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Inertial Velocity and Attitude Estimation for Quadrotors: Supplementary Material

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I. Parallel Transport on S^2

We now demonstrate that the parallel transport on S^2 with the Levi-Civita connection corresponding to the metric induced by $\mathbb{R}^3$ is equivalent to eq. (19) of the parent document, assuming the vector is transported along the geodesic from p to q. Without loss of generality, we will assume p is the north pole (i.e., the point $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ when the sphere is naturally embedded in $\mathbb{R}^3$) of the 2-sphere, since this manifold is symmetric under rotation.

Parallel transport is a linear operation on vectors because the covariant derivative is linear [1]

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \tag{1}$$

$$\nabla_X(fY) = f\nabla_X Y + \nabla_{fX} \cdot Y. \tag{2}$$

If f is a constant, $\nabla_{fX} = 0$, and thus for constants a and b:

$$\nabla_X(aY+bZ) = a\nabla_X Y + b\nabla_X Z. \tag{3}$$

If we denote the parallel transport of a vector $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$ from the tangent space at p to the tangent space at q through the geodesic from p to q by $\tau_{pq}(\mathbf{u})$, we have

$$\tau_{pq}(\mathbf{u}) = a\tau_{pq}(\mathbf{v}) + b\tau_{pq}(\mathbf{w}), \tag{4}$$

for $a, b \in \mathbb{R}$ and vectors $\mathbf{v}$ and $\mathbf{w}$ in the tangent space at p.

Hence, if we can show that, for some basis vectors $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ in the tangent space $\mathrm{T}_p S^2$,

$$\tau_{pq}(\mathbf{v}_\parallel) = R_{qp}\mathbf{v}_\parallel, \quad \tau_{pq}(\mathbf{v}_\perp) = R_{qp}\mathbf{v}_\perp, \tag{5}$$

then we have shown that eq. (19) of the parent document is true for any vector $\mathbf{v}_p$ in the tangent space at p. We will show this by first constructing differential equations from the parallel transport equation, then by showing that they are satisfied by the components of tangent vectors $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ moving according to eq. (19) of the parent document. We use stereographic coordinates during this process.

First, the vectorial representation $\mathbf{q}$ of the point q on the sphere is represented as a function of the stereographic coordinates

$$\mathbf{q}(t) = \frac{1}{1+s_x^2(t)+s_y^2(t)} \cdot \begin{bmatrix} 2s_x(t) \\ 2s_y(t) \\ 1-s_x^2(t)-s_y^2(t) \end{bmatrix}. \tag{6}$$

From now on, we suppress the dependence of $s_x(t)$ and $s_y(t)$ on t unless necessary. Differentiating this with respect to s_x and s_y gives us the tangent basis vectors, denoted $\mathbf{e}_x$ and $\mathbf{e}_y$

$$\mathbf{e}_x = \frac{1}{(1+s_x^2+s_y^2)^2} \cdot \begin{bmatrix} 2(1-s_x^2+s_y^2) \\ -4s_x s_y \\ -4s_x \end{bmatrix}, \tag{7}$$

$$\mathbf{e}_y = \frac{1}{(1+s_x^2+s_y^2)^2} \cdot \begin{bmatrix} -4s_x s_y \\ 2(1+s_x^2-s_y^2) \\ -4s_y \end{bmatrix} \tag{8}$$

By taking the dot products of these vectors, we obtain the components of the induced metric tensor

$$g_{xx} = g_{yy} = \frac{4}{(1+s_x^2+s_y^2)^2}, \tag{9}$$

$$g_{xy} = g_{yx} = 0. \tag{10}$$

The Christoffel symbols can be computed using the formula [2]

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{ \frac{\partial}{\partial s_i} g_{jk} + \frac{\partial}{\partial s_j} g_{ki} - \frac{\partial}{\partial s_k} g_{ij} \right\} g^{km}, \tag{11}$$

where $i, j, k, m \in \{x, y\}$ and g^{km} are the components of the inverse of the metric tensor g_{km}. The Christoffel symbols for the affine connection are

$$\Gamma_{ij}^k = \frac{2}{1+s_x^2+s_y^2} \cdot \begin{cases} s_k & i = j \neq k \\ -s_k & i \neq j \text{ or } i = j = k. \end{cases} \tag{12}$$

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Any vector $\mathbf{v}$ in the tangent space T_pS^2 can be constructed

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y. \tag{13}$$

The parallel transport equations are obtained by setting the covariant derivative of $\mathbf{v}$ to zero. This provides

$$\frac{dv_k}{dt} = -\sum_{i,j}\Gamma^k_{ij}v_j\frac{ds_i}{dt}, \quad k = 1,\ldots,n \tag{14}$$

or, after substituting the Christoffel Symbols,

$$\begin{aligned}\dot{v}_x &= \frac{2((s_y\dot{s}_x - s_x\dot{s}_y)v_y + (s_x\dot{s}_x + s_y\dot{s}_y)v_x)}{1+s_x^2+s_y^2}\\ \dot{v}_y &= \frac{2((s_x\dot{s}_y - s_y\dot{s}_x)v_x + (s_x\dot{s}_x + s_y\dot{s}_y)v_y)}{1+s_x^2+s_y^2}.\end{aligned} \tag{15}$$

Now, we construct $\mathbf{v}_\|$ and $\mathbf{v}_\perp$ and see that their components, in terms of $\mathbf{e}_x$ and $\mathbf{e}_y$, satisfy eq. (15). Let $\mathbf{r}(t)$ be the time-parameterized path on the geodesic from $\mathbf{p}$ to $\mathbf{q}$. Define $\mathbf{v}_\|$ as

$$\mathbf{v}_\| = \frac{d\mathbf{r}}{dt}|_{t=0} = [\boldsymbol{\omega}]_\times\mathbf{p}, \tag{16}$$

where $\boldsymbol{\omega}$ is an angular velocity vector that is orthogonal to both $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{v}_\|$ is transported according to eq. (19) of the parent document, then

$$\begin{aligned}\tau_{pq}(\mathbf{v}_\|) &= R_{qp}\mathbf{v}_\|\\ &= R_{qp}[\boldsymbol{\omega}]_\times\mathbf{p}\\ &= [\boldsymbol{\omega}]_\times R_{qp}\mathbf{p}\\ &= [\boldsymbol{\omega}]_\times\mathbf{q},\end{aligned} \tag{17}$$

where we have used the fact that, since $R_{qp} = \exp(\theta_{qp}[\boldsymbol{\omega}]_\times)$, it commutes with $[\boldsymbol{\omega}]_\times$. We also define $\mathbf{v}_\perp$

$$\mathbf{v}_\perp = [\mathbf{p}]_\times\mathbf{v}_\| = [\mathbf{p}]_\times[\boldsymbol{\omega}]_\times\mathbf{p}. \tag{18}$$

Then, as was the case with $\mathbf{v}_\|$, if the parallel transport of $\mathbf{v}_\perp$ on the geodesic is described by eq. (19) of the parent document

$$\begin{aligned}\tau_{pq}(\mathbf{v}_\perp) &= R_{qp}\mathbf{v}_\perp\\ &= R_{qp}[\mathbf{p}]_\times[\boldsymbol{\omega}]_\times\mathbf{p}\\ &= R_{qp}[\mathbf{p}]_\times R_{qp}^\top R_{qp}[\boldsymbol{\omega}]_\times\mathbf{p}\\ &= R_{qp}[\mathbf{p}]_\times R_{qp}^\top[\boldsymbol{\omega}]_\times R_{qp},\mathbf{p},\\ &= R_{qp}[\mathbf{p}]_\times R_{qp}^\top[\boldsymbol{\omega}]_\times\mathbf{q}\\ &= [R_{qp}\mathbf{p}]_\times[\boldsymbol{\omega}]_\times\mathbf{q}\\ &= [\mathbf{q}]_\times[\boldsymbol{\omega}]_\times\mathbf{q}\end{aligned} \tag{19}$$

where we used the identity that, for any rotation matrix $R \in \mathrm{SO}(3)$ and any vector $\mathbf{v} \in \mathbb{R}^3$,

$$[R\mathbf{v}]_\times = R[\mathbf{v}]_\times R^\top, \tag{20}$$

if $\boldsymbol{\omega} = \begin{bmatrix}\omega_1 & \omega_2 & 0\end{bmatrix}^\top$ (the third component is zero since $\boldsymbol{\omega}$ is orthogonal to $\mathbf{p}$, which is the north pole of the sphere), then, from (6) and (17)

$$\tau_{pq}(\mathbf{v}_\|) = \frac{1}{1+s_x^2+s_y^2}\cdot\begin{bmatrix}-\omega_2(s_x^2+s_y^2-1)\\ \omega_1(s_x^2+s_y^2-1)\\ 2(\omega_1 s_y - \omega_2 s_x)\end{bmatrix}, \tag{21}$$

and, if $\tau_{pq}(\mathbf{v}_\|) = v_{\|x}\mathbf{e}_x + v_{\|y}\mathbf{e}_y$, then, from (8), we can verify

$$\begin{aligned}v_{\|x} &= \frac{1}{2}\omega_2(1+s_x^2-s_y^2) - \omega_1 s_x s_y,\\ v_{\|y} &= \frac{1}{2}\omega_1(s_x^2-s_y^2-1) - \omega_2 s_x s_y.\end{aligned} \tag{22}$$

If we substitute (22) into (15) and simplify, we obtain

$$\begin{aligned}\frac{(\omega_1 s_x + \omega_2 s_y)\dot{s}_y}{1+s_x^2+s_y^2} &= 0,\\ \frac{(\omega_1 s_x + \omega_2 s_y)\dot{s}_x}{1+s_x^2+s_y^2} &= 0.\end{aligned} \tag{23}$$

But we know that $\boldsymbol{\omega}$ is orthogonal to $\mathbf{q}$. Hence, from (6), we have $\omega_1 s_x + \omega_2 s_y = 0$. Thus, eq.s 15 are satisfied by (22).

Now, consider $\mathbf{v}_\perp$. We have from (6) and (19)

$$\tau_{pq}(\mathbf{v}_\perp) = \begin{bmatrix}\omega_1 - \frac{4s_x(\omega_1 s_x+\omega_2 s_y)}{(1+s_x^2+s_y^2)^2}\\ \omega_2 - \frac{4s_y(\omega_1 s_x+\omega_2 s_y)}{(1+s_x^2+s_y^2)^2}\\ \frac{2(\omega_1 s_x+\omega_2 s_y)(s_x^2+s_y^2-1)}{(1+s_x^2+s_y^2)^2}\end{bmatrix}. \tag{24}$$

Again, one can verify that, if $\tau_{pq}(\mathbf{v}_\perp) = v_{\perp x}\mathbf{e}_x + v_{\perp y}\mathbf{e}_y$, then we have

$$\begin{aligned}v_{\perp x} &= \frac{1}{2}\omega_1(1-s_x^2+s_y^2) - \omega_2 s_x s_y,\\ v_{\perp y} &= \frac{1}{2}\omega_2(1+s_x^2-s_y^2) - \omega_1 s_x s_y.\end{aligned} \tag{25}$$

Substituting (25) into (15) and simplifying yields

$$\begin{aligned}\frac{(\omega_1 s_x + \omega_2 s_y)\dot{s}_x}{1+s_x^2+s_y^2} &= 0,\\ \frac{(\omega_1 s_x + \omega_2 s_y)\dot{s}_y}{1+s_x^2+s_y^2} &= 0.\end{aligned} \tag{26}$$

Again, since $\boldsymbol{\omega}$ is orthogonal to $\mathbf{q}$, then $\omega_1 s_x + \omega_2 s_y = 0$ and these equations are satisfied.

Now, we have shown that eq. (19) of the parent document satisfies eq. (15) for the basis vectors of the tangent space at p, $\mathbf{v}_\|$ and $\mathbf{v}_\perp$. Hence, this is how we parallel transport any vector on the 2-sphere.

II. Algorithms

The following algorithms summarize the implementation of the UKF on the sphere.

Algorithm 1 Riemannian UKF on S^2

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procedure UKF($\hat{\mathbf{x}}_{k-1}, \hat{P}_{k-1}, \mathbf{u}_{k-1}, \mathbf{y}_k, T_{k-1}$)
  $\hat{\mathbf{s}}_{k-1} \leftarrow \hat{\mathbf{x}}_{k-1}[0:1]$
  $\hat{\mathbf{s}}'_{k-1} \leftarrow \hat{\mathbf{x}}_{k-1}[2:10]$
  $L_{k-1} \leftarrow \sqrt{(n+\lambda)\hat{P}_{k-1}}$
  $\mathcal{X}_{0,k-1} \leftarrow \hat{\mathbf{x}}_{k-1}$
  for $i = 1, \ldots, n$ **do**
    $\boldsymbol{\delta}_{i,k-1} \leftarrow L_{k-1}[0:1, i]$
    $\boldsymbol{\delta}'_{i,k-1} \leftarrow L_{k-1}[2:10, i]$
    $\mathcal{X}_{i,k-1} \leftarrow \begin{bmatrix} \exp_{\hat{\mathbf{s}}_{k-1}}(T\boldsymbol{\delta}_{i,k-1}) \\ \hat{\mathbf{s}}_{i,k-1} + \boldsymbol{\delta}'_{i,k-1} \end{bmatrix}$
    $\mathcal{X}_{n+i,k-1} \leftarrow \begin{bmatrix} \exp_{\hat{\mathbf{s}}_{k-1}}(-T\boldsymbol{\delta}_{i,k-1}) \\ \hat{\mathbf{s}}_{i,k-1} - \boldsymbol{\delta}'_{i,k-1} \end{bmatrix}$
  end for
  for $i = 0, \ldots, 2n$ **do**
    $\mathcal{X}^-_{i,k} \leftarrow \mathbf{f}(\mathcal{X}_{i,k-1}, \mathbf{u}_{k-1})$
    $\mathcal{S}_{i,k} \leftarrow \mathcal{X}^-_{i,k}[0:1, i]$
    $\mathcal{S}'_{i,k} \leftarrow \mathcal{X}^-_{i,k}[2:10, i]$
  end for
  $\hat{\mathbf{s}}^-_k \leftarrow$ WeightedAvgSphere($\mathcal{S}_{0,k}, \ldots, \mathcal{S}_{2n,k}$)
  $\hat{\mathbf{s}}'^-_k \leftarrow \sum_{i=0}^{2n} w_i \mathcal{S}'_{i,k}$
  $\hat{\mathbf{x}}^-_k \leftarrow \begin{bmatrix} \hat{\mathbf{s}}^{-\top}_k & \hat{\mathbf{s}}'^{-\top}_k \end{bmatrix}^\top$
  $T^-_k \leftarrow$ ParallelTransport($T_{k-1}, \hat{\mathbf{s}}_{k-1}, \hat{\mathbf{s}}^-_k$)
  for $i = 0, \ldots, 2n$ **do**
    $\boldsymbol{\delta}^-_{i,k} \leftarrow T^{-\top}_k \log_{\hat{\mathbf{s}}^-_k} \mathcal{S}_{i,k}$
    $\boldsymbol{\delta}'^-_{i,k} \leftarrow \mathcal{S}'_{i,k} - \hat{\mathbf{s}}'^-_k$
  end for
  $\hat{P}^-_k \leftarrow \sum_{i=0}^{2n} w_i \begin{bmatrix} \boldsymbol{\delta}^-_{i,k} \\ \boldsymbol{\delta}'^-_{i,k} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}^{-\top}_{i,k} & \boldsymbol{\delta}'^{-\top}_{i,k} \end{bmatrix} + Q$
  for $i = 0, \ldots, 2n$ **do**
    $\mathcal{Y}_{i,k} \leftarrow \mathbf{h}(\mathcal{X}^-_{i,k})$
  end for
  $\hat{\mathbf{y}}_k \leftarrow \sum_{i=0}^{2n} w_i \mathcal{Y}_{i,k}$
  $\hat{P}_{yy,k} \leftarrow \sum_{i=0}^{2n} w_i (\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)(\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)^\top + R$
  $\hat{P}_{xy,k} \leftarrow \sum_{i=0}^{2n} w_i \begin{bmatrix} \boldsymbol{\delta}^-_{i,k} \\ \boldsymbol{\delta}'^-_{i,k} \end{bmatrix} (\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)^\top$
  $K_k \leftarrow \hat{P}_{xy,k} \hat{P}^{-1}_{yy,k}$
  $\Delta_{x,k} \leftarrow K_k(\mathbf{y}_k - \hat{\mathbf{y}}_k)$
  $\Delta_{s,k} \leftarrow \Delta_{x,k}[0:1]$
  $\Delta_{s',k} \leftarrow \Delta_{x,k}[2:10]$
  $\hat{\mathbf{s}}_k \leftarrow \exp_{\hat{\mathbf{s}}^-_k}(T^-_k \Delta_{s,k})$
  $\hat{\mathbf{s}}'_k \leftarrow \hat{\mathbf{s}}'^-_k + \Delta_{s',k}$
  $\hat{\mathbf{x}}_k \leftarrow \begin{bmatrix} \hat{\mathbf{s}}^\top_k & \hat{\mathbf{s}}'^\top_k \end{bmatrix}^\top$
  $\hat{P}_k \leftarrow \hat{P}^-_k - K_k P_{yy,k} K^\top_k$
  $T_k \leftarrow$ ParallelTransport($T^-_k, \hat{\mathbf{s}}^-_k, \hat{\mathbf{s}}_k$)
end procedure

Algorithm 2 Weighted average of points $p_1, \ldots, p_n$ on a sphere

---

procedure WeightedAvgSphere($p_1, \ldots, p_n$)
  $\bar{\mathbf{p}} \leftarrow \sum_{i=1}^{n} w_i \cdot$ PointToVector(p_i)
  $\bar{p} \leftarrow$ VectorToPoint($\bar{\mathbf{p}}$)
  $\Delta_p \leftarrow \sum_{i=1}^{n} w_i \log_{\bar{p}} p_i$
  while $\|\Delta_p\| > \epsilon$ **do**
    $\bar{p} \leftarrow \exp_{\bar{p}} \Delta_p$
    $\Delta_p \leftarrow \sum_{i=1}^{n} w_i \log_{\bar{p}} p_i$
  end while
  $\bar{p} \leftarrow \exp_{\bar{p}} \Delta_p$
return $\bar{p}$
end procedure

Algorithm 3 Parallel transport of the tangent basis T on the sphere from point p_1 to point p_2

---

procedure ParallelTransport(T, p_1, p_2)
  $\mathbf{p}_1 \leftarrow$ PointToVector(p_1)
  $\mathbf{p}_2 \leftarrow$ PointToVector(p_2)
  $\theta \leftarrow \cos^{-1}(\mathbf{p}_1 \cdot \mathbf{p}_2)$
  $\mathbf{u} \leftarrow (\mathbf{p}_1 \times \mathbf{p}_2)/\|\mathbf{p}_1 \times \mathbf{p}_2\|$
  $R = I + \sin\theta[\mathbf{u}]_\times + (1 - \cos\theta)[\mathbf{u}]^2_\times$
return RT
end procedure

Algorithm 4 Conversion of a point s on the sphere to a unit vector in $\mathbb{R}^3$

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procedure PointToVector(s)
  $s_x \leftarrow s[0]$
  $s_y \leftarrow s[1]$
  $x \leftarrow 2s_x/(1 + s^2_x + s^2_y)$
  $y \leftarrow 2s_y/(1 + s^2_x + s^2_y)$
  $z \leftarrow (1 - s^2_x - s^2_y)/(1 + s^2_x + s^2_y)$
return $\begin{bmatrix} x & y & z \end{bmatrix}^\top$
end procedure

Algorithm 5 Conversion of a unit vector $\mathbf{p}$ in $\mathbb{R}^3$ to a point on the sphere in stereographic coordinates

procedure VECTORTOPOINT($\mathbf{p}$)
  $x \leftarrow \mathbf{p}[0]$
  $y \leftarrow \mathbf{p}[1]$
  $z \leftarrow \mathbf{p}[2]$
  $s_x \leftarrow x/(1+z)$
  $s_y \leftarrow y/(1+z)$
return $\begin{bmatrix} s_x & s_y \end{bmatrix}^\top$
end procedure

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