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Daniel E. Koditschek
University of Pennsylvania, kod@seas.upenn.edu

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High Gain Feedback and Telerobotic Tracking

D.E. Koditschek*
Yale University
New Haven, CT 06520-2157

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Abstract

Asymptotically stable linear time invariant systems are capable of tracking arbitrary reference signals with a bounded error proportional to the magnitude of the reference signal (and its derivatives). It is shown that a similar property holds for a general class of nonlinear dynamical systems which includes all robots. As in the linear case, the error bound may be made "arbitrarily" small by increasing the magnitude of the feedback gains which stabilize the system.

1 Introduction

Tracking is the archetypal pursuit of the control theorist. Given a dynamical system,

$$
\begin{aligned}
\dot{x} &= f(x, u), \\
y &= h(x)
\end{aligned}
$$

and a specified "reference signal", $r(t)$, it is required to find a control, $u^*(t)$ such that the forced system, $\dot{x} = f(x, u^*)$ "tracks" r in some sense — usually $\lim_{t\to\infty} y = r$. Solutions to such problems generally involve pre-filtering the reference trajectory through a suitable "feedforward" algorithm, and then adding a compensating error driven "feedback" term to arrive at the input, u^*. If the reference signal is known à priori, then the feedforward algorithm may entail pure differentiation to "pre-compensate" for the lags introduced by the dynamical system itself. However, on-line differentiation of unknown and unpredictable signals has long been eschewed by control theorists as an unreliable technique for both theoretical as well as practical reasons.

This paper considers the problem of tracking in the context of telerobotic manipulators. It is shown that a general class of highly nonlinear control systems which includes all robot models admits tracking algorithms based upon high gain linear state variable feedback. The choice of a pure feedback based algorithm for tracking is surely not optimal in any sense of the word. However, the only other techniques which are known to guarantee tracking for this class of systems make use of feedback algorithms which attempt exact cancellation [1,2,3], (or "nearly" exact cancellation, e.g. [4,5]) of intrinsic nonlinear dynamical terms via feedback, and pure differentiation of the reference trajectory in the feedforward path. In robot applications admitting the use of a "high level" planner it is plausible that the entire future strategy might be made available at once to the "low level" controller in which case tracking schemes requiring pure differentiation of the reference signal might be acceptable. In telerobotic applications the reference signal is, by definition, à priori unknown: it is generated as a record of the unpredicted arbitrary motion of a human agent of control. Schemes which require pure differentiation will probably not be useful in this context.

In a sense, the result reported here simply represents another example of the similarity between general mechanical systems and second order linear systems. It is well known that asymptotically stable linear time invariant systems are capable of tracking arbitrary reference signals with a bounded error proportional to the magnitude of the reference signal (and its derivatives). For a fixed bound on this magnitude, the asymptotic tracking error may be made "arbitrarily" small by increasing the magnitude of the eigenvalues in the left half of the complex plane. In practice, this is accomplished by increasing the gain of linear feedback compensators. In this paper it is shown that the analogous property holds true for the more general class of nonlinear mechanical systems.

As in the theory of linear servomechanisms, a practical obstacle to the systematic use of high gain feedback techniques in telerobotic applications is the inevitable presence of actuator torque limitations. Practical tracking strategies which address this problem while maintaining convergence guarantees are very much needed. This important consideration is entirely ignored here. The problem of characterizing the transient response of feedback compensated nonlinear mechanical systems is the topic of a paper currently in progress.

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2 Preliminary Discussion

2.1 Notation and Definitions

If $f : \mathbf{R}^n \to \mathbf{R}^m$ has continuous first partial derivatives, denote its $m \times n$ jacobian matrix as Df. When we require only a subset of derivatives, e.g. when $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and we desire the jacobian of f with respect to the variables $x_1 \in \mathbf{R}^{n_1}$, as x_2 is held fixed, we may write

$$D_{x_1} f \triangleq Df \begin{bmatrix} I_{n_1 \times n_1} \\ 0 \end{bmatrix}.$$

If $A : J \to \mathbf{R}^{n \times n}$ is a smooth map taking matrix values then let

$$\mu(A) \triangleq \sup_{q \in J} \sup_{\|x\|=1} |x^T A x|$$

and

$$\nu(A) \triangleq \inf_{q \in J} \inf_{\|x\|=1} |x^T A x|.$$

If J is compact, or the entries of A are bounded then both $\nu(A), \mu(A)$ are non-negative real numbers. For any constant matrix, $\mu(A)$ is the square root of the eigenvalue of greatest magnitude, while $\nu(A)$ is the square root of the eigenvalue of least magnitude of $A^T A$, from which it follows that

$$\mu(A) = \sup_{q \in J} \|A(q)\| \qquad 1/\nu(A) = \sup_{q \in J} \|A^{-1}(q)\|,$$

where $\|\cdot\|$ denotes the operator norm induced by the eulcidean norm of $\mathbf{R}^n$.

Given a set $\mathcal{P}$, a smooth scalar valued map, $v : \mathcal{P} \to \mathbf{R}$ is said to be *positive definite at a point* $p \in \mathcal{P}$ if $v(p) = 0$, and $v > 0$ in some open neighborhood of p. Given a smooth (time invariant) vector field, f, on some phase, space, $\mathcal{P}$, we shall say that, v, a positive definite map at $p_d \in \mathcal{P}$, constitutes a *Lyapunov function for* f *at* p_d if the time derivative along any motion of the vector field is non-positive,

$$\dot{v} = D_p v \; f(p) \leq 0,$$

in some neighborhood of p_d, and that it constitutes a *strict Lyapunov function for* f if the inequality is strict [6,7]. The *domain* of v with respect to p_d is the largest neighborhood around p which is free of additional critical points and upon which the derivative is still non-positive.

The existence of a strict Lyapunov function at a point is a sufficient condition for asymptotic stability of that equilibrium state. If a strict Lyapunov function has not been found, asymptotic stability may, nevertheless, be assured if a further condition on the possible limiting set holds. This is "LaSalle's Invariance Principle" [7]. It is possible, as well, to draw conclusions about the tracking capability of a forced dynamical system in consequence of of the stability properties of the unforced vector field at a particular equilibrium state. However, this seems to require the use of a strict Lyapunov function.

It has been known for quite some time that the total energy of a mechanical system may be interpreted as a Lyapunov function [8]. Unfortunately, this choice of Lyapunov function is never strict. The central contribution of this paper rests upon the construction of a strict Lyapunov function for the general class of nonlinear mechanical systems described below, (1). The tracking results follow as a standard consequence.

2.2 Dynamical Equations of Kinematic Chains

The equations of motion of a kinematic chain have been extensively discussed in the robotics literature, and this paper will rely upon the standard rigid body model of an open chain with revolute joints. Thus, we consider a robot to be a particular member of the class of *mechanical systems*,

$$M[q]\ddot{q} + B[\dot{q}, q]\dot{q} + k(q) = \tau \tag{1}$$

where the generalized positions take values in a configuration space, $q \in J$, and M is a positive definite invertible symmetric matrix for all $q \in J$. As shown in the appendix, in the case of kinematic chains, M, the "inertial" terms, B, the "coriolis and centrifugal" terms, and k, the gravitational disturbance vector, all vary in q by polynomials of transcendental functions. It follows that $\nu(M) > 0$ and $\mu(M) < \infty$.

This system may be rewritten in the form

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= M^{-1}[Bq_2 + k - \tau] \end{aligned} \tag{2}$$

where the generalized positions and velocities take values $p \triangleq \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathcal{P} \triangleq TJ$ in *phase space* — the tangent bundle over J.

While M, k are always bounded, the coriolis and centripetal forces are quadratic in the velocity — i.e. B is linear in $\dot{q}$ — and, therefore, may become unbounded. It is, however, bounded with respect to q, as the following technical result shows.

Lemma 1 *For any $k \in \mathbb{R}^n$,*

$$k^T B q_2 = q_2^T \hat{M}(k) q_2$$

where

$$\hat{M}(k) \triangleq \begin{bmatrix} k^T D_{q_{11}} M \\ \vdots \\ k^T D_{q_{1n}} M \end{bmatrix} - \frac{1}{2}\sum_{i=1}^{n} \kappa_i D_{q_{1i}} M .$$

is bounded above.

Proof:

$$B q_2 = (q_2 \otimes I)^T D_q M^s \, q_2 - \frac{1}{2}\left[(q_2 \otimes I)^T D_q M^s\right]^T q_2$$

and, hence,

$$\begin{aligned} q_2^T B^T k &= q_2^T \left[D_q M^s\right]^T (q_2 \otimes I) k - \tfrac{1}{2} q_2^T (q_2 \otimes I)^T D_q M^s k \\ &= q_2^T \left[D_q M^s\right]^T \left(k q_2^T\right)^s - \tfrac{1}{2} q_2^T (q_2 \otimes I)^T D_q M^s k \\ &= q_2^T \left(\begin{bmatrix} k^T D_{q_{11}} M \\ \vdots \\ k^T D_{q_{1n}} M \end{bmatrix} - \tfrac{1}{2}\textstyle\sum_{i=1}^{n} \kappa_i D_{q_{1i}} M \right) q_2. \end{aligned}$$

Since M contains transcendental functions in q, all of its derivatives in q must be bounded.

□

It follows that for some $\hat{\mu} < \infty$,

$$\|\hat{M}(k)\| \leq \hat{\mu}\|k\|. \tag{3}$$

Corollary 2 *For all $p \in \mathcal{P}$,*

$$q_2^T[\frac{1}{2}\dot{M} - B]q_2 \equiv 0.$$

Proof:

$$\begin{aligned} e_2^T[\tfrac{1}{2}\dot{M} - B]e_2 &= e_2^T[\tfrac{1}{2}\dot{M} - \hat{M}(e_2)]e_2 \\ &= \tfrac{1}{2} q_2^T \left[\left[D_q M^s\right]^T (q_2 \otimes I) - (q_2 \otimes I)^T D_q M^s\right] q_2 \\ &\equiv 0. \end{aligned}$$

□

2.3 Stability Properties of "PD" Compensated Systems

Suppose we are presented with the mechanical system (2), and a desired point,

$$p_d \triangleq \begin{bmatrix} q_d \\ 0 \end{bmatrix} \in \mathcal{P}.$$

Choose two positive definite matrices, $K_1, K_2 > 0$, and define the "PD" algorithm

$$\tau = k(q) - K_1[q_d - q] - K_2\dot{q}. \tag{4}$$

In terms of the translated "error coordinate system" for $\mathcal{P}$,

$$e(p) \triangleq \begin{bmatrix} q - q_d \\ \dot{q} \end{bmatrix}, \tag{5}$$

the resulting closed loop system has the form

$$\begin{aligned} \dot{e} &= \begin{bmatrix} 0 & I \\ -M^{-1}K_1 & -M^{-1}(B + K_2) \end{bmatrix} e \\ &\triangleq A[\dot{q}, q]e. \end{aligned} \tag{6}$$

Proposition 3 *For all $\gamma_0 > 0$,*

$$\tilde{v}(e) = \frac{1}{2}e^T\tilde{P}(q)e \triangleq \frac{1}{2}e^T\begin{bmatrix} \gamma_0 K_1 & 0 \\ 0 & \gamma_0 M(q) \end{bmatrix} e$$

is a Lyapunov function for the closed loop system (6).

Proof: It is clear that $\tilde{v}$ is positive definite at the origin of the error system. Taking the time derivatives along the solutions of the closed loop system, (6),

$$\begin{aligned} \dot{\tilde{v}} &= \tfrac{1}{2}e^T[\tilde{P}A + A^T\tilde{P} + \dot{\tilde{P}}]e \\ &= -e^T\begin{bmatrix} 0 & 0 \\ 0 & \gamma_0 K_2 \end{bmatrix} e + \gamma_0 e_2^T[\tfrac{1}{2}\dot{M} - B]e_2. \end{aligned}$$

Noting that $e_2 \equiv q_2$, it follows from Corollary 2 , that the second term is identically zero.

□

There follows the desirable result that proportional and derivative linear state feedback stabilizes a mechanical system, after the gravitational disturbance torques have been removed.

Theorem 1 ([9,10,11]) *The origin of the closed loop error coordinate system (6) is asymptotically stable.*

Proof: The existence of a Lyapunov Function, $\tilde{v}$, assures stability. According to LaSalle's invariance principle, the attracting set is the largest invariant set contained in $\{(e_1, e_2) \in \mathcal{P} : \dot{v} \equiv 0\}$, which, evidently, is the origin, since the vector field is oriented away from $\{e_2 \equiv 0\}$ everywhere else on that hyperplane.

□

Notice that the proof of attractivity requires an appeal to LaSalle's invariance principle in consequence of the fact that $\tilde{v}$ is not a strict Lyapunov function. In order to obtain the desired extension to tracking problems it is necessary to construct one. Unfortunately, the constructions devised to date require the artificial limitation to decoupled PD feedback. Namely, in the sequel, it will be assumed that the gain matrices of (6) are specified as

$$K_1 \triangleq \omega^2 I; \qquad K_2 \triangleq 2\varsigma\omega I \tag{7}$$

given two positive real numbers, ω, ς.

2.4 A Strict Lyapunov Function for Nonlinear Mechanical Systems

The following technical lemma will be of use in the main result, below.

Lemma 4 *For $M(q)$ as in (1) and any positive scalars, $\alpha, \beta, \gamma \in \mathbb{R}^+$,*

$$\inf_{\|e\|=1} e^T\begin{bmatrix} \alpha I & \beta I \\ \beta I & \gamma M(q) \end{bmatrix} e \geq \nu(K)$$

where

$$K \triangleq \begin{bmatrix} \alpha & \beta \\ \beta & \gamma\nu(M) \end{bmatrix}.$$

In particular, the matrix is positive definite when

$$\alpha\gamma\nu(M) > \beta^2 \tag{8}$$

Proof: Since

$$\begin{bmatrix} \alpha I & \beta I \\ \beta I & \gamma M(q) \end{bmatrix} > \begin{bmatrix} \alpha I & \beta I \\ \beta I & \gamma\nu(M) \end{bmatrix} = K \otimes I,$$

it will suffice to show that

$$\nu(K \otimes I) = \nu(K).$$

This follows since all eigenvalues of $K \otimes I$ are eigenvalues of K, according to Lemma 12 in the appendix. The particular conclusion obtains by taking the determinant of K.

□

Proposition 5 *For all $p_d \in \mathcal{P}$ and $\omega, \varsigma > 0$, given any bounded set, $\mathcal{B} \subset \mathcal{P}$, containing p_d there exists a scalar $\gamma_0 > 0$ such that*

$$v(e) \triangleq \frac{1}{2}e^T P(q)e = \frac{1}{2}e^T\begin{bmatrix} \omega^2\gamma_0 I & \omega\varsigma I \\ \omega\varsigma I & \gamma_0 M(q) \end{bmatrix} e$$

is a strict Lyapunov Function for the closed loop system, (6) on the domain $\mathcal{B}$, assuming the decoupled feedback gain matrices specified in (7).

Proof: Letting

$$\beta \triangleq \sup_{e \in \mathcal{B}} \|e\|,$$

find some γ_0 satisfying

$$\gamma_0 > \max\left\{\frac{\varsigma}{\sqrt{\nu(M)}}, \frac{1}{\nu(M)}, \frac{\mu(\hat{M})\beta}{\nu(M)} + 1\right\}. \tag{9}$$

According to Lemma 4 and the inequality involving the first entry of the inferior set in (9), it follows that P is a positive definite matrix for all $q \in J$, hence v, is positive defnite at p_d.

Taking time derivatives along the solutions of system (6), we have

$$\dot{v} = \tfrac{1}{2}e^T[PA + A^TP + \dot{P}]e,$$

which may be expanded as

$$\begin{aligned}\dot{v} &= -\omega\varsigma e^T \begin{bmatrix} \omega^2 M^{-1} & \omega M^{-1} \\ \omega M^{-1} & \gamma_0 I \end{bmatrix} e \\ &\quad -\omega\varsigma(\gamma_0 - 1)e_2^T e_2 - \omega\varsigma e_1^T M^{-1} B e_2 \\ &\quad +\gamma_0 e_2^T[\tfrac{1}{2}\dot{M} - B]e_2.\end{aligned}$$

The term in the last line vanishes according to Corollary 2 . Moreover, the block matrix in the first line is positive definite according to the inequality (9) and the result of Lemma 4 since

$$\begin{bmatrix} \omega^2 M^{-1} & \omega M^{-1} \\ \omega M^{-1} & \gamma_0 I \end{bmatrix} = \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \omega^2 I & \omega I \\ \omega I & \gamma_0 M \end{bmatrix} \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix}.$$

Finally, according to Lemma 1 , the term in the middle may be rewritten as

$$\omega\varsigma[(\gamma_0 - 1)e_2^T e_2 + e_1^T M^{-1} B e_2] = \omega\varsigma e_2^T[(\gamma_0 - 1)I + \hat{M}(M^{-1}e_1)]e_2 > 0,$$

where $k \triangleq M^{-1}e_1$, and the result follows from the inequality involving the last entry of the set in (9).

□

3 Consequences for Tracking Unknown Reference Signals

Now consider the decoupled "PD" compensated system forced by a continuously differentiable reference signal, $q_d(t)$,

$$\tau = k(q) - \omega^2[q_d(t) - q] - 2\omega\varsigma\dot{q}. \tag{10}$$

Assume that the reference trajectory is "unpredictable" — i.e. its first and second derivatives are unknown — but there is available an à priori bound on the maximum rate of change,

$$\|\dot{q}_d\| \leq \rho_0.$$

Notice that the forced closed loop system may be written in the same error coordinates as (5), above,

$$\dot{e} = A[\dot{q}, q]e + d, \tag{11}$$

where $d \triangleq \begin{bmatrix} \dot{q}_d(t) \\ 0 \end{bmatrix}$, is a "disturbance" input due to the unknown but non-zero reference derivative.

Theorem 2 *The closed loop "disturbed" error system (11) has bounded trajectories which asymptotically approach the set*

$$\left\{e \in \mathcal{P} : \|e\| \leq \frac{\mu(M)}{\nu(K)} 2\rho_0\sqrt{(\gamma_0/\varsigma)^2 + 1/\omega^2}\right\}$$

where

$$K \triangleq \begin{bmatrix} \omega & 1 \\ 1 & \gamma_0\nu(M)/\omega \end{bmatrix}.$$

Proof: We have

$$\begin{aligned}\dot{v} &= \tfrac{1}{2}e^T[PA + A^TP + \dot{P}]e + e^TPd, \\ &\leq -\omega^2\varsigma e^T\left(\tfrac{1}{\mu(M)}K \otimes I\right)e + \omega^2\varsigma \begin{bmatrix} \gamma_0 e_1^T\dot{q}_d/\varsigma \\ e_2^T\dot{q}_d/\omega \end{bmatrix}. \\ &\leq -\tfrac{\omega^2\varsigma}{2\mu(M)}\|e\|\left[\|e\|\nu(K) - 2\rho_0\mu(M)\sqrt{(\gamma_0/\varsigma)^2 + 1/\omega^2}\right].\end{aligned}$$

This is negative whenever e is outside the set indicated in the statement of the theorem.

□

Corollary 6 *The asymptotic tracking bound may be made arbitrarily small by increasing the magnitudes of the feedback gains in (10)*

Proof: For a sufficiently large value of ω it is possible to choose two real numbers $\kappa_1, \kappa_2 \in (0,1)$ such that

$$\varsigma = \kappa_2 \gamma_0 \sqrt{\nu(M)} \qquad ; \gamma_0 = \kappa_1 \omega$$

and the inequality (9) still holds. Using these definitions and the results of the theorem, the attracting region is bounded by the magnitude

$$\frac{\mu(M)}{\nu(K)} 2\rho_0 \sqrt{(\kappa_2^2/\nu(M)) + 1/\omega^2},$$

Note that

$$\begin{aligned} \nu(K) &= \omega + \gamma_0 \nu(M)/\omega - \sqrt{(\omega - \gamma_0 \nu(M)/\omega)^2 + 4} \\ &= \omega + \kappa_1 \nu(M) - \sqrt{(\omega - \kappa_1 \nu(M))^2 + 4}, \end{aligned}$$

hence,

$$\frac{d\nu(K)}{d\omega} = 1 - \frac{\omega - \kappa_1 \nu(M)}{\sqrt{(\omega - \kappa_1 \nu(M))^2 + 4}} > 0$$

and $\nu(K)$ is bounded from below as ω increases. Since κ_2 may be made as small as desired without violating (9), the result follows.

□

A The Stack Representation

If $A \in \mathbf{R}^{n \times m}$, the "stack" representation of $A \in \mathbf{R}^{nm}$ formed by stacking each column below the previous will be denoted A^S [12].

If $B \in \mathbf{R}^{p \times q}$, and A is as above then the *kronecker product* of A and B is

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ a_{21}B & \dots & a_{2m}B \\ & \vdots & \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \in \mathbf{R}^{np \times mq}.$$

The kronecker product is not, in general, commutative. Note that while the transpose "distributes" over kronecker products,

$$(A \otimes B)^T = (A^T \otimes B^T),$$

the stack operator, in general, does not.

Lemma 7 *If $A \in \mathbf{R}^{n \times m}$ then there exists a nonsingular linear transformation of $\mathbf{R}^{nm}$, T, such that*

$$\left(A^T\right)^S = TA^S$$

Proof: For $p = nm$, let $\mathcal{B} \triangleq \{b_1, \dots, b_p\}$ denote the canonical basis of $\mathbf{R}^p$ — i.e., b_i is a column of p entries with a single entry, 1, in position i, and the other $p-1$ entries set equal to zero. The transpose operator is a reordering of the canonical basis elements, hence may be represented by the elementary matrix,

$$T \triangleq \left[b_1, b_{n+1}, b_{2n+1}, \dots, b_{(m-1)n+1}, b_2, b_{n+2}, b_{2n+2}, \dots, b_{(m-1)n+2}, \dots b_n, b_{2n}, b_{3n}, \dots, b_{mn}\right].$$

□

For $n = m$, if we define $P_+ \triangleq I + T$, $P_- \triangleq I - T$ then both operators are projections onto the set of "skew-symmetric", "symmetric" operators of $\mathbf{R}^n$, repsectively, since $P_\pm^2 = P_\pm$. Note that $\operatorname{Ker} P_\pm = \operatorname{Im} P_\mp$.

The kronecker product does "distribute" over ordinary matrix multiplication in the appropriate fashion.

Lemma 8 *If $A \in \mathbf{R}^{n \times m}, B \in \mathbf{R}^{p \times q}, C \in \mathbf{R}^{m \times k}, D \in \mathbf{R}^{q \times l}$ then*

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

Lemma 9 ([12]) *If $B \in \mathbf{R}^{m \times p}$, $A \in \mathbf{R}^{n \times m}$, and $C \in \mathbf{R}^{p \times q}$ then*

$$[ABC]^{s} = (C^{T} \otimes A)B^{s}.$$

Noting that for any column, $c \in \mathbf{R}^{p \times 1}$, we have

$$c^{s} = \left[(c)^{T}\right]^{s} = c,$$

there follows the corollary

Corollary 10 *If $B \in \mathbf{R}^{m \times p}$, $c \in \mathbf{R}^{p}$ then*

$$\begin{aligned} Bc &= Bc^{s} = (c^{T} \otimes I)B^{s} \\ &= \left([Bc]^{T}\right)^{s} = \left(c^{T}B^{T}\right)^{s} = (I \otimes c^{T})\left(B^{T}\right)^{s}. \end{aligned}$$

Noting, moreover, that

$$tr\ \{A\} = \left(I^{s}\right)^{T} A^{s},$$

there follows the additional result

Corollary 11 *If $A \in \mathbf{R}^{n \times m}, B \in \mathbf{R}^{p \times m}$ then*

$$tr\ \left\{AB^{T}\right\} = \left(A^{s}\right)^{T} B^{s}.$$

Proof:

$$\begin{aligned} tr\ \left\{AB^{T}\right\} &= \left(I^{s}\right)^{T}\left(AB^{T}\right)^{s} \\ &= \left(I^{s}\right)^{T}(B \otimes I)A^{s} \\ &= \left(A^{s}\right)^{T}(B^{T} \otimes I)I^{s} \\ &= \left(A^{s}\right)^{T} B^{s}. \end{aligned}$$

□

Lemma 12 *For any square array, $A \in \mathbf{R}^{n \times n}$, if I_m is the identity on $\mathbf{R}^{m}$ then the spectrum of $(A \otimes I_m)$ is contained in the spectrum of A.*

Proof: Suppose λ is an eigenvalue of $(A \otimes I_m)$. There must be some non-zero vector, $x \in \mathbf{R}^{mn}$ in the kernel of $\lambda(I_n \otimes I_m) - (A \otimes I)$ Since $x = X^{s} \in \mathbf{R}^{n \times m}$, it follows that

$$\begin{aligned} 0 &= [\lambda(I_n \otimes I_m) - (A \otimes I)]x \\ &= \left[\lambda X - XA^{T}\right]^{s} \\ &= \left[X(\lambda I_n - A^{T})\right]^{s}. \end{aligned}$$

This implies that $\mathrm{Im}\, X^{T} \subset \mathrm{Ker}\ \lambda I_n - A$, and since the former subspace has dimension at least 1 (according to the assumption that $X \neq 0$), the latter must as well. Thus, λ is an eigenvalue of A.

□

B General Robot Arm Dynamics

The rigid body model of robot arm dynamics may be most quickly derived by appeal to the lagrangian formulation of Newton's Equations. If a scalar function, termed a *lagrangian*, $\lambda = \kappa - \upsilon$, is defined as the difference between total kinetic energy, κ, and total potential energy, υ, in a system, then the equations of motion obtain from

$$\frac{d}{dt}D_{\dot{q}}\lambda - D_q\lambda = \tau^{\mathrm{T}},$$

where τ is a vector of external torques and forces [13,14].

First consider the kinetic energy contributed by a small volume of mass δm_i at position p in link $\mathcal{L}_i$,

$$\delta\kappa_i = \frac{1}{2}{}^0\dot{p}_i^{\mathrm{T}}\,{}^0\dot{p}_i\delta m_i$$

where ${}^0p_i = {}^0F_i\,{}^ip$ is the matrix representation of the position p in the base frame of reference, ${}^0F_i$ is the matrix representation of the frame of reference of link $\mathcal{L}_i$ in the base frame, and ${}^ip$ is the matrix representation of the point in the link frame of reference, and, hence, [1]

$$\dot{p}_i = \dot{F}_i\,{}^ip,$$

since the position in the body is independent of the generalized coordinates. The total kinetic energy contributed by this link may now be written

$$\begin{aligned}\kappa_i &= \int_{\mathcal{L}_i}\tfrac{1}{2}\left[\dot{F}_i\,{}^ip\right]^{\mathrm{T}}\dot{F}_i\,{}^ip\,dm_i\\ &= \int_{\mathcal{L}_i}\tfrac{1}{2}\,trace\{\dot{F}_i\,{}^ip\left[\dot{F}_i\,{}^ip\right]^{\mathrm{T}}\}dm_i,\\ &= \tfrac{1}{2}\,trace\{\dot{F}_i\int_{\mathcal{L}_i}^{i}p\,{}^ip^{\mathrm{T}}dm_i\left[\dot{F}_i\right]^{\mathrm{T}}\}\\ &= \tfrac{1}{2}\,trace\,\{\dot{F}_i\overline{P}_i\dot{F}_i^{\mathrm{T}}\},\end{aligned}$$

(since the frame matrix is constant over the integration), where $\overline{P}_i$ is a symmetric matrix of *dynamical parameters* for the link. Explicitly, if the link has mass, $\overline{\mu_i}$, center of gravity (in the local link coordinate system) $\overline{p_i}$, and inertia matrix, $\overline{J_i}$, then

$$\overline{P}_i \triangleq \begin{bmatrix}\overline{J_i} & \overline{\mu_i p_i}\\ \overline{\mu_i p_i}^{\mathrm{T}} & \overline{\mu_i}\end{bmatrix}.$$

Passing to the *stack representation* (refer to Appendix A)

$$\begin{aligned}2\kappa_i &= trace\,\{\dot{F}_i\overline{P}_i\dot{F}_i^{\mathrm{T}}\}\\ &= \left[(\dot{F}_i\overline{P}_i)^{\mathbf{s}}\right]^{\mathrm{T}}\dot{F}_i^{\mathbf{s}}\\ &= \left[(\overline{P}_i\otimes I)^{\mathrm{T}}\dot{F}_i^{\mathbf{s}}\right]^{\mathrm{T}}\dot{F}_i^{\mathbf{s}}\\ &= \left[\dot{F}_i^{\mathbf{s}}\right]^{\mathrm{T}}\tilde{P}_i\dot{F}_i^{\mathbf{s}}\\ &= \left[(D_qF_i^{\mathbf{s}})\dot{q}\right]^{\mathrm{T}}\tilde{P}_i(D_qF_i^{\mathbf{s}})\dot{q}\\ &= \dot{q}^{\mathrm{T}}M_i\dot{q},\end{aligned}$$

where we have implicitly defined

$$M_i(q) \triangleq \left[D_qF_i^{\mathbf{s}}\right]^{\mathrm{T}}\tilde{P}_iD_qF_i^{\mathbf{s}}; \qquad \tilde{P}_i \triangleq \overline{P}_i^{\mathrm{T}}\otimes I.$$

It follows that the total kinetic energy of the entire chain is given as

$$\kappa = \frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}; \qquad M(q) \triangleq \sum_{i=1}^{n}M_i(q).$$

The potential energy contributed by δm_i in $\mathcal{L}_i$ is

$$\delta\upsilon_i = z_0^{\mathrm{T}}F_i\,{}^ipg\delta m_i$$

where g is the acceleration of gravity, hence the potential energy contributed by the entire link is

$$\upsilon_i = z_0^{\mathrm{T}}F_i\int_{\lambda_i}{}^ipg\,dm_i = z_0^{\mathrm{T}}F_i\overline{p_i}g,$$

and $\upsilon = \sum_{i=1}^{n}z_0^{\mathrm{T}}F_i\overline{p_i}g$. [2]

To proceed with the computation, note that $D_{\dot{q}}\lambda = D_{\dot{q}}\kappa = \dot{q}^{\mathrm{T}}M(q)$, hence,

$$\frac{d}{dt}D_{\dot{q}}\lambda = \ddot{q}^{\mathrm{T}}M(q) + \dot{q}^{\mathrm{T}}\dot{M}(q).$$

[1] We will omit the prior superscript, 0, when it is clear the the coordinate system of reference is the base

[2] Assume that z_0 "points up" in a direction opposing the gravitational field.

Moreover,

$$\begin{aligned} D_q \kappa &= \tfrac{1}{2}\dot{q}^T D_q[M(q)\dot{q}] \\ &= \tfrac{1}{2}\dot{q}^T[\dot{q}^T \otimes I]D_q M^s , \end{aligned}$$

hence,if all terms from Lagrange's equation involving the generalized velocity are collected, we may express them in the form $\dot{q}^T B^T$, where

$$B(q,\dot{q})^T \triangleq \dot{M}(q) - \frac{1}{2}[\dot{q}^T \otimes I]D_q M^s .$$

Finally, by defining $k(q) \triangleq [D_q v]^T$, Lagrange's equation may be written in the form (1)

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + k(q) = \tau.$$

M, called the "inertia" matrix, may be shown to be positive definite over the entire workspace as well as bounded from above since it contains only polynomials involving transcendental functions of q. B contains terms arising from "coriolis" and "centripetal" forces, hence is linear in $\dot{q}$ (these forces are quadratic in the generalised velocities), and bounded in q, since it involves only polynomials of transcendental functions in the generalised position. Finally, k arises from gravitational forces, is bounded, and may be observed to have much simpler structure (still polynomial in transcendental terms involving q) than the other expressions. An important study of the form of these terms was conducted by Bejcsy [15].

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