the economy 87.03% of agents are still constrained and total consumption has fallen to \( q_r = 0.0834 \). In order to drive consumption back to \( q_r = 0.0843 \) the monetary authority reacts decreasing the interest rate by 14.75% to a level of 3.41%. Therefore, as we conjectured, interest rates have to be corrected by 2.25% more in the constrained economy in order to restore the same percentage decrease in consumption. In other words, interest rates seem to have a lower impact in economies where agents are all constrained than in economies where some are constrained and some are not. These results are available upon request.
3.5 Discussion

As opposed to papers like Kiyotaki and Moore (1997) or Ferraris and Watanabe (2007), where the asset used as collateral is also an input to production, land and capital respectively, in my paper this collateralizable asset will just be a storable good. Therefore, whereas agents in my model only borrow to finance consumption, agents in those other papers borrow also for investment purposes, and this is reflected in the equilibrium equations. While this modeling approach is in some sense simplifying, it has allowed me to focus on issues other than the optimal accumulation of capital and such, for which other frameworks are more suitable. This is closely related to the Mundell-Tobin effect. This theory suggests that in response to inflation people would hold less money balances and more of other assets, which would drive interest rates down. This is what would probably happen in models where financial assets or real assets that are inputs to production (capital, land) are used as collateral. In most of these models, the return of these assets is not directly linked to inflation as it is that of money. However, in this model, for a given distribution of $\delta$, inflation and interest rates usually move in the same direction.

More importantly, in response to inflation, people hold less money because it loses value, but also less of the other asset, which serves as collateral.

The reason is that in this paper the asset that serves as collateral and money are complements. That is, the asset is held partially for its return, but mainly for the purpose of acquiring money (loans). It has no other role relating to production or investment that makes it worth deviating resources from money balances and into those other assets in the presence of inflation. People will hold such asset, among other reasons, for its specific expected return. However, this return alone turns out not to be enough to hold the asset for pure investment purposes, as I explain below. Thus, if money loses value people are less willing to hold collateral for the motive of acquiring money. Notice also that I said that in most models where the Mundell-Tobin effect is present the return of
assets is not directly linked to inflation. Geromichalos, Licari, and Suárez-Lledó (2007) provide a model where that link is explicit and a direct relation between the return of assets and inflation is described. However, in that paper money and the real asset are substitutes and competing media of exchange. The authors show that when inflation increases the return of the asset has to decrease. In contrast, my model says that when inflation increases, \( i \) increases, the return of the asset is not affected. Both because the distribution of returns \( F(\delta) \) is given and because the price of the asset is 1. Nonetheless, it is the holdings of the asset will be affected by the value of money and the liquidity it can provide.

One of the central issues in the paper is that the valuation of the asset used as collateral is not conventional either. In fact that asset is already priced, and its price is that of the consumption good in the centralized market: 1. Moreover, in the asset pricing equation the value of the assets will reflect both its specific expected return and the liquidity that they can provide in contingencies when collateralized.

If we turn now our attention to the equilibrium equations we can observe that the right-hand side of equation (3.28) is always strictly positive. This implies that \( \gamma > \beta(1 + i) \), and since \( i > 0 \), then \( \gamma \) would always be greater than \( \beta \). This is so because there is a fraction of agents that are constrained. One might think that in such situation efficiency cannot be achieved and being away from the Friedman rule is the second best. However, as Rocheteau and Wright (2005) and Berentsen and Rocheteau (2003) argue, the theoretically optimal policy in that kind of framework would still be the Friedman rule. This conclusion goes through in my model as well. In fact, when \( \gamma = \beta \) everybody would just hold the amount of money that guarantees the efficient consumption.\(^{20}\) If this is the case no loans would be taken and agents would not even hold the asset for pure investment purposes since, according to equation (3.29), \( 1 > \beta \delta \).

\(^{20}\)Berentsen and Rocheteau (2003) maintain that the Friedman rule is optimal in search models with divisible money if buyers have all the bargaining power. In my model I contemplate take-it-or-leave-it offers by the buyer. The conclusions are the same.
This means that the specific expected return of the asset alone is not enough. In the case of other real or financial assets their price would also reflect the value of resale. This feature does not appear in the asset pricing equation in my model.

Proposition 3.4.1 proved analytically, and numerical simulations confirmed it, that unconstrained borrowers adjust their consumption significantly more than constrained borrowers do when the interest rate changes. The reason is that unconstrained agents can fully adjust their consumption to the new interest rate. The magnitude of the adjustment in unconstrained consumption is given by equation (3.20), \( u'(q_u) = 1 + i \). When \( i \) changes unconstrained borrowers can modify their choice of \( q \) until their marginal utility is again equal to the new marginal cost of borrowing. In other words, the variation in \( q \) by unconstrained is actually the maximum adjustment an agent can undertake. Constrained borrowers, however, see the correction in consumption limited by the value of their collateral, and in fact they will never be able to extract all the utility from the process of adapting to the new interest rate. Besides this consideration, another interesting advantage of having adopted interest rate targeting instead of money growth rate targeting is that I have been able to assess the direct influence of interest rates on real loan demand as one of the driving forces in my model, and whose mechanism included the tightening or loosening of borrowing constraints.

### 3.6 Conclusions

The purpose of this paper was to properly analyze the role of collateral and study its interaction with monetary policy an interest rates. In my model a credit market opens before trade occurs where idle balances can be deposited and so earn interest. Collateral is required in order to be able to borrow. In fact, the borrowing capacity depends on the value of the asset that is used as collateral. This value is specific to each agent, and in some cases it will be high enough to guarantee an optimal loan contract, and in other
cases agents will find themselves borrowing constrained.

A new asset is introduced that presents different features from those of fiat money and a standard financial asset and still coexists with money due to its value as a collateralizable asset in contingencies. This asset is less liquid but yields more return than fiat money, and is more liquid but with lower total return than a regular financial asset, but it is still valued for its "indirect" liquidity. In fact this makes asset pricing in this model a bit special, since the price of the asset will reflect not only its specific return but also the liquidity that can provide if collateralized. Money and the asset that serves as collateral are complements. Thus, in contrast with other models, when interest rates are higher the assets' return does not increase. Only the amount of asset held adjusts according to the value of money.

The Friedman rule is still the optimal policy. In fact, at $\gamma = \beta$ agents behave efficiently by taking no loans and storing nothing of the asset. Agents do not want to store for pure investment purposes since, according to equation (3.29), $1 > \beta^5$. That is, the specific expected return of the asset alone is not enough to motivate investment. Agents require an additional incentive to hold this asset, which is provided by its liquidity properties. On the other hand, the heterogeneity introduced on the borrower's side has important consequences. In the presence of a change in the interest rate, the consumption adjustment by unconstrained agents is greater than that of constrained borrowers. The reason is that unconstrained borrowers can fully adapt to a movement in the interest rate until their marginal utility equals the marginal cost of borrowing.

The main driving force behind this is the real loan demand by each type of agent. Nominal interest rate targeting by the monetary authority allows to capture the functioning of this mechanism and makes it a lot easier to solve the model. Numerical simulations confirm that the heterogeneity modeled in this paper may have implications for the behavior of interest rates. These have a larger impact in heterogeneous economies than in environments where we have a homogeneous population of constrained agents.
Chapter 4

Appendix I

Proof of Lemma 2.3.2. Using the definition of \( q(b) \) and applying the implicit function theorem in (2.3) yields

\[
\dot{q}'(b) = \frac{(R + \psi)}{z'(\dot{q})} = \frac{(R + \psi)(\theta u' + 1 - \theta)^2}{u'(\theta u' + 1 - \theta) - \theta(1 - \theta)(u - q)u''}.
\]

Hence, for all \( b < b^* \), we have \( \dot{q}'(b) > 0 \). Moreover, it can be easily checked that

\[
\lim_{b \to b^*} \dot{q}(b) = q^*.
\]

We conclude that \( \dot{q}(b) < q^* \) for all \( b < b^* \).

Proof of Lemma 2.3.3. The budget constraint of the centralized market implies that the agent can increase the hours worked in period \( t \) by \( dH_t \) and get \( db_{t+1} = \frac{dH_t}{\psi_t} \) units of the asset.\(^1\) In the next period the agent can decrease the amount of hours worked by \( dH_{t+1} = -(R + \psi_{t+1})db_{t+1} = -(R + \psi_{t+1}) \frac{dH_t}{\psi_t} \). The net utility gain of doing so is

\[
dU_t = -dH_t + \beta dH_{t+1} = -dH_t \left[ 1 - \beta \left( \frac{R + \psi_{t+1}}{\psi_t} \right) \right].
\]

\(^1\)The agent can either carry these assets into the decentralized market or keep them at home. For the purpose of this proof this choice is irrelevant, so we assume without loss of generality that she carries the asset into the decentralized market. Hence, we write \( db_{t+1} \) instead of \( da_{t+1} \).
Clearly, $\psi < \beta(R + \psi_{t+1})$ implies that $dU_t > 0$, and so in any equilibrium $\psi \geq \beta(R + \psi_{t+1})$. \hfill \Box

**Proof of Lemma 2.3.4.** From (3.8) and (2.4) we obtain

$$J_b(z_{t+1}) = \begin{cases} 
-\psi + \beta(R + \psi_{t+1}), & \text{if } b_{t+1} \geq b^* \\
-\psi + \beta(R + \psi_{t+1})e[\hat{q}_{t+1}(b_{t+1})], & \text{if } b_{t+1} < b^*.
\end{cases} \quad (4.1)$$

Lemma 2.3.3 implies that $J_b(z_{t+1}) \leq 0$ for every $b_{t+1} \geq b^*$. It is also straightforward to verify that $\lim_{b_{t+1} \to b^* -} J_b(z_{t+1}) < 0$ and $\lim_{b_{t+1} \to 0} J_b(z_{t+1}) > 0$. Moreover, for all $b_{t+1} < b^*$, we have

$$J_{b,b}(z_{t+1}) = \beta(R + \psi_{t+1}) e'(q) \hat{q}'(b) < 0,$$

where the inequality follows from Lemma 2.3.2. Combining these observations, we conclude that there exists a unique $b_{t+1} \in (0, b^*)$ that maximizes the objective function.

To show that the optimal choice also satisfies $b_{t+1} \leq \bar{b}$, we use the first-order condition which holds with equality,

$$\psi = \beta(R + \psi_{t+1})e[\hat{q}_{t+1}(b_{t+1})]. \quad (4.2)$$

Lemma 2.3.3 and (4.2) imply that $e[\hat{q}_{t+1}(b_{t+1})] \geq 1$, or equivalently $\hat{q}_{t+1}(b_{t+1}) \leq \bar{q}$. Then, using Lemma 2.3.2 and the definition of $\bar{q}$ we can conclude that $b_{t+1} \leq \bar{b}$.

Regarding the optimal choice of $a_{t+1}$, notice that this variable enters the objective function linearly and it is multiplied by the term $-\psi + \beta(R + \psi_{t+1})$. Hence, (2.5) is an immediate consequence of Lemma 2.3.3. \hfill \Box

**Proof of Lemma 2.4.4.** First, it is easy to verify that $\lim_{b_{t+1} \to 0} J_b(z_{t+1}) > 0$. Hence, $b_{t+1} > 0$, and so $\pi_{t+1} > 0$ regardless of the optimal choice of money holdings. If $m_{t+1} = 0$, then the proof coincides with that of Lemma 2.3.4. In what follows we consider solutions with

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Consider any pair \((m_{+1}, b_{+1})\) such that \(\pi_{+1} = \phi(m_{+1} + \mu M) + (R + \psi)b_{+1} \geq \pi^*\). From (2.14) and (2.15) we obtain

\[
\begin{align*}
J_m(z_{+1}) &= -\phi + \beta \phi_{+1} \leq 0, \\
J_b(z_{+1}) &= -\psi + \beta (R + \psi_{+1}) \leq 0,
\end{align*}
\]

where both inequalities follow from Lemma 2.4.3.

Next, consider any pair \((m_{+1}, b_{+1})\) such that \(\pi_{+1} < \pi^*\). For this range we have

\[
\begin{align*}
J_m(z_{+1}) &= -\phi + \beta \phi_{+1} e(q_{+1}), \\
J_b(z_{+1}) &= -\psi + \beta (R + \psi_{+1}) e(q_{+1}).
\end{align*}
\]

Since \(m_{+1}, b_{+1} > 0\), both first-order conditions hold with equality, implying that

\[
\frac{\phi}{\psi} = \frac{\phi_{+1}}{R + \psi_{+1}}.
\]

Using (4.5) we can re-write the objective function as

\[
J(z_{+1}) = [-\psi + \beta (R + \psi_{+1})]a_{+1} + \left(\beta - \frac{\phi}{\phi_{+1}}\right) \pi_{+1} + \beta \xi \left\{u[q_{+1}(\pi_{+1})] - \pi_{+1}\right\}.
\]

Differentiation with respect to \(\pi_{+1}\) yields

\[
J_{\pi}(z_{+1}) = \left(\beta - \frac{\phi}{\phi_{+1}}\right) + \beta \xi \left\{\frac{u'[q_{+1}(\pi_{+1})]}{x'[q_{+1}(\pi_{+1})]} - 1\right\}.
\]

Using the assumption that \(e'(q) < 0\) together with Lemma 2.4.2, it follows that the function \(\frac{u'[q_{+1}(\pi_{+1})]}{x'[q_{+1}(\pi_{+1})]}\) is strictly decreasing in \(\pi_{+1}\). This observation has two important implications. First, it can be shown that \(\lim_{\pi_{+1} \to \pi^*} J_{\pi}(z_{+1}) < 0\). Second, for all \(\pi_{+1} \in\)
(0, π∗), we have Jππ(z+1) < 0. Combining these results with (4.3) and (4.4), we can conclude that the optimal choice of π+1 is unique and it satisfies π+1 ∈ (0, π∗).

Finally, (4.6) indicates that at the optimum

\[ \beta \xi \left\{ \frac{u'(\bar{q}_{+1}(\pi+1))}{z'[\bar{q}_{+1}(\pi+1)]} \right\} = \frac{\phi}{\phi+1} - \beta \geq 0, \]

where the inequality follows from Lemma 2.4.3, and it implies that \( \bar{q}_{+1}(\pi+1) \leq \bar{q} \). Then, using Lemma 2.4.2 and the definition of \( \bar{q} \), it is straightforward to verify that the optimal choice of total real balances satisfies π+1 ≤ ̅π.

\[ \square \]

Proof of Lemma 2.4.5. In any equilibrium with \( a_{+1} = 0 \), we have

\[ \phi M + (R + \psi)T = z(q^S) = \phi_{+1} M_{+1} + (R + \psi_{+1})T, \]

which implies that

\[ M \left[ \phi - (1 + \mu)\phi_{+1} \right] + T(\psi - \psi_{+1}) = 0 \]  

holds in every period. In this paper we exclude cycles, i.e. equilibria in which (4.7) holds at every period and the signs of its two terms alternate with some specific frequency. Hence, (4.7) holds at every date either because both terms are zero, or one of these terms is positive, and the other one is negative (with the same absolute value) at every date. We next show that the latter case cannot be an equilibrium. Consider first the case in which \( \psi_{+1} > \psi \) in every period. This is not consistent with equilibrium unless the sequence \( \{\psi_t\}_{t=0}^{\infty} \) converges asymptotically to a finite real number. However, at steady state (2.17) implies that \( \psi_{+1} = -R + [\beta e(q^S)]^{-1} \psi \), i.e. the evolution of the price of the asset is given by a linear first-order difference equation. Hence, if \( \psi_{+1} > \psi \), the sequence \( \{\psi_t\}_{t=0}^{\infty} \) is unbounded and equilibrium collapses.

Using a similar argument it can be shown that if \( \psi_{+1} < \psi \) at every date, the sequence
\{((\phi M)_t)\}_{t=0}^{\infty} \text{ grows without bound. We conclude that for the class of equilibria under consideration, } \psi \text{ is constant across time and } \phi - (1 + \mu)\phi_{+1} = 0, \text{ or equivalently } \phi/\phi_{+1} = 1 + \mu. \qed
Chapter 5

Appendix II

In equilibrium \( q = \phi m_b \) for any \( i > 0 \). Suppose there exists an equilibrium with \( i > 0 \) and \( q < \phi m_b \), where \( m_b = m + b \). At any equilibrium like this the amount consumed \( q \) that solves the problem in market 2 is optimal. That is, \( q = q^* \) and is defined by \( u'(q^*) = c'(q^*) = 1 \). Remember that the problem to be solved in the centralized market is

\[
\max_{x,h,m_{+1},a_{+1}} U(x) - h + \beta V_1(m_{+1}, a_{+1}) \\
\text{s.t. } x + \phi m_{+1} + a_{+1} = h + \phi m - \phi(1 + i)l + \delta a.
\]

Also, the total utility in a certain period of time is given by

\[
U(x, h, q) = U(x) - h + nu(q) - (1 - n)q.
\]

Thus, since \( q = q^* < \phi m + \phi \delta \), and given the assumptions on the utility functions, we can always consider a stationary equilibrium in which someone decided to reduce her debt by an amount \( dl \), small enough such that she is still in the neighborhood of the optimal \( q \), therefore still able to consume \( q^* \). The budget constraint would still be non-binding.
and the variation in total utility would be
\[ dU = \phi(-dl) - \phi(1 + i)(-dl) = \phi(1 + i)dl - \phi dl = i \, dl \geq 0. \]

Then, \( dU > 0 \) for any \( i > 0 \). Therefore we could not have any equilibrium with \( i > 0 \) and \( q < \phi m_b \), and the buyer's budget constraint is always binding in equilibrium. 

\textbf{Proof of Lemma 3.3.1.} Define the following function,
\[ f(\delta) = u' \left( z + \frac{\delta a}{1 + i} \right) - 1 + i, \]
where the term \( 1 + i \) is constant, \( i \) is exogenously fixed in the model. From the maintained assumptions of the model we know that \( u'(0) = +\infty, u'(+\infty) = 0 \), and that \( u' \) is strictly decreasing in \( q \), since \( u'' < 0 \). Thus, for a sufficiently small \( \delta \) we can have a \( q \) small enough so that \( u'(z + \frac{\delta a}{1 + i}) > 1 + i \). Inversely, for a high enough \( \delta \) the corresponding \( q \) would be such that \( u'(z + \frac{\delta a}{1 + i}) < 1 + i \).

Therefore, since \( u' \) is strictly decreasing and continuous and \( 1 + i \) is constant, by the Intermediate Value Theorem there exists a value \( \delta_c \) such that \( f(\delta_c) = 0 \). This value is thus defined by
\[ u' \left( z + \frac{\delta_c a}{1 + i} \right) = 1 + i. \]

We also know from the equilibrium analysis that for agents whose \( \delta < \delta_c \) then \( u'(q) > 1 + i \), and as \( \delta \) is higher \( u'(z + \frac{\delta a}{1 + i}) \to 1 + i \). Thus for \( \delta < \delta_c \) we have \( f(\delta) > 0 \) and for \( \delta > \delta_c \) then \( f(\delta) < 0 \). The latter is the reason why those agents with high \( \delta \) always choose \( l_b = \frac{\delta_c a}{\phi(1+i)} \) and never higher.

Now, suppose \( \delta_c \) does not lie in the interior of the distribution, i.e. \( \hat{\delta} < \delta_c \). Then all the agents in the economy would be constrained. However, we know that it cannot be
the case that either \( \tilde{\delta} \), or \( i \), or both are such that

\[
u'(z + \frac{\tilde{\delta} a}{1 + i}) > 1 + i.
\]

For no matter how small the interest rate is a sufficiently high \( \tilde{\delta} \) can always be found such that, since \( q \) is decreasing in \( i \), \( u' \) is strictly decreasing in \( q \), and \( u'(\infty) = 0 \), then there will always be some agent for which \( f(\tilde{\delta}) < 0 \). Imagine the worst scenario where \( i \) is close to zero. Then we can always find economies with that level of interest rate whose \( \tilde{\delta} \) is large enough so that \( u'[q(\tilde{\delta})] \to 0 \), then \( u'[q(\tilde{\delta})] < 1 + i = 1 \). Therefore, there will always be a positive mass of unconstrained agents. It is those economies that I am interested in.

\[\square\]

**Proof of Proposition 3.3.1.** Since \( \gamma \) is uniquely pinned down by (3.28), the equilibrium boils down to two equations in two unknowns: (3.29) and (3.22) in \( z \) and \( a \). Thus, in order to prove existence and uniqueness of an equilibrium basically three things need to be shown: that these two curves are strictly monotonic and have different slopes, and that they actually intersect.

First, I want to obtain the slope \( dz/da \). It turns out that while total differentiation is enough to draw conclusions from (3.22), the implicit function theorem needs to be applied to equation (3.29). If we express equation (3.29) as

\[
r(z, a) = \int_{0}^{\delta(z,a)} \delta n \left( \frac{u'(z + \frac{\delta a}{1 + i})}{1 + i} - 1 \right) dF(\delta) - \frac{1 - \beta \delta}{\beta} = 0
\]

then, according to this theorem, such slope would be

\[
\frac{dz}{da} = -\frac{\frac{\partial r(z,a)}{\partial a}}{\frac{\partial r(z,a)}{\partial z}} = -\frac{\int_{0}^{\delta} \delta \frac{u'(z + \frac{\delta a}{1 + i})}{1 + i} dF(\delta)}{\int_{0}^{\delta} \delta n \frac{u''(z + \frac{\delta a}{1 + i})}{1 + i} dF(\delta)} < 0.
\]

We know that from the assumptions that \( u'' < 0 \). Then, both numerator and denominator
are strictly negative. This implies that the slope $dz/da$ is strictly negative. Therefore, I have shown that equation (3.29) has negative slope and is strictly monotonic.

Let us now turn to equation (3.22). Applying total differentiation to this expression and by the Leibniz's rule we obtain

$$\frac{dz}{da} = \frac{n}{1-n} \left[ \int_0^{\delta_c(z,a)} \frac{\delta}{1+i} dF(\delta) + \int_{\delta_c(z,a)}^{\delta} \frac{\delta_c}{1+i} dF(\delta) \right] > 0. \quad (5.2)$$

Therefore, it has been shown that equation (3.22) has positive slope and is strictly monotonic. Thus, these two curves have slopes with opposite sign and, as it is obvious from inspection of equations (5.1) and (5.2), the magnitude of the slopes is clearly different. It only remains to show that they actually intersect.

In the model the only feasible choices of real balances and asset holdings are those with positive values. However, we have to make sure that this is indeed the case and these two curves intersect in the first quadrant of the plane $(z,a)$. In addition, even though we know that one curve is upward sloping and the other one is downward sloping, we have to rule out cases where, for instance, the upward sloping curve would start above the other one and so they would not intersect. Notice that at $z = 0$, equation (3.22) becomes

$$\left[ \int_0^{\delta_c(z=0,a)} \frac{\delta}{1+i} dF(\delta) + \int_{\delta_c(z=0,a)}^{\delta} \frac{\delta_c}{1+i} dF(\delta) \right] a = 0,$$

and since the term in brackets is strictly positive, when $z = 0$ equation (3.22) tells us that $a = 0$. On the other hand, equation (3.29) can also be analyzed at $z = 0$:

$$\frac{1-\beta \delta}{\beta} = \int_0^{\delta_c(z=0,a)} \delta_n \left( \frac{u'}{1+i} - 1 \right) dF(\delta),$$

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which can be rewritten as

\[
\frac{1 - \beta \delta}{\beta} + \int_0^{\delta_c(z=0,a)} \delta n dF(\delta) = \frac{n}{1 + i} \int_0^{\delta_c(z=0,a)} \delta u' \left( \frac{\delta a}{1 + i} \right) dF(\delta).
\]

Notice that we integrate over \( \delta \) and \( u' \) is monotonically decreasing. Thus, \( a \) can always be factored out of the integral. Also, both sides of the equation are strictly positive. Therefore, at \( z = 0 \), we can conclude that \( a > 0 \). That is, the choice of \( a \) that satisfies equation (3.29) is positive. This proves that in the plane \((z, a)\) equation (3.29) starts from above (3.22) and strictly decreases, while (3.22) strictly increases. It turns out quite hard to develop the last expression any further in order to obtain a cleaner solution for \( a \). However, examples of how this last part of the proof would work with a particular functional form for \( u' \) can easily be provided.

There is still one last case that needs to be considered. It is the rare case of an asymptote. Even though the decreasing curve started from above the increasing one it could be that both converged to a horizontal asymptote that lies between them. If this was the case, then these curves would never intersect. Details of the proof that shows why such situation can be ruled out are available upon request. Therefore, we can finally conclude that the two curves will intersect and a unique equilibrium exists. \( \square \)

**Proof of Proposition 3.4.1.** Recall that, in equilibrium, \( q_u = z + l_u^R \), and \( q_c = z + l_c^R \); where \( l_u^R = \frac{\delta a}{1 + i} \), and \( l_c^R = \frac{\delta a}{1 + i} \). If I differentiate these equations I obtain

\[
\frac{dq_c}{di} = \frac{dz}{di} + \frac{\delta}{1 + i} \frac{da}{di} - \frac{\delta a}{(1 + i)^2} < 0,
\]

\[
\frac{dq_u}{di} = \frac{dz}{di} + \frac{\delta_c}{1 + i} \frac{da}{di} - \frac{\delta_c a}{(1 + i)^2} < 0.
\]

Both are clearly negative, given that \( \frac{dz}{di} < 0 \) and \( \frac{da}{di} < 0 \). Also, even though \( \delta_c \) may in principle depend on \( i \), and it would be analytically tedious to prove it, numerical
simulations show that $d\delta_c/di = 0$. Let us then compare those two values,

\[
\frac{dq_u}{di} - \frac{dq_c}{di} = \frac{dz}{di} + \frac{\delta_c}{1 + i} \frac{da}{di} - \frac{\delta_c a}{(1 + i)^2} \left( \frac{dz}{di} + \frac{\delta}{1 + i} \frac{da}{di} - \frac{\delta a}{(1 + i)^2} \right) \\
= \frac{da}{di} \left( \frac{\delta_c}{1 + i} - \frac{\delta}{1 + i} \right) + \frac{\delta a}{(1 + i)^2} - \frac{\delta_c a}{(1 + i)^2} < 0. \quad (5.5)
\]

As I showed before, the previous expression is always negative because for any constrained agent $\delta < \delta_c$. Notice that, from equations (5.3) and (5.4) above we know that $dq_u/di < 0$, and $dq_c/di < 0$. This together with (5.5) shows that

\[
\left| \frac{dq_u}{di} \right| > \left| \frac{dq_c}{di} \right|. 
\]
Bibliography


