2-27-2014

A Linear/Producer/Consumer Model of Classical Linear Logic

Jennifer Paykin
jpaykin@gmail.com

Stephan A. Zdancewic
University of Pennsylvania, stevez@cis.upenn.edu

University of Pennsylvania Department of Computer and Information Science Technical Report No. MS-CIS-14-03.

A Linear/Producer/Consumer Model of Classical Linear Logic

Abstract

This paper defines a new proof- and category-theoretic framework for classical linear logic that separates reasoning into one linear regime and two persistent regimes corresponding to ! and ?. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton's linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions that reflect the linear/producer/consumer structure. The paper's metatheoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into the category theory. The work also presents several concrete instances of the LPC model, including one based on finite vector spaces.

Disciplines

Computer Engineering | Computer Sciences

Comments

University of Pennsylvania Department of Computer and Information Science Technical Report No. MS-CIS-14-03.

A Linear/Producer/Consumer model of Classical Linear Logic

Jennifer Paykin Steve Zdancewic
University of Pennsylvania
MS-CIS-14-03

February 27, 2014

Abstract

This paper defines a new proof- and category-theoretic framework for classical linear logic that separates reasoning into one linear regime and two persistent regimes corresponding to ! and ?. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton's linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions that reflect the linear/producer/consumer structure. The paper's metatheoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into the category theory. The work also presents several concrete instances of the LPC model, including one based on finite vector spaces.

1 Introduction

Since its introduction by Girard in 1987, *linear logic* has been found to have a range of applications in logic, proof theory, and programming languages. Its notion of "resource consciousness" sheds light on topics as diverse as proof search (Liang and Miller 2009), memory management (Ahmed et al. 2007), alias control (Hicks et al. 2004), computational complexity (Gaboardi 2007), and security (Zdancewic and Myers 2002), among many others.

Linear logic's power stems from its ability to carefully manage resource usage: it makes a crucial distinction between *linear* (used exactly once) and *persistent* (unrestricted use) hypotheses, internalizing the latter via the ! connective. From a semantic point of view, the literature has converged (following Benton (1995)) on an interpretation of ! as a comonad given by $! = \mathrm{F} \circ \mathrm{G}$ where $\mathrm{F} \dashv \mathrm{G}$ is a symmetric monoidal adjunction between categories $\mathcal{L}$ and $\mathcal{P}$ arranged as shown below:

$$\begin{tikzcd} \mathcal{P} \arrow[rr, bend right, "\mathrm{F}"'] \arrow[rr, phantom, "\top"] & & \mathcal{L} \arrow[ll, bend right, "\mathrm{G}"'] \end{tikzcd}$$

Here, $\mathcal{L}$ (for "linear") is a symmetric monoidal closed category and $\mathcal{P}$ (for "persistent") is a cartesian category. This is, by now, a standard way of interpreting *intuitionistic* linear logic (for details, see the discussion in Melliès' article (Melliès 2009)).

If, in addition, the category $\mathcal{L}$ is $*$-autonomous, then the structure above is sufficient to interpret *classical* linear logic, where the monad ? is determined by $? = (\mathrm{F}^{op}\,(\mathrm{G}^{op}\,(-^{\perp})))^{\perp}$. While sound, this situation is not entirely satisfactory because it essentially commits to a particular implementation of ? in terms of $\mathcal{P}^{op}$, which, as we show, is not necessary.

With that motivation, this paper defines a proof- and category-theoretic framework for full *classical linear logic* that uses *two* persistent categories: one corresponding to ! and one to ?. The resulting categorical

$$\begin{tikzcd}
 & \mathcal{L} \arrow[loop above, "(-)^{\perp}"] \arrow[dl, bend right, "\lceil - \rceil"'] \arrow[dr, bend left, "\lfloor - \rfloor"] & \\
\mathcal{P} \arrow[ur, bend right, "\mathrm{F_P}"'] \arrow[ur, phantom, "\top"] \arrow[rr, bend right, "(-)_*"'] & & \mathcal{C} \arrow[ul, bend left, "\mathrm{F_C}"] \arrow[ul, phantom, "\perp"] \arrow[ll, bend right, "(-)^*"']
\end{tikzcd}$$

Figure 1: Categorical model with linear, producing and consuming categories.

structure is shown in Figure 1, where $\mathcal{P}$ now takes the place of the "producing", category, in duality with $\mathcal{C}$ as the "consuming" category. This terminology comes from the observations that:

$$\begin{array}{ccc} !A \vdash 1 & !A \vdash A & !A \vdash !A \otimes !A \\ \bot \vdash ?A & A \vdash ?A & ?A \mathbin{⅋} ?A \vdash ?A \end{array}$$

Intuitively, the top row means that $!A$ is sufficient to *produce* any number of copies of A and, dually, the bottom rows says that $?A$ can *consume* any number of copies of A.

Contributions. In Section 2 we survey Benton's linear/non-linear presentation of intuitionistic linear logic (Benton 1995). Section 3 defines a linear/producer/consumer (LPC) presentation of classical linear logic which, extending Benton's work, syntactically exhibits the decomposition of ! and ? into their constituent functors. We prove that cut and duality rules are admissible in the logic, and that LPC is consistent.

Section 4 develops the categorical model for LPC, relates it to other models from the literature, and considers how to interpret judgments of the LPC logic as morphisms in the appropriate categories. Section 5 presents several concrete example instances of the LPC categorical framework, and, in particular, gives an example in which $\mathcal{C}$ is not just $\mathcal{P}^{op}$.

We conclude the paper with a discussion of related work.

2 Linear/Non-Linear Logic

This section introduces the Linear/Non-Linear (LNL) model of intuitionistic linear logic presented by Benton (1995, 1994). In traditional presentatons of linear logic, the exponential $!A$ is a linear proposition with persistance. This means linearlity is the default state of the system, and persistance the exception. In the larger context of term calculi, it often is more natural to have two kinds of variables and propositions depending on whether the variable will be used linearly or persistantly (Barber 1996). This linear/non-linear or linear/persistent paradigm shifts the balance of power by placing linearity and persistance on equal footing. Categorically, such a relationship can be modelled as a symmetric monoidal adjunction between a cartesian (non-linear) category $\mathcal{P}$ and a symmetric monoidal closed (linear) category $\mathcal{L}$:

$$\begin{tikzcd}
\mathcal{P} \arrow[r, bend right, "\mathrm{F}"'] \arrow[r, phantom, "\top"] & \mathcal{L} \arrow[l, bend right, "\mathrm{G}"']
\end{tikzcd}$$

This categorical axiomatization will be revisited in Section 4.

The LNL logic is made up of a pair of sequents—one linear and one persistent. There are two classes of propositions, as follows:

$$\begin{aligned} A, B &:= \mathbf{I} \mid A \otimes B \mid A \multimap B \mid \mathrm{F}\, X \\ X, Y &:= 1 \mid X \times Y \mid X \rightarrow Y \mid \mathrm{G}\, A \end{aligned}$$

The linear sequent may reference either linear propositions A or persistent propositions X in order to prove a linear proposition; thus sequents are of the form $\Theta; \Gamma \vdash_{\mathcal{L}} A$ where Θ ranges over persistent propositions and Γ ranges over linear propositions. Meanwhile non-linear sequents are of the form $\Theta \vdash_{\mathcal{P}} X$, meaning that they may refer to only persistent propositions in order to prove a persistent proposition.

One of the principle properties of linear logic is that every hypothesis is used exactly once in a derivation. Consequently, the so-called "linear" rules for axioms, $\otimes$, $\times$, $\multimap$ and $\rightarrow$ always partition contexts instead of duplicating them. However, the expected behavior for non-linear propositions is that they can be used any number of times. This behavior is encoded explicitly by means of the weakening and contraction rules, which are applicable for non-linear propositions in either the linear or non-linear sequents:

$$\frac{\Theta, X, X; \Gamma \vdash_{\mathcal{L}} A}{\Theta, X; \Gamma \vdash_{\mathcal{L}} A}\ \mathrm{C}_{\mathsf{L}} \qquad \frac{\Theta; \Gamma \vdash_{\mathcal{L}} A}{\Theta, X; \Gamma \vdash_{\mathcal{L}} A}\ \mathrm{W}_{\mathsf{L}}$$

$$\frac{\Theta, X, X \vdash_{\mathcal{P}} Y}{\Theta, X \vdash_{\mathcal{P}} Y}\ \mathrm{C}_{\mathsf{C}} \qquad \frac{\Theta \vdash_{\mathcal{P}} Y}{\Theta, X \vdash_{\mathcal{P}} Y}\ \mathrm{W}_{\mathsf{C}}$$

The only way to move in between the linear and non-linear sequents is by passing through the F and G functors:

$$\frac{\Theta, X; \Gamma \vdash_{\mathcal{L}} A}{\Theta; \mathrm{F}\, X, \Gamma \vdash_{\mathcal{L}} A}\ \text{F-L} \qquad \frac{\Theta \vdash_{\mathcal{P}} X}{\Theta \vdash_{\mathcal{L}} \mathrm{F}\, X}\ \text{F-R}$$

$$\frac{\Theta; A, \Gamma \vdash_{\mathcal{L}} A}{\Theta, \mathrm{G}\, A; \Gamma \vdash_{\mathcal{L}} A}\ \text{G-L} \qquad \frac{\Theta \vdash_{\mathcal{L}} A}{\Theta \vdash_{\mathcal{P}} \mathrm{G}\, A}\ \text{G-R}$$

There are three forms of the cut rule, depending on the type of the cut term and sequent. These have the following form:

$$\frac{\Theta_1; \Gamma_1 \vdash_{\mathcal{L}} A \qquad \Theta_2; A, \Gamma_2 \vdash_{\mathcal{L}} B}{\Theta_1, \Theta_2; \Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} B}\ \textsc{Cut}_{\mathsf{L}}^{\mathcal{L}} \qquad \frac{\Theta_1 \vdash_{\mathcal{P}} X \qquad X, \Theta_2; \Gamma \vdash_{\mathcal{L}} A}{\Theta_1, \Theta_2; \Gamma \vdash_{\mathcal{L}} A}\ \textsc{Cut}_{\mathsf{L}}^{\mathcal{P}}$$

$$\frac{\Theta_1 \vdash_{\mathcal{P}} X \qquad X, \Theta_2 \vdash_{\mathcal{P}} Y}{\Theta_1, \Theta_2 \vdash_{\mathcal{P}} Y}\ \textsc{Cut}_{\mathsf{C}}^{\mathcal{P}}$$

We will see in Section 3 that the form of these rules carries over to the LPC logic defined in that section. In LNL, the formulation of these rules stems from the syntactic separation between linear and persistent propositions; linear sequents prove linear propositions and persistent sequents prove persistent propositions. The LPC cut rules follow the same pattern of linear versus persistent sequents, but the structural necessity of this formulation is less apparent in that context. Nevertheless, the formulation is semantically necessary with regards to cut admissibility in LPC.

3 Linear/Producer/Consumer Logic

The syntax of the LPC logic is made up of three syntactic forms for propositions (as opposed to Benton's two): linear propositions A, producer propositions P and consumer propositions C.

$$\begin{array}{rclllllll}
A & ::= & 0 & \mid & A_1 \oplus A_2 & \mid & \top & \mid & A_1 \mathbin{\&} A_2 \\
& \mid & 1_{\mathsf{L}} & \mid & A_1 \otimes A_2 & \mid & \bot_{\mathsf{L}} & \mid & A_1 \parr A_2 \\
& \mid & F_! \, P & \mid & F_? \, C & & & & \\
P & ::= & 1_{\mathsf{P}} & \mid & P_1 \otimes P_2 & \mid & \lceil A \rceil & & \\
C & ::= & \bot_{\mathsf{C}} & \mid & C_1 \parr C_2 & \mid & \lfloor A \rfloor & &
\end{array}$$

The syntactic form of a proposition is called its *mode*—linear L, producing P or consuming C. The meta-variable X ranges over propositions of any mode, and the tagged meta-variable X^{m} ranges over propositions of mode m. The term *persistent* refers to propositions that are either producers or consumers.

LPC replaces the usual constructors ! and ? with two pairs of connectives: $F_!$ and $\lceil - \rceil$ as well as $F_?$ and $\lfloor - \rfloor$. If A is a linear proposition, $\lceil A \rceil$ is a producer and $\lfloor A \rfloor$ is a consumer. On the other hand, a producer proposition P may be "frozen" into a linear proposition $F_! \, P$, effectively discarding its persistent characteristics. Similarly for a consumer C, $F_? \, C$ is linear. The linear propositions $!A$ and $?A$ are encoded in this system as $F_! \, (\lceil A \rceil)$ and $F_? \, (\lfloor A \rfloor)$ respectively.

The syntax does not include duality operators. Later on in this section we will define duality as a meta-operation on propositions and prove that duality rules are admissible in the logic.

The inference rules of the logic are given in Figures 2 to 5. There two sequent relations: the linear sequent $\Gamma \vdash \Delta$ and the persistent sequent $\Gamma \Vdash \Delta$. In the linear sequent, the contexts Γ and Δ may be made up of propositions of any mode: linear, producing, or consuming. In the persistent sequent, however, the contexts may contain only persistent propositions. The denotation Γ^{P} refers to contexts containing only producer propositions, and Δ^{C} refers to contexts containing only consumer propositions.

The linear inference rules in Figures 2 and 3 encompass rules for the units and the linear operators $\oplus$, $\&$, $\otimes$ and $\parr$. It is worth noting that the multiplicative product $\otimes$ is defined only on linear and producer propositions, while the multiplicative sum $\parr$ is defined only on linear and consumer propositions.

The structural inference rules are given in Figure 4. Weakening and contraction apply only for producers on the left and consumers on the right. The rules for the operators $F_!$, $F_?$, $\lceil - \rceil$ and $\lfloor - \rfloor$ are more interesting as they must be able to encode dereliction and promotion. On the left, the $F_!$ and $\lceil - \rceil$ rules can be applied freely to transform linear propositions into producers and vice versa. These rules emulate the dereliction rule for linear logic:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \text{versus} \qquad \cfrac{\cfrac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L}}{\Gamma, F_! \, \lceil A \rceil \vdash \Delta}\ F_!\text{-L}$$

On the right however, these rules can only be applied when the contexts are persistent and of the correct form, as in the ! introduction rule in linear logic:

$$\frac{\Gamma^! \vdash \Delta^?, A}{\Gamma^! \vdash \Delta^?, !A} \qquad \text{versus} \qquad \cfrac{\cfrac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! \, \lceil A \rceil}\ F_!\text{-R}$$

The $F_?$ and $\lfloor - \rfloor$ rules are dual to those of $F_!$ and $\lceil - \rceil$.

3.1 Displacement

The commas on the left-hand-side of both the linear and persistent sequents intuitively correspond to the multiplicative product $\otimes$, and the commas on the right correspond to the multiplicative sum $\parr$. This correspondence motivates the context restriction of the rules that move between the linear and persistent

$$\frac{}{X \vdash X}\ \mathrm{Ax}^{\vdash}$$

$$\frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\bot_{\mathsf{m}} \vdash \cdot}\ \bot^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, \bot_{\mathsf{m}}}\ \bot^{\vdash}\text{-R}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L2}$$

$$\frac{\Gamma \vdash \Delta, A_1 \quad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \,\&\, A_2}\ \&^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{}{\Gamma \vdash \Delta, \top}\ \top^{\vdash}_{\mathsf{L}}\text{-R}$$

$$\frac{}{\Gamma, 0 \vdash \Delta}\ 0^{\vdash}_{\mathsf{L}}\text{-L} \qquad \frac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \oplus A_2 \vdash \Delta}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R2}$$

$$\frac{\Gamma, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, {X_1}^{\mathsf{m}} \otimes {X_2}^{\mathsf{m}} \vdash \Delta}\ \otimes^{\vdash}\text{-L} \qquad \frac{\Gamma_1 \vdash \Delta_1, {X_1}^{\mathsf{m}} \quad \Gamma_2 \vdash \Delta_2, {X_2}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, {X_1}^{\mathsf{m}} \otimes {X_2}^{\mathsf{m}}}\ \otimes^{\vdash}\text{-R}$$

$$\frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, 1_{\mathsf{m}} \vdash \Delta}\ 1^{\vdash}\text{-L} \qquad \frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\cdot \vdash 1_{\mathsf{m}}}\ 1^{\vdash}\text{-R}$$

$$\frac{\Gamma_1, {X_1}^{\mathsf{m}} \vdash \Delta_1 \quad \Gamma_2, {X_2}^{\mathsf{m}} \vdash \Delta_2 \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma_1, \Gamma_2, {X_1}^{\mathsf{m}} \parr {X_2}^{\mathsf{m}} \vdash \Delta_1, \Delta_2}\ \parr^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, {X_1}^{\mathsf{m}} \parr {X_2}^{\mathsf{m}}}\ \parr^{\vdash}\text{-R}$$

Figure 2: Linear Inference Rules for Linear Sequent

$$\frac{}{P \Vdash P}\ \mathrm{Ax}^{\Vdash}_{\mathsf{P}} \qquad \frac{}{C \Vdash C}\ \mathrm{Ax}^{\Vdash}_{\mathsf{C}}$$

$$\frac{\Gamma, P_1, P_2 \Vdash \Delta}{\Gamma, P_1 \otimes P_2 \Vdash \Delta}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{\Gamma_1 \Vdash \Delta_1, P_1 \quad \Gamma_2 \Vdash \Delta_2, P_2}{\Gamma_1, \Gamma_2 \Vdash \Delta_1, \Delta_2, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, 1_{\mathsf{P}} \Vdash \Delta}\ 1^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{}{\cdot \Vdash 1_{\mathsf{P}}}\ 1^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma_1, C_1 \Vdash \Delta_1 \quad \Gamma_2, C_2 \Vdash \Delta_2}{\Gamma_1, \Gamma_2, C_1 \parr C_2 \Vdash \Delta_1, \Delta_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C_1, C_2}{\Gamma \Vdash \Delta, C_1 \parr C_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{}{\bot_{\mathsf{C}} \Vdash \cdot}\ \bot^{\Vdash}_{\mathsf{C}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \bot_{\mathsf{C}}}\ \bot^{\Vdash}_{\mathsf{C}}\text{-R}$$

Figure 3: Linear Inference Rules for Persistent Sequent

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, C}\ \mathrm{W}^{\vdash}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{W}^{\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, C}\ \mathrm{W}^{\Vdash}\text{-R}$$

$$\frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{C}^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C, C}{\Gamma \vdash \Delta, C}\ \mathrm{C}^{\vdash}\text{-R}$$

$$\frac{\Gamma, P, P \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{C}^{\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C, C}{\Gamma \Vdash \Delta, C}\ \mathrm{C}^{\Vdash}\text{-R}$$

$$\frac{\Gamma, P \vdash \Delta}{\Gamma, F_{!}\, P \vdash \Delta}\ F_{!}\text{-L} \qquad \frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_{!}\, P}\ F_{!}\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, F_{?}\, C \vdash \Delta^{\mathsf{C}}}\ F_{?}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, F_{?}\, C}\ F_{?}\text{-R}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L} \qquad \frac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}}, A \vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, \lfloor A \rfloor \Vdash \Delta^{\mathsf{C}}}\ \lfloor - \rfloor\text{-L} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash A, \lfloor A \rfloor}\ \lfloor - \rfloor\text{-R}$$

Figure 4: Structural Inference Rules

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\Vdash}_{\mathsf{C}}$$

Figure 5: Cut Inference Rules

regimes. The restriction ensures that almost all of the propositions have the "natural" mode—producers on the left and consumers on the right. We say almost because the principle formula in each of these rules defies this classification. We will call such propositions *displaced*.

Definition 1. *In a derivation of* $\Gamma \Vdash \Delta$*, a producer* P *is* displaced *if it appears in* Δ*. A consumer* C *is* displaced *if it appears in* Γ*.*

Lemma 2 (Displacement)**.** *For any* LPC *derivation* $\mathcal{D}$ *of* $\Gamma \Vdash \Delta$*,* $\mathcal{D}$ *contains exactly one displaced proposition.*

Proof. By straightforward induction on $\mathcal{D}$.

If $\mathcal{D}$ is a producer axiom, then the proposition on the right is displaced; if $\mathcal{D}$ is a consumer axiom, then the proposition on the left is displaced.

If the last rule in $\mathcal{D}$ is a left tensor rule then then principle formula is not displaced, so the inductive hypothesis applies directly to prove there is exactly one displaced formula in $\mathcal{D}$. Similarly, weakening and contraction occur only for non-displaced propositions.

If $\mathcal{D}$ is the 1_{P} right rule then 1_{P} occurs in a displaced position with no other propositions in the contexts.

Suppose the last rule in $\mathcal{D}$ is a $\otimes$-R rule as follows:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}_1}{\Gamma_1 \vdash \Delta_1, P_1} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2 \vdash \Delta_2, P_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, P_1 \otimes P_2} \otimes^{\vdash}_{\mathsf{P}}\text{-R}$$

The inductive hypotheses for $\mathcal{D}_1$ and $\mathcal{D}_2$ state that there is exactly one displaced formula in each of these derivations. Since P_1 is displaced in $\mathcal{D}_1$, none of the formulas in Γ_1 or Δ_1 may be displaced, and similarly for $\mathcal{D}_2$. Therefore the only displaced formula in $\mathcal{D}$ itself is $P_1 \otimes P_2$.

Suppose the last rule in Δ is a $\lceil - \rceil$-R rule:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, \lceil A \rceil} \lceil - \rceil\text{-R}$$

By construction, none of the propositions in Γ^{P} or Δ^{C} are displaced, although $\lceil A \rceil$ is.

The cases for $\bot_{\mathsf{C}}$, $⅋$ and $\lfloor - \rfloor$ are similar. ☐

3.2 Cut Rules

This section presents the cut rules of Figure 5 and proves they are admissible in LPC. These rules are perhaps unintuitive—a simpler cut formulation might include the following pair of rules:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_1 \vdash \Delta_1, P} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, P \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \textsc{Cut}_{\mathsf{P}}\text{-}\textsc{Bad1} \qquad\qquad \dfrac{\dfrac{\mathcal{D}_1}{\Gamma_1 \vdash \Delta_1, P} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, P \Vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \textsc{Cut}_{\mathsf{P}}\text{-}\textsc{Bad2}$$

Notice that these rules could not be structured to produce persistent judgments when at least one of the hypotheses are linear, because linear propositions are not possible in persistent judgments. To see why these rules are inadmissible in LPC, consider the simple derivations

$$\mathcal{D}_1 = \dfrac{\dfrac{}{\cdot \vdash 1_{\mathsf{P}}} 1^{\vdash}_{\mathsf{P}}\text{-R}}{1_{\mathsf{L}} \vdash 1_{\mathsf{P}}} 1^{\vdash}_{\mathsf{L}}\text{-L} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\dfrac{}{\cdot \vdash 1_{\mathsf{L}}} 1^{\vdash}_{\mathsf{L}}\text{-R}}{1_{\mathsf{P}} \vdash 1_{\mathsf{L}}} 1^{\vdash}_{\mathsf{P}}\text{-L}}{1_{\mathsf{P}} \Vdash \lceil 1_{\mathsf{L}} \rceil} \lceil - \rceil\text{-R}$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textsc{Cut+}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\Vdash}_{\mathsf{C}}$$

Figure 6: Cut+ Inference Rules

Then the cut rule would postulate a derivation of $1_{\mathsf{L}} \vdash \lceil 1_{\mathsf{L}} \rceil$. But there is no cut-free method of constructing this derivation as it is impossible to introduce a term of the form $\lceil - \rceil$ on the right-hand-side of a linear sequent.

The next logical step is to restrict the propositions in $\mathcal{D}_1$ to persistent ones when the cut term is a producer—this way the resulting cut could produce a persistent judgment as well. So consider the following rule:

$$\frac{\dfrac{\mathcal{D}_1}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, P} \quad \dfrac{\mathcal{D}_2}{\Gamma_2, P \vdash \Delta_2}}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}_{\mathsf{P}}\textsc{-Bad3} \qquad \frac{\dfrac{\mathcal{D}_1}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, P} \quad \dfrac{\mathcal{D}_2}{\Gamma_2, P \Vdash \Delta_2}}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}_{\mathsf{P}}\textsc{-Bad4}$$

Consider then the derivations

$$\mathcal{D}_1 = \dfrac{\dfrac{\dfrac{}{\cdot \vdash 1_{\mathsf{P}}}\ 1^{\vdash}_{\mathsf{P}}\text{-R}}{1_{\mathsf{L}} \vdash 1_{\mathsf{P}}}\ 1^{\vdash}_{\mathsf{L}}\text{-L}}{\lceil 1_{\mathsf{L}} \rceil \vdash 1_{\mathsf{P}}}\ \lceil - \rceil\text{-L} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{}{\bot_{\mathsf{C}} \Vdash \bot_{\mathsf{C}}}\ \textsc{Ax}^{\Vdash}}{\bot_{\mathsf{C}}, 1_{\mathsf{P}} \Vdash \bot_{\mathsf{C}}}\ 1^{\Vdash}_{\mathsf{P}}\text{-L}$$

Then the desired cut sequent would be of the form $\lceil 1_{\mathsf{L}} \rceil, \bot_{\mathsf{C}} \Vdash \bot_{\mathsf{C}}$, which is not provable in LPC without Cut.

Admissibility

To show admissibility of the Cut rules, it is sufficient to show admissibility of an equivalent set of rules, called Cut+, which are given in Figure 6. For linear cut terms, the Cut+ rule is identical to the corresponding Cut rule. For persistent cut terms, Cut+ uses the observation that when a persistent proposition is *not* displaced in a sequent, it can be replicated any number of times. That is, for any n the derivations

$$\frac{\Gamma, (P)_n \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \text{and} \qquad \frac{\Gamma \vdash \Delta, (C)_n}{\Gamma \vdash \Delta, C}$$

are admissible in LPC, and similarly for the persistent derivation. It is easy to see that the Cut and Cut+ rules are equivalent in strength.

Lemma 3 (Cut+ Admissibility)**.** *The* Cut+ *rules in Figure 6 are admissible in* LPC*.*

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the hypotheses of one of the cut rules. We proceed by induction on the cut term primarily and the sum of the depths of $\mathcal{D}_1$ and $\mathcal{D}_2$ secondly.

1. Suppose $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a weakening rule on the cut term. In the case where the cut term is a producer and $\mathcal{D}_2$ is a linear judgment, we have

$$\frac{\mathcal{D}_1}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P} \qquad \text{and} \qquad \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P)_n \vdash \Delta_2}}{\Gamma_2, (P)_{n+1} \vdash \Delta_2}\ \text{W-L}$$

By the inductive hypothesis on P, $\mathcal{D}_1$ and $\mathcal{D}_2'$, there exists a cut-free derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2$. The persistent judgment and consumer cases are similar.

2. Suppose $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a contraction rule on the cut term. Again, if the cut term is a producer and $\mathcal{D}_2$ is a linear judgment, then

$$\dfrac{\mathcal{D}_1}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P} \qquad \text{and} \qquad \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P)_{n+2} \vdash \Delta_2}}{\Gamma_2, (P)_{n+1} \vdash \Delta_2}\ \text{C-L}$$

By the inductive hypothesis on P, $\mathcal{D}_1$ and $\mathcal{D}_2'$, there exists a derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2$. The persistent judgment and consumer cases are similar.

3. If $\mathcal{D}_1$ or $\mathcal{D}_2$ is an axiom, the case is trivial.

4. Suppose the cut term is the principle formula in both $\mathcal{D}_1$ and $\mathcal{D}_2$. It suffices to exclude weakening and contraction rules, as these have already been covered.

(&)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_1 \vdash \Delta_1, A_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_1 \vdash \Delta_1, A_2}}{\Gamma_1 \vdash \Delta_1, A_1 \,\&\, A_2}\ \&_{\mathsf{L}}^{\vdash}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, A_1 \vdash \Delta_2}}{\Gamma_2, A_1 \,\&\, A_2 \vdash \Delta_2}\ \&_{\mathsf{L}}^{\vdash}\text{-L1}$$

By the inductive hypothesis on A_1, $\mathcal{D}_{11}$ and $\mathcal{D}_2'$ there exists a derivation of $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ as desired.

(1_{L})

$$\mathcal{D}_1 = \dfrac{}{\cdot \vdash 1_{\mathsf{L}}}\ 1_{\mathsf{L}}^{\vdash}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2 \vdash \Delta_2}}{\Gamma_2, 1_{\mathsf{L}} \vdash \Delta_2}\ 1_{\mathsf{L}}^{\vdash}\text{-L}$$

Then $\mathcal{D}_2'$ itself is the desired derivation.

($\otimes_{\mathsf{L}}$)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11} \vdash \Delta_{11}, A_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_{12} \vdash \Delta_{12}, A_2}}{\Gamma_{11}, \Gamma_{12} \vdash \Delta_{11}, \Delta_{12}, A_1 \otimes A_2}\ \otimes_{\mathsf{L}}^{\vdash}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, A_1, A_2 \vdash \Delta_2}}{\Gamma_2, A_1 \otimes A_2 \vdash \Delta_2}\ \otimes_{\mathsf{L}}^{\vdash}\text{-L}$$

By the inductive hypothesis on A_2, $\mathcal{D}_{12}$ and $\mathcal{D}_2'$, there exists a derivation $\mathcal{E}$ of $\Gamma_{12}, \Gamma_2, A_1 \vdash \Delta_{12}, \Delta_2$. By the inductive hypothesis on A_1, $\mathcal{D}_{11}$ and $\mathcal{E}$ we can then obtain the desired derivation of $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \vdash \Delta_{11}, \Delta_{12}, \Delta_2$.

($\otimes_{\mathsf{P}}$)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, P_1 \otimes P_2}\ \otimes_{\mathsf{P}}^{\Vdash}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P_1 \otimes P_2)_n, P_1, P_2 \vdash \Delta_2}}{\Gamma_2, (P_1 \otimes P_2)_{n+1} \vdash \Delta_2}\ \otimes_{\mathsf{P}}^{\vdash}\text{-L}$$

First of all, the inductive hypothesis on $P_1 \otimes P_2$, $\mathcal{D}_1$ itself and $\mathcal{D}_2'$ gives us a derivation

$$\dfrac{\mathcal{E}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}$$

Multiple applications of the inductive hypothesis give the following derivation:

$$\dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \qquad \dfrac{\dfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2} \qquad \dfrac{\mathcal{E}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}}{\Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1 \vdash \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \mathrm{IH}(P_2)}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \mathrm{IH}(P_1)$$

Because the replicated contexts are made up exclusively of non-displaced propositions, it is possible to apply contraction multiple times to obtain the desired sequent.

($F_!$)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P}}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, F_! \, P}\ F_!\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, P \vdash \Delta}}{\Gamma_2, F_! \, P \vdash \Delta}\ F_!\text{-L}$$

Because of the form of the $F_!$-R rule, $\mathcal{D}_1'$ has an acceptable form for which to apply the inductive hypothesis.

($\lceil - \rceil$)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A}}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_2, \lceil A \rceil \vdash \Delta_2}\ \lceil - \rceil\text{-L}$$

The desired derivation is given by the inductive hypothesis applied to A, $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

5. Suppose the cut term is not the principle formula in $\mathcal{D}_1$.

(0-L)

$$\mathcal{D}_1 = \dfrac{}{\Gamma_1, 0 \vdash \Delta_1, A}\ 0^{\vdash}_{\mathsf{L}}\text{-L} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, A \vdash \Delta_2}$$

The desired derivation of $\Gamma_1, 0, \Gamma_2 \vdash \Delta_1, \Delta_2$ can be obtain by a direct application of the 0-L rule.

($\oplus$-L)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_1, A_1 \vdash \Delta_1, B} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_1, A_2 \vdash \Delta_1, B}}{\Gamma_1, A_1 \oplus A_2 \vdash \Delta_1, B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, B \vdash \Delta_2}$$

The inductive hypothesis on B, $\mathcal{D}_{11}$ and $\mathcal{D}_2$ asserts the existence of a derivation $\mathcal{E}_1'$ of $\Gamma_1, A_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ and similarly the inductive hypothesis for A_2, $\mathcal{D}_{12}$ and $\mathcal{D}_2$ gives a derivation $\mathcal{E}_2'$ of $\Gamma_1, A_2, \Gamma_2 \vdash \Delta_1, \Delta_2$. The desired derivation is given by application of the $\oplus$-L rule:

$$\dfrac{\dfrac{\mathcal{E}_1}{\Gamma_1, A_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \qquad \dfrac{\mathcal{E}_2}{\Gamma_1, A_2, \Gamma_2 \vdash \Delta_1, \Delta_2}}{\Gamma_1, A_1 \oplus A_2, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L}$$

($\oplus$-R)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1 \vdash \Delta_1, A_1, B}}{\Gamma_1 \vdash \Delta_1, A_1 \oplus A_2, B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, B \vdash \Delta_2}$$

The inductive hypothesis on B, $\mathcal{D}_1'$ and $\mathcal{D}_2$ gives a derivation $\mathcal{E}$ of $\Gamma_1, \Gamma_2 \vdash \Delta_1, A_1, \Delta_2$, from which we can obtain

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A_1, \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A_1 \oplus A_2, \Delta_2}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1}$$

($1^{\Vdash}_{\mathsf{P}}$-L)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1 \Vdash \Delta_1, P}}{\Gamma_1, 1_{\mathsf{P}} \Vdash \Delta_1, P}\ 1^{\Vdash}_{\mathsf{P}}\text{-L}$$

As in the previous examples, application of the constructor commutes with application of the inductive hypothesis.

($\otimes^{\Vdash}_{\mathsf{P}}$-L)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1, P_1, P_2 \Vdash \Delta_1, Q}}{\Gamma_1, P_1 \otimes P_2 \Vdash \Delta_1, Q}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, (Q)_n \vdash \Delta_2}$$

By the inductive hypothesis on Q, $\mathcal{D}_1'$ and $\mathcal{D}_2$, there exists a derivation $\mathcal{E}$ of $\Gamma_1, P_1, P_2, \Gamma_2 \vdash \Delta_1, \Delta_2$. From this we can apply the $\otimes$-L rule to obtain

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma_1, P_1, P_2, \Gamma_2 \vdash \Delta_1, \Delta_2}}{\Gamma_1, P_1 \otimes P_2, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \otimes^{\vdash}_{\mathsf{P}}\text{-L}$$

($\otimes^{\vdash}_{\mathsf{L}}$-R)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11} \vdash \Delta_{11}, A_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_{12} \vdash \Delta_{12}, A_2, B}}{\Gamma_{11}, \Gamma_{12} \vdash \Delta_{11}, \Delta_{12}, A_1 \otimes A_2, B}\ \otimes^{\vdash}_{\mathsf{L}}\text{-R} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2, B \vdash \Delta_2}$$

The inductive hypothesis on B, $\mathcal{D}_{12}$ and $\mathcal{D}_2$ asserts the existence of a derivation $\mathcal{E}$ of $\Gamma_{12}, \Gamma_2 \vdash \Delta_{12}, A_2, \Delta_2$. Application of the $\otimes$-R rule on $\mathcal{D}_{11}$ and $\mathcal{E}$ gives the desired derivation.

($\cdot$) If the cut term is a producer, then $\mathcal{D}_1$ is a persistent judgment so it cannot be the case that the last rule of $\mathcal{D}_1$ is a $F_!$ rule or a $\lceil - \rceil$-L or $\lfloor - \rfloor$-R rule. But it also cannot be the case that the last rule in $\mathcal{D}_1$ is a in a $\lceil - \rceil$-R or $\lfloor - \rfloor$-L rule because there is a non-principle formula—namely, the cut formula—which is in a displaced position.

($F_!$-L)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1, P \vdash \Delta_1, (C)_n}}{\Gamma_1, F_!\, P \vdash \Delta_1, (C)_n}\ F_!\text{-L} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}}$$

Then the inductive hypothesis on C, $\mathcal{D}_1'$ and $\mathcal{D}_2$ states that there exists a derivation $\mathcal{E}$ of $\Gamma_1, P, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}$ from which we can obtain

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma_1, P, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}}{\Gamma_1, F_!\, P, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ F_!\text{-L}$$

($F_!$-R)

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, (C)_n, P}}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, (C)_n, F_!\, P}\ F_!\text{-R} \qquad \text{and} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}}$$

Then by the inductive hypothesis on C, $\mathcal{D}_1'$ and $\mathcal{D}_2$ there exists a derivation $\mathcal{E}$ of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P, \Delta_2^{\mathsf{C}}$. From this we may construct the following derivation:

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P, \Delta_2^{\mathsf{C}}}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, F_!\, P, \Delta_2^{\mathsf{C}}}\ F_!\text{-R}$$

($\lceil - \rceil$-L)

$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_1'}{\Gamma_1, A \vdash \Delta_1, B}}{\Gamma_1, \lceil A \rceil \vdash \Delta_1, B} \lceil - \rceil\text{-L} \qquad \text{and} \qquad \cfrac{\mathcal{D}_2}{\Gamma_2, B \vdash \Delta_2}$$

The inductive hypothesis on B, $\mathcal{D}_1'$ and $\mathcal{D}_2$ gives a derivation of $\Gamma_1, A \vdash \Delta_1, \Delta_2$, to which the left $\lceil - \rceil$ rule can be applied as usual.

($\lceil - \rceil$-R)

$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, (C)_n}}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, (C)_n} \lceil - \rceil\text{-R} \qquad \text{and} \qquad \cfrac{\mathcal{D}_2}{\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}}$$

By the inductive hypothesis on P, $\mathcal{D}_1'$ and $\mathcal{D}_2$, there is a derivation $\mathcal{E}$ of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, \Delta_2^{\mathsf{C}}$. Because the contexts in $\mathcal{D}_2$ were undisplaced, it is possible to apply the $\lceil - \rceil$-R rule to $\mathcal{E}$ to obtain

$$\cfrac{\cfrac{\mathcal{E}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, \Delta_2^{\mathsf{C}}}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, \Delta_2^{\mathsf{C}}} \lceil - \rceil\text{-R}$$

(W-L)

$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, Q}}{\Gamma_1^{\mathsf{P}}, P \Vdash \Delta_1^{\mathsf{C}}, Q} \text{W-L} \qquad \text{and} \qquad \cfrac{\mathcal{D}_2}{\Gamma_2, (Q)_n \vdash \Delta_2}$$

Weakening can be applied to any context, so here it commutes with the inductive hypothesis. The case for contraction is similar.

□

Corollary 4 (Cut Admissibility)**.** *The* Cut *rules in Figure 5 are admissible in* LPC.

3.3 Duality

Looking again at the LPC inference rules, it is easy to see that every rule has a dual. We take advantage of this implicit duality in our proofs to cut down the number of cases we have to consider, but we have not yet made this notion formal. Unlike standard presentations of linear logic, LPC does not consist of an explicit duality operator $(-)^{\perp}$, nor a linear implication $\multimap$ with which to encode duality. Instead, we will define $(-)^{\perp}$ to be a meta-operation on propositions and prove the following duality rules are admissible in LPC:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta} \; (-)^{\perp}\text{-L} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}} \; (-)^{\perp}\text{-R}$$

In fact, we define three versions of the duality operation: $(-)^{\perp}$ for linear propositions, $(-)^*$ for producers and $(-)_*$ for consumers. These operators have the property that for a linear proposition A, $A^{\perp}$ is linear, but for a producer P, P^* is a consumer, and for a consumer C, C_* is a producer. The duality rules are given in Figure 7.

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (-)^{\perp}\text{-R}$$

$$\frac{\Gamma \vdash \Delta, P}{\Gamma, P^{*} \vdash \Delta}\ (-)^{*\vdash}\text{-L} \qquad\qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \Delta, P^{*}}\ (-)^{*\vdash}\text{-L-R}$$

$$\frac{\Gamma \Vdash \Delta, P}{\Gamma, P^{*} \Vdash \Delta}\ (-)^{*\Vdash}\text{-L} \qquad\qquad \frac{\Gamma, P \Vdash \Delta}{\Gamma \Vdash \Delta, P^{*}}\ (-)^{*\Vdash}\text{-R}$$

$$\frac{\Gamma \vdash \Delta, C}{\Gamma, C_{*} \vdash \Delta}\ (-)_{*}{}^{\vdash}\text{-L} \qquad\qquad \frac{\Gamma, C \vdash \Delta}{\Gamma \vdash \Delta, C_{*}}\ (-)_{*}{}^{\vdash}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta, C}{\Gamma, C_{*} \Vdash \Delta}\ (-)_{*}{}^{\Vdash}\text{-L} \qquad\qquad \frac{\Gamma, C \Vdash \Delta}{\Gamma \Vdash \Delta, C_{*}}\ (-)_{*}{}^{\Vdash}\text{-R}$$

Figure 7: Duality inference rules

We define these duality operations as follows:

$$\begin{array}{rlcrl}
1_{\mathsf{L}}^{\perp} & := \perp_{\mathsf{L}} & \qquad & (F_{!}\, P)^{\perp} & := F_{?}\, P^{*} \\
(A \otimes B)^{\perp} & := A^{\perp} \parr B^{\perp} & & (F_{?}\, C)^{\perp} & := F_{!}\, C_{*} \\
\perp_{\mathsf{L}}^{\perp} & := 1_{\mathsf{L}} & & 1_{\mathsf{P}}^{*} & := \perp_{\mathsf{C}} \\
(A \parr B)^{\perp} & := A^{\perp} \otimes B^{\perp} & & (P \otimes Q)^{*} & := P^{*} \parr Q^{*} \\
\top^{\perp} & := 0 & & \lceil A \rceil^{*} & := \lfloor A^{\perp} \rfloor \\
(A \mathbin{\&} B)^{\perp} & := A^{\perp} \oplus B^{\perp} & & \perp_{\mathsf{C}*} & := 1_{\mathsf{P}} \\
0^{\perp} & := \top & & (C \parr D)_{*} & := C_{*} \otimes D_{*} \\
(A \oplus B)^{\perp} & := A^{\perp} \mathbin{\&} B^{\perp} & & \lfloor A \rfloor_{*} & := \lceil A^{\perp} \rceil
\end{array}$$

Lemma 5. *The following axioms hold in* LPC*:*

$$\frac{}{A, A^{\perp} \vdash \cdot} \qquad \frac{}{P, P^{*} \Vdash \cdot} \qquad \frac{}{C, C_{*} \Vdash \cdot}$$

Proof. We proceed by mutual induction on A, P and C.

- If $A = 0$ then the derivation of $0, \top \vdash \cdot$ is given by the 0-L rule.
- If $A = A_1 \oplus A_2$ then the inductive hypotheses give us derivations $\mathcal{D}_1$ of $A_1, {A_1}^{\perp} \vdash \cdot$ and $\mathcal{D}_2$ of $A_2, {A_2}^{\perp} \vdash \cdot$. The desired derivation can then be constructed as follows:

$$\frac{\dfrac{\dfrac{\mathcal{D}_1}{A_1, {A_1}^{\perp} \vdash \cdot}}{A_1, {A_1}^{\perp} \mathbin{\&} {A_2}^{\perp} \vdash \cdot}\ \&_{\mathsf{L}}^{\vdash}\text{-L1} \qquad \dfrac{\dfrac{\mathcal{D}_2}{A_2, {A_2}^{\perp} \vdash \cdot}}{A_2, {A_1}^{\perp} \mathbin{\&} {A_2}^{\perp} \vdash \cdot}\ \&_{\mathsf{L}}^{\vdash}\text{-L2}}{A_1 \oplus A_2, {A_1}^{\perp} \mathbin{\&} {A_2}^{\perp} \vdash \cdot}\ \oplus_{\mathsf{L}}^{\vdash}\text{-L}$$

- If $A = 1_{\mathsf{L}}$ then the following is a valid derivation of $1_{\mathsf{L}}, \perp_{\mathsf{L}} \vdash \cdot$:

$$\frac{\dfrac{}{\perp_{\mathsf{L}} \vdash \cdot}\ \perp_{\mathsf{L}}^{\vdash}\text{-L}}{1_{\mathsf{L}}, \perp_{\mathsf{L}} \vdash \cdot}\ 1_{\mathsf{L}}^{\vdash}\text{-L}$$

- If $A = A_1 \otimes A_2$ then the inductive hypotheses postulate the derivations $\mathcal{D}_1$ of $A_1, {A_1}^\perp \vdash \cdot$ and $\mathcal{D}_2$ of $A_2, {A_2}^\perp \vdash \cdot$. From these it is possible to obtain the following:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A_1, {A_1}^\perp \vdash \cdot} \qquad \dfrac{\mathcal{D}_2}{A_2, {A_2}^\perp \vdash \cdot}}{A_1, A_2, {A_1}^\perp \parr {A_2}^\perp \vdash \cdot}\;\parr^{\vdash}_{\mathsf{L}}\text{-L}}{A_1 \otimes A_2, {A_1}^\perp \parr {A_2}^\perp \vdash \cdot}\;\otimes^{\vdash}_{\mathsf{L}}\text{-L}$$

- If $A = F_! \, P$ then the inductive hypothesis for P gives a derivation $\mathcal{D}$ of $P, P^* \Vdash \cdot$. We then have

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{P, P^* \Vdash \cdot}}{P, F_? \, P^* \vdash \cdot}\;F_?\text{-L}}{F_! \, P, F_? \, P^* \vdash \cdot}\;F_!\text{-L}$$

- If $P = \lceil A \rceil$ then the inductive hypothesis provides a derivation $\mathcal{D}$ of $A, A^\perp \vdash \cdot$. Consider

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{A, A^\perp \vdash \cdot}}{\lceil A \rceil, A^\perp \vdash \cdot}\;\lceil - \rceil\text{-L}}{\lceil A \rceil, \lfloor A^\perp \rfloor \Vdash \cdot}\;\lfloor - \rfloor\text{-L}$$

The other cases are similar. $\square$

An isomorphic proof shows that $\cdot \vdash A, A^\perp$, $\cdot \Vdash P, P^*$ and $\cdot \Vdash C, C_*$ are axioms as well. This proof does *not* extend to the case of persistent propositions in a linear sequent, but it is nevertheless possible to construct proofs of $P, P^* \vdash \cdot$ and $\cdot \vdash P, P^*$ as follows:

$$\dfrac{\dfrac{}{P^* \vdash P^*}\;\textsc{Ax}^{\vdash} \dfrac{}{P, P^* \Vdash \cdot}}{P, P^* \vdash \cdot}\;\textsc{Cut}^{\vdash}_{\mathsf{C}} \qquad\qquad \dfrac{\dfrac{}{\cdot \Vdash P, P^*} \qquad \dfrac{}{P \vdash P}\;\textsc{Ax}^{\vdash}}{\cdot \vdash P, P^*}\;\textsc{Cut}^{\vdash}_{\mathsf{P}}$$

Theorem 6 (Admissibility of Duality)**.** *The duality inference rules given in Figure 7 are admissible in* LPC*.*

Proof. The linear rules can be encoded using cut and the lemmas above.

$$\dfrac{\Gamma \vdash \Delta, A \qquad \dfrac{}{A, A^\perp \vdash \cdot}}{\Gamma, A^\perp \vdash \Delta}\;\textsc{Cut}^{\vdash}_{\mathsf{L}} \qquad\qquad \dfrac{\dfrac{}{\cdot \vdash A, A^\perp} \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^\perp}\;\textsc{Cut}^{\vdash}_{\mathsf{L}}$$

Next, consider the producer duality rule for persistent sequents. The right rule can be implemented by means of a cut, but when applying cut for the left rule, the mode of the contexts are restricted, as follows:

$$\dfrac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \dfrac{}{P, P^* \Vdash \cdot}}{\Gamma^{\mathsf{P}}, P^* \Vdash \Delta^{\mathsf{C}}}\;\textsc{Cut}^{\Vdash}_{\mathsf{P}}$$

By the displacement theorem this restriction is actually redundant; P is the only displaced proposition in any derivation of $\Gamma \Vdash \Delta, P$.

The same restrictions on the cut rules lead to the following left duality rule for producers in linear sequents, which is *not* equivalent to the one in Figure 7:

$$\dfrac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \dfrac{}{P, P^* \vdash \cdot}}{\Gamma^{\mathsf{P}}, P^* \vdash \Delta^{\mathsf{C}}}\;\textsc{Cut}^{\vdash}_{\mathsf{P}}$$

In this case the hypothesis and conclusion are different types of sequent, and the restrictions exclude the occurrence of any linear terms in the contexts. We can avoid these restrictions by proving the more general form of the rule directly. Let $\mathcal{D}$ be any derivation of $\Gamma \vdash \Delta, P$. We will prove by induction on $\mathcal{D}$ that there is a derivation of $\Gamma, P^* \vdash \Delta$.

- If P is not the principle formula in $\mathcal{D}$ then we can apply the inductive hypothesis directly to obtain the desired sequent.
- If $\mathcal{D}$ is an axiom, then the desired sequent has the form $P, P^* \vdash \cdot$, which we constructed above.
- The last rule in $\mathcal{D}$ cannot be the $\lceil - \rceil$ right rule because $\mathcal{D}$ is a linear sequent.
- If $\mathcal{D}$ is the 1_{P} right rule then the desired sequent is $\bot_{\mathsf{C}} \vdash \cdot$, which is simply the $\bot_{\mathsf{C}}$ left rule.
- Finally, if

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}_1}{\Gamma_1 \vdash \Delta_1, P_1} \qquad \dfrac{\mathcal{D}_2}{\Gamma_2 \vdash \Delta_2, P_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, P_1 \otimes P_2} \otimes_{\mathsf{P}}^{\vdash}\text{-R}$$

  then the inductive hypotheses for $\mathcal{D}_1$ and $\mathcal{D}_2$ give us derivations $\mathcal{D}_1'$ of $\Gamma_1, {P_1}^* \vdash \Delta_1$ and $\mathcal{D}_2'$ of $\Gamma_2, {P_2}^* \vdash \Delta_2$. From this we can obtain

$$\dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1, {P_1}^* \vdash \Delta_1} \qquad \dfrac{\mathcal{D}_2'}{\Gamma_2, {P_2}^* \vdash \Delta_2}}{\Gamma_1, \Gamma_2, {P_1}^* \parr {P_2}^* \vdash \Delta_1, \Delta_2} \parr_{\mathsf{C}}^{\vdash}\text{-L}$$

The formulation of duality rules for consumers in linear sequents is similar.

□

3.4 Consistency

Using the cut and duality rules we can prove that LPC is consistent. Define the negation of a linear proposition to be $\neg A := A^{\perp} \parr 0$.

Lemma 7 (Consistency)**.** *There is no proposition A such that A and $\neg A$ are both provable in* LPC*.*

Proof. Suppose there were such an A, along with derivations $\mathcal{D}_1$ of $\cdot \vdash A$ and $\mathcal{D}_2$ of $\cdot \vdash A^{\perp} \parr 0$. Then we construct a derivation of $\cdot \vdash 0$ as follows:

$$\dfrac{\dfrac{\mathcal{D}_2}{\cdot \vdash A^{\perp}, 0} \qquad \dfrac{\dfrac{\mathcal{D}_1}{\cdot \vdash A}}{A^{\perp} \vdash \cdot}(-)^{\perp}\text{-L}}{\cdot \vdash 0} \textsc{Cut}_{\mathsf{L}}^{\vdash}$$

However, there is no cut-free proof of $\cdot \vdash 0$ in LPC, which contradicts cut admissibility. □

4 Categorical Model

In this section we will describe a categorical axiomatization of LPC based on the three-category picture in Figure 1. We start by defining each of the three categories and the necessary adjunctions between them. Then we show LPC's equivalence to other semantic models in the literature. Finally we define an interpretation of the logic into the category theory and demonstrate some properties about the interpretation.

4.1 The LPC Model

Monoidal Structures

We start with some basic definitions about symmetric monoidal structures.

Definition 8. *A* symmetric monoidal category *is a category $\mathcal{C}$ equipped with a bifunctor $\otimes$, an object 1, and the following natural isomorphisms:*

$$
\begin{aligned}
\alpha_{A_1,A_2,A_3} &: A_1 \otimes (A_2 \otimes A_3) \to (A_1 \otimes A_2) \otimes A_3 \\
\lambda_A &: 1 \otimes A \to A \\
\rho_A &: A \otimes 1 \to A \\
\sigma_{A,B} &: A \otimes B \to B \otimes A
\end{aligned}
$$

These must satisfy the following coherence diagrams:

$$
\begin{tikzcd}
A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) \arrow[r, "id_{A_1} \otimes \alpha_{A_2,A_3,A_4}"] \arrow[d, "\alpha_{A_1,A_2,A_3 \otimes A_4}"'] & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \arrow[dd, "\alpha_{A_1,A_2 \otimes A_3,A_4}"] \\
(A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[d, "\alpha_{A_1 \otimes A_2,A_3,A_4}"'] & \\
((A_1 \otimes A_2) \otimes A_3) \otimes A_4 \arrow[r, "{\alpha_{A_1,A_2,A_3}}^{-1} \otimes id_{A_4}"'] & (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \otimes (1 \otimes A_2) \arrow[rr, "\alpha_{A_1,1,A_2}"] \arrow[dr, "id_{A_1} \otimes \lambda_{A_2}"'] & & (A_1 \otimes 1) \otimes A_2 \arrow[dl, "\rho_{A_1} \otimes id_{A_2}"] \\
& A_1 \otimes A_2 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \otimes (A_2 \otimes A_3) \arrow[r, "\alpha_{A_1,A_2,A_3}"] \arrow[d, "id_{A_1} \otimes \sigma_{A_2,A_3}"'] & (A_1 \otimes A_2) \otimes A_3 \arrow[r, "\sigma_{A_1 \otimes A_2,A_3}"] & A_3 \otimes (A_1 \otimes A_2) \arrow[d, "\alpha_{A_3,A_1,A_2}"] \\
A_1 \otimes (A_3 \otimes A_2) \arrow[r, "\alpha_{A_1,A_3,A_2}"'] & (A_1 \otimes A_3) \otimes A_2 \arrow[r, "\sigma_{A_1,A_3} \otimes id_{A_2}"'] & (A_3 \otimes A_1) \otimes A_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes B \arrow[d, "\sigma_{A,B}"'] \arrow[dr, "id_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\sigma_{B,A}"'] & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes A \arrow[rr, "\sigma_{1,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes 1 \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}
$$

Definition 9. *Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \sigma)$ and $(\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho', \sigma')$ be symmetric monoidal categories. A* symmetric monoidal functor $\mathrm{F} : \mathcal{C} \Rightarrow \mathcal{C}'$ *is a functor along with a map $m^{\mathrm{F}}_1 : 1' \to \mathrm{F}\,1$ and a natural transformation $m^{\mathrm{F}}_{A,B} : \mathrm{F}\,(A) \otimes' \mathrm{F}\,(B) \to \mathrm{F}\,(A \otimes B)$ that satisfy the following coherence conditions:*

$$\begin{tikzcd}
(\mathrm{F}\,(A_1) \otimes' \mathrm{F}\,(A_2)) \otimes' \mathrm{F}\,(A_3) \arrow[r, "\alpha'_{A_1,A_2,A_3}"] \arrow[d, "m^{\mathrm{F}}_{A_1,A_2} \otimes' id_{\mathrm{F}\,(A_3)}"'] & \mathrm{F}\,(A_1) \otimes' (\mathrm{F}\,(A_2) \otimes' \mathrm{F}\,(A_3)) \arrow[d, "id_{\mathrm{F}\,(A_1)} \otimes' m^{\mathrm{F}}_{A_2,A_3}"] \\
\mathrm{F}\,(A_1 \otimes A_2) \otimes' \mathrm{F}\,(A_3) \arrow[d, "m^{\mathrm{F}}_{A_1 \otimes A_2, A_3}"'] & \mathrm{F}\,(A_1) \otimes' \mathrm{F}\,(A_2 \otimes A_3) \arrow[d, "m^{\mathrm{F}}_{A_1, A_2 \otimes A_3}"] \\
\mathrm{F}\,((A_1 \otimes A_2) \otimes A_3) \arrow[r, "\mathrm{F}\,(\alpha_{A_1,A_2,A_3})"'] & \mathrm{F}\,(A_1 \otimes (A_2 \otimes A_3))
\end{tikzcd}$$

$$\begin{tikzcd}
1' \otimes' \mathrm{F}\,(A) \arrow[r, "\lambda'_{\mathrm{F}\,(A)}"] \arrow[d, "m^{\mathrm{F}}_1 \otimes' id_{\mathrm{F}\,(A)}"'] & \mathrm{F}\,(A) \\
\mathrm{F}\,(1) \otimes' \mathrm{F}\,(A) \arrow[r, "m^{\mathrm{F}}_{1,A}"] & \mathrm{F}\,(1 \otimes A) \arrow[u, "\mathrm{F}\,(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}\,(A) \otimes' 1' \arrow[r, "\rho'_{\mathrm{F}\,(A)}"] \arrow[d, "id_{\mathrm{F}\,(A)} \otimes' m^{\mathrm{F}}_1"'] & \mathrm{F}\,(A) \\
\mathrm{F}\,(A) \otimes' \mathrm{F}\,(1) \arrow[r, "m^{\mathrm{F}}_{A,1}"] & \mathrm{F}\,(A \otimes 1) \arrow[u, "\mathrm{F}\,(\rho_A)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{F}\,(A) \otimes' \mathrm{F}\,(B) \arrow[r, "\sigma'_{\mathrm{F}\,(A),\mathrm{F}\,(B)}"] \arrow[d, "m^{\mathrm{F}}_{A,B}"'] & \mathrm{F}\,(B) \otimes' \mathrm{F}\,(A) \arrow[d, "m^{\mathrm{F}}_{B,A}"] \\
\mathrm{F}\,(A \otimes B) \arrow[r, "\mathrm{F}\,(\sigma_{A,B})"'] & \mathrm{F}\,(B \otimes A)
\end{tikzcd}$$

A functor $\mathrm{F} : \mathcal{C} \Rightarrow \mathcal{C}'$ is symmetric comonoidal *if it is equipped with a map $n^{\mathrm{F}}_1 : 1' \to \mathrm{F}\,1$ and natural transformation $n^{\mathrm{F}}_{A,B} : \mathrm{F}\,(A \otimes B) \to \mathrm{F}\,(A) \otimes' \mathrm{F}\,(B)$ such that the appropriate (dual) diagrams commute.*

Definition 10. *Let F and G be symmetric monoidal functors $\mathrm{F}, \mathrm{G} : \mathcal{C} \Rightarrow \mathcal{C}'$. A monoidal natural transformation $\tau : \mathrm{F} \to \mathrm{G}$ is a natural transformation satisfying*

$$\begin{tikzcd}
\mathrm{F}\,(A) \otimes' \mathrm{F}\,(B) \arrow[r, "m^{\mathrm{F}}_{A,B}"] \arrow[d, "\tau_A \otimes' \tau_B"'] & \mathrm{F}\,(A \otimes B) \arrow[d, "\tau_{(A \otimes B)}"] \\
\mathrm{G}\,(A) \otimes' \mathrm{G}\,(B) \arrow[r, "m^{\mathrm{G}}_{A,B}"'] & \mathrm{G}\,(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}\,(1) \arrow[rr, "\tau_1"] & & \mathrm{G}\,(1) \\
& 1' \arrow[ul, "m^{\mathrm{F}}_1"] \arrow[ur, "m^{\mathrm{G}}_1"'] &
\end{tikzcd}$$

For F and G symmetric comonoidal functors, a natural transformation $\tau : \mathrm{F} \to \mathrm{G}$ is comonoidal if it satisfies the appropriate dual diagrams.

Definition 11. *A* symmetric (co-)monoidal adjunction *is an adjunction $\mathrm{F} \dashv \mathrm{G}$ between symmetric (co-) monoidal functors F and G where the unit and counit of the adjunction are symmetric (co-)monoidal natural transformations.*

Linear Category

The linear category should interpret the inference rules from Figure 2 as well as the linear duality. Traditionally this fragment of linear logic is modeled by a *-autonomous category, where multiplicative product $\otimes$ and linear implication $\multimap$ are the primitive operators. In LPC however, the primitive multiplicative operators are $\otimes$ and $\parr$, while linear implication $\multimap$ is derived as $A \multimap B := A^{\perp} \parr B$. Therefore the linear category in the LPC axiomatization is a linearly distributive category, introduced by Cockett and Seely (1997), where the two monoidal structures are the primitive components.

Definition 12. *Let $\mathcal{L}$ be a category with two symmetric monoidal structures: $(\otimes, 1, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \sigma^{\otimes})$ and $(\parr, \perp, \alpha^{\parr}, \lambda^{\parr}, \rho^{\parr}, \sigma^{\parr})$. Let $\delta_{A_1,A_2,A_3} : A_1 \otimes (A_2 \parr A_3) \to (A_1 \otimes A_2) \parr A_3$ be a natural transformation in $\mathcal{L}$. We can derive the following variations on δ:*

$$\begin{tikzcd}
A_1 \otimes (A_2 \parr A_3) \arrow[r, "id \otimes \sigma^{\parr}"] \arrow[d, "\delta^{L,L} = \delta"'] & A_1 \otimes (A_3 \parr A_2) \arrow[r, "\sigma^{\otimes}"] \arrow[d, "\delta^{L,R}"'] & (A_3 \parr A_2) \otimes A_1 \arrow[r, "\sigma^{\parr} \otimes id"] \arrow[d, "\delta^{R,R}"'] & (A_2 \parr A_3) \otimes A_1 \arrow[d, "\delta^{R,L}"'] \\
(A_1 \otimes A_2) \parr A_3 \arrow[r, "\sigma^{\parr}"'] & A_3 \parr (A_1 \otimes A_2) \arrow[r, "id \parr \sigma^{\otimes}"'] & A_3 \parr (A_2 \otimes A_1) \arrow[r, "\sigma^{\parr}"'] & (A_2 \otimes A_1) \parr A_3
\end{tikzcd}$$

Then $\mathcal{L}$ is a symmetric linearly distributive category *if the following coherence conditions are satisfied:*

Distribution and Unit

$$\begin{tikzcd}
1 \otimes (A \parr B) \arrow[rr, "\delta^{L,L}"] \arrow[dr, "\lambda^{\otimes}"'] & & (1 \otimes A) \parr B \arrow[dl, "\rho^{\otimes} \otimes id"] \\
& A \parr B &
\end{tikzcd}$$

Distribution and Associativity

$$\begin{tikzcd}
A_1 \otimes (A_2 \otimes (A_3 \parr A_4)) \arrow[r, "\alpha^{\otimes}"] \arrow[d, "id \otimes \delta^{L,L}"'] & (A_1 \otimes A_2) \otimes (A_3 \parr A_4) \arrow[dd, "\delta^{L,L}"'] \\
A_1 \otimes ((A_2 \otimes A_3) \parr A_4) \arrow[d, "\delta^{L,L}"'] & \\
(A_1 \otimes (A_2 \otimes A_3)) \parr A_4 \arrow[r, "\alpha^{\otimes} \parr id"] & ((A_1 \otimes A_2) \otimes A_3) \parr A_4
\end{tikzcd}$$

Distribution and Distribution

$$\begin{tikzcd}
& (A_1 \parr A_2) \otimes (A_3 \parr A_4) \arrow[dl, "\delta^{R,R}"'] \arrow[dr, "\delta^{L,L}"] & \\
A_1 \parr (A_2 \otimes (A_3 \parr A_4)) \arrow[d, "id \parr \delta^{L,L}"'] & & ((A_1 \parr A_2) \otimes A_3) \parr A_4 \arrow[d, "\delta^{R,R} \parr id"] \\
A_1 \parr ((A_2 \otimes A_3) \parr A_4) \arrow[rr, "\alpha^{\parr}"'] & & (A_1 \parr (A_2 \otimes A_3)) \parr A_4
\end{tikzcd}$$

Coassociativity and Distribution

$$\begin{tikzcd}
A_1 \otimes (A_2 \parr (A_3 \parr A_4)) \arrow[r, "id \otimes \alpha^{\parr}"] \arrow[d, "\delta^{L,R}"'] & A_1 \otimes ((A_2 \parr A_3) \parr A_4) \arrow[d, "\delta^{L,L}"] \\
A_2 \parr (A_1 \otimes (A_3 \parr A_4)) \arrow[d, "id \parr \delta^{L,L}"'] & (A_1 \otimes (A_2 \parr A_3)) \parr A_4 \arrow[d, "\delta^{L,R} \parr id"] \\
A_2 \parr ((A_1 \otimes A_3) \parr A_4) \arrow[r, "\alpha^{\parr}"'] & (A_2 \parr (A_1 \otimes A_3)) \parr A_4
\end{tikzcd}$$

In addition, all of the above diagrams must hold when translated through the following symmetries:[1] *[op'] reverses the arrows and swaps* $\otimes$ *and* $\parr$*, as well as* 1 *and* $\perp$*.*

$$\begin{array}{lll}
\delta^{L,L} \leftrightarrow \delta^{R,R} & \alpha^{\otimes} \mapsto (\alpha^{\parr})^{-1} & \alpha^{\parr} \mapsto (\alpha^{\otimes})^{-1} \\
\delta^{L,R} \mapsto \delta^{L,R} & \lambda^{\otimes} \mapsto (\lambda^{\parr})^{-1} & \lambda^{\parr} \mapsto (\lambda^{\otimes})^{-1} \\
\delta^{R,L} \mapsto \delta^{R,L} & \rho^{\otimes} \mapsto (\rho^{\parr})^{-1} & \rho^{\parr} \mapsto (\rho^{\otimes})^{-1} \\
 & \sigma^{\otimes} \mapsto (\sigma^{\parr})^{-1} & \sigma^{\parr} \mapsto (\sigma^{\otimes})^{-1}
\end{array}$$

$\otimes'$ *swaps the arguments of the tensor* $\otimes$*:*

$$\begin{array}{ll}
\delta^{L,L} \leftrightarrow \delta^{R,L} & \alpha^{\otimes} \mapsto (\alpha^{\otimes})^{-1} \\
\delta^{L,R} \leftrightarrow \delta^{R,R} & \lambda^{\otimes} \leftrightarrow \rho^{\otimes} \\
 & \sigma^{\otimes} \mapsto (\sigma^{\otimes})^{-1}
\end{array}$$

$\parr'$ *swaps the arguments of the cotensor* $\parr$*:*

$$\begin{array}{ll}
\delta^{L,L} \leftrightarrow \delta^{L,R} & \alpha^{\parr} \mapsto (\alpha^{\parr})^{-1} \\
\delta^{R,L} \leftrightarrow \delta^{R,R} & \lambda^{\parr} \leftrightarrow \rho^{\parr} \\
 & \sigma^{\parr} \mapsto (\sigma^{\parr})^{-1}
\end{array}$$

The details of these symmetries are described in more detail by Cockett and Seely (1997).

Definition 13. *A symmetric linear distributive category* $\mathcal{L}$ *is said to* have negation *if there exists a map* $(-)^{\perp}$ *on objects of* $\mathcal{L}$*, and families of maps*

$$\gamma_A^{\perp} : A^{\perp} \otimes A \to \perp \qquad\qquad \gamma_A^{1} : 1 \to A \parr A^{\perp}$$

inducing maps

$$\gamma_A^{\perp\,\prime} : A \otimes A^{\perp} \xrightarrow{\sigma^{\otimes}_{A,A^{\perp}}} A^{\perp} \otimes A \xrightarrow{\gamma_A^{\perp}} \perp$$

$$\gamma_A^{1\,\prime} : 1 \xrightarrow{\gamma_A^{1}} A \parr A^{\perp} \xrightarrow{\sigma^{\parr}_{A,A^{\perp}}} A^{\perp} \parr A$$

which satisfy the following diagrams under the symmetries op', $\otimes'$ *and* $\parr'$*:*

[1] When not specified, the symmetry is the identity.

$$
\begin{tikzcd}
A \otimes 1 \arrow[r, "id_A \otimes {\gamma^1_{A^\perp}}'"] \arrow[drr, "\rho^\otimes_A"'] & A \otimes (A^\perp \parr A) \arrow[r, "\delta^{L,L}"] & (A \otimes A^\perp) \parr A \arrow[r, "{\gamma^\perp_{A^\perp}}' \parr id_A"] & \perp \parr A \arrow[dl, "\lambda^\parr_A"] \\
 & & A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A^\perp \otimes 1 \arrow[r, "id_{A^\perp} \otimes \gamma^1_A"] \arrow[drr, "\rho^\otimes_{A^\perp}"'] & A^\perp \otimes (A \parr A^\perp) \arrow[r, "\delta^{L,L}"] & (A^\perp \otimes A) \parr A^\perp \arrow[r, "\gamma^\perp_A \parr id_{A^\perp}"] & \perp \parr A^\perp \arrow[dl, "\lambda^\parr_{A^\perp}"] \\
 & & A^\perp &
\end{tikzcd}
$$

Lemma 14 (Cockett and Seely (1997)). *Symmetric linearly distributive categories with negation correspond to *-autonomous categories.*

As a corollary, $(-)^\perp$ extends to a contravariant involutive functor which is both monoidal and comonoidal.

To encode the additives, we require that the linear category has finite products $\&$ with unit $\top$, and finite coproducts $\oplus$ with unit 0.

The Persistent Categories

The producer and consumer categories must model weakening and contraction, but they must also be related via a categorical duality that respects the monoidal structures.

Definition 15. *Two symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, \parr)$ are in duality with each other if there exist contravariant functors $(-)^* : \mathcal{P} \Rightarrow \mathcal{C}$, which is comonoidal, and $(-)_* : \mathcal{C} \Rightarrow \mathcal{P}$, which is monoidal, and natural isomorphisms*

$$
\epsilon^*_{*\,C} : (C_*)^* \to C \qquad \textit{and} \qquad \eta^*_{*\,P} : P \to (P^*)_*
$$

where ϵ^*_* is comonoidal and η^*_* is monoidal.

Definition 16. *Let $(\mathcal{P}, \otimes, 1_{\mathsf{P}})$ be a symmetric monoidal category. A commutative comonoid in $\mathcal{P}$ is an object P in $\mathcal{P}$ along with two morphisms $e^\otimes : P \to 1_{\mathsf{P}}$ and $d^\otimes : P \to P \otimes P$ such that the following commuting diagrams are satisfied:*

$$
\begin{tikzcd}
 & P \arrow[dl, "d^\otimes"'] \arrow[dr, "d^\otimes"] & \\
P \otimes P \arrow[rr, "\sigma_{P,P}"'] & & P \otimes P
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "d^\otimes"] \arrow[dr, "{\lambda_P}^{-1}"'] & P \otimes P \arrow[d, "e^\otimes \otimes id_P"] \\
 & 1_{\mathsf{P}} \otimes P
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & P \otimes P \arrow[r, "id_P \otimes d^\otimes"] & P \otimes (P \otimes P) \arrow[d, "\alpha_{P,P,P}"] \\
P \arrow[ur, "d^\otimes"] \arrow[dr, "d^\otimes"'] & & \\
 & P \otimes P \arrow[r, "d^\otimes \otimes id_P"'] & (P \otimes P) \otimes P
\end{tikzcd}
$$

Dually, a commutative monoid in a symmetric monoidal category $(\mathcal{C}, ⅋, \bot_{\mathsf{C}})$ is an object C along with morphisms $e^{⅋} : \bot_{\mathsf{C}} \to C$ and $d^{⅋} : C ⅋ C \to C$ such that the appropriate diagrams commute.

The LPC model

Definition 17. *A linear/producing/consuming (*LPC*) model consists of the following components:*

1. *A symmetric weakly distributive category $(\mathcal{L}, \otimes, ⅋)$ with negation $(-)^{\perp}$, finite products & and finite coproducts $\oplus$.*
2. *Symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, ⅋)$, in duality with each other by means of contravariant functors*
$$(-)^* : \mathcal{P} \Rightarrow \mathcal{C} \qquad \text{and} \qquad (-)_* : \mathcal{C} \Rightarrow \mathcal{P}.$$
3. *Monoidal natural transformations*
$$e_P^{\otimes} : P \to 1_{\mathsf{P}} \qquad \text{and} \qquad d_P^{\otimes} : P \to P \otimes P$$
in $\mathcal{P}$, and comonoidal natural transformations
$$e_C^{⅋} : \bot_{\mathsf{C}} \to C \qquad \text{and} \qquad d_C^{⅋} : C ⅋ C \to C$$
in $\mathcal{C}$, interchanged under duality, such that
   (a) *for every object P in $\mathcal{P}$, $(P, d_P^{\otimes}, e_P^{\otimes})$ forms a commutative comonoid; and*
   (b) *for every object C in $\mathcal{C}$, $(C, d_C^{⅋}, e_C^{⅋})$ forms a commutative monoid.*
4. *Symmetric monoidal functors*
$$\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P} \qquad \text{and} \qquad F_! : \mathcal{P} \Rightarrow \mathcal{L}$$
and symmetric comonoidal functors
$$\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C} \qquad \text{and} \qquad F_? : \mathcal{C} \Rightarrow \mathcal{L}$$
that respect the dualities in that
$$(F_!\, P)^{\perp} \simeq F_?\, (P^*) \qquad \text{and} \qquad \lceil A \rceil \simeq \lfloor A^{\perp} \rfloor$$
and that form monoidal/comonoidal adjunctions
$$\lceil - \rceil \dashv F_! \qquad \text{and} \qquad F_? \dashv \lfloor - \rfloor.$$

We can make a few observations about the LPC characterization:

- The monoidal components $m^{F_!}$ of the $F_!$ functor are necessarily isomorphisms, whose inverses are as follows:
$$(m_{1_{\mathsf{L}}}^{F_!})^{-1} = n_{1_{\mathsf{L}}}^{F_!} : F_!\, 1_{\mathsf{P}} \xrightarrow{F_!\, m_{1_{\mathsf{L}}}^{\lceil - \rceil}} F_!\, \lceil 1_{\mathsf{L}} \rceil \xrightarrow{\epsilon_{1_{\mathsf{L}}}} 1_{\mathsf{L}}$$
$$\begin{aligned}(m_{P,Q}^{F_!})^{-1} = n_{P,Q}^{F_!} : F_!\, (P \otimes Q) &\xrightarrow{F_!\, (\eta_P \otimes \eta_Q)} F_!\, (\lceil F_!\, P \rceil \otimes \lceil F_!\, Q \rceil) \\ &\xrightarrow{F_!\, (m_{F_!\, P, F_!\, Q}^{\lceil - \rceil})} F_!\, \lceil F_!\, P \otimes F_!\, Q \rceil \\ &\xrightarrow{\epsilon_{F_!\, P \otimes F_!\, Q}} F_!\, P \otimes F_!\, Q\end{aligned}$$
  Thus $F_!$ is both monoidal and comonoidal, and similarly for $F_?$.
- The condition that every object in $\mathcal{P}$ forms a commutative comonoid is equivalent to the condition that $\mathcal{P}$ is cartesian. The long form of the definition here highlights the fact that the comonoid structures in $\mathcal{P}$ induce the respective structures in $\mathcal{L}$ for the comonad !. Similarly, Condition 3(b) is equivalent to stating that $\mathcal{C}$ is cocartesian.

4.2 LPC and other models of linear logic

As LPC is inspired by Benton's linear/non-linear paradigm, we would like to formalize the relationship between LPC and LNL.

Definition 18 (Melliés (2003))**.** *A linear/non-linear (*LNL*) model consists of*

1. *a symmetric monoidal closed category $\mathcal{L}$;*
2. *a cartesian category $\mathcal{P}$; and*
3. *functors $\mathrm{G} : \mathcal{L} \Rightarrow \mathcal{P}$ and $\mathrm{F} : \mathcal{P} \Rightarrow \mathcal{L}$ that form a symmetric monoidal adjunction* $\mathrm{F} \dashv \mathrm{G}$.

The LNL model given by Benton (1995) has the added condition that the cartesian category be cartesian closed.

Lemma 19. *Every* LPC *model is an* LNL *model.*

A $*$-autonomous category in a linear/non-linear model induces the consumer category in LPC:

Lemma 20. *If the category $\mathcal{L}$ in an* LNL *model is *-autonomous, then the categories $(\mathcal{L}, \mathcal{P}, \mathcal{P}^{op})$ form an* LPC *model.*

Next we prove that every LPC model contains a classical linear category as defined by Schalk (2004).

Definition 21. *A comonad $(!, \mu, \nu)$ consists of a functor $!$ and natural transformations*

$$\nu_A : !A \to A \qquad \textit{and} \qquad \mu_A : !A \to !!A$$

such that the following diagrams commute:

$$\begin{tikzcd}
!A \arrow[r, "\mu_A"] \arrow[d, "\mu_A"'] \arrow[dr, "id_{!A}"] & !!A \arrow[d, "\nu_{!A}"] \\
!!A \arrow[r, "!\,\nu_A"'] & !A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mu_A"] \arrow[d, "\mu_A"'] & !!A \arrow[d, "\mu_{!A}"] \\
!!A \arrow[r, "!\,\mu_A"'] & !!!A
\end{tikzcd}$$

Definition 22. *A comonad $(!, \mu, \nu)$ is monoidal if $!$ is a monoidal functor and μ and ν are monoidal natural transformations.*

Definition 23. *A symmetric monoidal category $\mathcal{L}$ has a linear exponential comonad if it has a monoidal comonad $(!, \mu, \nu)$ such that*

1. *There exist monoidal natural transformations $\mathrm{e}^!_A : !A \to A$ and $\mathrm{d}^!_A : !A \to !A \otimes !A$ such that for every object A in $\mathcal{L}$, $(!A, \mathrm{e}^!_A, \mathrm{d}^!_A)$ forms a commutative comonoid.*
2. *The morphisms $\mathrm{e}^!_A$ and $\mathrm{d}^!_A$ are coalgebra morphisms, meaning that they satisfy*

$$\begin{tikzcd}
!A \arrow[rr, "\mu_A"] \arrow[d, "d^!_A"'] & & !!A \arrow[d, "!\,d^!_A"] \\
!A \otimes !A \arrow[r, "\mu_A \otimes \mu_A"'] & !!A \otimes !!A \arrow[r, "m^!_{!A,!A}"'] & !(!A \otimes !A)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mu_A"] \arrow[d, "e^!_A"'] & !!A \arrow[d, "!\,e^!_A"] \\
1_{\mathsf{L}} \arrow[r, "m^!_{1_{\mathsf{L}}}"'] & !1_{\mathsf{L}}
\end{tikzcd}$$

3. *Every morphism μ_A is a morphism of comonoids, meaning that it satisfies*

$$\begin{tikzcd}
!A \arrow[d, "\mu_A"'] \arrow[dr, "e^!_A"] & \\
!!A \arrow[r, "e^!_{!A}"'] & 1_{\mathsf{L}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
!A \arrow[r, "d^!_A"] \arrow[d, "\mu_A"'] & !A \otimes !A \arrow[d, "\mu_A \otimes \mu_A"] \\
!!A \arrow[r, "d^!_{!A}"'] & !!A \otimes !!A
\end{tikzcd}$$

Definition 24 (Schalk (2004))**.** *A category $\mathcal{L}$ is a model for classical linear logic if and only if it*

1. *is *-autonomous;*
2. *has finite products & and thus finite coproducts $\oplus$;*
3. *has a linear exponential comonad ! and thus a linear exponential monad ?.*

Theorem 25. *The category $\mathcal{L}$ from the* LPC *model is a model for classical linear logic.*

Proof. Lemma 14 states that $\mathcal{L}$ is *-autonomous, so it suffices to show that $\mathcal{L}$ has a linear exponential comonad. The proof is similar to that of Benton (1994), so we will simply provide a proof sketch here.

The adjunction $F_! \dashv \lceil - \rceil$ is known to form a comonad $F_! \lceil - \rceil$ in $\mathcal{L}$ with components

$$\nu_A := F_! \lceil A \rceil \xrightarrow{\epsilon_A} A \qquad \text{and} \qquad \mu_A := F_! \lceil A \rceil \xrightarrow{F_! \, \eta_{\lceil A \rceil}} F_! \lceil F_! \lceil A \rceil \rceil$$

where ϵ is the unit of the adjunction, and η the counit. The comonad is monoidal because both functors, unit and counit are monoidal.

The monoid's components come from the monoid in $\mathcal{P}$ passed through the adjunction:

$$\begin{aligned}
e^!_A &:= F_! \lceil A \rceil \xrightarrow{F_! \, e_{\lceil A \rceil}} F_! \, 1_{\mathsf{P}} \xrightarrow{n^{F_!}_{1_{\mathsf{L}}}} 1 \\
d^!_A &:= F_! \lceil A \rceil \xrightarrow{F_! \, d_{\lceil A \rceil}} F_! \left(\lceil A \rceil \otimes \lceil A \rceil \right) \xrightarrow{n^{F_!}_{\lceil A \rceil, \lceil A \rceil}} F_! \lceil A \rceil \otimes F_! \lceil A \rceil
\end{aligned}$$

These natural transformations form a commutative comonoid because of the fact that $F_!$ is a monoidal functor and because (P, e_P, d_P) forms a commutative monoid in $\mathcal{P}$.

To show that $e^!_A$ is a coalgebra morphism, we can expand out the diagram as follows:

$$\begin{tikzcd}
F_! \lceil A \rceil \arrow[rrr, "F_! \, \eta_{\lceil A \rceil}"] \arrow[d, "F_! \, e_{\lceil A \rceil}"'] & & & F_! \lceil F_! \lceil A \rceil \rceil \arrow[d, "F_! \lceil F_! \, e_{\lceil A \rceil} \rceil"] \\
F_! \, 1_{\mathsf{P}} \arrow[rr, "F_! \, \eta_{1_{\mathsf{P}}}"] \arrow[d, "(m^{F_!}_{1_{\mathsf{L}}})^{-1}"'] \arrow[dr, "\mathrm{id}_{F_! \, 1_{\mathsf{P}}}"] & & F_! \lceil F_! \, 1_{\mathsf{P}} \rceil \arrow[r, "\mathrm{id}_{F_! \lceil F_! \, 1_{\mathsf{P}} \rceil}"] & F_! \lceil F_! \, 1_{\mathsf{P}} \rceil \arrow[d, "F_! \lceil (m^{F_!}_{1_{\mathsf{L}}})^{-1} \rceil"] \\
1_{\mathsf{L}} \arrow[r, "m^{F_!}_{1_{\mathsf{L}}}"'] & F_! \, 1_{\mathsf{P}} \arrow[rr, "F_! \, m^{\lceil - \rceil}_{1_{\mathsf{L}}}"'] & & F_! \lceil 1_{\mathsf{L}} \rceil \arrow[ul, "F_! \lceil m^{F_!}_{1_{\mathsf{L}}} \rceil"']
\end{tikzcd}$$

The rectangle commutes by the naturality of η, and the bottom parallelogram commutes because η is monoidal. Meanwhile, the two lower triangles are due to the fact that $n^{F_!}_{1_\mathsf{L}}$ is the inverse of $m^{F_!}_{1_\mathsf{L}}$.

The proof that $d^!_A$ is a coalgebra morphism is similar.

Finally, the fact that μ_A forms a morphism of comonoids stems easily from the facts that e, d, and $m^{F_!}$ are natural transformations. $\square$

4.3 Interpretation of the Logic

We define an interpretation of the LPC logic that maps propositions to objects in either $\mathcal{L}$, $\mathcal{P}$ or $\mathcal{C}$, and derivations to morphisms. For objects, the $[\![-]\!]_\mathsf{L}$ interpretation function is defined on all propositions, but $[\![-]\!]_\mathsf{P}$ and $[\![-]\!]_\mathsf{C}$ are defined only on persistent propositions. On the linear units and combinators, the interpretations act as expected. On the adjoint functors, the behavior is as follows:

$$
\begin{aligned}
[\![F_!\, P]\!]_\mathsf{L} &= F_!\, [\![P]\!]_\mathsf{P} & \qquad [\![F_?\, C]\!]_\mathsf{L} &= F_?\, [\![C]\!]_\mathsf{C} \\
[\![P]\!]_\mathsf{L} &= F_!\, [\![P]\!]_\mathsf{P} & [\![C]\!]_\mathsf{L} &= F_?\, [\![C]\!]_\mathsf{C} \\
[\![\lceil A \rceil]\!]_\mathsf{P} &= \lceil [\![A]\!]_\mathsf{L} \rceil & [\![\lfloor A \rfloor]\!]_\mathsf{C} &= \lfloor [\![A]\!]_\mathsf{L} \rfloor
\end{aligned}
$$

Finally, for persistent propositions of the opposite mode, we use duality to interpret the propositions:

$$
[\![C]\!]_\mathsf{P} = ([\![C]\!]_\mathsf{C})_* \qquad\qquad [\![P]\!]_\mathsf{C} = ([\![P]\!]_\mathsf{P})^*
$$

Contexts can be interpreted with the comma as either the tensor or cotensor in the linear category, though in the producer category there is no cotensor and vice versa for the consumer category.

$$
\begin{aligned}
[\![\cdot]\!]^{\otimes}_\mathsf{L} &= 1_\mathsf{L} & \qquad [\![X,\Gamma]\!]^{\otimes}_\mathsf{L} &= [\![X]\!]_\mathsf{L} \otimes [\![\Gamma]\!]^{\otimes}_\mathsf{L} \\
[\![\cdot]\!]^{\parr}_\mathsf{L} &= \bot_\mathsf{L} & [\![X,\Gamma]\!]^{\parr}_\mathsf{L} &= [\![X]\!]_\mathsf{L} \parr [\![\Gamma]\!]^{\parr}_\mathsf{L} \\
[\![\cdot]\!]_\mathsf{P} &= 1_\mathsf{P} & [\![X,\Gamma]\!]_\mathsf{P} &= [\![X]\!]_\mathsf{P} \otimes [\![\Gamma]\!]_\mathsf{P} \\
[\![\cdot]\!]_\mathsf{C} &= \bot_\mathsf{C} & [\![X,\Gamma]\!]_\mathsf{C} &= [\![X]\!]_\mathsf{C} \parr [\![\Gamma]\!]_\mathsf{C}
\end{aligned}
$$

Recall that a context is an unordered collection of propositions, while its interpretation is an ordered mapping. The interpretation function is well-defined up to isomorphism because all three categories are symmetric monoidal, but for a more rigorous treatment of the interpretation function, the isomorphisms should be made explicit.

A linear derivation $\mathcal{D}$ of the form $\Gamma \vdash \Delta$ will be interpreted as a morphism $[\![\mathcal{D}]\!]_\mathsf{L} : [\![\Gamma]\!]^{\otimes}_\mathsf{L} \to [\![\Delta]\!]^{\parr}_\mathsf{L}$, but this will not suffice for persistent derivations $\Gamma \Vdash \Delta$. When mapped into $\mathcal{P}$, the codomain cannot be interpreted as a $\parr$-separated list. We proved in Section 3.1 that every persistent derivation $\mathcal{D}$ contains exactly one displaced proposition. This means that $\mathcal{D}$ is either of the form $\Gamma^\mathsf{P} \Vdash \Delta^\mathsf{C}, P$ or $\Gamma^\mathsf{P}, C \Vdash \Delta^\mathsf{C}$. In the category $\mathcal{P}$, this derivation will be interpreted as a morphism

$$
[\![\mathcal{D}]\!]_\mathsf{P} : [\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes [\![\Delta^\mathsf{C}]\!]_\mathsf{P} \to [\![P]\!]_\mathsf{P} \qquad \text{or} \qquad [\![\mathcal{D}]\!]_\mathsf{P} : [\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes [\![\Delta^\mathsf{C}]\!]_\mathsf{P} \to [\![C]\!]_\mathsf{P},
$$

respectively. Similarly, in the category $\mathcal{C}$, the derivation will be interpreted as a morphism

$$
[\![\mathcal{D}]\!]_\mathsf{C} : [\![P]\!]_\mathsf{C} \to [\![\Delta^\mathsf{C}]\!]_\mathsf{C} \parr [\![\Gamma^\mathsf{P}]\!]_\mathsf{C} \qquad \text{or} \qquad [\![\mathcal{D}]\!]_\mathsf{C} : [\![C]\!]_\mathsf{C} \to [\![\Delta^\mathsf{C}]\!]_\mathsf{C} \parr [\![\Gamma^\mathsf{P}]\!]_\mathsf{C}.
$$

The interpretation is defined by mutual induction on the derivations.

Interpreting weakening and contraction in the persistent sequent is straightforward using the monoid in $\mathcal{C}$ and comonoid in $\mathcal{P}$. For weakening in the linear sequent, suppose we have the following derivation:

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma \vdash \Delta}}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L}
$$

The interpretation of $\mathcal{D}$ inserts the comonoid in $\mathcal{P}$ into the linear category.

$$\begin{aligned}
[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes F_! \, [\![P]\!]_{\mathsf{P}} \\
&\xrightarrow{[\![\mathcal{D}]\!]_{\mathsf{L}} \otimes F_! \, e^{\otimes}} [\![\Delta]\!]_{\mathsf{L}}^{\parr} \otimes F_! \, 1_{\mathsf{P}} \\
&\xrightarrow{\mathrm{id} \otimes (m^{F_!})^{-1}} [\![\Delta]\!]_{\mathsf{L}}^{\parr} \otimes 1_{\mathsf{L}} \\
&\xrightarrow{\rho^{\otimes}} [\![Q]\!]_{\mathsf{P}}
\end{aligned}$$

In the producer sequent, suppose we have a derivation

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, Q}}{\Gamma^{\mathsf{P}}, P \Vdash \Delta^{\mathsf{C}}, Q} \ \mathrm{W}^{\Vdash}\text{-L}$$

The interpretation in $\mathcal{P}$ is defined as follows:

$$\begin{aligned}
[\![\mathcal{D}]\!]_{\mathsf{P}} : ([\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![P]\!]_{\mathsf{P}}) \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \\
&\xrightarrow{(\mathrm{id} \otimes e^{\otimes}) \otimes \mathrm{id}} ([\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes 1_{\mathsf{P}}) \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \\
&\xrightarrow{\rho^{\otimes} \otimes \mathrm{id}} [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \\
&\xrightarrow{[\![\mathcal{D}']\!]_{\mathsf{P}}} [\![Q]\!]_{\mathsf{P}}
\end{aligned}$$

The case for contraction is similar. For a linear derivation with contraction such as

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma, P, P \vdash \Delta}}{\Gamma, P \vdash \Delta} \ \mathrm{C}^{\vdash}\text{-L}$$

it is possible to construct the morphim

$$\begin{aligned}
[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes F_! \, [\![P]\!]_{\mathsf{P}} \\
&\xrightarrow{\mathrm{id} \otimes F_! \, d^{\otimes}} [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes F_! \, ([\![P]\!]_{\mathsf{P}} \otimes [\![P]\!]_{\mathsf{P}}) \\
&\xrightarrow{\mathrm{id} \otimes (m^{F_!})^{-1}} [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes (F_! \, [\![P]\!]_{\mathsf{P}} \otimes F_! \, [\![P]\!]_{\mathsf{P}}) \\
&\xrightarrow{[\![\mathcal{D}']\!]_{\mathsf{L}}} [\![\Delta]\!]_{\mathsf{L}}^{\parr}
\end{aligned}$$

Next we move on to the rules for the adjoint functors. The $F_!$-L and $F_?$-R rules are interpreted directly by the inductive hypothesis; the $F_!$-R and $F_?$-L rules are a bit more complicated. Suppose $\mathcal{D}$ is the following derivation:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! \, P} \ F_!\text{-R}$$

The inductive hypothesis provides a morphism $[\![\mathcal{D}']\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![P]\!]_{\mathsf{P}}$. It is necessary to undo this duality transformation for interpretation in the linear category. Notice that for any persistent context Γ, there is an isomorphism $\pi : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \cong F_! \, [\![\Gamma]\!]_{\mathsf{P}}$ given by the monoidal components of $F_!$. Furthermore, there is an isomorphism τ between $(\llbracket\Gamma\rrbracket_{\mathsf{L}}^{\parr})^{\perp}$ and $F_{!}\,\llbracket\Gamma\rrbracket_{\mathsf{P}}$ given by the isomorphism $(F_{?}\,C)^{\perp} \cong F_{!}\,C_{*}$. Using these morphisms we define the interpretation of $\mathcal{D}$:

$$
\begin{aligned}
\llbracket\mathcal{D}\rrbracket_{\mathsf{L}} : \llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{L}}^{\otimes} &\\
&\xrightarrow{\rho^{\otimes}} \llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{L}}^{\otimes} \otimes 1_{\mathsf{L}}\\
&\xrightarrow{\mathrm{id}\otimes\gamma^{1}} \llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{L}}^{\otimes} \otimes ((\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})^{\perp} \parr \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})\\
&\xrightarrow{\pi\otimes(\tau\parr\mathrm{id})} F_{!}\,\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}} \otimes (F_{!}\,\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}} \parr \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})\\
&\xrightarrow{\delta^{\mathrm{L,L}}} (F_{!}\,\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}} \otimes F_{!}\,\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}}) \parr \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr}\\
&\xrightarrow{m^{F_{!}}\parr\mathrm{id}} F_{!}\,(\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}} \otimes \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}}) \parr \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr}\\
&\xrightarrow{F_{!}\,\llbracket\mathcal{D}'\rrbracket_{\mathsf{P}}\parr\mathrm{id}} F_{!}\,\llbracket P\rrbracket_{\mathsf{P}} \parr \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr}\\
&\xrightarrow{\sigma^{\parr}} \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr} \parr \llbracket F_{!}\,P\rrbracket_{\mathsf{L}}
\end{aligned}
$$

The $\lceil - \rceil$-L rule uses the unit of the adjunction in its interpretation. Suppose

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma, A \vdash \Delta}}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L}
$$

Then its interpretation can be defined as

$$
\begin{aligned}
\llbracket\mathcal{D}\rrbracket_{\mathsf{L}} : \llbracket\Gamma\rrbracket_{\mathsf{L}}^{\otimes} \otimes F_{!}\,(\lceil\llbracket A\rrbracket_{\mathsf{L}}\rceil) &\\
&\xrightarrow{\mathrm{id}\otimes\epsilon} \llbracket\Gamma\rrbracket_{\mathsf{L}}^{\otimes} \otimes \llbracket A\rrbracket_{\mathsf{L}}\\
&\xrightarrow{\llbracket\mathcal{D}'\rrbracket_{\mathsf{L}}} \llbracket\Delta\rrbracket_{\mathsf{L}}^{\parr}
\end{aligned}
$$

The $\lceil - \rceil$-R rule uses the counit of the adjunction, along with the isomorphisms π and τ defined previously. Let

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R}
$$

Its interpretation is defined as follows:

$$
\begin{aligned}
\llbracket\mathcal{D}\rrbracket_{\mathsf{P}} : \llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}} \otimes \llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}} &\xrightarrow{\eta\otimes\eta} \lceil F_{!}\,\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}}\rceil \otimes \lceil F_{!}\,\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}}\rceil\\
&\xrightarrow{m^{\lceil - \rceil}} \lceil F_{!}\,\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{P}} \otimes F_{!}\,\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{P}}\rceil\\
&\xrightarrow{\lceil\pi^{-1}\otimes\tau^{-1}\rceil} \lceil\llbracket\Gamma^{\mathsf{P}}\rrbracket_{\mathsf{L}}^{\otimes} \otimes (\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})^{\perp}\rceil\\
&\xrightarrow{\lceil\llbracket\mathcal{D}'\rrbracket_{\mathsf{L}}\otimes\mathrm{id}\rceil} \lceil(\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr} \parr \llbracket A\rrbracket_{\mathsf{L}}) \otimes (\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})^{\perp}\rceil\\
&\xrightarrow{\lceil\delta^{\mathrm{R,L}}\rceil} \lceil(\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr} \otimes (\llbracket\Delta^{\mathsf{C}}\rrbracket_{\mathsf{L}}^{\parr})^{\perp}) \parr \llbracket A\rrbracket_{\mathsf{L}}\rceil\\
&\xrightarrow{\gamma^{\perp}\parr\mathrm{id}} \lceil\perp_{\mathsf{L}} \parr \llbracket A\rrbracket_{\mathsf{L}}\rceil\\
&\xrightarrow{\lceil\lambda^{\parr}\rceil} \lceil\llbracket A\rrbracket_{\mathsf{L}}\rceil = \llbracket\lceil A\rceil\rrbracket_{\mathsf{P}}
\end{aligned}
$$

Admissible Rules

The admissible duality and cut rules from the LPC logic should correspond with the expected notions of duality and cut in the categorical model. In the simplest case, suppose a derivation $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash A} \qquad \dfrac{\mathcal{D}_2}{A \vdash \Delta}}{\Gamma \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

Property 26. $[\![\mathcal{D}]\!]_{\mathsf{L}} = [\![\mathcal{D}_2]\!]_{\mathsf{L}} \circ [\![\mathcal{D}_1]\!]_{\mathsf{L}}$.

We will sketch some of the cases from the cut admissibility proof to demonstrate that the property holds. First, suppose

$$\mathcal{D}_1 = \dfrac{}{\cdot \vdash 1_{\mathsf{L}}}\ 1^{\vdash}_{\mathsf{L}}\text{-R} \qquad\qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\cdot \vdash \Delta}}{1_{\mathsf{L}} \vdash \Delta}\ 1^{\vdash}_{\mathsf{L}}\text{-L}$$

By the definition of cut, we have $\mathcal{D} = \mathcal{D}_2'$. So our goal is to show that $[\![\mathcal{D}_2']\!]_{\mathsf{L}} = [\![\mathcal{D}_2]\!]_{\mathsf{L}} \circ [\![\mathcal{D}_1]\!]_{\mathsf{L}}$ which holds because $[\![\mathcal{D}_2]\!]_{\mathsf{L}} = [\![\mathcal{D}_2']\!]_{\mathsf{L}}$ and $[\![\mathcal{D}_1]\!]_{\mathsf{L}} = \mathrm{id}_{1_{\mathsf{L}}}$.

Next, suppose A has the form $A_1 \mathbin{\&} A_2$ and that

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma \vdash A_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma \vdash A_2}}{\Gamma \vdash A_1 \mathbin{\&} A_2}\ \&^{\vdash}_{\mathsf{L}}\text{-R} \qquad\qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{A_1 \vdash \Delta}}{A_1 \mathbin{\&} A_2 \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L}$$

By the definition of cut, $\mathcal{D}$ is equal to the inductive derivation

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma \vdash A_1} \qquad \dfrac{\mathcal{D}_2'}{A_1 \vdash \Delta}}{\Gamma \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

while we have

$$[\![\mathcal{D}_1]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}_{11}]\!]_{\mathsf{L}} \& [\![\mathcal{D}_{12}]\!]_{\mathsf{L}}} [\![A_1 \mathbin{\&} A_2]\!]_{\mathsf{L}}$$
$$[\![\mathcal{D}_2]\!]_{\mathsf{L}} : [\![A_1 \mathbin{\&} A_2]\!]_{\mathsf{L}} \xrightarrow{\pi_1} [\![A_1]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}_2']\!]_{\mathsf{L}}} [\![\Delta]\!]^{\parr}_{\mathsf{L}}$$

Finally, consider the case where A is not the principle formula in $\mathcal{D}_1$, and instead we have

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma, B \vdash A}}{\Gamma, \lceil B \rceil \vdash A}\ \lceil - \rceil\text{-L}$$

Working through the interpretation function, we know

$$[\![\mathcal{D}_1]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes F_! \lceil [\![B]\!]_{\mathsf{L}} \rceil \xrightarrow{\mathrm{id} \otimes \epsilon} [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes [\![B]\!]_{\mathsf{L}}$$
$$\xrightarrow{[\![\mathcal{D}_1']\!]_{\mathsf{L}}} [\![A]\!]_{\mathsf{L}}$$

In addition, since

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\mathcal{D}_1'}{\Gamma, B \vdash A} \qquad \dfrac{\mathcal{D}_2}{A \vdash \Delta}}{\Gamma, B \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}}{\Gamma, \lceil B \rceil \vdash \Delta}\ \lceil - \rceil\text{-L}$$

by definition we have

$$\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} : \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \otimes F_! \lceil \llbracket B \rrbracket_{\mathsf{L}} \rceil \xrightarrow{\mathrm{id} \otimes \epsilon} \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \otimes \llbracket B \rrbracket_{\mathsf{L}}$$
$$\xrightarrow{\llbracket \mathcal{D}_1' \rrbracket_{\mathsf{L}}} \llbracket A \rrbracket_{\mathsf{L}}$$
$$\xrightarrow{\llbracket \mathcal{D}_2 \rrbracket_{\mathsf{L}}} \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr}$$

5 Examples

This section provides some concrete instances of the LPC model. The following chart summarizes the three examples and their LPC categories:

	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{C}$
Vectors	FinVect	FinSet	FinSet^{op}
Relations	Rel	Set	Set^{op}
Boolean Algebras	FinBoolAlg	FinPoset	FinLat

5.1 Vector Spaces

The $\otimes$, $(-)^{\perp}$ and & operators of linear logic are easily interpreted by the notions of tensor product, duality and direct product for vector spaces. However, the exponentials ! and ? are not induced from the usual structures of linear algebra. In this section we will expand on this intuition to develop the category of finite-dimensional vector spaces (over a finite field) as the linear component of an LPC model.

We start with some preliminaries about vector spaces. For the entirety of this example, fix $\mathbb{F}$ to be a finite-dimensional field of dimension q.

Definition 27. *Let* FinVect($\mathbb{F}$) *be the category of finite-dimensional vector spaces over the finite field* $\mathbb{F}$*. That is, the objects of* FinVect($\mathbb{F}$) *are vector spaces and the morphisms are linear transformations.*

Definition 28. *The free vector space* **Free** (X) *of a finite set* X *over* $\mathbb{F}$ *is the vector space with vectors the formal sums* $\alpha_1 x_1 + \cdots + \alpha_n x_n$*, addition defined pointwise, and scalar multiplication defined by distribution over the* x_i*'s.*

A basis for **Free** (X) is the set $\{\delta_x \mid x \in X\}$ where δ_x is the free sum x.

The tensor product $\otimes$ is defined in terms of quotients of a free vector space.

Definition 29. *The tensor product of two vector spaces* U *and* V *is* $U \otimes V = \mathbf{Free}\,(U \times V)/W$ *where* W *is the subspace generated by the following elements:*

$$\begin{array}{cc} (u_1, v) + (u_2, v) - (u_1 + u_2, v) & \alpha\,(u, v) - (\alpha\, u, v) \\ (u, v_1) + (u, v_2) - (u, v_1 + v_2) & \alpha\,(u, v) - (u, \alpha\, v) \end{array}$$

The equivalence class of (u, v) is denoted $u \otimes v$. Notice that if B_1 is a basis for U and B_2 is a basis for V then $B \otimes B_2$ is a basis for $U \otimes V$. The unit 1_{L} of the tensor is the one-dimensional vector space generated by the basis $\{1V\}$.

The tensor product of two linear maps $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ is

$$f_1 \otimes f_2\,(u_1 \otimes v_1) = f_1\,(u_1) \otimes f_2\,(u_2)$$

Since there is no traditional interpretation for $\parr$ in linear algebra, we define $\parr$ to be exactly $\otimes$.

Definition 30. *The dual of a vector space* V *over* $\mathbb{F}$ *is the set* $V^{\perp}$ *of linear maps from* V *to* $\mathbb{F}$*.*

For any vector $v \in V$, we can define $\overline{v} \in V^{\perp}$ to be the linear map acting on basis elements $x \in B$ in the following way:

$$\overline{v}[x] = \begin{cases} 1 & x = v \\ 0 & x \neq v \end{cases}$$

Addition and scalar multiplication are defined pointwise. Then the set $\{\overline{x} \mid x \in B\}$ is a basis for $V^{\perp}$.

The additives & and $\oplus$ are embodied by the notions of the direct product and direct sum, which in the case of finite-dimensional vector spaces, coincide.

Lemma 31. *The category* $\textsc{FinVect}(\mathbb{F})$ *is a symmetric linearly distributive category with negation, products and coproducts.*

Proof. Since $\otimes$ and $\parr$ overlap, the natural transformation δ is simply associativity. The coherence diagrams for linear distribution then depend on the commutativity of tensor, associativity, and swap morphisms. To show the category has negation, we define $\gamma^{\perp}$ and γ^{1} as follows, where B is a basis for A:

$$\begin{aligned} \gamma_A^{\perp} &: A^{\perp} \otimes A \to \perp_{\mathsf{L}} & \qquad \gamma_A^{1} &: 1_{\mathsf{L}} \to A \parr A^{\perp} \\ \gamma_A^{\perp}\,(\delta_u \otimes v) &= \delta_u[v] \cdot \mathbb{1} & \qquad \gamma_A^{1}\,(\mathbb{1}) &= \sum_{v \in B} v \otimes \overline{v} \end{aligned}$$

It suffices to check that

$$\lambda \,\circ\, (\gamma^{\perp} \otimes \mathrm{id}) \,\circ\, \alpha \,\circ\, (\mathrm{id} \otimes \lambda) = \rho.$$

$\square$

So $\textsc{FinVect}$ will take the place of the linear category. The producer category will be the category $\textsc{FinSet}$ of finite sets and functions, and the consumer category will be its opposite category $\textsc{FinSet}^{op}$ of finite sets and inverse functions. It is not hard to see that the cartesian product in $\textsc{FinSet}$ along with unit $\{\emptyset\}$, is symmetric monoidal, and that the category admits the commutative comonoid, as follows:

$$\begin{aligned} e_X &: X \to \{\emptyset\} & \qquad d_X &: X \to X \times X \\ e_X\,(x) &= \emptyset & \qquad d_X\,(x) &= (x, x) \end{aligned}$$

Finally, we present the adjunctions. For now we will address only one of the adjunctions; the other can be inferred from the opposite category. Define $\lceil - \rceil : \textsc{FinVect} \Rightarrow \textsc{FinSet}$ to be the forgetful functor, which takes a vector space to its underlying set of vectors, and a linear map to the corresponding function. As a monoidal functor, it has the following components:

$$\begin{aligned} m_{1_{\mathsf{L}}}^{\lceil - \rceil} &: 1_{\mathsf{P}} \to \lceil 1_{\mathsf{L}} \rceil & \qquad m_{A,B}^{\lceil - \rceil} &: \lceil A \rceil \times \lceil B \rceil \to \lceil A \otimes B \rceil \\ m_{1_{\mathsf{L}}}^{\lceil - \rceil}\,(\emptyset) &= \lceil \mathbb{1} \rceil & \qquad m_{A,B}^{\lceil - \rceil}\,(\lceil u \rceil, \lceil v \rceil) &= \lceil u \otimes v \rceil \end{aligned}$$

On objects, the functor $F_! : \textsc{FinSet} \Rightarrow \textsc{FinVect}$ takes a set X to the free vector space generated by X. For a morphism $f : X_1 \to X_2$ in $\textsc{FinSet}$, we can define

$$\begin{aligned} F_!\,f &: \mathbf{Free}\,(X_1) \to \mathbf{Free}\,(X_2) \\ F_!\,f\,(\delta_x) &= \delta_{f\,(x)} \end{aligned}$$

This is a monoidal functor, with components

$$\begin{aligned} m_{1_{\mathsf{L}}}^{F_!} &: 1_{\mathsf{L}} \to F_!\,1_{\mathsf{L}} & \qquad m_{X_1,X_2}^{F_!} &: F_!\,X_1 \otimes F_!\,X_2 \to F_!\,(X_1 \times X_2) \\ m_{1_{\mathsf{L}}}^{F_!}\,(\mathbb{1}) &= \delta_{\emptyset} & \qquad m_{X_1,X_2}^{F_!}\,(\delta_{x_1} \otimes \delta_{x_2}) &= \delta_{(x_1,x_2)} \end{aligned}$$

Lemma 32. *The functors* $\lceil - \rceil$ *and* $F_!$ *form a symmetric monoidal adjunction* $\lceil - \rceil \dashv F_!$*.*

Proof. We define the unit and counit of the adjunction as follows:

$$\begin{aligned} \epsilon_A &: F_! \lceil A \rceil \to A & \qquad \eta_P &: P \to \lceil F_! \, P \rceil \\ \epsilon_A \left(\delta_{\lceil v \rceil}\right) &= v & \qquad \eta_P \left(x\right) &= \lceil \delta_x \rceil \end{aligned}$$

By simply unfolding the definitions, we can see that these families of morphisms are natural transformations. They form an adjunction because $\lceil \epsilon_A \rceil \circ \eta_{\lceil A \rceil}$ is the identity on FinSet:

$$\lceil \epsilon_A \rceil \circ \eta_{\lceil A \rceil} \left(\lceil v \rceil\right) = \lceil \epsilon_A \rceil \left(\lceil \delta_{\lceil v \rceil} \rceil\right) = \lceil v \rceil,$$

and because $\epsilon_{F_! \, P} \circ F_! \left(\eta_P\right)$ is the identity on FinVect:

$$\epsilon_{F_! \, P} \circ F_! \left(\eta_P\right) \left(\delta_x\right) = \epsilon_{F_! \, P} \left(\delta_{\eta_P \, (x)}\right) = \epsilon_{F_! \, P} \left(\delta_{\lceil \delta_x \rceil}\right) = \delta_x.$$

Finally, ϵ and η are both monoidal natural transformations. For ϵ, we must show that the following two diagrams commute:

$$\begin{tikzcd} F_! \lceil A \rceil \otimes F_! \lceil B \rceil \arrow[r, "m^{F_!}_{\lceil A \rceil, \lceil B \rceil}"] \arrow[d, "\epsilon_A \otimes \epsilon_B"'] & F_! \left(\lceil A \rceil \otimes \lceil B \rceil\right) \arrow[r, "F_! \, m^{\lceil - \rceil}_{A,B}"] & F_! \lceil A \otimes B \rceil \arrow[d, "\epsilon_{(A \otimes B)}"'] \\ A \otimes B \arrow[rr, "\mathrm{id}_{A \otimes B}"'] & & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} F_! \lceil 1_{\mathsf{L}} \rceil \arrow[r, "\epsilon_{1_{\mathsf{L}}}"] & 1_{\mathsf{L}} \\ F_! \, 1_{\mathsf{P}} \arrow[u, "F_! \, m^{\lceil - \rceil}_{1_{\mathsf{L}}}"'] & 1_{\mathsf{L}} \arrow[l, "m^{F_!}_{1_{\mathsf{L}}}"] \arrow[u, "\mathrm{id}_{1_{\mathsf{L}}}"'] \end{tikzcd}$$

Working out these definitions, we see that

$$\begin{aligned} \epsilon_{A \otimes B} \circ F_! \, m^{\lceil - \rceil}_{A,B} \circ m^{F_!}_{\lceil A \rceil, \lceil B \rceil} \left(\delta_{\lceil u \rceil} \otimes \delta_{\lceil v \rceil}\right) &= \epsilon_{A \otimes B} \circ F_! \, m^{\lceil - \rceil}_{A,B} \left(\delta_{(\lceil u \rceil, \lceil v \rceil)}\right) \\ &= \epsilon_{A \otimes B} \left(\delta_{m^{\lceil - \rceil}_{A,B} \, (\lceil u \rceil, \lceil v \rceil)}\right) \\ &= \epsilon_{A \otimes B} \left(\delta_{\lceil u \otimes v \rceil}\right) \\ &= u \otimes v \\ &= \epsilon_A \otimes \epsilon_B \left(\delta_{\lceil u \rceil} \otimes \delta_{\lceil v \rceil}\right) \end{aligned}$$

and

$$F_! \, m^{\lceil - \rceil}_{1_{\mathsf{L}}} \circ m^{F_!}_{1_{\mathsf{L}}} \left(\mathbb{1}\right) = F_! \, m^{\lceil - \rceil}_{1_{\mathsf{L}}} \left(\delta_{\emptyset}\right) = \delta_{m^{\lceil - \rceil}_{1_{\mathsf{L}}} \, (\emptyset)} = \delta_{\lceil \mathbb{1} \rceil} = \epsilon_{1_{\mathsf{L}}} \left(\mathbb{1}\right)$$

Similarly, η is a monoidal transformation. The following diagrams must commute:

$$\begin{tikzcd} P \times Q \arrow[rr, "\mathrm{id}_{P \times Q}"] \arrow[d, "\eta_P \times \eta_Q"'] & & P \times Q \arrow[d, "\eta_{P \times Q}"'] \\ \lceil F_! \, P \rceil \times \lceil F_! \, Q \rceil \arrow[r, "m^{\lceil - \rceil}_{F_! \, P, F_! \, Q}"] & \lceil F_! \, P \otimes F_! \, Q \rceil \arrow[r, "\lceil m^{F_!}_{P,Q} \rceil"'] & \lceil F_! \left(P \times Q\right) \rceil \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathsf{P}} \arrow[r, "\eta_{1_{\mathsf{P}}}"] \arrow[d, "\mathrm{id}_{1_{\mathsf{P}}}"] & \lceil F_! \, 1_{\mathsf{P}} \rceil \\ 1_{\mathsf{P}} \arrow[r, "m^{\lceil - \rceil}_{1_{\mathsf{L}}}"'] & \lceil 1_{\mathsf{L}} \rceil \arrow[u, "\lceil m^{F_!}_{1_{\mathsf{L}}} \rceil"'] \end{tikzcd}$$

These diagrams are witnessed by the following computations:

$$
\begin{aligned}
\lceil m^{F_!}_{P,Q} \rceil \circ m^{\lceil - \rceil}_{F_!\,P, F_!\,Q} \circ \eta_P \times \eta_Q\,(x,y) &= \lceil m^{F_!}_{P,Q} \rceil \circ m^{\lceil - \rceil}_{F_!\,P, F_!\,Q}\,(\lceil \delta_x \rceil, \lceil \delta_y \rceil) \\
&= \lceil m^{F_!}_{P,Q} \rceil\,(\lceil \delta_x \otimes \delta_y \rceil) \\
&= \lceil m^{F_!}_{P,Q}\,(\delta_x \otimes \delta_y) \rceil \\
&= \lceil \delta_{(x,y)} \rceil = \eta_{P \times Q}\,(x,y) \\
\lceil m^{F_!}_{1_{\mathsf{L}}} \rceil \circ m^{\lceil - \rceil}_{1_{\mathsf{L}}}\,(\emptyset) &= \lceil m^{F_!}_{1_{\mathsf{L}}} \rceil\,(\lceil \mathbb{1} \rceil) = \lceil m^{F_!}_{1_{\mathsf{L}}}\,(\mathbb{1}) \rceil \\
&= \lceil \delta_\emptyset \rceil = \eta_{1_{\mathsf{P}}}\,(\emptyset)
\end{aligned}
$$

□

Corollary 33. FinVect, FinSet, *and* FinSet^{op} *together form an* LPC *model.*

5.2 Relations

Let Rel be the category of sets and relations, and let Set be the category of sets and functions. (Notice that the sets in either category here may be infinite, unlike in the FinVect case.) It is easy to see that Rel is linearly distributive where the tensor and the cotensor are both cartesian product, and distributivity is just associativity. The unit is a singleton set; for concreteness assume $1_{\mathsf{L}} = \bot_{\mathsf{L}} = \emptyset$. Furthermore, negation on Rel is the identity on objects, along with the maps

$$
\begin{aligned}
\gamma^\bot_A &: A^\bot \otimes A \to \bot_{\mathsf{L}} & \gamma^1_A &: 1_{\mathsf{L}} \to A \parr A^\bot \\
\gamma^\bot_A &= \{((x,x),\emptyset) \mid x \in A\} & \gamma^1_A &= \{(\emptyset,(x,x)) \mid x \in A\}
\end{aligned}
$$

Set is cartesian and its opposite category Set^{op}, cocartesian. The $F_!$ and $F_?$ functors are the forgetful functors which interpret a function as a relation. The $\lceil - \rceil$ functor takes a set to its powerset. Suppose R is a relation between A and B. Then $\lceil R \rceil : \lceil A \rceil \to \lceil B \rceil$ is defined as

$$\lceil R \rceil(X) = \{y \in B \mid \exists x \in X, (x,y) \in R\}$$

with monoidal components

$$
\begin{aligned}
m^{\lceil - \rceil}_{1_{\mathsf{L}}} &: 1_{\mathsf{L}} \to \lceil 1_{\mathsf{L}} \rceil & m^{\lceil - \rceil}_{A,B} &: \lceil A \rceil \times \lceil B \rceil \to \lceil A \times B \rceil \\
m^{\lceil - \rceil}_{1_{\mathsf{L}}}\,(\emptyset) &= \emptyset & m^{\lceil - \rceil}_{A,B}\,(X_1, X_2) &= X_1 \times X_2
\end{aligned}
$$

The dual notion $\lfloor - \rfloor$ is just the inverse.

5.3 Boolean Algebras and the Birkhoff Duality

Next we examine an example of the LPC categories where $\mathcal{P}$ and $\mathcal{C}$ are related by a non-trivial duality. The relationship is based on Birkhoff's representation theorem (Birkhoff 1937), which can be interpreted as a duality between the categories of finite partial orders and order-preserving maps ($\mathcal{P}$) on the one hand, and finite distributive lattices with bounded lattice homomorphisms ($\mathcal{C}$) on the other hand.

The linear category is the category $\mathcal{L}$ of finite boolean algebras with bounded lattice homomorphisms. This $\mathcal{L}$ is symmetric weakly distributive where the products overlap with the coproducts, meaning that the model is somewhat degenerate. For the monoidal structure, the units are both the singleton lattice $\emptyset$, and the tensors $A \otimes B$ and $A \parr B$ are the boolean algebra with base set $A \times B$ and lattice structure as follows:

$$
\begin{aligned}
\bot &= (\bot, \bot) \\
(x_1, y_1) \vee (x_2, y_2) &= (x_1 \vee x_2, y_1 \vee y_2) \\
\neg(x,y) &= (\neg x, \neg y)
\end{aligned}
\qquad
\begin{aligned}
\top &= (\top, \top) \\
(x_1, y_1) \wedge (x_2, y_2) &= (x_1 \wedge x_2, y_1 \wedge y_2)
\end{aligned}
$$

Given a partially ordered set $(P, \leq)$, a subset $X \subseteq P$ is called *lower* if it is downwards closed with respect to $\leq$. The set of all lower sets of P forms a lattice with $\top = P$, $\bot_{\mathsf{L}} = \emptyset$, meet as union and join and intersection. Let P^* refer to this lattice.

Meanwhile, given a lattice C, an element x is join-irreducible if x is neither $\bot_{\mathsf{L}}$ nor the join of any two elements less than x. That is, $x \neq y \vee z$ for $y, z \neq x$. Let C_* be the partially ordered set with base set the join-irreducible elements of C, with the ordering

$$x \leq y \quad \text{iff} \quad x = y \wedge x$$

The operators $(-)^*$ and $(-)_*$ extend to functors that form a duality between $\mathcal{P}$ and $\mathcal{C}$. For the details of that categorical duality, refer to Stanley (2011).

The monoidal structure on $\mathcal{P}$ is given by the cartesian product with the following ordering:

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{iff} \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

The unit is the singleton order $\{\emptyset\}$. For every poset P in $\mathcal{P}$, the following components form a commutative comonoid:

$$\begin{array}{cc} e^{\otimes}_P : P \to 1_{\mathsf{P}} & d^{\otimes}_P : P \to P \otimes P \\ e^{\otimes}_P\,(x) = \emptyset & d^{\otimes}_P\,(x) = (x, x) \end{array}$$

Next, finite distributive lattices have a monoidal structure with the unit the singleton lattice $\{\emptyset\}$ and the tensor $C_1 \parr C_2$ the lattice where the base set is $C_1 \times C_2$ and the lattice structure is given by

$$\begin{array}{cc} \bot = (\bot, \bot) & \top = (\top, \top) \\ (x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2) & (x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) \end{array}$$

For every lattice C in $\mathcal{C}$ there exists a commutative monoid with the following components:

$$\begin{array}{cc} e^{\parr}_C : \bot_{\mathsf{C}} \to C & d^{\parr}_C : C \parr C \to C \\ e^{\parr}_C\,(\emptyset) = \bot & d^{\parr}_C\,(x, y) = x \wedge y \end{array}$$

The natural transformations $e^{\otimes}$ and $e^{\parr}$, and $d^{\otimes}$ and $d^{\parr}$ are in fact interchanged under the Birkhoff duality.

Next we define the symmetric monoidal functors. Define

$$\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P} \qquad \text{and} \qquad \lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C}$$

to be forgetful functors.

For a finite partial order P, define $F_!\,P$ to be the boolean algebra with base set $\mathcal{P}(P)$, and with top, bottom, join, meet and negation corresponding to P, $\emptyset$, union, intersection and complementation respectively. For a morphism $f : P_1 \to P_2$, define $F_!\,f : F_!\,P_1 \to F_!\,P_2$ to be

$$F_!\,f\,(X) = \{f\,(x) \mid x \in X\}.$$

Similarly, for a finite distributive lattice C, define $F_?\,C$ to be the boolean algebra with base set $\mathcal{C}$ with the same structure as above. For morphisms $f : C_1 \to C_2$ define $F_?\,f : F_?\,C_1 \to F_?\,C_2$ to again be

$$F_?\,f\,(X) = \{f\,(x) \mid x \in X\}.$$

These functors must respect the dualities in that

$$(F_!\,P)^{\perp} \simeq F_?\,P^* \qquad \text{and} \qquad \lceil A \rceil^* \simeq \lfloor A^{\perp} \rfloor$$

We will prove the first of these here. For any element x in a poset P, define $\overline{x} \in P^*$ to be the downward closed set $\{y \mid y \leq x\}$. Notice that for any x, $\bigvee \overline{x} = x$ and for any independently downward closed set X, $\overline{\bigvee X} = X$.

Define a morphism $f : (F_! \, P)^{\perp} \to F_? \, P^*$ by

$$f\,(X) = \{\overline{x} \mid x \in \neg X\}$$

and define $g : F_? \, P^* \to (F_! \, P)^{\perp}$ by

$$g\,(X) = \neg\{\bigvee Y \mid Y \in X\}$$

It is easy to check that these are both bounded lattice homomorphisms. To verify that $f \circ g \simeq \mathrm{id}_{F_? \, P^*}$ and $g \circ f \simeq \mathrm{id}_{(F_! \, P)^{\perp}}$, consider the following computations:

$$\begin{aligned}
f \circ g\,(X) &= f\,(\neg\{\bigvee Y \mid Y \in X\}) \\
&= \{\overline{x} \mid x \in \neg(\neg\{\bigvee Y \mid Y \in X\})\} \\
&\cong \{\overline{x} \mid x \in \{\bigvee Y \mid Y \in X\}\} \\
&= \{\overline{\bigvee Y} \mid Y \in X\} = \{Y \mid Y \in X\} = X \\
g \circ f\,(X) &= g\,(\{\overline{x} \mid x \in \neg X\}) \\
&= \neg\{\bigvee Y \mid Y \in \{\overline{x} \mid x \in \neg X\}\} \\
&= \neg\{\bigvee \overline{x} \mid x \in \neg X\} \\
&= \neg\{x \mid x \in \neg X\} = \neg(\neg X) \cong X
\end{aligned}$$

To construct the adjunction $F_! \dashv \lceil - \rceil$, it suffices to show a bijection of homomorphism sets $\mathrm{Hom}(F_! \, P, A) \cong \mathrm{Hom}(P, \lceil A \rceil)$. Suppose $f : F_! \, P \to A$ in $\mathcal{L}$. Then define $f^{\sharp} : P \to \lceil A \rceil$ by

$$f^{\sharp}\,(x) = f\,(\{z \in X \mid z \leq x\})$$

This morphism is in fact order-preserving. Next, for $g : P \to \lceil A \rceil$ define $g^{\flat} : F_! \, P \to A$ as follows:

$$g^{\flat}\,(X) = \bigvee_{x \in X} g\,(x)$$

Again it is easy to check that $g^{\flat}$ is a lattice homomorphism. To show $F_! \dashv \lceil - \rceil$ it suffices to show that $(f^{\sharp})^{\flat} = f$ and $(g^{\flat})^{\sharp} = g$. Let $f : F_! \, P \to A$ be a bounded lattice homomorphism of boolean algebras. Then

$$\begin{aligned}
(f^{\sharp})^{\flat}\,(X) &= \bigvee_{x \in X} f^{\sharp}\,(x) \\
&= \bigvee_{x \in X} f\,(\{z \mid z \leq x\}) \\
&= f\,(\bigvee_{x \in X} \{z \mid z \vee x = x\}) \\
&= f\,(\bigvee_{x \in X} \{x\} \cup \{z \mid z \vee x = x\}) \\
&= f\,(\bigvee_{x \in X} \{z \vee x \mid z \vee x = x\}) \\
&= f\,(\bigvee_{x \in X} \{x\}) = f\,(X)
\end{aligned}$$

Let $g : P \to \lceil A \rceil$ be an order-preserving map. Then

$$\begin{aligned}
(g^{\flat})^{\sharp}\,(x) &= g^{\flat}\,(\{z \mid z \leq x\}) \\
&= \bigvee_{z \leq x} g\,(z) = g\,(x)
\end{aligned}$$

From these definitions, the unit and counit of the adjunction are as follows:

$$\begin{aligned}
&\epsilon_A : F_! \lceil A \rceil \to A \\
&\epsilon_A (X) = \mathrm{id}^\flat_{\lceil A \rceil} (X) = \bigvee_{x \in X} \mathrm{id}_{\lceil A \rceil} (x) = \bigvee X \\
&\eta_P : P \to \lceil F_! P \rceil \\
&\eta_P (x) = (\mathrm{id}_{F_! P})^\sharp (x) = \{z \mid z \leq x\}
\end{aligned}$$

To show the adjunction is monoidal, it suffices to prove ϵ and η are monoidal natural transformations.

The proof of the monoidal adjunction $\lfloor - \rfloor \dashv F_?$ is similar.

6 Related Work

Girard (1987) first introduced linear logic to mix the constructivity of intuitionistic propositional logic with the duality of classical logic. Partly because of this constructivity, there has been great interest in the semantics of linear logic in both the classical and intuitionistic fragments. Consequently, there exist several categorical frameworks for its semantic models.

One influential framework is Benton *et al.*'s *linear category* (Benton et al. 1993), consisting of a symmetric monoidal closed category with products and a linear exponential comonad !. Other characterizations include the Seely category (Seely 1987), based on a distribution morphism between $!A \otimes !B$ and $!(A \,\&\, B)$. Wadler (1992) and Bierman (1994) proved that there was a disconnect between Seely's category and the popular term calculus due to Abramsky (1993). Bierman's response was a new Seely category, sound with respect to Abramsky's term language, which adds a symmetric monoidal adjunction between a Seely category and its co-Kleisli category.

Except for Seely's original formulation, these works deal with the intuitionistic fragment of linear logic. The multiplicative fragment (restricted to $\otimes$ and $⅋$) of classical linear logic is usually modeled by a *-autonomous category, introduced by Barr (1991). Schalk (2004) adapted linear categories to the classical case by requiring that the symmetric monoidal closed category be *-autonomous. The coproduct $⅋$ and coexponential ? are then induced from the duality.

Cockett and Seely (1997), seeking to study $\otimes$ and $⅋$ as independent structures unobscured by duality, introduced linearly distributive categories, which make up the linear category in the LPC model. The authors extended this motivation to the exponentials by modeling ! and ? as linear functors (Cockett and Seely 1999), meaning that ? is *not* derived from ! and $(-)^\perp$. The LPC model reflects that work by allowing ! and ? to have different adjoint decompositions.

Other variations of classical linear logic, notably Girard's Logic of Unity (Girard 1993), distinguish linear propositions from persistent ones. The sequent $\Gamma; \Gamma' \vdash \Delta'; \Delta$ is meant to be seen as a derivation where Γ' and Δ' are persistent and admit weakening and contraction. In Γ and Δ every proposition is purely linear.

Ramifying LU's separation (in the intuitionistic case), Benton (1995) developed the linear/non-linear logic and categorical model described in Section 2. Barber used this model as the semantics for a term calculus called DILL (Barber 1996). A Lafont category (Lafont 1988) is a canonical instance of an LNL model where $!A$ is the free commutative comonoid generated by A. This construction automatically admits an adjunction between automatically forms an adjunction between a linear category $\mathcal{L}$ and the category of commutative comonoids over $\mathcal{L}$. However, the LNL and LPC models have an advantage over Lafont categories by allowing a much greater range of interpretations for the exponential—Lafont's construction excludes traditional models of linear logic like coherence spaces and the category Rel.

References

S. Abramsky. Computational interpretations of linear logic. *Theoretical Computer Science*, 111:3–57, 1993.

A. Ahmed, M. Fluet, and G. Morrisett. L3: A linear language with locations. *Fundam. Inf.*, 77(4):397–449, 2007. ISSN 0169-2968.

A. Barber. Dual intuitionistic linear logic. Technical report, 1996.

M. Barr. *-autonomous categories and linear logic. *Mathematical Structures in Computer Science*, 1(2): 159–178, 1991.

N. Benton, G. M. Bierman, J. M. E. Hyland, and V. de Paiva. A term calculus for intuitionistic linear logic. In *Proceedings of the International Conference on Typed Lambda Calculi and Applications*, pages 75–90. Springer-Verlag LNCS 664, 1993.

P. N. Benton. A mixed linear and non-linear logic: proofs, terms and models (preliminary report). Technical Report 352, Computer Laboratory, University of Cambridge, September 1994.

P. N. Benton. A mixed linear and non-linear logic: proofs, terms and models. In *Proceedings of Computer Science Logic (CSL '94), Kazimierz, Poland.*, pages 121–135. Springer-Verlag, 1995.

G. M. Bierman. *On intuitionistic linear logic*. PhD thesis, University of Cambridge, Computer Laboratory, 1994.

G. Birkhoff. Rings of sets. *Duke Mathematical Journal*, 3(3):443–454, 1937.

J. Cockett and R. Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114(2):133 – 173, 1997. ISSN 0022-4049. doi: http://dx.doi.org/10.1016/0022-4049(95)00160-3.

J. Cockett and R. Seely. Linearly distributive functors. *Journal of Pure and Applied Algebra*, 143(13):155 – 203, 1999. ISSN 0022-4049.

M. Gaboardi. *Linearity: an Analytic Tool in the study of Complexity and Semantics of Programming Languages*. PhD thesis, Universita degli Studi di Torino-Institut National Polytechnique de Lorraine, 2007.

J.-Y. Girard. Linear logic. *Theoretical computer science*, 50(1):1–101, 1987.

J.-Y. Girard. On the unity of logic. *Annals of pure and applied logic*, 59(3):201–217, 1993.

M. Hicks, G. Morrisett, D. Grossman, and T. Jim. Experience with safe manual memory-management in Cyclone. In *ISMM '04: Proceedings of the 4th international symposium on Memory management*, pages 73–84, New York, NY, USA, 2004. ACM. ISBN 1-58113-945-4. doi: http://doi.acm.org/10.1145/1029873.1029883.

Y. Lafont. The linear abstract machine. *Theoretical Computer Science*, 59:157–180, 1988. Corrections in vol. 62, pp. 327–328.

C. Liang and D. Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009.

P. A. Melliés. Categorical models of linear logic revisited. 2003.

P.-A. Melliès. Categorical semantics of linear logic. *Panoramas et Syntheses*, 27, 2009.

A. Schalk. Whats is a categorical model of linear logic. Technical report, University of Manchester, 2004.

R. A. Seely. *Linear logic, *-autonomous categories and cofree coalgebras*. 1987.

R. P. Stanley. *Enumerative combinatorics*, volume 49. Cambridge university press, 2011.

P. Wadler. There's no substitute for linear logic. In *8th International Workshop on the Mathematical Foundations of Programming Semantics*, 1992.

S. Zdancewic and A. C. Myers. Secure information flow via linear continuations. *Higher Order and Symbolic Computation*, 15(2/3):209–234, 2002.