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The Hopf Fibrations are Characterized by Being Fiberwise Homogeneous

Abstract

The Hopf fibrations of S^{2n+1} by great circles, S^{4n+3} by great 3-spheres, and S^{15} by great 7-spheres are the prototypical examples of fibrations of spheres by subspheres, and have a number of features that make them attractive. For example, they have parallel fibers, and what's more, they are the only fibrations of spheres by subspheres which have this property. They also have a great deal of symmetry. They are fiberwise homogeneous: for any two fibers, there is an isometry of the total space taking the first given fiber to the second, and preserving fibers. The main result of this dissertation is that the Hopf fibrations are characterized by this property among all fibrations of round spheres by subspheres. That is, they deserve to be promoted to our attention for being so symmetric. We develop an intuition for the property of fiberwise homogeneity by classifying all the fibrations of Euclidean and Hyperbolic 3-space by geodesics having this feature. They are analogues of the Hopf fibration of the 3-sphere by great circles. We also find new nonstandard examples of fiberwise homogeneous fibrations of the Clifford Torus $S^3 \times S^3$ in the 7-sphere by great 3-spheres which are not restrictions of the Hopf fibration. Finally, we prove a local version of our main result: that open sets in the 3-sphere which are fibered by great circles and are locally fiberwise homogeneous are subsets of the Hopf fibration.

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Haggai Megged Nuchi

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ABSTRACT

THE HOPF FIBRATIONS ARE CHARACTERIZED BY BEING FIBERWISE HOMOGENEOUS

Haggai Megged Nuchi

Herman Gluck

The Hopf fibrations of S^{2n+1} by great circles, S^{4n+3} by great 3-spheres, and S^{15} by great 7-spheres are the prototypical examples of fibrations of spheres by subspheres, and have a number of features that make them attractive. For example, they have parallel fibers, and what's more, they are the only fibrations of spheres by subspheres which have this property. They also have a great deal of symmetry. They are *fiberwise homogeneous*: for any two fibers, there is an isometry of the total space taking the first given fiber to the second, and preserving fibers. The main result of this dissertation is that the Hopf fibrations are characterized by this property among all fibrations of round spheres by subspheres. That is, they deserve to be promoted to our attention for being so symmetric. We develop an intuition for the property of fiberwise homogeneity by classifying all the fibrations of Euclidean and Hyperbolic 3-space by geodesics having this feature. They are analogues of the Hopf fibration of the 3-sphere by great circles. We also find new nonstandard examples of fiberwise homogeneous fibrations of the Clifford Torus $S^3 \times S^3$ in the 7-sphere by great 3-spheres which are not restrictions of the Hopf fibration. Finally, we prove a local version of our main result: that open sets in the 3-sphere which are fibered by great circles and are locally fiberwise homogeneous are subsets of the Hopf fibration.

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1 Introduction

1.1 Background

In two papers published in 1931 and 1935, Heinz Hopf discovered several beautiful fibrations of spheres by great subspheres: fibrations of S^{2n+1} by great circles, S^{4n+3} by great 3-spheres, and S^{15} by great 7-spheres [10,11]. There are many fibrations of spheres by smooth subspheres, but Hopf's fibrations in particular have a number of interesting properties that promote them to our attention.

For one, their fibers are parallel. This property furnishes the ambient spaces with Riemannian submersions to base spaces, and the base spaces are spheres and projective spaces, all with their standard metrics. This property really is special to the Hopf fibrations. Escobales, Ranjan, and Wolf [6,15,21] independently proved that if a fibration of a round sphere by great subspheres has parallel fibers, then it's a Hopf fibration. Grove and Gromoll [8] and Wilking [20] extended that result and proved that any fibration of a round sphere by parallel fibers, not just one with great subsphere fibers, is a Hopf fibration. The take-home message is that the Hopf fibrations really are worthy of being brought to our attention: there is a mathematical property which picks them out, on top of their aesthetic appeal.

The Hopf fibrations also have a great deal of symmetry. While we expect a generic fibration of S^{2n+1} (respectively S^{4n+3} , S^{15}) by circles (respectively 3-spheres, 7-spheres) not to have many symmetries, the Hopf fibrations in these dimensions have symmetry

group $U(n+1) \cup_c U(n+1)$ (respectively $Sp(n+1) \times Sp(1)/\mathbb{Z}_2$, $Spin(9)$). The symmetry groups of the Hopf fibrations are large enough to take any point in the ambient space to any other point while taking fibers to fibers. In other words, the symmetry group of a Hopf fibration acts transitively on the total space. We call the Hopf fibrations *pointwise homogeneous*: for any two points in the total space, there is an isometry taking one to the other and preserving fibers. The Hopf fibrations also have the weaker symmetry property of being *fiberwise homogeneous*:

Definition 1.1.1. Let \mathcal{F} be a fibration of a Riemannian manifold (M, g) . We say that \mathcal{F} is *fiber-wise homogeneous*, abbreviated *FWH*, if for any two fibers $F_1, F_2 \in \mathcal{F}$, there is an isometry T of (M, g) satisfying

1. $\forall F \in \mathcal{F}, \exists F' \in \mathcal{F}$ s.t. $T(F) = F'$
2. $T(F_1) = F_2$

The main result of this thesis is that the Hopf fibrations are unique among fibrations of round spheres by subspheres in being fiberwise homogeneous. In other words, having a great deal of symmetry really is an appropriate reason to promote the Hopf fibrations to our attention, and we do not miss any other examples by paying attention to this property. We achieve several more results related to this theme, which we will shortly describe.

1.2 Organization

This thesis is organized in the following way. Later in this chapter we introduce some definitions and lemmas which will be useful throughout the rest of this work. In Chapter

2, we describe the fiberwise homogeneous fibrations of Euclidean and Hyperbolic 3-space by geodesics. We prove

Theorem A. *The FWH fibrations of Euclidean 3-space by geodesics form a 1-parameter family. The FWH fibrations of Hyperbolic 3-space by geodesics form a 2-parameter family, plus one additional fibration.*

Our motivation in studying these fibrations is to develop an intuition for what the property “fiberwise homogeneous” means. Spaces with dimension 3 and fibers of dimension 1 are simple enough that one has a chance of visualizing the objects under discussion.

In Chapter 3 we prove our main result.

Theorem B. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of S^n with its standard metric by subspheres of dimension k , i.e.*

- S^3, S^5, S^7, \dots by circles
- $S^7, S^{11}, S^{15}, \dots$ by 3-spheres
- S^{15} by 7-spheres.

Then \mathcal{F} is a Hopf fibration.

Technically, the regularity we need on the fibration \mathcal{F} , in order for the proof to work, is that the fibration should be continuous and the fibers should be differentiable: the proof requires that each fiber have a well-defined tangent space at each point.

To a small extent, it’s a little disappointing that Theorem B is true. If it were false, we would find ourselves with a beautiful new example of a highly symmetric fibration

of a sphere by subspheres. The following theorem, appearing in Chapter 4, is a partial consolation prize.

Theorem C. *There is a 1-parameter family of FWH fibrations of the Clifford Torus $S^3 \times S^3$ in S^7 by great 3-spheres which are not restrictions of the Hopf fibration of S^7 by great 3-spheres.*

Even though the only FWH fibrations of the 7-sphere by 3-spheres are the Hopf fibrations, when we restrict the total space to the Clifford Torus $S^3 \times S^3$, we find exciting new examples of FWH fibrations.

A much more desirable and stronger form of Theorem B would be the following local version. Take any open set in S^n with the standard metric, fibered by subspheres, and suppose that this fibration is locally fiberwise homogeneous: for any two fibers, there are open neighborhoods about each one such that there's an isometry taking the first neighborhood to the second, and taking the first fiber to the second, while preserving all the fibers in those neighborhoods. Then we would hope to prove that we were looking at an open subset of a Hopf fibration. This theorem seems presently out of reach. But in Chapter 5, we do prove the following partial result.

Theorem D. *Let \mathcal{F} be a locally FWH fibration of a connected open set in the 3-sphere by great circles. Then \mathcal{F} is a portion of a Hopf fibration.*

In the setting of fibrations of the 3-sphere by great circles, we have available to us a very useful tool due to Gluck and Warner [7], which has no analogue in higher dimensions. It is this tool which we use to prove Theorem D.

1.3 Preliminaries

We gave the definition of “fiberwise homogeneous” earlier. What follows are a few definitions and lemmas which will be useful to us.

Definition 1.3.1. Let \mathcal{F}_1 and \mathcal{F}_2 be two fibrations of a Riemannian manifold (M, g) . We say that \mathcal{F}_1 and \mathcal{F}_2 are *equivalent* if $\exists T \in \text{Isom}(M, g)$ such that $\mathcal{F}_2 = T(\mathcal{F}_1)$.

Lemma 1.3.2. *If \mathcal{F}_1 is equivalent to \mathcal{F}_2 and \mathcal{F}_1 is FWH, then \mathcal{F}_2 is also FWH. Moreover, the groups of isometries acting on \mathcal{F}_1 and \mathcal{F}_2 are conjugate to one another in the full isometry group of (M, g) .*

Proof. If $G \subset \text{Isom}(M, g)$ acts isometrically and transitively on the fibration \mathcal{F}_1 , and $\mathcal{F}_2 = T(\mathcal{F}_1)$, then TGT^{-1} acts isometrically and transitively on \mathcal{F}_2 . \square

The following lemma will be quite useful to us. If we have a FWH fibration \mathcal{F} of a Riemannian manifold M , it will allow us to assume that a group G acting transitively on \mathcal{F} is a connected Lie subgroup of $\text{Isom}(M)$.

Lemma 1.3.3. *Let M be a connected Riemannian manifold, and let G be a subgroup of $\text{Isom}(M)$. Denote by \overline{G}_0 the identity component of the closure of G . Suppose G acts transitively on a smooth fibration \mathcal{F} of M . Then, \overline{G}_0 acts transitively on \mathcal{F} as well.*

Proof. Let G preserve the fibers of \mathcal{F} . Let $\{g_n\}_{n=1}^\infty \subset G \subseteq \text{Isom}(M)$, and let $g_n \rightarrow g \in \text{Isom}(M)$. If each g_n preserves each fiber of \mathcal{F} , then the limit g clearly does as well. Thus \overline{G} preserves the fibers of \mathcal{F} . Also $G \subseteq \overline{G}$, so \overline{G} acts transitively on \mathcal{F} as well.

Now let G be a closed disconnected Lie subgroup of $\text{Isom}(M)$ acting transitively on \mathcal{F} . Let the manifold B be the base of \mathcal{F} . Then B has the structure of a connected

homogeneous space, and hence is diffeomorphic to G/H , where H is the isotropy subgroup of G fixing a point. Let G_i be the connected components of G , G_0 the identity component. The subgroup H intersects every G_i , or else B would be disconnected, so there are $g_i \in H \cap G_i$ for all i . Then $g_i H = H$, from which it follows that the image of G_0 intersects the image of every G_i in G/H . But as g_0 ranges across all elements of G_0 , $g_0 g_i$ ranges across all elements of G_i , so $g_0 g_i H = g_0 H$ and the image of G_0 is identical to the image of each G_i . Thus the image of G_0 is all of G/H . Therefore the identity component of G acts transitively on B and hence on \mathcal{F} . □

2 Fiberings the 3-dimensional space forms by geodesics

2.1 Introduction

In this chapter, we describe all the fiberwise homogeneous fibrations of Euclidean and Hyperbolic 3-space by geodesics, and prove that there are no others. We can think of these fibrations as analogues of the Hopf fibration of the 3-sphere by great circles: highly symmetric fibrations of a 3-dimensional space form by geodesics, where we take zero and negative constant curvature instead of positive curvature.

Later on, in Chapter 3, we will prove that the Hopf fibrations are the only fibrations of spheres by subspheres which are fiberwise homogeneous. In the lowest dimension, that statement reduces to saying that the Hopf fibration of the 3-sphere by great circles is the only FWH fibration of the 3-sphere by circles.

Our motivation is simply to develop an intuition for what the property of fiberwise homogeneity looks like. When our total space is 3-dimensional and the fibers are 1-dimensional, we have a better chance of visualizing concrete examples. We choose the simply connected 3-dimensional space forms as the total spaces, and geodesics as fibers, because they are the closest analogues of the Hopf fibrations.

The main theorem of this chapter is the following:

Theorem A. *The FWH fibrations of Euclidean 3-space by geodesics form a 1-parameter family. The FWH fibrations of Hyperbolic 3-space by geodesics form a 2-parameter family, plus one additional fibration.*

Remark 2.1.1. As part of the proof of Theorem A, we construct the fibrations explicitly. See Figure 2.1 for pictures of typical examples. Technically, we classify fibrations up to equivalence; we don't consider e.g. a fibration by parallel horizontal lines as different from a fibration by parallel vertical lines.

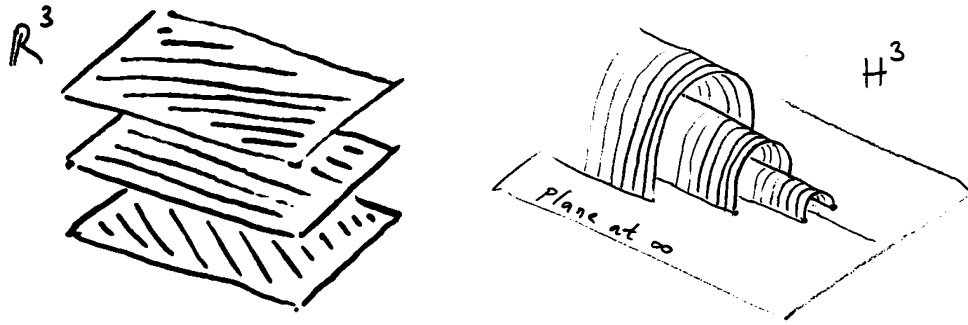


Figure 2.1: Typical FWH fibrations of \mathbb{R}^3 , left, and H^3 , right. In the pictured fibration of \mathbb{R}^3 , we layer Euclidean space by parallel planes, and fiber each plane by parallel lines whose direction changes at a constant rate as we move from plane to plane. In the pictured fibration of H^3 , geodesics form nested tunnels over a line in the plane at infinity in the upper half-space model of H^3 .

Here is a rough outline of the classification of the FWH fibrations of Euclidean and Hyperbolic 3-space by geodesics. Let X denote either E^3 or H^3 (the proof has a similar form in both cases). Start with a hypothetical FWH fibration \mathcal{F} of X by geodesics. Let G be a subgroup of $\text{Isom}(X)$ which acts transitively on the fibers of \mathcal{F} . By Lemma 1.3.3, we may assume without loss of generality that G is connected and closed. The conjugacy classes of closed connected subgroups of $\text{Isom}(X)$ are well understood in the literature. By letting G range across all closed connected subgroups of $\text{Isom}(X)$, we can analyze the geometry of G to discover which FWH fibrations by geodesics it could possibly act on.

For example, 1-parameter groups of isometries are too small to act transitively on \mathcal{F} , and the full isometry group is too large to preserve \mathcal{F} . Like Goldilocks in her interactions with the three bears, we check each intermediate group of isometries to see if it is “just right.” In this way we discover all FWH fibrations by geodesics. Finally, after eliminating redundancies in the list (some may be equivalent to one another), we arrive at the list of FWH fibrations of $\text{Isom}(X)$ by geodesics, up to equivalence.

2.2 Euclidean 3-space

In order to carry out the classification which was just briefly described, we need a description of the conjugacy classes of closed connected subgroups of the isometry group of E^3 . A full list can be found in [2]. We duplicate the list here, and describe each element in detail. After describing these groups, we will give the list of FWH fibrations of Euclidean 3-space, followed by the proof that our list is exhaustive.

In the following list, each item is written as “Name of group, dimension of group. Description of group.” For the name of the group, we either choose a name from [2], or invent our own; we try to give descriptive names where possible. In the description of the group, we choose a concrete representative of the conjugacy class, acting on the usual coordinates (x, y, z) of E^3 . We remark that [2] actually describes the subalgebras of the Lie algebra of $\text{Isom}(E^3)$, but these are in one-to-one correspondence with closed connected subgroups of $\text{Isom}(E^3)$ via exponentiation.

Closed connected groups of isometries of E^3 :

- $\{1\}$, dim 0. The trivial group.

- $T(1)$, dim 1. The group of translations along the z -axis.
- $SO(2)$, dim 1. The group of rotations around the z -axis.
- $\overline{SO(2)}_t$, dim 1. The group of screw-translations along the z -axis with pitch t . Distinct $t \neq 0$ give distinct conjugacy classes. On the Lie algebra level, this group is generated by $-t\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. The notation is meant to remind us that this group is the universal covering group of $SO(2)$. These are distinct for different t , except that $\overline{SO(2)}_t$ is conjugate to $\overline{SO(2)}_{-t}$ by an orientation-reversing isometry of E^3 .
- $SO(2) \times T(1)$, dim 2. The group of rotations around and translations along the z -axis.
- $T(2)$, dim 2. The group of translations in the xy -plane.
- $\overline{E(2)}_t$, dim 3. A universal covering group of the isometries of the Euclidean plane. It consists of translations in the xy -plane together with screw-translations along the z -axis with pitch t . It's generated by $T(2)$ and $\overline{SO(2)}_t$. Just as with $\overline{SO(2)}_t$, these are distinct for different t , except that $\overline{E(2)}_t$ is conjugate to $\overline{E(2)}_{-t}$ by an orientation-reversing isometry of E^3 .
- $T(3)$, dim 3. The group of translations of Euclidean 3-space.
- $E(2)$, dim 3. The group of translations and rotations of the xy -plane.
- $SO(3)$, dim 3. The group of rotations around the origin $(0, 0, 0)$.
- $E(2) \times T(1)$, dim 4. The group generated by translations and rotations of the xy -plane, and translations in the z -axis.

- $E(3)$, dim 6. The identity component of the full isometry group of E^3 . Generated by translations and rotations.

Now we describe all FWH fibrations of E^3 by geodesics. Let \mathcal{F}_t (See Figure 2.2) be the fibration consisting of the integral curves of the vector field

$$v_t = \cos(tz) \frac{\partial}{\partial x} + \sin(tz) \frac{\partial}{\partial y}.$$

In other words, layer E^3 by parallel planes. Fiber each plane by parallel straight lines, but vary the angle of the lines at a constant rate as we move from plane to plane. The variable t controls the rate of change of the angle.

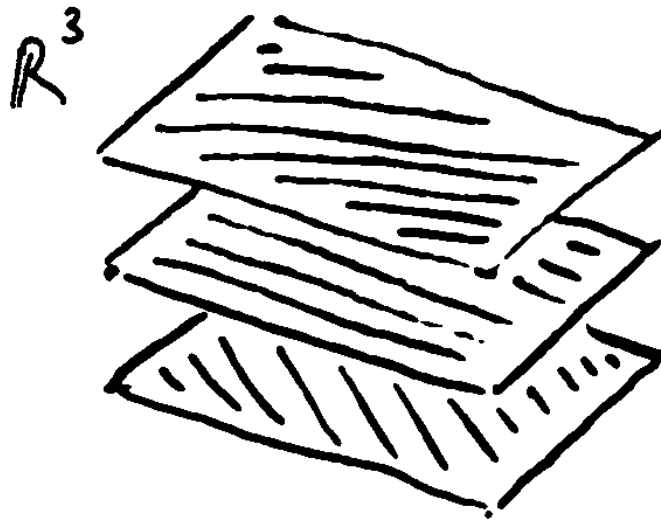


Figure 2.2: A fiberwise homogeneous fibration of \mathbb{R}^3 .

Theorem 2.2.1. *The fiber-wise homogeneous fibrations of Euclidean 3-space by straight lines, up to equivalence, are precisely \mathcal{F}_t for each $t \in \mathbb{R}, t \geq 0$.*

Proof. First we will show that each \mathcal{F}_t is FWH. Then we will show that no other fibrations are FWH. Then we will show that no \mathcal{F}_t is equivalent to $\mathcal{F}_{t'}$ for $t, t' \geq 0$ and $t \neq t'$.

First note that $T(3)$ (for example) acts transitively on \mathcal{F}_0 , and that $\overline{E(2)}_t$ acts transitively on \mathcal{F}_t for $t \neq 0$, so that these fibrations really are FWH.

Now, let \mathcal{F} be a FWH fibration \mathcal{F} by geodesics, and let G be a group of isometries of E^3 acting transitively on \mathcal{F} . Without loss of generality, by Lemma 1.3.3 we may assume that G is closed and connected. By replacing \mathcal{F} with an equivalent fibration if necessary (Lemma 1.3.2), we may also assume that G is exactly the representative of its conjugacy class which is listed in the table above. We will now show that \mathcal{F} must be \mathcal{F}_t for some t .

- Suppose $G = \{1\}$, $T(1)$, $SO(2)$, or $\overline{SO(2)}_t$. Then the dimension of the group is too small to act transitively on \mathcal{F} , because the image of a single fiber under G will be either 1- or 2-dimensional. This is a contradiction.
- Suppose $G = SO(2) \times T(1)$. Consider the fiber F through the origin. If F is the z -axis, then G fixes it and hence cannot act transitively on \mathcal{F} . If F is not the z -axis, then a rotation in G which fixes the origin will take F to a different line through the origin, and hence G does not preserve \mathcal{F} . This is a contradiction.
- Suppose $G = T(2)$, and consider the fiber F through the origin. If F lies in the xy -plane, then G cannot take F out of the xy -plane and hence does not act transitively on \mathcal{F} . If F is transverse to the xy -plane, then the image of F under G is a fibration by parallel straight lines and hence is equivalent to \mathcal{F}_0 .
- Suppose $G = \overline{E(2)}_t$. Consider the fiber F through the origin. If F is transverse to the xy -plane, then a screw-translation around the origin by an angle of π will take F to a fiber which intersects F transversely, and hence G does not preserve \mathcal{F} , a contradiction. If F lies in the xy -plane, then the image of F under G is equivalent

to the fibration \mathcal{F}_t : translations move it to its parallel translates in the xy -plane, and screw-translations move the xy -plane in the z -direction while controlling the angle of the lines.

- Suppose $G = T(3)$. Let F be any fiber of \mathcal{F} . The image of F under G is equivalent to \mathcal{F}_0 , so $\mathcal{F} = \mathcal{F}_0$.
- Suppose $G = E(2)$ or $G = E(2) \times T(1)$. Consider the fiber F through the origin. If F is not the z -axis, then a rotation of the xy -plane about the origin takes F to a line which intersects F transversely, a contradiction. Thus F is the z -axis, and the image of F under G is equivalent to \mathcal{F}_0 .
- Suppose $G = SO(3)$ or $G = E(3)$. Consider the fiber F through the origin. Apply a rotation about the origin which does not fix F . Then the image of F is a line which intersects F transversely, and so G does not preserve \mathcal{F} , a contradiction.

Thus we conclude that \mathcal{F} is equivalent to \mathcal{F}_t for some t .

Finally we show that no two such fibrations are equivalent. Observe that the unit vector field v_t along \mathcal{F}_t is an eigenfield for the curl operator on E^3 with eigenvalue t . Curl eigenfields remain curl eigenfields after an isometry and the eigenvalue is preserved up to sign (depending on the orientation of the isometry), so no two such fibrations are equivalent. □

2.3 Hyperbolic 3-space

We classify FWH fibrations of Hyperbolic 3-space by geodesics by the same method as for Euclidean 3-space. We start with a description of the closed connected groups of

isometries of Hyperbolic 3-space. The list below can be found in [17]. Just as with the analogous list for Euclidean 3-space, the list in [17] is actually of Lie subalgebras of the Lie algebra corresponding to the isometry group of Hyperbolic 3-space. To be precise, the theorem actually discusses the Lie algebra for the group $SO(1, 3)$, but this group is isomorphic to the isometry group of H^3 .

In the following list, each item is written as “Name of group, dimension of group. Description of group.” In the description of the group, we choose a concrete representative of the conjugacy class. We will use some basic facts of the geometry of Hyperbolic 3-space; see Thurston [19] for more information. Sometimes it will be useful for us to use the upper half-space model of Hyperbolic 3-space, which we give the coordinates $(z, x) \in \mathbb{C} \times \mathbb{R}^+$, and we denote points on the plane at infinity as $(z, 0)$. Other times we may also use the Poincaré disk model, which we give the coordinates $v = (x, y, z) \in \mathbb{R}^3$ with $\|v\| < 1$, and the sphere at infinity consisting of those v with $\|v\| = 1$.

We remind the reader that geodesics in either of those models consist of straight lines or arcs of circles, both of whose endpoints meet the plane at infinity or sphere at infinity orthogonally. In the upper half-space model, that includes vertical lines (i.e. sets of the form $\{z\} \times \mathbb{R}^+$). We also make liberal use of the extremely useful fact that orientation-preserving isometries of Hyperbolic 3-space extend to Möbius transformations of the sphere or plane at infinity (see [19]); in what follows we will describe a group of hyperbolic isometries by its effect on the plane or sphere at infinity.

Closed connected groups of isometries of H^3 :

- $\{1\}$, dim 0. The identity.

- *Hyp*, dim 1. The group of dilations of the plane at infinity in the upper half-space model. These are known as hyperbolic transformations.
- *Par*, dim 1. The group of translations of the plane at infinity in the upper half-space model by real directions. These are known as parabolic transformations.
- *Ell*, dim 1. The group of rotations of the plane at infinity about the origin. These are known as elliptic transformations.
- *Lox*, dim 1. The group of screw-dilations of the plane at infinity about the origin. The infinitesimal generator of this group is a linear combination of the infinitesimal generators for the hyperbolic and elliptic groups $H(1)$ and $E(1)$. There are actually an infinite number of conjugacy classes of groups of this type, corresponding to different ratios of this linear combination. “*Lox*” is for loxodromic.
- $T(2)$, dim 2. The group of translations of the plane at infinity.
- $\langle Hyp, Par \rangle$, dim 2. The group generated by the hyperbolic transformations and parabolic transformations: translations of the plane at infinity in real directions and dilations about points on the real axis.
- $\langle Ell, Hyp \rangle$, dim 2. The group of dilations and rotations about the origin in the plane at infinity.
- *Hom*, dim 3. The homothety group of the plane at infinity, consisting of translations and dilations.
- *ScrewHom*, dim 3. The screw-homothety group of the plane at infinity, generated

by translations as well as a loxodromic group (of type *Lox* above). Just as with the loxodromic groups, there are infinitely many nonequivalent groups of this type.

- $E(2)$, dim 3. The group of Euclidean transformations of the plane at infinity. Generated by translations and rotations.
- $H(2)$, dim 3. The identity component of the group of isometries of a totally geodesic Hyperbolic plane. Concretely, we choose the hyperbolic 2-plane in the Poincaré disk model with $z = 0$. This group happens to be isomorphic to $PSL(2, \mathbb{R})$, though we will not need this fact.
- $SO(3)$, dim 3. The group of orthogonal transformations of the sphere at infinity in the Poincaré disk model.
- Sim , dim 4. The group of similitudes of the plane at infinity. Generated by translations, rotations, and dilations.
- $H(3)$, dim 6. The identity component of the full isometry group of Hyperbolic 3-space.

That's the full list of conjugacy classes of connected closed subgroups of the isometry group of H^3 .

We now describe the FWH fibrations of Hyperbolic 3-space by geodesics. We use the upper half-space model. To start, there's one special one which is qualitatively different from the others. Let \mathcal{F}_∞ be the fibration consisting of vertical lines, i.e. lines of the form $\{z\} \times \mathbb{R}^+$. The others are similar to one another. Recall that we have identified the plane at infinity with $\mathbb{C} \times \{0\}$. Let's ignore the second coordinate and just identify the plane

at infinity with \mathbb{C} . Define the set $S \subset \mathbb{C}$ by

$$S = \{z \in \mathbb{C} : \text{Im } z \geq 1, \text{Re } z \geq 0\}.$$

That is, S is the first quadrant in the complex plane, shifted one unit in the positive imaginary direction. For each $z \in S$, we define an unparametrized geodesic γ_z in Hyperbolic 3-space by assigning its endpoints in \mathbb{C} to be $-i$ and z . Now let the group $\langle Hyp, Par \rangle$ act on γ_z . The image is all of H^3 and thus forms a fibration of H^3 , which we name \mathcal{F}_z . To see this, note that Par translates γ_z in real directions, so the image of γ_z under Par is an infinitely long tunnel over the real line in the plane at infinity. The image of this tunnel under Hyp is all of H^3 , because Hyp consists of dilations of the plane at infinity about points on the real line. Thus $\langle Hyp, Par \rangle$ acts transitively on the fibration \mathcal{F}_z . See Figure 2.3.

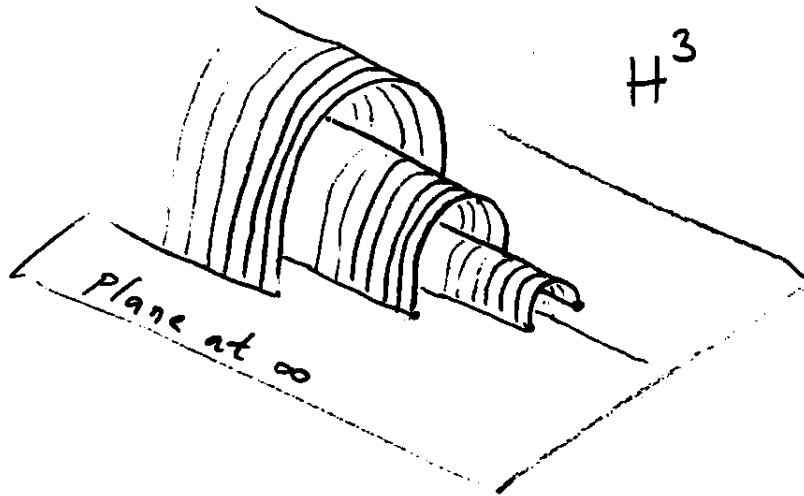


Figure 2.3: A typical picture of an \mathcal{F}_z in the upper half-space model of H^3 .

We're technically not restricted to choosing γ_z with z in S . We could just as easily

choose any z with $\text{Im} > 0$ and define γ_z in the same way. But then we would have some redundancies in our list: some γ_z would be equivalent to $\gamma_{z'}$ for $z \neq z'$.

Theorem 2.3.1. *The FWH fibrations of Hyperbolic 3-space by geodesics, up to equivalence, are precisely the fibrations \mathcal{F}_z for $z \in S$, together with \mathcal{F}_∞ .*

Proof. Just as with Euclidean 3-space, the proof of this theorem is in three parts. First, we demonstrate that the fibrations we list in the statement really are FWH, then we prove that there are no others, and then we prove that our list has no redundancies.

First, note that $\langle Hyp, Par \rangle$ acts transitively on each \mathcal{F}_z , by construction, so they are all FWH. Note also that Sim (the group of similitudes of the plane at infinity) acts transitively on \mathcal{F}_∞ , so it is FWH as well.

Now let \mathcal{F} be a FWH fibration of H^3 by geodesics, and let G be a group of isometries acting transitively on \mathcal{F} . By Lemma 1.3.3, we may assume G is closed and connected. By replacing \mathcal{F} by an equivalent fibration if necessary, we may assume that G appears on the list above of closed connected groups of isometries of H^3 .

- Suppose $G = \{1\}, Hyp, Par, Ell,$ or Lox . As in the Euclidean case, the image of a fiber under G will have dimension only 1 or 2 and hence cannot fill all of H^3 . This is a contradiction.
- Suppose $G = T(2), Hom, ScrewHom, E(2),$ or Sim . Let F be any fiber in \mathcal{F} , in the upper half-space model. Suppose F is a semicircle meeting the plane at infinity orthogonally. Then its endpoints form a line in the plane at infinity. Apply a translation to F along this line. Then F moves to another geodesic which intersects F transversely, so G does not preserve \mathcal{F} . This is a contradiction, so in fact F must

be a vertical line. Now the image of F under G is \mathcal{F}_∞ , so $\mathcal{F} = \mathcal{F}_\infty$.

- Suppose $G = \langle Hyp, Par \rangle$. Let F be any fiber. Suppose F is a vertical line in the upper half-space model. Then the image of F under G will fill out only points with positive imaginary part, or only points with negative imaginary part, or points with only zero imaginary part, depending on where the endpoint of \mathcal{F} is. Thus G does not act transitively on \mathcal{F} , a contradiction. Thus F must be a semi-circle in the upper half-space model. As we just observed, G preserves the set of points with positive (respectively negative, respectively zero) imaginary part, so F must have one endpoint with positive imaginary part and one endpoint with negative imaginary part on the plane at infinity. Then the image of F under G fill out H^3 and hence $\mathcal{F} = \mathcal{F}_z$ for some $z \in \mathbb{C}$. If z is not in S , then \mathcal{F} is equivalent to $\mathcal{F}_{z'}$ for $z' \in S$: simply apply to \mathcal{F} a rotation of π about the origin and/or a reflection in the imaginary line in the plane at infinity.
- Suppose $G = \langle Ell, Hyp \rangle$. Let F be the fiber of \mathcal{F} in the upper half-space model passing through $(0, 1) \in \mathbb{C} \times \mathbb{R}^+$. Suppose F is a semicircle. Then a rotation applied to F moves F to a semicircle intersecting F transversely. Thus G does not preserve \mathcal{F} , a contradiction. Therefore F must be a vertical line. But then G fixes F , rather than acting transitively on \mathcal{F} , also a contradiction.
- Suppose $G = H(2)$. Let F be the fiber through $0 \in \mathbb{R}^3$ in the Poincaré disk model. The isometry group of the hyperbolic 2-plane consisting of points $(x, y, z) \in \mathbb{R}^3$ with $z = 0$ contains the group of rotations about the z -axis. If F is not the portion of the z -axis with norm less than 1, then these rotations take F to a line segment

which intersects F transversely. Thus F is a portion of the z -axis. Now the image of F under G defines a fibration of H^3 consisting of all the geodesics orthogonal to a fixed hyperbolic 2-plane. Moving from the disk model to the upper half-space model, take this hyperbolic 2-plane to be the 2-plane over the real line in the plane at infinity. We see that \mathcal{F} is precisely the fibration \mathcal{F}_i .

- Suppose $G = SO(3)$ or $H(3)$. Consider the fiber F through the origin in the Poincaré disk model. Apply some rotation of the sphere at infinity which does not fix F ; that moves F to a line which intersects F transversely, a contradiction.

Thus \mathcal{F} is equivalent to some \mathcal{F}_z for $z \in S$, or \mathcal{F} is \mathcal{F}_∞ .

We close out the chapter by showing that no two FWH fibrations in our list are equivalent to one another. Whereas in the Euclidean case we made use of the fact that the unit vector fields along our fibrations were curl eigenfields with distinct eigenvalues, here we are unable to find so pretty an argument. What follows is rather technical and unenlightening.

Suppose that \mathcal{F}_z is equivalent to $\mathcal{F}_{z'}$ for $z \neq z'$ and both z and z' in S . It's clear that \mathcal{F}_∞ is not equivalent to \mathcal{F}_z for any z , and that \mathcal{F}_i is not equivalent to \mathcal{F}_z for $z \neq i$; by the above discussion, they have different symmetry groups. Thus $G = \langle Hyp, Par \rangle$ is the identity component of the symmetry group of \mathcal{F}_z and $\mathcal{F}_{z'}$.

Let T be an isometry of H^3 taking \mathcal{F}_z to $\mathcal{F}_{z'}$. Then $TGT^{-1}(\mathcal{F}_{z'}) = \mathcal{F}_{z'}$, so T is in the normalizer of G . If \mathcal{F}_z equals $\mathcal{F}_{z'}$ as fibrations, we're done, so assume that T is not in G . At the sphere of infinity in the disk model, G fixes exactly one point (if we identify the sphere at infinity with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with \mathbb{C} the plane at infinity in

the half-space model, then G fixes ∞). Therefore T must fix that point as well. Thus T acts as a similitude of the plane at infinity in the upper half-space model, possibly orientation-reversing.

The group G also preserves the real line (and no other affine line) in the plane at infinity, so T must preserve this line as well. The similitudes of the plane which accomplish this are generated by G and by the following transformations: rotations of the plane at infinity by π about 0, reflections in the real axis, and reflections in the imaginary axis. The action of G preserves \mathcal{F}_z , so we can focus on the effect of the last three transformations on \mathcal{F}_z .

Reflection in the imaginary axis takes the geodesic joining $-i$ to z to the geodesic joining $-i$ to $-\bar{z}$, so it takes \mathcal{F}_z to $\mathcal{F}_{-\bar{z}}$. If z and $-\bar{z}$ are both in S then we must have $z = -\bar{z} = z'$, with z on the imaginary axis.

Consider the line segment joining $-i$ to z in the plane at infinity. The real axis splits this line segment into two pieces, the ratio of whose sizes (top to bottom) is $\text{Im } z$ to 1.

Consider the geodesic fiber F in \mathcal{F}_z which has one endpoint at i , and call the other endpoint w . The real axis also cuts the line segment joining i and w , and the ratio of their lengths (top to bottom) is also $\text{Im } z$ to 1. Applying a rotation of the complex plane by π about the origin to \mathcal{F}_z , or applying a reflection in the real axis, moves F so that one endpoint is now at $-i$, and w is taken to z' .

Consider the line segment joining the endpoints of the geodesic joining $-i$ to z' . The ratio of the lengths (top to bottom) is now 1 to $\text{Im } z$, because we have flipped it upside down. Thus for a rotation by π or a reflection about the real axis, we have $\text{Im } z' = 1/\text{Im } z$. If z and z' are both in S , then we must have $\text{Im } z = \text{Im } z' = 1$. We also must have

just applied a rotation and not a reflection, because reflection takes z in S to z' with nonpositive real part, and we assumed $z \neq i$. Therefore the angle which the segment from $-i$ to z makes with the real axis is the same as the angle which the segment from $-i$ to z' makes, and we must have $z = z'$.

Thus no two of the FWH fibrations by geodesic in the statement of the theorem are equivalent. □

3 Hopf fibrations are characterized by being fiberwise homogeneous

3.1 Introduction

In this chapter, we prove that the Hopf fibrations are characterized by being FWH among all fibrations of spheres by smooth subspheres. The proof relies on the work of Montgomery-Samelson [12] and Borel [4], and generalized by Oniscik [13], who found all the compact Lie groups which act transitively and effectively on spheres and projective spaces. The proof also relies on the representation theory of compact Lie groups. In Section 3.2, we give the statement and proof of the main theorem. We leave the representation theory to Section 3.3.

3.2 Main Theorem

Theorem B. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of S^n with its standard metric by subspheres of dimension k , i.e.*

- S^3, S^5, S^7, \dots by circles
- $S^7, S^{11}, S^{15}, \dots$ by 3-spheres
- S^{15} by 7-spheres.

Then \mathcal{F} is a Hopf fibration.

Here is a summary of the proof. First we use the classification of homogeneous spaces to show that the base space of a FWH fibration \mathcal{F} is diffeomorphic to the base of a

Hopf fibration with the same total space and the same fiber dimension. We call this the Hopf model for \mathcal{F} . Then we use a theorem of Montgomery-Samelson and Borel, and generalized by Oniscik, to find a list of the compact Lie groups which can act transitively and effectively on the base space of \mathcal{F} . We find that these groups (or covers of them) are isomorphic to subgroups of the symmetry group of the Hopf model for \mathcal{F} . We then prove a lemma that the symmetry group of \mathcal{F} acts irreducibly on the Euclidean space in which the ambient round sphere is embedded. We then use the representation theory of compact Lie groups to show that the action of the symmetry group is standard; i.e., is the same as the action of the symmetry group of the Hopf model. Finally we show that \mathcal{F} must actually have a fiber in common with its Hopf model, and hence must be identical to it.

Proposition 3.2.1 (Follows from Theorem 7.50 in Besse [3], citing work of Wang, Borel, and Singh Varma). *Let \mathcal{F} be a FWH C^1 fibration of S^n by k -spheres. Then the base of \mathcal{F} is diffeomorphic to the base of the Hopf fibration of the same dimension.*

Proof. The cohomology ring of the base of \mathcal{F} is identical to that of its Hopf model. This follows from the Serre spectral sequence; see [9, Example 1.15] for an example computation. The base of \mathcal{F} also has the structure of a homogeneous space. By Theorem 7.50 in Besse [3], the base of \mathcal{F} is diffeomorphic to the base of its Hopf model. \square

Proposition 3.2.2 ([13], Theorem 6 parts (a), (b), and (f), and Table 2). *Let G be a compact connected Lie group acting transitively and effectively on M .*

- *Let $M = \mathbb{C}P^n$. Then $G = SU(n+1)/\mathbb{Z}_{n+1}$ or possibly $G = Sp(k+1)/\mathbb{Z}_2$ if $n = 2k+1$.*

- Let $M = \mathbb{H}P^n$. Then $G = Sp(n+1)/\mathbb{Z}_2$.

- Let $M = S^8$. Then $G = SO(9)$.

Notice that, if $M = S^2 (= \mathbb{C}P^1)$ then $G = SO(3) (= SU(2)/\mathbb{Z}_2 = Sp(1)/\mathbb{Z}^2)$, and if $M = S^4 (= \mathbb{H}P^1)$ then $G = SO(5) (= Sp(2)/\mathbb{Z}^2)$.

In what follows, whenever we write that G is a Lie group acting transitively on a fibration \mathcal{F} , we take G to be closed and connected. We may assume this because of Lemma 1.3.3.

Lemma 3.2.3. *Let \mathcal{F} be a FWH C^1 fibration of S^n by k -spheres. Let $G \subset SO(n+1)$ act transitively on \mathcal{F} . Then the action of G on \mathbb{R}^{n+1} is irreducible.*

Proof. Suppose we have a nontrivial splitting $\mathbb{R}^{n+1} = A \oplus B$, with A, B being G -invariant subspaces. Then WLOG we have $1 \leq \dim A \leq \frac{1}{2}(n+1)$. Every fiber of \mathcal{F} must meet the unit sphere $S(A)$ in A . This sphere satisfies $0 \leq \dim S(A) \leq \frac{1}{2}(n-1)$. But the base of \mathcal{F} has dimension at least $\frac{1}{2}(n+1) > \dim S(A)$ (we see this by checking each of the possibilities for k and n case by case). This is a contradiction because the fibers may not intersect one another. Thus there cannot be a nontrivial splitting by G -invariant subspaces, and so G acts irreducibly. \square

Now we give a Key Lemma.

Key Lemma. *Let G be a compact connected Lie group acting irreducibly on \mathbb{R}^d , and acting transitively on a FWH C^1 fibration \mathcal{F} of S^{d-1} by spheres. Let $H \triangleleft G$ be the normal subgroup (possibly discrete or disconnected) of G which takes each fiber of \mathcal{F} to itself.*

1. Suppose G acts on \mathbb{R}^{2n+2} , and $G/H \cong SU(n+1)/\mathbb{Z}_{n+1}$. Then G contains $SU(n+1)$ as a subgroup, acting in the standard way on \mathbb{R}^{2n+2} .
2. Suppose G acts on \mathbb{R}^{4n+4} , and $G/H \cong Sp(n+1)/\mathbb{Z}_2$. Then G contains $Sp(n+1)$ as a subgroup, acting in the standard way on \mathbb{R}^{4n+4} .
3. Suppose G acts on \mathbb{R}^{16} and $G/H \cong SO(9)$. Then G contains $Spin(9)$ as a subgroup, acting as the spin representation on \mathbb{R}^{16} .

The proof only involves standard representation theory of compact Lie groups. We defer the proof of the Key Lemma to Section 3.3 so as not to get bogged down. We now split Theorem B into four smaller cases, and prove each separately.

Theorem 3.2.4. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of the 3-sphere by circles (i.e. 1-spheres). Then \mathcal{F} is the Hopf fibration.*

Proof. Let G be a subgroup of $SO(4)$ which acts transitively on \mathcal{F} . Let H be the normal subgroup of G which takes each fiber of \mathcal{F} to itself. Then G/H acts transitively and effectively on the base of \mathcal{F} . By Proposition 3.2.1, the base of \mathcal{F} is diffeomorphic to S^2 . By Proposition 3.2.2, G/H is isomorphic to $SO(3)$.

By Lemma 3.2.3, G acts irreducibly on \mathbb{R}^4 . By the Key Lemma, case 1 (since $SO(3) = SU(2)/\mathbb{Z}_2$), G contains $SU(2) = Sp(1)$ as a subgroup, acting as left multiplication by unit quaternions on \mathbb{R}^4 .

Let E be the subbundle of TS^3 consisting of tangent lines to the fibers of \mathcal{F} . The group $Sp(1)$ preserves \mathcal{F} , and hence the field of tangent lines to \mathcal{F} is left-invariant. Therefore its integral curves (i.e. the fibers of \mathcal{F}) form a Hopf fibration. □

Theorem 3.2.5. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of the $2n + 1$ -sphere by circles, with $n \geq 2$. Then \mathcal{F} is the Hopf fibration.*

Proof. Let G be a subgroup of $SO(2n + 2)$ which acts transitively on \mathcal{F} . Let H be the normal subgroup of G which takes each fiber of \mathcal{F} to itself. Then G/H acts transitively and effectively on the base of \mathcal{F} . By Proposition 3.2.1, the base of \mathcal{F} is diffeomorphic to $\mathbb{C}P^n$. By Proposition 3.2.2, G/H is isomorphic to $SU(n + 1)/\mathbb{Z}_{n+1}$ or possibly to $Sp(k + 1)/\mathbb{Z}_2$ if $n = 2k + 1$. By Lemma 3.2.3, G acts irreducibly on \mathbb{R}^{2n+2} . Consider the two cases for G/H separately.

1. Suppose first that $G/H = SU(n + 1)/\mathbb{Z}_{n+1}$. By the Key Lemma, case 1, G contains $SU(n + 1)$ as a subgroup, acting on \mathbb{R}^{2n+2} in the standard way. We use the $SU(n + 1)$ action to identify \mathbb{R}^{2n+2} with \mathbb{C}^{n+1} . Fix $x \in S^{2n+1}$, and let F_x be the fiber of \mathcal{F} through x . The isotropy subgroup of $SU(n + 1)$ fixing x is isomorphic to $SU(n)$. Since $SU(n)$ preserves \mathcal{F} and preserves x , it must also preserve the tangent line to F_x through x . But the only way $SU(n)$ preserves the tangent line is if the tangent line points in the direction of ix . But x is arbitrary, and so for all $x \in S^{2n+1}$, the fiber through x points in the direction ix . The trajectories of this field of tangent lines form a Hopf fibration.

2. Suppose instead that $n = 2k + 1$ and that $G/H = Sp(k + 1)/\mathbb{Z}_2$. By the Key Lemma, case 2, G contains $Sp(k + 1)$ as a subgroup, acting on $\mathbb{R}^{2(2k+1)+2} = \mathbb{R}^{4k+4}$ in the standard way. Let $x \in S^{4k+3}$ be arbitrary, and F_x the fiber of \mathcal{F} through x . Identify \mathbb{R}^{4k+4} with \mathbb{H}^{k+1} so that $Sp(k + 1)$ acts as quaternionic linear transformations. The isotropy subgroup of $Sp(k + 1)$ is isomorphic to $Sp(k)$, fixes the \mathbb{R}^4 spanned by

x, xi, xj, xk , and does not fix any vector in the orthogonal \mathbb{R}^{4k} . Let v_x be a unit tangent vector to F_x at x . The isotropy subgroup $Sp(k)$ of x must fix v_x , so we can write $v_x = xp$ for some purely imaginary quaternion p . But then the group $Sp(k+1)$, preserving \mathcal{F} , takes x to any other point y on S^{4k+3} , and takes the vector v_x to the vector v_y tangent to F_y at y . Thus the field xp at x is tangent to \mathcal{F} , and so \mathcal{F} is in fact isometric to the Hopf fibration. □

Theorem 3.2.6. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of the $4n+3$ -sphere by 3-spheres. Then \mathcal{F} is the Hopf fibration.*

Proof. Let G be a subgroup of $SO(4n+4)$ which acts transitively on \mathcal{F} . Let H be the normal subgroup of G which takes each fiber of \mathcal{F} to itself. Then G/H acts transitively and effectively on the base of \mathcal{F} . By Proposition 3.2.1, the base of \mathcal{F} is diffeomorphic to $\mathbb{H}P^n$. By Proposition 3.2.2, G/H is isomorphic to $Sp(n+1)/\mathbb{Z}_2$. By Lemma 3.2.3, G acts irreducibly. By the Key Lemma, case 2, G contains $Sp(n+1)$ as a subgroup, acting in the standard way on \mathbb{R}^{4n+4} .

Identify \mathbb{R}^{4n+4} with \mathbb{H}^{n+1} so that $Sp(n+1)$ acts by quaternionic-linear transformations. Let $x \in S^{4n+3}$ be arbitrary, and let F_x be the fiber of \mathcal{F} through x . Let P_x be the tangent 3-plane to F_x at x . The isotropy subgroup of $Sp(n+1)$ fixing x is isomorphic to $Sp(n)$, and it also fixes the 4-dimensional subspace of \mathbb{H}^{n+1} spanned by x, xi, xj, xk . The isotropy subgroup $Sp(n)$ must also fix the tangent 3-plane P_x , so P_x must be spanned by xi, xj, xk . But this argument applies equally well to the 3-plane Q_x tangent to the Hopf fiber through x . So P_x coincides with Q_x . Thus \mathcal{F} is identical to the Hopf fibration. □

Theorem 3.2.7. *Let \mathcal{F} be a fiberwise homogeneous C^1 fibration of the 15-sphere by 7-spheres. Then \mathcal{F} is the Hopf fibration.*

Proof. Let G be a subgroup of $SO(16)$ which acts transitively on \mathcal{F} . Let H be the normal subgroup of G which takes each fiber of \mathcal{F} to itself. Then G/H acts transitively and effectively on the base of \mathcal{F} . By Proposition 3.2.1, the base of \mathcal{F} is diffeomorphic to S^8 . By Proposition 3.2.2, G/H is isomorphic to $SO(9)$.

By Lemma 3.2.3, G acts irreducibly on \mathbb{R}^{16} , and by the Key Lemma, G contains a subgroup isomorphic to $Spin(9)$ which acts as the spin representation on \mathbb{R}^{16} .

Let $x \in S^{15}$ be arbitrary, and let F_x be the fiber of \mathcal{F} through x . Let P_x be the tangent 7-plane to F_x through x . The isotropy subgroup of $Spin(9)$ which fixes x is $Spin(7)$, and it acts on the orthogonal \mathbb{R}^{15} as the sum of the $SO(7)$ action on an \mathbb{R}^7 and the spin representation on an \mathbb{R}^8 , see Ziller [22]. The tangent 7-plane P_x must be fixed by the isotropy action, and hence must lie in the \mathbb{R}^7 . But this argument applies equally well to the 7-plane Q_x tangent to the Hopf fiber through x . So P_x coincides with Q_x . Thus \mathcal{F} is identical to the Hopf fibration. □

3.3 Representation Theory

We have saved the proof of the Key Lemma for this section, so that we can black-box the representation theory of compact Lie groups. The proof is totally standard. Our use for it is unique enough that we are unlikely to find its exact statement in the literature. The main tool we will use is a comprehensive list of low-dimensional irreducible representations of the classical compact Lie groups, due to Andreev-Vinberg-Élashvili [1]. But we will also need to deal with the minor irritation that their list is of complex representations,

and for our purposes we need to know about real irreducible representations. We will also need to say a little about irreducible representations of product groups.

We make use of the fact that complex irreducible representations of a compact Lie group are of “real type” or “complex type” or “quaternionic type.” We leave as a black box the precise meanings of these terms. See Bröcker and tom Dieck [5] for more details.

Let G be a compact Lie group, and let $\rho : G \rightarrow GL(\mathbb{R}^n)$ be an irreducible representation. Let $\rho_{\mathbb{C}} : G \rightarrow GL(\mathbb{C}^n)$ be ρ followed by the natural inclusion $GL(\mathbb{R}^n) \hookrightarrow GL(\mathbb{C}^n)$. Conversely, let $\pi : G \rightarrow GL(\mathbb{C}^n)$ be a complex irreducible representation, and let $\pi_{\mathbb{R}} : G \rightarrow GL(\mathbb{R}^{2n})$ be the result of forgetting the complex structure on π .

Proposition 3.3.1 (Theorem 6.3 in [5]). *Let $\rho : G \rightarrow GL(n, \mathbb{R})$ be irreducible. One of two possibilities holds:*

1. $\rho_{\mathbb{C}}$ is a complex irreducible representation of G of dimension n , of real type.
2. ρ is equal to $\pi_{\mathbb{R}}$ for some complex irreducible representation π of G of dimension $n/2$, of complex or quaternionic type.

Proposition 3.3.2 (Lemma 4 and Table 1 in [1], Theorems IX.10.5-7 in [18]). *Let G be a compact simple Lie group. What follows is a complete list of the irreducible representations of G in $GL(\mathbb{C}^d)$ with dimension d less than $\dim G$, and their type (i.e. real or complex or quaternionic).*

- $G = SU(n+1)$, $\dim G = n^2 + 2n$. Let $n \geq 2$. One irreducible representation each of dimension $n+1$ (the defining representation), $(n+1)(n+2)/2$, $n(n+1)/2$. They’re all of complex type, except that when $n = 3$, the last is of real type. When $n = 2$, the last one is equivalent to the defining representation. When $n = 5, 6, 7$, there is

an additional representation of dimension $(n-1)n(n+1)/6$, which is quaternionic for $n=5$, and of complex type otherwise.

- $G = Sp(n)$, $\dim G = n(2n+1)$. One irreducible representation of dimension $2n$ of quaternionic type, and one of dimension $2n^2 - n - 1$ of real type. When $n=3$, there's an additional one of dimension 14 of quaternionic type.
- $G = Spin(9)$, $\dim G = 36$. One irreducible representation of dimension 9, one of dimension 16, both of real type.

Proposition 3.3.3 (Theorem 3.9 in Sepanski [16]). *Let G and H be compact Lie groups. A representation of $G \times H$ in $GL(n, \mathbb{C})$ is irreducible if and only if it is the tensor product of an irreducible representation of G with one of H .*

Proposition 3.3.4 (follows from Theorem 5.22 in Sepanski [16]). *Let H be a normal (possibly disconnected) subgroup of a compact connected Lie group G , and suppose $K := G/H$ is simple. Then there exist finite-sheeted covering groups G' , H' of G , H_0 (the identity component of H), such that $G' \cong H' \times \tilde{K}$, where \tilde{K} is the universal covering group of K .*

Let V be a complex irreducible representation of a compact Lie group G . Pick an arbitrary basis for V , so that we identify V with \mathbb{C}^n , and identify G with a subgroup of $GL(n, \mathbb{C})$. The character of V is a function $\chi_V : G \rightarrow \mathbb{C}$, defined by

$$\chi_V(g) = \text{Tr}(g).$$

The trace $\text{Tr}(g)$ is independent of our choice of identification of V with \mathbb{C}^n , so χ_V is well-defined.

Proposition 3.3.5 (Proposition 6.8 in Bröcker and tom Dieck [5]). *Let V be a complex irreducible representation of a compact Lie group G with character $\chi_V : G \rightarrow \mathbb{C}$. Then*

$$\int_G \chi_V(g^2) dg = \begin{cases} 1 & \Leftrightarrow V \text{ is of real type} \\ 0 & \Leftrightarrow V \text{ is of complex type} \\ -1 & \Leftrightarrow V \text{ is of quaternionic type} \end{cases}$$

Corollary 3.3.6. *Let V, W be complex irreducible representations of G, H respectively. Let $V \otimes W$ be the tensor product of V and W , an irreducible representation of $V \times W$ (see Proposition 3.3.3). Then*

- $V \otimes W$ is of real type if V and W are both of real type or both of quaternionic type.
- $V \otimes W$ is of complex type if at least one of V and W are of complex type.
- $V \otimes W$ is of quaternionic type if one of V and W is of real type and the other is of quaternionic type.

Proof. Observe that $\chi_{V \otimes W}(g, h) = \chi_V(g)\chi_W(h)$. Then

$$\begin{aligned} \int_{G \times H} \chi_{V \otimes W}(g^2, h^2) dg dh &= \int_{G \times H} \chi_V(g^2)\chi_W(h^2) dg dh \\ &= \left(\int_G \chi_V(g^2) dg \right) \left(\int_H \chi_W(h^2) dh \right). \end{aligned}$$

The result follows. □

Key Lemma. *Let G be a compact connected Lie group acting irreducibly on \mathbb{R}^d , and acting transitively on a FWH C^1 fibration \mathcal{F} of S^{d-1} by spheres. Let $H \triangleleft G$ be the normal subgroup (possibly discrete or disconnected) of G which takes each fiber of \mathcal{F} to itself.*

1. Suppose G acts on \mathbb{R}^{2n+2} , and $G/H \cong SU(n+1)/\mathbb{Z}_{n+1}$. Then G contains $SU(n+1)$ as a subgroup, acting in the standard way on \mathbb{R}^{2n+2} .
2. Suppose G acts on \mathbb{R}^{4n+4} , and $G/H \cong Sp(n+1)/\mathbb{Z}_2$. Then G contains $Sp(n+1)$ as a subgroup, acting in the standard way on \mathbb{R}^{4n+4} .
3. Suppose G acts on \mathbb{R}^{16} and $G/H \cong SO(9)$. Then G contains $Spin(9)$ as a subgroup, acting as the spin representation on \mathbb{R}^{16} .

Proof. In each of the three cases, the strategy is the same. We want to show that G is a simply connected Lie group acting in the standard way. We use Proposition 3.3.4 to find a covering group G' of G which we can write as a product. We pull back the action of G to an action of G' , and use Proposition 3.3.3 to decompose this action as a tensor product of irreducible representations. Then we use the list of low-dimensional representations in Proposition 3.3.2, together with Corollary 3.3.6 to show that the action of G' is standard, and in particular is nontrivial on its center, so that $G = G'$.

1. There exist covering groups G' , H' of G , H_0 for which $G' = H' \times SU(n+1)$. The action of G pulls back to an irreducible action of G' . By Proposition 3.3.1, there is either a complex irreducible representation of dimension $2n+2$ and of real type which restricts to the real action of G' , or there's a complex irreducible representation of dimension $n+1$ of complex or quaternionic type which equals the real action of G' after forgetting the complex structure.

Either way, the complex action of G' is the tensor product of irreducible representations of H' and $SU(n+1)$, by Proposition 3.3.3, and the latter is nontrivial (or else G/H could not act transitively on the base of \mathcal{F}). Suppose first that $n \geq 2$;

we'll return to $n = 1$ momentarily. When $n \geq 2$, we have $2n + 2 < n^2 + n$, and so we may find every complex irreducible representation of $SU(n + 1)$ of dimension at most $2n + 2$ on the list found in Proposition 3.3.2, case 1.

The only irreducible representations of $SU(n + 1)$ not of complex type are one of real type in dimension 6 when $n = 3$, and one of quaternionic type of dimension 20 when $n = 5$. These dimensions do not divide either $n + 1$ or $2n + 2$ in either case. By Corollary 3.3.6, the action of G' must be the result of forgetting the complex structure on the $(n + 1)$ -dimensional tensor product of a complex irreducible representation of H' with a complex irreducible representation of $SU(n + 1)$ of complex type. It follows that we must be looking at the tensor product of a 1-dimensional representation of H with the defining representation of $SU(n + 1)$, because every other representation of $SU(n + 1)$ has dimension larger than $n + 1$.

If $n = 1$, then the action of G' on \mathbb{R}^4 is either the restriction of an irreducible complex action on \mathbb{C}^4 of real type or it's an irreducible action on \mathbb{C}^2 , forgetting the complex structure, and of complex or quaternionic type. The group $SU(2)$ has precisely one complex irreducible representation of dimension 2 (the defining representation), and one of dimension 4, and they're both of quaternionic type. But if we take the tensor product of the 4-dimensional representation of $SU(2)$ with a 1-dimensional representation of H' , then the result cannot be of real type, applying Corollary 3.3.6 and observing that 1-dimensional representations of compact Lie groups are never of quaternionic type. (For semi-simple compact Lie groups, their only 1-dimensional representation is the trivial one, and thus is of real type, and for

tori, the 1-dimensional representations are always of complex type.)

Thus the action of G' always has a subgroup the defining representation of $SU(n+1)$.

This action is nontrivial on its center, so when the action of $SU(n+1)$ descends to the original action of G , we keep the full $SU(n+1)$ group. Thus G contains $SU(n+1)$ as a subgroup acting in the standard way.

2. There exist covering groups G', H' of G, H_0 for which $G' = H' \times Sp(n+1)$. The action of G pulls back to an action of G' . By Proposition 3.3.1, there's either a complex irreducible representation of G' of dimension $4n+4$ and of real type, restricting to the real action of G' , or a complex irreducible representation of G' of dimension $2n+2$ and of complex or quaternionic type which equals the real action of G' .

An irreducible representation of G' is the tensor product of irreducible representations of H' and $Sp(n+1)$, by Proposition 3.3.3. The irreducible representation of $Sp(n+1)$ must be nontrivial, or else G/H could not act transitively on the base of \mathcal{F} . It must also have dimension at most $4n+4$, which is less than $2n^2+5n+3$ for $n \geq 1$. Thus the irreducible representation of $Sp(n+1)$ appears in the list in Proposition 3.3.2. The dimension of the representation which is not the defining one is $2(n+1)^2 - (n+1) - 1$, or $2n^2+3n$. This quantity is greater than $4n+4$ for $n \geq 2$, and equals 5 for $n=1$, so it certainly does not divide either $2n+2$ or $4n+4$. So it cannot appear in the tensor product representation of G' . Similarly, the other representation of $Sp(3)$ of dimension 14 cannot appear in the tensor product representation of G' , because 14 does not divide 6 or 12.

Thus the irreducible representation of G' of dimension $2n + 2$ must contain the defining representation of $Sp(n + 1)$ as a subgroup. This representation is nontrivial on its center, so it projects down to the defining representation of $Sp(n + 1)$ as a subgroup of G .

3. There exist covering groups G', H' of G, H_0 for which $G' = H' \times Spin(9)$. The action of G pulls back to an action of G' . By Proposition 3.3.1, there's either a complex irreducible representation of G' of dimension 16 and of real type, restricting to the real action of G' , or there's a complex irreducible representation of G' of dimension 8 which equals the real action of G' . The action of G' is the tensor product of irreducible representations of H' and of $Spin(9)$, by Proposition 3.3.3, and the representation of $Spin(9)$ is nontrivial or else G/H could not act transitively on the base of \mathcal{F} .

There are only two nontrivial complex irreducible representations of $Spin(9)$ of dimension less than 36; the vector representation of dimension 9 and the spin representation of dimension 16. Thus the action of G' is the tensor product of a 1-dimensional representation of H' with the 16-dimensional spin representation of $Spin(9)$. This action is nontrivial on the center of $Spin(9)$, and so G contains a copy of $Spin(9)$ as well, acting as the spin representation on \mathbb{R}^{16} .

This concludes the proof of the Key Lemma. □

4 A nonstandard example of fiberwise homogeneous fibrations of $S^3 \times S^3$ by great 3-spheres

4.1 Introduction

In the last chapter, we showed that among all fibrations of spheres by subspheres, the Hopf fibrations are the only ones which are fiberwise homogeneous. In this chapter, we show that the Hopf fibrations are no longer unique if we consider their restriction to a subspace of the total space.

We focus on the 7-sphere, scaled to have radius $\sqrt{2}$. The 7-sphere is embedded in $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$. Consider the “Clifford Torus” $S^3 \times S^3$ in the 7-sphere, where each factor is the sphere of radius 1 in its respective \mathbb{R}^4 factor. The main result of this chapter is the following:

Theorem C. *There is a 1-parameter family of FWH fibrations of the Clifford Torus $S^3 \times S^3$ in S^7 by great 3-spheres which are not restrictions of the Hopf fibration of S^7 by great 3-spheres.*

Once we have developed the proper notation, we restate this theorem later as Theorem 4.4.1

4.2 Fibrations of $S^3 \times S^3$ by great 3-spheres

By a “great 3-sphere” in $M = S^3 \times S^3$ we mean a subset of M which, when included in S^7 as the Clifford Torus, is a great 3-sphere. Equivalently, a great 3-sphere M is the

graph of an isometry from one 3-sphere factor to the other.

John Petro classified fibrations of $S^3 \times S^3$ by great 3-spheres in [14]. The space of such fibrations has four components, each of which deformation retracts to an $\mathbb{R}P^3$. Two of those components can be identified with maps of S^3 into $SO(4)$ which, when lifted to the double cover $S^3 \times S^3$, are graphs of distance-decreasing functions from one factor to the other. The other two components consist of maps of S^3 into the other component of $O(4)$, which satisfy a similar condition. See [14, Section 4] for details.

We recall that $SO(4)$ is double-covered by $S^3 \times S^3$, which has an isometric action on S^3 via $(p, q) \cdot x = pxq^{-1}$, where p, x, q are unit quaternions. From now on, we will freely identify S^3 with the unit quaternions and denote multiplication simply by concatenation.

Given a distance-decreasing map $f : S^3 \rightarrow S^3$, we get a fibration of $S^3 \times S^3$ by great 3-spheres by considering the fibers

$$F_p = \{(x, px f(p)) \in S^3 \times S^3 : x \in S^3\}, p \in S^3.$$

Remark 4.2.1. The set of fibrations we get in this way form one component of the space of all such fibrations. We get another component by writing “ $f(p)xp$ ” instead, and we get the other two components by applying an orientation-reversing isometry to one of the S^3 factors.

Notice that if we take $f(p) \equiv 1$, we obtain the restriction of the Hopf fibration of S^7 by great 3-spheres to $S^3 \times S^3$. We see this by noticing that $F_p = \{(1, p) \cdot x : x \in S^3\}$ is the quaternionic line spanned by $(1, p)$.

Similarly, taking f to be any constant function also gives us a fibration which is isometric to the Hopf fibration.

4.3 Properties of the Hopf Fibration

The Hopf fibration of the 7-sphere by great 3-spheres is well-known to be characterized among all fibrations by smooth 3-spheres by having parallel fibers (See Gromoll and Grove [8]). Along those lines, the restriction of the Hopf fibration to $S^3 \times S^3$ is characterized among great 3-sphere fibrations by having parallel fibers.

To see this, take such a fibration and copy it to the other Clifford Tori in S^7 , namely $S^3(\cos \alpha) \times S^3(\sin \alpha)$ for $\alpha \in (0, \pi/2)$, as well as the two fibers $S^3 \times \{0\}$ and $\{0\} \times S^3$. A straightforward computation shows that the fibers of this new fibration are parallel, and hence the new fibration must be the Hopf fibration. Thus the original fibration of the Clifford torus was the restriction of the Hopf fibration.

The Hopf fibration of $S^3 \times S^3$ is also FWH: the isometry $(\mathbb{1}, L_q)$ takes each fiber F_p to the fiber F_{qp} . Here $\mathbb{1}$ is the identity on the first factor, and L_q is left multiplication by q on the second factor. The main result of this chapter is that the Hopf fibration is not characterized by this property: there are other FWH fibrations.

4.4 FWH fibrations of $S^3 \times S^3$

Fix α in the range $[0, \pi/6]$. Let $f_\alpha : S^3 \rightarrow S^3$ be defined by

$$f_\alpha(p) = pe^{i\alpha}p^{-1}.$$

Theorem 4.4.1. *For each $\alpha \in [0, \pi/6]$, the fibration \mathcal{F}_α whose fibers are given by*

$$F_p = \{(x, px f_\alpha(p)) \in S^3 \times S^3 : x \in S^3\}, p \in S^3,$$

is FWH. The fibrations are distinct for different α .

Proof. The image of S^3 under f_α is the round 2-sphere of radius $\sin \alpha$, consisting of unit quaternions whose real part is $\cos \alpha$. Thus each f_α is distinct, and so the fibrations are distinct.

That these really are fibrations follows from Petro's work and the following lemma.

Lemma 4.4.2. *For each $\alpha \in [0, \pi/6]$, f_α is distance-decreasing.*

Proof of Lemma. The image of any element of S^3 under conjugation sweeps out all elements with the same real part. So the image of every f_α is a round 2-sphere. Each f_α is also constant on the left cosets $q \cdot e^{i\theta}$. It follows that each f_α is a Hopf projection map to a round 2-sphere of radius $\sin \alpha$. When the radius is exactly $1/2$, then the Hopf projection is distance-nonincreasing. When the radius is less than $1/2$, then the Hopf projection is distance-decreasing.

Embedding a round 2-sphere into the 3-sphere as a sphere of constant real part is not totally geodesic; it's distance-decreasing if the radius of the 2-sphere is not 1. Thus the composition of a Hopf projection to a 2-sphere of radius $\leq 1/2$ followed by embedding that 2-sphere into the 3-sphere is distance-decreasing. So each f_α is distance-decreasing for $\alpha \in [0, \pi/6]$. □

Finally, we show that each \mathcal{F}_α is FWH. Let T_q be the isometry of $S^3 \times S^3$ given by

$$T_q = (R_{q^{-1}}, L_q \circ R_{q^{-1}}),$$

i.e. right multiplication by q^{-1} in the first factor and conjugation by q in the second

factor. Then we easily compute that T_q takes each fiber F_p to the fiber F_{qp} :

$$\begin{aligned}
T_q \cdot F_p &= (R_{q^{-1}}, L_q \circ R_{q^{-1}}) \cdot F_p \\
&= (R_{q^{-1}}, L_q \circ R_{q^{-1}}) \cdot \{(x, p x p e^{i\alpha} p^{-1}) \in S^3 \times S^3 : x \in S^3\} \\
&= \{(x q^{-1}, q p x p e^{i\alpha} p^{-1} q^{-1}) \in S^3 \times S^3 : x \in S^3\} \\
&= \{(y, q p y q p e^{i\alpha} p^{-1} q^{-1}) \in S^3 \times S^3 : x \in S^3\} \\
&= \{(y, (q p) y (q p) e^{i\alpha} (q p)^{-1}) \in S^3 \times S^3 : x \in S^3\} \\
&= F_{qp},
\end{aligned}$$

where we let $y = x q^{-1}$ in the fourth line. □

4.5 Geometry of the nonstandard examples

In this section we describe the geometry of the nonstandard examples. It's a challenge to visualize these examples. The base space of the fibrations is homeomorphic to a 3-sphere, but we are faced with the problem that the fibers are not parallel. As a result we cannot place a Riemannian metric on the base space in a natural way. However, we can define a distance function on the base space by using the Hausdorff distance function on fibers. The Hausdorff distance function d_H is defined on the space of closed subsets of a given metric space. Let $N_\epsilon(A)$ denote the open ϵ -neighborhood of a set A . Then

$$d_H(A, B) := \inf\{\epsilon \geq 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}.$$

Now define a distance function on the base of the fibration \mathcal{F}_α by

$$d_\alpha(p, q) := d_H(F_p, F_q), \text{ for all } p, q \in S^3.$$

We don't expect d_α to be induced by any Riemannian metric, but we can still ask e.g. what the isometry group is, or whether this metric resembles any familiar Riemannian metric on S^3 .

We won't go into details, but the reader can check that the group $SU(2) \times U(1)$ acts by isometries on $S^3 \times S^3$ and preserves \mathcal{F}_α , via the map $\iota : SU(2) \times U(1) \rightarrow SO(4) \times SO(4)$,

$$\iota(q, e^{i\theta}) = (L_{e^{i\theta}} R_{q^{-1}}, L_q R_{q^{-1}}),$$

and $(q, e^{i\theta}) \cdot F_p = F_{qpe^{-i\theta}}$.

Note that ι has an ineffective kernel of \mathbb{Z}_2 , so that this action on \mathcal{F}_α actually descends to an action of $U(2)$. Thus the base of \mathcal{F}_α , with its Hausdorff metric, has an isometric $U(2)$ action, and we can check further that this metric bears a strong resemblance to the Berger metric on the 3-sphere.

5 Locally fiberwise homogeneous fibrations of open sets in the 3-sphere by great circles are portions of Hopf fibrations

5.1 Introduction

In Chapter 3, we prove that a fibration of a round sphere by smooth subspheres which is fiberwise homogeneous must be the Hopf fibration.

A much stronger theorem would be the following local version: if we fiber an open subset of a round sphere by smooth subspheres, so that it's *locally fiberwise homogeneous* — so that any two fibers have open sets around them with an isometry taking one open set to the other, preserving fibers and taking the first given fiber to the second — then that fibration is a portion of a Hopf fibration.

The proof that a global FWH fibration of a round sphere by subspheres is a Hopf fibration relies heavily on the global structure of the fibration. To even get off the ground in proving the statement, we need to know exactly which manifold the base space of the fibration is diffeomorphic to. So it does not appear that we can prove our local theorem with the same methods we used to prove the global theorem.

In this chapter we prove the following theorem.

Theorem D. *Let \mathcal{F} be a locally FWH fibration of a connected open set in the 3-sphere by great circles. Then \mathcal{F} is a portion of a Hopf fibration.*

We prove our desired local theorem only for the lowest dimension, and only under the restriction that our fibers are great circles. The reason we can prove our theorem under

these circumstances is that we can make use of a moduli space for the space of great-circle fibrations of the 3-sphere.

5.2 Great circle fibrations of the 3-sphere

The following description of great circle fibrations of the 3-sphere is due to H. Gluck and F. Warner [7].

An oriented great circle in the 3-sphere can be identified with an oriented 2-plane through the origin in \mathbb{R}^4 . Thus a great circle in the 3-sphere is identified with a point in $G_2\mathbb{R}^4$, the Grassmann manifold of oriented 2-planes in \mathbb{R}^4 . A fibration of the 3-sphere by great circles is identified in this way with a 2-dimensional submanifold of $G_2\mathbb{R}^4$. We also may identify $G_2\mathbb{R}^4$ with the manifold $S^2 \times S^2$.

Theorem 5.2.1 (From [7]). *There is a one-to-one correspondence between great circle fibrations of the 3-sphere and submanifolds of $G_2\mathbb{R}^4 = S^2 \times S^2$ which are graphs of distance decreasing functions from one S^2 factor to the other.*

Similarly, there is a one-to-one correspondence between great circle fibrations of open sets in the 3-sphere, and submanifolds of $G_2\mathbb{R}^4 = S^2 \times S^2$ which are graphs of distance decreasing functions from an open set in one S^2 factor to the other factor.

The one-to-one correspondence is simply to identify a great circle with a point in $G_2\mathbb{R}^4$.

This moduli space for great circle fibrations of the 3-sphere allows us to more easily answer questions we have about these fibrations, by translating them into questions about distance-decreasing functions on the 2-sphere. In [7], Gluck and Warner use this method to show (for example) that any great circle fibration of the 3-sphere contains two orthogonal

fibers, and that the space of such fibrations deformation retracts to the subspace of Hopf fibrations.

We need to translate the ideas of “fiberwise homogeneous” and “locally fiberwise homogeneous” from the setting of fibrations to the setting of distance-decreasing maps on the 2-sphere. In the former setting, we have a subgroup of $SO(4)$ acting on a fibration in the 3-sphere; therefore, we need to know how the $SO(4)$ action on the 3-sphere translates to the setting of $S^2 \times S^2$, where the distance-decreasing functions live.

The group $SO(4)$ is double covered by $SU(2) \times SU(2)$, and in turn double covers $SO(3) \times SO(3)$. If we follow the identification of $G_2\mathbb{R}^4$ with $S^2 \times S^2$ (see [7]) for details), we find that the action of $SO(4)$ on S^3 induces an action of $SO(3) \times SO(3)$ on $S^2 \times S^2$, where the first (respectively second) $SO(3)$ factor acts by isometries on the first (resp. second) S^2 factor.

We call a subset S of $S^2 \times S^2$ homogeneous if some subgroup of the isometry group of $S^2 \times S^2$ preserves S and acts transitively on it. We say that S is locally homogeneous if for any $s_1, s_2 \in S$ there is an isometry of $S^2 \times S^2$ taking a neighborhood of s_1 isometrically to a neighborhood of s_2 .

Proposition 5.2.2. *Let \mathcal{F} be a fibration of (an open set in) S^3 by great circles. Let S be the corresponding graph in $S^2 \times S^2$ given by Theorem 5.2.1; i.e. identify each great circle with a point in $G_2\mathbb{R}^4$. Then \mathcal{F} is (locally) fiberwise homogeneous if and only if S is (locally) homogeneous in $S^2 \times S^2$.*

Proof. The proof is immediate from the definitions of FWH, locally FWH, homogeneous, locally homogeneous, and the identification of great circles with points in $G_2\mathbb{R}^4 = S^2 \times S^2$.

□

5.3 Locally FWH fibrations are subsets of Hopf fibrations

In this section we prove the main theorem of the chapter.

Theorem 5.3.1. *Let \mathcal{F} be a locally FWH fibration of a connected open set in the 3-sphere by great circles. Then \mathcal{F} is a portion of a Hopf fibration.*

The distance-decreasing functions associated to the Hopf fibrations are the constant functions. We will show that that the distance-decreasing map f of S^2 associated to the fibration is the constant map. We will assume that this map f is at least C^2 .

Proof. Let \mathcal{F} be a locally FWH fibration of an open set $W \subseteq S^3$ by great circles. Let $f : V \rightarrow S^2$ be the corresponding distance-decreasing function on an open set $V \subseteq S^2$ whose graph in $S^2 \times S^2$ consists of the great circles making up \mathcal{F} . We assume that f is C^2 . As a consequence, the differential df_x of f is defined at every point $x \in V$. The homogeneity of the graph of f is equivalent to the following: for every x and x' in V , there are isometries g_1 and g_2 of S^2 such that g_1 takes an open neighborhood U of x to an open neighborhood U' of x' (and takes x to x'), and such that

$$f \circ g_1 = g_2 \circ f$$

holds on U .

The homogeneity of f implies that the differential df is “similar” independent of $x \in V$. We would like to say something along the lines of: the homogeneity of the graph of f implies that the eigenvalues of df_x are independent of x . After all, df_x is a linear map

between 2-dimensional vector spaces. But df_x is not a map from a vector space to itself. So instead, we consider the image under df_x of the unit circle in the tangent space to x . The image of a circle centered at the origin under a general linear map will be some ellipse, possibly degenerate (possibly a circle or line segment or point), and when the ellipse has distinct axes, the preimages of the axes will be orthogonal.

The homogeneity of f implies that the image under df_x of the unit circle in the tangent space to x will be independent of x . That is, the magnitude of the major and minor axes of the ellipse will be constant (and possibly identical and/or zero) for all $x \in V$.

The image of a unit circle under df is either a circle (possibly with radius 0) or a proper ellipse with distinct major and minor axes. We show that the second possibility cannot happen; the axes must be identical.

Suppose that the ellipse has two different axes. We will derive a contradiction. In that case, the local homogeneity of f implies not only that the magnitudes of the axes of the ellipses are constant, but also that the local isometries preserve the preimages of the major and minor axes. In other words, let X, Y be unit vector fields along V which map via df to the major and minor axes of the ellipses in the tangent spaces of $f(V)$. Then the local isometries which commute with f also preserve X and Y . Now the following lemma applies to show that V must have nonpositive curvature.

Lemma 5.3.2. *Let F be a locally homogeneous surface, and suppose (X, Y) is an orthonormal frame along F which is preserved by the locally homogeneous structure (i.e. the isometries which take any point of F to any other also preserve X and Y). Then the sectional curvature of F is nonpositive.*

We save the proof for the end of this section. We have a contradiction, because Lemma 5.3.2 tells us V has nonpositive curvature, yet V is an open subset of the round sphere, which has positive curvature. Thus the image under df of the unit circle in a tangent space to V must be a circle of radius $r \geq 0$, with r independent of the point in V . Note first that $r \leq 1$, because f is distance-decreasing. In fact we must have that r is strictly less than 1. Even though distance-decreasing functions might preserve distance infinitesimally, if $r = 1$ then f preserves all geodesic distances and hence is not distance-decreasing. Now we show that we must have $r = 0$. Suppose $r > 0$. Then f multiplies all geodesic distances by r , and hence multiplies curvature by $1/r^2$, which is less than 1. But the image of V lies in a 2-sphere of the same radius as the domain, so this is impossible. Therefore we must have $r = 0$, and hence $df \equiv 0$. It follows that f is a constant map, because V is connected, and therefore our locally fiberwise homogeneous fibration is a portion of a Hopf fibration. □

Proof of Lemma 5.3.2. We denote sectional curvature of a plane spanned by an orthonormal basis by $K(X, Y) = \langle R(X, Y)X, Y \rangle$, where R is the Riemannian curvature tensor, and we denote by ∇ the Riemannian connection on F . Note that any real-valued function on F depending only on X and Y is constant, because of the local homogeneous structure

on F preserving X and Y . We have

$$\begin{aligned}
K(X, Y) &= \langle R(X, Y)X, Y \rangle \\
&= \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X, Y]} X, Y \rangle \\
&= \langle \nabla_Y \nabla_X X, Y \rangle + \langle \nabla_X \nabla_Y X, Y \rangle + \langle \nabla_{[X, Y]} X, Y \rangle \\
&= Y \langle \nabla_X X, Y \rangle - \langle \nabla_X X, \nabla_Y Y \rangle \\
&\quad + X \langle \nabla_Y X, Y \rangle - \langle \nabla_Y X, \nabla_X Y \rangle \\
&\quad + \langle \nabla_{[X, Y]} X, Y \rangle \\
&= 0 + 0 + \langle \nabla_{[X, Y]} X, Y \rangle.
\end{aligned}$$

In the fourth equality, we use the compatibility of the Riemannian connection with the metric. In the fifth equality, we use the knowledge that functions of X and Y are constant to show that the first and third terms are equal to 0. We find that the second and fourth terms are equal to 0 by computing that $\nabla_X X$ points along Y and $\nabla_Y Y$ points along X ; similarly $\nabla_Y X$ points along Y and $\nabla_X Y$ points along X . A sample computation along these lines is:

$$\langle \nabla_X X, X \rangle = \frac{1}{2} X \langle X, X \rangle = 0,$$

and hence $\nabla_X X$, being orthogonal to X , points along Y . The other computations are similar.

Now that we have shown $K(X, Y) = \langle \nabla_{[X, Y]} X, Y \rangle$, we will show that the latter is nonpositive. The bracket $[X, Y]$ is preserved by the local isometries of F , so we can write

$[X, Y] = aX + bY$ for constant a, b . Then,

$$\begin{aligned} K(X, Y) &= \langle \nabla_{[X, Y]} X, Y \rangle \\ &= \langle \nabla_{aX + bY} X, Y \rangle \\ &= a \langle \nabla_X X, Y \rangle + b \langle \nabla_Y X, Y \rangle. \end{aligned}$$

Using compatibility of the connection with the metric again, together with symmetry of the connection and the fact that functions of X and Y are constant, we quickly arrive at:

$$\langle \nabla_X X, Y \rangle = -\langle X, [X, Y] \rangle = -\langle X, aX + bY \rangle = -a,$$

$$\langle \nabla_Y X, Y \rangle = -\langle Y, [X, Y] \rangle = -\langle Y, aX + bY \rangle = -b,$$

from which it follows that

$$K(X, Y) = a \langle \nabla_X X, Y \rangle + b \langle \nabla_Y X, Y \rangle = -a^2 - b^2 \leq 0.$$

So the curvature of F is nonpositive. □

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