Coordinated Navigation of Multiple Independent Disk-Shaped Robots

C. Serkan Karagoz
Ford Motor

H. Isil Bozma
Bogazici University

Daniel E. Koditschek
University of Pennsylvania, kod@seas.upenn.edu

Follow this and additional works at: http://repository.upenn.edu/ese_papers
Part of the Electrical and Computer Engineering Commons, and the Systems Engineering Commons

Recommended Citation

The attached version is a pre-print version of the article.
For the post-print version click here.

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/ese_papers/711
For more information, please contact repository@pobox.upenn.edu.
Coordinated Navigation of Multiple Independent Disk-Shaped Robots

Abstract
This paper addresses the coordinated navigation of multiple independently actuated disk-shaped robots—all placed within the same disk-shaped workspace. Assuming perfect sensing, shared-centralized communications and computation, as well as perfect actuation, we encode complete information about the goal, obstacles, and workspace boundary using an artificial potential function over the configuration space of the robots’ simultaneous nonoverlapping positions. The closed-loop dynamics governing the motion of each (velocity-controlled) robot take the form of the appropriate projection of the gradient of this function. We impose (conservative) restrictions on the allowable goal positions that yield sufficient conditions for convergence: We prove that this construction is an essential navigation function that guarantees collision-free motion of each robot to its destination from almost all initial free placements. The results of an extensive simulation study investigate practical issues such as average resulting trajectory length and robustness against simulated sensor noise.

For more information: Kod’Lab

Disciplines
Electrical and Computer Engineering | Engineering | Systems Engineering

Comments
The attached version is a pre-print version of the article.

For the post-print version click here.
Coordinated Navigation of Multiple Independent Disk-Shaped Robots

C. Serkan Karagöz, H. İsıl Bozma, Member, IEEE, and Daniel E. Koditschek, Fellow, IEEE

Abstract—This paper addresses the coordinated navigation of multiple independently actuated disk-shaped robots - all placed within the same disk-shaped workspace. Assuming perfect sensing, shared centralized communications and computation, as well as perfect actuation, we encode complete information about the goal, obstacles and workspace boundary using an artificial potential function over the configuration space of the robots simultaneous non-overlapping positions. The closed-loop dynamics governing the motion of each (velocity-controlled) robot take the form of the appropriate projection of the gradient of this function. We impose (conservative) restrictions on the allowable goal positions, that yield sufficient conditions for convergence: we prove that this construction is an essential navigation function that guarantees collision-free motion of each robot to its destination from almost all initial free placements. The results of an extensive simulation study investigate practical issues such as average resulting trajectory length and robustness against simulated sensor noise.

Index Terms—Artificial potential functions, feedback-based navigation, coordinated motion, configuration spaces, reactive systems, swarm robots, autonomous robots.

I. INTRODUCTION

This paper addresses a geometrically simplified version of coordinated motion planning [1]. A collection of disk-like robots inhabits a two-dimensional disk-shaped workspace. Each velocity-controlled robot can move simultaneously with and independently of the other robots. Moreover, each has a specified goal location in which it needs to end up. The ensemble of these locations encodes the overall task. Departing from the classical coordinated motion planning paradigm in the manner of [5], [6], we further require that each robot’s control strategy be reactive. By this, we mean that all motion is generated by a vector field — a function of the instantaneous ensemble of locations, parametrized (in part) by the fixed ensemble of goals that returns at each instant a direction of motion for each robot. In this reactive setting, each robot must start from its arbitrary initial placement, confront the other robots as required dynamically and eventually end up in its goal position. Reactive planners offer the usual benefits of feedback relative to the traditional open-loop planners in their sensitivity to execution time disturbances and thus promise more efficient and robust performance. Of course, improperly designed feedback schemes can cause instability, hence the central problem is to demonstrate convergence.

This paper presents a formulation of the problem nearly identical to that of [5], [6] and proposes a similarly close solution2. As before, we assume complete centralized information about all the robots’ instantaneous positions as well as a fixed goal location assigned to each one. Again, we use this information to construct an artificial potential function and apply its gradient as a centralized controller communicated accurately and instantaneously to the fully actuated robot ensemble. However, now we offer the missing convergence proof, guaranteeing from almost every initial condition within the connected component the movement of all robots to their destinations without any collisions along the way. The coupled closed loop gradient dynamics governing the motion of the robot ensemble projects onto the coordinate slice corresponding to each individual robot a vector field sensitive to its own position as well as those of all the other robots. Although this approach is in principle applicable with complete generality to any navigation problem over a known configuration space [7], [8], and the construction for this very specific class of problems has essentially been in place for over two decades [6], [9], the present paper offers the first formal demonstration of its correctness. Analogous constructions have been shown to be correct in simpler, related versions of the problem [5], [10], [11]. But despite favorable simulation experience, the possibility of spurious local minima on which the system might get stuck has remained an open question. In summary, this paper shows for the first time that the line of reasoning and strategy originating in [8] can be extended constructively to coordinated navigation of disk-shaped robots in a disk-shaped workspace with complete information. Provided certain constraints on the allowed goal positions are satisfied, obstacle-free navigation to the goal placements from almost every initial placement of the robots lying in the connected component of the configuration space is guaranteed.

A. Coordinated Motion Planning

Traditionally, the coordinated motion problem has been viewed as a special case of the general open-loop motion plan-

1The extension of our control solution beyond this ‘quasi-static’ or ‘generalized damper’ [2] to the dynamical setting of a second order mechanical system (a motion where controlled forces generate changes in velocity) is almost immediate, according to the procedures discussed in [3], [4].

2The major advance beyond [5], [6] in this present formulation is that we can now handle a compact workspace via the imposition of an additional outer boundary as defined by Eq. 2 that makes the problem more broadly applicable but considerably harder.
ning problem. In this tradition, the kinematics of planning are separated from the dynamics of execution [12]. A geometric planner produces a trajectory in the joint configuration space of the ensemble of robots connecting a pre-specified initial condition to the fixed goal configuration (the total degrees of freedom are given by the sum of the individual machines’) [13]. This plan is then ‘guarded’ in real time execution by a local tracking controller. In these open-loop approaches, the focus is on developing computational geometric means that are assured of finding a path in the configuration space that does not violate any of the hypersurfaces encoding the constraints on the robots’ degrees of freedom [14], [15]. Most geometric approaches are based on roadmaps or cell decomposition [16], [17]. Furthermore, depending on how the planning is achieved, they are either classified as being centralized or decentralized [18]. Unfortunately, the computational complexity of the coordinated motion planning has proved to be PSPACE-hard even in two dimensional environments where only translations are allowed and when the final configuration specifying the final positions of all movable objects is known [1], [19].

This result has been viewed as a guide to calibration of problem difficulty and has led researchers to consider the more tractable, but restricted classes of the problem [13], [20], [21]. Against this backdrop, researchers have approached the problem by proposing heuristic or approximate schemes [13]. Centralized approaches propose various solutions such as transforming the problem into a series of subproblems [14], reducing the search dimensionality [22] or introducing additional constraints [23], [24]. Alternatively, in decentralized approaches, the path planner is distributed among the robots that possibly communicate [18], [25]. Intermediate problem formulations (mixing elements of centralized and distributed planning) have also been considered [26], [27], [28]. For all of these feedforward problem formulations, when there is any change in the robots’ objectives or the environment, complete recalculation of paths is required. Moreover, in obvious consequence of the heuristic nature of these schemes, there is no guarantee of completeness.

We take an approach within the extreme opposite paradigm: purely feedback-based motion planning. Despite the long established guaranteed existence of such planners in general [7], [8], [10], specific algorithms with provable properties for specific problem settings have been slow to appear. A good summary account of the many heuristic vector field planners that appeared in that decade (e.g., [29], [30], [31], [32], [8], [33]) can be found in [34], and a tutorial account of the following decade’s work in this vein (all of which is heuristic and suffers from possibility of local minima) can be found in [16], [17]. A major boost to the theoretical foundations of reactive planning has been contributed by the definition and formal toolbox of topological complexity [35] (which has been determined for this problem in [36], [37]).

In recent years, the construction of provably correct vector field planners has progressed along two major axes. First, a variety of general algorithmic approaches have been recently advanced by assuming the availability of a convex (e.g., cubical [38], or simplicial [39]) cellular decomposition. Notably, in [39], a smooth ($C^0$) global vector field is achieved by interpo-

lating local vector fields defined over each simplex, ensuring asymptotic convergence to the goal position while guaranteeing collision avoidance. The forbidding complexity of even algebraic [15], [40] much less convex cellular decomposition in the setting of general motion planning problems must give some pause in pursuing this direction. Some preliminary work [41] suggests that the regularity of multi-body configuration spaces such as arise in this problem may render convex cellular decompositions viable for low numbers of cooperating robots - but such computations must inevitably increase geometrically with the degrees of freedom. In contrast, that same regularity permits the use of the closed form expressions we study here, entailing merely quotients of quadratic functions and their gradients - a major benefit of the global analytical approach of this paper\(^3\).

A second direction of recent work on reactive planning has re-examined versions of the multiple disk navigation problem we treat here in response to the two decade old extension [5], [6] of the original navigation function solution to the single disk problem [8]. An excellent review of this more contemporary literature is provided in the most recent of these papers [43] and in [44] which also come the closest in their aims and methods to those of this paper. The chief difference of our work from [44] (and its extension to nonholonomically constrained disks [45]) is their focus on a partially decentralized problem version: all agents have global, instantaneous knowledge of all others’ positions, but an agent’s ultimate destination is known only to itself. Their navigation function has much greater complexity, apparently in consequence. Both this paper and [43] follow the original construction [6] and analysis [8] in their concern to exhibit a provably correct navigation function for multiple, fully actuated first order disk navigation under the assumption of noise-free global sensing and inter-agent communication, affording recourse to a completely centralized computation and exact, deterministic implementation of the associated gradient field as a control law. In [43], the construction departs in significant ways from that of [6], most notably by recourse to a continuous but non-differentiable navigation function, yet the pattern of analysis introduced in [8] is presented in nearly identical form, modulo the introduction of methods from nonsmooth analysis [46]. In this paper, our construction is similar to [6] with the addition that the workspace is bounded by an a priori specified radius in which all the robots are required to remain. Furthermore, notwithstanding the major overlap with the mode of analysis introduced in [8], we are forced to depart from that pattern at certain essential junctures as explained throughout the paper. In contrast to [43], our construction is smooth on the interior of the free space but of course cannot be smooth on the (non-smooth, sub-analytic) boundary\(^4\). Beyond the intrinsic interest in smooth controllers articulated originally in [7], a parallel

\(^3\) This tradeoff between analytically intricate, computationally simple vs. analytically simple, computationally intensive global representations in reactive motion planning seems to echo a more general pattern in computational topology [42].

\(^4\) Please see the discussion in Section I-D where we address this issue by relaxing the requirement for nondegeneracy over the closed freespace to merely over the interior.
literature initiated around the same time [4] employs the lift of a navigation function as a key component of obstacle-avoiding controllers for second order plants: in some important application settings this lift will require the jacobian of the original gradient field — for example see [47] for a very nice recent example of this approach applied to the dynamical version of the present setting of multiple coordinated vehicles.

\[ \text{C. The Problem Statement} \]

Consider a collection of \( p \) disk shaped robots lying on the same two dimensional workspace bounded by an outer disk. Each robot has two completely actuated degrees of freedom in this workspace, is assigned to a goal position vector and can move independently of the others. Thus each robot becomes an obstacle — possibly moving — for the remaining other robots. We assume that:

(i) Each robot has a ‘perfect’velocity controller that can achieve exactly and instantaneously any desired bounded planar velocity command vector;

(ii) At every instant, each robot has perfect real time knowledge of its own position; and

(iii) At every instant, each robot knows exactly the sizes and the locations of all the other robots at that instant.

(iv) For all time, each robot knows exactly its own goal location as well as that of all the other robots.

If there are \( p \) individual planar robots, then let \( b \in \mathbb{R}^{2p} \) denote the augmented state vector of all robots and \( g \in \mathbb{R}^{2p} \) denote the augmented state vector of all goal positions. As assumed in (i), above, we consider the simplest control setting and model their change of state \( \dot{b} \) according to control law: \( \dot{b} = u \). As discussed above, we will set the control input, \( u \), to be the gradient vector of an appropriate smooth map, \( \varphi : F \to [0, 1] \) on the free robot configuration space \( F \subset \mathbb{R}^{2p} \) (to be formally defined below) so that \( u = -\nabla \varphi \). The equilibria \( b(\infty) \) of this system constitute its fixed points. This task is successfully completed if \( b(\infty) = g \) or successfully terminated if \( b(\infty) \neq g \) (i.e., the system cannot cycle but must eventually converge to some critical point - the wrong one only from an initial condition set of measure zero [7], [8]).

\[ \text{D. Navigation Functions} \]

Since the basin of a point attractor is a topological ball [51], and the free space is not contractible [52] there clearly cannot exist vector fields that take every point \( b \in F \) to the goal \( g \). However, there is no such obstruction to smooth vector fields with a point attractor whose basin includes the connected component of the goal in \( F \) excluding a set of zero measure. We believe that the disadvantage of ‘losing the way’ on an ‘invisible’subset of freespace is offset by the many considerable advantages that dynamical systems based motion planning enjoy, as reviewed, for example in [10], hence our interest in the following class of scalar valued functions, originally defined in [7]. A map \( \varphi : F \to [0, 1] \) is a navigation function if it is:

\[ \text{Note that simple potential-dissipative controllers} \ [3], [4] \ \text{can lift an unmodified gradient field to achieve an asymptotically stable second order system with no need for further derivatives. However these simpler constructions do not achieve the same performance as required in applications such as [47], which follows a more aggressive approach originally proposed in [48], and developed in the subsequent literature [49], [50]. Intuitively, the difference in performance is akin to that between an underdamped vs. a critically damped LTI system, and the ability to regulate the transients in this manner is often quite important in practical settings.} \]

\[ \text{In these cases, when performance considerations motivate controlling the graph error [47]-[50], we know of no alternative to the sort of smooth construction we pursue here since even Lipschitz continuous non-smooth gradients yield unacceptable discontinuous lifts.} \]

\[ \text{The outer boundary which encloses an area four times that illustrated is not shown so as not to lose the desired detail of visualization.} \]
1) Analytic on $F$;
2) Admissible on $F$ — that is, it attains its maximum on the boundary $\partial F$.
3) Polar on $F$ — that is, its unique minimum occurs at the goal configuration $g \in F$;
4) Morse on $F$ — that is, all critical points are non-degenerate;

If the negative gradient of $\varphi$ is transverse on the boundary and directed inwards, all solutions of the gradient system approach the critical points where the gradient vanishes. If $\varphi$ is a Morse function (critical points are non-degenerate), then critical points are isolated, and the unstable equilibria attract a set of points whose measure is zero. In particular, if $g$ is a unique minimum of $\varphi$, then almost all points in the connected component of the goal, $g$, move toward it and asymptotically achieve it. Thus, an appropriately constructed $\varphi$ solves the geometric path planning problem. Moreover, if $\varphi$ is interpreted as an artificial potential function, then the gradient vector field leads to the automated generation of robots’ control velocities. Furthermore, within certain constraints, the robots’ limiting behavior is identical to that of the vector field.

We will find it convenient to relax point 4) of the definition above, and introduce the notion of an essential navigation function, by stipulating instead that $\varphi$ be:

4) Morse on $\bar{F}$ — All interior critical points are non-degenerate;

While the freespace interior is smooth, its boundary cannot be — there arises the familiar problem of ‘corner points’ [54] over which set the Hessian is undefined. Rather than introducing the machinery of non-smooth analysis as in [43], we simply relax the condition because it confers no advantage on the boundary. In other words, while degeneracy might possibly occur on $\partial \bar{F}$, no open set of initial conditions can be attracted to such critical points since $\varphi$ cannot increase along the motion of $-\nabla \varphi$.

E. Contribution of the Paper

The main contribution of the paper is to show that our construction (Eqs. 4-5) is indeed an essential navigation function. For in the present case of disk-shaped robots all moving independently in a disk-shaped workspace, this guarantees an exact coordinated navigation algorithm that employs feedback to drive all robots to their respective goals with no collisions along the way from almost every initial configuration in the connected component of the goal. More precisely, we show that with some conservative but readily computed restrictions on the goal positions, the constructed artificial potential function can be made to be an essential navigation function — by suitable assignment of the parameters that we prescribe exactly in Theorem 1 as a function of the known problem geometry.

II. THE CANDIDATE POTENTIAL FUNCTION

A. Notation

We will index the collection of $p \in \mathbb{Z}^+$ robots by the set $P = \{1, \ldots, p\}$. Each robot $i \in P$ is located by its center point $b_i \in \mathbb{R}^2$, parametrized by its radius $\rho_i \in \mathbb{R}^+$ and assigned a goal position $g_i \in \mathbb{R}^2$. The state $b \in \mathbb{R}^{2p}$ of all the robots is defined as $b \triangleq \sum_{i \in P} b_i \otimes e_i$, where $e_1, e_2, \ldots, e_p \in \mathbb{R}^p$ are the unit base vectors in $\mathbb{R}^P$. The aggregate goal vector $g \in \mathbb{R}^{2p}$ is defined by $g \triangleq \sum_{i \in P} g_i \otimes e_i$.

Now, define the index set of robot pairs $Q = \{(i, j) | i < j \}$. The cardinality of $Q$ is denoted by $q \triangleq |Q| = (p^2-p)/2$. For all robot pairs $(i, j) \in Q$, define their distance difference $d_{ij} \in \mathbb{R}^2$ as $d_{ij} \triangleq b_i - b_j$. Note that by definition $d_{ij} = (I_2 \otimes e_i^T)$, where $I_n$ is the $n$ dimensional identity matrix and $e_i \triangleq e_i - e_j$. The robots’ pairwise relative distance is $\delta_{ij} \triangleq ||d_{ij}||$. Similarly, their relative pairwise distance difference at the goal is $g_{ij} \in \mathbb{R}^2$ defined by $g_{ij} \triangleq g_i - g_j$. Again, by definition $g_{ij} = (I_2 \otimes e_i^T) g$. Let $Q^0$ denote the index set of robot pairs including the workspace boundary as a zeroth disk, that is, $Q^0 \triangleq Q \cup \{(0, i) | i \in P \}$.

The robots cannot overlap, so we require that:

$$\delta_{ij} \geq \rho_{ij} \triangleq \rho_i + \rho_j \quad \forall (i, j) \in Q \quad (1)$$

Differing from the original construction [6], the workspace is bounded by radius $\rho_0 \in \mathbb{R}^+$, hence each robot $i$ must remain inside a disk of radius $\rho_{0i} \triangleq \rho_0 - \rho_i$, that is:

$$||b_i|| \leq \rho_{0i} \quad \forall i \in P \quad (2)$$

The free robot configuration space $\bar{F}$, is defined as the subset of robot positions in $\mathbb{R}^{2p}$ which satisfy (1) and (2).

$$\bar{F} \triangleq \{ b \in \mathbb{R}^{2p} \mid (\forall i \in P, ||b_i|| \leq \rho_{0i}) \land (\forall (i, j) \in Q, \delta_{ij} \geq \rho_{ij}) \} \quad (3)$$

In other words, we are concerned with the closure of non-contacting placements. For the reader’s convenience, we have included in the Appendix a summary table of the principal notation introduced in this section as well as in Section III.

B. Construction

Following the recipes in [8] and [6], the candidate function $\varphi : F \to [0, 1]$ is constructed as the composition:

$$\varphi(b) = \sigma \circ \sigma \circ \tilde{\varphi}(b) \quad (4)$$

The function $\tilde{\varphi} : F \to [0, \infty)$ encodes the goal point and the obstacles of all the robots using the quotient of two functions $\gamma : F \to [0, \infty)$ and $\beta : F \to [0, \infty)$:

$$\tilde{\varphi}(b) \triangleq \frac{\gamma(k(b))}{\beta(b)} \quad k \in \mathbb{Z}^+ \quad (5)$$

The numerator $\gamma(b) \triangleq (b - g)^T(b - g)$ encodes the Euclidean distance from the goal. The denominator encodes the distance from freespace boundary and is defined as $\beta(b) \triangleq \prod_{(i, j) \in Q^0} \beta_{ij}(b)$, where $\forall (i, j) \in Q^0, \beta_{ij}(b) = \delta_{ij}^2 - \rho_{ij}^2$ and $\forall i \in P, \beta_{0i}(b) = \rho_{0i}^2 - ||b_i||^2$. The freespace boundary $\partial F$ is the zero level set of $\beta^{-1}(0)$ and entails robots touching each other or the workspace boundary. The parameter $k$ is a design parameter that determines the relative weight of these two terms. As will be seen in the sequel, $k$ plays a critical

\footnote{Here, $\otimes$ denotes the Kronecker product, where, if $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{r \times q}$, then $A \otimes B \in \mathbb{R}^{nr \times mq}$ with an $ij^{th}$ block of size $p \times q$ specified by $a_{ij} B$.}
role in ensuring that the function \( \varphi \) is an essential navigation function.

Since \( \hat{\varphi} \) blows up on \( \partial F \), it is not admissible. In order to make \( \hat{\varphi} \) admissible, it is squashed by the function \( \sigma : [0, \infty) \to [0, 1] \), defined by \( \sigma(x) = \frac{1}{1 + e^{-x}} \). The resulting function becomes admissible but the goal point \( g \) is a degenerate critical point. In order to restore the goal point’s non-degeneracy, the sharpening function \( \sigma_1 : [0, 1] \to [0, 1] \) is applied, given by \( \sigma_1(x) = x^{1/k} \). Thus, the resulting function \( \varphi \) becomes admissible and has non-degenerate minimum at \( b = g \).

C. Restriction on Goal Locus - \( g \)

Our proof requires a few natural restrictions on allowable goal positions \( g \). Similar constraints have been introduced for the different, but related versions of the problem in earlier studies. For example, to retain the geometry as well as the topology of a ‘sphere world’ in the freespace, the robot is defined as a point mass object in [8]. In [55], the minimal gap between any pair of obstacles is restricted to be larger than the diameter of the robot and the mated object. Our assumptions constrain how closely the robots may be commanded to locate finally with respect to each other and to the outer boundary in their goal positions. The goal \( g \) is allowed to be chosen from a subset of \( F \) subject to two assumptions given in the sequel.

First, it is helpful to introduce a classification of the freespace that is \( \epsilon \) away from the boundary by defining a notion of robot neighborhoods and their associated ‘clusters’. Much past research on the coordination of multiple robots has encountered the need to decompose a neighborhood of the configuration space boundary into a hierarchy of variously arranged clusters, the earliest mention of this idea known to us having been contributed in [56]. Most closely related to our present formulation of robot neighborhoods and their associated ‘robot clusters’ is the introduction in [44] of a family of ‘relation verification’ functions whose members roughly correspond to each of these different possible ‘clusters’ and, like ours, are indexed over all possible partitions of the set of agents. The cardinality of the collection of partitions grows super-exponentially in the cardinality of the base set. Fortunately, in our problem formulation, these clusters do not enter into the controller itself but only play a role in the analysis of correctness, specifically in Prop. 3.6. In contrast, likely because of their focus on the more challenging decentralized version of the problem, the obstacle term in the navigation functions that generate the controllers of [44] explicitly include each of these super-exponentially many factors.

Robot Neighborhoods: Let \( \epsilon \in \mathbb{R}^+ \) be an arbitrarily small design parameter that determines robot neighborhoods. In particular, its value is set as to ensure that

\[
0 < \epsilon < \rho'' \quad \text{where} \quad \rho'' = \min_{i \in P} \{ \rho_0 \} \tag{6}
\]

\( \forall i \in P \), define an \( \epsilon \)-neighbor set \( N_\epsilon(b, i) \subseteq P \) to be the indices of its closest neighbors – namely \( N_\epsilon(b, i) \triangleq \{ j \in P \mid 0 < \beta_{ij}(b) \leq \epsilon \} \). Now, recursively define the \( n \)th \( \epsilon \)-neighbor sets \( N_\epsilon^n(b, i) \subseteq P \) as \( N_0(b, i) := \{ i \} \) and

\[
N_\epsilon^{n+1}(b, i) := \left( \bigcup_{j \in N_\epsilon^n(b, i)} N_\epsilon(b, j) \right) \cap N_\epsilon^n(b, i) \tag{7}
\]

According to this definition, each \( (n + 1) \)st neighbor of robot \( i \) is \( \epsilon \) close to some \( n \)th neighbor of robot \( i \), but no closer - i.e. it is not \( \epsilon \) close to any \( (n - 1) \)st neighbor. The process is stopped when \( N_\epsilon^{n+1}(b, i) = \emptyset \).

Robot Clusters: Specify a partition \( \{ P_1(b), ..., P_s(b) \} \) where \( P_i(b) \in 2^P \) and \( s(b) \) is the number of cells in this partition using a recursively defined function \( P_1(b) \) and its complementary function \( P_1(b) \) as follows: The base step is given by

\[
r_1 := 1, \quad P_1(b) := \bigcup_{j=0}^{p-1} N_\epsilon^j(b, r_1) \tag{8}
\]

and the recursive step is given by

\[
r_{n+1} := \min \left( \bigcap_{j=0}^{p-1} \tilde{P}_j(b), \right) \quad P_{n+1}(b) := \bigcup_{j=0}^{p-1} N_\epsilon^j(b, r_{n+1}) \tag{9}
\]

stopping when \( \bigcap_{j=0}^{p-1} \tilde{P}_j(b) = \emptyset \). At each configuration this partition divides up the robots into distinctive clusters of “closest neighbors”. For convenience, we wish to keep track of the partition cell index set \( S(b) \triangleq \{ i \in P \mid i \leq s(b) \} \). It can be verified that \( \bigcup_{i \in S(b)} P_i(b) \) is a partition over the robot index set [57].

Next, consider an arbitrary cluster \( P' \subseteq P \) containing at least two elements \( |P'| \geq 2 \). Associate with it \( F' \subseteq F \)

\[
F' \triangleq \{ b \in F \mid \exists i \in S(b), P_i(b) = P' \} \tag{10}
\]

Let \( Q' \subseteq Q \) be the corresponding pair index set defined as:

\[
Q' \triangleq \{ (i, j) \in Q \mid i, j \in P' \} \tag{11}
\]

Finally define two derived problem parameters \( \Lambda' \) and \( \Lambda'' \) defined as follows:

\[
\Lambda' \triangleq \max_{b \in F'} \left\{ \sum_{(i, j) \in Q'} \delta_{ij} + \frac{2|P'| - 2}{\rho'} \left\| \sum_{n \in P'} J (b_n - \tilde{g}') \otimes e_n \right\| \right\} \tag{12}
\]

and

\[
\Lambda'' \triangleq \max_{b \in F', i = \arg \max_{n \in P'} \left\| b_n \right\|} \left\{ \sum_{j \in P'} \delta_{ij} \right\} \tag{13}
\]

where \( \rho' \triangleq \min_{i, j \in Q} \{ \rho_{ij} \} \), \( J \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) is the 90° planar rotation matrix and \( \tilde{g}' \triangleq \frac{1}{|P'|} \sum_{i \in P'} b_i \) is the centroid of the robots in the cell \( P' \).

With these definitions in place we are now ready to introduce the assumptions that restrict the allowable goal configurations. The first states that for any robot cluster, the goal positions of the robots in this group are separated from each other by a value of \( \Lambda' \). This term is the maximum value

\[11\text{We will denote by an overbar the complementary index set so that, for example, } \bar{N}_\epsilon(b, i) = P - N_\epsilon(b, i).\]

\[12\text{The symbol } \bigcup \text{ denotes disjoint union [54].}\]
of a function of the pairwise distances between the robots and their centroid. This maximization is over any cell containing these robots. Figure 2(Left) shows a workspace configuration containing three robots (big circles) which might block the way of each other while navigating to their goal positions (dark points) since the goal points are not separated enough according to Assumption 1.

Assumption 1: \( \forall P^i \in 2^P \) where \( |P^i| \geq 2 \)

\[ \sum_{(i,j) \in Q'} \|g_{ij}\| > \Lambda' \]

where \( Q' \) and \( \Lambda' \) are calculated according to the Eq. 7-8. □

The second assumption states that for any robot group, each goal position is not allowed to be located closer to the workspace boundary more than a value of \( \Lambda'' \). This term is the maximum value of the sum of the distances between the closest robot to the workspace boundary and the other robots. This maximization is over any cell containing these robots. Figure 2(Right) illustrates a disconnected free space as the robot radii are too large with respect to that of the workspace which is an infeasible goal position according to Assumption 2.

Assumption 2: \( \forall P' \in 2^P \) where \( |P'| \geq 2 \)

\[ |P'|\sqrt{\rho'^2 - \varepsilon} - \Lambda'' - \sum_{i \in P'} \|g_i\| > 0 \]

where \( \Lambda'' \) is calculated according to the Eq. 9. □

These assumptions, introduced to facilitate the proof as remarked above, are sufficient for the desired result, but involve bounds that have proven to be conservative in the simulations. For example, it seems clear that they guarantee a completely connected free space, but the dependence of the homotopy type of \( \mathcal{F} \) (including the conditions for its connectedness) on the disk radii is a delicate issue of great importance — indeed touching on such longstanding questions as the ancient sphere packing problem\(^{13}\) [58] — whose characterization goes far beyond the scope of the present paper. Nonetheless, for formal guarantees to hold, the goals would need to satisfy the two assumptions and the tuning parameter \( k \), would indeed need to be set as a function of these bounds.

\(^{13}\)For example, authors of [58] point out that determining conditions on the disk radii yielding a non-empty free space (e.g. the smallest radius precluding any free placement of movable uniform disks) rephrases the sphere packing problem in a bounded region. See [59] for a nice overview of the history of this problem which, as the author shows, stretches back at least a millennium prior to Kepler’s famous conjecture in 1611 [60].

III. THE CANDIDATE IS AN ESSENTIAL NAVIGATION FUNCTION

A. Statement of Main Theorem

If \( \varphi \) is a navigation function, then its associated gradient field automatically generates velocity control policies for each of the robots under whose joint influence they all achieve the desired goal, \( g \), from almost all initial conditions in its connected component of the freespace with the guarantee of no collisions along the way [61].

Theorem 1: For any goal \( g \) satisfying assumptions 1 and 2, there exists a positive integer \( K^* \in \mathbb{Z}^+ \) such that for every \( k > K^* \), the real-valued function

\[ \varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left( \frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)} \right)^{1/k} \]  

(10)

is an essential navigation function.

Proof: By definition, \( \varphi \) is analytic and admissible on \( \mathcal{F} \). By Proposition 3.1, assumptions 1 and 2 imply that there exists a positive integer \( K \in \mathbb{Z}^+ \) such that for every \( k > K \), \( \varphi \) is polar in \( \mathcal{F} \). By Proposition 3.2, assumptions 1 and 2 imply that there exists a positive integer \( N \in \mathbb{Z}^+ \) such that for every \( k > N \), \( \varphi \) is Morse on \( \mathcal{F} \). Taking \( K^* = \max\{K, N\} \), the result thus follows. □

B. Proof of Correctness

Consider the partition of the free configuration space \( \mathcal{F} \) into five disjoint subsets - following a line of reasoning inspired by that of [8]:

1) the goal point \( \{g\} \)
2) the boundary of the free space \( \partial \mathcal{F} = \beta^{-1}(0) \)
3) the set near the outer boundary \( \mathcal{F}_0(\varepsilon) = \{b \in \mathcal{F} | \exists i \in S(b), \exists j \in P_i(b), 0 < \delta_{ij}(b) \leq \varepsilon\} - (\{g\} \cup \partial \mathcal{F}) \)
4) the set near the internal obstacles \( \mathcal{F}_1(\varepsilon) = \{b \in \mathcal{F} | \exists i \in S(b), |P_i(b)| \geq 2\} - (\{g\} \cup \partial \mathcal{F} \cup \mathcal{F}_0(\varepsilon)) \)
5) the set away from the obstacles \( \mathcal{F}_2(\varepsilon) = \mathcal{F} - (\{g\} \cup \partial \mathcal{F} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon)) \)

Note that because the goal is held away from \( \partial \mathcal{F} \), \( \varepsilon \) is a design parameter as stated in Section II-C. Let \( C_{\varphi} \triangleq \{b \in \mathcal{F} | ||D\varphi(b)|| = 0\} \) denote the set of critical points of the function \( \varphi \). Let \( T : \mathcal{F} \to 2^{\mathcal{F}^o} \) denote the pair touching function — that is

\[ T(b) \triangleq \{(i,j) \in Q | \delta_{ij} = \rho_{ij}\} \cup \{(0,i) | i \in P | ||b_i|| = \rho_{bi}\} \]

The following proposition shows the absence of the local minima of function \( \varphi \).

Proposition 3.1: For any free robot configuration space \( \mathcal{F} \) constrained by Assumptions 1 and 2, there exists a positive integer \( K \in \mathbb{Z}^+ \) such that for every \( k > K \), the real-valued function

\[ \varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left( \frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)} \right)^{1/k} \]  

(11)

has unique minimum point at \( g \), that is, \( \hat{\varphi} \) is polar on \( \mathcal{F} \).

Proof: The polarity of \( \varphi \) is analyzed in each subset of \( \mathcal{F} \). Note that the functions \( \varphi \) and \( \hat{\varphi} \) have the same critical points with
the same type (minimum, maximum or a saddle) except at 
\( \partial F \).

1) By definition, \( \varphi(g) = \frac{\gamma(g)}{(\gamma^k(g) + \beta(g))^{1/k}} \). Taking the gradient 
\( \nabla \gamma(b) = 2(b - g) \) and noting that \( \gamma(g) = 0 \) and 
\( \nabla \gamma(g) = 0 \), 
\[
\nabla \varphi(g) = \left(\frac{\gamma^k(g) + \beta(g)}{\gamma^k(g)}\right)^{1/k} \times \left(\frac{\gamma^k(g) + \beta(g)}{\gamma^k(g) + \beta(g)}\right)^{1/k} \nabla \gamma(g) - \gamma(g)\nabla \left(\frac{\gamma^k(g) + \beta(g)}{\gamma^k(g) + \beta(g)}\right)^{1/k} = 0
\]

Then \( g \) is a critical point of \( \varphi \). Since \( \gamma(g) = 0 \), \( \varphi(g) = 0 \). 
Furthermore, by construction, \( \varphi: F \rightarrow [0,1] \), then \( g \) is a minimum point of \( \varphi \).

2) Next, consider \( \varphi \) on \( \partial F \). By definition, at least two 
robots must touch to each other or one robot must touch to the 
workspace boundary. Partition \( \partial F = \{ b \in \partial F : |T(b)| = 1 \} \cup \{ b \in \partial F : |T(b)| > 1 \} \). 
There are no critical points in \( \{ b \in \partial F : |T(b)| = 1 \} \) 
by Proposition 3.3 given in Section III-C. The critical 
points in \( \{ b \in \partial F : |T(b)| > 1 \} \) are maxima by 
Proposition 3.4.

3) \( \hat{\varphi} \) has no critical points in \( F_0(\varepsilon) \) by Proposition 3.5 - 
which asserts that for a given design parameter \( \varepsilon \), 
there exists a lower bound on the parameter \( k, K_3(\varepsilon) > 0 \), 
such that, if \( k > K_3(\varepsilon) \), then \( C_\varepsilon \cap F_0(\varepsilon) = \emptyset \).

4) The critical points in \( F_2(\varepsilon) \) are not minima by 
Proposition 3.6 – which asserts the following: For a given 
design parameter \( \varepsilon \), there exists a lower bound on the 
parameter \( k, K_3(\varepsilon) > 0 \), such that, if \( k > K_3(\varepsilon) \) then \( \hat{\varphi} \) 
has no minimum in any set \( F_1(\varepsilon) \).

5) \( \hat{\varphi} \) has no critical points in \( F_2(\varepsilon) \) by Proposition 3.7 – 
which asserts that for a given design parameter \( \varepsilon \) there 
extists a lower bound on the parameter \( k, K_1(\varepsilon) > 0 \), 
such that if \( k \geq K_1(\varepsilon) \) then \( C_\varepsilon \cap F_2(\varepsilon) = \emptyset \).

The proof of Proposition 3.1 is completed by choosing lower 
bound \( K > 0 \) on the parameter \( k \) as follows, 
\[
K = \max \{ K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon) \}
\] 
(12)

\( \Box \)

Non-degeneracy, the Morse property, is established by the next 
result, Proposition 3.2.

**Proposition 3.2:** For any free robot configuration space \( \hat{F} \) 
subject to Assumptions 1 and 2 and for a given design 
parameter \( \varepsilon \), there exists a positive integer \( N(\varepsilon) \in \mathbb{Z}^+ \) 
such that for every \( k > N(\varepsilon) \), the real-valued function, 
\[
\varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left(\frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)}\right)^{1/k}
\] 
(13)

has non-degenerate critical points, that is, \( \varphi \) is Morse in \( \hat{F} \).

**Proof:** The function \( \varphi \) is analyzed in each disjoint region of \( \hat{F} \).

1) The goal point \( g \) is a non-degenerate minimum point by 
Proposition 3.8.

2) There are no critical points in \( F_0(\varepsilon) \) by Proposition 3.5.

3) By Proposition 3.9, there exists a lower bound \( N(\varepsilon) > 0 \) 
on the parameter \( k \) such that if \( k > N(\varepsilon) \), then \( D^2 \hat{\varphi} \) 
restricted to \( F_1(\varepsilon) \) is non-singular.

4) There are no critical points in \( F_2(\varepsilon) \) by Proposition 3.7. 
If the parameter \( k \) is chosen accordingly, the result follows. \( \Box \)

Proposition 3.1 and Proposition 3.2 follow mostly a line of rea-
soning similar to their counterparts in [8]. However, they also 
depart from the respective analysis. First, in Proposition 3.1, 
we define partitions over the robot index set and use the 
robot clusters to “find” the unstable tangent direction. Second, 
Proposition 3.2 invokes Proposition 3.9 wherein we depart 
necessarily from the approach taken in [8]. In that problem 
setting, every saddle is associated with two complementary 
subspaces where the Hessian matrix is sign-definite with the 
corresponding negative and positive cones explicitly revealed 
by computation [7]. In contrast, the present problem introduces 
a configuration space of dimension \( 2p \) (with \( p > 1 \)), which 
is known to have nonzero Betti numbers [62] for every interme-
diate dimension [58]. Hence, according to the Morse inequalities 
[63], there must now be saddles of every index and the hope 
of explicitly revealing the corresponding positive and negative 
cones of each different type seems hopeless. Instead, here we 
abandon that geometric approach and instead focus directly on 
satisfying algebraic conditions for nonsingularity by appeal 
to notions of diagonal dominance. Specifically, we use a theorem 
to this effect by Sherman, Morrison and Woodbury [64] along 
with some related results in linear algebra [65].

C. **Polarity**

The details of proof of Proposition 3.1 are presented in 
this section. Due to space restrictions, some of the very 
detailed computations supporting the proofs of some of the 
constituent lemmas cannot be included in this paper. However 
they are available in [61], [57].

1) **The Free Space Boundary:** \( \partial F = \beta^{-1}(0) \): Referring to 
the definition of the pair-touching function \( T \) (defined in 
Section III-B), \( |T(b)| = 0 \) means no robots are touching each 
other and none of them is touching the workspace boundary. 
The free space boundary \( \partial F \) will be investigated for two cases: 
(i) Case 1: \( |T(b)| = 1 \), (ii) Case 2: \( |T(b)| = 2 \). The following 
propagation proves that there are no critical points on \( \partial F \) 
for Case 1.

**Proposition 3.3:** If \( |T(b)| = 1 \), then \( C_\varepsilon \cap \partial F = \emptyset \).

**Proof:** If \( |T(b)| = 1 \), then only one of the \( \beta \) is zero. 
Call this term \( \beta_{ln} \), \( (l,n) \in Q^0 \). Then, all the summation terms 
of \( \nabla \varphi \) vanish except the ones that containing \( \beta_{ln} \neq 0 \) and 
\( \nabla \beta_{ln} \neq 0 \). Hence, \( \nabla \varphi_{|\partial F} = \frac{1}{\nabla \beta_{ln} \nabla \beta_{ln}} \neq 0 \).

The following proposition proves that \( \varphi \) admits maximum 
valued critical points on \( \partial F \) for Case 2.

**Proposition 3.4:** If \( |T(b)| \geq 2 \), then \( C_\varepsilon \cap \partial F \) contains 
only maximum valued critical points.

**Proof:** Since \( |T(b)| \geq 2 \), \( \exists (i,j), (l,n) \in T \), such that 
\( \beta_{ij} = \beta_{ln} = 0 \). Then, all the summation terms of \( \nabla \varphi \) 
vanish except the ones containing \( \beta_{ij} \neq 0 \) or \( \beta_{ln} \neq 0 \), 
resulting in \( \nabla \varphi_{|\partial F} = \frac{1}{\nabla \beta_{ij} \nabla \beta_{ln}} = 0 \). But \( \varphi: F \rightarrow [0,1] \) 
and \( \varphi_{|C_\varepsilon \cap \partial F} = \frac{1}{\nabla \beta_{ij} \nabla \beta_{ln}} = 1 \), which means that those critical 
points achieve the maximum value of \( \varphi \). \( \Box \)

2) **The Set Near the Outer Boundary:** \( F_0(\varepsilon) \): The following 
propagation shows that there are no critical points in \( F_0(\varepsilon) \) - 
the subspace of \( F \) that is close to the outer boundary.
Proposition 3.5: For a given design parameter \( \varepsilon \), there exists a lower bound on the parameter \( k \), \( K_1(\varepsilon) > 0 \), such that, if \( k > K_1(\varepsilon) \), then \( C_{\varepsilon} \cap \mathcal{F}_0(\varepsilon) = \emptyset \).

Proof: (By contradiction) By definition, \( \forall b \in \mathcal{F}_0(\varepsilon) \) if \( \phi(b) = \bigcup_{i \in S(b)} P_i(b) \) is the corresponding partition then \( \exists i \in S(b) \) such that \( \exists j \in P_i, \delta_{ij} \leq \varepsilon \). In other words, there exists at least one cell consisting of at least one robot close to the workspace boundary.

First, denote the cell which is arbitrarily chosen from the cells consisting of at least one robot close to the boundary by \( P_z \). Let \( z' \) refer to the index of the closest robot to the boundary in the cell \( P_z \), that is, \( z' = \arg\max_{i \in P_z, \delta_{iz} \leq \varepsilon} \{||b_i||\} \). If \( b \) is a critical point, then \( k \beta \nabla \gamma = \gamma \nabla \beta \). After expanding the terms \( \nabla \gamma \) and \( \nabla \beta \), using the definitions \( b \) and \( g \), letting \( \alpha_{ij} = \frac{1}{\kappa^2} \gamma((i,j) \in Q^0, \xi \triangleq \{(i,j) \in Q | i,j \in P_z\} \) and \( Q_{\xi} \triangleq Q \setminus Q_z \) decompose the summation over \( Q \) and \( P \) respectively and simplify as:

\[
\sum_{(i,j) \in Q_{\xi}} (1 + \alpha_{0n})b_{0n} = \sum_{n \in P_z} g_n + \sum_{(i,j) \in Q_{\xi}} \alpha_{ij} d_{ij}
\]

After taking the magnitude of both sides and applying the triangle inequality, using \( b_n = b_z + \delta_{nz} \) on the left-hand side and maximizing \( \delta_{ij}/\beta_{ij} \), taking minimum of left-hand side and finally using \( \forall n \in P_z, \alpha_{0n} > 0 \)

\[
\sum_{n \in P_z} (||b_z|| - \delta_{nz}) \leq \sum_{n \in P_z} ||g_n|| + \sum_{(i,j) \in Q_{\xi}} \sqrt{\rho^2 + \varepsilon}.
\]

Recall that \( \rho'' = \min_{i \in P} \{\rho_{bi}\} \). Using \( \min_{\xi \in \mathcal{F}_0(\varepsilon)} \{||b_z||\} = \sqrt{\rho''^2 - \varepsilon} \) and minimizing left-hand side,

\[
|P_z|\sqrt{\rho''^2 - \varepsilon} - \sum_{n \in P_z} \delta_{nz} = \sum_{n \in P_z} ||g_n|| \leq \frac{\sqrt{\sum_{(i,j) \in Q_{\xi}} \rho^2} + \varepsilon}{\kappa^2}
\]

Using Assumption 2, if \( g \) is chosen appropriately the left-hand side of the above inequality will be positive. If \( k \) is chosen as,

\[
k > \frac{\gamma \sum_{(i,j) \in Q_{\xi}} \sqrt{\rho^2 + \varepsilon}}{|P_z|\sqrt{\rho''^2 - \varepsilon} - \sum_{n \in P_z} \delta_{nz} - \sum_{n \in P_z} ||g_n||} \triangleq K_1(\varepsilon)
\]

then \( b \) cannot be a critical point. Thus, \( \phi \) has no critical points in \( \mathcal{F}_0(\varepsilon) \). Further details can be found in [61] or [57].

3) The Set Near the Internal Obstacles: \( F_1(\varepsilon) \): The following proposition shows that \( \phi \) has no minimum in \( F_1(\varepsilon) \) - the subset of \( F \) that is close to the internal obstacles.

Proposition 3.6: For a given design parameter \( \varepsilon \), there exists a lower bound on the parameter \( k \), \( K_2(\varepsilon) > 0 \), such that, if \( k > K_2(\varepsilon) \) then \( \phi \) has no minimum in any set \( F_1(\varepsilon) \).

Proof: It is sufficient to show that for \( C_{\varepsilon} \cap F_1(\varepsilon) \), \( \forall b \in R^{2p} \) such that \( v^T D^2 \phi v < 0 \). By definition, \( \forall b \in F_1(\varepsilon) \), there is a partition \( \bigcup_{i \in S(b)} P_i(b) \) such that \( \exists i \in S(b) \) where \( |P_i(b)| \geq 2 \). Pick arbitrarily a cell consisting of at least two robots and denote by \( P_z \) - that is \( |P_z| \geq 2 \). Now consider the following vector, \( v_2 \triangleq \sum_{n \in P_z} J(b_n - \hat{g}_z) \otimes e_n \) where \( \hat{g}_z \) denotes the centroid of the robots in the cell \( P_z \), \( \hat{g}_z \triangleq \frac{1}{|P_z|} \sum_{n \in P_z} b_n \). We have chosen this vector based on our following observation in the simulations: When the robots are getting close to each other, each starts moving in a direction perpendicular to line between their center and the cell centroid. Recall that \( Q_z \triangleq \{(i,j) \in Q | i,j \in P_z\} \) and \( P_z \triangleq P \setminus P_z \).

\[
A \triangleq \left( \sum_{(i,n) \in Q_z} ||g_n|| - \sum_{(i,n) \in Q_z} \delta_{in} - \frac{2p_z - 2}{\rho'} \frac{v_2}{||v_2||^2} \right)^2
\]

Doing some manipulations and grouping the terms on the right-hand side as follows, \( \frac{\rho''}{\gamma} \varepsilon v^T D^2 \phi v \leq \frac{\gamma}{2p_z - 2} \frac{A - \gamma(p_z-1)}{2\kappa^2} \frac{(||p_z-p_z-1)|| + p_z}{\delta_{ij} \beta_{ij}} \)

\[
+ \sum_{j \in P_z} \left( \frac{2}{\beta_{ij}} \left[ J(b_j - \hat{g}_z) \right] \frac{1}{\beta_{ij}} \frac{||b_j - \hat{g}_z||^2}{\sigma_3}\right)
\]

\[
+ \sum_{j \in P_z} \left( \frac{2}{\beta_{ij}} \left[ J(b_j - \hat{g}_z) \right] \frac{1}{\beta_{ij}} \frac{||b_j - \hat{g}_z||^2}{\sigma_2}\right)
\]

Let \( \sigma_2 \triangleq \sigma_2' + \sigma_2'' \). Note that \( Q_z \triangleq Q \setminus Q_z \). If \( g \) is chosen according to Assumption 1, then term \( A > 0 \). If \( k \) is chosen as,

\[
k > \max_{\forall b \in F_1(\varepsilon)} \left\{ \frac{\gamma(p_z-1)}{2p_z-2} \frac{(||p_z-p_z-1)|| + p_z}{\delta_{ij} \beta_{ij}} \right\} \triangleq K_{21}(\varepsilon)
\]

then \( \sigma_1 > 0 \). Thus, a sufficient condition to make \( \varepsilon v^T D^2 \phi v \leq 0 \), is

\[
k > \max_{\forall b \in F_1(\varepsilon)} \left\{ \frac{\sigma_2 + \sigma_3}{\sigma_1} \right\} \triangleq K_{22}(\varepsilon)
\]

Finally, the proof is completed by choosing, \( K_2(\varepsilon) = \max\{K_{21}(\varepsilon), K_{22}(\varepsilon)\} \).

4) The Set Away From the Obstacles: \( F_2(\varepsilon) \): The following proposition shows that for sufficiently large \( k \) values, there are no critical points in \( F_2(\varepsilon) \).

Proposition 3.7: For a given design parameter \( \varepsilon \) there exists a lower bound on the parameter \( k \), \( K_3(\varepsilon) > 0 \), such that if \( k \geq K_3(\varepsilon) \) then \( C_{\varepsilon} \cap F_2(\varepsilon) = \emptyset \).

Proof: \( \forall b \in \mathcal{C}_\varepsilon \), \( \beta \gamma \nabla \gamma = \gamma \nabla \beta \). Taking the norm of the both sides and re-arranging terms in \( 2k/\beta = \sqrt{\gamma}||\nabla \beta|| \)

\[
k = \frac{\sqrt{\gamma}||\nabla \beta||}{2\beta} \tag{14}
\]
If \( k \) is selected to have value,
\[
 k > \max_{\phi \in \mathcal{F}_2(\epsilon)} \sqrt{\sum_{(i,j) \in Q} \sqrt{\beta_{ij}^2 + \epsilon} + \sum_{i \in P} \sqrt{\beta_{ii}^2 - \epsilon}}
\]
\[
\triangleq K_2(\epsilon)
\]
then, Eq. 14 does not hold which in turn implies that there are no critical points in \( \mathcal{F}_2(\epsilon) \). \( \square \)

D. Nondegeneracy

The details of Proposition 3.2 are given in this section. Again, due to space restrictions, some of the very most detailed computations supporting the proofs are available in [57].

1) Goal point \( \{q\} \):

**Proposition 3.8:** The goal point, \( g \) is a non-degenerate minimum of \( \varphi \).

**Proof:** It can be shown that
\[
 D^2\varphi|_C = \frac{1}{(\gamma + \beta)^{1/2}} 2 I_{2p} - \gamma D^2(\gamma + \beta)^{1/2}
\]
Noting that \( \gamma|_q = 0 \) and \( \nabla\gamma|_q = 2(b - g) = 0; \)
\[
 D^2\varphi|_q = \frac{2}{\beta + \gamma} I_{2p}
\]
implies that \( g \) is a non-degenerate minimum of \( \varphi \). \( \square \)

2) The Set Near the Internal Obstacles: \( \mathcal{F}_1(\epsilon) \): There are no critical points in \( \{ b \in \partial \mathcal{F} : |T(b)| = 1 \} \) by Proposition 3.3 given in Section III-C. The critical points in \( \{ b \in \partial \mathcal{F} : |T(b)| > 1 \} \) are maxima by Proposition 3.4. \( \hat{\varphi} \) has no critical points in \( \mathcal{F}_0(\epsilon), \mathcal{F}_2(\epsilon) \) by Proposition 3.5 and Proposition 3.7 respectively. Now let us consider the critical points of \( \hat{\varphi} \) that are in \( \mathcal{F}_1(\epsilon) \).

**Proposition 3.9:** \( \exists N(\epsilon) \) such that for \( k > N(\epsilon) \), \( D^2\hat{\varphi} \) restricted to \( \mathcal{F}_1(\epsilon) \) is non-singular.

**Proof:** Define \( L_0 \triangleq -\frac{1}{\sqrt{\beta_{ij}}}(I_{2p} \circ e_i^T), \forall i \in P \) and \( L_{ij} \triangleq \frac{1}{\sqrt{\beta_{ij}}}(I_{2p} \circ e_i^T), \forall (i,j) \in Q \).

Let \( L_0 \) be the \( 2p \times 2p \) matrix \( L_0 \triangleq [L_{01} \ldots L_{0p}] \) and \( L_1 \) be the \( 2p \times 2q \) matrix \( L_1 \triangleq [L_{1T} \ldots L_{pT-1,T}] \). Let \( L \) be the \( 2p \times 2(p + q) \) matrix \( L \triangleq [L_0 L_1] \) and \( o \) be the \((p + q) \times 1\) vector \( o \triangleq \left[ \frac{1}{p} \ldots \frac{1}{q} \right]^T \). Note that \( L \) has rank \( 2p \) in \( \mathcal{F}_1(\epsilon) \). Let \( M \) be the \((2p + q) \times (p + q)\) block diagonal matrix,
\[
 M \triangleq \begin{bmatrix}
 L_{01}b & 0 & \ldots & 0 \\
 0 & \ddots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \ddots \\
 0 & \ldots & \ldots & \ldots & L_{p-1,p}b
\end{bmatrix}
\]

It can be shown that (Lemma E.13 [57]) \( \forall b \in C_\phi \cap \mathcal{F}_1(\epsilon) \)
\[
\beta \int_0^\epsilon \, D^2\hat{\varphi} = \frac{\beta}{\pi} D^2\hat{\varphi} = \frac{k}{\sqrt{\beta}} I_{2p} + 2LMMTL^T - \frac{2}{k} LMM^T M^T L^T - L_{11} L_{1} + L_0 L_0^T
\]
\[
 M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}
\]

where \( M_{11} \) is a \( 2p \times p \) matrix, \( M_{12} \) is a \( 2p \times q \) matrix, \( M_{21} \) is a \( 2q \times p \) matrix, and \( M_{22} \) is a \( 2q \times q \) matrix respectively. Hence, it can be shown that
\[
 M^TAM = \begin{bmatrix}
 M_{11}^T M_{11} & 0_{p \times q} \\
 0_{q \times p} & -M_{22}^T M_{22} \end{bmatrix}
\]

Letting \( A = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \), we may re-write the previous equation as:
\[
\frac{\beta}{\pi} D^2\hat{\varphi} = \frac{k}{\sqrt{\beta}} I_{2p} + L \begin{bmatrix} I_{q+p} - \frac{1}{k} oo^T \end{bmatrix} M^T + A L^T
\]

Next consider \( I_{q+p} - \frac{1}{k} oo^T \). By construction,
\[
 I_{q+p} - \frac{1}{k} oo^T = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} - \frac{1}{k} \begin{bmatrix} 1_{p \times p} & -1_{p \times q} \\ -1_{q \times p} & 1_{q \times q} \end{bmatrix}
\]

Now let \( V = 2 \left( I_{q+p} - \frac{1}{k} oo^T \right) \). We show that \( V \) is full rank via considering its elements:
\[
 |v_{ij}| = \begin{cases} 2 \left(1 - \frac{1}{k}\right) & \text{if } i = j \\ \frac{1}{k} & \text{otherwise} \end{cases}
\]

Note that for each row of \( V \),
\[
 \sum_{j \neq i} |v_{ij}| = 2 \left(1 - \frac{1}{k}\right) > 2 \frac{p + q - 1}{k} = \sum_{j \neq i} |c_{ij}|
\]

Hence, since \( V \) is strictly diagonally dominant, it follows that \( V \) is of full rank by Levy-Desplanques theorem [66] and we have,
\[
 \text{rank}(V) = p + q
\]

Hence the result holds for its inverse. It can be shown that each \( v_{ij}^{-1} \) entry of \( V^{-1} \) has the following form:
\[
 v_{ij}^{-1} = \begin{cases} \frac{1}{k} \left( 1 - \frac{p + q - 1}{p} \right) & \text{if } i = j \\ \frac{1}{k} \left( 1 - \frac{p + q}{p} \right) & \text{if } i, j \leq p \text{ and } i \neq j \\ \frac{1}{k} \left( 1 - \frac{p + q}{p} \right) & \text{if } i, j > p \text{ and } i \neq j \\ \frac{1}{k} \left( 1 - \frac{p + q}{p} \right) & \text{otherwise} \end{cases}
\]

Now consider \( \text{rank}(A) \). According to a theorem by Sherman, Morrison and Woodbury, \( MM^T + A \) is invertible if \( MT A^{-1} M + V^{-1} \) is invertible [64]. Namely,
\[
 \text{rank}(A) = 2(p + q)
\]

iff
\[
 \text{rank}(MT A^{-1} M + V^{-1}) = p + q
\]

Now consider \( \text{rank}(A) \). Note that by construction \( A^{-1} = A \) and \( \text{rank}(A) = 2(p + q) \). Let \( M \) be represented as a block matrix as
\[
 M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}
\]

where \( M_{11} \) is a \( 2p \times p \) matrix, \( M_{12} \) is a \( 2p \times q \) matrix, \( M_{21} \) is a \( 2q \times p \) matrix, and \( M_{22} \) is a \( 2q \times q \) matrix respectively. Hence, it can be shown that
\[
 M^T AM = \begin{bmatrix} M_{11}^T M_{11} & 0_{p \times q} \\ 0_{q \times p} & -M_{22}^T M_{22} \end{bmatrix}
\]
By construction, both \( M^2_{TT}M_{11} \) and \(-M^2_{TT}M_{12}\) are diagonal matrices. Furthermore, if \( S \) is the ordered set of permutations of \( P \) and \( \epsilon \) denotes the lexicographic order of a given permutation \( ln \), each diagonal entry \( m_{ii} \) of \( MT \cdot AM \) is defined as

\[
m_{ii} = \begin{cases} \|L_{0i}\|_2^2 & \text{if } i \leq p \\ -\|L_{1i}\|_2^2 & \text{if } i = p + \epsilon(\ln) \end{cases}
\]

where it should be recalled that \( \|L_{0i}\|_2^2 = \frac{b_i}{\rho_i} \) and \( \|L_{1i}\|_2^2 = \frac{(b_i - b_j)(b_i - b_j)}{\rho_i^2} \).

Let \( X = M^2\cdot A^{-1}\cdot M + V_{ii} \), where \( x_{ij} \) denotes the elements of \( x \). Next, we show that \( X \) is a nonsingular matrix via diagonal dominance. First, note that each diagonal element \( x_{ii}, i = 1, \ldots, p + q \) is equal to:

\[
x_{ii} = \begin{cases} \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\beta_{0i}^2}{\beta_{ii}} & \text{if } i \leq p \\ \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} & \text{if } i > p \text{ and } i = p + \epsilon(\ln) \\ \end{cases}
\]

On the other hand, each off-diagonal element \( x_{ij}, i \neq j \) is equal to:

\[
x_{ij} = \begin{cases} \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} & \text{if } i, j \leq p \text{ and } i \neq j \\ \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} & \text{if } i, j > p \text{ and } i \neq j \\ \end{cases}
\]

Consider \( \sum_{j \neq i} |x_{ij}| \).

\[
\sum_{j \neq i} |x_{ij}| = \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\beta_{0i}^2}{\beta_{ii}}
\]

First consider \( i \leq p \). It can shown that

\[
|x_{ii}| - \sum_{j \neq i} |x_{ij}| = \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\beta_{0i}^2}{\beta_{ii}} - \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} - \frac{\beta_{0i}^2}{\beta_{ii}} = \frac{p^2 - p + k + 2}{2} \cdot \frac{\beta_{0i}^2}{\beta_{ii}} - \frac{p^2 - p + k + 2}{2} + \frac{\beta_{0i}^2}{\beta_{ii}}
\]

Since the first term on the rhs is an increasing function of \( k \) and \( \frac{\beta_{0i}^2}{\beta_{ii}} > 1, \exists K_4(\epsilon) \), such that for \( k > K_4(\epsilon) \)

\[
|x_{ii}| > \sum_{j \neq i} |x_{ij}|
\]

Now consider \( i > p \). First note that either \( 0 < \beta_{1in} < \epsilon \) (Case 1) or \( \epsilon < \beta_{1ln} < (2\rho_{0} - \rho_{1n})^2 - \rho_{1n}^2 \) (Case 2). The first case holds for all robot pairs that are within \( \epsilon \) neighborhood of each other while the second case holds for all the other remaining pairs since the workspace is bounded. Of course, by assumption, as we consider \( F_1(\epsilon) \), there exists at least one \( (l,n) \in Q \) such that \( 0 < \beta_{1ln} \leq \epsilon \). Hence

\[-\infty < -\frac{\beta_{1ln}^2}{\beta_{1ln}} \leq -\frac{\rho_{1n}^2}{\epsilon}
\]

It can be shown that \( |x_{ii}| \) is bounded as:

\[
\left| \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\rho_{1n}^2}{\epsilon} \right| \leq |x_{ii}| < \infty
\]

Now let us consider \( |x_{ii}| = \sum_{j \neq i} |x_{ij}| \) with the lower bound on \( |x_{ii}| \) which is equal to

\[
\left| \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\rho_{1n}^2}{\epsilon} \right| \leq |x_{ii}| < \infty
\]

Since \( \epsilon \) is an arbitrarily small design parameter as discussed in Section II-C, the term \( \frac{\rho_{1n}^2}{\epsilon} \) will dominate in Eq. 16 and hence \( |x_{ii}| > \sum_{j \neq i} |x_{ij}| \).

This implies \( \frac{1}{2} \cdot \frac{k - (p + q) + 1}{2} + \frac{\rho_{1n}^2}{\epsilon} \). Hence, since \( x \) is strictly diagonally dominant, hence according to Levy-Desplanques theorem [66]:

\[
\text{rank}(X) = \text{rank}(M^2 \cdot A^{-1} \cdot M + V^{-1}) = p + q
\]

This in turn implies that

\[
\text{rank}(MV^T + A) = 2(p + q)
\]

Recalling that \( B = L(MV^T + A)L^T \), since \( \text{rank}(L) = \text{rank}(L^T) = 2p \), according to lower and upper bounds on the rank of product of matrices [65], the following holds true:

\[
2p \leq \text{rank}(B) \leq 2p
\]

Hence, \( B \) is ensured of being full rank and hence non-singular.

If \( B = DAD^{-1} \) be an eigendecomposition of \( B \) where the \( A \) is a diagonal matrix with eigenvalues \( \lambda_i \) and \( \bar{U} \) is the matrix of eigenvectors, then

\[
|B| = |UDU^{-1} + k_{2p}^2| = |U|D + k_{2p}^2|U|^{-1}
\]

\[
= D + k_{2p}^2
\]

\[
= \prod_{l=1}^{2p} (\lambda_l + k_{2p})
\]

(17)

Recall that by Prop. 3.6, there exists at least one negative eigenvalue. If \( b \in C_{\phi} \), then by definition

\[
k_{2p} = \frac{\gamma \nabla \beta}{k_{2p}^2}
\]

Thus, Eq. 18 is equivalently expressed as:

\[
\nabla \gamma = \frac{\gamma}{k_{2p}^2} \nabla \beta
\]
Recall that since \( \gamma(b) \overset{\Delta}{=} (b - g)^T (b - g) \) by definition, equivalently \( \gamma(b) = \frac{1}{2} \nabla \gamma^T \nabla \gamma \). Hence, at a critical point, \( \gamma \) is equal to:

\[
\gamma = \frac{\gamma^2}{4 k^2 \beta^2} \nabla \beta^T \nabla \beta = \frac{\gamma^2}{k^2 \Omega}
\]

with \( \Omega \overset{\Delta}{=} \nabla \beta^T \nabla \beta \) which implies that

\[
\gamma = \frac{k^2}{\Omega} \tag{19}
\]

Rewriting \( \frac{k}{\gamma} \) after substituting for \( \gamma \) using Eq. 20 and simplifying

\[
\frac{k}{\gamma} = \frac{\Omega}{k}
\]

Thus, for \( b \in C_{\varphi} \), \( B + \frac{k}{\gamma} I_{2p} \) is equal to \( B + \frac{\Omega}{k} I_{2p} \). Using Eq. 17

\[
\left| B + \frac{k}{\gamma} I_{2p} \right| = \sum_{i=1}^{2p} (\lambda_i + \frac{\Omega}{k})
\]

\( F_1(\varepsilon) \) can be partitioned into two subsets - \( F_{0h} \) and \( F_{0c} = F_1(\varepsilon) - F_{0h} \). Now consider the negative eigenvalue of \( B \) having smallest magnitude and denote it by \( \lambda'(B) \). Consider the closure of \( F_{0c} \) - namely \( F_{0c} \). As \( F_{0c} \) is compact, let

\[
\lambda^* = \inf_{\lambda \in F_{0c}} |\lambda'(B)|
\]

Finally, choose

\[
k > \sup_{\lambda \in F_{0c}} \frac{\Omega}{\lambda^*}
\]

Thus, if \( b \in C_{\varphi} \cap \bar{F}_1(\varepsilon) \), then \( \frac{k}{\gamma} I_{2p} + B \) is nonsingular. The proof is completed by choosing

\[
N(\varepsilon) = \max \{ K_4(\varepsilon), K_5(\varepsilon), K_6(\varepsilon) \}
\]

IV. SIMULATIONS

We now report on simulations of the flows associated with the construction to suggest the nature and quality of the motion planning resulting from the artificial potential function \( \varphi \). A workspace tightness measure \textit{tight} is defined as:

\[
\text{tight} = \frac{100}{\log_{10} \left( \prod_{(i,j) \in Q} \| g_{ij} - \rho_{ij} \|^2 \right)}
\]

Note that this measure of tightness captures the difficulty of the task. The closer the robots need to be packed together the more careful and precise the robots have to be in their movements. We will summarize performance by means of the measures originally introduced in [6]. The first performance measure is the normalized robot path length measure \( \text{nrl} \) which is the total distance traveled by the robots normalized by the sum of the Euclidean distances between initial and final positions of the robots,

\[
\text{nrl} = \frac{\sum_{i \in P} \int_{t_f}^{t_i} \| b_i(t) \| dt}{\sum_{i \in P} \| b_i(0) - g_i \|}
\]

Here, \( t_f \) denotes the duration of a simulation, \( b_i(t) \) denotes the position vector of robot \( i \) at time \( t \) and \( b_i(0) \) denotes the initial position of robot \( i \). The second measure is the design parameter \( k \) of function \( \varphi \). Recall in case of accurate positional data, the robots are ensured of moving without any collisions along the way.

A. Circular Formations

We first study a problem involving six robots and five different randomly chosen goal configurations of circular formations with increasing tightness as shown in Figure 3. Figure 4(left) shows the variation of \( \text{nrl} \) as a function of goal tightness measure \textit{tight}. In this graphic, each bar represents the mean and the standard deviation of 30-40 sample runs with random initial configurations. \( k \) is taken to be 60. The effect of \( k \) is discussed in the following section. Unlike [6], we observe that the general trend and the deviation of \( \text{nrl} \) values increase with increasing workspace tightness. This result is expected since the closer the robots need to pack together, the more times will encounter each other, thus requiring longer paths that move around each other in order to reach their goal positions. It is seen that in the most complex workspace, path length is on average 1.25 - 25 percent longer than the (typically infeasible) Euclidean straight line between initial and final configurations. In the easiest workspace, this value decreases to 1.08.
1) For small $k$ values, in the constructed potential function, the term for obstacle avoidance dominates. The robots attempt to increase their proximity to nearby robots as much as possible. Consequently, the paths taken by the robots get longer. Still, the maximum mean $nrl$ value is 1.68 when $k = 20$. Furthermore, the moving task is not accomplished for $k$ values smaller than 20 in the simulations starting from some initial configurations. This fact is expected since there is a lower bound on $k$ for convergence to the goal positions.

2) For large $k$ values, the robots are concerned with pointing towards their goal positions rather than avoiding each other. In this case, a robot may try to pass through the spaces between the other robots which are only 1-2 cm larger than its diameter. Therefore, the paths taken by the robots become shorter.

B. Array-like Formations

We next study a problem involving ten robots and four different randomly chosen goal configurations of array-like formations with increasing tightness as shown in Figure 5. The variation of $nrl$ with respect to goal tightness is as shown in Figure 6. Again, it is observed that with increased tightness, there is a tendency for the path lengths traveled by the robots to increase as well. In this simulation, we then consider the tightest goal and assume that sensor measurements are subject to noise. The noisy state observations $\hat{b}_i$ are generated as $\hat{b}_i = b_i + \eta_s$, where $\eta_s$ represents the position measurement noise. It is assumed to be Gaussian $\eta_s \sim N(0, \Sigma_s)$ where the covariance $\Sigma_s$ are known. In our simulations, different noise levels are considered: Low($\sigma = 0.1$), moderate ($\sigma = 0.5$) and high ($\sigma = 1$). Figure 7 shows $nrl$ vs $\sigma$—where it is observed that although $nrl$ increases dramatically, the tasks still can be completed. However, it should be noted that with higher levels of noise, the probability of collisions between the robots increase as expected since there is a discrepancy between where each robot is actually and where it thinks it is. Let us note that in this case, the performance of robots can be improved by resorting to state estimation methods as has been shown in a different, but related task of parts’ moving.

![Fig. 5. Array-like formations of increasing tightness: a) tight = 0.504, b) tight = 0.538, c) tight = 0.560, d) tight = 0.611.](image)

![Fig. 6. Normalized robot path length vs. workspace tightness in array-like formations.](image)

![Fig. 7. Normalized robot path length vs. noise $\sigma$ in array-like formations.](image)

![Fig. 8. Random goal positions for varying packedness (low, medium, high) and for varying population (20,30,40) robot teams.](image)
C. Random Goal Positions

Finally, we consider randomly positioned goal locations of varying tightness for robot populations of 20, 30 and 40 as seen in Fig. 8. The variation of $nrl$ with respect to the number of robots is as given in Fig. 9 where the results are average values for 20 runs with random initial positions. It is observed that increase in the number of robots does not affect $nrl$ much.

![Normalized robot path length vs. number of robots in random goal positions.](image)

Finally, despite a large number of numerical experiments with goals, $g \in \mathcal{F}$ that violate Assumptions 1) and 2) of Section II-C conditions we have not been able to find goal configurations that are not attainable, bolstering our strong sense that these assumptions, while convenient to our proof, are pessimistically conservative and not necessary for the desired result. Numerous successful simulations run on very “tight” goal configurations certainly belie their difficulty and we suspect that only very specific goal “shapes” may give trouble. Provided that $k$ is set high enough and numerical overflow/underflow problems are eliminated, the goals have always proven to be attainable. However, even in the worst case, if some goals “tight” enough to violate Assumptions 1) and 2) do not yield a successful navigation function, our construction (4) gives rise to safe (guaranteed no collisions) non-degenerate gradient systems which have only isolated point attractors. Hence “blocked” initial conditions would reach unacceptable equilibrium positions rather than exhibiting oscillatory (some more exotic, undetectable) behavior.

V. CONCLUSION

This paper extends the navigation function methodology [7] to the coordinated navigation of independent disk-shaped robots moving in a disk-shaped planar workspace as first proposed over two decades ago [6]. Intuitively, the source of difficulty that characterizes this problem arises because each robot becomes a dynamic obstacle for the remaining robots. Since this is a real time dynamical systems based planner, there can be no a priori knowledge of robots’ trajectories. However, by making assumptions i) - iv) in Section I-C, we adopt the framework of encoding complete information about the goal, dynamic obstacles and workspace boundary. The main contribution is to establish that our proposed construction is indeed an essential navigation function - namely it satisfies the properties 1) - 4) listed in Section I-D. The analysis yields closed-form expressions that depend on the goal configuration and the $k$ parameter of this construction. First, lower bounds constrain the allowable goal proximity of among robot pairs as well as to the workspace boundary to be “reasonable”. Next, suitable parameter values are found sufficient to ensure the construction indeed holds the required properties. As a consequence of its defining properties, the gradient field resulting from an essential navigation function yields a flow guaranteed to bring almost every initial condition in the connected component to the goal with no collision along the way. The recourse to an online feedback based planner lends robustness against the unanticipated changes in workspace configuration (state stability) and inevitable sensor and actuator inaccuracies (structural stability). Even if disk-shaped robots treated here constitute a very small portion of the general coordinated navigation problem of arbitrary robots in arbitrary workspaces, we expect that this construction will advance the design of artificial potential functions for scenarios that are progressively more realistic respecting geometry, actuation, sensing and distributed information.

ACKNOWLEDGMENTS

The authors would like to thank the editor and the anonymous reviewers for their careful reading as well as their many helpful comments and suggestions for improving the original manuscript. We also would like to acknowledge the contributions of Mehmet Akgül and Haluk Bayram for the simulation software. The second author thanks Ferit Öztürk for discussions on Morse theory. The third author thanks Michael Farber for discussions on topological complexity and Yuliy Baryshnikov and Fred Cohen for discussions on the topology of configuration spaces.

APPENDIX

The following list presents the most commonly used the definitions in the paper where the third column indicates the place of first introduction.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \in \mathbb{Z}^+$</td>
<td>The number of robots</td>
<td>II-A</td>
</tr>
<tr>
<td>$P = {1, \ldots, p}$</td>
<td>Robot index set</td>
<td>II-A</td>
</tr>
<tr>
<td>$b_i \in \mathbb{R}^2$</td>
<td>Center of robot $i$</td>
<td>II-A</td>
</tr>
<tr>
<td>$\rho_i \in \mathbb{R}^+$</td>
<td>Radius of robot $i$</td>
<td>II-A</td>
</tr>
<tr>
<td>$g_i \in \mathbb{R}^2$</td>
<td>Goal of robot $i$</td>
<td>II-A</td>
</tr>
<tr>
<td>$b \in \mathbb{R}^{2p}$</td>
<td>$\triangleq \sum_{i \in P} b_i \otimes e_i$</td>
<td>II-A</td>
</tr>
<tr>
<td>$g \in \mathbb{R}^{2p}$</td>
<td>$\triangleq \sum_{i \in P} g_i \otimes e_i$</td>
<td>II-A</td>
</tr>
<tr>
<td>$\rho_{ij}$</td>
<td>$\triangleq \rho_i - \rho_j$</td>
<td>II-A</td>
</tr>
<tr>
<td>$Q$</td>
<td>$\triangleq {(i, j)</td>
<td>i, j \in P, i &lt; j}$</td>
</tr>
<tr>
<td>$Q^0$</td>
<td>$\triangleq Q \cup {(0, i) \forall i \in P}$</td>
<td>II-A</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n$ dimensional identity matrix</td>
<td>II-A</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>$\triangleq e_i - e_j$</td>
<td>II-A</td>
</tr>
<tr>
<td>$d_{ij} \in \mathbb{R}^2$</td>
<td>$\triangleq b_i - b_j = (I_2 \otimes e_{ij}) b$</td>
<td>II-A</td>
</tr>
<tr>
<td>$\delta_{ij}$</td>
<td>$\triangleq</td>
<td></td>
</tr>
<tr>
<td>$g_{ij} \in \mathbb{R}^2$</td>
<td>$\triangleq g_i - g_j = (I_2 \otimes e_{ij}) g$</td>
<td>II-A</td>
</tr>
<tr>
<td>$\gamma(b)$</td>
<td>$\triangleq (b - g)^T (b - g)$</td>
<td>II-B</td>
</tr>
</tbody>
</table>


---

C. Serkan Karagöz received the B.Sc. degree in Electrical and Electronics Engineering, the M.Sc. degree in Systems and Control from the Bogazici University, Istanbul, Turkey, in 1991 and 1993 respectively. He worked for Honeywell Automation and Controls between 1993-1996 where he was involved with various industrial automation projects. He started his doctoral studies in the Electrical and Electronics Engineering Department in the same university. During his doctoral studies in 2000, he was a visiting researcher at the Artificial Intelligence Laboratory, University of Michigan, Ann Arbor, MI. As part of his doctoral studies, he was responsible for the design and building of a mobile robot EDAR. He joined Ford of Europe, Manufacturing Engineering Department in 2002 and currently works in John Andrews Technical Centre in Cologne, Germany. He is a certified black belt engineer and is leading the vehicle testing projects at European plants of Ford Motor Company.

H. Ipl Bozma has been professor of Electrical and Electronics Engineering at Bogazici University, Istanbul, Turkey since 2004. She has received her B.S. degree (with honors) from Bogazici University in 1983, M.S. and Ph.D. degrees from Yale University, New Haven, USA, in 1986, and 1992, respectively. She then joined the faculty of Electrical and Electronics Engineering at Bogazici University. In 1996, she has co-founded the Intelligent Systems Laboratory in Electrical Engineering and has been the director of the lab since then. Her research interests are multi-robot systems, robot vision and cognition, attentive robots and game theory. She is the author or co-author of more than sixty papers in refereed journals or conference proceedings. Among her awards are: Francis Erbsmann Award in IPMI’91, Wye, UK, 1997 Texas Instruments European Universities DSP Challenge, 2nd Runner-up Award, TESISD (Turkish Electronic Industrialists Society) Innovation Award (2004) and Bogazici University Academic Incentive Awards.

Daniel E. Koditschek is the Alfred Filer Moore Professor of Electrical and Systems Engineering at the University of Pennsylvania School of Engineering and Applied Science. Dr. Koditschek received his bachelors degree in Engineering and Applied Science and his M.S. and Ph.D. degrees in Electrical Engineering in 1981 and 1983, all from Yale University. He served on the Yale Faculty in Electrical Engineering until moving to the University of Michigan a decade later. In January 2005, he moved to the University of Pennsylvania as Chair of the Electrical Engineering Department, a position he held until 2012. Dr. Koditscheks research interests include robotics and, more generally, the application of dynamical systems theory to intelligent mechanisms. He has authored more than 200 archival journal and refereed conference publications that have appeared in a broad spectrum of venues ranging from the Transactions of the American Mathematical Society through The Journal of Experimental Biology, with a concentration in several of the IEEE journals and related transactions. Various aspects of this work have received mention in general scientific publications such as Scientific American and Science as well as in the popular and general lay press such as The New York Times and Discover Magazine. Dr. Koditschek is a member of AMS, ACM, MAA, SIAM, SICB and Sigma Xi and is a Fellow of the IEEE and AAAS. Dr. Koditschek holds secondary appointments within the School of Engineering and Applied Science in the departments of Computer and Information Science and Mechanical Engineering.