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Information and Learning in Markets

FEI LI
University of Pennsylvania, lifei1019@gmail.com

Information and Learning in Markets

Abstract

This dissertation consists of three essays. I investigate information dynamics under different settings. In Chapter 1, I consider a market with a profit-maximizing monopolist seller who has K identical goods to sell before a deadline. At each date, the seller posts a price and the quantity available but cannot commit to future offers. Over time, potential buyers with different reservation values enter the market. Buyers can strategically time their purchases, trading off (1) a possibly lower price in the future with the risk of being rationed and (2) the current price without competition. I analyze equilibrium price paths and buyers' purchase behavior. I show that incentive compatible price paths decline smoothly over the time period between sales and jump up immediately after a transaction. In equilibrium, high value buyers purchase immediately on arrival. Crucially, before the deadline, the seller may periodically liquidate part of his stock via a fire sale to secure a higher price in the future. Intuitively, these sales allow the seller to `commit' to high prices going forward. The possibility of fire sales before the deadline implies that the allocation may be inefficient. The inefficiency arises from the scarce good being misallocated to low value buyers, rather than the withholding inefficiency that is normally seen with a monopolist seller. In chapter 2, I study dynamic bargaining with uncertainty over the buyer's valuation and the seller's outside option. A long-lived seller makes offers to a long-lived buyer whose value is private information. There may exist a short-lived buyer, whose value is higher than that of the long-lived buyer. The arrival of a short-lived buyer, if she exists, is determined by a Poisson process. I characterize the unique perfect Bayesian equilibrium. The equilibrium displays price fluctuations: in some periods, the seller charges a high price unacceptable to the long-lived buyer, in the hope that the short-lived buyer appears in that period; in the other periods, he offers a price attractive to some values of the long-lived buyer. The price dynamics result from the interaction between two learning processes: exogenous learning about the existence of short-lived buyers, and endogenous learning about the long-lived buyer's value. In chapter 3, I study the dynamics of workers' on-the-job search behavior and its consequences in an equilibrium labor market. In a model with both directed search and learning about the match quality of firm-worker pairs, I highlight the job search target effect of learning: as a worker updates the evaluation of his current job, he adjusts his on-the-job search target, which results in a different job finding rate. I show that this model generates a non-monotonic relation between the employment-to-employment transition rate and tenure, which provides a new explanation of the hump-shaped separation rate-tenure profile.

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INFORMATION AND LEARNING IN MARKETS

Fei Li

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Supervisor of Dissertation

____

George J. Mailath, Walter H. Annenberge Professor in the Social Science

Graduate Group Chairperson

____

George J. Mailath, Walter H. Annenberge Professor in the Social Science

Dissertation Committee

Hanming Fang, Professor of Economics

Guido Menzio, Associate Professor of Economics

Mallesh Pai, Assistant Professor of Economics

INFORMATION AND LEARNING IN MARKETS

To Can Tian

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ABSTRACT

INFORMATION AND LEARNING IN MARKETS

Fei Li

George Mailath

This dissertation consists of three essays. I investigate information dynamics under different settings. In Chapter 1, I consider a market with a profit-maximizing monopolist seller who has K identical goods to sell before a deadline. At each date, the seller posts a price and the quantity available but cannot commit to future offers. Over time, potential buyers with different reservation values enter the market. Buyers can strategically time their purchases, trading off (1) a possibly lower price in the future with the risk of being rationed and (2) the current price without competition. I analyze equilibrium price paths and buyers' purchase behavior. I show that incentive compatible price paths decline smoothly over the time period between sales and jump up immediately after a transaction. In equilibrium, high value buyers purchase immediately on arrival. Crucially, before the deadline, the seller may periodically liquidate part of his stock via a fire sale to secure a higher price in the future. Intuitively, these sales allow the seller to 'commit' to high prices going forward. The possibility of fire sales before the deadline implies that the allocation may be inefficient. The inefficiency arises from the scarce good being misallocated to low value buyers, rather than the withholding inefficiency that is normally seen with a monopolist seller. In chapter 2, I study dynamic bargaining with uncertainty over the

buyer's valuation and the seller's outside option. A long-lived seller makes offers to a long-lived buyer whose value is private information. There may exist a short-lived buyer, whose value is higher than that of the long-lived buyer. The arrival of a short-lived buyer, if she exists, is determined by a Poisson process. I characterize the unique perfect Bayesian equilibrium. The equilibrium displays price fluctuations: in some periods, the seller charges a high price unacceptable to the long-lived buyer, in the hope that the short-lived buyer appears in that period; in the other periods, he offers a price attractive to some values of the long-lived buyer. The price dynamics result from the interaction between two learning processes: exogenous learning about the existence of short-lived buyers, and endogenous learning about the long-lived buyer's value. In chapter 3, I study the dynamics of workers' on-the-job search behavior and its consequences in an equilibrium labor market. In a model with both directed search and learning about the match quality of firm-worker pairs, I highlight the job search target effect of learning: as a worker updates the evaluation of his current job, he adjusts his on-the-job search target, which results in a different job finding rate. I show that this model generates a non-monotonic relation between the employment-to-employment transition rate and tenure, which provides a new explanation of the hump-shaped separation rate-tenure profile.

Contents

Chapter 1

Revenue Management Without Commitment

This chapter is a joint work with Francsec Dilme.

1.1 Introduction

Many markets share the following characteristics: (1) goods for sale are (almost) identical, and all expire and must be consumed at a certain point of time, (2) the initial number of goods for sale is fixed in advance, and (3) consumers have heterogeneous reservation values and enter the market sequentially over time. Such markets include the airline, cruise-line, hotel and entertainment industries. The revenue management literature studies the pricing of goods in these markets, and these techniques are reported to be quite valuable in many industries, such as airlines (Davis (1994)), retailers (Friend and Walker (2001)), etc. The standard assumptions in this literature are that sellers have perfect commitment power and buyers are impatient. That is, buyers cannot time their purchases and sellers can commit to the future price path or mechanism. In contrast, this paper studies a revenue management problem in which buyers are *patient* and sellers are endowed with *no commitment power*.

We consider the profit-maximizing problem faced by a monopolist seller who has K identical goods to sell before a deadline. At any date, the seller posts a price and the quantity available (capacity control) but *cannot commit* to future offers. Over time, potential buyers with different reservation values (either high or low) privately enter the market. Each buyer has a single-unit demand and can time her purchase. Goods are consumed at the fixed deadline, and all trades happen before or at that point.

Our goal is to show that the seller can sometimes use fire sales before the deadline to credibly reduce his inventory and so charge higher prices in the future. We accordingly consider settings where the seller does not find it profitable to only sell at the deadline and then only to high-value buyers, with the accompanying possibility of unsold units. In such settings, we explore the properties of a pricing path in which, at the deadline, if the seller still has unsold goods, he sets the price sufficiently low that all remaining goods are sold for sure. For most of the time before the deadline, the seller posts the highest price consistent with high-value buyers purchasing immediately on arrival, and occasionally, he posts a fire sale price that is affordable to low-value buyers. By holding fire sales, the seller reduces his inventory quickly, and therefore, he can induce high-value buyers to accept a higher price in the future. Intuitively, these sales allow the seller to 'commit' to high prices going forward. Once the transaction happens, whether at the discount price or not, the seller's inventory is reduced, and the price jumps up instantaneously. Hence, in general, a highly fluctuating path of realized sales prices will appear, which is in line with the observations in many relevant industries.[1]

The suboptimality of only selling at the deadline to high-value buyers could occur for many reasons. For example, at the deadline, the seller may expect that there will be little effective high-value demand in the market. This may be because the arrival rate of high-value buyers is low, or because buyers may also leave the market without making a purchase, or because buyers face inattention frictions and so they may miss the deadline, which we discuss in detail below.

The equilibrium price path relies on the seller's lack of commitment and buyers' intertemporal concern. An intuitive explanation is as follows. At the deadline, due to the insufficient effective demand, the seller holding unsold goods sets a low price to clear his inventory, which is known as the last-minute deal.[2] Before the deadline, since a last-minute deal is expected to be posted shortly, buyers have the incentive to wait for the discount price.[3] However, waiting for a deal is risky due to competition at the low price, from both newly arrived high-value buyers and low-value ones who are only willing to pay a low price. By weighing the risk of losing the competition and so the deal, a high-value buyer is willing to make her purchase immediately

[1]For example, McAfee and te Velde (2008) find that airfares' fluctuation is too high to be explained by the standard monopoly pricing models.

[2]In the airline industry, sellers do post last-minute deals. See *Wall Street Journal*, March 15, 2002, "Airlines now offer 'last minute' fare bargains weeks before flights," by Kortney Stringer.

[3]In the airline industry, many travelers are learning to expect possible discounts in the future and strategically time their purchase. See the *Wall Street Journal*, July 2002, "A Holiday for Procrastinators: Booking a Last-Minute Ticket," by Eleena de Lisser.

at a price higher than the discount one. We name the highest price she is willing to pay to avoid the competition as her reservation price. For any such high-value buyer, her reservation price is decreasing in time, since the arrival of competition shrinks as the deadline approaches, and decreasing in the current inventory size, since the probability that she will be rationed at deal time depends on the amount of remaining goods. To maximize his profit, the seller posts the high-value buyer's reservation price for most of the time and, at certain times before the deadline, may hold fire sales to reduce his inventory and to charge a higher price in the future.

Figure 1.1 illustrates this idea in the simplest case with only two items for sale at the beginning. Suppose the seller serves high-value buyers only before the deadline, allowing discounts at the deadline only. Conditional on the inventory size, the price declines in time. The high-value buyer's acceptable price in the two-unit case is lower than the price in the one-unit case, and the price difference indicates the difference in the probability that a high value buyer is rationed at the last minute in different cases. If a high-value buyer enters the market early and buys a unit immediately, the seller can sell it at a relatively high price and earn a higher profit than he could earn from running fire sales. However, if no such buyer ever shows up, then the time will eventually come when selling one unit via a fire sale and then following the one-unit pricing strategy is more profitable to the seller. To see the intuition, consider the seller's benefit and cost of liquidating the first unit via a fire sale. The *benefit* is that, by reducing one unit of stock, the seller can charge the high-value buyer who arrives next a higher price for his last unit. On the other hand, the (opportunity) *cost* is that, if more than one high-value buyer arrives before the deadline, the seller cannot

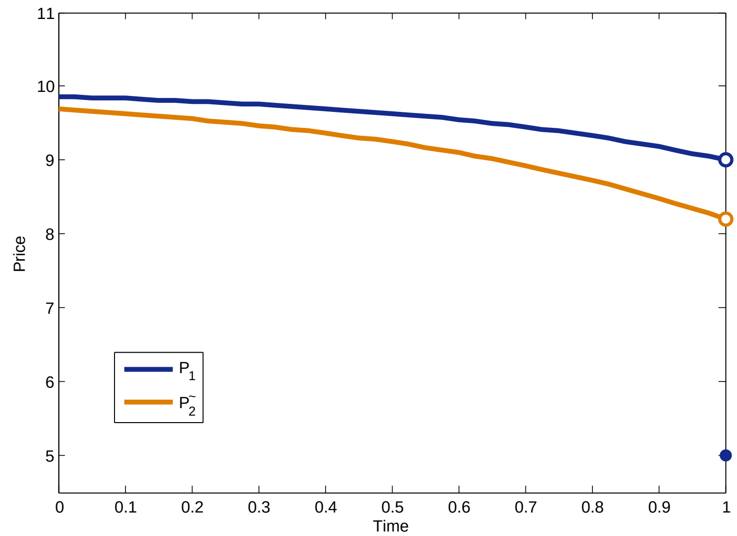

Figure 1.1: Necessity of FIre Sales Before the Deadline in the Two-Unit Case. The solid (dashed) line shows how the list price will change in the case of one (two) unit of initial stock if low-value buyers are served at the deadline only.

serve the second one, who is willing to pay a price higher than the fire sale price. Since a new high-value buyer arrives independently, as the deadline approaches, the probability that more than one high-value buyer arrives before the deadline goes to zero much faster than the probability that one high-value buyer arrives. Thus, the opportunity cost is negligible compared to the benefit, and therefore, the seller has the incentive to liquidate the first unit via a fire sale.

Analyzing a dynamic pricing game with private arrivals is complicated for the following reason. Since the seller can choose both the price and quantity available at any time, he may want to sell his inventory one-by-one. Thus, some buyers may be rationed when demand is less than supply before the game ends. Suppose a buyer was rationed at time t and the seller still holds unsold units. The rationed buyer privately learns that demand is greater than supply at time t and uses the information to update her belief about the number of remaining buyers. Buyers who arrive after this transaction have no such information. As a result, belief heterogeneity among buyers naturally occurs based on their private histories, and buyers' strategies may depend on their private beliefs non-trivially. Such belief heterogeneity evolves over time and becomes more complicated as transactions happen one after another, making the problem intractable.

To overcome this technical challenge, we assume that buyers face inattention frictions. That is, in each "period" with a positive measure of time, instead of assuming that buyers can observe offers all the time, we assume that each buyer notices the seller's offer and makes her purchase decision at her attention times only. In each "period," a buyer independently draws one attention time from an atomless distribution.[4] In addition, buyers' attention can be attracted by an offer with sufficiently low price, that is, a fire sale.[5] This implies that (1) at any particular time, the probability that a buyer observes a non fire sale offer is zero, (2) the probability that more than one buyer observes a non-fire-sale offer at the same time is zero too, and (3) all buyers observe a fire-sale-offer when it is posted. As a result, high-value buyers would not be rationed except at deal time. Furthermore, we focus on equilibria where high-value buyers make their purchases upon arrivals. Therefore,

[4]In the airline ticket example, it is natural to assume each buyer checks the price once or twice per day instead of looking at the airfare website all the time.

[5]In practice, this extra chance is justified by consumers' attention being attracted by advertisements of deals sent by a third party: low price alert e-mails from intermediate websites that offer airfares such as http://www.orbitz.com and http://www.faredetective.com.

a high-value buyer being rationed at deal time attributes failure of her purchase to the competition with low-buyer buyers instead of other high-value buyers, so she cannot infer extra information about the number of buyers in the market. As we will show, there is an equilibrium in which buyers' strategies do not depend on their private histories.[6]

As we described earlier, we are interested in the environment where the seller finds selling only at the deadline and serving only high-value buyers to be suboptimal. In the presence of inattention frictions, the seller cannot guarantee that the high-value buyers will be available at the deadline. Hence, at the deadline, to maximize his profit, the seller has to post a last-minute deal to draw full attention of the market, which naturally leads the seller to start selling early.[7]

We believe that the importance of revenue management studies without commitment is at least threefold. First, in the literature, reputation concerns are commonly cited as a justification of the perfect commitment power of sellers. However, for such a reputation mechanism to work and to act as a legitimate defense of commitment, one needs to understand the benefit and cost of sustaining the commitment price path. Obviously, an in-depth understanding of a world without commitment must be the basis for building the cost of the seller's deviation. Second, studying a model

[6]The idea that, in a continuous-time environment, decision times arrive randomly is not new. See for example, Perry and Reny (1993) and Ambrus and Lu (2010) in bargaining models, and Kamada and Kandori (2011) in revision games. In macroeconomics, there is a large literature analyzing the role of inertia information on sticky prices. See the text-book treatment by Veldkamp (2011). However, none of those papers employ such an assumption to avoid the complexities of private beliefs.

[7]Notice that our economic prediction on the price path does not depend on the presence of inattention frictions. As we mentioned before, a low arrival rate of buyers or the disappearance of present buyers can also exclude the trivial case where the seller is willing to sell at the deadline only. We explore the possibility of disappearing buyers in the extension.

without commitment can help us to evaluate how crucial the perfect commitment assumption is and to what degree the insights we have gained depend on it. Last, a non-commitment model should be the starting benchmark to understand the role of certain selling strategies with the feature of price commitment in reality. For instance, in both the airline and the hotel industries, sellers use the best price guarantee or best available rate policy. That is, if the buyer finds a cheaper price than what he paid within a certain time period, the seller commits to refund the difference and gives the buyer some extra compensation. In a perfect commitment model, it is hard to see the role of these selling policies.

1.1.1 The Literature

This paper is closely related to two streams of the literature. First, there is a large *revenue management* literature that has examined the market with sellers who need to sell finitely many goods before a deadline and impatient buyers who arrive sequentially.[8] However, as argued by Besanko and Winston (1990), mistakenly treating forward-looking customers as myopic may have an important impact on sellers' revenue. Board and Skrzypacz (2010) characterize the revenue-maximizing mechanism in a model where agents arrive in the market over time. In the continuous time limit, the revenue-maximizing mechanism is implemented via a price-posting mechanism, with an auction for the last unit at the deadline.

In the works mentioned above, perfect commitment of the seller is typically assumed. Little has been done to discuss the case in which a monopolist with scarce

[8] See the book by Talluri and van Ryzin (2004).

supply and no commitment power sells to forward-looking customers. Aviv and Pazgal (2008) consider a two-period case, and so do Jerath, Netessine, and Veeraraghavan (2010). Deb and Said (2012) study a two-period problem where a seller faces buyers who arrive in each period. They show that the seller's optimal contract pools low-value buyers, separates high-value ones, and induces intermediate ones to delay their purchase.

To the best of our knowledge, Chen (2012) and Hörner and Samuelson (2011) have made the first attempt to address the non-commitment issue in a revenue management environment using a multiple-period game-theoretic model. They assume that the seller faces a fixed number of buyers who strategically time their purchases. They show that the seller either replicates a Dutch auction or posts unacceptable prices up to the very end and charges a static monopoly price at the deadline. However, as argued by McAfee and te Velde (2008), arrival of new buyers seems to be an important driving force of many observed phenomena in a dynamic environment. As we will show, the sequential arrival of buyers plays a critical role in the seller's optimal pricing and fire sale decision.

Additionally, our model is also related to the *durable goods* literature in which the seller without capacity constraint sells durable goods to strategic buyers over an infinite horizon. As Hörner and Samuelson (2011) show, the deadline endows the seller with considerable commitment power, and the scarcity of the good changes the issues surrounding price discrimination, with the impetus for buying early at a high price now arising out of the fear that another buyer will snatch the good in the meantime. In the standard durable goods literature, the number of buyers is fixed. However, some papers consider the arrival of new buyers. Conlisk, Gerstner and Sobel (1984) allows a new cohort of buyers with binary valuation to enter the market in each period and show that the seller will vary the price over time. In most periods, he charges a price just to sell immediately to high-value buyers. Periodically, he charges a sales price to sell to accumulated low-value buyers.

In contrast to most durable goods papers, Garrett (2011) assumes that a seller with full commitment power faces a representative buyer who arrives at a random time. Once the buyer arrives, her valuation changes over time. He shows that the optimal price path involves fluctuations over time. Similar to Conlisk, Gerstner and Sobel (1984), most of the time, the seller charges a price just to sell immediately to the arrived buyer when her valuation is high. No transaction implies that either (1) the buyer did not arrive, or (2) she arrived but her valuation is low. After a long time with no transactions, the seller is more and more convinced that the latter is true. As a result, he charges a price acceptable to the arrived buyer with low valuation. Even though, similar to both Conlisk, Gerstner and Sobel (1984) and Garrett (2011), new arrivals and heterogeneous valuation are also the driving force of fire sales in our model, the economic channels are very different. In their papers, the seller has discounting a cost, so charges low price to sell to accumulated low-value buyers in order to reap some profit and avoid delay costs. However, in our model, the seller does not discount and can ensure a unit profit as the fire sales income at the deadline for all inventory. Since the buyers face scarcity, the seller liquidates some goods to convince future buyers to accept higher prices.

The rest of this paper is organized as follows. In Section 1.2, we present the model setting and define the solution concept we are going to use. In Section 1.3, we derive an equilibrium in the single-unit case. In Section 1.4, the multi-units case is studied. In Section 1.5, we discuss some modelling choices, applications and possible extensions of the baseline model. Section 1.6 concludes. The discussion the set of admissible strategies and the solution concept in this game, and all proofs are in Appendix.

1.2 Model

Environment. We consider a dynamic pricing game between a single seller who has K identical and indivisible items for sale and many buyers. Goods are consumed at a fixed time that we normalize to 1, and deliver zero value after. Time is continuous. The seller has the interval $[0, 1]$ of time in which to trade with buyers. There is a parameter Δ such that $1/\Delta \in \mathbb{N}$. The time interval $[0, 1]$ is divided into periods: $[0, \Delta)$,$[\Delta, 2\Delta)$, ...$[1 - \Delta, 1]$. The seller and the buyers *do not* discount.

Seller. The seller can adjust the price and supply at each moment: at time t, the seller posts the price $P(t) \in \mathbb{R}$, and capacity control $Q(t) \in \{1, 2, ..K(t)\}$, where $K(t) \in \mathbb{N}$ represents the amount of goods remaining at time t, and $K(0) = K$.[9] The seller has a zero reservation value on each item, so his payoff is the summation of all transaction prices.

Buyers. There are two kinds of buyers: low-value buyers (L-buyers, henceforth) and high-value buyers (H-buyers, henceforth). Each buyer has a single unit of de-

[9] We assume $Q(t) \neq 0$. However, the seller can post a price sufficiently high to block any transactions.

mand. Let v_L denote an L-buyer's reservation value of the unit, and v_H that of an H-buyer, where $v_H > v_L > 0$. A buyer who buys an item at price p gets payoff $v - p$ where $v \in \{v_L, v_H\}$.

Population Dynamics. The population structure of buyers changes differently over time. At the beginning, there is no H-buyer in the market. As time goes on, H-buyers arrive privately at a constant rate $\lambda > 0$. Let $N(t)$ be the number of H-buyers. An H-buyer leaves only if her demand is satisfied.[10] For tractability, we assume that the population structure of L-buyers is relatively predictable and stationary. At the beginning of each period, M L-buyers arrive in the market, where $M \in \mathbb{N}$ is common knowledge. When an L-buyer's demand is not satisfied, she leaves the market, and at the end of each period, all L-buyers leave.[11] We assume $M \geq K(0)$.

Transaction Mechanism. If the amount of demand at price $P(t)$ is less than or equal to $Q(t)$, all demands are satisfied; otherwise, $Q(t)$ randomly selected buyers are able to make purchases, and the rest are rationed. A price lower than v_L is always dominated by v_L. Thus, L-buyers do not face non-trivial purchase time decisions. To save notation, we assume that they are non-strategic and will accept any price no higher than v_L. We define such a price as a *deal*.

Definition 1. *A* ***deal*** *is an offer with* $P(t) \leq v_L$.

If $i \leq Q(t)$ goods are sold at time t, the seller's inventory goes down. In other words, $\lim_{t' \searrow t} K(t') = K(t) - i$. Over time, as buyers make purchases, the inventory

[10] Our results continue to hold when H-buyers leave the market at a rate $\rho \geq 0$.

[11] An added value of this assumption is that it allows us to highlight our channel to generate fire sales. In Conlisk, Gerstner and Sobel (1984), the presence of periodic sales is driven by the arrival and accumulation of low-value buyers. By assuming that the population structure of low-value buyers is stationary, their classical explanation of a price cycle does not work in our model.

decreases. Hence, $K : t \to \mathbb{N}$ is a left continuous and non-increasing function. Once $K(t)$ hits zero or time reaches the deadline, the game ends.

Inattention Frictions. We assume that buyers, regardless of their reservation value and arrival times, face inattention frictions. At the beginning of each period, all buyers, regardless of their value, randomly draw an attention time τ, which is uniformly distributed in the time interval of the current period.[12] For an H-buyer who arrives in the period, her attention time in the current period is her arrival time. In the period where the seller posts a *deal* at time τ, each buyer has an additional attention time at time τ in the current period. In the rest of this paper, we call these random attention times exogenously assigned by Nature *regular attention times*, while we call the additional attention time *deal attention times*. A buyer observes the offer posted, $P(t), Q(t)$ and the seller's inventory size, $K(t)$ at her attention time *only*. At that time, she can decide to accept or reject the offer. Rejection is not observed by the seller and other buyers. Since, without deal announcements, each buyer draws her attention time independently, once a buyer observes and decides to take an available offer $P(t) > v_L$, she will not be rationed. Thus the competition among buyers is always intertemporal when $P(t) > v_L$. At deal times when $P(t) \leq v_L$, buyers observe the offer at the same time, so there is direct competition among buyers. Notice that Δ capture the inattention fictions of buyers, and we focus on the case where Δ is small.

History. A non-trivial *seller history* at time t, $h_S^t = (P(\tau), Q(\tau), K(\tau))_{0 \leq \tau < t}$, is a history such that the game is not over before t and it summarizes all relevant

[12] Our results hold for any atomless distribution with full support.

transactions and information about offers in the past. Let $\mathcal{H}_S$ be the set of all seller's history. The seller's strategy σ_S determines a price $P(t)$ and capacity control $Q(t)$ given a seller history h_S^t. Due to the buyers' inattention frictions, at any time before the deadline, the seller believes that more than one buyer notices an offer with probability zero. As a result, we focus on the seller's strategy space in which $Q(t) = 1$ for $P(t) > v_L$ without loss of generality.

Let $a(t)$ be an index function such that it is 1 at an H-buyer's attention times, and 0 otherwise. Thus, $a^t = \{a(\tau)\}_{\tau=0}^{t}$ records the history of an H-buyer's past attention times up to t. A non-trivial *buyer history*, $h_B^t = \left\{a^t, \{P(\tau), Q(\tau), K(\tau)\}_{\tau: a(\tau)=1 \text{ and } \tau\in[0,t]}\right\}$. In other words, a buyer remembers the prices, capacity and inventory size she observed at her past attention times. Let $\mathcal{H}_B$ denote the set of all history of an H-buyer. Following Chen (2012) and Hörner and Samuelson (2011), we focus on symmetric equilibria in which an H-buyer's strategy depends only on her history not on her identity. That is to say, the H-buyer's strategy σ_B determines the probability that she will accept the current price $P(t)$ given a buyer's history h_B^t. We focus on a pure strategy profile, so $\sigma_B \in \{0, 1\}$.

1.2.1 On Continuous Time Games

We choose a continuous time model in this project, since it has technical advantages in answering our questions. Specifically, the determination of the optimal timing for fire sales is in fact an optimal stopping time problem; therefore, the continuous-time properties of this problem make the analysis easier.

However, continuous time raises obstacles to the analysis of dynamic games. First, it is well known that, in a continuous time game, a well-defined strategy may not induce a well-defined outcome. This is analyzed by Simon and Stinchcombe (1989) and Bergin and MacLeod (1993). The reason is that there is no well-defined "last" or "next" period in a continuous time game; hence, players' actions at time t may depend on information arriving instantaneously before t. For example, in our model, one seemingly possible pricing strategy is that the seller sets $P(t) = 10$ if $t = 0$ or $P(s) = 10$ for $s \in [0, t)$; otherwise, $P(t) = 1$. Intuitively, this strategy should imply a price outcome $P(t) = 10$ for any $t \in [0, 1]$. However, any for $t^* \in (0, 1)$, an outcome $P(t) = 10$ for $t \in [0, t^*]$ and $P(t) = 1$ when $t \in (t^*, 1]$ is compatible with the strategy above. See Simon and Stinchcombe (1989) for more examples.

Therefore, to make this game well-defined, we must impose additional restrictions on the set of strategies. Following Bergin and MacLeod (1993), we restrict the seller's choices in the admissible strategy space. The formal restriction is presented in Appendix A, and we provide the intuition here. To construct the set of admissible strategies, we first restrict the strategy to the inertia strategy space. Intuitively speaking, an inertia strategy is such that instead of an instantaneous response, a player can change her decision only after a very short time lag; hence, such strategy cannot be conditional on very recent information. The set of all inertia strategies includes strategies with arbitrarily short lags, so it may not be complete. To capture the instantaneous response of players, we complete the set and use the completion as the feasible strategy set of our game. For each instantaneous response strategy, we identify its associated outcome as follows. First, we find a sequence of inertia strategies converging to the instantaneous strategy. In such a sequence, each inertia strategy has a well-defined outcome, which gives us a sequence of outcomes. Second, we identify the limit of the outcome sequence as the outcome of this instantaneous response strategy. Let Σ_S^* as the admissible strategy space of the seller. Since H-buyers face inattention frictions, they cannot revise their decision instantaneously, so we do not need to impose any restriction on their strategy; let Σ_B^* denote the set of strategies of H-buyers, and let $\Sigma^* = \Sigma_S^* \times \Sigma_B^*$ be the strategy space we study.

1.2.2 Payoff and Solution Concept

In general, a player's strategy depends on his or her private history. A perfect Bayesian equilibrium in our game is a strategy profile of the seller and the buyers, such that given other players' strategy, each player has no incentive to deviate, and players update their belief via Bayes' rule where possible. However, the set of all perfect Bayesian equilibria of this game is hard to characterize.

We instead look for simple but intuitive *no-waiting equilibria* that satisfy the following properties. First, the equilibrium strategy profile must be simple; that is, players' equilibrium strategies depend on their histories only through the state variables specified later. Second, on the path of play, H-buyers make their purchases once they arrive. Third, we impose a restriction on buyers' beliefs about the underlying history off the path of play: each H-buyer believes that there are no other previous H-buyers presently in the market.

Note that some H-buyers may wait because of the deviation of the seller: the seller can post an unacceptable price for a time interval of positive measure in which H-buyers have to wait for future offers. However, each buyer can observe only finitely many offers at her past attention times and, for the rest of time, she has to form a belief about the underlying history. The perfect Bayesian equilibrium concept does not impose any restriction on those beliefs where the Bayes' rule does not apply. To support a no-waiting equilibrium, we assume that each H-buyer believes that no other H-buyers are waiting in the market. The justification of this refinement can be found in Appendix A.

1.2.2.1 Payoff

To define the equilibrium, we need to specify an H-buyer's payoff given she believes that no previous H-buyers are waiting in the market. Given a seller's continuation strategy $\tilde{\sigma}_S \in \Sigma_S^*$, other H-buyers' symmetric continuation strategy $\tilde{\sigma}_B \in \Sigma_B^*$, and a buyer's history h_B^t, an H-buyer's payoff from choosing a strategy $\tilde{\sigma}'_B \in \Sigma_B^*$ at her attention time is defined as

$$U\left(\tilde{\sigma}'_B, \tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t\right) = \mathbb{E}_{\tau|t}\left[v_H - P\left(\tau\right)\right]$$

where $\tau \in [t,1] \cup \{2\}$ is H-buyers' transaction time which is random and depends on the other players' strategies and the population dynamics of buyers. When $\tau = 2$, the buyer does not obtain the good because the seller's stock is sold out before she decides to place an order. In this case, $P(2) = v_H$. At time t, an H-buyer employs a cutoff strategy where she accepts a price if it is less than or equal to some reservation price p, and this reservation price is pinned down by the buyer's indifference condition:

$$v_H - p = \mathbb{E}_{\tau|t}\left[v_H - P\left(\tau\right)\right].$$

Suppose all H-buyers play a symmetric $\tilde{\sigma}_B \in \Sigma_B^*$. The payoff to the seller with stock k from a strategy $\tilde{\sigma}_S \in \Sigma_S^*$ is given by

$$\Pi_k \left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_S^t\right) = \mathbb{E}_\tau \left[P\left(\tau\right) + \Pi_{k-1}\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_S^\tau\right)\right],$$

where h_S^t is the seller's history, $\Pi_0 = 0$. Because buyers face inattention frictions, by posting any price $P\left(1\right) > v_L$, the seller expects no buyer notices the offer, and his expected profit is zero; by posting a deal price, the seller can sell all of his inventory. Hence, we have

$$\Pi_k \left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_S^1\right) = \begin{cases} 0, & \text{if } P\left(1\right) > v_L, \\ kv_L, & \text{otherwise,} \end{cases}$$

Note that the seller may or may not believe that there are previously arrived H-buyers waiting in the market. His belief about the number of H-buyers depends on the price he posted before.

1.2.2.2 (No-Waiting) Markov Perfect Equilibrium

We focus on Markov equilibria where an H-buyer makes her purchase decision based on two state variables: calendar time and inventory size, and she makes the purchase on her arrival time on the equilibrium path. The seller's equilibrium strategy depends on calendar time, inventory size, and his estimated number of present H-buyers. Specifically, based on the his realized history, the seller forms a belief about the number of H-buyers, $N\left(t\right)$. Let $\Phi\left(t\right)$ be the seller's belief over $N\left(t\right)$ where $\Phi_n\left(t\right)$ represents the probability that the seller believes that $N\left(t\right) = n$. Furthermore, the seller needs to distinguish between H-buyers whose attention times were before t, and those whose attention times are equal to or after t in the current period. Let $N^{-}(t)$ denote the number of H-buyers whose attention times were before t, and let $N^{+}(t)$ denote those whose attention times are equal to or after t in the current period. Let $\Phi^{-}(t)$ and $\Phi^{+}(t)$ be the seller's beliefs over $N^{-}(t)$ and $N^{+}(t)$, where $\Phi_{n}^{-}(t)$ (and $\Phi_{n}^{+}(t)$) represent that the seller believes that $N^{-}(t) = n$ (and $N^{+}(t) = n$) at time t. Given $\Phi^{-}(t)$ and $\Phi^{+}(t)$, we can calculate the seller's belief as follows: for any $n \in \mathbb{N}$, $\Phi_{n}(t) = \sum_{i=0}^{n} \Phi_{i}^{-}(t) \Phi_{n-i}^{+}(t)$.

Definition 2. *The set $\Xi_{S} \subset [0,1]^{\infty}$ is a collection of seller's beliefs $[\Phi^{-}(t), \Phi^{+}(t)]$ such that can be reached after any seller history.*

As we mentioned before, we restrict the strategy space such that $Q(t) = 1$ for $P(t) > v_{L}$. Hence, the seller only needs to choose the price. We define a Markovian strategy profile as follows.

Definition 3. *A strategy profile (σ_{S}, σ_{B}) is Markovian if and only if*

1. *the seller's strategy σ_{S} depends on the seller's history via $(t, K(t), \Phi^{-}(t), \Phi^{+}(t))$ only, and*

2. *the H-buyer's strategy σ_{B} depends on the buyer's history via $(t, K(t))$ only.*

In the definition, the H-buyer's strategy is a function of the calendar time and the seller's inventory size, but it does not imply that the number of other H-buyers is payoff irrelevant to an H-buyer. In fact, an H-buyer's continuation value does depend on her belief about the number of other H-buyers. However, we focus on no-waiting equilibria where each H-buyer believes that no other H-buyer is waiting in the market; thus, her strategy does not depend on her belief about the number of other H-buyers non-trivially.

Furthermore, we can define the solution concept in this game.

Definition 4. *A (no-waiting) Markov perfect equilibrium (henceforth equilibrium) consists of a (pure) strategy profile* (σ_B^*, σ_S^*) *such that, for any seller's history* h_S^t*, and for any buyer's history* h_B^t*,*

1. *given the seller's strategy* σ_S^**, other buyers' strategy* σ_B^**,*

$$U\left(\sigma_B^*, \sigma_B^*, \sigma_S^*, h_B^t\right) \geq U\left(\tilde{\sigma}_B, \sigma_B^*, \sigma_S^*, h_B^t\right)$$

   for any admissible $\tilde{\sigma}_B$*,*

2. *given buyers' strategy* σ_B^**,*

$$\Pi_k\left(\sigma_B^*, \sigma_S^*, h_S^t\right) \geq \Pi_k\left(\sigma_B^*, \tilde{\sigma}_S, h_S^t\right)$$

   for any admissible $\tilde{\sigma}_S$*,* $k \in \{1, 2, ..K\}$*,*

3. *the seller's belief is consistent with the seller's history and* (σ_B^*, σ_S) *for any admissible strategy* $\sigma_S \in \Sigma_S^*$*, and*

4. (σ_B^*, σ_S^*) *is Markovian.*

Nonetheless, note that potential deviations strategy can be either Markovian or non-Markovian.

Over time, the seller's belief evolves based on the realized history. We leave the formal law of motion of $\Phi^+(t)$ and $\Phi^-(t)$ to the Appendix B but provide some intuitive description here. The seller's belief updating is driven by four forces. First, at any time t, there are exogenous arrivals. When the price is too high to be accepted by newly arrived H-buyers, they have to wait and therefore $N^-(t)$ increases. Second, since each H-buyer independently draws her attention time, in a small but non-trivial time interval, an H-buyer, if she is in the market and her attention time in the current period does not pass, observes the offer posted with positive probability. As a result, if an equilibrium offer is posted but the time without transactions grows, H-buyers are likely to be fewer, and therefore, the seller adjusts his belief about $N^+(t)$. Alternatively, if the offer posted is not acceptable to H-buyers, the seller believes that some H-buyers may have observed but rejected it, so $N^-(t)$ increases but $N^+(t)$ decreases. Third, as time goes to the end of the period, all buyers' attention time passes, so $N^+(t)$ converges to zero, and $N^-(t)$ converges to $N(t)$. At the beginning of each period, all remaining buyers can draw a new attention time within the current period, so $N^+(t^+) = N^-(t^-)$ when $t = l\Delta$ for $l = 0, 1, 2, ...1/\Delta - 1$. Last, the seller's belief jumps after each transaction because of the endogenous departure of buyers. The first two forces make the seller's belief smoothly update, but the last two make it jump.

Notice that in many dynamic price discrimination games, the seller's equilibrium pricing strategy is history dependent rather than Markovian, which makes the problem less tractable. In a two-period model, Fudenberg and Tirole (1983) show that there is no Markov equilibria. The non-existence of Markov equilibria continues to hold in an infinite horizon dynamic pricing game. See the discussion by Gul, Sonnenschein and Wilson (1986) in a durable goods environment. The reason is that if a buyer rejects an offer at a particular time, the continuation belief about the buyer's type would change dramatically. In our model, thanks to the presence of inattention frictions, the seller cannot infer any information if a particular offer is not accepted, since the probability the offer was observed by a buyer is zero. As we will show, there is a Markov equilibrium.

1.3 Single Unit

We start by analyzing the game where $K(0) = 1$, the seller has one unit to sell. Deriving equilibria in this game is the first step forward the analysis of more general games. We first provide an intuitive conjecture on an equilibrium of this game and verify our conjecture. Furthermore, we show that the equilibrium we proposed is the unique equilibrium.

The first observation is that the seller can ensure a profit v_L because there are M L-buyers at the deadline. An intuitive conjecture of the seller's strategy is to serve the H-buyers only before the deadline to obtain a profit higher than v_L and charge v_L at the deadline if no H-buyer arrives. Since an H-buyer would like to avoid a competition with (1) L-buyers at the deadline, and (2) other H-buyers who may arrive before the deadline, she is willing to forgo some surplus and accept a price higher than v_L. Moreover, as deadline approaches, the competition coming from newly arrived H-buyers becomes less and less intense, and therefore the H-buyer's *reservation price* declines.

Specifically, we conjecture that in equilibrium, the seller charges a price such that: (1) H-buyers accept it on arrivals, and (2) low type buyers make their purchases only at the deadline if the good is still available. The optimality of the seller's pricing rule implies that, before the deadline, an H-buyer is indifferent between purchasing at time t and waiting: on the one hand, if the H-buyer strictly prefers to purchase the good immediately, the seller can raise the price a little bit to increase his profit; on the other hand, if the price is so high that the H-buyer strictly prefers to wait, the transaction will not happen at time t and all H-buyers wait in the market. Furthermore, we will show that accumulating H-buyers is suboptimal for the seller because the H-buyers' reserve prices are declining over time. At the deadline, the seller will charge the price v_L to clean out his stock since he believes that there are no H-buyers left.

We give a heuristic description of the equilibrium in the main text and leave the formal analysis to the Appendix B. At the deadline, the H-buyer's reservation price is v_H. However, the probability that an H-buyer's regular attention time is at the deadline is zero; thus, the dominant pricing strategy for the seller is to post a deal price v_L to obtain a positive profit. As a result, in any equilibrium, $P(1) = v_L$. For the rest of the time, we denote $p_1(t)$ as an H-buyer's reservation price at her attention time $t < 1$ and the inventory size $K(t) = 1$. Consider an H-buyer with an attention time $t \in [1 - \Delta, 1)$; thus, the probability that new H-buyers arrive before the deadline is $1 - e^{-\lambda(1-t)}$. Suppose this H-buyer understands that on the path of play, no H-buyer who has arrived before her waited. Therefore, she believes that she is the only H-buyer in the market. She then faces the following trade-off:

1. if she accepts the current offer, she gets the good for sure at a price which is higher than v_L;

2. if she does not accept the current offer, the seller will believe that no H-buyer arrived and to obtain a positive profit, he will charge a price v_L to liquidate the good at the deadline. In the latter situation, the H-buyer has to compete with M L-buyers for the item, and the probability she is not rationed is $\frac{1}{M+1}$.

These considerations can pin down an H-buyer's reservation price, $p_1(t)$, at which she is indifferent between accepting the offer or not at time t. Specifically, the indifference condition of an H-buyer whose attention time is t is given as follows:

$$v_H - p_1(t) = e^{-\lambda(1-t)} \frac{1}{M+1} (v_H - v_L). \tag{1.1}$$

The left-hand side represents the H-buyer's payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits, which is risky because (1) other H-buyers may arrive in $(t, 1)$ with a probability $1 - e^{-\lambda(1-t)}$, and (2) she has to compete with M L-buyers at the deadline. Differentiating equation (1.1) with respect to t, we have $\dot{p}_1(t) = -\lambda [v_H - p_1(t)]$.

Letting $t \to 1$, we obtain the limit price right before the deadline,

$$p_1\left(1^-\right) = \frac{M}{M+1} v_H + \frac{1}{M+1} v_L. \tag{1.2}$$

Hence, if M is large, the limit price right before the deadline is very close to v_H. Note that $p_1(1^-)$ is different from the H-buyer's actual reservation price at the deadline, v_L. Let $U_{1-\Delta}$ denote an H-buyer's expected utility at the beginning of the last period. Since her attention time, $\tilde{t}$, is a random variable, we have

$$\begin{aligned} U_{1-\Delta} &= \int_{1-\Delta}^{1} \frac{1}{\Delta} e^{-\lambda\left(\tilde{t}-1+\Delta\right)} \left[v_H - p_1\left(\tilde{t}\right)\right] d\tilde{t} \\ &= \int_{1-\Delta}^{1} \frac{1}{\Delta} \left[e^{-\lambda\Delta} \frac{v_H - v_L}{M+1}\right] d\tilde{t}. \end{aligned} \tag{1.3}$$

Notice that, for each $\tilde{t}$, the H-buyer's ex ante payoff, by considering the risk of the arrival of new buyers and the price declining until $\tilde{t}$, is $e^{-\lambda\Delta}\frac{v_H - v_L}{M+1}$, which implies that an H-buyer at the beginning of the last period, is indifferent between being assigned any attention time in the current period. Hence, $U_{1-\Delta} = v_H - p_1(1-\Delta)$.

Now, consider the H-buyer's reservation price at an earlier time. Note that, when $K(0) = 1$, the seller can ensure a profit v_L at any time by charging the fire sale price. However, he expects to charge a higher price to H-buyers who arrive early and want to avoid competition with H-buyers who arrive in the future and L-buyers. As a result, the fire sale price v_L is charged only at the deadline. At any other time t, the seller targets H-buyers only and offers a price $p_1(t)$. Consider an H-buyer whose attention time is $t \in [1-2\Delta, 1-\Delta)$. Her indifference condition is given by

$$v_H - p_1(t) = e^{-\lambda(1-\Delta-t)} U_{1-\Delta}. \tag{1.4}$$

where the left-hand side represents the H-buyer's payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits, with probability

$e^{-\lambda(1-\Delta-t)}$, she is still in the market at the beginning of the next period and the good is still available; so she can draw a new attention time in the last period and expect a payoff $U_{1-\Delta}$. Differentiating equation (1.4) with respect to t, we have $\dot{p}_1(t) = -\lambda[v_H - p_1(t)]$. As t goes to $1-\Delta$, $v_H - p_1(t)$ converges to $U_{1-\Delta}$. As a result, $p_1(t)$ is differentiable in $[1-2\Delta, 1)$. Repeating the argument above for $1/\Delta$ times, we have the ordinary differential equation (ODE, henceforth) for the H-buyers' reservation price $p_1(t)$ such that

$$\dot{p}_1(t) = -\lambda(v_H - p_1(t)) \text{ for } t \in [0,1), \tag{1.5}$$

with a boundary condition (1.2). In our conjectured equilibrium, the price the seller charges is $p_1(t)$ for $t \in [0,1)$ and it jumps down to v_L at the deadline.

Similarly, we can derive the seller's payoff $\Pi_1(t)$. At the deadline, $\Pi_1(1) = v_L$ since the good is sold for sure at the fire sale price. Before the deadline, for a small $dt > 0$, the profit follows the following recursive equation:

$$\begin{aligned}\Pi_1(t) &= p_1(t)\lambda dt + (1-\lambda dt)\,\Pi_1(t+dt) + o(dt)\,, \\ &= p_1(t)\lambda dt + (1-\lambda dt)\left[\Pi_1(t) + \dot{\Pi}_1(t)\,dt\right] + o(dt)\,,\end{aligned}$$

where an H-buyer arrives and purchases the good at time t with probability λdt, and no H-buyer arrives with a complementary probability. By taking $dt \to 0$, the seller's profit must satisfy the following ODE:

$$\dot{\Pi}_1(t) = \lambda[\Pi_1(t) - p_1(t)]\,, \tag{1.6}$$

with a boundary condition $\Pi_1(1) = v_L$. Note that, even though the equilibrium price is not continuous in time at the deadline, the seller's profit is because the probability that the transaction happens at a price higher than v_L goes to zero as t approaches the deadline.

In short, in our conjectured equilibrium, H-buyers accept a price not higher than their reservation price $p_1(t)$, and the seller posts such price for any $t < 1$, and v_L at the deadline. No H-buyer waits on the path of play. The next question is whether players have the incentive to follow the conjectured equilibrium strategies. A simple observation is that no H-buyer has the incentive to deviate since she is indifferent between taking and leaving the offer at any attention time. What about the seller? Does the seller have the incentive to do so and accumulate H-buyers for a while before the deadline? The answer is again no. This is because each buyer believes that no previous buyers are waiting in the market, and the seller is going to follow the equilibrium pricing rule in the continuation play. Since the H-buyer's reservation price declines over time, the seller always wants to serve the earliest H-buyer. Hence, the seller's equilibrium expected payoff at t is given by

$$\Pi_1(t) = \int_t^1 e^{-\lambda(s-t)} \lambda p_1(s)\, ds + e^{-\lambda(1-t)} v_L.$$

Formally,

Proposition 1. *Suppose $K = 1$. There is a unique equilibrium in which,,*

1. *for any non-trivial seller's history, the seller posts a price,* $P(t)$ *s.t.*

$$P(t) = \begin{cases} p_1(t), & \text{if } t \in [0,1) \\ v_L, & \text{if } t = 1, \end{cases}$$

   where

$$p_1(t) = v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)},$$

2. *an H-buyer accepts a price at her attention time* $t \in [0,1)$ *if and only if it is less than or equal to* $p_1(t)$ *and she accepts any price no higher than* v_H *at the deadline.*

Notice that neither $p_1(t)$ not $\Pi_1(t)$ depends on Δ because each H-buyer makes her purchase once she arrives but does not draw additional attention time on the path of play.

Fire sales appear with positive probability at the deadline only, that is, the last-minute deal. With probability $e^{-\lambda}$, no H-buyer arrives in the market and the seller posts the last-minute deal. The good is not allocated to an L-buyer unless no H-buyer arrives. As a result, the allocation rule is efficient.

1.4 Multiple Units

In this section we consider the general case in which the seller has $K > 1$ units to sell. Since most intuition can be explained for the two-unit case, we provide a heuristic description of the equilibrium in a two-unit case, and we then state the equilibrium for $K > 2$.

1.4.1 The Two-Unit Case

Consider the case where $K = 2$. A simple observation is that, after the first transaction at time τ, $K(t) \leq 1$ for $t \in (\tau, 1]$, and what happens afterwards is characterized by Proposition 1. The question is how the first transaction happens: what is the sale price and when does the H-buyer accept the offer? Note that the seller always has a choice to post a price v_L at any t. Since this price is so low that L-buyers can afford it, a transaction will happen for sure and the seller's stock switches to $K(t^+) = K(t) - 1$. In equilibrium, the earliest time at which the seller is willing to sell the first item at the price v_L is denoted by t_1^*. In principle, when $K(t) = 2$, t_1^* can be any time before or at the deadline. As we have shown in Proposition 1, in any continuation game with $K(t) = 1$, on the equilibrium path, the seller charges the price v_L only at the deadline; hence, the last equilibrium fire sale time is always $t_0^* = 1$. However, it is not clear yet when the first equilibrium fire sale time is. Note that, because of the scarcity of the goods at the price v_L, an H-buyer may be rationed at t_1^*. Consequently, she is willing to pay a higher price before t_1^*.

We conjecture that the equilibrium should satisfy the following properties. Before t_1^*, the seller posts a price such that an H-buyer is willing to purchase the good once she arrives. Once an H-buyer buys the good, the amount of stock held by the seller jumps to one. From that moment on, the equilibrium is described by Proposition 1. Similar to the single-unit case, when $K(t) = 2$, an H-buyer's reservation price at

$t \le t_1^*$, $p_2(t)$, satisfies the following ODE:

$$\dot{p}_2(t) = -\lambda\left[p_1(t) - p_2(t)\right] \text{ for } t \in [0, t_1^*) \tag{1.7}$$

The intuition is as follows. Suppose, at $t < t_1^*$, an H-buyer sees the price $p_2(t)$. It is risky for her to wait because a new H-buyer arrives at rate λ and gets the first good at price $p_2(t)$, in which case the original buyer can get the second good only at price $p_1(t)$. At her attention time t, the H-buyer is indifferent between taking the current offer and waiting only if the price declining effect, measured by $\dot{p}_2(t)$, can compensate the possible loss.

Since the seller may obtain a higher unit-profit by selling a good to an H-buyer instead of to an L-buyer, a reasonable conjecture is as follows. In equilibrium, the seller does not run any fire sales prior to the deadline. In other words, the first fire sale time is $t_1^* = 1$, and the seller's optimal price path, $P(t)$, is such that (1) $P(t) > v_L$ for $t < 1$, (2) an H-buyer takes the offer once she arrives, and (3) the seller runs a clearance sale at the deadline. Now that $K(t) = 2$, the equilibrium price satisfies the ODE (1.7) with $t_1^* = 1$. At the deadline, the seller has to post v_L, and an H-buyer can obtain a good at the deal price with probability $\frac{2}{M+1}$; thus, the boundary condition of the ODE (1.7) at $t = 1$ is $p_2(1^-) = \frac{2}{M+1}v_H + \frac{M-1}{M+1}v_L$. This strategy profile, however, is not an equilibrium!

Lemma 1. *In any equilibrium, $t_1^* < 1$.*

Lemma 1 rules out the aforementioned conjecture. To see why, first note that $p_2(t) < p_1(t)$ for $t < 1$ since an H-buyer is more likely to get the good when the supply is 2. As t approaches the deadline, the probability that a new H-buyer arrives before the deadline becomes smaller and smaller. The probability that only one H-buyer arrives before the deadline is approximated by $\lambda(1-t)$. In this case,

1. if the seller naively posts price $p_2(t)$, his profit is $p_2(\tau)+v_L$ where τ is the H-buyer's arrival time.

2. Alternatively, if the seller runs a one-unit fire sale before the arrival, he can ensure a payoff of v_L immediately and expect a price $p_1(\tau) > p_2(\tau)$ in future.

When t is close to the deadline, the *benefit* of price cutting is approximated by $p_1(1)-p_2(1)$. On the other hand, there is an *opportunity cost* to holding a fire sale before the deadline. More than one H-buyer may arrive before the deadline and the probability of this event is approximated by $\lambda^2(1-t)^2$. In this case, if the seller naively posts price $p_2(t)$ and $p_1(t)$ to the end but does not post v_L, his profit is approximated by $p_2(1)+p_1(1)$. Thus the opportunity cost of the fire sale is approximated by $p_2(1)-v_L$ when t is close to the deadline. As t goes to 1, $\lambda^2(1-t)^2$ goes to zero at a higher speed than $\lambda(1-t)$; thus, the cost is dominated by the benefit for t close enough to 1, and therefore, the seller will post the fire sale price v_L to liquidate one unit at $t_1^* < 1$ to raise future H-buyers' reservation price. In other words, the fire sale plays the role of a commitment device.

We leave the formal equilibrium construction to the Appendix B but illustrate the idea here to provide intuition. Suppose Δ is small enough; thus, a buyer can make her next purchase decision soon after one rejection. Suppose buyers believe that the fire sale time is t_1^*. For $t < t_1^*$, and $K(t)=2$, an H-buyer's reservation price satisfies the ODE (1.7); for $t \in [t_1^*, 1)$ and $K(t) = 2$, H-buyers believe that the seller is going to post v_L immediately, and thus their reservation prices satisfies the following equation

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1}[v_H - p_1(t)],$$

where the left-hand side of the equation is the H-buyer's payoff by accepting her reservation price and obtaining the good now, and the right-hand side is her expected payoff by rejecting the current offer. With probability $\frac{1}{M+1}$, the H-buyer gets the good at the deal price right after time t, and with a complementary probability, an L-buyer gets the deal and the H-buyer has to take $p_1(t)$ at her next attention time. Since Δ is small, one can ignore the arrivals and the time difference between two adjacent attention times of the H-buyer, and therefore, the H-buyer's reservation price at $t \in [t_1^*, 1)$ is given by

$$p_2(t) = \frac{1}{M+1}v_L + \frac{M}{M+1}p_1(t).$$

The incentive-compatible condition of the H-buyer implies that $p_2(t)$ must be continuous at t_1^*, and thus the boundary condition of the ODE (1.7) is

$$p_2(t_1^*) = \frac{1}{M+1}v_L + \frac{M}{M+1}p_1(t_1^*). \tag{1.8}$$

As a result, an H-buyer's reservation price at t when $K(t) = 2$ critically depends on her belief about t_1^*.

Given H-buyers' common beliefs about t_1^*, and their reservation prices when $K(t) = 2$, the seller's problem is to choose his optimal fire sale time to maximize his profit; i.e.:

$$\Pi_2(t) = \max_{t_1} \int_t^{t_1} e^{-\lambda(s-t)} \lambda \left[p_2(s) + \Pi_1(s)\right] ds + e^{-\lambda(t_1 - t)} \left[v_L + \Pi_1(t_1)\right].$$

In equilibrium, buyers' belief is *correct*, so the seller's optimal fire sale time is t_1^* itself. The first-order-condition of the seller's problem at t_1^* is:

$$\lambda \left[p_2(t_1^*) - v_L\right] + \dot{\Pi}_1(t_1^*) = 0. \tag{1.9}$$

At t_1^*, a transaction happens at price v_L for sure, so we have

$$\Pi_2(t_1^*) = \Pi_1(t_1^*) + v_L, \tag{1.10}$$

which is the well-known *value-matching condition* in an optimal stopping time problem.

For $t < t_1^*$, and $K(t) = 2$, the seller posts the H-buyer's reservation price, $p_2(t)$, and his expected profit is given by

$$\Pi_2(t) = \lambda dt \left[p_2(t) + \Pi_1(t + dt)\right] + (1 - \lambda dt)\, \Pi_2(t + dt) + O\left(dt^2\right).$$

Taking $dt \to 0$, the seller's profit satisfies the following *Hamilton-Jacobi-Bellman* (henceforth, HJB) equation

$$\dot{\Pi}_2(t) = -\lambda\left[p_2(t) + \Pi_1(t) - \Pi_2(t)\right]. \tag{1.11}$$

Combining (1.9), (1.10) and (1.11) at t_1^* yields

$$\dot{\Pi}_2(t_1^*) = \dot{\Pi}_1(t_1^*), \tag{1.12}$$

which is known as the *smooth-pasting condition*.

As a result, at the equilibrium fire sale time t_1^*, three necessary conditions (1.8), (1.10), and (1.12) must hold. The necessity of the value-matching condition (1.10) and the smooth-pasting condition (1.12) comes from the optimal stopping time property of the interior fire sale time, and condition (1.8) results from the H-buyers' incentive-compatible condition. When time is arbitrarily close to t_1^*, the probability that new H-buyers arrive before t_1^* shrinks, and the H-buyer needs to choose between taking the current offer and waiting to compete with the L-buyers for the deal. Therefore, her reservation price must make the H-buyer indifferent between taking it and rejecting it. If t is not close to t_1^*, the competition from newly arrived H-buyers before t_1^* is non-trivial, and therefore, to convince an H-buyer to accept the price, it must satisfy the ODE (1.7) with a boundary condition (1.8) at t_1^*. The seller's equilibrium profit when $K(t) = 2$ is given by

$$\Pi_2(t) = \begin{cases} \Pi_1(t) + v_L, & t \geq t_1^* \\ \int_t^{t_1^*} e^{-\lambda(s-t)}\lambda\left[p_2(s) + \Pi_1(s)\right]ds + e^{-\lambda\left(t_1^*-t\right)}\left[v_L + \Pi_1(t_1^*)\right], & t < t_1^* \end{cases}$$

where t_1^* satisfies conditions (1.8),(1.10) and (1.12), $\Pi_1(t)$ is characterized in Proposition 1, and $p_2(t)$ satisfies ODE (1.7) with a boundary condition (1.8).

The following proposition formalizes our heuristic description of the equilibrium.

Proposition 2. *Suppose $K(0) = 2$. There is a $\bar{\Delta} > 0$ such that when $\Delta \in (0, \bar{\Delta})$, there exists a unique equilibrium. In this equilibrium, there is a fire sale time $t_1^* \in [0,1)$ such that:*

1. *on the path of play, the seller posts*

$$P(t) = \begin{cases} p_1(t), & \text{when } t < 1 \text{ and } K(t) = 1, \\ p_2(t), & \text{when } t < t_1^* \text{ and } K(t) = 2, \\ v_L, & \text{otherwise.} \end{cases}$$

   where

$$p_2(t) = \begin{cases} v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[e^{\lambda\left(1-t_1^*\right)} + \frac{M}{M+1} + \lambda\left(t_1^* - t\right) \right], & t \in [0, t_1^*), \\ \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t), & t \in [t_1^*, 1), \end{cases}$$

   and $p_1(t)$ is specified in Proposition 1,

2. *an H-buyer's reservation price is $p_1(t)$ and $p_2(t)$ when $t < 1$, $K(t) = 1$ and 2, respectively, and v_H at $t = 1$.*

Note that the first fire sale time t_1^* always exists, even though for some parameters it is not an interior solution, i.e., $t_1^* = 0$. In that case, the seller is so pessimistic about the arrival of H-buyers that he prefers to liquidate the first unit at the very beginning. Figure 1.2 shows a simulated equilibrium price path.

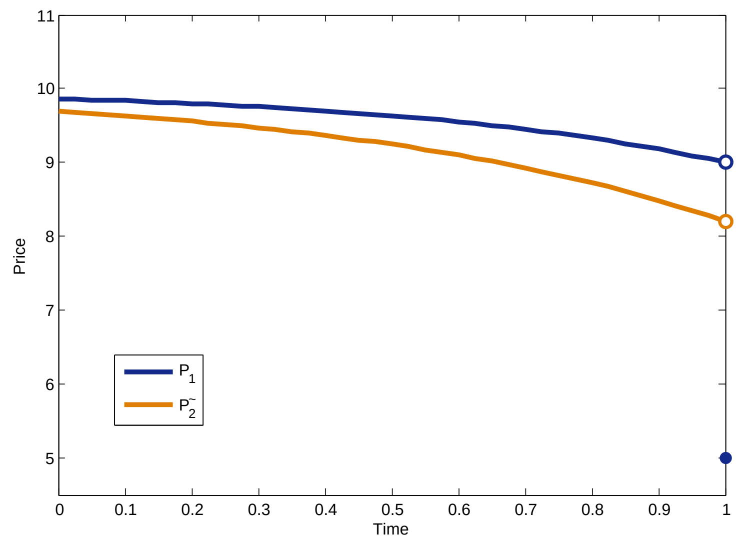

Figure 1.2: The equilibrium price path for the two-unit case. The solid line is the equilibrium price when $K(t) = 1$, while the dashed line is that when $K(t) = 2$. The first fire sale time is $t_1^* = 0.84$. When $t \geq t_1^*$ and $K(t) = 2$, the seller posts the deal price, v_L, to liquidate the first unit immediately. The parameter values are $v_H = 1, v_L = 0.7, M = 3$, and $\lambda = 2$.

In the equilibrium, for $t < t_1^*$, the price is $p_2(t)$, and it jumps up to $p_1(t)$ once a transaction happens. If there is no transaction before t_1^*, the price jumps down to v_L, and one unit is sold immediately; it then jumps up to the path of $p_1(\cdot)$. The first fire sale actually happens at t_1^* with probability $e^{-\lambda\left(1-t_1^*\right)}$. Since two or more H-buyers arrive after t_1^* with positive probability, the allocation is *inefficient.* However, in contrast to the standard monopoly pricing game where the inefficiency results from the seller's withholding, the inefficiency in this game arises from the scarce good being misallocated to L-buyers when many H-buyers arrive late.

It is worth noting that our equilibrium prediction on the fire sale critically depends on two assumptions: (1) H-buyers are forward-looking, and (2) the number of L-buyers is finite. First, suppose each H-buyer can draw at most one attention time, and thus she cannot strategically time her purchase. As a result, for any $t \in [0,1]$ and $k \in \mathbb{N}$, the H-buyers' reservation price is always $p_k(t) = v_H$ for any k. Hence, the optimal price path $P(t) = v_H$ when $t < 1$ and $P(t) = v_L$ when $t = 1$ for any $k \in \mathbb{N}$. In this particular model, the price is constant until $t = 1$. In a more general model, for example, buyers may have a heterogeneous reservation value $v \in [v_L, v_H]$. Talluri and van Ryzin (2004) consider many variations of this model. In these models, the result does not depend on the seller's commitment power. Second, when the number of L-buyers, M, is finite, an H-buyer can get a good at the deal price with positive probability. However, if M is infinity, the probability that an H-buyer can get a good at the deal price is zero. Hence, the difference between $p_1(t)$ and $p_2(t)$ disappears. In fact, an H-buyer cannot expect any positive surplus and is willing to accept a price v_H at any time.

1.4.2 The General Case

In general, the seller has K units where $K \in \mathbb{N}$. In the equilibrium, the seller may periodically post a deal price before the deadline. Specifically, there is a sequence of fire sale times, $\{t_k^*\}_{k=1}^{K-1}$, such that $t_{k+1}^* \leq t_k^*$ for $k \in \{1, 2, ..K-1\}$. When $t \in [t_1^*, 1)$, if $K(t) = 1$, the seller posts $p_1(t)$; if $K(t) > 1$, the seller liquidates $K(t) - 1$ units via a fire sale immediately and makes his inventory size jump to 1. When $t \in [t_2^*, t_1^*)$, if $K(t) = k$, the seller posts $p_k(t)$ for $k = 1, 2$ and serves H-buyers; if $K(t) > 2$, he liquidates $K(t) - 2$ units via a fire sale. By the same logic, for any $k \in \{2, ...K-1\}$, when $t \in [t_k^*, t_{k-1}^*)$, the seller's equilibrium pricing strategy is as follows: if $K(t) \leq k$, the seller serves H-buyers only by posting a price $P(t) = p_k(t)$; if $K(t) > k$, the seller posts a deal price and liquidates $K(t) - k$ units of stock immediately.

We derive the equilibrium by induction. Suppose in the $K-1$-unit case, H-buyers' reservation price is $p_k(t)$ for $k \in \{1, 2, ..K-1\}$, and the seller's equilibrium strategy is consistent with the description above. The seller's equilibrium profit is represented by $\Pi_k(t)$ for $k \in \{1, 2, ..K-1\}$. Now we construct the H-buyers' reservation price and the seller's pricing strategy and payoff in the K-units case. To satisfy the H-buyers' incentive-compatible condition, the equilibrium price at t when $K(t) = k \in \mathbb{N}$ satisfies the following differential equation:

$$\dot{p}_K(t) = -\lambda \left[p_{K-1}(t) - p_K(t)\right] \text{ for } t \in [0, t_{K-1}^*), \tag{1.13}$$

where t^*_{K-1} is the first equilibrium fire sale time when $K(t) = K$, and

$$p_K(t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{K-i}(t) \text{ for } t \in [t^*_{K-i}, t^*_{K-i-1})$$

where $i = 1, 2, ...K-1$ and $t^*_0 = 1$. The incentive-compatible condition of the H-buyer implies that $p_K(t)$ must be continuous at t^*_{K-1}; thus, the boundary condition of the ODE (1.13) is given by $p_K\left(t^*_{K-1}\right) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}\left(t^*_{K-1}\right)$, and therefore, the H-buyer's best response is specified for any $t \in [0,1]$ and $k \in \{1, 2, ...K\}$.

The seller's problem is to choose the optimal fire sale time and quantity to maximize his profit. Formally,

$$\Pi_K(t) = \max_{t_{K-1} \in [0,1]} \int_t^{t_{K-1}} e^{-\lambda(\tau - t)} \lambda \left[p_K(s) + \Pi_{K-1}(s)\right] ds \\ + e^{-\lambda(t_{K-1} - t)} \left[v_L + \Pi_{K-1}(t_{K-1})\right].$$

In equilibrium, buyers' beliefs are correct, so the seller's optimal fire sales time when $K(t) = K$ is t^*_{K-1}, which satisfies the value-matching and the smooth-pasting conditions.

If there exists an interior solution, t^*_K is pinned down as follows. At t^*_{K-1},

$$p_K\left(t^*_{K-1}\right) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}\left(t^*_{K-1}\right), \tag{1.14a}$$

$$\Pi_K\left(t^*_{K-1}\right) = \Pi_{K-1}\left(t^*_{K-1}\right) + v_L, \tag{1.14b}$$

$$\dot{\Pi}_K\left(t^*_{K-1}\right) = \dot{\Pi}_{K-1}\left(t^*_{K-1}\right). \tag{1.14c}$$

In equilibrium, we have $t^*_{K-1} \leq t^*_{K-2}$. The intuition is simple. In a no-waiting equilibrium, no previous arrived H-buyers are waiting in the market; thus, the demand from H-buyers shrinks as the deadline approaches. What is more, the probability that more than k H-buyers arrive before the deadline is approximated by $\lambda^k (1-t)^k$ when the current time t is close to the deadline. Apparently, the higher k is, the smaller the probability is. Hence, the seller who holds more units has the incentive to liquidate part of his inventory early. What is more, when Δ is small, on the path of play, the seller does not run more than one fire sale in the same period.

For a history in which $K(t) = k \in \{1, 2, ..K\}$ and $t \in [t_{k'}, t_{k'-1})$ for $k' < k-1$, the seller would try to liquidate multiple units of goods as soon as possible. The seller's profit when $K(t) = k$ is given by

$$\Pi_k(t) = \begin{cases} \{\int_t^{t^*_{k-1}} e^{-\lambda(s-t)} \lambda \left[p_k(\tau) + \Pi_{k-1}(\tau)\right] d\tau & \text{if } t < t^*_{k-1} \\ \quad + e^{-\lambda\left(t^*_{k-1} - t\right)} \left[v_L + \Pi_{k-1}\left(t^*_{k-1}\right)\right]\}, & \\ v_L(k-k') + \Pi_{k'}(t), & \text{if } t \in [t^*_{k'}, t^*_{k'-1}) \\ k v_L, & \text{if } t = 1 \end{cases}$$

where $k > k' \in \{1, 2, ...K-1\}$, and t^*_{k-1} satisfies conditions (1.14a), (1.14b) and (1.14c).

The following proposition formalizes our heuristic equilibrium description.

Proposition 3. *Suppose $K \in \mathbb{N}$. There is a $\bar{\Delta} > 0$ such that when $\Delta \in \left(0, \bar{\Delta}\right)$, there is a unique equilibrium in which there is a sequence of fire sale times $\{t^*_k\}_{k=1}^{K-1}$ such that:*

1. *$t^*_{k+1} \leq t^*_k$, and $t^*_k - t^*_{k+1} > \Delta$ when $t^*_k > \Delta$,*

2. *the H-buyers' reservation price is* $p_k(t)$ *for* $t < 1$ *and* $K(t) = k \in \{1, 2, ...K(0)\}$ *and* v_H *at* $t = 1$,

3. *on the path of play when* $K(t) = k$, *the seller posts*

$$P(t) = \begin{cases} p_k(t), & \text{if } t < t^*_{k-1}, \\ v_L, & \text{if } t \geq t^*_{k-1} \text{ and } K(t) \geq k. \end{cases}$$

In equilibrium, when $K(t) = k$, the price is $p_k(t)$ for $t < t^*_{k-1}$. Without any transaction, the price smoothly declines and jumps up to $p_{k-1}(t)$ once a transaction happens at t. If there is no transaction before t^*_{k-1}, the price jumps down to v_L, and the price path jumps back to $p_{k-1}(.)$ after a transaction at t^*_{k-1}. Consequently, a highly fluctuating price path can be generated. In Figure 1.3, we provide some simulation of equilibrium price path.

1.5 Discussion

In this section, we briefly discuss some possible extensions and applications of our baseline model.

1.5.1 Application: Best Available Rate

In the baseline model, we assume the seller has no commitment power. What if the seller has partial commitment power? In practice, sellers in both the airline and the hotel industries sometimes employ a best available rate (BAR) policy and

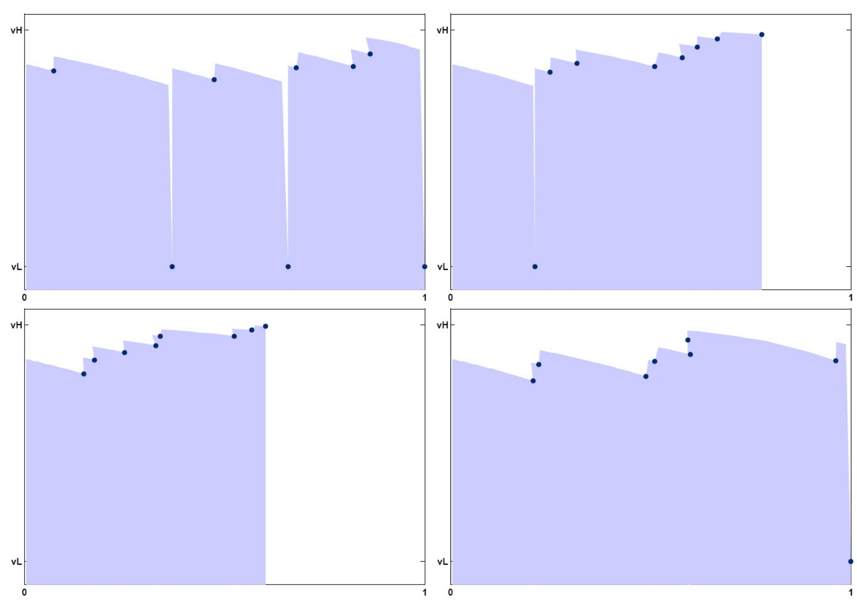

Figure 1.3: Simulated price path for different realizations of H-buyers' arrival in the 8-unit case. The upper edge of the shaded area describes the equilibrium list price, and dots indicate transactions. The parameter values are $v_H = 1$, $v_L = 0.7$, $M = 10$, $K = 8$ and $\lambda = 7$.

commit to not posting price lower than this best rate in the future. Does the seller have the incentive to do so in our model? Suppose the seller can commit to not posting a deal before the deadline. Then the seller may benefit. The intuition is as follows. An H-buyer's reservation price depends on the next fire sale time. If there is a deal soon, the reservation price is low, since there is a non-trivial probability that an L-buyer can obtain a good at the fire sale price. At the beginning of the game, if the seller can employ a BAR and commit to not posting v_L before the deadline, he can charge a higher price conditional on the inventory size. To illustrate the idea, we can consider the two-unit case. The seller's payoff by committing $P(t) > v_L$ for $t < 1$ is

$$\Pi_2^{BAR} = \int_0^1 e^{-\lambda s} \lambda \left[p_2(s) + \Pi_1(s) \right] ds + e^{-\lambda} 2 v_L,$$

such that $p_2(t)$ satisfies the ODE (1.7) with a boundary condition $p_2(1^-) = \frac{M-1}{M+1} v_H + \frac{2}{M+1} v_L$. By committing to no fire sale before the deadline, the seller can ask a higher price when $K(t) = 2$. As a result, $\Pi_2^{BAR} > \Pi_2(0)$ for certain parameters. In Figure 1.4, we plot the profit with BAR, $\Pi_2^{BAR}(t)$ and that without it, $\Pi_2(t)$. In the beginning $\Pi_2^{BAR}(t) > \Pi_2(t)$. As time goes on, the difference between them vanishes and becomes negative when the time is very close to the deadline.

1.5.2 Extension: Disappearing H-Buyers

In the baseline model, we assume an H-buyer leaves the market only when her demand is satisfied. Our results do not qualitatively change if buyers leave at a non-trivial rate over time. Suppose a buyer leaves the market at a rate $\rho > 0$ at any time, and her payoff by leaving the market without making a purchase is zero. If

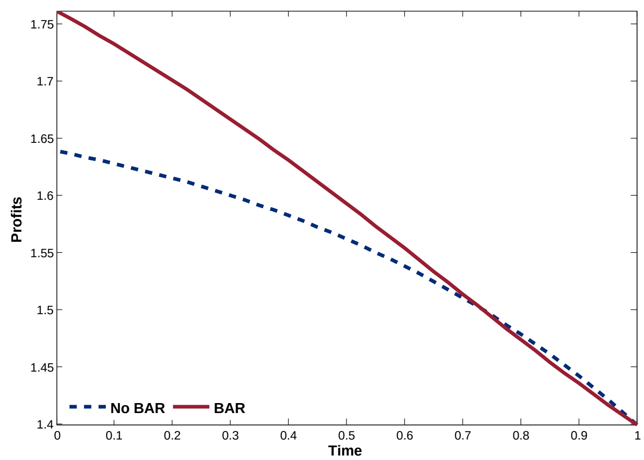

Figure 1.4: The solid line is the profit with BAR, while the dashed line is that without BAR. When t is close to 0, the profit with BAR is higher than that without BAR. The parameter values are $v_H = 1, v_L = 0.7, M = 3$, and $\lambda = 2$.

a buyer chooses to wait in the market, she faces the risk of exogenous leaving. In particular, when $K = 1$, an H-buyer's reservation price satisfies the following ODE

$$\dot{p}_1(t) = -(\lambda + \rho)[v_H - p_1(t)] \text{ for } t \in [0, 1),$$

with the boundary condition (1.2). By rejecting the current offer, an H-buyer needs to take into account two risks: (1) another H-buyer arrives and purchases the first units before her next attention time, and (2) her exogenous departure. Her payoff is zero if either happens.

In the two-unit case, for $t < t_1^*$, the H-buyer's reservation price follows

$$\dot{p}_2(t) = -\lambda[p_1(t) - p_2(t)] - \rho[v_H - p_2(t)],$$

and for $t \geq t_1^*$, the form of $p_2(t)$ is identical to that in the baseline model. The intuition behind it is as follows. For $t < t_1^*$, by rejecting a current offer, an H-buyer needs to take into account the risk that (1) another H-buyer arrives before her next attention time, and (2) she exogenously leaves the market. In the former case, she has to pay $p_1\left(\tilde{t}\right)$ instead of $p_2\left(\tilde{t}\right)$ at her next attention time $\tilde{t} > t$; in the latter case, she obtains a payoff of zero, which is equivalent to paying a price v_H. Since the risk of exogenous departure will only change the H-buyer's reservation price qualitatively, our main results still hold.

1.6 Other Related Literature and Conclusion

1.6.1 Other Literature

In the *revenue management* literature, in addition to the papers we discuss in section 1.1, there are numerous papers that have examined similar problems in different environments. Gershkov and Moldovanu (2009) extend the benchmark model to the heterogeneous objects case. The standard assumption maintained in these works is that buyers are impatient, and therefore cannot strategically time their purchases. However, as argued by Besanko and Winston (1990), mistakenly treating forward-looking customers as myopic may have an important impact on sellers' revenue. Hence, the revenue management problem with patient buyers draws the economists' attention. For example, Wang (1993) considers the case in which a seller has one object for sale and buyers arrive according to a Poisson distribution and experience a flow delay cost. He shows that with an infinite horizon, the profit-maximizing mechanism is to post a constant price and it may induce a delay of purchases on the path of play.

In a framework similar to that of Board and Skrzypacz (2010), Li (2012) considers a similar model and characterizes the allocation policy that maximizes the expected total surplus and its implementation. Mierendorff (2011a) assumes that buyers randomly arrive and their valuation depends on the time at which the good is sold and characterizes the efficient allocation rule as a generalization of the static Vickrey auction. Pai and Vohra (2010) consider a model without discounting where agents privately arrive and leave the market over time. They show that the revenue-maximizing allocation rule can be characterized as an index rule: each buyer can be assigned an index, and the allocation rule allots the good to a buyer if her index exceeds some threshold. Mierendorff (2011b), on the other hand, considers a similar environment but studies the optimal mechanism design problem when the regularity condition fails. Shneyerov (2012) studies a single-unit revenue management problem where the seller is more patient than the buyers. Su (2007) studies a model where buyers are heterogeneous in both valuation and patience and derives the optimal pricing policy. Deneckere and Peck (2012) study a perfect competitive price posting model where buyers arrive over time. They show that buyers endogenously sort themselves efficiently, with high valuations purchasing first.

In the *durable goods* literature, Stokey (1979, 1981) provides an early discussion of the monopolist's dynamic pricing problem. To consider the issue of new arrivals, Sobel (1991) considers a model with a more general setting and shows that the Coase conjecture does not hold. Sobel (1984) extends the model of Conlisk, Gerstner and

Sobel (1984) by considering a multi-seller case. He shows that, in some equilibria, all seller lower their price at the same time and to the same level. Board (2008) allows the entering generations to differ over time. Fuchs and Skrzypacz (2010) study a Coasian bargaining model in which exogenous events (for example, new buyers) may arrive according to a Poisson process. They show that the possibility of arrivals leads to delay. Huang and Li (2012) allow the existence of new arrivals to be initially uncertain but it can be learned by players over time. They show that the interaction between screening and learning about new arrivals can generate frequent price fluctuations when the seller's commitment power vanishes. Mason and Valimaki (2011) study a monopoly pricing problem where a seller faces a sequence of short-lived buyer whose arrival rate is unknown and can be learned over time. Biehl (2001) and Deb (2010) study a durable goods model where consumers' reservation value may change over time. Said (2012) studies a monopoly pricing problem of perishable goods, where buyers arrive over time. He shows that the seller can implement the efficient allocation using a sequence of ascending auctions. McAfee and Wiseman (2008) consider a durable good selling model where the seller can choose the capacity and they show that the Coase conjecture fails. Fuchs and Skrzypacz (2011) study the role of deadlines in a Coasian bargaining model where the seller has a single unit to sell. Dudine, Hendel and Lizzeri (2006) consider a durable good model where demand changes over time and buyers can purchase and store goods in advance. They find that if the seller cannot commit, the prices are higher than in the case in which he can commit, which is inconsistent with the prediction of the standard Coase conjecture literature.

Instead of pricing, many authors study other mechanisms a seller can use to sell her product to strategic buyers. McAfee and Vincent (1997) assume that the seller can run a sequence of auctions and adjust his reservation price over time. Skreta (2006) examines the case where the seller faces one buyer with private valuation in a finite horizon model, allowing the seller to use general mechanisms, and shows that posted prices are revenue-maximizing among all mechanisms. Skreta (2011) extends the model to the case where the seller faces many buyers.

In the *industrial organization* literature, some papers study the role of different kinds of sales. Lazear (1986) studies firms' pricing strategy in a two-period model and provides the first justification of clearance sales. Nocke and Peitz (2007) allow the seller to optimally choose his capacity and price in a two-period model and show that clearance sales may be optimal under certain conditions. Möller and Watanabe (2009) investigate a monopolist's profit-maximizing selling strategy when buyers face uncertainty about their demands. They show that, when aggregate demand exceeds capacity, both advance purchase discounts and clearance sales may be optimal. Lazarev (2012) studies the time paths of prices for airline tickets offered on monopoly routes in the U.S. Using estimates of the model's demand and cost parameters, he compares the welfare consumers receive under the current ticketing system to several alternative systems. In an oligopoly market where sellers face capacity constraint, Kreps and Scheinkman (1983) show that a mixed pricing strategy profile is supported as the equilibrium under certain conditions. Maskin and Tirole (1988) study a duopoly market where firms adjust their price alternately and show that in a Markov perfect equilibrium, the price pattern satisfies the Edgeworth cycle:

each firm cuts its price successively to increase its market share until the price war becomes too costly, at which point some firm increases its price. The other firms then follow suit, after which price cutting begins again. In a consumer search model, Varian (1980) justifies the role of sales by a mixed pricing strategy, and Armstrong and Zhou (2011) investigate the role of exploding offers and buy-now discounts.

1.6.2 Conclusion

This paper makes two contributions. First, we highlight a new channel for generating the periodic fire sales. When the deadline is approaching, the seller, if he still has a large inventory, does not expect many arrivals of high-value buyers, so he has the incentive to liquidate part of his stock via a sequence of fire sales to increase future H-buyers' reservation price. This insight can justify the price fluctuations in industries such as airlines, cruise-lines and hotel services. Second, by introducing the inattention frictions of buyers, we provide a tractable framework to study dynamic pricing problems in both finite and infinite horizon games. On the theory side, by introducing the inattention frictions of buyers, one can study a relatively simple equilibrium, the (no-waiting) Markov perfect equilibrium in such games. We believe that the inattention frictions can simplify the analysis in more general environments. On the application side, one can investigate the role of commitment associated with selling strategies, such as the best price guarantee, which is meaningless in a perfect commitment model.

There are many future research projects one can pursue following our work.

Multiple Buyer-Types. In general, considering buyers' multiple reservation values is complicated in our model. Nevertheless, we can discuss a conjecture equilibrium in the three-type case. Specifically, a new buyer arrives with rate λ. Conditional on arrival, the buyer's reservation value of the good is v_H with probability η, and it is v_M with probability $1-\eta$, where $v_H > v_M > v_L$. Similar to the Coase conjecture literature, a skimming property holds; that is, if a price p is acceptable to an M-buyer, it must be acceptable to an H-buyer as well. Define a θ-buyer's reservation price when $K(t) = k$ as $p_k^{\theta}(t)$. The skimming property implies that $p_k^H(t) \geq p_k^M(t)$. At equality, the seller can serve both H-buyers and M-buyers at the same price. Otherwise, the seller can post either $p_k^M(t)$ to serve both, or $p_k^H(t)$ to serve H-buyers only and potentially accumulate M-buyers for a positive measure of time. Over time, if there is no transaction at $p_k^H(t)$, the seller is more and more convinced that there are some M-buyers. If the seller holds a large number of goods and t is close to the deadline, he has the incentive to charge $p_k^M(t)$ to sell a unit to the M-buyer. Similar logic is adopted by Conlisk, Gerstner, and Sobel (1984), albeit in a stationary model. In the case of continuous reservation values, $v \in [v_L, v_H]$, we conjecture that the seller screens buyers smoothly.

Outside Offers and Competition Among Sellers. In the baseline model, we assume that there is a single seller and we extend our results by considering buyers' exogenous departure. However, in the real world, a buyer may leave the market because she finds a better outside offer. Suppose for each buyer, other offers arrive at rate γ, and each offered price $\tilde{p}$ is drawn from a commonly known distribution

$F_t(\cdot)$. Hence, an H-buyer's indifference condition at her attention time t implies that

$$\dot{p}_k(t) = -\lambda\left[p_{k-1}(t) - p_k(t)\right] - \gamma F_t(p_k(t))\left\{\mathbb{E}\left[\tilde{p}|\tilde{p} \leq p_k(t)\right] - p_k(t)\right\},$$

where the additional term $\gamma F_t(p_k(t))\left\{\mathbb{E}\left[\tilde{p}|\tilde{p} \leq p_k(t)\right] - p_k(t)\right\}$ reads that: at a rate γ, an outside offer with a price $\tilde{p}$ is realized and, with probability $F_t[p_k(t)]$, it is cheaper than the current price, in which case the buyer takes the offer. Hence, one can easily extend our basic model to consider the effect of outside offers. Furthermore, one can endogenize the distribution by considering a general equilibrium model in which many sellers and buyers randomly match, the arrival and departure rates are interpreted as the search frictions, and the outside offer distribution is given in equilibrium. We conjecture that our mechanism to generate fire sales still holds as long as the competition is not perfect.

Overbooking Policy. In the multi-unit case, we show that the allocation mechanism is generally inefficient. Some L-buyers can obtain goods via fire sales but the H-buyers who arrive late may not. One possible selling strategy to overcome this inefficiency is to allow overbooking. The seller can sell more than he has at a higher price to H-buyers and buy back some goods previously sold to L-buyers. See Courty and Li (2000), Ely, Garrett and Hinnosaar (2012) and Fu, Gautier and Watanabe (2012) for studies of related issues in different environments.

Transparency Policy. In our baseline model, the inventory size is observable. In practice, the inventory size is the seller's private information. However, we can imagine a similar game where the seller can provide verifiable information about his current inventory size without paying any cost. Since the smaller the current inventory size, the higher the price H-buyers are willing to pay, Milgrom's (1981) full disclosure theorem can justify the symmetric information of the inventory size. If the information disclosure is costly, a seller has the incentive to disclose his inventory size only if it is small enough. Another natural question to ask is, if the seller can choose the transparency of his inventory size and past price, is it ex ante optimal to hide this information or not? Is the optimal ex ante transparency policy time-consistent? Recently, Hörner and Vieille (2009), Kim (2012), and Kaya and Liu (2012) study the role of transparency of past prices in different environments and show that it has a significant impact on the formation of future prices.

The Presence of the Secondary Market. In our baseline model, buyers cannot trade with each other. This assumption applies in the airline, cruise and hotel-booking industries, but not in other markets such as sport tickets and theater tickets. We believe that it will be interesting to discuss the role of a secondary market in our framework. See Sweeting (2012) for an empirical analysis of the price dynamics in the secondary markets for major league baseball tickets.

Chapter 2

Bargaining While Learning About New Arrivals

This chapter is a joint work with Chong Huang.

2.1 Introduction

The arrival of new buyers has a great impact on the market where payoffs are determined through bargaining. The seller may have an incentive to delay the trade and wait for other buyers to arrive, whereas the present buyer will lose bargaining power due to the competition that arises from the arrival of new buyers. Further, in many situations, the likelihood of new arrivals may be uncertain.

Consider the following story, for instance. Suppose a seller and a buyer are negotiating on the price of a house. There may be another buyer entering the market at some point. This new buyer, if he exists, places a very high value on the house and has an urgent demand. However, it is not clear whether such a new buyer exists. The possibility of new arrivals depends on the desirability of the current house, the activity in the local housing market, and other factors that players may not be able to foresee. It is initially unknown but can be learned over time. However, the likelihood of new arrivals is initially uncertain and could be learned over time. When the seller is very optimistic about the existence of new arrivals, she wants to wait for such a new buyer by charging a high price that the current buyer would never take. As time elapses and no new buyer shows up, the seller becomes pessimistic, and so she begins to treat the current buyer seriously. Then in this environment, how do the seller's exogenous learning about new arrivals and her endogenous learning about the current buyer's value affect the transaction time and the equilibrium pricing path?

In this paper, we study a bargaining model, highlighting the interaction between the seller's exogenous learning about the existence of new arrivals and her endogenous learning about the current buyer's value. A long-lived seller, possessing a single unit of an indivisible durable good, makes a price offer in each period. There is a long-lived buyer whose reservation value is his private information. Observing the seller's offer, the long-lived buyer decides whether to accept it. In addition, a short-lived buyer with a high reserve price may exist. If such a short-lived buyer exists, his arrival is governed by a Poisson process. If the current offer is rejected by the long-lived buyer, the arriving short-lived buyer makes the purchase decision immediately and then leaves the market. The game ends once the good is sold.

We show that the model has a unique perfect Bayesian equilibrium and in the equilibrium, the standard Coase conjecture fails when learning about new arrivals is non-trivial. [1] That is, when the seller can make offers arbitrarily frequently, the initial price is bounded away from the seller' reservation value. This is because waiting for new arrivals serves as a non-trivial outside option of the seller. As long as the likelihood of arrivals is non-trivial, terminating the bargaining by offering a very low price is suboptimal to him. Since the learning speed of the existence of new arrivals is exogenous, the value of the outside option smoothly declines over time. Therefore, the seller slowly screens the long-lived buyer, which results in a strategic delay of the transaction.

We also characterize the equilibrium price dynamics. We show that the interaction between the seller's exogenous learning about the existence of new arrivals and her endogenous learning about the current buyer's value determines the equilibrium price dynamics. When the seller is optimistic about the new arrivals, she charges a price equal to the short-lived buyer's value. Such a high price offer is effectively made to the potential short-lived buyer only. We call such an offer a *waiting offer*. Charging a waiting offer forever, however, is suboptimal. If the good is not sold by charging a waiting offer for a very long time, the posterior belief about the existence of new arrivals drops below some threshold point. Then the seller finds that is is more valuable to screen the long-lived buyer by cutting the price. We call such an offer a *screening offer*. Of course, the screening offer may be accepted by an arriving short-lived buyer, if it is rejected by the long-lived buyer. So if the good is still not sold at a screening offer, both the seller and the long-lived buyer will adjust downward their beliefs about the existence of the new arrivals. Apparently, when

[1] See Coase (1972), Bulow (1982), Gul, Sonnenschein and Wilson (1986) and Ausubel and Deneckere (1989).

the likelihood of new arrivals is high enough, the seller chooses to post a waiting offer. When the likelihood of new arrivals is relatively low, the seller chooses to post a screening offer.

Specifically, Proposition 9 shows that if the time interval between two offers is sufficiently small, screening offers cannot be charged in more than two consecutive periods. The logic is similar to the Coase conjecture: when the seller can make offers arbitrarily frequently, he always prefers to speed up the screening of the long-lived buyer. As a result, after being rejected once or twice, the seller believes that the long-lived buyer's value is too low so that he switches to wait for new arrivals.

Dynamic bargaining models allowing new arrivals have not drawn economists' attentions until recently. Sobel (1990) considers a dynamic pricing model in which buyers keep arriving over time. In an equilibrium, the seller serves high value buyers most of the time and periodically serves low value buyers, so the equilibrium price fluctuates over time. However, in Sobel (1990), the price fluctuation is driven by the accumulation of arrivals instead of learning. Inderst (2008) assumes that the seller can choose whether to keep the original buyer or to switch to the new long-lived buyer who arrives with a known probability. If he switches, he starts a new screening, and thus the Coase conjecture is robust in this perturbation. Fuchs and Skrzypacz (2010) also consider a bargaining game between a seller and a buyer for a trade of one unit of an indivisible good, in which a stochastic event arrives in each period with a known probability. If no event arrives, the seller posts a price to the buyer whose value is his private information. Conditional on arrival, the seller's expected value depends on his belief about the buyer's value. One example of the stochastic event is the arrival of a new buyer whose value is unknown. Conditional on arrival, two buyers bid for the good in an auction, and the seller's expected revenue from the auction depends on the current buyer's value. They show that there is a delay in equilibrium and the seller slowly screens out buyers with higher valuations. They characterize the continuous time limit of the price path, which turns out to be very tractable. The existence of the arrival is commonly known in these models, so equilibrium prices are strictly decreasing. As we show in our model, learning about the existence of the short-lived buyer leads to price fluctuations in equilibrium.

In a parallel study by Faingold, Liu and Shi (2011), the buyer has an outside option whose existence is learned over time. The belief about the existence of the buyer's outside option decreases over time, and the seller smoothly changes the price. Our paper is different from Faingold, Liu and Shi (2011) in that the seller posts a price at the very beginning of each period, and conditional on arrival, the seller has to commit to the price. We show that the combination of learning and this marginal commitment power generates price fluctuations in our model.

In addition, a number of papers have studied the problem of a seller who learns about the demand it faces.[2] Mason and Valimaki (2011) studied a monopolist who faces a sequence of short-lived buyers with an unknown arrival rate and a commonly known value distribution. The seller posts a price in each period and learns the buyers' arrival rate, which depends on both an unknown parameter of the Poisson process and the posted price. As time passes without a sale, the seller becomes more pessimistic about the arrival rate and therefore smoothly lowers the posted price. In

[2]See Rothschild (1974), McLennan (1984), Bergemann and Valimaki (1994, 2006), among others.

this paper, the seller faces not only a sequence of short-lived buyers with degenerated value but an unknown arrival rate but also a long-lived buyer with privately known value. Hence, we see Mason and Valimaki (2011) as a complement to our research, rather than a substitute.

The rest of this paper is organized as follows. In sections 2.2 and 2.3, we introduce the bargaining model with learning about new arrivals. Section 2.4 is devoted to the no learning case as a benchmark. In section 2.5, we construct the unique equilibrium of the bargaining game with learning and show the price fluctuation as an inevitable phenomenon. Section 2.6 concludes. All omitted proofs are presented in the appendix.

2.2 The Model

Time is discrete and is indexed by $t \in \{0, 1, 2, \dots\}$. Any period has the same length Δ. A long-lived seller has one unit of a durable good for sale. The value of the good to the seller is 0. There is a long-lived buyer, and the value of the good to this buyer, v, is a random variable with support $[\underline{v}, 1]$. In this paper, we assume $\underline{v} > 0$, so we focus on the "gap case" bargaining model. The value v is drawn by Nature at the beginning of the game according to a commonly known distribution, and the long-lived buyer privately knows it. Let $F(v)$ and $f(v)$ be the cumulative distribution function and the density function of v, respectively. Assume that $f(v) \in (0, \bar{f}]$ for all $v \in [\underline{v}, 1)$. In addition, we assume the density function satisfies $f(1) = f'(1) = 0$.[3] There may exist a short-lived buyer whose value of the good is 1. The long-lived seller and the long-lived buyer share a common prior belief about the existence of the short-lived buyer, denoted by $\alpha_0 \in (0, 1]$.[4] Conditional on his existence, the arrival of the short-lived buyer is determined by a Poisson process. Specifically, in each period, the short-lived buyer enters the market with probability $\lambda\Delta$ if he exists. Intuitively, when the initial likelihood of arrival is small enough, the seller may ignore the possibility of arrival and focus on screening the long-lived buyer. To avoid a trivial case, we make the following assumption.

Assumption 1 (non-trivial learning)**.** $\underline{v} < \frac{\lambda\alpha_0}{\lambda\alpha_0 + r}$.

In each period t, the seller first announces a price p_t. The long-lived buyer, observing p_t, decides whether to accept or reject the offer. If he rejects it, the short-lived buyer may arrive (with probability $\lambda\Delta$) if he exists. Conditional on his arrival, the short-lived buyer decides whether to accept the current price if the good is available. If there is no transaction in period t, the game enters period $t+1$. Once the good is sold, the game ends.

The long-lived seller and the long-lived buyer share the same discount factor $e^{-r\Delta}$, where $r > 0$. If the transaction happens with price p_t in period t, the seller's payoff is $e^{-tr\Delta}p_t$. If the game ends with the long-lived buyer buying the good at price p_t, then the long-lived buyer's payoff is $e^{-tr\Delta}(v - p_t)$. If the game ends with the short-lived buyer accepting an offer, the long-lived buyer's payoff is zero. The short-lived buyer's

[3]This assumption helps us to show the existence and uniqueness of the equilibrium, so that we can focus on the discussion of equilibrium properties.

[4]When $\alpha_0 = 0$, the game is the canonical Coasian bargaining model.

payoff is

$$v_s = \begin{cases} 1 - p_t, & \text{if the good is available and he accepts it,} \\ 0, & \text{otherwise.} \end{cases}$$

Define $\mathcal{H}^t$ as the set of period t histories such that no transaction happens, so $h^t \in \mathcal{H}^t$ is a sequence of price $\{p_\tau\}_{\tau=0}^{t-1}$, which has not been accepted by either the long-lived or the short-lived buyer. Let $\mathcal{H} \equiv \cup_t \mathcal{H}^t$. A strategy of the seller is a map from the histories of rejected prices to price offers in the current period. It is obvious that any price strictly greater than 1 is dominated by the price 1 and thus is suboptimal, because neither the long-lived buyer nor the potential short-lived buyer will accept a price strictly greater than 1. Hence, we restrict the price space on $[0, 1]$. Denote a pure strategy of the seller by the mapping $P : \mathcal{H} \to [0, 1]$. A behavior strategy of the long-lived buyer specifies whether to accept the offer in any period t, given the price p_t, past rejected prices $\{p_\tau\}_{\tau=0}^{t-1}$ and his value v. Formally, let $A : \mathcal{H} \times [0, 1] \times [\underline{v}, 1] \to \{0, 1\}$, where $A(h^t, p, v) = 0$ means that the long-lived buyer with value v and history h^t rejects the offer p_t at period t. The short-lived buyer accepts any offer $p_t \leq 1$ if it is available.

2.3 Equilibrium

2.3.1 Weak Markov Equilibrium

A perfect Bayesian equilibrium (henceforth "PBE") in this game consists of a strategy profile, a system of beliefs about the long-lived buyer's value, and a system of beliefs about the existence of the short-lived buyer. In a PBE, given the systems of beliefs, all players behave sequentially rationally in any information set; and given the strategy profile, the system of beliefs is calculated by Bayes' rule, whenever it can be applied.

Because whether the short-lived buyer shows up is publicly observable to both the long-lived seller and the long-lived buyer, they share the same belief about the existence of the short-lived buyer after any history h^t. Denote such a belief at the beginning of period t by α_t, then if no short-lived buyer arrives and no transaction happens in period t, both the long-lived seller and the long-lived buyer update their beliefs by Bayes' rule as

$$\alpha_{t+1} = \frac{\alpha_t(1-\lambda\Delta)}{1-\alpha_t\lambda\Delta}. \tag{2.1}$$

In the rest of the paper, we sometimes use α and α' to denote the belief about the existence of the short-lived buyer in the current period and in the next period, respectively. Similarly, primes are used to denote next-period values of other variables. Fixing a $\Delta > 0$, the updating rule of α implies that there are countably many realizations of α. We denote by $B_\Delta(\hat{\alpha})$ the set of realizations of α, which can be generated from $\hat{\alpha}$ by the Bayes' rule given Δ.

The belief α_{t+1} summarizes the information about the short-lived buyer's existence from the no transaction history at the end of period t. The following Lemma, extending the insight of Fudenberg, Levine, and Tirole (1985) to this model, claims that the long-lived seller's belief about the long-lived buyer's value after any no transaction history is a truncated sample of the original distribution. As a consequence, the long-lived seller's posterior belief about the long-lived buyer's value is summarized by the upper bound of the truncated distribution.

Lemma 2 (Conditional Skimming Property)**.** *In any perfect Bayesian equilibrium, conditional on the current α, if the long-lived buyer with value v accepts an offer, any long-lived buyer with $v' > v$ must accept it.*

The conditional skimming property implies that in any perfect Bayesian equilibrium, after any relevant history of offered prices, there exists a value k such that the long-lived buyer rejects all of these offers if and only if his value $v \leq k$. The intuition is as follows. The long-lived buyer's benefit of waiting comes from the decline in the future price, which does not depend on his value. But the cost of waiting results from postponing consumption, which is increasing in the long-lived buyer's value v. Put differently, it is more costly for the high value long-lived buyer to delay his consumption than it is for the low value buyers.

The conditional skimming property implies that in a PBE, the long-lived buyer's action in any period t depends only on his value, the current price, and the current belief about the existence of short-lived buyers. In particular, after any history h^t in a PBE, if the long-lived buyer with value k is indifferent between taking the current offer and waiting for future offers, the long-lived buyer with a value larger than k strictly prefers the current offer. Hence, the cutoff k summarizes the payoff relevant information about the long-lived buyer's value, and it is common knowledge that the seller's belief about the long-lived buyer's value is distributed according to a truncated distribution of $F(v)$ with support $[\underline{v}, k]$. As a result, it is natural to consider that in a PBE, players will condition their actions in any information set only on two state variables (k, α), the highest value at which the long-lived buyer has not bought and the belief about the existence of the short-lived buyer. In the literature, a PBE with such conditions is called a strong Markov equilibrium. However, as we will explain in more details in section 2.3.2, a strong Markov equilibrium does not generally exist. Therefore, in this paper, we use the weak-Markov equilibrium as the solution concept of the model.

Definition 5. *A strategy profile (P, A) is a weak Markov equilibrium (henceforth equilibrium), if it is a PBE and there exist two functions*

$$\sigma : [\underline{v}, 1] \times B_\Delta(\alpha_0) \times [0,1] \to [0,1] \quad \textit{and} \quad \kappa : [0,1] \times B_\Delta(\alpha_0) \to [\underline{v}, 1],$$

such that, in any period t,

1. *$P(h^t) = \sigma(k_t, \alpha_t, p_{t-1})$, if $h^t \in \mathcal{H}^t$ induces (k_t, p_{t-1}); and*
2. *for any $h^t \in \mathcal{H}^t$ and any price p, $A(h^t, p, v) = 0$ if and only if $v \leq \kappa(p, \alpha_t)$.*

We say (σ, κ) describes an equilibrium.

The functions σ and κ describe the seller's equilibrium pricing strategy and the long-lived buyer's equilibrium acceptance rule, respectively. The first requirement of the equilibrium definition states that the seller's price in period t depends only on her belief about the long-lived buyer's value, her belief about the existence of the short-lived buyer, and the last period rejected price p_{t-1}. In addition, we will show that on the equilibrium path, the continuation play depends only on (k_t, α_t), so we abuse notation by treating σ as a function of (k_t, α_t) but not p_{t-1} on the equilibrium path. However, we keep in mind that on the off-equilibrium path, σ may depend on

p_{t-1}. The second requirement of the equilibrium definition shows that the long-lived buyer will employ a cutoff rule, on and off the equilibrium path. Given any price and the belief about the existence of the short-lived buyer in period t, the long-lived buyer at value $\kappa(p, \alpha_t)$ will be indifferent between taking the offer p and waiting.

2.3.2 Why Not a Strong Markov Equilibrium?

A weak Markov equilibrium differs from a strong Markov equilibrium mainly in that on the off-equilibrium path, the seller's price may depend on the previous prices. Though allowing the seller to set a price conditional on the previously rejected offers seems unnatural, it is necessary for the existence of an equilibrium. In this section, we briefly discuss the non-existence of a strong Markov equilibrium, and more details will be shown when we construct the equilibrium of the model.

To complete a description of a seller's strategy, after any price is rejected, a belief about the long-lived buyer's value must be assigned. Because of the conditional skimming property, we need to assign a k' after any price is rejected. In an equilibrium, such a k' must have the property that the k'-value long-lived buyer is indifferent between taking the rejected price and waiting. However, if the continuation play depends only on (k', α'), there may be some $\tilde{p}$ such that the long-lived buyer's indifference condition is violated, no matter what his value is. Therefore, for us to assign a $\tilde{k}$ after this deviating price $\tilde{p}$ is rejected, the $\tilde{k}$-value long-lived buyer's continuation payoff has to depend on $\tilde{p}$. Suppose this is not the case. Then, if the prescribed continuation play provides the $\tilde{k}$ long-lived buyer a high value, the $\tilde{k}$ long-lived buyer will strictly prefer rejecting $\tilde{p}$; if the prescribed continuation play provides the $\tilde{k}$-value long-lived buyer a low value, the $\tilde{k}$-value long-lived buyer will strictly prefer accepting $\tilde{p}$. So $\tilde{k}$ cannot be indifferent. As a result, for some $\tilde{k}$ long-lived buyer to be indifferent, he must be facing some uncertainty for future plays; that is, the seller must randomize after $\tilde{p}$ is rejected. But how the seller randomizes will depend on what the deviating price is. So, the seller's off-equilibrium price may depend on her previous prices. To illustrate this intuition, let's consider the following two-period two-type example.

Ignore the short-lived buyer first. Consider that the buyer's value is either 1 or 3. The prior belief is $\mu_1 = Pr(v = 3) = 1/2$. Let the discount factor equal 1/2. The seller makes one offer in each of the two periods. In the unique PBE, $p_1 = 2$ and $p_2 = 1$. The high type buyer takes p_1, and the low type buyer takes p_2. If p_1 is rejected, the belief becomes $\mu_2 = 0$. This is an equilibrium, since the high type buyer is indifferent between taking $p_1 = 3$ in the first period and taking $p_2 = 1$ in the second period. Now, suppose the seller deviates to $\tilde{p}_1$, which is greater than 2 but smaller than 3. If the high type buyer takes it with probability 1, in the continuation game, $\mu_2 = 0$ and $p_2 = 1$. Given this continuation play, the high type buyer should not take any price greater than 2 in the first period. Conversely, if the high type buyer rejects it with probability 1, $\mu_2 = \mu_1$ and $p_2 = 3$. Given this continuation play, the high type buyer should take $\tilde{p}_1$. So to assign a belief on the high value, the only possibility is to let the high type buyer randomize.

Suppose the high type buyer takes $\tilde{p}_1$ with probability 1/2. If the price is rejected, $\mu_2 = 1/3$ by Bayes' rule. In the second period, the seller is indifferent between charging 1 and 3. That is, in the continuation play in the second period, if the belief is the Markov state variable, we have two pure strategy Markov equilibria and infinite many mixed strategy Markov equilibria. However, we also need the high type buyer to be indifferent between taking and rejecting $\tilde{p}_1$ in the first period. If he takes it, he gets $3 - \tilde{p}_1$; if he rejects it, the seller may randomize between 3 and 1. If the seller charges 1 with probability γ in the continuation game, the high value buyer's expected payoff is $3 - [\gamma + 3(1-\gamma)] = 2\gamma$. To make the high value buyer indifferent, $\gamma = \frac{3-\tilde{p}_1}{2}$. As a result, the seller's price in the second period after a deviation depends on the price charged in the first period.

In this example, after any deviation in the first period, the seller must randomize, which implies that the seller's belief about the high value buyer must be equal to 1/3. This is a crucial point that allows us to show that a strong Markov equilibrium does not exist in this game. However, this unique off-equilibrium path belief results from the assumption that the buyer has only two types. In the case that the buyer's value is distributed over the support $[1, 3]$, in this two-period game, a strong Markov equilibrium exists. In our model, the long-lived buyer's value is distributed over the support $[\underline{v}, 1]$, but the bargaining is a potentially infinite horizon game. Due to the complication of the equilibrium construction, we are unable to show whether a strong Markov equilibrium exists. Hence, in this paper, we use weak Markov equilibrium as the solution concept of the model.

2.3.3 Screening Offer and Waiting Offer

Before moving to the formal analysis of the model, in this section, we introduce some preliminary analysis, as well as some notations. Let's first consider the belief about the existence of the short-lived buyer. Conditional on his existence, the short-lived buyer arrives in any period with probability $\lambda\Delta$. Therefore, no transaction in period t will make both the seller and the long-lived buyer shift downward their belief about the existence of the short-lived buyer according to (2.1).

Second, in the standard bargaining models without the short-lived buyer, the equilibrium price sequence is strictly decreasing over time because of the skimming property. Consequently, the upper limit of the support of the seller's belief about the long-lived buyer's value is also strictly decreasing over time. In our model, however, the seller may charge a high price in equilibrium, which the long-lived buyer rejects for sure, because the potential short-lived buyer places a higher value on the good and thus takes the high price offer. Hence, the upper limit of the support of the seller's belief about the long-lived buyer's value may not be strictly decreasing. If the seller decides to charge such a price, he will set a price at 1, the short-lived buyer's value, and the long-lived buyer rejects the offer for sure.

Definition 6. *A price is a screening offer if and only if the long-lived buyer with some value $v \in [\underline{v}, k]$ may accept it, given α. The price $p = 1$ is a waiting offer.*

If the seller provides a waiting offer, the long-lived buyer will reject it in an equilibrium, no matter what her value is. Therefore, after a waiting offer, the seller's belief about the long-lived buyer's value does not change. If the seller provides a screening offer, there is a positive measure of values with which the long-lived buyer will accept it. As a result, no transaction causes the seller to update his belief about the long-lived buyer's value as

$$k' = \begin{cases} \kappa(p, \alpha), & \text{if } p \text{ is a screening offer,} \\ k, & \text{if } p = 1 \text{ is the waiting offer.} \end{cases} \tag{2.2}$$

For a given state variable vector (k, α), denote the seller's value from optimally charging a screening offer by $V(k, \alpha)$. If the seller chooses a waiting offer, we denote his value by $J(k, \alpha)$. Therefore, given (k, α), the seller charges the optimal screening offer if and only if $V(k, \alpha) \geq J(k, \alpha)$. Specifically, given the buyer's cutoff strategy $\kappa(p, \alpha)$, the seller's problem is

$$R(k, \alpha) = \max\{J(k, \alpha), V(k, \alpha)\}, \tag{2.3}$$

where

$$J(k, \alpha) = \lambda\alpha\Delta + (1 - \lambda\alpha\Delta)e^{-r\Delta}R(k', \alpha') \tag{2.4}$$

and

$$\begin{aligned} & V(k, \alpha) \\ = \quad & \max_p\{[\frac{F(k) - F(\kappa(p, \alpha))}{F(k)} + \frac{F(\kappa(p, \alpha))}{F(k)}\alpha\lambda\Delta]p \\ & \qquad + \frac{F(\kappa(p, \alpha))}{F(k)}(1 - \alpha\lambda\Delta)e^{-r\Delta}R(\kappa(p, \alpha), \alpha')\}. \end{aligned} \tag{2.5}$$

Denote the optimal screening offer given (k, α) by $\sigma^s(k, \alpha)$ (that is, $\sigma^s(k, \alpha)$ is the optimal policy for equation (2.5)), then

$$\sigma(k,\alpha)=\begin{cases}\sigma^s(k,\alpha), & \text{if} \quad V(k,\alpha)\geq J(k,\alpha),\\ 1, & \text{if} \quad V(k,\alpha)<J(k,\alpha).\end{cases}$$

Using the highest price accepted by the long-lived buyer with value v given α, we reformulate (2.5) as

$$\begin{aligned} & V(k,\alpha) \\ = \quad & \max_{\underline{v}\leq k'\leq k}\{[\frac{F(k)-F(k')}{F(k)}+\frac{F(k')}{F(k)}\alpha\lambda\Delta]\kappa^{-1}(k'|\alpha) \\ & \qquad +\frac{F(k')}{F(k)}(1-\alpha\lambda\Delta)e^{-r\Delta}R(k',\alpha')\}. \end{aligned} \tag{2.6}$$

Hence, the policy correspondence of the seller's problem is

$$T(k,\alpha)=\begin{cases}k, & \text{if} \quad \sigma(k,\alpha)=1,\\ T_s(k,\alpha), & \text{otherwise},\end{cases}$$

where $T_s(k,\alpha)$ is the maximum of the set of solutions to problem (2.6).

By definition, when the seller charges a waiting offer, the long-lived buyer will reject it for sure, no matter what her value is. When the seller posts a screening offer, the best response of the long-lived buyer with value v can be characterized by the following indifferent condition:

$$v-\sigma(k,\alpha)=e^{-r\Delta}(1-\alpha\lambda\Delta)U(v,k',\alpha'|\sigma),$$

where

$$U(v, k, \alpha|\sigma) = \max\{v - \sigma(k, \alpha), e^{-r\Delta}(1 - \alpha\lambda\Delta)U(v, k', \alpha'|\sigma)\}$$

denotes the continuation value of the long-lived buyer with value v, when the state variable vector is (k, α) and the seller follows pricing strategy P. As argued above, the equilibrium prices may not be strictly decreasing over time, because the seller may charge waiting offers. Thus, given a price sequence, the indifferent condition of the long-lived buyer can be written as

$$v - p_t = e^{-(n+1)r\Delta}\left[\prod_{i=t}^{t+n}(1 - \alpha_i\lambda\Delta)\right](v - p_{t+n+1}),$$

where on the right-hand side, the long-lived buyer waits for $n + 1$ periods and takes the risk that the short-lived buyer arrives between period t and period $t + n$.

2.4 Bargaining Without Learning

In this section, we assume $\alpha_0 = 1$. That is, it is common knowledge that the short-lived buyer exists. Though the arrival time of the short-lived buyer is still random, there is no learning about the existence of the short-lived buyer. Therefore, the analysis in this section provides a good benchmark for us to show the effect of learning about the existence of the short-lived player in Section 2.5.

Because the seller is sure that the short-lived buyer exists, he can charge the waiting offer forever and expect a time-invariant value. This is a non-trivial outside option for him. Denote such an outside option by J_0, then

$$J_0 = \lambda\Delta + (1 - \lambda\Delta)e^{-r\Delta} J_0.$$

When Δ is small, $e^{-r\Delta}$ can be approximated by $1 - r\Delta$, so the value of waiting for arrivals J_0 can be approximated by $\frac{\lambda}{r+\lambda}$ for small $\Delta > 0$. Furthermore, by Assumption 1, we have $J_0 > \underline{v}$. That is to say, the seller prefers to wait for arrivals rather than trading at an extremely low price (close to $\underline{v}$) immediately. Consequently, once the seller believes the long-lived buyer's value is sufficiently low, he prefers to stop screening him and wait for arrivals by using waiting offers.

Once the seller charges a waiting offer in period t, his belief about the long-lived buyer's value does not change. Hence, if there is no transaction in period t, his belief about the long-lived buyer's value in period $t+1$ is the same as that in period t. Therefore, if it is optimal for the seller to charge the waiting offer in period t, it is optimal for him to charge the waiting offer in period $\tau > t$. Consequently, the seller's problem of when to charge the waiting offer is a stopping time problem. Furthermore, $\frac{\lambda}{r+\lambda} < 1$, so the seller will charge screening offers before switching to the waiting offer forever. Hence, before switching to the waiting offer, the seller's problem is almost identical to that in the canonical Coase bargaining problem. The following Proposition summarizes this intuition.

Proposition 4. *When $\alpha_0 = 1$, an equilibrium exists. Generically, the equilibrium is the unique PBE of the model. Furthermore, there exists an integer N such that, in the equilibrium,*

1. *the seller posts screening offers in the first N periods,*

2. *the price decreases in the first N periods, and*

3. *from the $N+1$ period on, the seller switches to the waiting offer forever.*

Proposition 4 implies that, in equilibrium, there are two phases: the screening phase and the waiting phase. That is to say, the seller first screens the long-lived buyer for finitely many periods by gradually cutting the price. Once the price reaches a certain cutoff level, the seller believes that the value of the long-lived buyer is sufficiently low so that he will give up the long-lived buyer and wait for the arrivals by charging a high price in the future. In the following proposition, we show that the screening process ends very quickly when the time interval of two consecutive periods converges to zero and the initial price converges to $\frac{\lambda}{\lambda+r}$. Hence, a modified Coase conjecture holds, and there is no strategic delay in the equilibrium.

Proposition 5. *In the equilibrium, the following properties hold:*

1. $p_t \geq \frac{\lambda}{\lambda+r}$ *for any* $\Delta > 0, t \geq 0$.

2. *(No Strategic Delay)* p_0 *goes to* $\frac{\lambda}{\lambda+r}$ *as* Δ *goes to zero.*

2.5 Bargaining While Learning About Arrivals

In this section, we study the bargaining game when $\alpha_0 \in (0,1)$. In this case, the interaction between the seller's exogenous learning about the existence of the short-lived buyer and his endogenous learning about the long-lived buyer's value has significant effects on the equilibrium characterization, the equilibrium bargaining outcome, and the equilibrium pricing path.

2.5.1 Equilibrium Characterization

Different from the no learning case, posting the waiting offer forever from some period on cannot be part of an equilibrium. The reason is as follows. By posting a waiting offer, the seller learns the likelihood of arrival only but not the value of the long-lived buyer. After finitely many periods, the seller believes that the likelihood of arrival is negligible, and his expected payoff from waiting longer for a new arrival is almost zero! As a result, the seller can simply charge a price $p = \underline{v}$ to trade with the long-lived buyer immediately. Hence, the "waiting offers forever" strategy cannot be part of an equilibrium in any continuation game. In addition, the seller's equilibrium payoff in any continuation game with α is bounded below by $\max\{J_0(\alpha), \underline{v}\}$, so for sufficiently large $\lambda\alpha$, the equilibrium screening price is bounded away from $\underline{v}$.

Define

$$\alpha^\dagger \equiv \max\{\alpha|\alpha \in B_\Delta(\alpha_0) \text{ and } \underline{v} \geq \alpha\lambda\Delta + (1-\alpha\lambda\Delta)e^{-r\Delta}\underline{v}\}$$

to be the largest realization of α, at which it is better for the seller to set a price at $\underline{v}$ (to finish the bargaining) than to charge the waiting offer once and then set a price at $\underline{v}$.

Lemma 3. *For any $k \in [\underline{v}, 1]$, if $\alpha \in B_\Delta(\alpha^\dagger) = B_\Delta(\alpha_0) \cap \{\alpha|\alpha \leq \alpha^\dagger\}$, then the seller does not charge the waiting offer in equilibrium.*

Lemma 3 implies that when α is small enough, the seller charges only screening offers in any equilibrium. Therefore, the waiting offer, if charged in an equilibrium, is a temporary phenomenon. Because when $\alpha > \alpha^{\dagger}$, the seller will not charge a price $p = \underline{v}$ to finish the bargaining and, there is a positive probability that some $\alpha \leq \alpha^{\dagger}$ is reached on the equilibrium path. Therefore, to analyze players' equilibrium behaviors when α is large, we first need to characterize the equilibrium behavior when $\alpha \leq \alpha^{\dagger}$. The following Proposition provides a detailed characterization of the continuation equilibrium for such small α's.

Proposition 6. *In any continuation game starting at $(k, \alpha) \in [\underline{v}, 1] \times B_{\Delta}\left(\alpha^{\dagger}\right)$, an equilibrium exists, and it satisfies the following properties:*

1. *It is the unique PBE of the continuation game.*

2. *The game ends in finitely many periods.*

When $\alpha \leq \alpha^{\dagger}$, the belief about the existence of the short-lived buyer is so low that the long-lived seller will charge only screening offers. We can construct the unique equilibrium in the same way as in the no learning case. First, for each $\alpha \in B_{\Delta}(\alpha^{\dagger})$, we define $k^0(\alpha)$ as the maximum k such that a seller at (k, α) optimally charges $\underline{v}$ immediately rather than making other screening and waiting offers, and therefore, the game ends in one period. The reason is that the seller is so pessimistic about both the long-lived buyer's value and the short-lived buyer's arrival that he prefers to obtain $\underline{v}$ immediately. As a result, the game ends within one period. Similarly, fixing any $\alpha \in B_{\Delta}(\alpha^{\dagger})$, we define $k^1(\alpha)$, such that when $k \in (k^0(\alpha], k^1(\alpha))$, the seller's optimal screening offer is a price $\sigma(k, \alpha)$, and there is a marginal type $k' \in [\underline{v}, k^0(\alpha'))$ who is

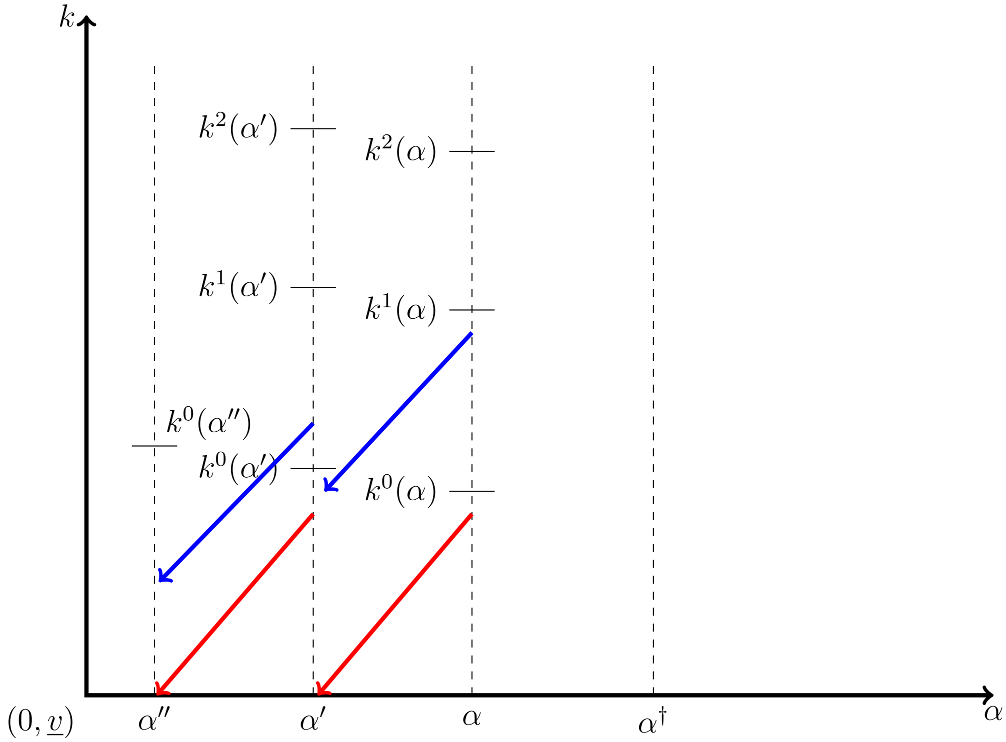

Figure 2.1: equilibrium construction in the learning case

indifferent between taking $\sigma(k,\alpha)$ and waiting for one more period. We show there is an integer N, such that inductively applying this construction method for N times, we have $k^N(\alpha) > 1$ for any $\alpha \in B_\Delta(\alpha^\dagger)$. This idea of equilibrium construction is illustrated in Figure 2.1.

In the original game, it takes finitely many periods for the belief to decrease from $\alpha_0 < 1$ to $\alpha^\dagger$ regardless of the players' strategy profile. Hence, Corollary 1 immediately results from Proposition 6.

Corollary 1. *In any perfect Bayesian equilibrium, the game ends in finitely many periods.*

The next Proposition extends the equilibrium construction to the space of (k, α), where $(k, \alpha) \in [\underline{v}, 1] \times B_\Delta(\alpha_0) \setminus B_\Delta(\alpha^\dagger)$, and shows that it is essentially the unique PBE.

Proposition 7. *The equilibrium exists. Generically, it is the unique PBE.*

The equilibrium construction and the proof of its uniqueness are presented in the Appendix. Here we present only the idea of construction as follows. First, since the unique equilibrium strategy profile, (σ, κ), is constructed for any continuation game starting at $(k, \alpha^\dagger)$ $(k \in [\underline{v}, 1])$, the equilibrium payoff of the long-lived buyer with any value is specified under (σ, κ). Second, extend the long-lived buyer's strategy profile $\kappa(p, \alpha)$ to $\alpha^2 = \min\{\alpha \in B_\Delta(\alpha_0), \alpha > \alpha^\dagger\}$ such that $\kappa(p, \alpha^2)$ is indifferent between taking p in this period and obtaining his continuation payoff, $U\left(\kappa(p, \alpha^2), \kappa(p, \alpha^2), \alpha^\dagger | \sigma, \kappa\right)$, in the next period. Third, extend the seller's screening strategy $\sigma^s(\cdot, \alpha)$ to α^2. Fourth, for any $k \in [\underline{v}, 1]$, compare the seller's values $V(k, \alpha^2)$ induced by the optimal screening offer and $J(k, \alpha^2)$ induced by a waiting offer, and define $R(k, \alpha^2) = V(k, \alpha^2), \sigma(k, \alpha^2) = \sigma^s(k, \alpha^2)$ if $V(k, \alpha^2) \geq J(k, \alpha^2)$, and $R(k, \alpha^2) = J(k, \alpha^2), \sigma(k, \alpha^2) = 1$ otherwise. Finally, compute the buyer's payoff, $U(k, k, \alpha^2 | \sigma, \kappa)$ for each $k \in [\underline{v}, 1]$. Since it takes finitely many steps to go from α_0 to $\alpha^\dagger$, we can repeat the above construction method for finitely many times and extend (σ, κ) to α_0. We can essentially find a unique path of $\{k_t, \alpha_t\}_{t=0}^{N^*}$ from $(1, \alpha_0)$ to $(\underline{v}, \alpha_\tau)$ where $\tau \leq N^*$.

2.5.2 Strategic Delay and Price Fluctuation

In this subsection, we analyze properties of the equilibrium and demonstrate the role of the interaction between the seller's exogenous learning about the existence of the short-lived buyer and his endogenous learning about the long-lived buyer's value. When the likelihood of new arrivals is sufficiently high, waiting for arrivals is a non-trivial outside option, so that the seller has no incentive to post a sufficiently low price to ensure a trade with the long-lived buyer takes place immediately, even though the time interval between the two offers is arbitrarily small. The following Proposition formalizes this intution. Therefore, with the potential new arrivals, there are strategic delays of trades.

Proposition 8 (Strategic Delay with Non-trivial Learning). *For any Δ and $T > 0$, there is an $\bar{\alpha} < 1$ such that, for any $\alpha_0 \in (\bar{\alpha}, 1)$, the equilibrium price $P_t > \underline{v}$ for any $t < T$.*

From Lemma 3, we know that, after finitely many periods, the seller charges only screening offers in the equilibrium. Hence, in equilibrium, there are two phases. In the second phase, the seller believes that the likelihood of arrivals is small and uses screening offer only. In the first phase, the seller may use both waiting offers and screening offers. Since the waiting price is high, the equilibrium price dynamics in the first phase may exhibit jumps. Intuitively, when the likelihood of new arrivals measured by α is large, the waiting offer may be optimal, because the expected revenue from selling the good to the short-lived buyer is high. When charging the waiting offer, only the belief about the arrivals changes, and the seller's belief about the long-lived buyer's value does not change. Hence, after charging waiting offers for a while, α goes down over time, the waiting offer becomes less attractive, and the seller will switch to screening offers. However, after several periods with screening offers, α may become relatively large (compared with k), so the waiting offer may become the optimal choice for the seller again. Therefore, the equilibrium price may exhibit fluctuation: some decreasing screening prices are followed by the waiting offer in a number of periods, and then even lower screening prices are charged. Unfortunately, it is well known that fully analyzing the equilibrium strategies in a dynamic pricing game is in genearl impossible. Hence, we focus on the games that satisfy certain conditions. First, following Fuchs and Skrzypacz (2010), we focus on the games with atomless limits.

Condition 1 (Atomless Limit)**.** *In the equilibrium, for all $t > 0$, if the seller uses screening offers in period t, $t+1$ and $t+2$, then $\kappa_{t+1} - \kappa_{t+2}$ is $O(\Delta)$.*

Condition 1 simply means that, when the seller consecutively uses screening offers, the possibility of trade in each period is not very large. That is to say, the seller will smoothly screen the long-lived buyer. Second, we impose a smoothness condition.

Condition 2 (Smoothness)**.** *For all $t > 0$, $\kappa(p, \alpha) = \kappa(p', \alpha')$, we have $p - p'$ is $O(\Delta)$.*

Condition 2 means that when the likelihood of arrivals changes slightly, the long-lived buyer's equilibrium strategy does not dramatically change. Under both conditions, we can further characterize the equilibrium price dynamics in the first phase: when the time interval between two consecutive periods is small, in the first phase of the equilibrium, the seller posts screening offers for one or two periods, then switches to waiting offers for some periods, and then switches back to screening offers for one or two periods, and then switch back to waiting offers. This frequent switching between waiting offers and screening offers stops at the end of the first phase. The following Proposition formally shows this pricing pattern in the first phase.

Proposition 9 (Price Fluctuation)**.** *Suppose* Δ *is small and* $\alpha_0 \in (\alpha^\dagger, 1)$*. Suppose also both condition 1 and condition 2 hold. In equilibrium, when* $alpha_t$ *is larger than* $\alpha^\dagger$*, (at most) two consecutive screening offers must be followed by a waiting offer.*

The idea is that, when the seller can frequently revise the price, he always prefers to speed up the screening of the long-lived buyer. Hence, after being rejected once or twice, the seller believes that the long-lived buyer's value is too low so that he starts to post waiting offers. After several periods, the seller's belief about the existence of arrivals shifts downward, but his belief about the long-lived buyer's value remains; so he starts to screen the long-lived buyer again, and the screening process is very fast again. As a result, the seller frequently switches between waiting offers and screening offers in equilibrium.

2.6 Concluding Remarks

In this paper, we study a dynamic bargaining game between a long-lived seller and a long-lived buyer. The seller makes all offers, and the long-lived buyer has private information about his value. There may exist a short-lived player whose value is commonly known to be high. Conditional on his existence, the short-lived buyer's arrival is determined by a Poisson process. We characterize the unique equilibrium, which exhibits strategic delays and price fluctuations. In particular, when the seller is optimistic about the existence of the short-lived buyer, he charges a waiting offer, which is never accepted by the long-lived buyer. By making the waiting offer, the seller only adjusts her belief about the existence of short-lived buyers. Therefore, a waiting offer not only exploits the value of a potential new arrival but also controls the speed of learning. When the seller becomes sufficiently pessimistic about the existence of the short-lived buyer, she offers a price that is acceptable to the long-lived buyer with some values. If such offers are not accepted, the seller's beliefs about both the existence of new arrivals and the long-lived buyer's value change. The interaction between these two learning processes is the driving force of the price fluctuation.

We restrict our study to the situation where the short-lived buyer's value is commonly known and equals 1. One can easily extend the result to any $v_s < 1$. In addition, one can assume that there may be a sequence of short-lived buyers whose value are i.i.d. and who share the same distribution with the long-lived buyer's value. An arrival is observed only when the short-lived buyer takes the offer. In a model without the long-lived buyer, Mason and Valimaki (2011) show that the equilibrium waiting price declines over time. With the long-lived buyer, our conjecture is that the equilibrium price sequence frequently switches between two price paths: a screening price path and a waiting price path. There is another interesting extension of the model. Assume that there may be a sequence of short-lived buyers whose values are independent. Each short-lived buyer's value is either high or low. Only the seller can observe the arrival of the short-lived buyer. This is a difficult problem, because the seller and the long-lived buyer may have different beliefs on the equilibrium path. We leave this question for future research.

Chapter 3

Efficient Learning And Job Turnover in the Labor Market

3.1 Introduction

An enormous number of employment-to-employment (EE) transitions take place in the U.S. labor market. Based on the estimation by Nagypal (2008), 2.2% of employed workers leave for a job in a different firm, and the flow of EE transitions accounts for 49% of all exits from employers, versus 20% of separations that are employment-to-unemployment (EU) transitions and 31% that are transitions from employment to being out of the labor force. In the labor search literature, EE transitions are generally accomplished through employed workers' on-the-job search (OJS) behavior. Numerous empirical studies show that workers' OJS behavior and its performance vary regarding tenure, the motivation for the OJS and other factors. In this paper, I develop an equilibrium search model to study the dynamics of workers' OJS behavior and its consequences.[1]

Specifically, I consider a directed search model in which firms post contracts and workers search for jobs. Once a worker and a firm meet, they form a one-to-one match. Their pair-specific match quality is initially unknown (with the same prior on both sides) and is revealed gradually over time. When the match is believed to be bad with very high probability, the firm destroys the job to avoid further loss and the worker becomes unemployed. Employed workers can search for a new job and so can unemployed workers, and the optimal search strategy depends on a worker's evaluation (belief) of his current match quality. Since a job with an extremely bad evaluation is going to be destroyed, an employed worker has an incentive to search on the job because (1) he is willing to find a new job with better pay and (2) he is afraid of losing his current job in the future.

Over time, a worker and his employer adjust their evaluation of the current job match quality based on the worker's past job performance. The diversity of individual histories results in ex post heterogeneity in the evaluation of the current match and therefore in the job search behavior of workers. This learning mechanism has two conflicting effects on the tenure-EE transitions profile. On the one hand, there is a standard *selection effect*, which was initially highlighted by Jovanovic (1979a,b and 1984). Over time, the match quality will be learned. Known good matches are kept and known bad matches are destroyed. Consequently, the proportion of good matches raises over time. For a particular worker, the longer his tenure, the

[1] See, for examples, Mincer and Jovanovic (1982), Flinn (1986), Farber (1999), Fujita (2012) and Bjelland, Fallick, Haltiwanger and McEntarfer (2011).

higher the probability that his current match is good. Since good matches will not be destroyed, workers in good matches have less incentive to engage in OJS. Hence the selection effect suggests a negative relation between tenure and the EE transition rate. On the other hand, I show that there is a *job search target effect* of learning. Some workers believe their current job's match quality is good, and they don't need to worry about being fired in the near future, so they are attracted only by well-paying jobs. Other workers believe their current job's match quality is not good enough, so they are afraid of losing their current job. As a result, they are less selective and target their search to jobs with lower pay. A matched worker with a long tenure but who has not revealed the type of his current match would be treated as being working in a bad match with high probability. Hence these workers are afraid of losing their job in the near future, and they have strong incentives to switch to a new job as soon as possible. In a frictional labor market, a worker with a low evaluation of his current job can adjust his OJS strategy to raise the probability of transition. This *job search target effect* would raise the possibility of EE transitions on average.

In general, this problem is hard to analyze in an equilibrium search framework. In a standard search model, as Burdett and Mortenson (1998) and Shi (2009) show, firms may post different wage schemes, which induces two dimensions of ex post heterogeneity among employed workers: (1) their evaluation of the current match quality, and (2) the wage scheme promised by their current employer. A worker's job search behavior depends on both of them, and therefore it is hard to analyze the worker's OJS dynamics in an equilibrium model. To obtain a tractable model, I follow

Menzio and Shi (2011) and consider the socially efficient allocation, and implement the efficient solution by allowing agents to sign complete contracts. By focusing on a model with complete contracts, not only can I describe the interaction between the selection effect and the job search target effect and its empirical implications in a tractable model, but also I can separate their impact on labor markets from that of other mechanisms', such as the lack of agents' commitment and the particular form of wage formation, both of which have been well studied in the literature.

To characterize the socially efficient allocation, I start with the social planner's problem. A social planner decides (1) the separation rule of existing matches and (2) the search strategy for each worker. The efficient separation rule is given as a cutoff belief about match quality. When the belief about match quality is higher than the cutoff level, the planner keeps the underlying match. Otherwise, the planner destroys the match and naturally stops learning about its quality. Following the literature on directed search (Acemoglu and Shimer 1999, Moen 1997), I assume that there are numerous locations in the economy. A match forms only if a worker meets a firm at the same location. To workers, locations differ from each other in terms of the probability of finding a new job and promised pay. The efficient choice of a searching location is determined by the current state of a worker. If an employed worker is in a good match, he does not search for a new job. If an employed worker's current match quality is uncertain, he is sent to a specific location to find a new job, and the probability of getting a new job is non-increasing in the belief about his current match quality. An unemployed worker searches for a job at the location with the highest job-finding probability.

Under the efficient allocation, the interaction between job search and learning has a nontrivial impact on workers' job turnover. For matches with a short tenure, the job search target effect dominates the selection effect, so the EE transition rate is increasing in tenure. For matches with a long tenure, the selection effect dominates the job search effect, so the EE transition rate is decreasing in tenure. When the tenure is long enough, all uncertainty is resolved, and only good matches are kept and workers in those matches do not OJS anymore. As a result, the EE transition rate as a function of tenure first increases at low tenure levels, then decreases, and eventually becomes constant. Since the job separation rate is the sum of the EE transition rate and the EU transition rate, the separation rate-tenure profile also has a hump shape. These theoretical results are roughly consistent with a variety of stylized facts emerging from the data at both the micro and macro levels. For example, Farber (1994) find that the separation rate increases early in tenure and decreases later, but in the end, the separation rate becomes constant, and Menzio, Telyukova and Visschers (2012) find that the EE transition rate increases in tenure in the first four months, and decreases thereafter.

The current model also generates an implication for the relation between the OJS target of a worker and his motivation for OJS. When a worker has a high evaluation of his current match quality, he is not too afraid of losing his current job, so he looks only for promising new job if he searches on the job. Since promising jobs are also competitive, his job finding rate is low. On the other hand, when a worker has a low evaluation of his current match quality, he is afraid of being fired in the near future, so he wants to find a new job as soon as possible. As a result, his target job is less promising and also less competitive, and therefore, his job finding rate is high. This prediction is roughly consistent with a new empirical finding by Fujita (2012). He finds that (1) some workers engage in OJS because they fear losing their current job, while others search on the job because they are unsatisfied with their current job, and (2) the unsatisfied on-the-job searchers have a lower job finding rate and a higher wage growth due to the job transition.

My paper is closely related to Moscarini (2005), who nests Jovanovic's model (1984) into an equilibrium search model. There are several main differences. First, the economic intuitions of the separation-tenure relation are different in the two papers. In Moscarini (2005), the initial period in which the separation rate is increasing in tenure is called the wait-and-see phase, whose existence relies on the properties of the learning process. In particular, following Jovanovic (1984), Moscarini (2005) assumes that the production signal follows a diffusion process and therefore the sample path of the posterior of the match quality is continuous. Hence, an endogenous separation cannot be instantaneous but "kicks in" only after some time.[2] Thus, on average, the separation rate initially increases with tenure. However, the initial wait-and-see phase disappears if the learning process has no continuous sample path.[3] In my model, the hump shape of the separation rate results from the combination of the job search target effect and the selection effect. Should the tenure-varying job search target effect be missing, both the EE transition rate and the separation rate would be decreasing in the beginning. Second, in Moscarini (2005), wage is determined over

[2]Mathematically, the separation rate is zero for a new match. Since the sample path of diffusion process is continuous and the separation rate is non-negative, the separation rate must increase for a while and then decrease.

[3]For example, the learning process is a Poisson process, as in my model.

time by Nash bargaining between a worker and his employer, which is not efficient in general, while in my model, the learning and search allocation are efficient, which implies that the hump shape of the separation rate does not rely on the inefficiency of wage formation or workers' OJS behavior.

The rest of this paper is organized as follows. Section 2 presents the basic environment, individual payoff and learning process. I characterize the social planner's problem in Section 3. Section 4 considers a simple contract to implement the social planner's allocation in a frictional labor market. Section 5 discusses the empirical implications. A number of extensions are discussed in Section 6. All technical proofs can be found in the Appendix.

3.2 The Model

Time is continuous. The economy is populated by a continuum of workers of measure one and by a continuum of firms of measure greater than 1. Each worker has the utility function $\int e^{-rT} C_T dT$, where $C_T \in \mathbb{R}$ is the worker's consumption at time T and r is his discount rate. Each firm has the payoff function$\int e^{-rT} \Pi_T dT$, where $\Pi_T \in R$ is the firm's profit at T. Each firm has one vacancy and can hire at most one worker. Vacant firms or unemployed workers are unproductive.

There is a continuum of locations indexed by a real number $l \in [0, 1]$. A vacant firm and a worker can match only if they are searching in the same location. In each period, both firms and workers decide which location to enter. A location is interpreted as a submarket if there are firms and workers there. Different submarkets can be indexed by the promised value to the worker, $x \in \mathbb{R}$, posted by firms in that market. I denote mapping $\Upsilon : [0,1] \to \mathbb{R} \cup \varnothing$ as the submarket assignment function. In other words, $x = \Upsilon(l)$ is the promised value to the worker specified by the contract offered at location l, while $\Upsilon(l) = \varnothing$ means there is no submarket at location l. At location l with $\Upsilon(l) \neq \varnothing$, the ratio between the number of jobs that are vacant and the number of searching workers is denoted by $\tilde{\theta}(l) \in \mathbb{R}^+$. I refer to $\theta(x)$ as the tightness of the submarket at location l such that $x = \Upsilon(l)$. In other words, I do not distinguish between the two markets $l \neq l'$ with the same x.

All submarkets are subject to search frictions. In particular, workers and firms that are searching in the same location are brought into contact by a meeting technology with constant returns to scale that can be described in terms of the market tightness $\theta \in \mathbb{R}^+$. In particular, at any time a worker finds a vacant job with probability $p(\theta(l))$ at location l, where $\theta(l)$ is the market tightness at location l and function $p : \mathbb{R}^+ \to [0,1]$ is twice continuously differentiable, strictly increasing, strictly concave, which satisfies (i) $p(0) = 0$, (ii) $\lim\limits_{\theta \to 0} p'(\theta) = \infty$, and $\lim_{\theta \to \infty} p(\theta)$ is bounded by a finite number. Similarly, a vacancy meets a worker with rate $q(\theta(l))$ in location l where $q : \mathbb{R}^+ \to [0,1]$ is a twice continuously differentiable, strictly decreasing function such that $q(\theta) = p(\theta)/\theta$ when $\theta > 0$, and $q(0)$ is bounded. When a firm and a worker meet, a new match is formed, and the worker's old match, if any, is destroyed.

Each firm chooses to enter at most one submarket by paying a maintenance flow cost k at any time and posts an employment contract x, which is the promised value to the worker. All workers, whether employed or unemployed, observe all available offers in the labor market and choose one submarket to enter and search for a new job. Different wage dynamics are allowed given the identical initial expected promise. In general, a worker's individual wage dynamics can depend on both the aggregate market variables and the match-specific payoff history.

The match between a firm and a worker is either good or bad. If the match is good, at any time, the matched firm receives 1 unit of payoff at a rate λ; if the match is bad, a matched firm receives nothing. Initially, a matched worker-firm pair shares symmetric information about the match quality with a common prior $\alpha_0 \in (0, 1)$ that the current match is good. They observe the outcomes and hold common posterior beliefs α_t throughout time, where α_t denotes the belief that they assign to the match being good at t, where t denotes the worker's tenure in his current job. For simplicity, no extra flow payoff is generated by a match. A match is destroyed exogenously at a rate δ at any time. An unemployed worker enjoys a flow payoff of $b > 0$, which can be interpreted as his home production. To avoid a trivial case where the endogenous separation is never optimal, assume $\alpha_0\lambda > b > 0$, that is, a new match is better than no match, but no match is better than a bad match.

For a match with $\alpha = 1$, no belief adjustment happens regardless of its current period output. For a match with $\alpha \leq \alpha_0$, if the unit of payoff is received in tenure period t, α_{t^+} jumps to 1; otherwise, by standard Bayes' rule updating, the evolution of α_t follows

$$\dot{\alpha}_t = -\lambda(1 - \alpha_t)\alpha_t, \tag{3.1}$$

which is 0 if α_t equals either 0 or 1.

The worker's search strategy may depend on both the social state and his individual state. The former includes the unemployment rate and the distribution of the current match quality. The latter includes: (1) whether the worker is employed, and (2) the belief about the current match quality if he is employed. Formally, define $\Omega = [0, \alpha_0] \cup \{u\} \cup \{1\}$ as a worker's individual state space. A worker's state $\omega \in \Omega$ can be interpreted as follows. For an uncertain matched worker, his type $\omega \in [0, \alpha_0]$ is the belief about the current match quality. For a matched worker who has sent a good signal before, $\omega = 1$. For an unemployed worker, $\omega = u$. Denote the probability measure μ_T over Ω as the social state of the economy. Let $\Xi = \Delta(\Omega)$ denote the set to which μ_T belongs for all T. In this paper, I focus on the steady state, so $\mu_T = \mu^*$.

3.3 Efficient Allocation

To characterize the efficient allocation in the steady state, I solve the social planner's problem in the steady state first. Since I focus on the steady state, $\mu_T = \mu^*$, the planer's strategy depends on workers' individual states only.

For an unemployed worker, he enjoys a flow payoff, b. The planner sends him to search for a new job in submarket θ. In such a submarket, he finds that a new job arrives at a rate $p(\theta)$. To support the market tightness θ, the planner sends θ firms to this submarket for each unemployed worker. From the perspective of the planner, an unemployed worker's problem is governed by the following HJB function:

$$rS(u) = b + p(\theta(u))[S(\alpha_0) - S(u)] - k\theta(u), \tag{3.2}$$

where his efficient search strategy is pinned down by

$$k = p'(\theta(u))[S(\alpha_0) - S(u)]. \tag{3.3}$$

The left-hand side of (3.3) is the social marginal cost of vacancy creation, and the right-hand side is the social marginal benefit.

For an existing match, a good signal arrives at a rate of either λ or 0, which depends on the match quality, and an exogenous separation shock arrives at a rate δ. Given the belief of the current match quality, the planner chooses (1) the worker's on-the-job search strategy $\theta(\alpha)$, and (2) the separation strategy of the current match $z(\alpha) \in \{0, 1\}$. For any α, one can solve the associated value $S(\alpha)$ and policy function $\theta(\alpha)$, $z(\alpha)$.

Apparently, when $\alpha = 1$, it is inefficient to ask the worker to search on the job, so $\theta(\alpha) = \varnothing$, and $z(\alpha) = 0$. Thus the social value of a good match is pinned down by the following equation.

$$rS(1) = \lambda + \delta[S(u) - S(1)]. \tag{3.4}$$

For an uncertain match, $\alpha \in (0, \alpha_0]$. When $z(\alpha) = 0$, the social value of this match satisfies the following HJB function:

$$\begin{aligned} rS(\alpha) &= \alpha\lambda + \lambda\alpha(S(1) - S(\alpha)) - \lambda\alpha(1-\alpha)S'(\alpha) \\ &\quad + \delta(S(u) - S(\alpha)) + p(\theta(\alpha))[S(\alpha_0) - S(\alpha)] - k\theta(\alpha), \end{aligned} \tag{3.5}$$

and the optimal on-the-job search strategy is pinned down by

$$k = p'(\theta(\alpha))[S(\alpha_0) - S(\alpha)]; \tag{3.6}$$

When $z(\alpha) = 1$, I have $S(\alpha) = S(u)$. The following lemma shows that, given the social value of a unemployed worker, the socially optimal separation strategy $z(\alpha)$ can be characterized by a cutoff strategy.

Lemma 4. *Fixing $S(u)$, the constrained socially optimal problem of an employed worker satisfies the follows:*

1. *the optimal separation strategy is given by*

$$z(\alpha) = \begin{cases} 1 & \textit{if } \alpha < \alpha_* \\ 0 & \textit{otherwise} \end{cases}$$

2. *the cutoff belief α_* is the* largest *α such that*

$$S(\alpha_*) = S(u) \tag{3.7}$$

$$S'(\alpha_*) = 0 \tag{3.8}$$

   and therefore $S(\alpha)$ is the solution of ODE (3.5) with the boundary conditions (3.7, 3.8).

3. *the on-the-job search is characterized by (3.6).*

The proof directly follows the exponential bandit literature, so it is omitted.[4] Given $S(\alpha)$, one can calculate the value of $S(u)$ by using equation (3.2). Hence, the efficient solution, $S(u), S(\alpha)$ can be solved as a fixed point of the system (3.2, 3.5) and satisfies (3.7, 3.8). The following proposition characterizes the properties of the efficient steady state. The stationary distribution μ^* is presented in the supplementary materials.

Proposition 10. *The socially efficient allocation uniquely exists and it satisfies the following properties:*

1. *$S(\alpha)$ is convex for all $\alpha \in [\alpha_*, \alpha_0]$.*

2. *$S(\alpha)$ is strictly increasing for all $\alpha \in [\alpha_*, \alpha_0]$,*

3. *$\theta(\alpha)$ is strictly decreasing for all $\alpha \in [\alpha_*, \alpha_0]$.*

4. *$\lim_{\alpha \to \alpha_*} \theta'(\alpha) = 0$ and $\lim_{\alpha \to \alpha_0} \theta(\alpha) = 0$.*

The efficient job search strategy $\theta(\alpha)$ is strictly decreasing in α. The Bayes' rule of learning, equation (3.1), implies that α is decreasing in t, so the job finding rate of an employed worker with a current belief α_t, $p(\theta(\alpha_t))$ is increasing in his tenure t. When a worker starts to work at a firm, his evaluation of the current job is high, and he is encouraged to look only at promising jobs. Since promising jobs are also competitive, the job finding rate is low. When a worker stays at a firm for a long time and does not have a good record, his evaluation of the current match is low,

[4]See Keller, Rady and Cripps (2005). Also see chapter 4 of Dixit and Pindyck (1994) for an intuitive discussion.

so he is encouraged to look at less promising jobs to leave current position before it is destroyed. Since less promising jobs are less competitive, the job finding rate is high. As a result, I construct a theoretical link between a worker' evaluation of his current job and his efficient job finding rate.

Remark 1. *When $p(\theta) = \min\{\theta, \bar{\theta}\}$ and $\bar{\theta}$ is finite, there are no labor frictions in the market, and the on-the-job search decision problem is a linear programming problem with a corner solution: $\theta(\alpha) = \bar{\theta}$ if $S(\alpha_0) - S(\alpha) > k$, $\theta = 0$ otherwise.*

Remark 2. *When the match quality is known, only a good match is created, so $\alpha_0 = 1$. The absence of learning implies that matched workers' and firms' values are constant over time and on-the-job search is not efficient. Hence, there is only one active submarket in the social planner's solution.*

The intuition of the two remarks above is as follows. The social planner's fundamental trade-off is between the replacement premium of an existing match and the cost of creating a vacancy. When match quality is common knowledge, neither learning nor on-the-job search has value; hence, the optimal allocation is a corner solution. When the market is frictionless, the fundamental trade-off becomes a linear programming problem, and distinguishing a match with a different belief is not necessary. Hence, the optimal allocation is a corner solution as well. The two remarks imply that, in such an environment, under efficient allocation, nontrivial OJS dynamics can only result from the interaction between learning and search friction.

3.4 Decentralization

In this section, I consider the implementation of the social planner's solution. I assume that the contracts offered by firms to workers are bilaterally efficient in the sense that they maximize the joint value of the match, that is, the sum of the worker's expected lifetime utility and the firm's expected lifetime profits. I make this assumption because there are a variety of specifications of the contract space under which the contracts that maximize the firm's profits are, in fact, bilaterally efficient. As Menzio and Shi (2009) show in a similar environment, the profit-maximizing contracts are bilaterally efficient if the contract space is complete in the sense that a contract can specify the promised utility to the worker, x, the separation probability z and the worker's on-the-job search strategy θ. This result is intuitive. The firm maximizes its profits by choosing the contingencies z, x so as to maximize the joint value of the match and by choosing the contingencies for w so as to deliver the promised value x.[5]

In order to decentralize the social planner's optimal allocation, I first define the joint surplus of an uncertain existing match $M(\alpha)$. Note that $M(\alpha)$ is *not* the social surplus generated by the match since the matched worker and firm do not take into account the wage posting cost paid by the worker's potential new employer.

[5]Moreover, one can prove that the profit-maximizing contracts are bilaterally efficient if they can specify the wage only as a function of tenure and productivity (while the separation and search decisions are made by the worker). This result is also intuitive. The firm maximizes its profits by choosing the wage when it meets a worker so as to deliver the promised value x and by choosing the wage as a function of the belief about the match so as to induce the worker to maximize the joint value of the match (by setting the wage equal to the product of the match). Alternatively, profit-maximizing contracts are bilaterally efficient if they can specify severance transfers that induce the worker to internalize the effect of his separation and search decisions on the firm's profits. See Moen and Rosen (2004), and Menzio and Shi (2009, 2011) for more examples.

Labor market supply side. First, consider an employed worker at the beginning of the search and matching stage. Since the contract is bilaterally efficient, given the equilibrium market tightness function θ, the worker chooses to search in the submarket with promised value $x(\theta)$ to maximize the continuation value of his current match, which is given by

$$rM(1) = \lambda + \delta[V_u - M(1)] \tag{3.9}$$

$$\begin{aligned} rM(\alpha) &= \alpha\lambda + \lambda\alpha(M(1) - M(\alpha)) - \lambda\alpha(1-\alpha)M'(\alpha) \\ &\quad +\delta(V_u - M(\alpha)) + p(\theta(\alpha))[x(\theta) - M(\alpha)], \end{aligned} \tag{3.10}$$

and the optimal OJS strategy is

$$\theta(\omega) = \arg\max p(\theta)[x(\theta) - M(\omega)] \text{ for } \omega \in [0, \alpha_0] \cup \{1\} \tag{3.11}$$

By the same logic, an unemployed worker chooses to search in the submarket with tightness $\theta(x_u)$ and promised value x_u to maximize his value, which is given by

$$rV_u = b + p(\theta(u))[x(\theta(u)) - V_u] \tag{3.12}$$

where the optimal search strategy is

$$\theta(u) = \arg\max p(\theta)[x(\theta) - V_u] \tag{3.13}$$

with a similar interpretation.

Labor market demand side. Firms without a match are on the demand side of the labor market. They choose whether to enter the labor market and which submarket to enter. The competition in the labor market implies that firms' expected discounted profit is zero, and there is no difference between any of the submarkets for any firm. A firm may form a new match at a rate $q(\theta)$, which depends on the tightness of the market the firm is in, and its expected profit is given by $M(\alpha_0) - x$.

By posting a new job, a firm needs to pay a flow cost k . Hence, the firm's free-entry condition is given by

$$q(\theta)[M(\alpha_0) - x(\theta)] = k. \tag{3.14}$$

Labor market equilibrium. The labor market equilibrium consists of (1) $M(\alpha), V_u, \theta(\alpha), \theta(u)$ and $x(\theta)$, which satisfy equations (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14), and (2) a stationary distribution, μ^*, which is consistent with $\theta(\alpha)$ and $\theta(u)$. The firm's free-entry condition implies that $M(\alpha_0) - x = k/q(\theta)$ for any submarket. The fact that $q(\theta) = p(\theta)/\theta$ yields that

$$p(\theta)[M(\alpha_0) - x] = k\theta \tag{3.15}$$

Plugging (3.14) into (3.10) and (3.12) implies that

$$\begin{aligned} rM(\alpha) \;&=\; \alpha\lambda + \lambda\alpha(M(1) - M(\alpha)) - \lambda\alpha(1-\alpha)M'(\alpha) \\ &\quad +\delta(V_u - M(\alpha)) + p(\theta(\alpha))[M(\alpha_0) - M(\alpha)] - k\theta, \end{aligned} \tag{3.16}$$

and

$$rV_u = b + p(\theta_u)[M(\alpha_0) - V_u] - k\theta_u \tag{3.17}$$

So in the equilibrium $M(\alpha) = S(\alpha)$, $V_u = S_u$, and therefore the equilibrium search strategy is efficient. Given the unique equilibrium strategy θ, one can uniquely pin down the individual's state transition and therefore the stationary distribution μ^*. The following proposition summarizes the analysis above.

Proposition 11. *A stationary equilibrium exists and it is efficient.*

3.5 Empirical Implications

Selection Effect and OJS Probability. In this model, learning about match quality generates a number of interesting empirical implications. First, as in Jovanovic (1984), there is a selection effect of learning. Specifically, over time, good matches should send signals with higher probability. Hence, the probability that the match quality is good and is known is increasing in the tenure. Since workers in good matches do not engage in OJS, the selection effect implies that the OJS probability of workers is decreasing in their tenure. Consider a randomly picked worker with tenure t. Without knowing his signal history, one does not know for sure whether this match is good or whether this worker is searching on the job. Nonetheless, it is possible to find the ex ante probability that a randomly chosen worker is searching on the job, as a function of t, which is

$$\sigma_t \equiv \begin{cases} \alpha_0 \Pr(\tau > t) + (1 - \alpha_0) & t < t^*, \\ 0 & t \geq t^*, \end{cases}$$

where τ is a random variable representing the time at which the good signal occurs, so the probability that it has not happened by t is $\Pr(\tau > t) = \exp(-\lambda t)$, which is decreasing in t. The critical cutoff time is defined as $t^* = \inf\{t > 0 | \alpha_0 - \alpha_* = \int_0^{t^*} \lambda \alpha_s (1 - \alpha_s) ds\}$, at which point the belief hits α_* and the firm optimally destroys the current uncertain match, so the worker becomes unemployed. Before this point, a match is bad with ex ante probability $1 - \alpha_0$, and the worker always searches for a new job. With complementary probability a_0, the match is good, and the worker searches only when a good signal has not arrived; hence, the quality remains uncertain. Therefore, the model predicts that the OJS probability is at first decreasing in workers' tenure but eventually becomes constant. This negative relation between the probability of OJS and tenure is supported by many empirical findings, for example, Pissarides and Wadsworth (1994).

OJS Target Effect. Departing from the standard learning literature, I highlight another effect of learning: the job search target effect. If a match has not generated good signal before, the belief about the current match decreases over time. As the separation time t^* approaches, the worker is afraid of being unemployed soon, so he adjusts his OJS strategy to raise the job finding rate. However, to raise the job finding rate, the worker must lower his OJS target because in an equilibrium labor market, only less promising jobs are less competitive and so have higher job finding rates. The following proposition formalizes the intuition above.

Proposition 12. *In the stationary equilibrium, the promised utility of the job targeted by a worker's OJS is increasing in his belief about his current match being good, and his job finding rate is decreasing in his belief.*

Owing to the lack of the data of workers' belief, it is hard to directly test the prediction in Proposition 12. However, this prediction is indirectly supported by the recent empirical finding by Fujita (2012). Fujita (2012) distinguishes workers who are searching on the job based on their motivation: some of them are unsatisfied about their current job, and others are afraid of losing their current jobs. He finds that (1) the job finding rate of the former is higher than that of the latter, and (2) after job transitions, the former' wage increment is significantly higher than the latter's.

EE Transition. An EE transition takes place only if an employed worker actively searches for and gets a new job. At any moment, this tenure-dependent transition rate is defined by

$$\xi_t^{ee} \equiv \sigma_t p\left(\theta\left(\alpha_t\right)\right).$$

The EE transition happens only if a worker is looking for another job. Given his tenure, the probability that a worker is engaging in OJS is σ_t. Conditional on that, the probability that he actually finds a new job is $p(\theta(\alpha_t))$. The tenure effect on ξ_t^{ee} is driven by two forces, the *job search target effect* and the *selection effect* in opposite directions. The former implies that the probability of OJS declines over tenure, and so does the EE transition rate; while the latter implies that the job finding rate increases over tenure, and so does the EE transition rate. Hence, these two effects drive the EE transition rate in opposite directions over tenure. In the following proposition, I show that the job search target effect dominates in the early stage of a worker's tenure, while the selection effect dominates later. The intuition is that, in the early stage of a worker's tenure, his belief is high, so in his target market of OJS, market tightness θ is very small. By the Inada condition of the matching function, $p' \to \infty$ for small θ. As a result, a tiny change in the worker's OJS strategy, θ, has a huge impact on his job finding rate, and therefore the job search target effect dominates the selection effect in the beginning and the EE transition rate is increasing in tenure. When it comes to the worker with a long tenure, the job search target effect is diminished, so the EE transition rate is decreasing in tenure. When a worker's tenure is long enough, all uncertainty is resolved and only good matches are kept, so the match quality must be good and therefore the EE transition rate is zero.

Proposition 13. *There is $\underline{t} > 0$, $\bar{t} < t^*$ such that (1) $\bar{t} > \underline{t}$, and the EE transition rate ξ_t^{ee} is increasing in tenure for $t < \underline{t}$, decreasing for $t \in [\bar{t}, t^*]$, and zero for $t > t^*$.*

EU Transition. The EU transition rate as a function of tenure is $\xi_t^{eu} = \delta$ for any $t \neq t^*$, when the EU transition happens only as a result of exogenous separation. At $t = t^*$, in addition to exogenous separations, all matches that did not send a good signal will be endogenously destroyed, the measure of which is positive. The atom of the EU rate results from the assumption of a precise and uniform learning process. If the learning process is heterogeneous as a consequence of either different priors or noisy observations, such an atom can be eliminated. The mass point in the EU transition rate showing at a particular tenure point is considered empirically irrelevant. However, it fits the observation in the academic job market, where learning is based on relatively uniform and precise information on the quality and quantity of research publications.

Separation Rate. Separation of an existing match may result from either EE or EU transition; hence, the separation rate of an existing match with tenure t, ξ_t must be $\xi_t = \xi_t^{ee} + \xi_t^{eu}$. When $t \neq t^*$, the EU transition rate δ is tenure-free, the tenure effect on the separation rate is almost identical to that on the EE rate. At $t = t^*$, a positive measure of matches will be separated. Just as for that of the EU transition rate, this mass point of separation hazard at t^* is also empirically irrelevant.

Corollary 2. *Generically, the job separation rate is increasing in tenure for $t < \underline{t}$, decreasing for $t \in [\bar{t}, t^*]$, and equal to δ for $t > t^*$.*

The prediction on the separation rate-tenure profile is consistent with the previous empirical literature. For example, using weekly data, Farber (1994) finds that in the first six months, the separation rate is increasing in tenure. In the current paper, to focus on the dynamics of the EE transition rate on the job separation rate, I assume that the EU rate is constant for most t, and the hump-shaped job separation rate results from the similar shape of the EE transition rate. To the best of my knowledge, there is no empirical study using weekly data to estimate the EE and the EU transition rate-tenure profile. However, there is empirical evidence that supports the non-monotone EE transition rate-tenure profile in the analysis of monthly data sets. Using the U.S. Census' Survey of Income and Program Participation (SIPP), Menzio, Telyukova and Visschers (2012) find that (1) the EU transition rate is decreasing in tenure, and (2) the EE transition rate increases in tenure in the first four months (from 3% to 5%), and decreases thereafter. Since the EE transition makes up 49% of all separations, while the EU transition makes up only 20%, the shape of the EE transition rate should contribute more to the separation rate. Hence, it is reasonable to believe that the hump-shaped separation rate-tenure profile is mainly due to the similar shape of the EE-tenure profile.

3.6 Concluding Remark

I conclude the paper by discussing some possible extensions.

Bad News Cases. In this paper, I focus on the perfect good news learning process. This is perfect reasonable for some industries, for example, academia. However, an obvious and natural alternative is to study the perfect bad news learning process. Assume that a match generates an quality-independent flow payoff is $y > 0$, and a good match generates no extra loss but an bad one generates a 1-unit loss at a rate λ. Furthermore, as in the good news model, I assume that a good match is better than no match, and no match is better than a bad match to avoid a trivial case where endogenous separation is never optimal, i.e., $y - (1 - \alpha_0)\lambda > b > y - \lambda$. In this case, when bad news is realized, the firm learns that the match is bad and therefore fires the worker immediately. By observing a history with no bad news, a matched firm and worker become more and more optimistic about their match quality. In this economy, any existing match has a belief α higher than α_0 in equilibrium; thus on-the-job search is not valuable. The equilibrium has only one labor market with market tightness $\theta\,(u)$.

Imperfect Good News Cases. In the benchmark model, I assume that a bad match cannot generate any profit, which seems restrictive. What if it can generate one unit of reward at a lower rate $\lambda_b \in (0, \lambda)$? To avoid a trivial case, I assume $\lambda_b < b < \alpha_0\lambda + (1-\alpha_0)\lambda_b$. In other words, a new match is better than no match, but no match is better than a bad match. Then, given no reward arriving in $[t, t+dt)$, the belief at the end of that time period is $\alpha_{t+dt} = \frac{\alpha_t \exp(-\lambda dt)}{\alpha_t \exp(-\lambda dt)+(1-\alpha_t)\exp(-\lambda_b dt)}$ by Bayes' rule. Yet, if one reward is realized in $[t, t+dt)$, the belief about the match quality jumps up from α_t to

$$\alpha_{t+dt} = \frac{\alpha_t[1-\exp(-\lambda dt)]}{\alpha_t[1-\exp(-\lambda dt)] + (1-\alpha_t)[1-\exp(-\lambda_b dt)]},$$

by Bayes' rule. When dt goes to zero, the updating can be approximated by

$$\dot{\alpha}_t = \lim_{dt\to 0} \frac{\alpha_{t+dt} - \alpha_t}{dt} = \begin{cases} -(\lambda - \lambda_b)\alpha_t(1-\alpha_t) & \text{no reward at } t, \\ \frac{\lambda\alpha_t}{\lambda\alpha_t + \lambda_b(1-\alpha_t)} & \text{one reward at } t, \end{cases}$$

and the probability that more than one reward is realized is $O(dt^2)$, which is negligible when dt is small. By the same logic, one can solve the social planner's optimal stopping belief and OJS strategy. Over time, good matches can survive with higher probability than bad ones due to the dynamics of endogenous separation driven by learning; thus, the empirical implications for job transitions still hold qualitively. However, the implications are slightly different from those in the benchmark model in the following sense: (1) No match is believed to be good for sure, and therefore, the endogenous separation will not disappear even for the match with long tenure. (2) The arrival of a reward can increase the belief about match quality; thus, it is possible that a belief $\alpha_t \in (\alpha_0, 1)$ appears in equilibrium. Clearly, it is inefficient to destroy a match with belief higher than α_0, and therefore, employed workers with belief $\alpha > \alpha_0$ will not search on-the-job under a bilaterally efficient contract. At the beginning of a match, the job search target effect works as well and dominates the selection effect. When the difference between λ and λ_b is sufficiently large, the learning process is similar to that in the perfect good news model. Hence, the non-monotonicity of the EE transition rate is preserved.

Informative Interview. In the benchmark model, the match is modeled as an experience good whose quality needs to be slowly learned over time. Yet, in some situations, the employer can extract non-trivial information about the match quality through an interview. Suppose a firm can draw an informative signal of the match quality and update its belief about the match quality through an interview before the match is formed. The signal is drawn from a match quality dependent distribution that satisfies the monotone likelihood ratio property (MLRP), and the updated posterior $\tilde{\alpha}_0 \in [\underline{\alpha}_0, \bar{\alpha}_0]$, where $0 < \underline{\alpha}_0 < \bar{\alpha}_0 < 1$. In this extension, the social planner will form a new match only if the updated posterior $\tilde{\alpha}_0$ is higher than a cutoff level that depends on the worker's current state. For an unemployed worker, this cutoff is the stopping time belief α_*. For an employed worker, this cutoff is the belief α_t about the worker's current match quality. Let $\Pr(\tilde{\alpha}_0 > \alpha_t)$ be the ex ante probability that the posterior is larger than the worker's current belief. Hence, the on-the-job search policy is determined by

$$\max_{\theta} p(\theta) \Pr(\tilde{\alpha}_0 > \alpha_t)\{\mathbb{E}[S(\tilde{\alpha}_0)|\tilde{\alpha}_0 > \alpha_t] - S(\alpha_t)\} - k\theta.$$

It is clear that both $\Pr(\tilde{\alpha}_0 > \alpha_t)$ and $\mathbb{E}[S(\tilde{\alpha}_0)|\tilde{\alpha}_0 > \alpha_t] - S(\alpha_t)$ are non-increasing in α_t; thus the optimal policy $\theta(\alpha)$ is non-increasing in α, which is similar to that in the benchmark model. Hence the empirical implications for workers' turnover predicted by the benchmark model are qualitively preserved.

Costly On-the-Job Search. Suppose workers' on-the-job search requires a flow cost ϵdt. To avoid a trivial case where OJS is always suboptimal, I assume that ϵ is small enough. Since the gain from on-the-job search $\max_\theta\{p(\theta)[S(\alpha_0) - S(\alpha)] - k\theta\}$ is increasing in the job replacement premium, $[S(\alpha_0) - S(\alpha)]$, for small enough ϵ, there exists a cut-off belief α^ϵ such that

$$\begin{aligned}
\max_{\theta} p(\theta)[S(\alpha_0) - S(\alpha)] - k\theta &\leq \epsilon \text{ if } \alpha \geq \alpha^\epsilon, \\
\max_{\theta} p(\theta)[S(\alpha_0) - S(\alpha)] - k\theta &> \epsilon \text{ if } \alpha < \alpha^\epsilon.
\end{aligned}$$

In other words, matched workers would search on-the-job only if they believed the match quality is low enough. When $\alpha < \alpha^\epsilon$, the social planner's problem is unchanged, and therefore, it is obvious that introducing costly OJS does not change the main result but reduces the social surplus $S(\alpha)$ for each α and therefore shortens the duration of experimentation.

Appendix A

Appendix for Chapter 1

A.1 Appendix: Strategy and Equilibrium

A.1.1 Admissible Strategy Space

In general, the seller's strategy is a mapping from the set of the seller's history to the price and target sold number.

$$\sigma_S : \mathcal{H}_S \to \mathbb{R}^+ \times \mathbb{N}.$$

For each H-buyer i, let the index function, $a_i(t)$, denote her attention status at t. It is equal to 1 at her attention time, and 0 at other times. At her attention time, the H-buyer can decide to purchase the good or not. At the time she decides to purchase the good, we let the index function, $b_i(t) = 1$; at other times, $b_i(t) = 0$. Let $\omega_i(t) = \{a_i(t), b_i(t)\}$, Ω_i be the set of all $\omega_i(t)$, and $\Omega =_{i=1}^{\infty} \Omega_i$. A non-trivial private history of an H-buyer i at her attention time t is

$$h_{Bi}^{t}=\left\{\left\{a_{i}\left(\tau\right)\right\}_{\tau=0}^{t},\left\{P\left(\tau\right),Q\left(t\right),K\left(\tau\right)\right\}_{\tau\in\left\{\tau'|\tau'\in[0,t],a_{i}\left(\tau'\right)=1\right\}}\right\}$$

and let $\mathcal{H}_B$ represent the set of all non-trivial private histories of an H-buyer. The H-buyers' strategy at their attention time is

$$\sigma_B:\mathcal{H}_B\rightarrow[0,1].$$

Denote the underlying outcome by $o\left(t\right)=\left\{P\left(t\right),Q\left(t\right),K\left(t\right),\left\{\omega_{i}\left(t\right)\right\}_{i=1}^{\infty}\right\}$, and let o^t be the underlying history. Given an underlying outcome, players' expected payoff can be calculated.

A metric on the sets of the seller's history is defined as

$$D\left(h_{S}^{t},\tilde{h}_{S}^{t}\right)=\int_{0}^{t}\left[||P\left(s\right),\tilde{P}\left(s\right)||+||Q\left(s\right),\tilde{Q}\left(s\right)||+||K\left(s\right),\tilde{K}\left(s\right)||\right]ds.$$

define the metric on the sets of Ω as follows: for $\omega,\tilde{\omega}\in\Omega$,

$$D_{i}\left(\omega_{i},\tilde{\omega}_{i},[0,t]\right)=\int_{0}^{t}\left[||b_{i}\left(s\right),\tilde{b}_{i}\left(s\right)||+||a_{i}\left(s\right),\tilde{a}_{i}\left(s\right)||\right]ds,$$

and

$$D\left(\omega,\tilde{\omega},[0,t]\right)=\sum_{i=1}^{\infty}D_{i}\left(\omega_{i},\tilde{\omega}_{i},[0,t]\right).$$

A metric on the sets of the underlying outcome is defined as

$$D\left(o^t, \tilde{o}^t\right) = D\left(h_S^t, \tilde{h}_S^t\right) + D\left(\omega, \tilde{\omega}, [0,t]\right)$$

where $||\cdot||$ is the Euclidean norm. Let $\mathcal{B}_{\mathcal{H}_S}, \mathcal{B}_{\mathcal{H}_B}$ be Boreal σ-algebra determined by D.

Condition 3. *σ_S is a $\mathcal{B}_{\mathcal{H}_S}$ measurable function and σ_B is a $\mathcal{B}_{\mathcal{H}_B}$ measurable function.*

Condition 4. *For all $t \in [0,1]$ and $h_S^t, \tilde{h}_S^t \in \mathcal{H}_S$ such that $D\left(h_S^t, \tilde{h}_S^t\right) = 0$, $\sigma_S\left(h^t\right) = \sigma_S\left(\tilde{h}^t\right)$.*

The first condition is a technical one, and the second condition implies that, if two seller histories are the same almost surely, the strategy should specify the same price and target supply.

Definition 7. *A seller's strategy σ_S satisfies the **inertia condition** if given $t \in [0,1)$, there exists an $\varepsilon > 0$ and a constant pricing and supply rule that*

$$\sigma_S\left(\tilde{h}_S^{t'}\right) = \sigma_S\left(h_S^t\right) \quad \forall t' \in (t, t+\varepsilon)$$

for every $\tilde{h}_S^{t+\varepsilon} \in \mathcal{H}$ such that $D\left(h_S^t, \tilde{h}_S^t\right) = 0$.

Note ε is time dependent. This assumption requires that at every time, players must follow a fixed rule specified by σ_S for a small period of time. The key is that the seller's action in $[t, t+\varepsilon)$ cannot depends on each others' action directly almost everywhere. This implies that the seller can vary his actions rule a countably number of times. Let Σ be the set of all σ satisfying inertia conditions.

The following proposition shows that if $\sigma_S \in \Sigma$, a strategy profile (σ_S, σ_B) determines a unique distribution on the underlying outcome space. The spirit of the proof is similar to the proof of Theorem 1 in Bergin and MacLeod (1993).

Proposition 14. *A strategy profile σ generates a unique distribution over the underlying outcome if $\sigma_S \in \Sigma$.*

Proof. Fixed an underlying outcome o, and $t \in [0,1]$, we want to show that there is a unique distribution $\Gamma \in \Delta\left(\{o_s\}_{s\in[t,1]}\right)$ generated by σ. When $t = 1$, it is trivially given by condition 2. When $t < 1$, the underlying outcome o^t and associated history $h_S^t, \{h_{Bi}^t\}_{i=1}^{\infty}$ are given. Let A_τ be the set of distribution on $\{o_s\}_{s\in[t,\tau)}$ which can be generated by σ. Since players' actions are determined by $\sigma_S\left(h_S^t\right)$ and $\sigma_B\left(h_B^t, P(h_S^t), K(s)\right)$ and the uncertainty is driven by the arrival of new buyers and the selection of the seller when there are sales, A_τ is a singleton when $\tau \in (t, t+\varepsilon)$.

Now we claim A_τ is a singleton for any $\tau \in [t,1]$. Suppose not; then there exists a t^* which is the largest τ such that there is a unique distribution on $\{o_s\}_{s\in[t,\tau)}$ which is generated by σ. By the inertia condition, there is an $\varepsilon > 0$ which depends on τ such that there is a unique distribution on $\{o_s\}_{s\in[\tau,\tau+\varepsilon)}$ generated by σ, which is a contradiction with the definition of t^*.

Since A_τ is nonempty, by the inertia condition, there is a $\varepsilon > 0$ such that $A_{\tau+\varepsilon}$ is a singleton. Proceed iteratively in this way, constructing a unique outcome on $[0,t_1), [0,t_2), \ldots$ with $t_j > t_j{-}1$. The upper bound on this process is at 1; otherwise, there is a contradiction by the definition of the inertia strategy. As a result, there is a unique distribution $\Gamma \in \Delta\left(\{o_s\}_{s\in[t,1]}\right)$ generated by σ.

□

However, the seller cannot respond "instantaneously" to any defection by playing the inertia strategy. So we take the completion of Σ with respect to a metric that measures the underlying outcome distribution induced by a different strategy.

$$d\left(\sigma_S, \sigma_S'\right) = \sup_{t \in [0,1]} \left\{ \sup_{B_t \in \mathcal{F}_t} \left| \Gamma\left(B_t\right) - \Gamma'\left(B_t\right) \right| \right\}$$

where $\{\mathcal{F}_t\}_{t \in [0,1]}$ is the filtration generated by the underlying outcome, so $\mathcal{F}_t$ is the σ-algebra describing the time t outcome, and B_t is a measurable event at time t.

Two inertia strategies are equivalent if they generate the same distribution over the underlying outcome space. Denote by Σ_S^* the completion of Σ_s relative to d. Hence, each strategy $\sigma_S \in \Sigma^*$ corresponds to some Cauchy sequence in Σ. Theorem 2 in Bergin and MacLeod (1993) immediately implies that, for each $\sigma_S \in \Sigma_S^*$, there is a sequence $\{\sigma_S^n\}$ where $\sigma_S^n \in \Sigma_S$ and $\sigma_S^n \to \sigma_S$, such that there is a sequence of distribution $\{\Gamma^n\}$ where $\Gamma^n \in \Delta\left(\{o_\tau\}_{\tau \in [t,1]}\right)$ is the unique distribution generated by σ_S^n, σ_B and $\Gamma^n \to \Gamma \in \Delta\left(\{o_\tau\}_{\tau \in [t,1]}\right)$. Hence, we say σ can be identified with a unique outcome $\Gamma \in \Delta\left(\{o_s\}_{s \in [t,1]}\right)$. We say a strategy σ is *admissible* if and only if $\sigma \in \Sigma^*$. *Q.E.D.*

A.1.2 On No-Waiting Equilibria

We focus on no-waiting equilibria where buyers believe that no previous H-buyers are waiting in the market both on and off the equilibrium path. We justify this assumption in the following two cases of deviation: "wrong" price and "wrong" inventory size. First, when an H-buyer observes one or more deviation prices, she believes that the seller posts the equilibrium prices always except for at some of her past attention times and the seller's estimation about the population structure of buyers is still the equilibrium one, and therefore the seller would follow the equilibrium pricing rule in the continuation play. The second case is one in which the seller is supposed to post a deal at a sales time. Since there exist many L-buyers who can take the deal immediately, the seller can ensure that his inventory size is consistent with the equilibrium requirement at all times by following the equilibrium strategy. If an H-buyer observes a "wrong" inventory size after the supposed sales time, she knows that there has been a deviation on the seller's side for a positive measure of time, but does not know what prices the seller has been posting. If the prices have been acceptable to H-buyers, there will be no other H-buyers waiting in the market; otherwise, there may be. In fact, we assume that, once an H-buyer observes a "wrong" inventory size, she always believes that the deviation prices have been acceptable to previous H-buyers, and thus, off the path of play, she still believes that no other H-buyers are waiting in the market, and the seller's continuation play is going to be consistent with the equilibrium strategy.

A.2 Appendix: Proofs

A.2.1 Belief Updating

In this subsection, we derive the law of motion of the seller's belief, $\Phi^{+}(t)$ and $\Phi^{-}(t)$.

At $t = l\Delta$, for $n \in \mathbb{N}$, $\Phi_n^{-}(t) = 0$ for any $l \in \{0, 1, 2, ..1/\Delta\}$. For any $t \in$

$((l-1)\Delta, l\Delta)$, the updating depends on whether the price at time t is acceptable to the H-buyer who notices it. Given H-buyer's strategy σ_B. As a result, if the price is *not acceptable* to H-buyers in $[t, t+dt)$, Bayes' rule implies that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\Phi_n^-(t+dt) &= \Phi_{n-1}^-(t)\left\{\lambda dt + \sum_{n'=1}^{\infty} \Phi_{n'}^+(t)\binom{n'}{1}\left(\frac{dt}{l\Delta - t}\right)^1\left(1-\frac{dt}{l\Delta - t}\right)^{n'-1}\right\} \\
&\quad +\Phi_n^-(t)\left\{1-\lambda dt - \sum_{n'=1}^{\infty} \Phi_{n'}^+(t)\binom{n'}{1}\left(\frac{dt}{l\Delta - t}\right)^1\left(1-\frac{dt}{l\Delta - t}\right)^{n'-1}\right\} \\
&\quad +o(dt),
\end{aligned}$$

where $\binom{i}{j}\left(\frac{dt}{l\Delta-t}\right)^j\left(1-\frac{dt}{l\Delta-t}\right)^{i-j}$ denotes the probability that j of i H-buyers whose attention times are in $[t, t+dt)$. They notice the offer but decide not to purchase the good. Thus we have the endogenous updating equation of $\Phi^-(t)$:

$$\dot{\Phi}_n^-(t) = \lim_{dt\to 0}\frac{\Phi_n^-(t+dt) - \Phi_n^-(t)}{dt} = \left[\sum_{n'=1}^{\infty}\frac{n'}{l\Delta - t}\Phi_{n'}^+(t) + \lambda\right]\left[\Phi_{n-1}^-(t) - \Phi_n^-(t)\right].$$

If the price is *acceptable*, by assuming that buyers follow the equilibrium strategy, there is no change in $\Phi_n^-(t)$.

At $t = l\Delta$, for any $n \in \mathbb{N}$, $\Phi_n^+(t) = \Phi_n^-(t^-)$ for any $l \in \{0, 1, 2, ..1/\Delta\}$. For any $t \in ((l-1)\Delta, l\Delta)$, if there is no transaction, we can derive the law of motion of $\Phi_n^+(t)$ similar to deriving that of $\Phi_n^-(t)$. Also, the law of motion of $\Phi_n^+(t)$ depends on whether the price is acceptable to an H-buyer or not. If the price is *acceptable*, but there is no transaction, then for any $n \in \mathbb{N}$,

$$\begin{aligned}
\dot{\Phi}_n^+(t) &= \lim_{dt \to 0} \frac{\Phi_n^+(t+dt) - \Phi_n^+(t)}{dt} \\
&= \lim_{dt \to 0} \left\{ \frac{\Phi_n^+(t)\left[1 - \frac{dt}{l\Delta - t}\right]^n - \Phi_n^+(t)\,\Sigma_{n'=0}^{\infty}\Phi_{n'}^+(t)\left[1 - \frac{dt}{l\Delta - t}\right]^{n'}}{dt\sum_{n'=0}^{\infty}\Phi_{n'}^+(t)\left[1 - \frac{dt}{l\Delta - t}\right]^{n'}} \right\} \\
&= \lim_{dt \to 0} \left\{ \frac{-\Phi_n^+(t)}{l\Delta - t} \frac{n - \Sigma_{n'=0}^{\infty}\Phi_{n'}^+(t)\,n' + o(dt)}{\Sigma_{n'=0}^{\infty}\Phi_{n'}^+(t)\left[1 - \frac{n'dt}{l\Delta - t}\right] + o(dt)} \right\} \\
&= \frac{-\Phi_n^+(t)}{l\Delta - t}\left[n - \mathbb{E}N^+(t)\right].
\end{aligned}$$

If the price is *not acceptable*, then

$$\begin{aligned}
\dot{\Phi}_n^+(t) &= \lim_{dt \to 0} \frac{\Phi_n^+(t+dt) - \Phi_n^+(t)}{dt} \\
&= \lim_{dt \to 0} \left\{ \frac{\Phi_{n+1}(t)\binom{n+1}{1}\left(\frac{dt}{l\Delta - t}\right)^1\left(1 - \frac{dt}{l\Delta - t}\right)^n + \Phi_n(t)\left(1 - \frac{dt}{l\Delta - t}\right)^n - \Phi_n^+(t) + o(dt)}{dt} \right\} \\
&= \frac{1}{l\Delta - t}\left[(n+1)\,\Phi_{n+1}(t) - n\Phi_n(t)\right].
\end{aligned}$$

If there is a transaction at a price $P(t) > v_L$, it must be an H-buyer who makes the purchase, so we have a belief jump following Bayes' rule:

$$\begin{aligned}
\Phi_n^+\left(t^+\right) &= \lim_{dt \to 0} \frac{\Phi_{n+1}^+(t)\,\frac{(n+1)dt}{l\Delta - t} + \Phi_n^+(t)\,\lambda dt + o(dt)}{\sum_{n'=0}^{\infty}\Phi_{n'}^+(t)\left(\frac{n'dt}{l\Delta - t} + \lambda dt\right) + o(dt)} \\
&= \frac{(n+1)\,\Phi_{n+1}^+(t) + \Phi_n^+(t)\,\lambda\,(l\Delta - t)}{\sum_{n'=0}^{\infty}\left[\Phi_{n'}^+(t)\,n'\right] + \lambda\,(l\Delta - t)} \\
&= \frac{(n+1)\,\Phi_{n+1}^+(t) + \Phi_n^+(t)\,\lambda\,(l\Delta - t)}{\mathbb{E}\left[N^+(t)\right] + \lambda\,(l\Delta - t)},
\end{aligned}$$

for any $n \in \mathbb{N}$.

If there is a transaction at a deal price $P(t) \leq v_L$, it may be an L-buyer or an H-buyer who made the purchase, so the updating of $\Phi(t)$ would depend on the current belief of the number of L-buyers. Let $\Upsilon_m(t)$ denote the seller's belief that $M(t) = m$ at time t. At the beginning of each period, $\Upsilon_M(t) = 1$, but $M(t)$ may change within a period because some L-buyers may leave the market by making their purchases at deal prices. Within a period, after the first deal at time t, we have

$$\Upsilon_M\left(t^+\right) = \sum_{n=0}^{\infty} \Phi_n(t) \frac{n}{M+n}, \Upsilon_{M-1}\left(t^+\right) = \sum_{n=0}^{\infty} \Phi_n(t) \frac{M}{M+n}, \tag{A.1}$$

and $\Upsilon_{M-i}(t^+) = 0$ for $i = 2, 3, ...M$. After the k^{th} deal at time t, we have:

$$\Upsilon_M\left(t^+\right) = \Upsilon_M(t) \sum_{n=0}^{\infty} \Phi_n(t) \frac{n}{M+n}, \tag{A.2}$$

$$\Upsilon_{M-i}\left(t^+\right) = \Upsilon_{M-i}(t) \sum_{n=0}^{\infty} \Phi_n(t) \frac{n}{M-i+n} + \Upsilon_{M-i+1}(t) \sum_{n=0}^{\infty} \Phi_n(t) \frac{M-i+1}{M-i+1+n}, \tag{A.3}$$

for $i = 1, 2, ...k$, and

$$\Upsilon_{M-k-i}\left(t^+\right) = 0 \text{ for } i = 1, 2, ..M-k. \tag{A.4}$$

Similarly, the belief of $N^{-}(t)$ and $N^{+}(t)$ will also jump as follows:

$$\begin{aligned}\Phi_{n}^{+}\left(t^{+}\right) &= \Phi_{n}^{+}(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^{-}(t)\Upsilon_{m}(t)\frac{m+n'}{n+m+n'}\\ &\quad +\Phi_{n+1}^{+}(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^{-}(t)\Upsilon_{m}(t)\frac{n+1}{n+1+m+n'},\\ \Phi_{n}^{-}\left(t^{+}\right) &= \Phi_{n}^{-}(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^{+}(t)\Upsilon_{m}(t)\frac{m+n'}{n+m+n'}\\ &\quad +\Phi_{n+1}^{-}(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^{+}(t)\Upsilon_{m}(t)\frac{n+1}{n+1+m+n'},\end{aligned}$$

for any $n \in \mathbb{N}$.

The law of motion of the seller's belief can be summarized in the following proposition. Denote t_{k}^{d} be the d^{th} deal times within the period.

Proposition 15. *Let $P(t)$ be the price at t. The seller's beliefs $\Phi^{+}(t)$ and $\Phi^{-}(t)$ update as follows: for any $n \in \mathbb{N}$,*

1. *at $t = l\Delta$, $\Phi_{n}^{-}(t) = 0$. For any $t \in ((l-1)\Delta, l\Delta)$, $\Phi^{-}(t)$ smoothly evolves s.t.*

$$\dot{\Phi}_{n}^{-}(t) = \left[1-\sigma_{B}(P(t))\right]\left[\sum_{n'=1}^{\infty}\frac{n'}{l\Delta - t}\Phi_{n'}^{+}(t)+\lambda\right]\left[\Phi_{n-1}^{-}(t)-\Phi_{n}^{-}(t)\right],$$

2. *at $t = l\Delta$, $\Phi_{n}^{+}(t) = \Phi_{n}^{-}(t^{-})$. For any $t \in ((l-1)\Delta, l\Delta)$, if there is no transaction, $\Phi^{+}(t)$ smoothly evolves s.t.*

$$\begin{aligned}\dot{\Phi}_{n}^{+}(t) &= \left[1-\sigma_{B}(P(t))\right]\left\{\frac{1}{l\Delta - t}\left[(n+1)\Phi_{n+1}(t)-n\Phi_{n}(t)\right]\right\}\\ &\quad -\sigma_{B}(P(t))\frac{\Phi_{n}^{+}(t)}{l\Delta - t}\left[n-\mathbb{E}N^{-}(t)\right],\end{aligned}$$

3. *at $t \in [0,1)$, if there is a transaction at a price $P(t) \leq v_L$, $\Phi^+(t)$ and $\Phi^-(t)$ jump as follow:*

$$\begin{aligned}
\Phi_n^+\left(t^+\right) &= \Phi_n^+(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^-(t)\,\Upsilon_m(t)\,\frac{m+n'}{n+m+n'} \\
&\quad +\Phi_{n+1}^+(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^-(t)\,\Upsilon_m(t)\,\frac{n+1}{n+1+m+n'}, \\
\Phi_n^-\left(t^+\right) &= \Phi_n^-(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^+(t)\,\Upsilon_m(t)\,\frac{m+n'}{n+m+n'} \\
&\quad +\Phi_{n+1}^-(t)\sum_{n'=0}^{\infty}\sum_{m=0}^{M}\Phi_{n'}^+(t)\,\Upsilon_m(t)\,\frac{n+1}{n+1+m+n'}
\end{aligned}$$

where $\Upsilon_m(t)$ is the seller's belief about $M(t)$ and its law of motion is given in (A.1), (A.2), (A.3),and (A.4) for the k^{th} deal, and

4. *for any $l = 1, 2, ..1/\Delta$, $t \in [(l-1)\Delta, l\Delta)$, if there is a transaction at a price $P(t) > v_L$, $\Phi^+(t)$ jumps as follows:*

$$\Phi_n^+\left(t^+\right) = \frac{(n+1)\,\Phi_{n+1}^+(t) + \Phi_n^+(t)\,\lambda\,(l\Delta - t)}{\mathbb{E}\left[N^+(t)\right] + \lambda\,(l\Delta - t)},$$

where $\Phi_{-1}^-(t) = 0$, $l \in \{0, 1, 2, ..1/\Delta\}$.

A.2.2 Proofs for the Single-Unit Case

A.2.2.1 Equilibria Construction

We construct an equilibrium such that the following conditions hold: (1) the seller posts a price $P(t)$ such that an H-buyer is indifferent between taking and leaving it for $t < 1$, (2) an H-buyer makes the purchase once she arrives, and (3) $P(1) = v_L$ is posted at the deadline. Construct the H-buyers' reservation price. At the deadline, it is obviously v_H. Since the seller posts $P(1) = v_L$ in any equilibrium, at $t \in [1-\Delta, 1)$, the H-buyers' reservation price is

$$\begin{aligned} v_H - p_1(t) &= e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1}, \\ \dot{p}_1(t) &= -\lambda [v_H - p_1(t)]. \end{aligned}$$

As $t \to 1$, $p_1(t) \to p_1(1^-)$. Differentiating $p_1(t)$ yields

$$\dot{p}_1(t) = -\lambda e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1} = -\lambda [v_H - p_1(t)]$$

with a boundary condition $p_1(1^-)$ at $t = 1$. Let $U_{1-\Delta}$ be an H-buyer's expected payoff at the beginning of the last period. The expectation is over the random attention time, and the risk of arrival of new buyers. Hence

$$\begin{aligned} U_{1-\Delta} &= \int_{1-\Delta}^{1} \frac{1}{\Delta} e^{-\lambda(s-1+\Delta)} [v_H - p_1(s)] \, ds \\ &= e^{-\lambda\Delta} \frac{v_H - v_L}{M+1} = v_H - p_1(1-\Delta). \end{aligned}$$

Consider a t that is smaller than but arbitrarily close to $1-\Delta$. At this attention time, an H-buyer's reservation price is

$$\begin{aligned} v_H - p_1(t) &= e^{-\lambda(1-\Delta-t)} U_{1-\Delta}, \\ \dot{p}_1(t) &= -\lambda [v_H - p_1(t)]. \end{aligned}$$

As $t \to 1-\Delta$, we have $\lim_{t \nearrow 1-\Delta} p_1(t) = p_1(1-\Delta)$, thus $p_1(t)$ is differentiable at $1-\Delta$. Repeating the above argument for $1/\Delta$ times, the reservation price $p_1(t)$ is differentiable in $[0,1)$ and satisfies the ODE (1.5) with the boundary condition (1.2).

The deal price is posted at the deadline only, and H-buyers do not delay their purchases, so neither the H-buyers' reservation price nor the seller's equilibrium profit depends on Δ. The closed-form solution of $p_1(t)$ and $\Pi_1(t)$ are given by

$$\begin{aligned} p_1(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)}, \\ \Pi_1(t) &= \left[1 - e^{-\lambda(1-t)}\right] v_H + e^{-\lambda(1-t)} v_L - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \lambda (1-t). \end{aligned}$$

In sum, the equilibrium strategy profile (σ_S^*, σ_B^*) is given as follows. The seller's equilibrium strategy $\sigma_S^*(t, \Phi^-(t), \Phi^+(t)) = p_1(t)$ for any $[\Phi^-(t), \Phi^+(t)] \in \Xi_S$ and $t < 1$, and $\sigma_S^*(1) = v_L$. The H-buyers' equilibrium strategy σ_B^* satisfies $\sigma_B^* = \mathbf{1}_{\{P(t) \le p_1(t), t \in [0,1)\}} + \mathbf{1}_{[P(1) \le v_H]}$.

A.2.2.2 The Proof of Proposition 1

We prove Proposition 1 step by step. A simple observation is that, given the seller's equilibrium strategy, H-buyers do not have an incentive to deviate since they are indifferent everywhere. To ensure the existence of the conjecture equilibrium, we only need to rule out deviations by the seller. We show that the seller has no incentive to post unacceptable prices for a positive measure of time. As a result, the seller has no profitable deviation. Since the construction of $p_1(t)$ is unique, there is no other equilibria in addition to the equilibrium we proposed.

Suppose the seller follows the equilibrium strategy. His expected profit satisfies the following equation:

$$\Pi_1(t) = \int_t^1 e^{-\lambda(s-t)} \lambda p_1(s)\, ds + e^{-\lambda(1-t)} v_L.$$

Taking the derivative with respect to time yields,

$$\dot{\Pi}_1(t) = -\lambda e^{-\lambda(1-t)} (v_H - v_L) \frac{M + \lambda(1-t)}{M+1} < 0.$$

Now we show that the seller's best response is indeed to post $P(t) = p_1(t)$ for $t < 1$ and $P(1) = v_L$. The proof is given step by step.

Step 1. At the deadline, it is the seller's dominant strategy to post $P(1) = v_L$. **Step 2**. At any time, a price $P(t) < p_1(t)$ is dominated by $p_1(t)$. **Step 3**. We claim that the seller has no incentive to post unacceptable prices for a positive measure of time. Suppose not, and the seller posts $P(t) > p_1(t)$ for $t \in [t', t'')$. We claim that such strategy is dominated by an alternative strategy: replacing $P(t)$ by $p_1(t)$ for $t \in [t', t'')$ but keep playing the original continuation strategy. To see the reason, consider two cases. **Case 1**: there is no arrival in $[t, t')$. In this case, the seller is indifferent between two strategies. **Case 2**: some H-buyers arrive in $[t, t')$. In this case, if the seller adopts the original strategy, his expected payoff is less than $p_1(t'')$, since (1) the H-buyer's reserve (acceptable) price before the deadline, $p_1(\cdot)$, is decreasing in time and (2) the seller's payoff at the deadline is v_L. If the seller adopts the alternative strategy, his expected payoff is $p_1(\tau_1)$ where τ_1 is a random time at which the first H-buyer arrives in $[t, t')$. Since $p_1(\tau_1) \geq p_1(t'')$, for any history, the original strategy is dominated by the alternative one. In general, the argument ensures that the seller has no incentive to post unacceptable prices in finite many positive measure time-intervals. Hence, the seller has no incentive to adopt the deviation strategy by posting unacceptable price for a positive measure of time. Apparently, any $P(t) < p_1(t)$ for a positive measure of time is dominated by the equilibrium pricing rule. Consequently, it is the seller's best response to post $P^*(t) = p_1(t)$ for $t < 1$, and $P^*(t) = v_L$, and our conjecture equilibrium is an equilibrium. By construction, $p_1(t)$ is unique, so there is no other equilibrium.

Q.E.D.

A.2.3 Proofs for the Two-Unit Case

At any t such that $K(t) = 1$, the problems are the same as in the case where $K = 1$; hence, $p_1(t)$ and $\Pi_1(t)$ remain in the same form, and so does $U^1_{l\Delta}$. At $t = 1$ and $K(1) = 2$, the seller posts $p_2(1) = v_L$ for sure. Now we need to look at the case where $t < 1$ and $K(t) = 2$.

A.2.3.1 The Proof of Lemma 1

Suppose not. Since v_L is posted only at the deadline, the seller's equilibrium profits at the deadline are given by

$$\Pi_k(1) = kv_L, k = 1, 2.$$

and $p_k(t)$, the reservation price at $k = 1, 2$, is post to serve H-buyers only at any $t < 1$. Specifically,

$$\begin{aligned} p_2(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[2 + \lambda(1-t)\right], \text{ and} \\ p_1(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \end{aligned}$$

Define $\tilde{\Pi}_2(t)$ as the seller's profit if $p_2(t)$ is always posted when $t < 1$ and $K(t) = 2$, then

$$\begin{aligned} \tilde{\Pi}_2(t) &= \int_t^1 \lambda e^{-\lambda(s-t)} \left[p_2(s) + \Pi_1(s)\right] ds + 2v_L e^{-\lambda(1-t)} \\ &= 2v_H - 2(v_H - v_L) e^{-\lambda(1-t)} \\ &\quad - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[\lambda(1-t)(M+3) + \lambda^2(1-t)^2\right]. \end{aligned}$$

Immediately,

$$\begin{aligned} & \tilde{\Pi}_2(t) - \left[v_L + \Pi_1(t)\right] \\ = \; & (v_H - v_L)\left(1 - 2e^{-\lambda(1-t)}\right) \\ & + \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[M + 1 - \lambda(1-t)(M+2) - \lambda^2(1-t)^2\right]. \end{aligned}$$

Though this difference is not monotone, using a Taylor expansion and algebra, there are two cases: (i) either $\tilde{\Pi}_2(t) - \left[v_L + \Pi_1(t)\right] < 0$ for all $t < 1$ when $\tilde{\Pi}_2(0) < v_L + \Pi_1(0)$, (ii) or, if $\tilde{\Pi}_2(0) > v_L + \Pi_1(0)$, $\exists\, t^* < 1$ s.t. $\tilde{\Pi}_2(t^*) = v_L + \Pi_1(t^*)$ and $\tilde{\Pi}_2(t) < v_L + \Pi_1(t)$ for $t \in (t^*, 1)$. *Q.E.D.*

A.2.3.2 The Proof of Proposition 2

Equilibrium Construction. We first construct the H-buyers' reservation price. Suppose that all buyers believe that the seller posts a deal $p_2(t_1^*) = v_L$ at $t_1^* < 1$ if $K(t_1^*) = 2$, the H-buyer's reservation price $p_2(t)$ before t_1^* if $K(t) = 2$, and $p_1(t)$ at any t s.t. $K(t) = 1$. Any H-buyer believes that she is the only one in the market and she accepts any price that is not higher than the reservation price. Similar to the single unit case, the H-buyer's reservation price $p_2(t)$ when $t \in [t_1^*, 1)$ satisfies:

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1} e^{-\lambda(l\Delta - t)} U_{l\Delta}^1,$$

where, as in the single-unit case, $U_{l\Delta}^1 = e^{-\lambda(1-l\Delta)} \frac{v_H - v_L}{M+1}$ and $t \in [(l-1)\Delta, l\Delta)$ for some $l < 1/\Delta$, hence

$$\begin{aligned} p_2(t) &= \frac{Mv_H + v_L}{M+1} - \frac{M}{(M+1)^2}(v_H - v_L) e^{-\lambda(1-t)} \\ &= \frac{M}{M+1} p_1(t) + \frac{1}{M+1} v_L, \text{ for } t \in [t_1^*, 1-\Delta). \end{aligned}$$

Observe that $\dot{p}_2 = M/(M+1)\dot{p}_1$ for $t \in [t_1^*, 1-\Delta)$. For $t \in [1-\Delta, 1)$,

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1} e^{-\lambda(1-t)} \frac{v_H - v_L}{M},$$

and $p_2(t) < \frac{M}{M+1} p_1(t) + \frac{1}{M+1} v_L$ since no new L-buyer will enter and an H-buyer's reservation price is $\tilde{p}_1(t) = v_H - e^{-\lambda(1-t)} \frac{v_H - v_L}{M}$ in this case. To construct the equilibrium, we study the auxiliary problem in which $p_2(t) = \frac{M}{M+1} p_1(t) + \frac{1}{M+1} v_L$ for $t \in [t_1^*, 1)$, and show that the seller's optimal fire sale time is $t_1^* < 1 - \Delta$ in this auxiliary problem when Δ is small. Furthermore, we argue that the seller's optimal fire sale time is also equal to t_1^* in the problem where $p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1}e^{-\lambda(1-t)}\frac{v_H - v_L}{M}$ for $t \in [1-\Delta, 1)$.

If $t < t_1^*$, then for some l, $t \in [(l-1)\Delta, l\Delta) \cap [0, t_1^*)$. If $l\Delta \geq t_1^*$, then $p_2(t)$ satisfies:

$$v_H - p_2(t) = e^{-\lambda\left(t_1^* - t\right)}U_{t_1^*}^2 + \lambda\left(t_1^* - t\right)e^{-\lambda\left(t_1^* - t\right)}e^{-\lambda\left(l\Delta - t_1^*\right)}U_{l\Delta}^1,$$

where $U_{t_1^*}^2 = v_H - p_2(t_1^*)$. Otherwise, if $l\Delta < t_1^*$, then

$$v_H - p_2(t) = e^{-\lambda(l\Delta - t)}U_{l\Delta}^2 + \lambda(l\Delta - t)e^{-\lambda(l\Delta - t)}U_{l\Delta}^1,$$

where $U_{l\Delta}^2 = v_H - p_2(l\Delta)$. In either of the two cases, we have

$$p_2(t) = v_H - \frac{v_H - v_L}{M+1}e^{-\lambda(1-t)}\left[e^{\lambda\left(1-t_1^*\right)} + \frac{M}{M+1} + \lambda\left(t_1^* - t\right)\right] < p_1(t),$$

for $t \in [0, t_1^*)$, and $\dot{p}_2(t) = -\lambda(p_1(t) - p_2(t))$ for $t \in [0, t_1^*)$. Note that $p_2(\cdot)$ is continuous on $[0, 1]$.

In fact, for any buyer's belief on t_1^*, $p_2(\cdot)$ depends on t_1^* through the boundary condition at t_1^* only but does not depend on Δ.

Now we consider the seller's problem. Given the buyer's reservation price $p_2(\cdot)$ based on the belief of t_1^*, the seller chooses the actual deal time, with $p_2(\cdot)$ forced to be the pricing strategy before the deal time. Hence,

$$\Pi_2(t) = \max_{t_1} \int_t^{t_1} e^{-\lambda(s-t)} \lambda \left[p_2(s) + \Pi_1(s)\right] ds + e^{-\lambda(t_1-t)} \left[v_L + \Pi_1(t_1)\right]. \tag{A.5}$$

In equilibrium, the buyers' belief is correct, so the seller's optimal choice is indeed t_1^*. The first derivative w.r.t. t_1 at $t_1^*$is

$$\begin{aligned} & e^{-\lambda\left(t_1^*-t\right)} \lambda[p_2(t_1^*) - v_L] + e^{-\lambda\left(t_1^*-t\right)} \dot{\Pi}_1(t_1^*) \\ = \; & \lambda e^{-\lambda\left(t_1^*-t\right)} \left[p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*)\right] = 0 \end{aligned}$$

Or equivalently, $p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*) = 0$.

Define $f(\cdot)$ on $[0,1]$ as follows:

$$f(t) = p_2(t) - v_L - p_1(t) + \Pi_1(t).$$

For $t \geq t_1^*$, we have $p_2(t_1) - p_1(t_1) = \frac{1}{M+1}\left[v_L - p_1(t_1)\right]$, then

$$\begin{aligned} f(t) &= \Pi_1(t) - v_L - \frac{p_1(t) - v_L}{M+1} \\ &= \frac{v_H - v_L}{M+1} \left\{ M - e^{-\lambda(1-t)} \left[M + \frac{M}{M+1} + \lambda(1-t) \right] \right\} \end{aligned}$$

Obviously, $\dot{f}(t) < 0$ and $f(1) = -M/(M+1) < 0$. Define t_1^* as the unique solution to $f(t) = 0$ if it exists, otherwise define $t_1^* = 0$. By construction, for $t \in (t_1^*, 1)$, the optimal solution of (A.5) is t; thus, the seller does not have any incentive to choose a deal time later than t_1^* in the auxiliary problem. If $t_1^* > 0$ i.e. $f(t_1^*) = 0$, for $t < t_1^*$,

$\dot{p}_2(t) = -\lambda(p_1(t) - p_2(t))$, hence

$$f(t) = \frac{1}{\lambda}\dot{p}_2(t) + \Pi_1(t) - v_L,$$

and $\dot{f}(t) = \dot{p}_2(t) + \dot{\Pi}_1(t) - \dot{p}_1(t)$ in which both $\dot{p}_2(t) < 0$ and $\dot{\Pi}_1(t) - \dot{p}_1(t)$ are negative therefore $\dot{f}(t) < 0$ for $t < t_1^*$. Since p_2, p_1 and Π_1 are all continuous over $[0,1]$, we have a continuous $f(t)$ and $\lim_{t \nearrow t_1^*} f(t) = f(t_1^*) = 0$, consequently $f(t) > 0$ for $t < t_1^*$; thus, the seller does not have any incentive to choose a deal time earlier than t_1^*.

Suppose Δ is small; thus, after the fire sale at t_1^*, new L-buyers enter and their number is M at the deadline. Hence, after the fire sale, the H-buyer's reservation price for $t \in (t_1^*, 1)$ is $p_1(t)$. Off the path of play, the story is different. Case 1. Suppose the seller holds the fire sales at $t < 1 - \Delta$. Then new L-buyers enter before the deadline, and the H-buyer's reservation price is still $p_1(t)$. Case 2. Suppose the seller runs the fire sale at $t \in (1 - \Delta, 1)$. Then there is no new L-buyer enters after the sales. Hence, the H-buyer's reservation price is $\tilde{p}_1(t) = v_H - e^{-\lambda(1-t)} \frac{v_H - v_L}{M}$ after the fire sale, and $\tilde{p}_2(t) = v_H - \frac{1}{M+1}(v_H - v_L) - \frac{M}{M+1} e^{-\lambda(1-t)} \frac{v_H - v_L}{M}$ before the fire sale. Since $\tilde{p}_k(t) < p_k(t)$, the seller's profit by running the fire sale after $1 - \Delta$ is strictly less than that in the auxiliary problem. Hence, it is strictly dominated. Consequently, in the real problem, the seller does not have any incentive to choose a deal time later than t_1^*.

Verification of the Conjecture. Next, we need to verify, when the seller can freely choose any price at any time, whether our conjecture equilibrium is indeed an equilibrium. By construction, H-buyers have no incentive to deviate. First, we show that the seller has no incentive to deviate from $p_2(t)$ when $K(t) = 2$. We then show that there is no other equilibrium in addition to equilibrium we proposed.

First, by the proof of proposition 1, when $K(t)$ jumps to 1, the continuation play of the seller in any equilibrium is $P(t) = p_1(t)$ for $t < 1$ and $P(1) = v_L$. Second, a simple observation is that, when $K(t) = 2$, any strategy induces $P(t) \in (v_L, p_2(t))$ is dominated by $p_2(t)$, so it is suboptimal. Third, conditional on k, the H-buyer's reservation price declines over time. Different from the single-unit case, the seller can enhance the H-buyer's reservation price in future by reducing his inventory. As a result, the seller may have the incentive to accumulate H-buyers by charging $P(t) > p_2(t)$ when $K(t) = 2$, and charge them $p_1(\cdot)$ after a fire sale. However, we claim that, when $K(t) = 2$, the seller has no incentive to choose a strategy with a price $P(t) > p_2(t)$ when $K(t) = 2$ at any positive measure of time.

Given any seller's strategy, denote $t_1^d = \inf\{t | K(t) = 2, P(t) = v_L\}$. Consider the last period first. For $t \in [1 - \Delta, 1]$, it is obvious that the seller's optimal price is either $p_2(t)$ or v_L when $K(t) = 2$. The reason is that it is the last chance that H-buyers will accept a price greater than v_L, and there is no benefit to posting unacceptable prices. Next we claim that, given H-buyers' reservation price, the seller's best response satisfies the following properties: for $t < t_1^d$, $P(t) = p_2(t)$ when $K(t) = 2$. We verify this step by step.

Step 1. Suppose in the seller's best response, $t_1^d \in [(l-1)\Delta, l\Delta)$. We call this period the fire sale period. We claim that for $t \in [(l-1)\Delta, t_1^d)$, $P(t) = p_2(t)$ when $K(t) = 2$. Suppose not. Then there are countably many time intervals with a positive measure in the current period in which the seller posts $P(t) > p_2(t)$

when $K(t) = 2$. We call them non-selling time intervals. **Case 1**. $P(t) > p_2(t)$ for $t \in [(l-1)\Delta, t_1^d)$. By doing so, the benefit is to accumulate H-buyers whose attention times are in such intervals and induce them to accept high prices after the fire sale. However, such a pricing strategy is dominated by the following one: posting v_L at $(l-1)\Delta$ and $p_1(t)$ for $t > (l-1)\Delta$. The reasons are that (1) $p_1(t)$ is decreasing over time, (2) an H-buyer who arrives at $t \in ((l-1)\Delta, t_1^d)$ is the only H-buyer in the market, and he may take the deal at t_1^d instead of paying $p_1\left(\tilde{t}\right)$ at her next attention time with positive probability. Hence, we have a contradiction! **Case 2**. Suppose there is a $t' \in [(l-1)\Delta, t_1^d)$ such that $P(t) = p_2(t)$ for $t \in [(l-1)\Delta, t')$ and $P(t) > p_2(t)$ for $t \in [t', t_1^d)$ when $K(t) = 2$. Similar to the argument in case 1, the seller can post v_L at t' instead of at t_1^d and earn extra benefit. **Case 3**. There are countably many mutually exclusive subintervals of $[(l-1)\Delta, t_1^d)$ in which $P(t) > p_2(t)$. Then there is a $t'' \in [(l-1)\Delta, t_1^d)$ such that $t_1^d - t''$ equals the measure of the sum of those in the non-selling time intervals. Each H-buyer's attention time follows an independent uniform distribution, and newly arrived H-buyers' arrival rate is time-independent, so the population structure of H-buyers whose attention times are in $[t'', t_1^d)$ is identical to that in the non-selling time intervals. Since both $p_2(t)$ and $p_1(t)$ decrease over time, the original pricing strategy is dominated by the following one at $t = (l-1)\Delta$: the seller posts $p_2(t)$ for $t \in [(l-1)\Delta, t')$ and $P(t) > p_2(t)$ for $t \in [t', t_1^d)$. Then, by the logic of case 2, we have a contradiction! In short, the seller does not post $P(t) > p_2(t)$ for $t \in [(l-1)\Delta, t_1^d)$.

Step 2. Now we claim that for any $t \in [(l-2)\Delta, (l-1)\Delta)$, the seller's best response satisfies that $P(t) = p_2(t)$. Suppose not. By the same argument in case 3 of step 1, we can focus on the strategy where $P(t) = p_2(t)$ for $t \in [(l-2)\Delta, t'')$ and $P(t) > p_2(t)$ for $t \in (t'', (l-1)\Delta)$. Then there must exist a $t' \in [(l-1)\Delta, t'')$ such that at time t the expected distribution of the number of H-buyers whose attention times are in $[t', \frac{t'+t_d^1}{2})$ equals that in $[\frac{t'+t_d^1}{2}, t_1^d)$. Two intervals have the same length, so the process of the attention times is identical too. As a result, we claim that the original strategy is dominated by the following one at time t: the seller posts $p_2(t)$ for $t \in [t', \frac{t'+t_d^1}{2})$ but $P(t) > p_2(t)$ for $t \in [\frac{t'+t_d^1}{2}, t_1^d)$. Again, by the logic of case 2 in step 1, we have a contradiction!

Step 3. In the period before the fire sales period, the price is acceptable and each H-buyer observes the price in that period. Thus, for periods before the seller has no incentive to post $P(t) > p_2(t)$. Hence, in the seller's best response, $P(t) = p_2(t)$ or v_L when $K(t) = 2$. By the construction of the auxiliary problem, we know the optimal fire sale time is t_1^*, thus the seller has no incentive to deviate from his equilibrium strategy.

Uniqueness. Since $t_1^*$is uniquely constructed, there is no other equilibrium. *Q.E.D.*

A.2.4 Proof of Proposition 3

A.2.4.1 Equilibria Construction

We construct the equilibrium by induction. Suppose there is a unique equilibrium for the game where $K(0) = K$ in which there exists a sequence of $\{t_k^*\}_{k=1}^{K-1}$, and $p_k(t)$ for $k = \{1, 2, ...K\}$, such that $t_{k+1}^* < t_k^*$, $p_{k+1} < p_k$, and $\dot{p}_k < 0$ where differentiable. Consequently, by the indifference conditions of an H-buyer's reservation price and uniform distributed attention time in a period, we can define $U^k_{l\Delta} = v_H - p_k(l\Delta)$ as the expected utility of an H-buyer if her next attention time is in next period starting from $l\Delta < t^*_{k-1}$ and $K(l\Delta) = k$, and $U^k_{t^*_{k-1}} = v_H - p_k\left(t^*_{k-1}\right)$ the expected utility if the next attention time is t^*_{k-1} and $K\left(t^*_{k-1}\right) = k$. We construct the candidate equilibrium for the game where $K(0) = K+1$, which includes: the H-buyers reservation price $p_{K+1}(t)$, the equilibrium first fire sale time t^*_K, and the seller' pricing strategy.

When $t^*_{K-1} = 0$, $t^*_K = 0$ as well. When $t^*_{K-1} < 0$, similar to the two-unit case, we can construct a fire sale time $t^*_K \in [0, t^*_{K-1}]$. Suppose buyers believe that the seller posts deals at $0 \le t^*_K \le t^*_{K-1} < ... < t^*_1 < 1$ when $K(t^*_k) > k$ and posts the H-buyer's reservation price $p_k(t)$ when $K(t) = k$, $k = 1, ..., K+1$. We consider the case where Δ is small enough. We assume $\forall k$, $\exists l_k$ s.t. $t^*_k < l_k\Delta < t^*_{k-1}$, that is, there is at most one deal time in a period. We will verify this hypothesis later.

First, consider $t \ge t^*_K$. Trivially, the seller will post $p_K(1) = v_L$ at the deadline and the reservation price of an H-buyer is v_H. When $t \in [t^*_{K-i}, t^*_{K-i-1})$, and $K(t) = K+1$, an H-buyer expects the seller to post v_L immediately and to reduce his inventory to $K-i$, hence

$$p_{K+1}(t) = \frac{i+1}{M+1} v_L + \frac{M-i}{M+1} p_{k-i}(t)$$

for $i = 0, ..., K-1$ and $t^*_0 := 1$. Note that, when $t > t^*_K$, $p_{K+1}(t)$ is decreasing but not continuous because $\lim_{t \nearrow t^*_k} p_{K+1}(t) > p_{K+1}(t^*_k)$, $\forall k < K$ and $\dot{p}_{K+1} = (M-i)/(M+1)\,\dot{p}_{k-i} < 0$ where it exists.

Now consider $t < t^*_K$. If $(l-1)\Delta \le t < t^*_K < l\Delta$, the H-buyer's indifference condition is:

$$\begin{aligned} v_H - p_{K+1}(t) &= e^{-\lambda\left(t_K^* - t\right)} U_{t_K^*}^{K+1} \\ &\quad + \sum_{k=1}^{K} \lambda^k e^{-\lambda(l\Delta - t)} \sum_{i=1}^{k} \frac{\left(t_K^* - t\right)^i}{i!} \frac{\left(l\Delta - t_K^\Delta\right)^{k-i}}{(k-i)!} U_{l\Delta}^{K+1-k}; \end{aligned}$$

or if $(l-1)\Delta \leq t < l\Delta \leq t_K^*$, the condition becomes:

$$v_H - p_{K+1}(t) = \sum_{k=0}^{K} e^{-\lambda(l\Delta - t)} \frac{\lambda^k (l\Delta - t)^k}{k!} U_{l\Delta}^{K+1-k}.$$

The continuation values $U_{l\Delta}^k$ and $U_{t_K^*}^{K+1}$, defined in the same fashion as before, are the expected utilities of an H-buyer if her next attention time is in the next period or at t_K^*, whichever comes first. The analytical expression for $p_{K+1}(t)$ is then obtained using the continuation values in a recursive way. It is straightforward to show that $p_{K+1}(t)$ is continuous at t_K^*. In addition, we have

$$\dot{p}_{K+1}(t) = -\lambda\left(p_K(t) - p_{K+1}(t)\right) \text{ for } t < t_K^*. \tag{A.6}$$

By construction, it is immediate that $p_{K+1}(t) < p_K(t)$, $\forall t < 1$, hence $\dot{p}_{K+1}(t) < 0$ and $\ddot{p}_{K+1}(t) = -\lambda^2\left(p_{K-1} - p_{K+1}\right) < 0$ where differentiable.

Second, we show some properties of the H-buyers' reservation price. The results are summarized in the following lemma.

Lemma 5. *For $t < t_k^*$, $\dot{p}_{k+1} - \dot{p}_k < 0$ where $k = \{1, 2, ...K\}$.*

Proof. We solve the closed-form solution of $p_{k+1} - p_k$. Simple algebra implies that

$$\dot{p}_{k+1}(t) - \dot{p}_k(t) = \lambda(p_{k+1} - p_k) + \lambda(p_{k-1} - p_k),$$

which is equivalent to

$$\begin{aligned} & [\dot{p}_{k+1}(t) - \dot{p}_k(t) - \lambda(p_{k+1} - p_k)]\, e^{-\lambda t} \\ = \; & \frac{d}{dt}\left[(p_{k+1} - p_k)\, e^{-\lambda t}\right] = -\lambda(p_k - p_{k-1})\, e^{-\lambda t}. \end{aligned}$$

Recursively, we have

$$\frac{d^k}{dt^k}\left[(p_{k+1} - p_k)\, e^{-\lambda t}\right] = -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda},$$

so

$$p_{k+1} - p_k = -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^k}{k!} e^{\lambda t} + \sum_{i=1}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t}$$

where C_i is a constant number for each i, and

$$\begin{aligned} \dot{p}_{k+1} - \dot{p}_k &= -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} e^{\lambda t} \\ &\quad + \sum_{i=1}^{k-1} C_i \frac{t^{k-i-1}}{(k-i-1)!} e^{\lambda t} + (\lambda + \rho)(p_{k+1} - p_k) \\ &= (p_k - p_{k-1}) + \lambda(p_{k+1} - p_k) \\ &\quad + (\lambda + 1)(-\lambda)^{k-1} \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} e^{\lambda t}. \end{aligned}$$

Hence, when $k \in \{2, 4, 6, 8, ...\}$, we have $\dot{p}_{k+1} - \dot{p}_k < 0$. By the same logic, we have

$$p_2 - p_1 = \frac{\lambda (v_H - v_L)}{M+1} e^{-\lambda(1-t)} t + e^{\lambda t} C_1 < 0$$

$$\frac{d^{k-1}}{dt^{k-1}} \left[(p_{k+1} - p_k) e^{-\lambda t} \right] = (-\lambda)^{k-1} \left[\frac{\lambda (v_H - v_L)}{M+1} e^{-\lambda} t + C_1 \right]$$

so

$$p_{k+1} - p_k = (-\lambda)^{k-1} \left[\frac{\lambda (v_H - v_L)}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} + C_1 \frac{t^{k-2}}{(k-2)!} \right] e^{\lambda t} + \sum_{i=2}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t}$$

and

$$\begin{aligned} \dot{p}_{k+1} - \dot{p}_k &= \lambda (p_{k+1} - p_k) + (p_k - p_{k-1}) \\ &\quad - (\lambda + 1) (-\lambda)^{k-2} \frac{t^{k-3}}{(k-2)!} \left[\frac{\lambda (v_H - v_L)}{M+1} e^{-\lambda} t + C_1 (k-2) \right] e^{\lambda t} \end{aligned}$$

hence when $k \in \{3, 5, 7, 9...\}$, $\dot{p}_{k+1} - \dot{p}_k < 0$. In short, $\dot{p}_{k+1} - \dot{p}_k < 0$ for any $k \in \mathbb{N}$. $\square$

Next, we consider the problem faced by the seller in which he chooses the fire sale time t_K^*, but is forced to post $p_{K+1}(t)$ when $t < t_K$ and $K(t) = K + 1$, and the seller's problem is to choose the optimal deal time.

$$\Pi_{K+1}(t) = \max_{t_K} \int_t^{t_K} e^{-\lambda(s-t)} \lambda \left[p_{K+1}(s) + \Pi_K(s) \right] ds + e^{-\lambda(t_K - t)} \Pi_c(t_K), \qquad \text{(A.7)}$$

where the continuation payoff $\Pi_c(t_K)$ is given as follows

$$\Pi_c(t_K) = \begin{cases} v_L + \Pi_K(t_K) & t_K < t^*_{K-1} \\ iv_L + \Pi_{K+1-i}(t_K) & t \in [t^*_{K+1-i}, t^*_{K-i}) \end{cases}$$

for $i = 2, 3, ...K-1$. In equilibrium, the H-buyers' belief is correct, so the seller's optimal choice is indeed $t^*_k(\Delta)$. The first order derivative to t_K is

$$\begin{aligned} & e^{-\lambda(t_K-t)}\lambda[p_{K+1}(t_K) - v_L] + e^{-\lambda(t_K-t)}\dot{\Pi}_K(t_K) \\ = \ & \lambda e^{-\lambda(t_K-t)}\left[p_{K+1}(t_K) - v_L - p_K(t_K) + \Pi_K(t_K) - \Pi_{K-1}(t_K)\right] \end{aligned}$$

At t^*_k, we have

$$p_{K+1}(t^*_k) - v_L = p_K(t^*_k) + \Pi_{K-1}(t^*_k) - \Pi_K(t^*_k).$$

Third, we show that there is a unique t^*_K that determines the auxiliary equilibrium. At t^*_K, we have $p_{K+1}(t^*_K) - v_L - p_K(t^*_K) + \Pi_K(t^*_K) - \Pi_{K-1}(t^*_K) = v_L + \Pi_K(t^*_K) - \Pi_{K-1}(t^*_K) - \frac{p_K(t^*_K)-v_L}{M+1}$. Let

$$f^1_K(t) = v_L + \Pi_K(t) - \Pi_{K-1}(t) - \frac{p_K(t) - v_L}{M+1} \text{ for } t \in [t^*_k, t^*_{K-1})$$

Similar to the two-unit case, a simple observation is that $\lim_{t\to t^*_{K-1}} \Pi_K(t) - \Pi_{K-1}(t) \to v_L$ and $\lim_{t\to t^*_{K-1}}\left[\mathbb{E}_{\tilde{t}|t}\left[e^{-\lambda(\tilde{t}-t)}p_K(t)\right] - v_L\right] > 0$ and both $\Pi_K(t) - \Pi_{K-1}(t)$ and $p_K(t)$ are continuous function, so $f_K(t) < 0$ for t close to t^*_{K-1}. If $f_K(t) < 0$ for any $t \in [0, t^*_{K-1})$, we claim that $t^*_k = 0$. Otherwise, we let $t^*_k = \sup\{t | t \leq t_{K-1},$

$f_K(t) = 0$ and $\exists\varepsilon > 0$ s.t. $f_K(t') > 0$ for $t' \in (t-\varepsilon, t)\}$. By construction, for $t \in \left(t_k^*, t_{K-1}^*\right)$, the optimal solution of (A.7) is t, and when $t < t_k^*$, the seller prefers t_k^* to any $t_K \in \left(t_k^*, t_{K-1}^*\right)$. What about $t_K \in \left(t_{K-i}^*, t_{K-i-1}^*\right)$ for $i = 1, 2, ..K-2$ and $t_K \in (t_1^*, 1)$? The first derivative of the seller's objective function is given by

$$p_{K+1}(t_K) + \Pi_K(t_K) - iv_L - \Pi_{K-i}(t_K) - [p_{K-i}(t) + \Pi_{K-i-1} - \Pi_{K-i}]$$

for $t_K \in \left(t_{K-i}^*, t_{K-i-1}^*\right)$. By construction of p_k, we have

$$p_{K+1}(t_K) = \frac{i+1}{M+1}v_L + \frac{M-i}{M+1}p_{K-i}(t),$$

where $\tilde{t}$ is the H-buyer's next regular attention time. Let

$$\begin{aligned}
f_K^i(t) &= \Pi_K(t_K) - iv_L - \Pi_{K-i-1}(t_K) + [p_{K+1}(t_K) - p_{K-i}] \\
&= [\Pi_K(t_K) - iv_L - \Pi_{K-i-1}(t_K)] \\
&\quad + \left\{\frac{i+1}{M+1}v_L + \frac{M-i}{M+1}p_{K-i}(t) - p_{K-i}(t)\right\} \\
&= [\Pi_K(t_K) - iv_L - \Pi_{K-i-1}(t_K)] + \frac{i+1}{M+1}[v_L - p_{K-i}(t)].
\end{aligned}$$

And by construction of Π_k, $\Pi_K(t_K) - iv_L - \Pi_{K-i-1}(t_K) = 0$ for $t_K \in (t_{K-i}, t_{K-i-1})$. So $f_K^i < 0$. As a result, the seller, holding $K+1$ units, prefers to sell the first unit via a fire sale at t_k^* to any $t_K \in (t_k^*, 1]$.

Now let us verify whether, at $t < t_k^*$, the seller's optimal choice is t_k^* if $K(t) =$

$K+1$. The first derivative is given by

$$p_{K+1}(t) - v_L - p_K(t) + \Pi_K(t) - \Pi_{K-1}(t)$$

and we know that it equals zero at t_k^*. We want to show that, for any $t < t_k^*$, the first derivative is positive. The reason is simply that both $\dot{p}_{K+1}(t) - \dot{p}_K(t)$ and $\dot{\Pi}_K(t) - \dot{\Pi}_{K-1}(t)$ are negative. The first term is proved in Lemma 5, and the second term is shown as follows. We know that $\dot{\Pi}_2(t) - \dot{\Pi}_1(t) < 0$. Now at t_k^*, $\dot{\Pi}_K(t_k^*) - \dot{\Pi}_{K-1}(t_k^*) = 0$, and the $\lim_{t \searrow t_k^*} \left[\ddot{\Pi}_K(t) - \ddot{\Pi}_{K-1}(t)\right] = -\lambda \left[\dot{p}_K(t) - \dot{p}_{K-1}(t) + \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t)\right] > 0$. Hence $\dot{\Pi}_K(t) - \dot{\Pi}_{K-1}(t) < 0$ for $t \in (t_k^* - \varepsilon, t_k^*)$ where ε is small but positive. If $\dot{\Pi}_K(t) - \dot{\Pi}_{K-1}(t) > 0$ for some t, by continuity of $\dot{\Pi}_K(\cdot) - \dot{\Pi}_{K-1}(\cdot)$, there must be a $\hat{t}$ s.t. $\dot{\Pi}_K\left(\hat{t}\right) - \dot{\Pi}_{K-1}\left(\hat{t}\right) = 0$ and $\dot{\Pi}_K(t) - \dot{\Pi}_{K-1}(t) < 0$ for any $t \in \left(\hat{t}, t_k^*\right)$. However, $\ddot{\Pi}_K\left(\hat{t}\right) - \ddot{\Pi}_{K-1}\left(\hat{t}\right) > 0$, which is a contradiction!

In short, the equilibrium of the auxiliary problem uniquely exists. Similar to the two-unit case, off the path of play, if the fire sale is postponed such that (1) $K(t) = k$ and (2) t and t_{k-2}^* are in the same period, the H-buyer's reservation price is lower than $p_k(t)$ and $p_{k-1}(t)$ when $K(t) = k$ and $k-1$ respectively. However, the seller's profit by running the fire sale in such a period is strictly less than that in the auxiliary problem. Hence, it is strictly dominated. Consequently, in the real problem, the seller does not have any incentive to choose a deal time later than t_{k-1}^* when $K(t) = k$.

A.2.4.2 Verification of the Conjecture

First, we verify that in the candidate equilibrium, two sales are not in the same period.

Lemma 6. *When Δ is small, and $t_k^*, t_{k+1}^* > \Delta$, $t_k^* - t_{k+1}^* > \Delta$, for all $k \in \mathbb{N}$*

Proof. If $t_k^*, t_{k+1}^* > \Delta$, they are both interior solutions. By construction, we have that $t_k^* > t_{k+1}^*$. We claim that $t_k^*, t_{k+1}^*$are not in the same period when Δ is small. Suppose not, we have $t_k^* - t_{k+1}^* \leq \Delta$. And at t_k^*, we have

$$\begin{aligned}
\Pi_{k+1}\left(t_k^*\right) &= \Pi_k\left(t_k^*\right) + v_L, \dot{\Pi}_{k+1}\left(t_k^*\right) = \dot{\Pi}_k\left(t_k^*\right), \\
\dot{\Pi}_{k+1}\left(t_k^*\right) &= -\lambda\left[p_{k+1}\left(t_k^*\right) + \Pi_k\left(t_k^*\right) - \Pi_{k+1}\left(t_k^*\right)\right], \\
\dot{\Pi}_k\left(t_k^*\right) &= -\lambda\left[p_k\left(t_k^*\right) + \Pi_{k-1}\left(t_k^*\right) - \Pi_k\left(t_k^*\right)\right],
\end{aligned}$$

so

$$p_{k+1}\left(t_k^*\right) - v_L = p_k\left(t_k^*\right) + \Pi_{k-1}\left(t_k^*\right) - \Pi_k\left(t_k^*\right).$$

but at t_{k+1}^*, we have

$$\begin{aligned}
p_{k+2}\left(t_{k+1}^*\right) - v_L &= p_{k+1}\left(t_{k+1}^*\right) + \Pi_k\left(t_{k+1}^*\right) - \Pi_{k+1}\left(t_{k+1}^*\right) \\
&= p_{k+1}\left(t_{k+1}^*\right) - v_L + \underbrace{\int_{t_{k+1}^*}^{t_k^*}\left[\dot{\Pi}_k(s) - \dot{\Pi}_{k+1}(s)\right] ds}_{O(\Delta)}.
\end{aligned}$$

However, $p_{k+1}\left(t_k^*\right) - p_{k+2}\left(t_{k+1}^*\right)$ is bounded away from zero for any Δ. So, when $\Delta \to 0$, we have a contradiction! Thus, when Δ is small, we have the desired result. $\square$

Second, we verify that the equilibrium path $\left\{p_k\left(t\right), t_{k-1}^*\right\}_{k=1}^{K+1}$ can be supported as a equilibrium. The proof is almost identical to the two-unit case. In any histories with $K\left(t\right) = k$, the seller posts either $p_k\left(t\right)$ or v_L, thus $\Phi_0\left(t\right) = 1$. Since t_k^* is the optimal fire sale time when $K\left(t\right) = k$ and $\Phi_0\left(t\right) = 1$, the seller has no incentive to deviate. So the conjecture equilibrium is indeed an equilibrium.

Uniqueness. Similar to the $K = 2$ case. In any period, there is no non-selling time intervals. Given that, the only possible equilibrium is the equilibrium we proposed. *Q.E.D.*

Appendix B

Appendix for Chapter 2

B.1 Equilibrium Construction

Proof of Lemma 2:

If v-buyer accepts p_τ in period τ, then $v - p_\tau \geq e^{-r\Delta}(1-\lambda\alpha\Delta)U(v, \alpha_\tau, H_\tau)$. We want to show that $v' - p_\tau > e^{-r\Delta}(1-\lambda\alpha\Delta)U(v', \alpha_\tau, H_\tau)$. Since from $\tau+1$ on, the v player can always adopt the optimal strategy of v'-buyer, that is, accept exactly when v'-buyer accepts, then

$$\begin{aligned} U(v, \alpha_\tau, H_\tau) &\geq \sum_{j=0}^{\infty} e^{-jr\Delta} \prod_{i=0}^{j} (1-\lambda\alpha_{\tau+i}\Delta)\gamma_{\tau+1+j}(v', \alpha, H_{\tau+1+j})(v - p_{\tau+j+1}), \\ U(v', \alpha_\tau, H_\tau) &= \sum_{j=0}^{\infty} e^{-jr\Delta} \prod_{i=0}^{j} (1-\lambda\alpha_{\tau+i}\Delta)\gamma_{\tau+1+j}(v', \alpha, H_{\tau+1+j})(v' - p_{\tau+j+1}), \end{aligned}$$

where $\gamma(v', \alpha_\tau, H_\tau)$ is the probability conditional on H_τ and α_τ that agreement is reached at time $\tau+1+j$ and the buyer use v'-buyer's optimal strategy from time

$\tau + 1$ on. In other words,

$$\begin{aligned} & U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau) \\ \leq\ & \sum_{j=0}^{\infty} e^{-jr\Delta} \prod_{i=0}^{j} (1 - \lambda\alpha_{\tau+j}\Delta)\gamma_{\tau+1+j}(v', H_\tau)(v' - v) \\ \leq\ & v' - v. \end{aligned} \tag{B.1}$$

Since $v - p_\tau \geq e^{-r\Delta}(1 - \lambda\alpha_\tau\Delta)U(v, \alpha_\tau, H_\tau)$, we have

$$\begin{aligned} & v' - p_\tau - e^{-r\Delta}U(v', \alpha_\tau, H_\tau) \\ \geq\ & v' - (v - e^{-r\Delta}U(v, \alpha_\tau, H_\tau)) - e^{-r\Delta}U(v', \alpha_\tau, H_\tau) \\ =\ & (v' - v) - e^{-r\Delta}[U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau)] \\ =\ & (v' - v) - [U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau)] \\ & \qquad + (1 - e^{-r\Delta})[U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau)]. \end{aligned}$$

By (B.1),

$$(v' - v) - [U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau)] \geq 0,$$

and

$$(1 - e^{-r\Delta})[U(v', \alpha_\tau, H_\tau) - U(v, \alpha_\tau, H_\tau)] > 0,$$

we have $v' - p_\tau > e^{-r\Delta}(1 - \lambda)U(v', \alpha_\tau, H_\tau)$. *Q.E.D.*

Proof of Proposition 4:

We first show the screening ends in finitely many periods in any equilibrium. Then, we construct the unique equilibrium by induction. The following lemma shows that screening offers are suboptimal, when the seller is sufficiently pessimistic about the long-lived buyer's value.

Lemma 7. *There is a $K^* > \underline{v}$ such that screening is not optimal when $k \in [\underline{v}, K^*)$.*

Proof. The seller can switch to the strategy "waiting offer forever" and obtain the expected value J_0 in any period. We first show that once the seller optimally posts one period waiting offer, he would not use screening offers in future. Suppose the seller posts a waiting offer in period t, it must be true that

$$R(k) = J(k) > V(k).$$

The game goes to period $t + 1$ if no short-lived buyers appears. By posting the waiting offer, the seller cannot screen the long-lived buyer's value. Thus, $k' = k$, and $V(k') = V(k) < J(k) = J(k')$. Hence, the seller will post the waiting offer in period $t+1$. As a result, once the seller posts a waiting offer in period t, he will post the waiting offer until the game is over. Hence, $J(k) = J_0$, when $J(k) > V(k)$.

Now we want to show that there is a $K^* \in (\underline{v}, 1]$ such that the seller will switch to the waiting offer when $k < K^*$. Suppose not, then for any $k \in [\underline{v}, 1]$, $J(k) \leq V(k)$. However, $V(k) \leq k$, so when k is sufficiently close to $\underline{v}$, $V(k)$ is strictly less than J_0.

There are two cases. First, $J_0 > V(1)$, then $K^* = 1$. The seller will posts waiting offer from period 1 on. In this case, the equilibrium is trivial. In the second case, $J_0 \leq V(1)$. By monotonicity of $V(k)$, there exists a K^* such that $V(k) \geq J_0$ if and only if $k \in [K^*, 1]$. Given the existence of K^*, the seller will finally switch to the waiting offer in any equilibrium, if it exists. $\square$

Now we show the existence of the equilibrium and its uniqueness by construction. Because the strategy "waiting offer forever" provides a lower bound of the seller's value, which is strictly higher than $\underline{v}$, this proof is essentially same as that in Ausubel and Deneckere (1989). However, the existence of the short-lived buyer makes the proof more complicated.

Step 1. We set

$$W^0(k) = J_0 \quad \text{and} \quad \sigma^0(k) = 1,$$

where $W^0(k)$ is the seller's payoff from charging the waiting offer 1, and $\sigma^0(k)$ is seller's optimal choice correspondence, which is single valued in this zero step screening game by construction. Denote by $\beta^1(k)$ the price that k-value long-lived buyer is indifferent between taking and having no trade. Then $\beta^1(k)$ is such that

$$k - \beta^1(k) = 0.$$

That is, suppose the seller's strategy is is to make a screening offer in the current period and begin to charge the waiting offer forever if the current offer is rejected, the k-value long-lived buyer is indifferent between accepting $\beta^1(k)$ and having no trade. Accordingly, if this last screening offer p is rejected, the upper bound of the seller's belief about the long-lived buyer's value is $\kappa^1(p) = p$. Obviously, $(\beta^1)^{-1}(p) = \kappa^1(p)$. By construction, $\beta^1(k)$ and $\kappa^1(p)$ are continuous and increasing.

Denote by $W^1(k)$ the seller's value from charging one screening offer and then switching to the waiting offer forever, so

$$W^1(k) = \max_{k' \le k} [\frac{F(k)-F(k')}{F(k)} + \frac{F(k')}{F(k)}\lambda\Delta]p + \frac{F(k')}{F(k)}(1-\lambda\Delta)e^{-r\Delta}J_0.$$

Let $T^1(k)$ be the maximum of the set of solutions to this optimization problem, and $T^1(k)$ exists by the maximum theorem in Ausubel and Deneckere (1993). Then we can define $\sigma^1(k) = \beta^1(T^1(k))$ to be the seller's price at state variable k in this constructed game. Define $k^0 = \max\{k \in [\underline{v}, 1] | W^1(k) = W^0(k) = J_0\}$. By definition, because of the existence of K^*, k^0 is well defined. By this definition, for any $k \le k^0$, the seller will switch to the waiting offer, and the long-lived buyer will reject such an offer, no matter what his value is. With $(\sigma^1, \kappa^1; W^1, \beta^1, T^1, k^0)$, we can begin our equilibrium construction.

Step 2. Set $k^{n+1} > k^n$ such that

$$F\left(k^{n+1}\right) = \min\{1, F\left(k^n\right)(1-\lambda\Delta) + (\lambda\Delta + r\Delta)W^n\left(k^n\right)F\left(k^n\right)\}$$

thus when $k^{n+1} < 1$,

$$F\left(k^{n+1}\right) - F\left(k^n\right)(1-\lambda\Delta) = F\left(k^n\right)(\lambda\Delta + r\Delta)W^n\left(k^n\right).$$

Let $\beta^{n+1}(k)$ be the price will makes the k-value long-lived buyer indifferent between taking the current offer and waiting for the next period offer. Then,

$$k - \beta^{n+1}(k) = (1-\lambda\Delta - r\Delta)(k - \sigma^n(k)).$$

Here, $\beta^{n+1}(k)$ is non-decreasing in k. With this definition, we can define $W^{n+1}(k)$ be the value of the following optimization problem:

$$W^{n+1}(k) = \max_{k' \leq k} \frac{F(k) - F(k')(1-\lambda\Delta)}{F(k)} \beta^{n+1}(k') + \frac{F(k')(1-\lambda\Delta)}{F(k)} e^{-r\Delta} W^{n}(k'),$$

Denote by $T^{n+1}(k)$ the supremum of the set of solutions to this optimization problem for $k \in [\underline{v}, k^{n+1}]$. We now claim that $T^{n+1} \leq k^{n}$.

Suppose not, then there is a $k \in [\underline{v}, k^{n+1}]$ such that $v = T^{n+1}(k)$ and $v \in (k^{n}, k^{n+1}]$. By approximating $e^{-r\Delta}$ by $1 - r\Delta$ and ignoring $O(\Delta^2)$ terms, we have

$$\begin{aligned}
& \frac{F(k) - F(v)(1-\lambda\Delta)}{F(k)} \beta^{n+1}(v) + (1 - \lambda\Delta - r\Delta) \frac{F(v)}{F(k)} W^{n+1}(v) \\
< \; & \frac{F(k^{n+1}) - F(k^{n})(1-\lambda\Delta)}{F(k)} + (1 - \lambda\Delta - r\Delta) W^{n+1}(k) \\
\leq \; & (\lambda\Delta + r\Delta) W^{n}(k^{n}) + (1 - \lambda\Delta - r\Delta) W^{n+1}(k) \\
\leq \; & (\lambda\Delta + r\Delta) W^{n+1}(k) + (1 - \lambda\Delta - r\Delta) W^{n+1}(k) \\
= \; & W^{n+1}(k).
\end{aligned}$$

This leads to a contradiction! Let $\sigma^{n+1}(k) = \beta^{n+1}[T^{n+1}(k)]$. $\beta^{n+1}(k)$ may have finitely many jumps, and it is right continuous. Define $\kappa^{n+1}(p)$ as follows: (1) $\kappa^{n+1}(p) = (\beta^{n+1})^{-1}(p)$ when $\beta^{n+1}(k)$ is invertible, or (2) if there is $\hat{k}$ such that $\lim_{k \searrow \hat{k}} \beta^{n+1}(k) = p^{+} \neq \lim_{k \nearrow \hat{k}} \beta^{n+1}(k) = p^{-}$, $\kappa^{n+1}(p) = \hat{k}$ for $p \in (p^{-}, p^{+}]$. Therefore, by construction, $\kappa^{n+1}(p)$ is a continuous function.

Step 3. The construction of κ^{n+1} in **Step 2** suggests to us that there may be more than one price, which induces the same state variable k' in the next period. On the

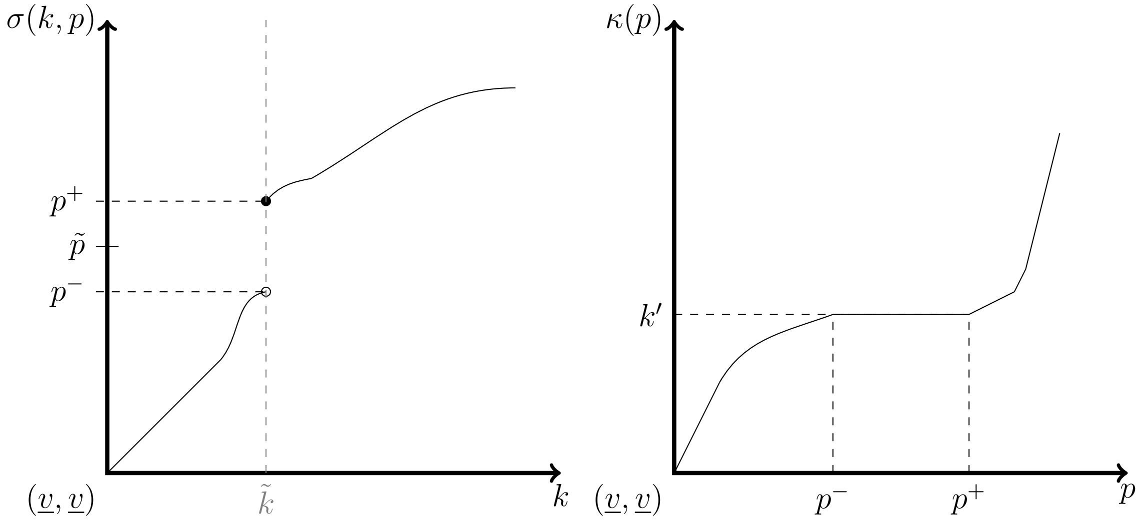

Figure B.1: Pricing function σ and cutoff function κ.

equilibrium path, this does not cause any problem, because there is a unique sequence of prices, and for each price p in this sequence, we have assigned an appropriate k, and given k the continuation play is described by (σ^n, κ^n), where σ^n does not depend on p. However, to complete a strategy profile, we also have to describe the continuation play after any deviation of the seller. In this case, the flat part of the constructed κ^{n+1} causes σ^n to depend on the deviating price. Figure B.1 illustrates this case.

When we construct κ^{n+1}, if there is $\hat{k}$ such that

$$\lim_{k \searrow \hat{k}} \beta^{n+1}(k) = p^+ \neq \lim_{k \nearrow \hat{k}} \beta^{n+1}(k) = p^-,$$

$\kappa^{n+1}(p) = \hat{k}$ for $p \in (p^-, p^+]$. This κ is illustrated in part (b) of Figure B.1. When the seller deviates to a price $\hat{p} \in (p^-, p^+)$, the equilibrium construction assigns $\hat{k}$ as the next period upper bound of the seller's belief about the long-lived buyer's value.

For this belief to be correct, the $\hat{k}$-value long-lived buyer must be indifferent between taking the current offer $\hat{p}$ and waiting for future offers. However, if the seller chooses p'^{+} in the next period given $\hat{k}$, the $\hat{k}$-value long-lived buyer will strictly prefer the current offer; and if the seller chooses p'^{-} in the next period given $\hat{k}$, the $\hat{k}$-value long-lived buyer will strictly prefer waiting. This implies that to make the $\hat{k}$-value long-lived buyer indifferent, the seller's price strategy in the next period given $\hat{k}$ must be mixed. For the seller to randomize given $\hat{k}$, the seller must be indifferent between the continuation strategies in the support of the mixed strategy in the next period. This requires that any continuation strategies in the support of the seller's mixed strategy in the next period given $\hat{k}$ must be part of an equilibrium at the state variable $\hat{k}$. Therefore, after a deviating price $\hat{p}$, the seller's strategy depends on the value of $\hat{p}$, in the way that the seller appropriately randomizes among equilibrium continuation strategies given $\hat{k}$. Hence, in an equilibrium, it is possible that on the off-equilibrium path, the price depends on the last period price. This reconciles the discussion in section 2.3.2 about weak-Markov equilibrium and strong-Markov equilibrium.

In **Step 2**, we specify that on the equilibrium path, to reach the next period state variable k', the seller charges the highest possible price. If there are other prices also leading to the same k', the seller's continuation value is the same, but the seller gets higher current payoff. If there is no other price leading to the same k', the construction itself shows that the seller's strategy is the best response to the long-lived buyer's strategy. Therefore, if (σ^n, κ^n) describes an equilibrium for all $k \in [\underline{v}, k^n]$, $(\sigma^{n+1}, \kappa^{n+1})$ describes an equilibrium for all $k \in [\underline{v}, k^{n+1}]$.

Step 4. To complete the argument that the constructed (σ, κ) describes an equilibrium, we need to show that there is $N \in \mathbb{N}$, such that $k^{N+1} \geq 1$. Recall that

$$F\left(k^{n+1}\right) - F\left(k^{n}\right) = F\left(k^{n}\right)\left[-\lambda\Delta + (\lambda\Delta + r\Delta)W^{n}\left(k^{n}\right)\right].$$

Because $W^{n}(k) \geq W^{n-1}(k) \geq ...W^{1}(k) > J_0 = \frac{\lambda\Delta}{r\Delta+\lambda\Delta}$ for any $k > k^0$, $F(k^{n+1}) - F(k^n) > 0$ and increases in n. Since the density function is bounded above by $\bar{f}$, there exists $N \in \mathbb{N}$, such that $k^{N+1} \geq 1$. *Q.E.D.*

Proof of Proposition 5:

By construction, the sequence of prices $\{p_t\}$ is decreasing for all $t \leq N$, and from period $N+1$ on, $p_t = 1$. So to show the lower bound of the equilibrium prices, we just need to show that $p_N \geq \frac{\lambda}{\lambda+\gamma} = J_0$. Suppose $p_N < \frac{\lambda}{\lambda+\gamma}$. Then if p_N is taken, the seller's payoff is $p_N < J_0$. If p_N is not taken, the seller will charge the waiting offer from the next period. In this case, her payoff is also strictly smaller than J_0. Therefore, in period N, the seller would like to begin to charge the waiting offer from period N on. This is contradicted to Proposition 4. Therefore, $p_t \geq \frac{\lambda}{\lambda+\gamma}$. The second part of the statement comes from Coasian conjecture in gap case. The proof is omitted since it is standard. *Q.E.D.*

Proof of Lemma 3:

Since we haven't compared $R(k, \alpha)$ and $R(k, \alpha')$, we discuss two cases. In the first case, assume $R(k, \alpha) \leq R(k, \alpha')$. By definition, $J(k, \alpha) = \lambda\alpha\Delta + (1 - \lambda\alpha\Delta)(1 -$

$r\Delta)R(k,\alpha')$, which is no larger than $R(k,\alpha)$ and $R(k,\alpha')$ (and $R(k,\alpha) \geq \underline{v}$). The waiting offer is strictly suboptimal if

$$\lambda\alpha\Delta \leq \frac{R(k,\alpha) - e^{-r\Delta}R(k,\alpha')}{1 - e^{-r\Delta}R(k,\alpha')} \leq \frac{R(k,\alpha') - e^{-r\Delta}R(k,\alpha')}{1 - e^{-r\Delta}R(k,\alpha')} = \frac{1 - e^{-r\Delta}}{1/R(k,\alpha') - e^{-r\Delta}}.$$

The right hand side is increasing in $R(k,\alpha')$. Since $R(k,\alpha') \geq \underline{v}$, simple algebra implies that waiting offer will not be posted if $\alpha \leq \alpha^\dagger = \frac{1}{\lambda\Delta}\frac{1-e^{-r\Delta}}{1/\underline{v}-e^{-r\Delta}}$.

In the second case, assume $R(k,\alpha) > R(k,\alpha')$. A sufficient condition for

$$R(k,\alpha) = V(k,\alpha) > \lambda\alpha\Delta + (1 - \lambda\alpha\Delta - r\Delta)R(k,\alpha') \tag{B.2}$$

is

$$R(k,\alpha) \geq \lambda\alpha\Delta + (1 - \lambda\alpha\Delta - r\Delta)R(k,\alpha).$$

This holds if and only if $(\lambda\alpha + r)R(k,\alpha) \geq \lambda\alpha$, which is true if and only if $\frac{rR(k,\alpha)}{1-R(k,\alpha)} \geq \lambda\alpha$. Since $R(k,\alpha) \geq \underline{v}$, it is obvious that if $\alpha < \alpha^\dagger$, the equation (B.2) holds. *Q.E.D.*

Proof of Proposition 6:

We first show the existence of the equilibrium and its uniqueness of a continuation game starting from (k,α), where $\alpha \in B_\Delta(\alpha^\dagger)$.

Step 1. For each $\alpha \in B_\Delta\left(\alpha^\dagger\right)$, define for all (k,α) with $k \in [\underline{v}, 1]$ and $\alpha \in B_\Delta(\alpha^\dagger)$,

$$W^0(k,\alpha) = \underline{v}, \quad \sigma^0(k,\alpha) = \underline{v}, \quad \text{and} \quad \kappa^0(\underline{v},\alpha) = \underline{v},$$

where $W^0(k,\alpha)$ is the seller's value from posting the offer $\sigma^0(k,\alpha) = \underline{v}$, and the long-lived buyer with a value bigger than or equal to $\underline{v}$ accepts this offer. Now define $\beta^1(k,\alpha)$ and $\kappa^1(p,\alpha)$ such that

$$\begin{aligned} k-\beta^1(k,\alpha) &= (1-\lambda\alpha\Delta-r\Delta)(k-\underline{v}), \\ \kappa^1(p,\alpha)-p &= (1-\lambda\alpha\Delta-r\Delta)\left(\kappa^1(p,\alpha)-\underline{v}\right); \end{aligned}$$

thus, $\kappa^1(\beta^1(k,\alpha),\alpha)=k$. Moreover, define

$$\begin{aligned} & W^1(k,\alpha) \\ = \; & \max_{k'\leq k} \frac{F(k)-F(k')(1-\lambda\alpha\Delta)}{F(k)}\beta^1(k',\alpha) \\ & \qquad + \frac{F(k')(1-\lambda\alpha\Delta)}{F(k)}e^{-r\Delta}W^0(k',\alpha'). \end{aligned}$$

Let $T^1(k,\alpha)$ be the maximum of the set of solutions to this optimization problem, we can define $\sigma^1(k,\alpha)=\beta^1(T^1(k,\alpha),\alpha')$.

Define

$$k^0(\alpha)=\max\left\{k|W^1(k,\alpha)=\underline{v}\right\}.$$

So when $k<k^0(\alpha)$, the seller will charge price $\sigma(k,\alpha)=\underline{v}$. Obviously, in this constructed zero stage game, (σ^0,κ^0) describes an equilibrium for $k\in[\underline{v},k^0(\alpha))$.

Step 2. With $(\sigma^1,\kappa^1;W^1,\beta^1,T^1,k^0)$, we now construct the equilibrium by induction. For each $k^n(\alpha')$, let $k^{n+1}(\alpha)$ be the largest k such that

$$F(k)=\min\{1,F(k^n(\alpha'))[1+\frac{\lambda\alpha\Delta(1-\beta^{n+1}(k,\alpha))}{\beta^{n+1}(k,\alpha)-\underline{v}}]\},$$

for all $n \geq 0$.

$$F(k^{n+1}(\alpha)) \leq F(k^n(\alpha'))[1 + \frac{\lambda\alpha\Delta(1-\beta^{n+1}(k,\alpha))}{\beta^{n+1}(k,\alpha) - \underline{v}}].$$

For each $\alpha \in B_\Delta\left(\alpha^\dagger\right)$, if the seller chooses a cutoff value k as the next period state variable, she need to make the k-value long-lived buyer indifferent between the current offer and the offer in the next period. Since (k, α') will be the next period state variable, in the next period offer will be, by definition, $\sigma^n(k, \alpha')$. Therefore, the current offer, which makes the k-value long-lived buyer indifferent would be $\beta^{n+1}(k, \alpha)$ such that

$$k - \beta^{n+1}(k,\alpha) = (1-\lambda\alpha\Delta)e^{-r\Delta}(k - \sigma^n(k,\alpha')).$$

Using this definition, we can define $W^{n+1}(k, \alpha)$ as the value of the following optimization problem:

$$\begin{aligned} & W^{n+1}(k,\alpha) \\ = \; & \max_{k' \leq k} \frac{F(k) - F(k')(1-\lambda\alpha\Delta)}{F(k)} \beta^{n+1}(k',\alpha) \\ & \qquad + \frac{F(k')(1-\lambda\alpha\Delta)}{F(k)} e^{-r\Delta} W^n(k',\alpha'). \end{aligned} \tag{B.3}$$

We claim that for any $k \in [k^n(\alpha'^{n+1}(\alpha))$, the maximum of the set of solutions to the optimization problem B.3 $T^{n+1}(k,\alpha) < k^n(\alpha')$.

Suppose not, then let $v = T^{n+1}(k,\alpha) \geq k^n(\alpha')$. Consider the following two conditions. First, because $\alpha \leq \alpha^\dagger$,

$$W^{n+1}(v,\alpha) > \lambda\alpha\Delta + (1-\lambda\alpha\Delta)W^{n+1}(v,\alpha').$$

Second, since the seller at the state variable (k,α) can employ the strategy using at (v,α), we have

$$F(k)W^{n+1}(k,\alpha) \geq F(v)W^{n+1}(v,\alpha) + p(F(k)-F(v)),$$

where p is the price making the long-lived buyer with value $T^{n+1}(v,\alpha)$ indifferent between taking the offer p and waiting for another period. These two conditions imply

$$\begin{aligned} & F(k)W^{n+1}(k,\alpha) \\ > \; & \lambda\alpha\Delta F(v) + (1-\lambda\alpha\Delta - r\Delta)F(v)W^{n+1}(v,\alpha') + p(F(k)-F(v)), \end{aligned}$$

which in turn implies that

$$\begin{aligned} & (1-\lambda\alpha\Delta - r\Delta)F(v)W^{n+1}(v,\alpha') \\ < \; & F(k)W^{n+1}(k,\alpha) - \lambda\alpha\Delta F(v) - p(F(k)-F(v)). \end{aligned}$$

Now, we have

$$
\begin{aligned}
& F(k)W^{n+1}(k,\alpha) \\
= \;& [F(k)-F(v)(1-\lambda\alpha\Delta)]\beta^{n+1}(v,\alpha)+F(v)(1-\lambda\alpha\Delta-r\Delta)\underline{v} \\
\leq \;& [F(k)-F(v)(1-\lambda\alpha\Delta)]\beta^{n+1}(v,\alpha) \\
& \qquad\qquad +F(v)(1-\lambda\alpha\Delta-r\Delta)W^{n+1}(v,\alpha') \\
< \;& [F(k)-F(v)(1-\lambda\alpha\Delta)]\beta^{n+1}(v,\alpha) \\
& \qquad\qquad +F(k)W^{n+1}(k,\alpha)-\lambda\alpha\Delta F(v)-p(F(k)-F(v)) \\
= \;& [F(k)-F(v)](\beta^{n+1}(v,\alpha)-p)+F(k)W^{n+1}(k,\alpha)-\lambda\alpha\Delta F(v)[1-\beta^{n+1}(v,\alpha)] \\
\leq \;& [F(k^{n+1}(\alpha))-F(k^{n}(\alpha'^{n+1}(v,\alpha)-\underline{v}) \\
& \qquad\qquad +F(k)W^{n+1}(k,\alpha)-\lambda\alpha\Delta F(k^{n}(\alpha'^{n+1}(v,\alpha)] \\
= \;& F(k)W^{n+1}(k,\alpha).
\end{aligned}
$$

This leads to the contradiction.

Step 3. Similar to the equilibrium construction in the no learning case, we have to specify the continuation play after the seller's deviation. For any k', which is induced by a unique price p in the current period, the continuation play is as constructed in **Step 2**. So the construction in **Step 2** shows that deviating to such a price is not profitable. If multiple prices induce a same k', the seller must randomize among continuation strategies, each of which is part of an equilibrium of the game beginning at (k', α'). In addition, how the seller randomizes depends on the deviating price, that is, the seller will choose an appropriate mixed strategy to make the k'-value long-lived buyer indifferent between taking the current deviating price and waiting for the next period offer. On the equilibrium path, the seller charges the highest price inducing k', so after the deviation, the seller's continuation payoff is the same as that on the equilibrium path. But the seller has higher current payoff by following the equilibrium price, so such a deviation is not profitable either. Therefore, if (σ^n, κ^n) describes an equilibrium for all $\alpha \in B_\Delta(\alpha^\dagger)$ and all $k \in [\underline{v}, k^n(\alpha))$, $(\sigma^{n+1}, \kappa^{n+1})$ describes an equilibrium for all $\alpha \in B_\Delta(\alpha^\dagger)$ and all $k \in [\underline{v}, k^{n+1}(\alpha))$.

Step 4. Now we show that a finite number of repetitions of the above argument extends $\left(\sigma^{N^*_\alpha}, \kappa^{N^*_\alpha}; W^{N^*_\alpha}, \beta^{N^*_\alpha}, T^{N^*_\alpha}, k^{N^*_\alpha - 1}\right)$ to $k \in [\underline{v}, 1]$ for any $\alpha \in B_\Delta\left(\alpha^\dagger\right)$. By the construction of sequences of $\{k^n(\alpha)\}$ for all $\alpha \in B_\Delta(\alpha^\dagger)$, suppose for any finite number N_α, $k^{N_\alpha}(\alpha) < 1$. There are two cases. In the first case, there is an $\epsilon > 0$, such that $k^n(\alpha) < 1 - \epsilon$. But then

$$\begin{aligned} & F(k^{n+1}(\alpha)) - F(k^n(\alpha')) \\ = \; & F(k^n(\alpha))\lambda\alpha\Delta\frac{1 - \beta^{n+1}(k^{n+1}(\alpha), \alpha)}{\beta^{n+1}(k^{n+1}(\alpha), \alpha) - \underline{v}} \\ \geq \; & F(k^n(\alpha))\lambda\alpha\Delta\frac{\epsilon}{1 - \epsilon - \underline{v}}. \end{aligned}$$

Because the density function is bounded above by $\bar{f}$, we can find the contradiction. In the second case, for any $\epsilon > 0$, there is N such that for all $n \geq N$, $k^n(\alpha) > 1 - \epsilon$. Since

$$\begin{aligned} & F(k^{n+1}(\alpha)) - F(k^n(\alpha')) \\ = \; & F(k^n(\alpha))\lambda\alpha\Delta\frac{1 - \beta^{n+1}(k^{n+1}(\alpha), \alpha)}{\beta^{n+1}(k^{n+1}(\alpha), \alpha) - \underline{v}}, \end{aligned}$$

$f'(1) = 0$ implies $k^{n+1}(\alpha) - k^n(\alpha')$ is bounded away from 0. So we also get the contradiction. *Q.E.D.*

Proof of Proposition 7:

By Lemma 3, the waiting offer will not be posted when $\alpha_t \in B_\Delta(\alpha^\dagger)$, and any continuation game, which starts with (k, α) such that $k \in [\underline{v}, 1]$, $\alpha \in B_\Delta(\alpha^\dagger)$, ends in $N^*_{\alpha^\dagger}$ periods. Hence, the seller's continuation value and associated pricing policy, $R(k, \alpha^\dagger)$ and $\sigma(k, \alpha^\dagger)$ can be calculated for all $k \in [\underline{v}, 1]$. And the k-buyer's equilibrium payoff $U\left(k, k, \alpha^\dagger | \sigma, \kappa\right)$ under the continuation play can also be calculated.

We move to (k, α^2) where $k \in [\underline{v}, 1]$. First, any $(k, \alpha^\dagger)$ can be reached from (k, α^2) by one period waiting offer. By making a waiting offer, the seller's payoff is given by

$$J\left(k, \alpha^2\right) = \lambda \alpha^2 \Delta + \left(1 - \lambda \alpha^2 \Delta - r\Delta\right) R\left(k, \alpha^\dagger\right),$$

the buyer has no relevant choice to make, and the game ends in $N^*_{\alpha^\dagger} + 1$ periods.

Second, construct $\beta\left(k, \alpha^2\right)$ as follows.

$$k - \beta\left(k, \alpha^2\right) = \left(1 - \lambda \alpha^2 - r\Delta\right) U\left(k, k, \alpha^\dagger | \sigma, \kappa\right),$$

where $\beta\left(k, \alpha^2\right)$ is the highest price such that k-buyer is indifferent between taking it and waiting to obtain a continuation payoff $U\left(k, k, \alpha^\dagger | \sigma, \kappa\right)$. Following Ausubel and Deneckere (1989), without loss of generality, assume $\sigma^s\left(k, \alpha^2\right)$ is monotone and right continuous. Define $\kappa\left(\beta\left(k, \alpha^2\right), \alpha^2\right) = k$ if it is well defined. As Ausubel and Deneckere (1989) noted, $\beta\left(k, \alpha^2\right)$ may have jumps at finitely many points. For any jump point $\hat{k}$ such that $\beta\left(\hat{k},\alpha^2\right) = \hat{p}_+$ and $\lim_{k\to\hat{k}^-}\beta\left(k,\alpha^2\right) = \hat{p}_-$, where $\hat{p}_- < \hat{p}_+$, let $\kappa\left(p,\alpha^2\right) = \hat{k}$ for all $p \in [\hat{p}_-,\hat{p}_+]$. Hence, by construction, $\kappa\left(p,\alpha^2\right)$ is continuous with respect to p, and $\kappa^{-1}\left(k|\alpha^2\right) = \beta\left(k,\alpha^2\right)$ is defined.

For any $k \in [\underline{v}, 1]$, we define the seller's value by posting an optimal screening offer as follows:

$$V(k,\alpha^2) = \max_{k'\in k}[1 - \frac{F(k')}{F(k)}(1-\lambda\alpha^2\Delta)]\beta\left(k',\alpha^2\right) + \frac{F(k')}{F(k)}(1-\lambda\alpha^2\Delta)e^{-r\Delta}R(k',\alpha^{\dagger}).$$

Let $T^s(k,\alpha^2)$ be the maximum of the set of solutions to this optimization problem, and $\sigma^s(k,\alpha^2) = \beta(T^s(k,\alpha^2),\alpha^2)$. Since both $R(k,\alpha^{\dagger})$ and $\beta(k,\alpha^2)$ is well defined, $V(k,\alpha^2)$ and $T^s(k,\alpha^2)$ are well defined. By posting an optimal screening offer $\sigma^s(k,\alpha^2)$ in the current period, the seller obtains $V(k,\alpha^2)$, and the game ends in at most $N^*_{\alpha^{\dagger}} + 1$ periods.

Define $R\left(k,\alpha^2\right) = \max\{V\left(k,\alpha^2\right), J\left(k,\alpha^2\right)\}$ as the seller's value by choosing an optimal offer with a price

$$\sigma(k,\alpha) = \begin{cases} \sigma^s\left(k,\alpha\right), & \text{if} \quad V\left(k,\alpha\right) \geq J\left(k,\alpha\right) \\ 1, & \text{otherwise} \end{cases}$$

Hence, we extend the equilibrium to (k,α^2) for any $k \in [\underline{v}, 1]$. The game ends in at most $N^*_{\alpha^{\dagger}} + 1$ periods regardless of the choice of offers in the current period. Given the continuation payoff at (k,α^2) for any $k \in [\underline{v}, 1]$, the k-buyer's continuation payoff, $U\left(k,k,\alpha^2|\sigma,\kappa\right)$ can be calculated.

Since, from α_0 to $\alpha_t = \alpha^{\dagger}$, there are finitely many periods, we can apply the above arguments for finitely many times to extend the equilibrium to (k, α_0) for any $k \in [\underline{v}, 1]$. *Q.E.D.*

B.2 Equilibrium Properties

Proof of Proposition 8:

The value of outside option by waiting forever is given by

$$\begin{aligned} J_0(\alpha_t) &= \lambda\alpha_t\Delta + (1-\lambda\alpha_t\Delta)\, e^{-r\Delta}\lambda\alpha_{t+\Delta}\Delta \\ &\quad + (1-\lambda\alpha_t\Delta)(1-\lambda\alpha_{t+\Delta}\Delta)\, e^{-2r\Delta}\lambda\alpha_{t+2\Delta}\Delta + ... \end{aligned}$$

Since α_t is strictly dicreasing, for any α_t, we have

$$\begin{aligned} J_0(\alpha_t) &< \lambda\alpha_t\Delta + (1-\lambda\alpha_t\Delta)\, e^{-r\Delta}\lambda\alpha_t\Delta \\ &\quad + (1-\lambda\alpha_t\Delta)(1-\lambda\alpha_t\Delta)\, e^{-2r\Delta}\lambda\alpha_t\Delta + ... \end{aligned}$$

or

$$J_0(\alpha_t) < \frac{\lambda\alpha_t\Delta}{1-e^{-r\Delta}(1-\lambda\alpha_t\Delta)}.$$

Aslo, $J_0(\alpha_t)$ satisfies the following recursive equation:

$$J_0(\alpha_t) = \lambda\alpha_t\Delta + (1-\lambda\alpha_t\Delta)\, e^{-r\Delta} J_0(\alpha_{t+\Delta}).$$

Hence, we have

$$\begin{aligned} J_0(\alpha_t) - J_0(\alpha_{t+\Delta}) &= \lambda\alpha_t\Delta + \left[(1-\lambda\alpha_t\Delta)e^{-r\Delta} - 1\right] J_0(\alpha_{t+\Delta}) \\ &> \lambda\alpha_t\Delta - \lambda\alpha_{t+\Delta}\Delta > 0 \end{aligned}$$

and therefore J_0 is strictly increasing in α_t. For small α, $J_0(\alpha) < \underline{v}$, but there is a $\underline{\alpha}(\Delta)$ such that $J_0(\alpha) > \underline{v}$ for all $\alpha > \underline{\alpha}(\Delta)$. When $J_0(\alpha)$ is greater than $\underline{v}$, the seller does not want to post $p = \underline{v}$. Hence, fix Δ and T, there is a sufficiently large $\alpha_0 < 1$ such that $\alpha_T > \underline{\alpha}(\Delta)$. Thus, $p_t > \underline{v}$. Since the updating process of α_t is exogenous, for any finite T, we can find an α_0 large enough but smaller than one such that $\alpha_T > \underline{\alpha}(\Delta)$. Take the smallest α_0 such that $\alpha_T > \underline{\alpha}(\Delta)$ as $\bar{\alpha}$.

Q.E.D.

Proof of Proposition 9:

Suppose not. Consider the following equilibrium outcome induced by an equilibrium strategy profile which can be described by (σ, κ), the state evolves as follows: on the path of play, suppose that a screening offer p_{-1} induces a state (k, α). The seller optimally posts two consecutive screening offers p_1 and p_2 in the following two periods, which induce state variables (k', α') and (k'', α'') respectively. We will construct a profitable deviation such that (1) the seller posts screening offer $\tilde{p}$ instead of p_1, and induces (k'', α').

By using three consecutive screening offers, the seller's payoff can be decomposed into three parts: the flow payoff in two periods, and the continuation payoff conditional on no trade.

The flow payoff in the first period is

$$[(\frac{F(k)-F(k')}{F(k)})+\frac{F(k')}{F(k)}\lambda\Delta\alpha]p_1,$$

where $(\frac{F(k)-F(k')}{F(k)})$ is the probability that the long-lived buyer takes the offer, and $\frac{F(k')}{F(k)}\lambda\Delta\alpha$ is the probability that a short-lived buyer takes the offer in the first period.

The flow payoff in the second period is

$$\frac{F(k')}{F(k)}(1-\lambda\Delta\alpha)\times(1-r\Delta)\,[\frac{F(k')-F(k'')}{F(k')}+\frac{F(k'')}{F(k')}\lambda\Delta\alpha'']p_2,$$

where $\frac{F(k')}{F(k)}(1-\lambda\Delta\alpha_t)$ is the probability that no trade happens in the first period, $\frac{F(k')-F(k'')}{F(k')}$ is the conditional probability that the long-lived buyer takes the offer p_2, and $\frac{F(k'')}{F(k')}\lambda\Delta\alpha'$ is the conditional probability that the short-lived buyer takes the offer in the second period.

The continuation value is

$$\frac{F(k')}{F(k)}\frac{F(k^{''})}{F(k')}(1-\lambda\Delta\alpha)(1-\lambda\Delta\alpha')\,(1-r\Delta)^2\,R(k'',\alpha''),$$

where $\frac{F(k')}{F(k)}\frac{F(k^{''})}{F(k')}(1-\lambda\Delta\alpha)(1-\lambda\Delta\alpha')$ is the probability that no trade happens in the first two periods, and $R(k'',\alpha'')$ is the seller's continuation value conditional on his belief on both long-lived buyer's value and short-lived buyer's existence.

Consider an alternative pricing strategy which we described above. By construction, κ is right continuous in p for any α; thus, the seller can always find the price $\tilde{p}$ such that k'-long-lived buyer is indifferent between taking $\tilde{p}$ given α_t and waiting.

The seller's payoff can be decomposed into two parts again: the flow payoff in the first period and the continuation payoff.

The flow payoff in the first period is

$$[(\frac{F(k)-F(k'')}{F(k)})+\frac{F(k'')}{F(k)}\Delta\lambda\alpha]\tilde{p},$$

where $\tilde{p}$ is the deviation price which can induce k''-long-lived buyer to be the marginal type.

Discounted continuation value is

$$\frac{F(k'')}{F(k)}(1-\lambda\Delta\alpha)\,(1-r\Delta)\,R\,(k'',\alpha')\,.$$

By definition, $R\,(k'',\alpha')\geq J\,(k'',\alpha')$; thus the lower bound of deviation payoff is

$$\tilde{R}\,(k,\alpha)=[(\frac{F(k)-F(k'')}{F(k)})+\frac{F(k'')}{F(k)}\Delta\lambda\alpha]\tilde{p}+\frac{F(k'')}{F(k)}(1-\lambda\Delta\alpha)\,(1-r\Delta)\,J\,(k'',\alpha')\,.$$

We are going to show that this lower bound is greater than the equilibrium payoff $R\,(k,\alpha)$ when Δ is small. Note that $J\,(k'',\alpha')$ can be decomposed into two parts: the flow payoff in the second period,

$$\frac{F(k'')}{F(k)}(1-\lambda\Delta\alpha)\,(1-r\Delta)\,\lambda\Delta\alpha'\times 1,$$

where the price is 1, and the continuation value,

$$\frac{F(k'')}{F(k)}(1-\lambda\Delta\alpha)(1-\lambda\Delta\alpha')\,(1-r\Delta)^2\,R(k'',\alpha'').$$

Multiplying $F(k)$ to the lower bound of the difference of the payoff induced by two strategy yields

$$
\begin{aligned}
&(F(k)-F(k'))(\tilde{p}-p_1)+(F(k')-F(k''))(\tilde{p}-(1-r\Delta)\,p_2)\\
&+\Delta\lambda\alpha(F(k'')\tilde{p}-F(k')p_1)+F(k'')(1-\lambda\alpha)\,(1-r\Delta)\,\lambda\Delta\alpha'(1-p_2).
\end{aligned}
$$

The sum of first three terms of the difference is

$$
\begin{aligned}
&(F(k)-F(k'))(\tilde{p}-p_1)+(F(k')-F(k''))(\tilde{p}-(1-r\Delta)\,p_2)\\
&+\lambda\alpha\Delta(F(k'')\tilde{p}-F(k')p_1).
\end{aligned}
$$

We claim it is $O(\Delta^2)$ because the following reason. In each period, there is a marginal type of the long-lived buyer who is indifferent between taking the offer now and waiting for a lower price latter. Hence, we have

$$
k'-p_1=(1-r\Delta)\,(1-\lambda\alpha_t\Delta)\,(k'-p_2)\,,
$$

in equilibrium; thus p_1-p_2 is $O(\Delta)$ from simple algebra. Same logic, $p_{-1}-p_1$ is also $O\,(\Delta)$. By condition 1, both $k-k'$ and $k'-k''$ are also at most $O(\Delta)$; thus, $F(k)-F(k^{'})\leq(k-k')\bar{f}$ and $F(k')-F(k^{''})\leq(k'-k'')\bar{f}$ are also at most $O(\Delta)$.

To induce k'' rather than k' in the first period, the seller has to decrease the price to $\tilde{p}<p_1$. By condition 2, we have that the difference between p_2 and

The last term of the difference is

$$F(k'')(1-\lambda\Delta\alpha)\,(1-r\Delta)\,\lambda\Delta\alpha'(1-p_2) > 0,$$

which is $O(\Delta)$ since k'' is bounded away from zero, and $p_2 < k' < 1$. Keep applying the above logic if $\sigma(k'', \alpha') \neq 1$. A finite number of repetitions of the above argument complete the proof.

Keep applying the above logic can prove that no more than N consecutive screening offer will show up in equilibrium when Δ is small. Note the cutoff Δ_α depends on α! Yet, on the path of play, there are finitely many realization of $\alpha \in B_\Delta(\alpha_0) \setminus B_\Delta\left(\alpha^\dagger\right)$, hence we can find the smallest $\bar{\Delta} = \min\{\Delta_\alpha\}$ for all possible $\alpha \in B_\Delta(\alpha_0) \setminus B_\Delta\left(\alpha^\dagger\right)$ such that when $\Delta \in (0, \bar{\Delta})$, the statement holds. *Q.E.D.*

Appendix C

Appendix for Chapter 3

C.1 Omitted Proof

Proof of Proposition 10

Existence and Uniqueness. The proof follows the standard contraction mapping fixed-point argument. Given any $S(u) \in [\frac{b}{r}, \frac{\lambda}{r}]$, from (3.2,3.4,3.5), there exists a unique solution $S(\alpha)$ such that

$$S(\alpha) = S(u) + \int_{\alpha_*}^{\alpha} h(x, S(x))dx, \tag{C.1}$$

where $h(\alpha, S(\alpha))$ is the right-hand side of (3.5) divided by r.

And $\theta(u) = \arg\max\{p(\theta)[S(\alpha_0) - S(\alpha)] - k\theta\}$. For an unemployed worker,

$$rS(u) = b + \max_{\theta}\{p(\theta)[S(\alpha_0) - S(u)] - k\theta\},$$

or

$$S\left(u\right)=\frac{b+p(\theta\left(u\right))S(\alpha_0)-k\theta\left(u\right)}{r+p(\theta\left(u\right))}. \tag{C.2}$$

The envelop theorem implies that

$$\frac{dS\left(u|S(\alpha_0)\right)}{dS(\alpha_0)}=\frac{p(\theta\left(u|S(\alpha_0)\right))}{r+p(\theta\left(u|S(\alpha_0)\right))}\in(0,1). \tag{C.3}$$

Combining (C.1) and (C.2) yields

$$S(\alpha)=\frac{b+p(\theta\left(u\right))S(\alpha_0)-k\theta\left(u\right)}{r+p(\theta\left(u\right))}+\int_{\alpha_*}^{\alpha}h(x,S(x))dx.$$

Define an operator $\boldsymbol{T}_p : C[0,\alpha_0]\to C[0,\alpha_0]$ where $S(\alpha)\in C[0,\alpha_0]$ is any bounded continuous differentiable function and $\boldsymbol{T}_pS=\frac{b-p(\theta(u))S(\alpha_0)-k\theta(u)}{r+p(\theta(u))}+\int_{\alpha_*}^{\alpha}h(x,S(x))dx$ where S is such that (C.1) and (C.2) hold. To prove the uniqueness of $S(\omega)$, one needs to verify whether $\boldsymbol{T}_p$ is a contraction mapping. For the second part of $\boldsymbol{T}_pS$, by the standard contraction mapping argument of the existence and uniqueness to the solution in the problem of an ordinary differential equation with initial value, $\boldsymbol{T}_p^2S=\int_{\alpha_*}^{\alpha}h(x,S(x))dx$ satisfies the Blackwell sufficient condition. Moving to the first part, $\boldsymbol{T}_p^1$, one needs to check whether the Blackwell sufficient condition holds.

Let $S_1>S_2$, then $S\left(u|S_1\left(\alpha_0\right)\right)>S\left(u|S_2(\alpha_0\right)$ following (C.3); thus, the monotonicity of $\boldsymbol{T}_p^1$ is proved. Move to the discounting property. Let $n\in\mathbb{R}^+$, and $S_3=S_1+n$. Following (C.3), $\frac{dS(u|S(\alpha_0))}{dS(\alpha_0)}<1$ for all $S\left(\alpha_0\right)\in\left[S_1\left(\alpha_0\right),S_1\left(\alpha_0\right)+n\right]$; thus the discounting property of $\boldsymbol{T}_p^1$ holds. Hence, $\boldsymbol{T}_p=\boldsymbol{T}_p^1+\boldsymbol{T}_p^2$ satisfies the Blackwell sufficient condition, and therefore it is a contraction mapping on a complete functional space, $C[0, \alpha_0]$. There exists a unique solution $S(\alpha)$ such that $S(\alpha) = \boldsymbol{T}_p S(\alpha)$, and $S(u)$ is also determined uniquely!

Convexity. Consider two α_1, α_2 such that (1) $\alpha_1, \alpha_2 \in (0, \alpha_0]$, and (2) $\alpha_* < \alpha_2 < \alpha_1 \leq \alpha_0$. Let $\alpha^\zeta = \zeta\alpha_2 + (1-\zeta)\alpha_1$ where $\zeta \in (0,1)$. I want to show that for all α_1, α_2 and α^ζ, $S(\alpha^\zeta) \leq \zeta S(\alpha_2) + (1-\zeta)S(\alpha_1)$. Denote $\theta^\zeta(\alpha)$ as the path of optimal on-the-job search starting from α^ζ during the current match. Denote by $S^g(\theta)$ the expected surplus from an arbitrary path of on-the-job search θ conditional on the true match quality being good and similarly for $S^b(\theta)$. Then $S(\alpha^\zeta) = \alpha^\zeta S^g(\theta^\zeta) + (1-\alpha^\zeta)S^b(\theta^\zeta)$. And it holds that

$$
\begin{aligned}
S(\alpha_1) &\geq \alpha_1 S^g(\theta^\zeta) + (1-\alpha_1)S^b(\theta^\zeta), \\
S(\alpha_2) &\geq \alpha_2 S^g(\theta^\zeta) + (1-\alpha_2)S^b(\theta^\zeta),
\end{aligned}
$$

since θ^ζ is a feasible price path. Hence

$$
\begin{aligned}
\zeta S(\alpha_2) + (1-\zeta)S(\alpha_1) &\geq \left\{ \begin{array}{c} \zeta[\alpha_1 S^g(\theta^\zeta) + (1-\alpha_1)S^b(\theta^\zeta)] \\ +(1-\zeta)[\alpha_2 S^g(\theta^\zeta) + (1-\alpha_2)S^b(\theta^\zeta)] \end{array} \right\} \\
&= (\zeta\alpha_1 + (1-\zeta)\alpha_2)S^g(\theta^\zeta) \\
&\quad +(1-\zeta\alpha_1 - (1-\zeta)\alpha_2)S^b(\theta^\zeta) \\
&= \alpha S^g(\theta^\zeta) + (1-\alpha)S^b(\theta^\zeta) = S(\alpha),
\end{aligned}
$$

which contradicts the fact that the solution of the HJB function maximizes the planner's individual worker problem. This proves the claim.

Monotonicity of $S(\alpha)$. Since $S'' \geq 0$, and $S'(\alpha_*) = 0$, $S' \geq 0$ for all $\alpha \in [\alpha_*, \alpha_0]$.

There are three cases. The first one is $S' = 0$ for all $\alpha \in [\alpha_*, \alpha_0]$, which implies that $\alpha_* = \alpha_0$. The second one is that $S' = 0$ for all $\alpha \in [\alpha_*, \hat{\alpha}]$ but $S' > 0$ for $\alpha \in (\hat{\alpha}, \alpha_0]$. But in this case, $S(\alpha) = S(u)$ and $S' = 0$ for $\alpha \in (\alpha_*, \hat{\alpha}]$, which contradicts the definition of α_*. The third one is that $S' > 0$ for all $\alpha \in (\alpha_*, \alpha_0]$.

Properties of $\theta(\alpha)$. The optimal on-the-job search decision satisfies $p'(\theta(\alpha))[S(\alpha_0) - S(\alpha)] = k$. When $\alpha \to \alpha_0$, $S(\alpha) \to S(\alpha_0)$, so $p'(\theta(\alpha)) \to 0$ and $\theta(\alpha) \to 0$. Differentiating $p'(\theta(\alpha))[S(\alpha_0) - S(\alpha)] = k$ yields

$$p''(\theta(\alpha))[S(\alpha_0) - S(\alpha)]\theta'(\alpha) = p'(\theta)S'(\alpha),$$

since $S(\alpha_0) - S(\alpha) > 0$ for any $\alpha < \alpha_0$, $p'' < 0$, I have $\theta'(\alpha) < 0$. Also, p', p'' and $S(\alpha_0) - S(\alpha)$ is bounded. When α goes to α_*, $S'(\alpha)$ goes to zero, and therefore θ' goes to zero. *Q.E.D.*

Proof of Proposition 12

The optimality of workers' OJS search implies that $\theta(\alpha)$ must satisfy the following first-order condition: $p'(\theta)[x(\theta) - M(\alpha)] + p(\theta)x'(\theta) = 0$, which implies that

$$x'(\theta(\alpha)) = -\frac{p'(\theta(\alpha))[x(\theta(\alpha)) - M(\alpha)]}{p(\theta(\alpha))} < 0.$$

By Proposition 10, $\theta'(\alpha) < 0$; thus, I have $\frac{dx}{d\alpha} = x'(\theta(\alpha))\theta'(\alpha) > 0$. *Q.E.D.*

Proof of Proposition 13

Differentiate ξ_t^{ee} yields

$$\dot{\xi}_t^{ee} = \underbrace{p'\theta'\dot{\alpha}\sigma_t}_{(+)} - \underbrace{\alpha_0\lambda\exp(-\lambda t)p(\theta(\alpha))}_{(+)}.$$

The first term of the right hand side is that conditional on the match not having sent a good signal before, the matched worker becomes more pessimistic over time, and therefore, his on-the-job search becomes more aggressive. Thus the probability of getting a new job becomes greater, and this raises the EE rate. The second one that lowers the EE rate is simply the decreasing probability of a good match not having sent a good signal. When t is small, by assumption of the matching function, $p'(\theta)$ is large, so the first force dominates the second. As t approaches t^*, α goes to α_*. By Lemma 10, $\theta'(\alpha)$ goes to zero, which implies that the effect of the first force goes to zero. Hence, the second one becomes dominant. Yet, if a random match's tenure is greater than t^*, only the good match can survive, in which case the EE transition rate is zero. *Q.E.D.*

C.2 Stationary Distribution

In a large labor market with a continuum population of workers, by assuming "the law of large number" holds, the invariant distribution, if it exists, can be interpreted as the stationary cross-sectional distribution of workers' state. In particular, the following Proposition shows that the market equilibrium has the unique stationary wage distribution μ^* : There are two mass points at $\omega = 1$ and $\omega = u$. For $\omega \in [\alpha_*, \alpha_0]$, the probability density function is well defined. Denote the p.d.f. of stationary belief distribution as $\phi(\alpha)$ for $\alpha \in [\alpha_*, \alpha_0]$, $\beta = \mu^*(1)$ the probability mass at $\omega = 1$, and $\upsilon = \mu^*(u)$ at $\omega = u$.

Proposition 16. *The stationary distribution of workers' state, μ^*, is characterized by (υ, β, ϕ) where*

$$\upsilon = \mu^*(u), \beta = \mu^*(1), \phi(\alpha) \text{ is the pdf of } \mu^* \text{ for } \alpha \in [\alpha_*, \alpha_0],$$

the probability density function $\phi(\alpha)$ is given by

$$\phi(\alpha) = \kappa(\alpha)/A \text{ for } \alpha \in (\alpha_*, \alpha_0], \tag{C.4}$$

$$\phi(\alpha_*) = 0, \tag{C.5}$$

and β, υ such that

$$\beta = \frac{1}{\delta A} \int_{\alpha_*}^{\alpha_0} \lambda\alpha\kappa(\alpha)d\alpha, \text{ and } \upsilon = \frac{\int_{\alpha_*}^{\alpha_0} \lambda\alpha\kappa(\alpha)d\alpha + \int_{\alpha_*}^{\alpha_0} \delta\kappa(\alpha)d\alpha + 1}{Ap(\theta\left(u\right))}, \tag{C.6}$$

where

$$A = \int_{\alpha_*}^{\alpha_0} \kappa(\alpha)d\alpha + \frac{\int_{\alpha_*}^{\alpha_0} (\lambda\alpha + \delta)\kappa(\alpha) + 1}{\delta p(\theta\left(u\right))} \int_{\alpha_*}^{\alpha_0} \lambda\alpha\kappa(\alpha),$$

and

$$\kappa(\alpha) = \exp\left[\int_{\alpha_*}^{\alpha} \frac{\lambda s + \delta + p(\theta(s))}{\lambda s(1-s)} ds\right].$$

The stationary distribution can be calculated by making the inflow equal outflow at any $\omega \in \Omega^*$. The density at α_* is zero since the Markov process is right continuous with respect to calendar time. For interior point α, the only inflow comes from match with belief $\alpha' > \alpha$ that survives but has not sent an good signal, while the outflow is $\mu(\alpha)$. In the steady state, $\mu_{T_1}(\alpha) = \mu_{T_2}(\alpha) = \phi(\alpha)$ for any $T_1, T_2 \geq 0$.

$$\lambda\alpha(1-\alpha)\frac{d}{d\alpha}\phi(\alpha) = [\lambda\alpha + \delta + p(\theta(\alpha))]\phi(\alpha), \tag{C.7}$$

where $\phi(\alpha)$ is the probability density at α.

At α_0, the inflow comes from matched and unemployed workers who successfully find a new job; outflow is $\phi(\alpha_0)$, in the steady state, $\mu_{T_1}(\alpha_0) = \mu_{T_2}(\alpha_0)$ for any $T_1, T_2 \geq 0$, and thus I have

$$\int_{\alpha_*}^{\alpha_0} p(\theta(\alpha))\phi(\alpha)d\alpha + \upsilon p(\theta\,(u)) = \phi(\alpha_0), \tag{C.8}$$

where υ is the measure of u-workers. Both the left-hand side and right-hand side of (C.8) are finite, and therefore there is no mass point at α_0.

For unemployed workers, the inflow comes from the separation of an existing match, while the outflow is the measure of unemployed workers who find a job. Letting inflow equal outflow, I have

$$\nu p(\theta\,(u)) = \beta\delta + \int_{\alpha_*}^{\alpha_0} \delta\phi(\alpha)d\alpha + \phi_u, \tag{C.9}$$

where ϕ_u is the density of workers who have just been fired, β is the measure of 1-workers.

Combining (C.8) and (C.9) yields

$$\phi(\alpha_0) = \int_{\alpha_*}^{\alpha_0} [p(\theta(\alpha) + \delta]\phi(\alpha)d\alpha + \phi_u + \beta\delta. \tag{C.10}$$

For a successful match, the inflow comes from a new good signal sent by an existing uncertain match, and the outflow comes from the exogenous separation. Inflow equals outflow implies that

$$\beta = \frac{1}{\delta}\int_{\alpha_*}^{\alpha_0} \lambda\alpha\phi(\alpha)d\alpha. \tag{C.11}$$

Given the equilibrium $\theta(\alpha), \alpha_*$, and matching function $p(\cdot)$, one can obtain a general solution of the ODE (C.7), which is given by

$$\phi_{\tilde{A}}(\alpha) = \frac{1}{\tilde{A}}\exp[\int_{\alpha_*}^{\alpha} \frac{\lambda s + \delta + p(\theta(s))}{\lambda s(1-s)}ds], \tag{C.12}$$

where $1/\tilde{A}$ is a constant positive number to ensure $\phi > 0$. To fix A, one needs to use a boundary condition implied by the fact that ϕ is a density function and ν, β are the probability. The condition is given by

$$\int_{\alpha_*}^{\alpha_0} \phi(\alpha)d\alpha = 1 - \upsilon - \beta. \tag{C.13}$$

Plugging (C.9) and (C.11) into (C.13) yields

$$\int_{\alpha_*}^{\alpha_0} \phi(\alpha)d\alpha = 1 - \frac{\int_{\alpha_*}^{\alpha_0}(\lambda\alpha+\delta)\phi(\alpha)d\alpha + \phi(\alpha_*)}{p(\theta\,(u))} - \frac{1}{\delta}\int_{\alpha_*}^{\alpha_0} \lambda\alpha\phi(\alpha)d\alpha.$$

Let $\kappa(\alpha) = \exp[\int_{\alpha_*}^{\alpha} \frac{\lambda s+\delta+p(\theta(s))}{\lambda s(1-s)} ds]$, and $\phi_{\tilde{A}}(\alpha) = \kappa(\alpha)/\tilde{A}$. Then the $\tilde{A} = A$ satisfying the boundary condition (C.13) is given by

$$A = \int_{\alpha_*}^{\alpha_0} \kappa(\alpha) d\alpha + \frac{\int_{\alpha_*}^{\alpha_0} (\lambda\alpha + \delta)\kappa(\alpha) d\alpha + 1}{\delta p(\theta\,(u))} \int_{\alpha_*}^{\alpha_0} \lambda\alpha\kappa(\alpha) d\alpha.$$

Given the solution $\phi(\alpha)$, $\phi(\alpha_0)$, $\phi_u = \lim_{\alpha\to\alpha_*} \phi(\alpha) = 1/A$, and υ, β can be solved by (C.9) and (C.11). Since (C.4) and (C.6) uniquely pin down μ^*, the stationary distribution is unique. *Q.E.D.*

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