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Robust Invariants From Functionally Constrained Motion

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Abstract

We address in the problem of control-based recovery of robot pose and the environmental lay-out. Panoramic sensors provide us with an 1D projection of characteristic features of a 2D operation map. Trajectories of these projections contain the information about the position of *a priori* unknown landmarks in the environment. We introduce here the notion of spatiotemporal signatures of projection trajectories. These signatures are global measures, like area, characterized by considerably higher robustness with respect to noise and outliers than the commonly applied point correspondence. By modeling the 2D motion plane as the complex plane we show that by means of complex analysis our method can be embedded in the well-known affine reconstruction paradigm.

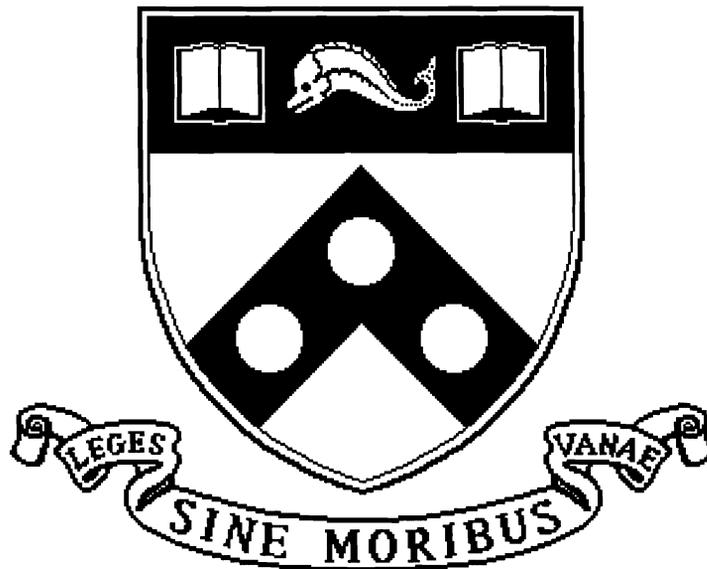
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Robust Invariants from Functionally Constrained Motion

Technical Report MS-CIS-98-08*

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July 1, 1998

Abstract

We address in the problem of control-based recovery of robot pose and the environmental lay-out. Panoramic sensors provide us with an 1D projection of characteristic features of a 2D operation map. Trajectories of these projections contain the information about the position of a priori unknown landmarks in the environment. We introduce here the notion of spatiotemporal signatures of projection trajectories. These signatures are global measures, like area, characterized by considerably higher robustness with respect to noise and outliers than the commonly applied point correspondence. By modeling the 2D motion plane as the complex plane we show that by means of complex analysis our method can be embedded in the well-known affine reconstruction paradigm.

1 Introduction

The problem of obtaining metrical information about a scene by moving a camera is well-known one, and has many different forms. For example, the ego motion may or may not be considered an unknown. In some robotics applications the ego motion may be accurately known by the use of sensors other than vision, e.g. by encoder data. Nevertheless, the problem of localizing a robot accurately is a hard one in its own right, and it seems prudent to generally consider ego motion an unknown. On the other hand, one could make assumptions about the *curve* on which the motion occurs without saying anything about the exact position of the robot. For example, a mobile robot with a single camera may fixate on a target and move towards it in a straight line. While it may

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be hard to accurately estimate the robots position, we do know that we can restrict the motion of the robot to a line if it uses a decent control law. Notice that this doesn't mean or assume that we know any of the parameters that describe this line in a global Cartesian reference frame, but in principle there is some information here that can be exploited. Another example is provided by a four wheeled vehicle, which may accurately travel in a circular motion. This example differs from the first in that the restriction comes about by mechanical means.

As stated above, this type of constraint does not assume that we have geometric information about the curve. It is a constraint of "higher order". In other words, not only does the robot not know where it is, but it does not even know exactly what curve it is on. What it does know is that the curve lies in a member of some prescribed space of curves, e.g. circles or lines.

We have described two examples in which such constraints occur. Below we describe a generalization of the example in which the constraint is obtained by control. This topic is intimately related to the subject of *image coordinates*.

This work has its origin in the study of panoramic vision systems for use in robot games. Using such a system, one may consider some vision problems as two dimensional, since the panoramic system essentially gives an overhead view of the scene. We will begin with a description of a such a system built in our lab in section 2. Section 3 describes the image coordinates that are naturally associated with the panoramic system. The main results of this paper are in section 4. This is the most technical section, and the one in which we demonstrate how to extract metrical information from motion. Finally, in section 5, we return to image coordinates and discuss some issues related to invariance under coordinate changes.

2 Panoramic Vision

Recently, some researchers in the robotics and vision community have begun to investigate the use of curved mirrors to obtain panoramic and omni-directional views¹ These systems generally consist of a curved mirror suspended above an upward pointing camera.

How to interpret and make use of the visual information obtained by such systems, e.g. how it can be used to control robots, is not immediately clear. Nayar ([4]) constructed an omni-directional sensor using a parabolic mirror. In this case, the goal was to reconstruct the "normal" views that a camera would have, and so this system allows one to "pan the camera in software". Mouaddib and Pegard ([5]) use a conical mirror to estimate a robot's pose. This is done via a search for radial lines, which correspond to vertical lines in the world. The locations of the vertical lines in the world are not known, but by moving the robot and solving the correspondence problem for the radial lines, the robot's pose can be estimated by solving a non-linear equation. Ishiguro, Ueda, Tsuji... . Betke and Gavruta ([1]) investigated the localization problem, motivated by the Siemens mobile robot *Ratbot*. *Ratbot*, like our robots, used a spherical mirror. In this case the positions of a collection of landmarks are known. By measuring the angles between the landmarks, the robot's pose can be determined. As few as three landmarks are necessary to solve this problem and [1] investigates what can be done with noisy data if there are many landmarks available.

¹We will distinguish omni-directional sensors, i.e. sensors that see in *all* directions, from panoramic sensors, which can see only a 360 degree cross section of the world.

3 Image Coordinates

Consider the a 2-d situation where a robot can detect and track three fixed points in the plane and measure the angles between them, θ and ϕ as in figure 3.

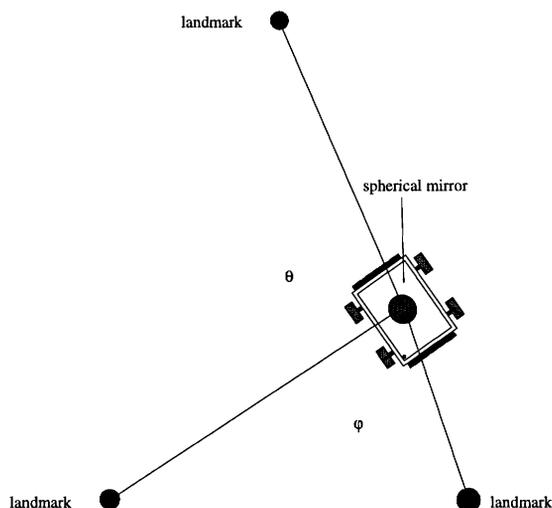


Figure 1: The angles between three fixed landmarks provide a natural coordinate system

In the formal sense, θ and ϕ define a local coordinate system on the plane, and so even if the robot does not have a means of measuring its Cartesian coordinates with respect to some frame, it can measure it's (θ, ϕ) coordinates. Notice that this is exactly what a panoramic vision system does when measuring the angles between vertical lines. Thus, in some sense, the panoramic vision system can be considered as an overhead camera. Considering that cameras naturally measure angles well, and remembering a lesson from differential geometry not to be prejudiced against strange coordinates, it seems likely that the (θ, ϕ) coordinate system is a good one for the robot to use.

What do the coordinate lines look like in this angular system ? The theorem of Thales that if one fixes two points on a circle A, B , then for any third point C on the circle, the angle $\angle ACB$ is a constant. A familiar special case occurs when A and B lie on opposite sides of the circle - the angle is then 90° . Thus the coordinate curves, e.g. $\theta = constant$, are circles through two of the landmarks (see figure 3). Such a family is sometimes referred to as a **pencil of circles**.

Therefore the "coordinate grid" of the angular coordinate system is formed by two families of intersecting circles (see 3). Of course, these coordinates are singular if the robot lies on the circle determined by the three landmarks, which we will refer to as the **singular circle** of the landmarks. Thus if the robot is near the singular circle, small perturbations in θ and ϕ correspond to large changes in the position of the robot.

It is important to remember that we will consider the landmarks as distinguishable and defining θ and ϕ by coming in a fixed order. Otherwise, it can appear that the angular coordinates do not determine a unique point in the plane.

A natural question, especially from the point of view of controlling a robot is how to convert from angular coordinates to Cartesian coordinates. This of course, assumes that the positions of

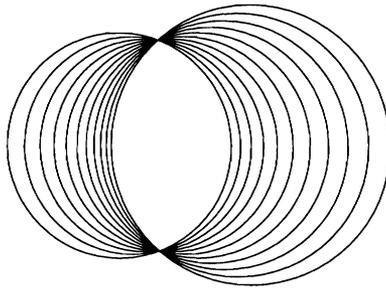


Figure 2: The pencil of circles through two fixed points

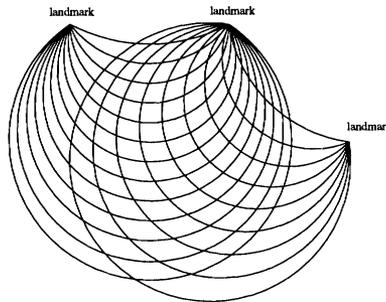


Figure 3: The coordinate grid induced by three landmarks

the landmarks are known. This is an old problem in fact, familiar to sailors² and surveyors. An analytical solution can be found in [2], for example. One approach is simply to write down the equation of each of the three circles that contain the robot and two of the landmarks and the position of the robot is where they all intersect. Despite the fact that this method does not provide an explicit, compact formula for converting from angular to Cartesian coordinates, it works well on a computer. The authors have been able to use this method to obtain extremely precise pose estimates (better than 1 %) for a small mobile robot. This must be qualified though, by pointing out that we kept our robot “away” from the singular circle. For more precise statements about this problem, the reader is referred to [1]. Our main point here though, is not just that panoramic systems have the ability to very accurately estimate pose, but that we have experimental evidence that the angles θ and ϕ can be measured very accurately by our panoramic system.

An interesting feature of the angular coordinate system is that it provides measurable coordinates *even if the locations of the landmarks are unknown in a Cartesian reference frame*. Thus one could consider a reconstruction problem in which the Cartesian coordinates of both the robot and the landmarks are unknown, but several measurements of the angular coordinates have been made and the goal is to find the Cartesian coordinates of the landmarks and the robot. In this case, it is necessary to have *more than* three landmarks. One can write down the equations that must hold, and view the problem as an over-determined non-linear system. This method is employed in [3].

If the above method is attempted for three landmarks, there are more unknowns than equations. Here we consider the problem where there are only three landmarks, but we constrain the motion in the manner described in the introduction.

²The sailors solved this problem with an analog device!

4 Relating the Landmarks

We consider the problem of determining the positions of three landmarks, given that a robot moves in a complete circle, making a measurement of its angular coordinates with respect to the landmarks at each instant in time. Without loss of generality, we take the circle to be the unit circle centered at the origin of the complex plane, and let z_1, z_2 and z_3 be the three landmarks. We also assume that the position of the robot at time t is e^{it} . (It will become clear below that the dynamics of the robot are not relevant.) Consider the trace of the curve $t \rightarrow (\theta(t), \phi(t))$. The trace is a closed planar curve, and we may experimentally approximate it by simply having the robot record its coordinates as it moves in the circle. We then pose the question “what can be said about z_1, z_2 and z_3 , given this curve?”

There are several natural quantities associated with such a curve, and some of these quantities are relatively insensitive to error in measurement. For example, the area bounded by the curve, or its barycenter. Notice that once the data has been gathered, it may be transformed, e.g. instead of considering the trace of $t \rightarrow (\theta(t), \phi(t))$, we can also consider the trace of $t \rightarrow (\cos(\theta(2t)), \phi(t))$. Our plan is as follows: for this last curve, compute a general formula for the area A , bounded by the curve in terms of z_1, z_2 and z_3 . Our main result is then

Theorem 1 *The area function $A = A(z_1, z_2, z_3)$ for the region bounded by the curve $t \rightarrow (\cos(\theta(t)), \phi(t))$ can be expressed as a rational function of z_1, z_2 and z_3 and their conjugates.*

To apply this result, one can experimentally compute the area and set it equal to $A(z_1, z_2, z_3)$, determining a relation between z_1, z_2 and z_3 .

Proof The area bounded by a closed curve C and parameterized by $t \rightarrow (x(t), y(t))$ may be expressed as the line integral

$$\int_C xdy = \int_C ydx. \quad (1)$$

Therefore we must express θ and ϕ in this manner. Up until this point, we have not been explicit about how to define θ and ϕ - one must address the usual sticky issues involved in choosing a branch of the arctangent function. We will return to this issue later, for now simply writing

$$\begin{aligned} \theta(t) &= \arctan(y - y_1, x - x_1) - \arctan(y - y_2, x - x_2), \\ \phi(t) &= \arctan(y - y_2, x - x_2) - \arctan(y - y_3, x - x_3), \end{aligned}$$

where $z_n = x_n + iy_n, n = 1, 2, 3$. If one then tries to plug in these equations to (1), a seemingly intractable integral results. This is where the methods of complex analysis enter. Taking $x + iy = z = z(t) = e^{it}$ we see that

$$\begin{aligned} \arctan(y - y_1, x - x_1) &= \text{Im}(\log(z_1 - z)), \\ \arctan(y - y_2, x - x_2) &= \text{Im}(\log(z_2 - z)), \end{aligned}$$

where we use the complex logarithm defined by $\log(re^{i\psi}) = r + i\psi, -\pi \leq \psi < \pi$. The identity $\text{Im}(w) = \frac{i}{2}(w - \bar{w})$ applied to the above yields

$$\theta(t) = -\frac{i}{2} \log\left(\frac{z - z_1}{z - z_2} \cdot \frac{\bar{z} - \bar{z}_2}{\bar{z} - \bar{z}_1}\right), \quad (2)$$

$$\phi(t) = -\frac{i}{2} \log\left(\frac{z - z_2}{z - z_3} \cdot \frac{\bar{z} - \bar{z}_3}{\bar{z} - \bar{z}_2}\right). \quad (3)$$

Our goal is now to express $A = \int_C 2\cos(2\theta)d\phi$ as a complex integral over S^1 . Therefore we need to express θ and $\frac{d\phi}{dt}$ as functions of z . It is convenient to let $f(z) = \frac{z - z_1}{z - z_2}$, and $g(z) = \frac{z - z_2}{z - z_3}$, so that

$$\theta = -\frac{i}{2} \log\left(\frac{f}{\bar{f}}\right), \quad (4)$$

$$\phi = -\frac{i}{2} \log\left(\frac{g}{\bar{g}}\right). \quad (5)$$

\bar{f} and \bar{g} may be expressed as rational functions of z since for unit z we have that $\bar{z} = \frac{1}{z}$. So we can express θ as a function of z and using the fact that $\cos(t) = \frac{e^{it} + e^{-it}}{2}$, we have that

$$\cos(2\theta) = \frac{1}{2} \left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f} \right),$$

which is a rational function of z .

Since

$$\frac{d\phi}{dt} = -\frac{i}{2} \left(\frac{d\bar{g}}{dz} - \frac{dg}{dz} \right) \frac{dz}{dt}$$

and $\frac{dz}{dt} = \frac{d}{dt}(e^{it}) = ie^{it} = iz$, it follows that

$$\int_C \theta d\phi = -\frac{i}{4} \int_{S^1} \left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f} \right) \left(\frac{d\bar{g}}{g} - \frac{dg}{g} \right) dz, \quad (6)$$

which has a rational function as an integrand. At this point, we employed the symbol manipulator, Maple, to analyze this integral. The integrand has singularities at z_1, z_2, z_3 and $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$. Assuming that the motion of the robot does not encircle any of the landmarks, the residue theorem says that the integral will be equal to

$$2\pi i(R_1 + R_2 + R_3),$$

where R_1, R_2 and R_3 are the respective residues of the integrand at $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$. The resulting expression for the area is quite large (see figure 4). This concludes the proof of our main result.

In order to construct further relations between z_1, z_2 and z_3 , the data may be transformed into another curve. For example, rather than considering $t \rightarrow (\cos(2\theta(t)), \phi)$, the curve $t \rightarrow (\sin(2\theta(t)), \phi)$ could have also been considered. Generically, one expects the area function from two different such representations to be independent, and so in principle one needs only to generate three such functions to solve for z_1, z_2 and z_3 . Unfortunately the only rational functions that the authors have found so far are very large.

Notice that area is only one of several quantities that may be chosen for such constructions. Arclength, winding number, barycenters, etc. are all possible choices, although some of these quantities are expected to be more stable than others. Arclength for example, is a quantity that involves derivatives, and so computing accurately from real data is trickier (and presumably less robust) than computing a quantity such as the barycenter.

5 Circles are Invariant

In this section we address the subject of angular coordinates further, and demonstrate how they may aid in gathering the data required to make use of the results of the above type.

As mentioned earlier, an interesting feature of angular coordinates is that they can be measured without knowing anything about the location of the landmarks that define them. This allows one to define a curve in angular coordinates that can be followed by a robot, without knowing an expression for the curve in Cartesian coordinates. Additionally, there are some families of curves that are invariant under the transformation from one angular system to another. In order to demonstrate this, we begin by reviewing some complex analysis.

A **fractional linear transformation** of the (extended) complex plane is a mapping of the form

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

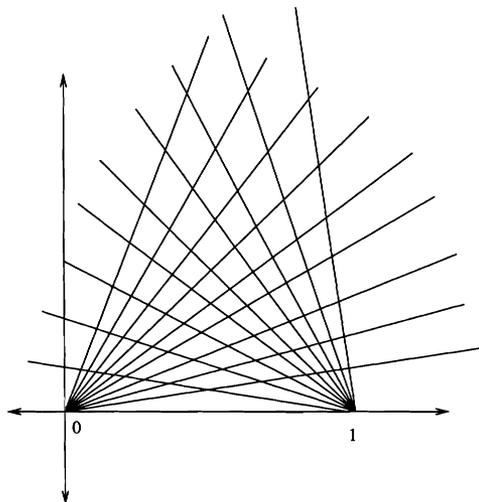
These transformations have many nice properties. To begin with, they form a group. The inverse of the above transform is given by

$$z \rightarrow \frac{dz - b}{-cz + a}.$$

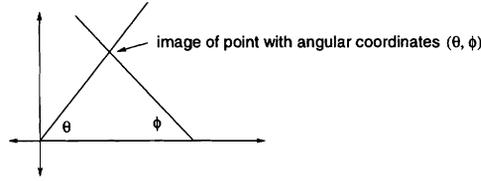
Since these maps are analytic, they are conformal, i.e. they preserve angles. But in fact have the much stronger property that they takes circles and lines to circles and lines (if one views the extended complex plane as the Riemann sphere then this just means that circles on the sphere are taken to circles on the sphere). Finally, they are **triply transitive**, i.e. given an ordered triple of three distinct points in the extended complex plane, (z_1, z_2, z_3) they may be mapped to any other such triple. To see this observe that the above triple is mapped to $(0, \infty, 1)$ by the transformation

$$z \rightarrow \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}. \quad (7)$$

These transformations are well suited for dealing with circles in the complex plane. Suppose that as above, z_1, z_2 and z_3 are our landmarks. Using the transform (7), we see that the pencil of circles through z_1 and z_2 are mapped to lines through the origin (since $z_2 \rightarrow \infty$, circles through z_2 becomes lines, lines being circles of infinite radius).



Similarly, circles through z_2 and z_3 are mapped to lines through the real number 1. Thus the coordinate grid in figure 3 looks like figure 5. Since angles are preserved between the circles under this transformation, a point with coordinates (θ, ϕ) in the angular system is mapped to a point which lies on a line through 0 making an angle of θ with the x -axis, and also on a line through 1 that makes an angle of ϕ with the x -axis.



Therefore the mapping described in Cartesian to angular coordinates is

$$(\theta, \phi) \rightarrow \left(\arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right), \arccos\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \right)$$

. Since $x = r\cos(t) + x_0, y = r\sin(t) + y_0$ describes a circle in the xy -plane, and because fractional linear transforms take circles to circles, the image of the this circle will also be a circle. It can be described explicitly as

$$\theta(t) = \arccos\left(\frac{r\cos(t) + x_0}{\sqrt{(r\cos(t) + x_0)^2 + (r\sin(t) + y_0)^2}}\right),$$

$$\phi(t) = \arccos\left(\frac{r\sin(t) + y_0}{\sqrt{(r\cos(t) + x_0)^2 + (r\sin(t) + y_0)^2}}\right).$$

Notice that these equations do not refer to the landmarks at all, i.e. the equation of a circle is invariant under the choice of z_1, z_2 and z_3 . This means that it is should be possible for a robot to execute a circular motion in angular coordinates without knowing the locations of z_1, z_2 and z_3 .

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