10-2012

Targeted Marketing and Seeding Products with Positive Externality

Arastoo Fazeli
University of Pennsylvania

Ali Jadbabaie
University of Pennsylvania, jadbabai@seas.upenn.edu

Fazeli, A. & Jadbabaie, A. (2012). Targeted Marketing and Seeding Products with Positive Externality. *Proceedings of the 50th Annual Allerton Conference on Communication, Control, and Computing.*

Abstract

We study a strategic model of marketing in social networks in which two firms compete for the spread of their products. Firms initially determine the production cost of their product, which results in the payoff of the product for consumers, and the number and the location of the consumers in a network who receive the product as a free offer. Consumers play a local coordination game over a fixed network which determines the dynamics of the spreading of products. Assuming myopic best response dynamics, consumers choose a product based on the payoff received by actions of their neighbors. This local update dynamics results in a game-theoretic diffusion process in the network. Utilizing earlier results in the literature, we derive a lower and an upper bound on the proportion of product adoptions which not only depend on the number of initial seeds but also incorporate their locations as well. Using these bounds, we then study which consumers should be chosen initially in a network in order to maximize product adoptions for firms. We show consumers should be seeded based on their eigenvector centrality in the network. We then consider a game between two firms aiming to optimize their products adoptions while considering their fixed budgets. We describe the Nash equilibrium of the game between firms in star and k-regular networks and compare the equilibrium with our previous results.

Keywords

Game Theory, Social Networks, Viral Marketing, Stochastic Process, Optimization

Targeted Marketing and Seeding Products with Positive Externality

Arastoo Fazeli† Ali Jadbabaie†

Abstract— **We study a strategic model of marketing in social networks in which two firms compete for the spread of their products. Firms initially determine the production cost of their product, which results in the payoff of the product for consumers, and the number and the location of the consumers in a network who receive the product as a free offer. Consumers play a local coordination game over a fixed network which determines the dynamics of the spreading of products. Assuming myopic best response dynamics, consumers choose a product based on the payoff received by actions of their neighbors. This local update dynamics results in a game-theoretic diffusion process in the network. Utilizing earlier results in the literature, we derive a lower and an upper bound on the proportion of product adoptions which not only depend on the number of initial seeds but also incorporate their locations as well. Using these bounds, we then study which consumers should be chosen initially in a network in order to maximize product adoptions for firms. We show consumers should be seeded based on their eigenvector centrality in the network. We then consider a game between two firms aiming to optimize their products adoptions while considering their fixed budgets. We describe the Nash equilibrium of the game between firms in star and k-regular networks and compare the equilibrium with our previous results.**

I. Introduction

Many recent studies have investigated the role of social networks in individual purchasing decisions [1]–[3]. As a result, marketing firms have become more interested in exploiting research on social networks in order to promote their products and spread their innovation and technologies. In particular, assuming the relationship between people in social networks and their rational choices, firms are interested to utilize the dynamics of adoptions in order to optimize their business decisions in a competitive market and achieve a higher profit.

Many products and services have positive network externality meaning that decision of an individual in adopting to a product or technology has a positive impact on her peers' decisions. There are many examples for such products or services. New technologies and innovations (e.g., cell phones), network goods and services (e.g., fax machines, email accounts), online games (e.g., Warcraft), social network web sites (e.g., Facebook, Twitter) and online dating services (e.g., OkCupid) are among examples in which people have a higher profit in adopting to a common strategy. Firms might be interested to exploit this positive externality effect of their products and services and the relationship among people in order to achieve a higher profit. Therefore, it is desirable for firms to design their products intelligently and target a specific set of people in social networks in order to maximize the spread of their products and achieve a larger share of the market.

The problem of influence and spreading in networks has been extensively studied in the past few years [4]–[10]. Especially, diffusion of new behaviors and strategies through local coordination games has been an active field of research [11]–[18]. Inspired by the work of Kearns and Goyal in [19] and Montanari and Saberi in [17], we study strategic competition between two firms which simultaneously allocate their fixed budgets to a set of costumers embedded in a social network. The payoff of firms is the expected fraction of people adopting their products. This adoption is determined through a game-theoretic diffusion process among costumers in the network. Therefore, firms shall provide enough incentives for consumption and spread of their product by the payoff that people receive by consuming it. To this end and considering their budgets, firms should strategically design their products and know *how many* and *who* they should free offer their products to in order to initially seed the network and promote their products.

Recently, a game theoretic model of competitive contagion and product adoption is proposed in [20]. In this model two firms initially decide on the production cost of their products (which results in the quality of their products and incentive for people to consume them) and the number of consumers they initially free offer their products to. Then neighboring consumers play a local coordination game which determines the dynamics of the spreading. By analyzing this local coordination game and utilizing earlier results in the literature, a lower and an upper bound are found for the number of consumers adopting each product at each time. These bounds depend on the payoff of products for consumers and the number of initial seeds of the network, however, they do not incorporate the location of initial seeds. It is desirable for firms to have targeted marketing strategies in which they decide on not only *how many* people, but also *who* they offer the products to. Therefore, in this paper we derive new bounds for the spread of products which depend on the location of initial seeds. These bounds are tighter than bounds in [20] in which only the size of the initial adoptions matters. This is due to the fact that new bounds incorporate the extra information about the location of initial seeds as well.

Utilizing new bounds, we study an optimization problem in which firms try to maximize the lower bound of their products adoptions, while considering their fixed budget.

†Department of Electrical and Systems Engineering and GRASP Laboratory at University of Pennsylvania. arastoo@seas.upenn.edu and jadbabai@seas.upenn.edu. This research was supported by ONR MURI HUNT, ONR MURI N000140810747 and AFOSR Complex Networks Program.

We first find initial seeds vectors which maximize lower bounds of firms. This answers to the question of who is more important in a network to be offered a free product. Using optimal initial seeds vectors, we define a game between firms where their objective is to maximize the lower bound of their products adoptions and their strategy is their production costs. While this game seems to be intractable in a general network, we characterize the Nash equilibrium of the game in star and k-regular networks and study the tradeoff between investing more money on improving the quality of a product versus seeding it with more people in a network.

The rest of this paper is organized as follows: In section II, we introduce our model and dynamics updates for consumers playing a local coordination game and find a lower and an upper bound on the spread of products in a network. In section III, we study the product adoptions optimization problem and analyze the game played between firms in star and k-regular networks. Finally, in section IV, we conclude the paper.

II. The Spread Dynamics

The model considered is based on a game theoretic diffusion model proposed in [21]. The spread dynamics is the same as the one discussed in [20]. There are n consumers $V = \{1, \ldots, n\}$ in a social network. The relationship among consumers is represented by an undirected graph $G = (V, E)$. Consumers $i, j \in V$ are neighbors if $(i, j) \in E$. The adjacency matrix of the graph G is denoted by A where $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. We assume $a_{ii} = 0$ meaning there is no self loop. We denote the degree of node i by d_i and the diagonal matrix of degrees of the graph G by $D = diag(\vec{d})$. The i-th largest eigenvalue of the row stochastic matrix $D^{-1}A$ is represented by λ_i. We also assume that there are two competing firms a and b producing products a and b. These two firms initially offer their products to a set of consumers in the network. Let the binary variable $x_i(t)$ denotes the choice of consumer i at time t. We assume $x_i(t) = 0$ if consumer i chooses the product a and $x_i(t) = 1$ if consumer i chooses the product b. Therefore, the state of consumers at time t is represented by a vector $\vec{x}(t)$. Denote by $\vec{S}_a$ the vector of initial seeds of firm a for which the i-th element is equal to 1 if product a is initially offered to consumer i . The vector $\vec{S}_b$ is defined similarly. Denote the norm of these vectors by $S_a = \|\vec{S}_a\|_1$ and $S_b = \|\vec{S}_b\|_1$. Initially all consumers are seeded either by firm a or firm b, therefore, $\vec{S}_a + \vec{S}_b = \vec{1}$. The two products have some payoffs for neighboring consumers depending on their states. If two neighbors in the graph choose the product a they receive a payoff of p_a, if they both choose the product b they receive a payoff of p_b and they receive zero if they choose different products. Therefore, the payoff of the interaction between consumer i and consumer j can be displayed as the following local coordination game:

	$x_j = 0$	$x_j = 1$
$x_i = 0$	p_a	0
$x_i = 1$	0	p_b

Thus, the total payoff of a consumer is simply the sum of her payoffs obtained from her interactions with her neighbors

$$u_i(x_i) = \sum_{j \in \mathcal{N}_i} u_i(x_i, x_j),$$

where $\mathcal{N}_i$ is the set of neighbors of consumer i. We assume consumers repeatedly apply myopic best response. This means that each consumer considering her neighbors, chooses a product that gives her the most payoff. For example, consumer i already adopted to the product a switches to the product b if enough of her neighbors have already adopted to the product b. For consumer i already adopted to the product a, the payoff of choosing the product a and b can be written as

$$u_i(x_i = 0) = p_a \sum_{j \in \mathcal{N}_i} (1 - x_j) \quad \text{for product a,}$$
$$u_i(x_i = 1) = p_b \sum_{j \in \mathcal{N}_i} x_j \qquad \text{for product b.}$$

Consumer i will switch to the product b if we have $u_i(x_i = 0) < u_i(x_i = 1)$ that is

$$\frac{\sum_{j \in \mathcal{N}_i} x_j}{d_i} > \frac{p_a}{p_a + p_b}, \tag{1}$$

similarly, consumer i already adopted to the product b will switch to the product a if we have $u_i(x_i = 1) < u_i(x_i = 0)$ that is

$$\frac{\sum_{j \in \mathcal{N}_i} (1 - x_j)}{d_i} > \frac{p_b}{p_a + p_b}. \tag{2}$$

We can define the right hand side of equations (1) and (2) as

$$r_a := \frac{p_b}{p_a + p_b} \qquad r_b := \frac{p_a}{p_a + p_b}. \tag{3}$$

Note that r_a and r_b in (3) are the degree of risk dominance of actions a and b respectively. This means that if for consumer i already adopted to the product a, the fraction of her neighbors adopting to the product b is greater than r_b, then consumer i's best response is to switch to the product b. We can explain r_a similarly. This myopic best response dynamics yields to a continuous time stochastic process $\vec{x}(t)$ in which each consumer i updates her state upon arrival of a Poisson clock of rate one and switches to the state that gives her the most payoff from interaction with her neighbors. Note that although the rules of updates are deterministic, this process is stochastic due to the randomness in arrival of a random Poisson clock. Employing the results of [20], the lower and upper bounds of the process $\vec{x}(t)$ with myopic best response dynamics can be shown to be

$$1 - \sqrt{\frac{S_a}{n}} \exp(\frac{t}{r_a}) \leq \frac{\sum_{i=1}^{n} \mathbb{E}(x_i(t))}{n} \leq \sqrt{\frac{S_b}{n}} \exp(\frac{t}{r_b}). \tag{4}$$

As it can be seen, these bounds depend on the numbers of initial seeds and degrees of risk dominance of actions. In this section we find tighter bounds for the process which not only take into account the size of initial seeds but also incorporate their locations. Later in this paper we use these bounds in order to show how initial seeds vectors should be chosen and demonstrate the trade off between initial seeds and production costs in Nash equilibrium.

Theorem 1: Consider the continuous time process $\vec{x}(t)$ with a random Poisson clock of rate one and the initial condition $\vec{x}(0)$ and dynamics

$$x_i(t) : 0 \to 1 \quad \text{if} \quad \sum_{j=1}^{n} a_{ij} x_j > r_b d_i,$$
$$x_i(t) : 1 \to 0 \quad \text{if} \quad \sum_{j=1}^{n} a_{ij} (1 - x_j) > r_a d_i.$$

For this process we have

$$1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_a}{n} \leq \frac{\sum_{i=1}^{n} \mathbb{E}(x_i(t))}{n} \leq \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_b}{n}. \tag{5}$$

Proof: Here we use an approach similar to the one used in [22] and [23]. Consider the continuous time Markov process $\vec{\tilde{x}}(t)$ with the same initial condition, i.e. $\vec{x}(0) = \vec{\tilde{x}}(0)$, and

$$\tilde{x}_i(t) : 0 \to 1 \quad \text{at rate} \quad \mathbf{1}(\sum_{j=1}^{n} a_{ij} \tilde{x}_j(t) > r_b d_i).$$

Since $\tilde{x}_i(t)$ does not go from one to zero, we can see that $\mathbb{E}(\vec{x}(t)) \leq \mathbb{E}(\vec{\tilde{x}}(t))$ for all $t \geq 0$. Now define the Markov process $\vec{y}(t)$ with $\vec{x}(0) = \vec{y}(0)$

$$y_i(t) : k \to k + 1 \quad \text{at rate} \quad \frac{\sum_{j=1}^{n} a_{ij} y_j(t)}{r_b d_i}.$$

Since $\mathbf{1}(\sum_{j=1}^{n} a_{ij} y_j(t) > r_b d_i) < \frac{\sum_{j=1}^{n} a_{ij} y_j(t)}{r_b d_i}$, standard coupling arguments implies $\mathbb{E}(\vec{\tilde{x}}(t)) \leq \mathbb{E}(\vec{y}(t))$. Now since $\mathbb{E}(\vec{x}(t)) \leq \mathbb{E}(\vec{\tilde{x}}(t))$ we get $\mathbb{E}(\vec{x}(t)) \leq \mathbb{E}(\vec{y}(t))$. Notice that the process $\mathbb{E}(y_i(t))$ takes value in $\mathbb{R}$. For this process we get the following differential equation

$$\frac{d}{dt}\mathbb{E}(\vec{y}(t)) = (\frac{D^{-1}A}{r_b})\mathbb{E}(\vec{y}(t)).$$

Computing the solution, we have

$$\mathbb{E}(\vec{y}(t)) = \exp((\frac{D^{-1}A}{r_b})t)\vec{y}(0).$$

Hence, the expected fraction of adoptions to the product b is bounded by

$$\frac{\sum_{i=1}^{n} \mathbb{E}(x_i(t))}{n} \leq \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{x}(0)}{n}. \tag{6}$$

Similarly, for the process $\vec{z}(t) = \vec{1} - \vec{x}(t)$ we have

$$\frac{\sum_{i=1}^{n} \mathbb{E}(z_i(t))}{n} \leq \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{z}(0)}{n}.$$

Therefore,

$$1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)(\vec{1} - \vec{x}(0))}{n} \leq \frac{\sum_{i=1}^{n} \mathbb{E}(x_i(t))}{n}. \tag{7}$$

Equation (6) and (7) and the fact that $\vec{S}_b = \vec{x}(0)$ and $\vec{S}_a = \vec{1} - \vec{x}(0)$ implies

$$1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_a}{n} \leq \frac{\sum_{i=1}^{n} \mathbb{E}(x_i(t))}{n} \leq \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_b}{n}.$$

■

Note that if $\vec{S}_a = \vec{0}$, then the lower bound in (5) becomes one, which implies all consumers adopt to product b for all time. This makes sense since there is no consumer initially with product a to spread it in the network. Similarly, if $\vec{S}_b = \vec{0}$, then the upper bound in (5) becomes zero, which implies no consumers adopts to product b at any time. This is also reasonable since there is no consumer initially with product b to spread it in the network. The other important point about bounds in (5) is that these bounds become loose when t goes to infinity. However, it is already known that all consumers adopt either product a or b in steady state depending on which one has a higher payoff. Therefore, only transeint behaviour of the dynamics for a fixed time horizon is interesting. Equation (5) also implies

$$1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_b}{n} \leq \frac{\sum_{i=1}^{n} (1 - \mathbb{E}(x_i(t)))}{n} \leq \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_a}{n}.$$

Note that a major difference between bounds (4) and (5) is that in (5) the position of the initial seeds plays a role while in (4) only aggregate number of initial seeds matters.

In the next section we discuss how firms can exploit these bounds in order to maximize the spread of their products in the network and also discuss the trade off between initial seeds and production cost in the Nash equilibrium of the game played between two firms.

III. Optimizing Product Adoption Using Bounds

As we mentioned in section II, firms initially offer their products to a subset of consumers. This can be viewed as an initial free offer to seed the network and promote each firm's product. We assume producing and offering each unit of product a and b costs c_a and c_b for firms a and b respectively. We also assume that the payoff of products for consumers in the social network is an increasing function of the firm's cost, i.e. $p = p(c)$. The rationale here is that in order to produce higher quality products for consumers in a social network, firms would have to spend more money on their products. In this section we study firms optimization problem in which they maximize the lower bound of their product adoptions (as it was found in section II) with respect to their decision variable which is their per unit production cost. Since the decision of each firm affects the lower bounds of the other firm, this optimization problem defines a game. Analyzing this game, we show that always there exists a Nash equilibrium in which the trade off between initial seeds and the production cost (and as a result quality of products) can be found. We also show how to find the most important nodes in the network to be seeded by firms. To this end, we study the optimization problem of firms in two settings. First, when only aggregate number of initial seeds is considered and bounds in (4) is used. Second, when the location of initial seeds also matters and bounds in (5) is employed.

A. Number of Initial Seeds

As it is shown in [20] when only aggregate number of initial seeds is considered firms solve the following optimization problem to maximize the lower bound of their product adoptions

$$\begin{aligned}
&\max_{c_a^{min}\le c_a\le c_a^{max}} \quad U_a = 1-\sqrt{\frac{S_b}{n}}\exp(\frac{t}{r_b})\\
&\text{Subject to} \quad S_a c_a = K_a,\\
&\max_{c_b^{min}\le c_b\le c_b^{max}} \quad U_b = 1-\sqrt{\frac{S_a}{n}}\exp(\frac{t}{r_a})\\
&\text{Subject to} \quad S_b c_b = K_b,
\end{aligned} \tag{8}$$

where K_a and K_b are the total budgets of firms a and b to initially seed the network and c_a and c_b are optimization variables. Assuming firms have payoff functions in the form of $p_a(c_a) = d_a c^{\alpha_a}$ and $p_b(c_b) = d_b c_b^{\alpha_b}$,where d_a, d_b, α_a and α_b are some constants, the utility functions of firms in the optimization problem (8) depend on both firms production costs. Therefore, we have a game in which strategies of firms is represented by c_a and c_b. Note that using the budget constraints we can see that $S_a = \frac{K_a}{c_a}$ and $S_b = \frac{K_b}{c_b}$, hence, S_a and S_b are dependent variables. It is shown in [20] that if there exists a Nash equilibrium in the game played between firms, which does not happen in boundaries of the feasible set, then we have

$$\frac{c_b^{2\alpha_b-1}}{c_a^{2\alpha_a-1}} = (\frac{K_a}{K_b})(\frac{\alpha_b}{\alpha_a})(\frac{d_a}{d_b})^2,$$

also when firms have the same constants of α and d in their payoff functions we obtain

$$\frac{c_b^*}{c_a^*} = (\frac{K_a}{K_b})^{\frac{1}{2\alpha-1}}, \qquad \frac{S_b^*}{S_a^*} = (\frac{K_b}{K_a})^{\frac{2\alpha}{2\alpha-1}}. \tag{9}$$

Equation (9) implies that If $\alpha > \frac{1}{2}$ then in Nash equilibrium firms with bigger budget seed a large number of people while lowering their investment on their product. However, if $\alpha < \frac{1}{2}$, firms with bigger budget improve the quality of their product rather than seeding it in a large number. This clarifies the trade off between production cost of a product and the number of initial seeds. The extent to which this trade off is depends on the relative total budget of firms and the exponent of production cost in product functions. This exponent indicates how sensitive the quality of the product is with respect to the cost of its production. Also, note that (9) is a necessary condition for a Nash equilibrium if it does not happen in boundaries. However, it does not guarantee the existence of a Nash equilibrium or its uniqueness. The next lemmas provide sufficient conditions for the existence and uniqueness of a Nash equilibrium in pure strategy.

Lemma 1: A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ if for all $i = 1, \ldots, I$:
i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.
ii) $u_i(s_1, \ldots, s_l)$ is continues in $(s_1, \ldots, s_l)$ and quasiconcave in s_i.

Proof: Proof can be found in [24]. ■

Lemma 2: Assume for all $i \in I$ the strategy sets S_i are given by

$$S_i = \{x_i \in \mathbb{R}_i^m | h_i(x_i) \ge 0\},$$

where $h_i : \mathbb{R}_i^m \mapsto \mathbb{R}$ is a concave function, and there exists some $\tilde{x}_i \in \mathbb{R}_i^m$ such that $h_i(\tilde{x}_i) > 0$. Assume also that the payoff functions $(u_1, \ldots, u_l)$ are diagonally strictly concave for all $x \in S$, i.e. the symmetric matrix $U(x) + U^T(x)$ is negative definite where

$$U(x) = \begin{pmatrix} \frac{\partial^2 u_1(x)}{\partial x_1^2} & \frac{\partial^2 u_1(x)}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 u_2(x)}{\partial x_2 \partial x_1} & \ddots & \vdots \\ \vdots & \cdots & \frac{\partial^2 u_I(x)}{\partial x_I^2} \end{pmatrix}.$$

Then the game has a unique pure strategy Nash equilibrium.

Proof: Proof can be found in [24]. ■

Using lemma 1 and 2, in the next theorem we show that a unique Nash equilibrium always exists in the game defined by the optimization problems in (8).

Theorem 2: If $0 < c_a^{min}, c_b^{min}$ and $p = dc^\alpha$ the game in (8) always has a unique Nash equilibrium in pure strategy.

Proof: We prove in two steps:
a) Existence: First note that strategy space in $\mathbb{R}^2$ is nonempty, convex, closed and bounded (and therefore compact) so it satisfies the requirement of lemma 1. Now we show that utility functions in (8) are concave and therefore quasiconcave. For this purpose, we show that $\tilde{U}_a = \log(1-U_a)$ is convex (this guarantees $1-U_a$ is convex and therefore U_a is concave). We have

$$\tilde{U}_a = \log(1-U_a) = \frac{1}{2}\log(\frac{S_b}{n}) + \frac{t}{r_b}.$$

Taking derivative with respect to c_a and using the definition of r_b in (3) we get

$$\frac{\partial^2 \tilde{U}_a}{\partial c_a^2} = \frac{t\alpha_a(\alpha_a+1)d_b c_b^{\alpha_b}}{d_a c_a^{\alpha_a+2}} > 0.$$

Therefore, $\tilde{U}_a$ is convex. The same argument applies to $\tilde{U}_b$. Also, $\tilde{U}_a$ and $\tilde{U}_b$ are both continues in c_a and c_b. As a result, the existence of a Nash equilibrium in pure strategy is guaranteed.
b) Uniqueness: For the uniqueness analysis of the Nash equilibrium in (8) first note that since $c_a \in [c_a^{min}, c_a^{max}]$ any positive and concave function h_a in this interval satisfies the requirement of lemma 2. Now consider the game

$$\begin{aligned}
&\min_{c_a^{min}\le c_a\le c_a^{max}} \tilde{U}_a = \log(1-U_a) = \frac{1}{2}\log(\frac{S_b}{n}) + \frac{t}{r_b},\\
&\min_{c_b^{min}\le c_b\le c_b^{max}} \tilde{U}_b = \log(1-U_b) = \frac{1}{2}\log(\frac{S_a}{n}) + \frac{t}{r_a}.
\end{aligned} \tag{10}$$

Since $\log$ is a monotone function, the uniqueness of the Nash equilibrium in both games of (8) and (10) is equivalent (except that the matrix $\tilde{U}$ should be positive definite instead of negative definite, because we have minimization in (10) instead of maximization in (8)). For convenience we analyze the game in (10). For the matrix $\tilde{U}$ we have

$$\tilde{U} = \begin{pmatrix} \frac{t\alpha_a(\alpha_a+1)d_b c_b^{\alpha_b}}{d_a c_a^{\alpha_a+2}} & -\frac{t\alpha_a\alpha_b d_b c_b^{\alpha_b-1}}{d_a c_a^{\alpha_a+1}} \\ -\frac{t\alpha_b\alpha_a d_a c_a^{\alpha_a-1}}{d_b c_b^{\alpha_b+1}} & \frac{t\alpha_b(\alpha_b+1)d_a c_a^{\alpha_a}}{d_b c_b^{\alpha_b+2}} \end{pmatrix}.$$

In order to determine if $\tilde{U}$ is positive definite, we study whether its leading principal minors are all positive or not. We have $\frac{\partial^2 \tilde{U}_a}{\partial c_a^2} > 0$. We also have $\det(\tilde{U}) > 0$, therefore, $\tilde{U}$ is positive definite. This implies $\tilde{U}^T$ is positive definite as well, and as a result $\tilde{U} + \tilde{U}^T$ is positive definite and diagonally strictly convex. Therefore, the game in (10) has a unique Nash equilibrium. ■

B. Seeding As a Function of Nodes

In this subsection, we consider the firms optimization problems where not only the size but also the location of initial seeds plays a role and firms maximize their lower bounds as defined in (5). This bound is tighter compared the bound in (4) since the information about the location of initial seeds is incorporated as well. Using the bounds in (5), we study how the initial seeds vector $\vec{S}_a$ and $\vec{S}_b$ should be chosen for a certain network. Afterwards, we study the trade off between initial seeds and production cost for star and regular networks.
To this end, we fix c_a and c_b and see how firms maximize the lower bounds of their utility functions with respect to initial seeds vectors:

$$\begin{aligned} &\max_{\vec{S}_a} U_a = 1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_b}{n} \\ &\text{Subject to} \quad c_a \vec{1}^T \vec{S}_a = K_a, \\ &\max_{\vec{S}_a} U_b = 1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_a}{n} \\ &\text{Subject to} \quad c_b \vec{1}^T \vec{S}_b = K_b. \end{aligned}$$

Since $\vec{S}_a + \vec{S}_b = \vec{1}$, the above maximization problem is equivalent to

$$\begin{aligned} &\max_{\vec{S}_a} U_a = \vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_a \\ &\text{Subject to} \quad \vec{1}^T \vec{S}_a \leq \frac{K_a}{c_a}, \\ &\max_{\vec{S}_b} U_b = \vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_b \\ &\text{Subject to} \quad \vec{1}^T \vec{S}_b \leq \frac{K_b}{c_b}. \end{aligned}$$

This is a standard integer programming problem with the solution

$$\begin{aligned} \vec{S}_a^i(0) &= \begin{cases} 1, & [\frac{K_a}{c_a}] \text{ largest elements of } \vec{1}^T \exp((\frac{D^{-1}A}{r_b})t) \\ 0, & \text{Otherwise,} \end{cases} \\ \vec{S}_b^i(0) &= \begin{cases} 1, & [\frac{K_b}{c_b}] \text{ largest elements of } \vec{1}^T \exp((\frac{D^{-1}A}{r_a})t) \\ 0, & \text{Otherwise.} \end{cases} \end{aligned} \tag{11}$$

In the next theorem we see how the initial seed selection can depend on centrality of nodes in the network.

Theorem 3: If $\lambda_2(D^{-1}A) \ll \lambda_1(D^{-1}A)$ and t large enough, then nodes with highest elements in the eigenvector corresponding to λ_1 are seeded by firms.

Proof: If we use eigenvector decomposition, we will have

$$\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t) = \sum_{k=1}^{n} \exp(\frac{\lambda_k t}{r_a})(\vec{1}^T \vec{u}_k)\vec{u}_k^T.$$

If $\lambda_2 \ll \lambda_1$, then by Perron-Frobenius theorem we have

$$\begin{aligned} \vec{1}^T \exp((\frac{D^{-1}A}{r_a})t) &\simeq \exp(\frac{\lambda_1 t}{r_a})(\vec{1}^T \vec{u}_1)\vec{u}_1^T, \\ \vec{1}^T \exp((\frac{D^{-1}A}{r_b})t) &\simeq \exp(\frac{\lambda_1 t}{r_b})(\vec{1}^T \vec{u}_1)\vec{u}_1^T. \end{aligned}$$

Therefore, as we can see nodes with the largest elements in $\vec{u_1}$ (which is the eigenvector centrality of nodes in the network) are chosen by both firms a and b. ■
Note that if both firms a and b choose to seed a specific node, the node is seeded randomly with some probabilities of q_a and q_b respectively (q_a+q_b=1). Using the initial seeds vectors in (11), the Nash equilibrium is the fixed point of the following optimization problem.

$$\begin{aligned} &\max_{c_a^{min} \leq c_a \leq c_a^{max}} U_a = 1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)\vec{S}_b}{n} \\ &\text{Subject to} \quad c_a \vec{1}^T \vec{S}_a = K_a, \\ &\max_{c_b^{min} \leq c_b \leq c_b^{max}} U_b = 1 - \frac{\vec{1}^T \exp((\frac{D^{-1}A}{r_a})t)\vec{S}_a}{n} \\ &\text{Subject to} \quad c_b \vec{1}^T \vec{S}_b = K_b. \end{aligned} \tag{12}$$

In the next theorem we show that a Nash equilibrium always exists in this game.

Theorem 4: If $0 < c_a^{min}, c_b^{min}$ and $p = dc^{\alpha}$ the game in (12) always has a Nash equilibrium in pure strategy.

Proof: In order to show the existence of a Nash equilibrium we use lemma 1. The requirement i of lemma 1 can be shown as in the proof of theorem 2, so we only need to check quasiconcavity of utility functions. It is easy to show that $\partial^2 \exp((\frac{D^{-1}A}{r_b})t)/\partial c_a^2$ is a non-negative matrix and as a result $\frac{\partial^2 U_a}{\partial c_a^2} < 0$. Therefore, U_a is concave and so is quasiconcave. U_a is continuous in c_a as well. The same argument applies to U_b. Therefore, requirement ii of lemma 1 is also satisfied and the game in (12) always has a Nash equilibrium. ■

Here as opposed to aggregate initial seed game it is not easy to find a closed form solution for the Nash equilibrium. However, in some special cases the Nash equilibrium can be found. To this end, we study the Nash equilibrium in star networks and k-regular networks.

1) Star Network: For a star network, the normalized adjacency matrix is in the form of

$$D^{-1}A = \begin{pmatrix} 0 & \frac{\vec{1}^T_{n-1}}{n-1} \\ \vec{1}_{n-1} & 0_{(n-1)*(n-1)} \end{pmatrix}.$$

Also, it is easy to show that $(D^{-1}A)^{2k} = (D^{-1}A)^2$ and $(D^{-1}A)^{2k-1} = D^{-1}A$ for all integer k. Using Taylor series, it can be shown that $\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t)$ is equal to

$$\begin{pmatrix} \cosh(\frac{t}{r_b}) + (n-1)\sinh(\frac{t}{r_b}), & (\cosh(\frac{t}{r_b}) + \frac{\sinh(\frac{t}{r_b})}{n-1})\vec{1}^T_{n-1} \end{pmatrix}.$$

Similar equations can be written for firm b. Therefore, both firms a and b choose the central node of star and $S_a - 1$ and $S_b - 1$ nodes with degree 1. If the size of the network is large enough, after simplification the utility functions in (12) become

$$\begin{aligned} U_a &\simeq 1 - q_a \sinh(\frac{t}{r_b}) - (\frac{S_b}{n})\cosh(\frac{t}{r_b}), \\ U_b &\simeq 1 - q_b \sinh(\frac{t}{r_a}) - (\frac{S_a}{n})\cosh(\frac{t}{r_a}). \end{aligned} \tag{13}$$

Also, when t is small enough, $\sinh(\frac{t}{r_b}) \simeq 0$ bounds become

$$\begin{aligned} U_a &\simeq 1 - (\frac{S_b}{n})\cosh(\frac{t}{r_b}), \\ U_b &\simeq 1 - (\frac{S_a}{n})\cosh(\frac{t}{r_a}). \end{aligned} \tag{14}$$

Since $\frac{S_b}{n} < \sqrt{\frac{S_a}{n}}$ and $\cosh(\frac{t}{r_b}) < \exp(\frac{t}{r_b})$ bounds in (14) are better than bounds in (4). This is due to knowing the network structure and employing the position of initial seeds. In order to find the Nash equilibrium, the derivative of each utility function in (13) should be taken and be set to zero. If we assume $p = dc^\alpha$ and t small enough, then after simplification in Nash equilibrium we will have

$$\frac{c_b^{2\alpha_b - 1}}{c_a^{2\alpha_a - 1}} = (\frac{K_a}{K_b})(\frac{q_b}{q_a})(\frac{\alpha_b}{\alpha_a})(\frac{d_a}{d_b})^2.$$

Also when firms have the same constants of α and d in their payoff functions we obtain

$$\frac{c_b^*}{c_a^*} = (\frac{q_b K_a}{q_a K_b})^{\frac{1}{2\alpha - 1}}, \qquad \frac{S_b^*}{S_a^*} = (\frac{q_a K_b}{q_b K_a})^{\frac{2\alpha}{2\alpha - 1}},$$

which is very similar to the Nash equilibrium in (9). The reason is that if t is small enough, it is too soon for positions of initial seeds to play a role, therefore, only size of the initial seeds matters and we get the same solution as in the scalar initial seeds case in (9).

2) k-Regular Network: In this subsection we study the Nash equilibrium in a k-regular network in which each node has exactly k neighbors. For a k-regular network, we have $\vec{1}^T D^{-1} A = \vec{1}^T$. Therefore,

$$\vec{1}^T \exp((\frac{D^{-1}A}{r_b})t) = \exp(\frac{t}{r_b})\vec{1}^T.$$

Hence, as expected, all nodes have the same value for firms and it does not matter which node is selected. Therefore, the optimization problem in (12) becomes

$$\begin{aligned} &\max_{c_a^{min} \le c_a \le c_a^{max}} U_a = 1 - (\frac{S_b}{n})\exp(\frac{t}{r_b}) \\ &\text{Subject to} \quad c_a S_a = K_a, \\ &\max_{c_b^{min} \le c_b \le c_b^{max}} U_b = 1 - (\frac{S_a}{n})\exp(\frac{t}{r_a}) \\ &\text{Subject to} \quad c_b S_b = K_b. \end{aligned} \tag{15}$$

Since $\frac{S_b}{n} < \sqrt{\frac{S_a}{n}}$ bounds in (15) are also better than bounds in (4). It can be seen that this problem has exactly the same answer as in (8) in which only the aggregate number of initial seeds matters. This is not surprising because in a k-regular network nodes have no preference in position.

IV. Conclusion

In this paper we studied a strategic model of marketing in social networks. This model studies two firms competing to maximize the adoptions of their products in a social network. Considering their fixed budgets, firms initially decide on their production cost and also how many and who they free offer their products to. The dynamics of the spread is determined by a local coordination game among consumers in which consumers act myopic rationally to maximize their profits. We found lower and upper bounds on the proportion of products adoptions which depend on the payoff of products offered by firms, the initial number and also the locations of adoptions. We showed that the optimal locations of initial seeds is chosen by firms based on nodes eigenvector centrality. Given optimal initial seeds vectors, we studied the game between firms in which they try to maximize the lower bounds of their product adoptions. We analyzed the Nash equilibrium of this game for star and k-regular networks and compared the Nash equilibrium with the one found in the scalar initial seeds setting.

References

[1] L. Feick and L. Price, "The market maven: A diffuser of marketplace information," *The Journal of Marketing*, pp. 83–97, 1987.

[2] P. Reingen, B. Foster, J. Brown, and S. Seidman, "Brand congruence in interpersonal relations: A social network analysis," *Journal of Consumer Research*, pp. 771–783, 1984.

[3] D. Godes and D. Mayzlin, "Using online conversations to study word-of-mouth communication," *Marketing Science*, pp. 545–560, 2004.

[4] C. Ballester, A. Calvó-Armengol, and Y. Zenou, "Who's who in networks. wanted: the key player," *Econometrica*, vol. 74, no. 5, pp. 1403–1417, 2006.

[5] S. Bharathi, D. Kempe, and M. Salek, "Competitive influence maximization in social networks," *Internet and Network Economics*, pp. 306–311, 2007.

[6] A. Galeotti and S. Goyal, "Influencing the influencers: a theory of strategic diffusion," *The RAND Journal of Economics*, vol. 40, no. 3, pp. 509–532, 2009.

[7] D. Kempe, J. Kleinberg, and É. Tardos, "Maximizing the spread of influence through a social network," in *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 137–146, ACM, 2003.

[8] D. Kempe, J. Kleinberg, and É. Tardos, "Influential nodes in a diffusion model for social networks," *Automata, Languages and Programming*, pp. 99–99, 2005.

[9] G. Chasparis and J. Shamma, "Control of preferences in social networks," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, pp. 6651–6656, IEEE, 2010.

[10] A. Vetta, "Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions," in *Foundations of Computer Science, 2002. Proceedings. The 43rd Annual IEEE Symposium on*, pp. 416–425, IEEE, 2002.

[11] G. Ellison, "Learning, local interaction, and coordination," *Econometrica: Journal of the Econometric Society*, pp. 1047–1071, 1993.

[12] M. Kandori, G. Mailath, and R. Rob, "Learning, mutation, and long run equilibria in games," *Econometrica: Journal of the Econometric Society*, pp. 29–56, 1993.

[13] J. Harsanyi and R. Selten, "A general theory of equilibrium selection in games," *MIT Press Books*, vol. 1, 1988.

[14] H. Young, "The evolution of conventions," *Econometrica: Journal of the Econometric Society*, pp. 57–84, 1993.

[15] H. Young, *Individual strategy and social structure: An evolutionary theory of institutions*. Princeton Univ Pr, 2001.

[16] H. Young, "The diffusion of innovations in social networks," *The Economy As an Evolving Complex System III: Current Perspectives and Future Directions*, p. 267, 2006.

[17] A. Montanari and A. Saberi, "The spread of innovations in social networks," *Proceedings of the National Academy of Sciences*, vol. 107, no. 47, p. 20196, 2010.

[18] J. Kleinberg, "Cascading behavior in networks: Algorithmic and economic issues," *Algorithmic game theory*, pp. 613–632, 2007.

[19] S. Goyal and M. Kearns, "Competitive contagion in networks," 2011.

[20] A. Fazeli and A. Jadbabaie, "Game Theoretic Analysis of a Strategic Model of Competitive Contagion and Product Adoption in Social Networks," in *Decision and Control (CDC), 2012 51th IEEE Conference on*, 2012.

[21] S. Morris, "Contagion," *The Review of Economic Studies*, vol. 67, no. 1, p. 57, 2000.

[22] A. Ganesh, L. Massoulié, and D. Towsley, "The effect of network topology on the spread of epidemics," in *INFOCOM 2005. 24th Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings IEEE*, vol. 2, pp. 1455–1466, IEEE, 2005.

[23] H. Amini, M. Draief, and M. Lelarge, "Marketing in a random network," *Network Control and Optimization*, pp. 17–25, 2009.

[24] A. Mas-Colell, M.D. Whinston, and J.R. Green, "Microeconomic theory" *Oxford university press New York*, vol. 1, 1995.