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Nima Moshtagh, Ali Jadbabaie, Kostas Daniilidis

Abstract— **We study the problem of flocking and coordination of a group of kinematic nonholonomic agents in 2 and 3 dimensions. By analyzing the velocity vectors of agents on a circle (for planar motion) or sphere (for 3D motion), we develop geodesic control laws that minimize a misalignment potential based on graph Laplacians resulting in velocity alignment. The proposed control laws are distributed and will provably result in flocking when the underlying proximity graph which represents the neighborhood relation among agents is connected. Furthermore, we develop a vision based control law that does not rely on heading measurements, but only requires measurement of bearing, optical flow and time-to-collision, all of which can be efficiently measured.**

I. INTRODUCTION

Cooperative control of multiple autonomous agents has become a vibrant part of control theory research. The main underlying theme of this line of research is to analyze and/or synthesize spatially distributed control architectures that can be used for motion coordination of large groups of autonomous vehicles. Each vehicle is assumed to be capable of local sensing and communication, and there is often a global objective, such as swarming, or reaching a stable formation, etc. A non-exhaustive list of relevant research in control theory and robotics includes [1], [3]–[6], [8], [10], [12], [13], [15], [17], [18].

On the other hand, such problems of distributed coordination have also been studied in areas as diverse as statistical physics and dynamical systems (in the context of synchronization of oscillators and alignment of self propelled particles [21], [24]), in biology, and ecology, and in computer graphics in the context of artificial life and simulation of social aggregation phenomena.

Most of the above cited research on distributed control of multi-vehicle systems has been focused on fully actuated systems [16], [22], or planar under-actuated systems [10], [19], [23]. Our goal here is to develop motion coordination algorithms that can be used for distributed control of a group of nonholonomic vehicles in 2 and 3 dimensions. Using results of Bullo *et al.* [2] we develop *geodesic control laws* that result in flocking and velocity alignment for nonholonomic agents in 3 dimensions.

In order to introduce the idea of a geodesic control law to the reader, we start with the special case of planar motion in section III. We will show that the planar version of such a control law (where the velocity vector is restricted to stay on a circle) is exactly the well-known Kuramoto model of coupled nonlinear oscillators [9], [19], [20]. Such a control law is a gradient controller that minimizes a potential function which represents the aggregate "misalignment energy" between all agents. In section IV we return to the general case of 3D motion and we develop control laws that result in stable coordination and velocity alignment of a group of agents with a fixed connectivity graph.

One application of the introduced geodesic control law is presented in section V, where a vision-based control law is developed. It is shown that flocking and formation control is possible using visual sensing, even if there is no communication between nearby agents, but each agent can sense certain information about its neighbors. Finally, in section VI we provide simulations that show the effectiveness of the designed controllers.

II. GRAPH THEORY PRELIMINARIES

In this section we introduce some standard graph theoretic notation and terminology. For more information, the interested reader is referred to [7].

An (undirected) graph $\mathbb{G}$ consists of a vertex set, $\mathcal{V}$, and an edge set $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathbb{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be adjacent, or neighbors and we denote this by writing $x \sim y$. The number of neighbors of each vertex is its valence. A path of length r from vertex x to vertex y is a sequence of $r+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathbb{G}$, then $\mathbb{G}$ is said to be connected. If there is such a path on a directed graph ignoring the direction of the edges, then the graph is weakly connected.

The adjacency matrix $A(\mathbb{G}) = [a_{ij}]$ of an (undirected) graph $\mathbb{G}$ is a symmetric matrix with rows and columns indexed by the vertices of $\mathbb{G}$, such that $a_{ij} = 1$ if vertex i and vertex j are neighbors and $a_{ij} = 0$, otherwise. The valence matrix $D(\mathbb{G})$ of a graph $\mathbb{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the

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valence of vertex i. The (un)directed graph of a (symmetric) matrix is a graph whose adjacency matrix is constructed by replacing all nonzero entries of the matrix with 1.

The symmetric singular matrix defined as:

$$L(\mathbb{G}) = D(\mathbb{G}) - A(\mathbb{G})$$

is called the Laplacian of $\mathbb{G}$. The Laplacian matrix captures many topological properties of the graph. The Laplacian L is a positive semidefinite M-matrix (a matrix whose off-diagonal entries are all nonpositive) and the algebraic multiplicity of its zero eigenvalue (i.e., the dimension of its kernel) is equal to the number of connected components in the graph. The n-dimensional eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}$.

Given an orientation of the edges of a graph, we can define the edge-vertex *incidence matrix* of the graph to be a matrix B with rows indexed by vertices and columns indexed by edges with entries of 1 representing the source of a directed edge and -1 representing the sink. The Laplacian matrix of a graph can be also represented in terms of its incidence matrix as $L = BB^T$ independent of the orientation of the edges.

III. Distributed Control of Planar Nonholonomic Vehicles

Consider a group of N agents on a plane. Each agent is capable of sensing some information from its neighbors as defined by:

$$\mathcal{N}_i \doteq \{j | i \sim j\} \subseteq \{1, \ldots, N\} \backslash \{i\}. \tag{1}$$

The neighborhood set of agent i, $\mathcal{N}_i$, is a set of agents that can share their heading (orientation) information with agent i. The size of the neighborhood depends on the characteristics of the communication device. We therefore assume that there is a predetermined radius R which determines the neighborhood relationship. The location of agent $i, (i = 1, \ldots, N)$ in the world coordinates is given by (x_i, y_i) and its velocity is $v_i = (\dot{x}_i, \dot{y}_i)^T$. The heading or orientation of agent i is θ_i and is given by: $\theta_i = \text{atan2}\big(\dot{y}_i, \dot{x}_i\big)$.

Without loss of generality, it is assumed that all agents move with constant unit speed. Thus, the kinematic model of each agent can be written as

$$\begin{aligned} \dot{x}_i &= \cos\theta_i \\ \dot{y}_i &= \sin\theta_i \\ \dot{\theta}_i &= \omega_i \qquad\qquad i = 1, \ldots, N \end{aligned} \tag{2}$$

The goal is to design the control input ω_i so that the group of mobile agents flock in the sense of following definition:

*Definition 3.1: (**Flocking**) A group of mobile agents is said to (asymptotically) flock, when all agents attain the same velocity vectors and distances between the agents are asymptotically stabilized to constant values. The state where all the headings are the same is called the consensus state.*

We consider the case where the neighboring relations among agents are represented by a fixed weighted graph.

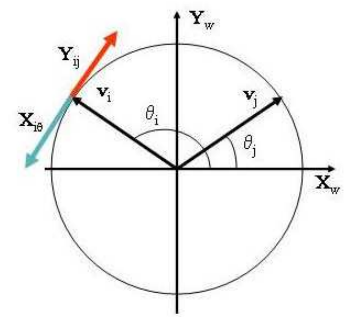

Fig. 1. Velocity vectors on the unit circle

Definition 3.2: The proximity graph $\mathbb{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ is a weighted graph consisting of:

- *a set of vertices $\mathcal{V}$ indexed by the set of mobile agents;*
- *a set of edges $\mathcal{E} = \{e_{ij} = (v_i, v_j) \mid v_i, v_j \in \mathcal{V}$, and $i \sim j\}$;*
- *a set of weights $\mathcal{W}$, over the set of edges $\mathcal{E}$.*

In order to design the desired control law for agent i, let us view all the velocity vectors of neighbors of agent i in a unit circle as shown in Figure 1. Each velocity vector v_i can be written in terms of the heading angle θ_i (measured in a fixed inertial frame) as follows $v_i = [\cos\theta_i \ , \ \sin\theta_i]^T, \ i \in \{1, \ldots, N\}$. As the velocity vector v_i changes, we can write the dynamic equation of agent i as $\dot{v}_i = \omega_i X_{i\theta}$ where vector $X_{i\theta}$ is tangent to v_i and given by $X_{i\theta} = [-\sin\theta_i \ , \ \cos\theta_i]^T$.

Let α_{ij} be the angle between two velocity vectors v_i and v_j, $\alpha_{ij} = |\theta_i - \theta_j|$. When v_i and v_j are neither equal nor opposite $(0 < \alpha_{ij} < \pi)$, we can define a unit vector Y_{ij} tangent to v_i such that it is pointing towards the velocity vector v_j. This unit-length vector is defined as:

$$Y_{ij} \doteq \frac{v_j^{\perp}}{|v_j^{\perp}|} = \frac{(v_i \times v_j) \times v_i}{\| \, (v_i \times v_j) \times v_i \, \|} = \frac{v_j - < v_i, v_j > v_i}{\sin\alpha_{ij}} \tag{3}$$

where $v_j^{\perp}$ is the component of v_j orthogonal to v_i. Now, we can prove the following theorem for the distributed control of the velocity vectors of a group of N agents.

Theorem 3.3: Consider the system of N equations $\dot{v}_i = \omega_i X_{i\theta}, \ i = 1, \ldots, N$. If the proximity graph is fixed and connected, then by applying the control law

$$\omega_i = \sum_{j \in \mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\theta} >= \sum_{j \in \mathcal{N}_i} < v_j, X_{i\theta} > \tag{4}$$

all trajectories converge to the set of equilibrium points given by $\dot{\theta} = 0$. Furthermore, consensus state is locally asymptotically stable, which means the N-agent group flocks in the sense of Definition 3.1.

Proof: We observe that on the unit circle $Y_{ij} = X_{i\theta}$ or $Y_{ij} = -X_{i\theta}$, depending on the orientations of v_i and v_j. Hence we write the input (4) as

$$\omega_i = -\sum_{j \in \mathcal{N}_i} \sin(\theta_i - \theta_j). \tag{5}$$

which as shown in [9], [13], [20] it is exactly the one used in the Kuramoto model of coupled nonlinear oscillators .

Assume an arbitrary orientation for the edges of graph $\mathbb{G}$. Consider the $N \times e$ incident matrix, B, of this oriented graph with N vertices and e edges. Then, we can write (5) as:

$$\dot{\theta} = \omega = -B \sin(B^T \theta) \tag{6}$$

where $\theta = [\theta_1, \ldots, \theta_N]^T$. Equation (6) can be written in a more compact form of:

$$\dot{\theta} = \omega = -BW(\theta)B^T\theta = -L_w(\theta)\theta, \tag{7}$$

where $W(\theta) = \text{diag}\{\text{sinc}(\theta_i - \theta_j) \mid (i,j) \in \mathcal{E}\}$. is a diagonal matrix whose entries are the edge weights for $\mathbb{G}$, and $L_w(\theta) = BW(\theta)B^T$.

When $\text{sinc}(\theta_i - \theta_j) = \sin(\theta_i - \theta_j)/(\theta_i - \theta_j)$ is positive, L_w can be thought of as the weighted Laplacian of $\mathbb{G}$. For this to hold θ should belong to the open cube $(-\pi/2, \pi/2)^N$, where N is the number of vertices of the graph. In other words, over any compact subset of the cube $(-\pi/2, \pi/2)^N$, the dynamics can be represented by a state-dependent weighted Laplacian.

Now consider the Lyapunov function

$$U = \frac{1}{2}\sum_{j\sim i} \parallel v_i - v_j \parallel^2 = \frac{1}{2}[e^{j\theta}]^* L [e^{j\theta}] = \sum_{j \sim i} 1 - \cos(\theta_i - \theta_j) \tag{8}$$

where the sum is over all the neighboring pairs, denoted by $i \sim j$, L is the Laplacian of the graph, and $[e^{j\theta}]$ is the stack of velocity vectors in complex notation. The above sum represents the total misalignment energy between velocity vectors. Since we have $U = e - \mathbf{1}^T \cos(B^T\theta)$ and because of (7), the time derivative of U becomes

$$\dot{U} = \nabla U \ \dot{\theta} = -\theta^T L_w^2 \theta = -\dot{\theta}^T \ \dot{\theta} \leq 0$$

Using LaSalle's invariance principle, we conclude that all trajectories converge asymptotically to equilibria corresponding to $\dot{\theta} = 0$. Furthermore, a simple quadratic Lyapunov function $V = \frac{1}{2}\theta^T\theta$, and a compact set $\Omega_c = \{\theta \mid V \leq c\}$ which is characterized by the largest level set of V that is contained inside the cube $(-\pi/2, \pi/2)^N$ can be used to show that the synchronized state is the only equilibrium within the set $E = \{\theta \in \Omega_c \mid \dot{V} = 0\}$. This is true since $\dot{V} = -\theta^T L_w \theta \leq 0$. Thus, equilibrium points are the set of solutions of $L_w\theta = 0$. If graph $\mathbb{G}$ is connected, within Ω_c the null space of weighted Laplacian L_w is the span of the vector $\mathbf{1} \doteq [1, \ldots, 1]^T$. Thus, the solution is Null$\{L_w\}$, which is the set $S = \{\theta \mid \theta \in \text{span}\{\mathbf{1}\}\}$. This suggests that all agents reach the same heading as $t \to \infty$. ■

Remark 3.4: When the proximity graph $\mathbb{G}$ has the *ring topology* (i.e. all agents have exactly two neighbors), there are two sets of equilibrium: $\theta \in \text{span}\{\mathbf{1}\}$ and $B^T\theta \in \text{span}\{\mathbf{1}\}$ where the former corresponds to the set $\{\theta_i = \theta_j, \ \forall i \neq j\}$ and the latter corresponds to $\{\theta_i - \theta_j = \frac{2\pi}{N}, \ \forall i \neq j\}$.

Remark 3.5: Local asymptotic stability of the consensus state can be established even when the proximity graph changes with time [14] so long as a weak connectivity notion called *joint connectivity* [8] holds.

IV. Distributed Coordination of Nonholonomic Agents in 3D

Consider a group of N agents in the 3 dimensional space. Our goal in this section is to design a control law for each agent such that it guarantees flocking in the sense of Definition 3.1.

Each agent is capable of communicating some information with its neighbors, defined by (1). The neighborhood set of agent i, $\mathcal{N}_i$, is a set of agents that can share their headings and attitudes (orientation) information with agent i. As before, it is assumed that there is a predetermined sphere with radius R which determines the neighborhood relationship. The location of agent i in the fixed world coordinates is given by (x_i, y_i, z_i) and its velocity is $v_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i)^T$. The orientation of the velocity vector of agent i can be characterized by specifying two angles θ_i (heading) and ϕ_i (attitude) relative to the world frame, and they are defined as:

$$\theta_i = \text{atan2}\big(\dot{y}_i, \dot{x}_i\big), \ 0 \leq \theta_i \leq 2\pi \tag{9}$$

$$\phi_i = \text{atan2}\Big(\sqrt{\dot{x}_i^2 + \dot{y}_i^2}, \dot{z}_i\Big), \ 0 < \phi_i < \pi \ . \tag{10}$$

Without loss of generality, it is assumed that all agents move with a constant unit speed. The velocity of agent i in 3 dimensions is given by:

$$v_i = \begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i \sin\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\phi_i \end{pmatrix}$$

Hence, all velocity vectors are on a unit sphere $\mathbf{S}^2 \doteq \{p = (x,y,z) \in \mathbb{R}^3 \ : \ \parallel p \parallel = 1\}$ (see Figure 2). We represent each vector v_i as a point on this unit sphere. As the direction of the velocity vector of agent i changes, the corresponding point v_i will move along a curve on the sphere. The tangent vector to this curve at $v_i \in \mathbf{S}^2$ can be uniquely represented as a vector $\dot{v}_i \in \mathbb{R}^3$ such that $\dot{v}_i \perp v_i$ and $\dot{v}_i \in T_i\mathbf{S}^2$ where $T_i\mathbf{S}^2$ is the tangent plane at v_i. A basis for the tangent space $T_i\mathbf{S}^2$ can be obtained by differentiating v_i, and thus $\dot{v}_i$ can be written as

$$\dot{v}_i = U_{i\theta}X_{i\theta} + U_{i\phi}X_{i\phi} \ \in T_i\mathbf{S}^2$$

where $\mathbf{B}_i = \{X_{i\theta}, X_{i\phi}\}$ is an *orthonormal basis* for the tangent plane $T_i\mathbf{S}^2$, and

$$X_{i\theta} = \begin{pmatrix} -\sin\theta_i \\ \cos\theta_i \\ 0 \end{pmatrix}, \ X_{i\phi} = \begin{pmatrix} \cos\theta_i\cos\phi_i \\ \sin\theta_i\cos\phi_i \\ -\sin\phi_i \end{pmatrix} \ .$$

The control inputs $U_{i\theta}$ and $U_{i\phi}$ are related to $\dot{\theta}_i$ and $\dot{\phi}_i$:

$$U_{i\theta} = \dot{\theta}_i \sin\phi_i, \qquad U_{i\phi} = \dot{\phi}_i \ . \tag{11}$$

When points v_i and v_j are neither equal nor opposite, a vector $Y_{ij} \in T_i\mathbf{S}^2$ called the *geodesic versor* can be defined to show the geodesic direction from v_i to v_j (see Figure 2). The unit length geodesic versor is defined by equation (3). The difference from the 2-dimensional case is that on the

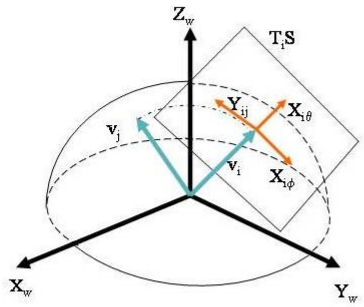

Fig. 2. The sphere and its tangent plane.

sphere the angle α_{ij} is the radian distance between points v_i and v_j over the great circle path.

Now, we can prove the following theorem for the geodesic control of the velocity vectors of a group of N agents, which is a generalization of Theorem 2 in [2] to arbitrary number of agents and connected topologies.

Theorem 4.1: Consider the system of N equations $\dot{v}_i = U_{i\theta}X_{i\theta} + U_{i\phi}X_{i\phi},\ i \in \{1,\ldots,N\}$. If the proximity graph of the agents is fixed and connected, then by applying the control laws

$$U_{i\theta} = \sum_{j\in\mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\theta} >= \sum_{j\in\mathcal{N}_i} < v_j, X_{i\theta} > \quad (12)$$

$$U_{i\phi} = \sum_{j\in\mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\phi} >= \sum_{j\in\mathcal{N}_i} < v_j, X_{i\phi} > \quad (13)$$

all trajectories converge to equilibria. Furthermore, the consensus state is locally asymptotically stable. As a result the N-agent group flocks in the sense of definition 3.1.

Proof: Convergence to equilibria can be established using the same Lyapunov function given in (8). To prove the stability of the consensus state for the system with the control laws given in Theorem 4.1, we need to write (12) and (13) in terms of the heading and attitude angles. Using (11) we obtain expression for $\dot{\theta}_i$ and $\dot{\phi}_i$:

$$\dot{\theta}_i = -\sum_{j\in\mathcal{N}_i} \frac{\sin\phi_j}{\sin\phi_i}\sin(\theta_i-\theta_j) \quad (14)$$

$$\dot{\phi}_i = -\sum_{j\in\mathcal{N}_i} \sin\phi_i\cos\phi_j - \sin\phi_j\cos\phi_i\cos(\theta_i-\theta_j)\ . \quad (15)$$

Let $\theta = [\theta_1,\ldots,\theta_N]^T$ denote the heading vector and write (14) as

$$\dot{\theta} = -B\sin(B^T\theta) = -BW(\theta)B^T\theta \quad (16)$$

where $W(\theta)$ is a weight matrix given by

$$W(\theta) = \mathrm{diag}\{\frac{\sin\phi_j}{\sin\phi_i}\mathrm{sinc}(\theta_i-\theta_j) \mid (i,j)\in\mathcal{E}\}.$$

When $\theta_i,\theta_j \in (-\pi/2,\pi/2)$ the function $\mathrm{sinc}(\theta_i-\theta_j)$ is positive . Also, from the definition of the attitude angle (10), we know that $\phi_i,\phi_j \in (0,\pi)$, (excluding the poles) therefore, $W(\theta)$ is a valid weight matrix.

Let $r = \cos(\theta_i-\theta_j)$. Since $r \in [-1,\ 1]$ and $\phi_i \in (0,\pi/2)$ we observe that

$$r\ \sin\phi_j\cos\phi_i \le \sin\phi_j\cos\phi_i$$

from which we can conclude $\dot{\phi}_i \le -\sum_{j\in\mathcal{N}_i}\sin(\phi_i-\phi_j)$. Let ϕ denote the vector of attitudes $\phi = [\phi_1,\ldots,\phi_N]^T$. In matrix notation we can write the above inequality as

$$\dot{\phi} \le -B\sin(B^T\phi) = -BW(\phi)B^T\phi \quad (17)$$

where the diagonal weight matrix is given by $W(\phi) = \mathrm{diag}\{\mathrm{sinc}(\phi_i-\phi_j) \mid (i,j)\in\mathcal{E}\}$. $W(\phi)$ is a valid weight matrix, because by restricting angles ϕ_i to $(0,\pi/2)$ the function $\mathrm{sinc}(\phi_i-\phi_j)$ is always positive.

Now consider the quadratic Lyapunov function

$$V \doteq \frac{1}{2}\theta^T\theta + \frac{1}{2}\phi^T\phi.$$

By using (16) and (17) we can show that $\dot{V}$ is nonpositive:

$$\dot{V} = \theta^T\dot{\theta} + \phi^T\dot{\phi} \le -\theta^T L_\theta\theta - \phi^T L_\phi\phi \le 0$$

where $L_\theta = BW(\theta)B^T$ and $L_\phi = BW(\phi)B^T$ are the weighted Laplacians.

The compact set $\Omega_c = \{(\theta,\ \phi) \mid V \le c\}$ is now positively invariant for the largest value of c such that $\Omega_c \subset (-\pi/2,\pi/2)^N \times (0,\pi/2)^N$.

By LaSalle's invariance principle any trajectory starting in Ω_c converges to the largest invariant set, S, contained in $E = \{(\theta,\ \phi) \mid \dot{V} = 0\}$. The invariant set of this system is when $L_\theta\theta = 0$ and $L_\phi\phi = 0$, or when vector $\theta \in \mathrm{Null}\{L_\theta\}$ and $\phi \in \mathrm{Null}\{L_\phi\}$. Hence $S = \{(\theta,\ \phi) \mid \theta \in \mathrm{span}\{\mathbf{1}\},\ \phi \in \mathrm{span}\{\mathbf{1}\}\}$ is the solution. ■

This analysis shows that geodesic controllers (12) and (13) will result in stable flocking, so long as initial conditions are inside Ω_c.

V. Application: Vision-based Formation Control

We now would like to extend the above results to the case where there is no communication between nearest neighbors, but agents are equipped with visual sensors capable of sensing information from their neighbors. While the nearest neighbor interactions have been shown to be biologically plausible and have been observed in schools of fish and flocks of birds, the assumptions about knowledge of relative headings and distances are not, at least when vision is the main sensing modality.

The simplest assumption we can make is that such systems have only monocular vision and that they have basic visual capabilities like the estimation of *optical flow* and *time-to-collision*. Experimental evidence [25] suggest that several animal species, including pigeons, are capable of estimating time-to-collision .

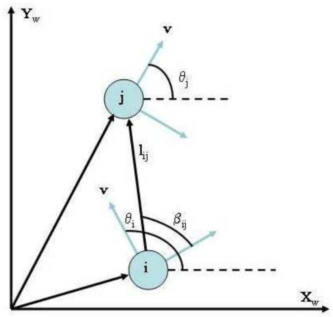

Fig. 3. Configuration of 2 agents.

A. System Model

Consider a group of N agents on a plane with kinematics given by (2). The neighborhood set of agent i, $\mathcal{N}_i$ is now a set of agents that can be seen by agent i. By "seeing" we mean that each agent can measure

- β_{ij} or the relative bearing in agent i's reference frame
- $\dot{\beta}_{ij}$ or "optical flow": the rate of change in bearing
- τ_{ij} or "time-to-collision"

for any agent j in the set of its neighbors $\mathcal{N}_i$. Note that measurement of τ_{ij} is not equivalent to measurement of the relative distance between agents as is usually the case in visual motion problems. This is due to the fact that time-to-collision can only recover the distance up to an unknown factor which in our case is different for every agent. The reader should also note that only one optical flow vector per rigid body is observed. Thus, making it impossible to rely on structure from motion algorithms.

Bearing β_{ij} and relative distance l_{ij} between agents i and j are given by (see Figure 3):

$$l_{ij}^2 = (x_j - x_i)^2 + (y_j - y_i)^2 \quad (18)$$

$$\beta_{ij} = \text{atan2}\big(y_j - y_i, x_j - x_i\big) - \theta_i + \frac{\pi}{2} \quad (19)$$

It can be shown that the time-to-collision between agents i and j denoted by τ_{ij} can be measured as the rate of growth of the image area [11], i.e. the relative change in the area A_{ij} of projection of agent j on the image plane of agent i. In other words

$$\tau_{ij} = \frac{A_{ij}}{\dot{A}_{ij}} = \frac{l_{ij}}{\dot{l}_{ij}}.$$

B. The vision-Based Distributed Control Law

We saw in section III, Theorem (3.3) that a controller of the form

$$\omega_i = -\sum_{j\in\mathcal{N}_i} \sin(\theta_i - \theta_j) \quad (20)$$

results in a stable flocking according to definition (3.1). The question is how to generate a distributed control law based on measured quantities, such that it is equivalent to the desired control law (20). Consider any pair of agents i and any of its neighbors j. By differentiating (18) we get

$$\frac{1}{\tau_{ij}} = \frac{\dot{l}_{ij}}{l_{ij}} = \frac{2}{l_{ij}}\sin(\frac{\theta_i-\theta_j}{2})\cos(\beta_{ij} + \frac{\theta_i-\theta_j}{2}) \quad (21)$$

and by differentiating (19) we obtain

$$\dot{\beta}_{ij} + \omega_i = -\frac{2}{l_{ij}}\sin(\frac{\theta_i-\theta_j}{2})\sin(\beta_{ij} + \frac{\theta_i-\theta_j}{2}) \quad (22)$$

A straightforward computation, using trigonometry identities, shows that the following relation between (21) and (22) holds:

$$\sum_{j\in\mathcal{N}_i}\frac{1}{\tau_{ij}}\cos\beta_{ij} - \sum_{j\in\mathcal{N}_i}(\omega_i + \dot{\beta}_{ij})\sin\beta_{ij} = \sum_{j\in\mathcal{N}_i}\frac{1}{l_{ij}}\sin(\theta_i - \theta_j). \quad (23)$$

where we have summed both sides over all the neighbors of agent i. Equation (23) reveals that the right-hand side has the form of the desired control law (20). By plugging (20) in (23), we get:

$$\omega_i = \frac{1}{1-\sum_{j\in\mathcal{N}_i}\sin\beta_{ij}}\sum_{j\in\mathcal{N}_i}\Bigg(\dot{\beta}_{ij}\sin\beta_{ij} - \frac{1}{\tau_{ij}}\cos\beta_{ij}\Bigg). \quad (24)$$

This is the desired control law in terms of the measured quantities only.

VI. Simulations

In this section we numerically show that the distributed control law (5), for the planar case, and the geodesic control laws (12) and (13), for the three dimensional case, can force a group of agents to flock according to definition (3.1). Figures 4 and 5 are for the two dimensional case and they show snapshots of the motion of the group at times $t = 0$ and $t = 100$ seconds, respectively. The initial position and heading of all agents are generated randomly within a pre-specified area. The neighboring radius is chosen large enough so that agents form a connected graph at time $t = 0$. The arrows on each agent show the directions of the velocity vectors.

Simulations show that agents *smoothly* adjust their headings and after a reasonable amount of time they converge to a formation, and their relative distances stabilizes. For the 3D case, Figures 6 and 7 depict the headings of all agents at times $t = 0$ and $t = 100$ seconds. They show that all agents will eventually be aligned.

VII. Conclusions and Future Work

We provided a coordination scheme which resulted in flocking of a collection of kinematic agents in 2 and 3 dimensions. The control law was based on nearest neighbor sensing. It can be shown that flocking is possible despite possible changes in the topology of the proximity graph representing the neighborhood relationship. We used the introduced planar control law to design a vision-based distributed controller that only relies on measurements from a vision sensor. Another generalization is to develop results similar to [22] for dynamic models, by using artificial potential functions [17].

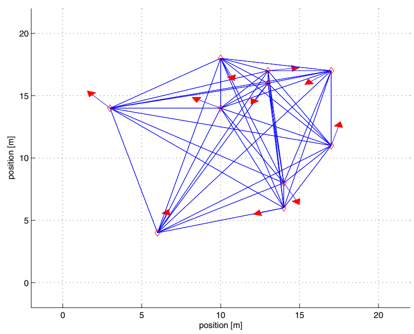

Fig. 4. 2D: At T = 0 sec agents form a connected graph.

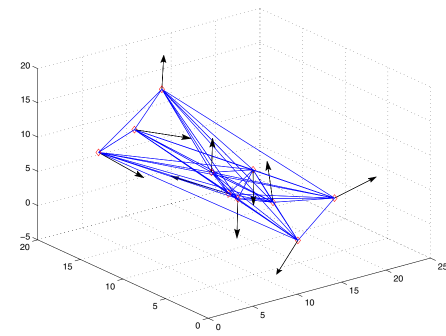

Fig. 6. 3D: The initial position and headings of 10 agents at t=0 sec.

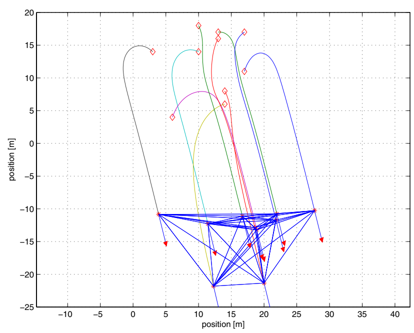

Fig. 5. 2D: At T = 100 sec group reaches a stable formation

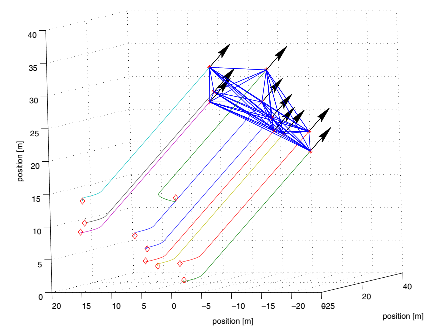

Fig. 7. 3D: The velocity vectors converge to a common direction.

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