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Adding Large Numbers by Computer

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Cyberneticians are often confronted with very large numbers. Already when relatively small sets of objects, states or attributes are considered, the enumeration of their possible permutations, orderings, relations, patterns, partitions, etc. quickly exceeds astronomical values. This fact has been demonstrated by Ashby [1] who suggested to call numbers larger than 10^{100} combinatorial numbers.

As in the preceeding sentence, it is convenient to express such numbers by their exponents, information theory being a prime example of choosing 2 as their base. For the manipulation of numbers that are in excess of what a given computer can store, this possibility might be the only way. Unfortunately the operation of addition is then undefinable, making summations impossible. However, if one is willing to tolerate a certain error, this limitation can be bypassed by the following function.

Let b^m be the largest number that a given computer can manipulate. The logarithm of the sum of two positive numbers, b^X and b^Y, can be approximated by:

$$\log_b(b^X + b^Y) \cong X \oplus Y = \max(X, Y) + \begin{cases} \text{if } |X - Y| \leqslant m & \log_b(1 + b^{-|X-Y|}) \\ \text{if } |X - Y| > m & 0 \end{cases}$$

The error of this approximation does not exceed the largest computable number:

$$b^m > (b^X + b^Y) - b^{X \oplus Y}$$

The proof is fairly simple. First we factor out the largest of the two summands:

$$\log_b(b^X + b^Y) = \log_b(b^{\max(X,Y)}(b^{X-\max(X,Y)} + b^{Y-\max(X,Y)}))$$

The larger of the two summands in the embedded parentheses has zero as its exponent and the smaller of the two summands has $-|X - Y|$ as its exponent. Hence:

$$\log_b(b^X + b^Y) = \max(X, Y) + \log_b(1 + b^{-|X-Y|}) \qquad \text{QED}$$

To appreciate what is involved, let the largest manipulable number be 10^{75}. The sum of 10^{120} which exceeds this limit by 45 decimal places and 10^{50} is a number starting with a 1 followed by 69 zeros, a 1 and 50 zeros. This is quite a large number and obviously not much different from 10^{120}. Correspondingly, $120 \oplus 50 = 120 + \log_{10}(1 + 10^{-70})$, the logarithmic expression being a positive number and not much different from zero. On the other hand, the sum $10^{120} + 10^{-50}$ exceeds the limit of error-free computation. According to the above function $120 - (-50) \gg 75$ and $120 \oplus -50 = 120$. Thus the error is exactly 10^{-50} which is a number with 49 zeros after the decimal point followed by a 1. This is a very small number indeed.

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To keep the accumulation of errors in sequential computations at a minimum it is advisable to sum the numbers in their order of magnitude, starting with the smallest number.

References

[1] W. Ross Ashby, "Introductory remarks at panel discussion," Chapter 12, pp. 165-169 in Mihajlo D. Mesarovic (Ed.) *Views on General Systems Theory; Proceedings of The Second Systems Symposium at Case Institute of Technology.* New York: Wiley, 1964.

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