



10-2018

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Recommended Citation

James B. Svacha Jr, Kartik Mohta, Michael Watterson, Giuseppe Loianno, and Vijay Kumar, "Inertial Velocity and Attitude Estimation for Quadrotors: Supplementary Material", . October 2018.

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Inertial Velocity and Attitude Estimation for Quadrotors: Supplementary Material

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Inertial Velocity and Attitude Estimation for Quadrotors: Supplementary Material

James Svacha¹, Kartik Mohta¹, Michael Watterson¹, Giuseppe Loianno², and Vijay Kumar¹

I. PARALLEL TRANSPORT ON S^2

We now demonstrate that the parallel transport on S^2 with the Levi-Civita connection corresponding to the metric induced by \mathbb{R}^3 is equivalent to eq. (19) of the parent document, assuming the vector is transported along the geodesic from p to q . Without loss of generality, we will assume p is the north pole (i.e., the point $[0 \ 0 \ 1]^\top$ when the sphere is naturally embedded in \mathbb{R}^3) of the 2-sphere, since this manifold is symmetric under rotation.

Parallel transport is a linear operation on vectors because the covariant derivative is linear [1]

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \quad (1)$$

$$\nabla_X(fY) = f\nabla_X Y + \nabla_{fX} \cdot Y. \quad (2)$$

If f is a constant, $\nabla_{fX} = 0$, and thus for constants a and b :

$$\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z. \quad (3)$$

If we denote the parallel transport of a vector $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$ from the tangent space at p to the tangent space at q through the geodesic from p to q by $\tau_{pq}(\mathbf{u})$, we have

$$\tau_{pq}(\mathbf{u}) = a\tau_{pq}(\mathbf{v}) + b\tau_{pq}(\mathbf{w}), \quad (4)$$

for $a, b \in \mathbb{R}$ and vectors \mathbf{v} and \mathbf{w} in the tangent space at p .

Hence, if we can show that, for some basis vectors \mathbf{v}_\parallel and \mathbf{v}_\perp in the tangent space $T_p S^2$,

$$\tau_{pq}(\mathbf{v}_\parallel) = R_{qp}\mathbf{v}_\parallel, \quad \tau_{pq}(\mathbf{v}_\perp) = R_{qp}\mathbf{v}_\perp, \quad (5)$$

then we have shown that eq. (19) of the parent document is true for any vector \mathbf{v}_p in the tangent space at p . We

This work was supported by the ARL grants W911NF-08-2-0004, W911NF-17-2-0181, ONR grants N00014-07-1-0829, N00014-14-1-0510, ARO grant W911NF-13-1-0350, NSF grants IIS-1426840, IIS-1138847, DARPA grants HR001151626, HR0011516850.

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will show this by first constructing differential equations from the parallel transport equation, then by showing that they are satisfied by the components of tangent vectors \mathbf{v}_\parallel and \mathbf{v}_\perp moving according to eq. (19) of the parent document. We use stereographic coordinates during this process.

First, the vectorial representation \mathbf{q} of the point q on the sphere is represented as a function of the stereographic coordinates

$$\mathbf{q}(t) = \frac{1}{1 + s_x^2(t) + s_y^2(t)} \cdot \begin{bmatrix} 2s_x(t) \\ 2s_y(t) \\ 1 - s_x^2(t) - s_y^2(t) \end{bmatrix}. \quad (6)$$

From now on, we suppress the dependence of $s_x(t)$ and $s_y(t)$ on t unless necessary. Differentiating this with respect to s_x and s_y gives us the tangent basis vectors, denoted \mathbf{e}_x and \mathbf{e}_y

$$\mathbf{e}_x = \frac{1}{(1 + s_x^2 + s_y^2)^2} \cdot \begin{bmatrix} 2(1 - s_x^2 + s_y^2) \\ -4s_x s_y \\ -4s_x \end{bmatrix}, \quad (7)$$

$$\mathbf{e}_y = \frac{1}{(1 + s_x^2 + s_y^2)^2} \cdot \begin{bmatrix} -4s_x s_y \\ 2(1 + s_x^2 - s_y^2) \\ -4s_y \end{bmatrix} \quad (8)$$

By taking the dot products of these vectors, we obtain the components of the induced metric tensor

$$g_{xx} = g_{yy} = \frac{4}{(1 + s_x^2 + s_y^2)^2}, \quad (9)$$

$$g_{xy} = g_{yx} = 0. \quad (10)$$

The Christoffel symbols can be computed using the formula [2]

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left\{ \frac{\partial}{\partial s_i} g_{jk} + \frac{\partial}{\partial s_j} g_{ki} - \frac{\partial}{\partial s_k} g_{ij} \right\} g^{km}, \quad (11)$$

where $i, j, k, m \in \{x, y\}$ and g^{km} are the components of the inverse of the metric tensor g_{km} . The Christoffel symbols for the affine connection are

$$\Gamma_{ij}^k = \frac{2}{1 + s_x^2 + s_y^2} \cdot \begin{cases} s_k & i = j \neq k \\ -s_k & i \neq j \text{ or } i = j = k. \end{cases} \quad (12)$$

Any vector \mathbf{v} in the tangent space $T_p S^2$ can be constructed

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y. \quad (13)$$

The parallel transport equations are obtained by setting the covariant derivative of \mathbf{v} to zero. This provides

$$\frac{dv_k}{dt} = - \sum_{i,j} \Gamma_{ij}^k v_j \frac{ds_i}{dt}, \quad k = 1, \dots, n \quad (14)$$

or, after substituting the Christoffel Symbols,

$$\begin{aligned} \dot{v}_x &= \frac{2((s_y \dot{s}_x - s_x \dot{s}_y)v_y + (s_x \dot{s}_x + s_y \dot{s}_y)v_x)}{1 + s_x^2 + s_y^2} \\ \dot{v}_y &= \frac{2((s_x \dot{s}_y - s_y \dot{s}_x)v_x + (s_x \dot{s}_x + s_y \dot{s}_y)v_y)}{1 + s_x^2 + s_y^2}. \end{aligned} \quad (15)$$

Now, we construct \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} and see that their components, in terms of \mathbf{e}_x and \mathbf{e}_y , satisfy eq. (15). Let $\mathbf{r}(t)$ be the time-parameterized path on the geodesic from \mathbf{p} to \mathbf{q} . Define \mathbf{v}_{\parallel} as

$$\mathbf{v}_{\parallel} = \left. \frac{d\mathbf{r}}{dt} \right|_{t=0} = [\boldsymbol{\omega}]_{\times} \mathbf{p}, \quad (16)$$

where $\boldsymbol{\omega}$ is an angular velocity vector that is orthogonal to both \mathbf{p} and \mathbf{q} . If \mathbf{v}_{\parallel} is transported according to eq. (19) of the parent document, then

$$\begin{aligned} \tau_{pq}(\mathbf{v}_{\parallel}) &= R_{qp} \mathbf{v}_{\parallel} \\ &= R_{qp} [\boldsymbol{\omega}]_{\times} \mathbf{p} \\ &= [\boldsymbol{\omega}]_{\times} R_{qp} \mathbf{p} \\ &= [\boldsymbol{\omega}]_{\times} \mathbf{q}, \end{aligned} \quad (17)$$

where we have used the fact that, since $R_{qp} = \exp(\theta_{qp} [\boldsymbol{\omega}]_{\times})$, it commutes with $[\boldsymbol{\omega}]_{\times}$. We also define \mathbf{v}_{\perp}

$$\mathbf{v}_{\perp} = [\mathbf{p}]_{\times} \mathbf{v}_{\parallel} = [\mathbf{p}]_{\times} [\boldsymbol{\omega}]_{\times} \mathbf{p}. \quad (18)$$

Then, as was the case with \mathbf{v}_{\parallel} , if the parallel transport of \mathbf{v}_{\perp} on the geodesic is described by eq. (19) of the parent document

$$\begin{aligned} \tau_{pq}(\mathbf{v}_{\perp}) &= R_{qp} \mathbf{v}_{\perp} \\ &= R_{qp} [\mathbf{p}]_{\times} [\boldsymbol{\omega}]_{\times} \mathbf{p} \\ &= R_{qp} [\mathbf{p}]_{\times} R_{qp}^{\top} R_{qp} [\boldsymbol{\omega}]_{\times} \mathbf{p} \\ &= R_{qp} [\mathbf{p}]_{\times} R_{qp}^{\top} [\boldsymbol{\omega}]_{\times} R_{qp} \mathbf{p}, \quad (19) \\ &= R_{qp} [\mathbf{p}]_{\times} R_{qp}^{\top} [\boldsymbol{\omega}]_{\times} \mathbf{q} \\ &= [R_{qp} \mathbf{p}]_{\times} [\boldsymbol{\omega}]_{\times} \mathbf{q} \\ &= [\mathbf{q}]_{\times} [\boldsymbol{\omega}]_{\times} \mathbf{q} \end{aligned}$$

where we used the identity that, for any rotation matrix $R \in \text{SO}(3)$ and any vector $\mathbf{v} \in \mathbb{R}^3$,

$$[R\mathbf{v}]_{\times} = R[\mathbf{v}]_{\times} R^{\top}, \quad (20)$$

if $\boldsymbol{\omega} = [\omega_1 \ \omega_2 \ 0]^{\top}$ (the third component is zero since $\boldsymbol{\omega}$ is orthogonal to \mathbf{p} , which is the north pole of the sphere), then, from (6) and (17)

$$\tau_{pq}(\mathbf{v}_{\parallel}) = \frac{1}{1 + s_x^2 + s_y^2} \cdot \begin{bmatrix} -\omega_2(s_x^2 + s_y^2 - 1) \\ \omega_1(s_x^2 + s_y^2 - 1) \\ 2(\omega_1 s_y - \omega_2 s_x) \end{bmatrix}, \quad (21)$$

and, if $\tau_{pq}(\mathbf{v}_{\parallel}) = v_{\parallel x} \mathbf{e}_x + v_{\parallel y} \mathbf{e}_y$, then, from (8), we can verify

$$\begin{aligned} v_{\parallel x} &= \frac{1}{2} \omega_2 (1 + s_x^2 - s_y^2) - \omega_1 s_x s_y, \\ v_{\parallel y} &= \frac{1}{2} \omega_1 (s_x^2 - s_y^2 - 1) - \omega_2 s_x s_y. \end{aligned} \quad (22)$$

If we substitute (22) into (15) and simplify, we obtain

$$\begin{aligned} \frac{(\omega_1 s_x + \omega_2 s_y) \dot{s}_y}{1 + s_x^2 + s_y^2} &= 0, \\ \frac{(\omega_1 s_x + \omega_2 s_y) \dot{s}_x}{1 + s_x^2 + s_y^2} &= 0. \end{aligned} \quad (23)$$

But we know that $\boldsymbol{\omega}$ is orthogonal to \mathbf{q} . Hence, from (6), we have $\omega_1 s_x + \omega_2 s_y = 0$. Thus, eq.s 15 are satisfied by (22).

Now, consider \mathbf{v}_{\perp} . We have from (6) and (19)

$$\tau_{pq}(\mathbf{v}_{\perp}) = \begin{bmatrix} \omega_1 - \frac{4s_x(\omega_1 s_x + \omega_2 s_y)}{(1 + s_x^2 + s_y^2)^2} \\ \omega_2 - \frac{4s_y(\omega_1 s_x + \omega_2 s_y)}{(1 + s_x^2 + s_y^2)^2} \\ \frac{2(\omega_1 s_x + \omega_2 s_y)(s_x^2 + s_y^2 - 1)}{(1 + s_x^2 + s_y^2)^2} \end{bmatrix}. \quad (24)$$

Again, one can verify that, if $\tau_{pq}(\mathbf{v}_{\perp}) = v_{\perp x} \mathbf{e}_x + v_{\perp y} \mathbf{e}_y$, then we have

$$\begin{aligned} v_{\perp x} &= \frac{1}{2} \omega_1 (1 - s_x^2 + s_y^2) - \omega_2 s_x s_y, \\ v_{\perp y} &= \frac{1}{2} \omega_2 (1 + s_x^2 - s_y^2) - \omega_1 s_x s_y. \end{aligned} \quad (25)$$

Substituting (25) into (15) and simplifying yields

$$\begin{aligned} \frac{(\omega_1 s_x + \omega_2 s_y) \dot{s}_x}{1 + s_x^2 + s_y^2} &= 0, \\ \frac{(\omega_1 s_x + \omega_2 s_y) \dot{s}_y}{1 + s_x^2 + s_y^2} &= 0. \end{aligned} \quad (26)$$

Again, since $\boldsymbol{\omega}$ is orthogonal to \mathbf{q} , then $\omega_1 s_x + \omega_2 s_y = 0$ and these equations are satisfied.

Now, we have shown that eq. (19) of the parent document satisfies eq. (15) for the basis vectors of the tangent space at p , \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} . Hence, this is how we parallel transport any vector on the 2-sphere.

Algorithm 1 Riemannian UKF on S^2

```

procedure UKF( $\hat{\mathbf{x}}_{k-1}, \hat{P}_{k-1}, \mathbf{u}_{k-1}, \mathbf{y}_k, T_{k-1}$ )
   $\hat{\mathbf{s}}_{k-1} \leftarrow \hat{\mathbf{x}}_{k-1}[0 : 1]$ 
   $\hat{\mathbf{s}}'_{k-1} \leftarrow \hat{\mathbf{x}}_{k-1}[2 : 10]$ 
   $L_{k-1} \leftarrow \sqrt{(n + \lambda) \hat{P}_{k-1}}$ 
   $\mathcal{X}_{0,k-1} \leftarrow \hat{\mathbf{x}}_{k-1}$ 
  for  $i = 1, \dots, n$  do
     $\delta_{i,k-1} \leftarrow L_{k-1}[0 : 1, i]$ 
     $\delta'_{i,k-1} \leftarrow L_{k-1}[2 : 10, i]$ 
     $\mathcal{X}_{i,k-1} \leftarrow \begin{bmatrix} \exp_{\hat{\mathbf{s}}_{k-1}}(T \delta_{i,k-1}) \\ \hat{\mathbf{s}}_{i,k-1} + \delta'_{i,k-1} \end{bmatrix}$ 
     $\mathcal{X}_{n+i,k-1} \leftarrow \begin{bmatrix} \exp_{\hat{\mathbf{s}}_{k-1}}(-T \delta_{i,k-1}) \\ \hat{\mathbf{s}}_{i,k-1} - \delta'_{i,k-1} \end{bmatrix}$ 
  end for
  for  $i = 0, \dots, 2n$  do
     $\mathcal{X}_{i,k}^- \leftarrow \mathbf{f}(\mathcal{X}_{i,k-1}, \mathbf{u}_{k-1})$ 
     $\mathcal{S}_{i,k} \leftarrow \mathcal{X}_{i,k}^- [0 : 1, i]$ 
     $\mathcal{S}'_{i,k} \leftarrow \mathcal{X}_{i,k}^- [2 : 10, i]$ 
  end for
   $\hat{\mathbf{s}}_k^- \leftarrow \text{WEIGHTEDAVGSPHERE}(\mathcal{S}_{0,k}, \dots, \mathcal{S}_{2n,k})$ 
   $\hat{\mathbf{s}}'_k \leftarrow \sum_{i=0}^{2n} w_i \mathcal{S}'_{i,k}$ 
   $\hat{\mathbf{x}}_k^- \leftarrow [\hat{\mathbf{s}}_k^{-\top} \quad \hat{\mathbf{s}}_k'^{-\top}]^\top$ 
   $T_k^- \leftarrow \text{PARALLELTRANSPORT}(T_{k-1}, \hat{\mathbf{s}}_{k-1}, \hat{\mathbf{s}}_k^-)$ 
  for  $i = 0, \dots, 2n$  do
     $\delta_{i,k}^- \leftarrow T_k^{-\top} \log_{\hat{\mathbf{s}}_k^-} \mathcal{S}_{i,k}$ 
     $\delta'_{i,k}^- \leftarrow \mathcal{S}'_{i,k} - \hat{\mathbf{s}}_k'^-$ 
  end for
   $\hat{P}_k^- \leftarrow \sum_{i=0}^{2n} w_i \begin{bmatrix} \delta_{i,k}^- \\ \delta'_{i,k}^- \end{bmatrix} \begin{bmatrix} \delta_{i,k}^{-\top} & \delta'_{i,k}^{-\top} \end{bmatrix} + Q$ 
  for  $i = 0, \dots, 2n$  do
     $\mathcal{Y}_{i,k} \leftarrow \mathbf{h}(\mathcal{X}_{i,k}^-)$ 
  end for
   $\hat{\mathbf{y}}_k \leftarrow \sum_{i=0}^{2n} w_i \mathcal{Y}_{i,k}$ 
   $\hat{P}_{yy,k} \leftarrow \sum_{i=0}^{2n} w_i (\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)(\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)^\top + R$ 
   $\hat{P}_{xy,k} \leftarrow \sum_{i=0}^{2n} w_i \begin{bmatrix} \delta_{i,k}^- \\ \delta'_{i,k}^- \end{bmatrix} (\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k)^\top$ 
   $K_k \leftarrow \hat{P}_{xy,k} \hat{P}_{yy,k}^{-1}$ 
   $\Delta_{x,k} \leftarrow K_k (\mathbf{y}_k - \hat{\mathbf{y}}_k)$ 
   $\Delta_{s,k} \leftarrow \Delta_{x,k} [0 : 1]$ 
   $\Delta_{s',k} \leftarrow \Delta_{x,k} [2 : 10]$ 
   $\hat{\mathbf{s}}_k \leftarrow \exp_{\hat{\mathbf{s}}_k^-}(T_k^- \Delta_{s,k})$ 
   $\hat{\mathbf{s}}'_k \leftarrow \hat{\mathbf{s}}_k'^- + \Delta_{s',k}$ 
   $\hat{\mathbf{x}}_k \leftarrow [\hat{\mathbf{s}}_k^\top \quad \hat{\mathbf{s}}_k'^\top]^\top$ 
   $\hat{P}_k \leftarrow \hat{P}_k^- - K_k P_{yy,k} K_k^\top$ 
   $T_k \leftarrow \text{PARALLELTRANSPORT}(T_k^-, \hat{\mathbf{s}}_k^-, \hat{\mathbf{s}}_k)$ 
end procedure

```

II. ALGORITHMS

The following algorithms summarize the implementation of the UKF on the sphere.

Algorithm 2 Weighted average of points p_1, \dots, p_n on a sphere

```

procedure WEIGHTEDAVGSPHERE( $p_1, \dots, p_n$ )
   $\bar{\mathbf{p}} \leftarrow \sum_{i=1}^n w_i \cdot \text{POINTTOVECTOR}(p_i)$ 
   $\bar{p} \leftarrow \text{VECTORTOPOINT}(\bar{\mathbf{p}})$ 
   $\Delta_p \leftarrow \sum_{i=1}^n w_i \log_{\bar{p}} p_i$ 
  while  $\|\Delta_p\| > \epsilon$  do
     $\bar{p} \leftarrow \exp_{\bar{p}} \Delta_p$ 
     $\Delta_p \leftarrow \sum_{i=1}^n w_i \log_{\bar{p}} p_i$ 
  end while
   $\bar{p} \leftarrow \exp_{\bar{p}} \Delta_p$ 
return  $\bar{p}$ 
end procedure

```

Algorithm 3 Parallel transport of the tangent basis T on the sphere from point p_1 to point p_2

```

procedure PARALLELTRANSPORT( $T, p_1, p_2$ )
   $\mathbf{p}_1 \leftarrow \text{POINTTOVECTOR}(p_1)$ 
   $\mathbf{p}_2 \leftarrow \text{POINTTOVECTOR}(p_2)$ 
   $\theta \leftarrow \cos^{-1}(\mathbf{p}_1 \cdot \mathbf{p}_2)$ 
   $\mathbf{u} \leftarrow (\mathbf{p}_1 \times \mathbf{p}_2) / \|\mathbf{p}_1 \times \mathbf{p}_2\|$ 
   $R \leftarrow I + \sin \theta [\mathbf{u}]_\times + (1 - \cos \theta) [\mathbf{u}]_\times^2$ 
return  $RT$ 
end procedure

```

Algorithm 4 Conversion of a point s on the sphere to a unit vector in \mathbb{R}^3

```

procedure POINTTOVECTOR( $s$ )
   $s_x \leftarrow s[0]$ 
   $s_y \leftarrow s[1]$ 
   $x \leftarrow 2s_x / (1 + s_x^2 + s_y^2)$ 
   $y \leftarrow 2s_y / (1 + s_x^2 + s_y^2)$ 
   $z \leftarrow (1 - s_x^2 - s_y^2) / (1 + s_x^2 + s_y^2)$ 
return  $[x \quad y \quad z]^\top$ 
end procedure

```

Algorithm 5 Conversion of a unit vector \mathbf{p} in \mathbb{R}^3 to a point on the sphere in stereographic coordinates

procedure VECTORTOPPOINT(\mathbf{p})

$x \leftarrow \mathbf{p}[0]$

$y \leftarrow \mathbf{p}[1]$

$z \leftarrow \mathbf{p}[2]$

$s_x \leftarrow x/(1+z)$

$s_y \leftarrow y/(1+z)$

return $[s_x \ s_y]^\top$

end procedure

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