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Abstract
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FAMILIES OF SAMPLE MEANS CONVERGE SLOWLY

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The uniform empirical integral differences of Sethuraman's large deviation theorem are proved to converge arbitrarily slowly.

1. Introduction. The purpose of this note is to answer a question of N. Glick [2, page 1380] concerning the rate of convergence to zero of a family of sample means as introduced by Sethuraman [1].

To introduce Glick's question we let (Ω, X, P) be a probability space, and let X_i, $i = 1, 2, \cdots$ be a sequence of M-valued Borel measurable random variables which are independent and identically distributed. We will suppose also that M is a complete separable metric space. The distribution of the $X_i(\omega)$ will be denoted by $\mu(\cdot)$ and the sample probability measure of $X_1(\omega), X_2(\omega), \cdots, X_n(\omega)$ will be denoted by $\mu(n, \omega, \cdot)$. (That is, $\mu(n, \omega, \cdot)$ is the measure which places mass $1/n$ at each of the points $X_1(\omega), X_2(\omega), \cdots, X_n(\omega)$.)

Numerous authors have considered properties of the random variable defined by

$$\sup_{f \in \mathscr{F}} |\textstyle\int f(x)\mu(n, \omega, dx) - \int f(x)\mu(dx)| \tag{1.1}$$

where $\mathscr{F}$ is a suitable class of functions on M. Noteworthy among such results both for its precision and generality is the large deviation theorem of Sethuraman [1] and [2]. This result says, under certain restrictions on $\mathscr{F}$, one has for $\varepsilon > 0$ that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log P(\sup_{f \in \mathscr{F}} |\textstyle\int f(x)\mu(n, \omega, dx) - \int f(x)\mu(dx)| \geqq \varepsilon) \\ = \log \rho(\mathscr{F}, \varepsilon) \end{aligned} \tag{1.2}$$

where $0 < \rho(\mathscr{F}, \varepsilon) < 1$. The conditions which Sethuraman requires of $\mathscr{F}$ in (1.2) are only two:

(S_1) $\mathscr{F}$ is a class of functions from M to the real line which is pre-compact in the topology of uniform convergence on compact sets;

(S_2) There is a function $g(x)$ such that $|f(x)| \leqq g(x)$ for all $f \in \mathscr{F}$ and such that $\int \exp(tg(x))\mu(dx) < \infty$ for all t.

The question posed by N. Glick [2, page 1330] is to ascertain the truth or falsity of

$$n^{\frac{1}{2}} \sup_{f \in \mathscr{F}} |\textstyle\int f(x)\mu(n, \omega, dx) - \int f(x)\mu(dx)| = O_p(1) \tag{1.3}$$

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where $\mathscr{F}$ satisfies the Sethuraman conditions S_1 and S_2. (Here one writes $Z_n = O_p(1)$ to mean, as usual, that $\max_n P(Z_n \geqq \lambda) \to 0$ as $\lambda \to \infty$.)

The fact that (1.3) is false is one immediate consequence of the main theorem of this note. Much more pointedly, it is proved that the *random variables* (1.1) *converge to zero arbitrarily slowly.*

2. Main result.

THEOREM 1. *Let* X_i, $i = 1, 2, \cdots$ *be* i.i.d. *and uniformly distributed on* [0, 1]. *For any sequence* c_n *such that* $c_n \to \infty$ *there is a class of functions* $\mathscr{F}$ *which is* (a) *uniformly bounded by* 1 *and* (b) *compact in the topology of uniform convergence such that*

$$c_n \sup_{f \in \mathscr{F}} |\textstyle\int f(x)\mu(dx) - \int f(x)\mu(n, \omega, dx)| \to \infty \tag{1.4}$$

with probability one.

PROOF. To construct the class $\mathscr{F}$ suppose that $0 < h_n \leqq 1$ is a sequence of reals such that $h_n \to 0$ and k_n is a sequence of integers such that $k_n > n$. For each n a finite set $\mathscr{F}_n$ of functions is defined by the following procedure.

First [0, 1] is divided into k_n intervals of equal length. Next a function f on [0, 1] is defined by (1) choosing n of the k_n intervals and defining f to be zero on the chosen intervals; (2) defining f to be zero on the first and the last of the intervals; (3) defining f to be h_n on the intervals where f is not yet defined and which are not adjacent to intervals where f has been defined, and (4) defining f by linear interpolation on all remaining intervals.

Finally $\mathscr{F}_n$ is taken to be the set of all f defined by the preceding process. We note the elements of $\mathscr{F}_n$ are bounded by h_n and that $\mathscr{F}_n$ contains at most $\binom{k_n}{n}$ distinct elements.

The class $\mathscr{F}$ is defined to be the union of all the $\mathscr{F}_n$ and the function which is identically zero. Since $h_n \to 0$ and since each $\mathscr{F}_n$ is finite, any infinite sequence from $\mathscr{F}$ is seen to contain a uniformly convergent subsequence. Since the uniform topology is metric, this shows $\mathscr{F}$ is compact.

To provide estimates for (1.4) we suppose that k_n is chosen so large that for $A_n = \{\omega : |X_i - X_j| \leqq 1/k_n \text{ for some } i \neq j,\ i, j \leqq n\}$ we have $P(A_n) \leqq 1/n^2$. It will also be supposed that k_n is taken so large that $2(n+2)/k_n \leqq h_n/2$ for our future convenience.

Now for each $\omega \in A_n^c$ one can select an $f^* \in \mathscr{F}_n$ such that f^* is zero at each of the points $X_1(\omega), X_2(\omega), \cdots, X_n(\omega)$. Consequently one has $\int f^*(x)\mu(n, \omega, dx) = 0$. On the other hand, by elementary geometry one has

$$\textstyle\int f\mu(dx) \geqq h_n - 2(n+2)/k_n$$

for any $f \in \mathscr{F}_n$. Hence by the choice which was made for k_n, one has for $\omega \in A_n^c$,

$$\begin{aligned} \sup_{f \in \mathscr{F}} &|\textstyle\int f(x)\mu(dx) - \int f(x)\mu(n, \omega, dx)| \\ &\geqq |\textstyle\int f^*(x)\mu(dx) - \int f^*(x)\mu(n, \omega, dx)| \geqq h_n/2\,. \end{aligned} \tag{1.5}$$

Since $P(A_n \text{ i.o.}) = 0$ by the Borel–Cantelli lemma we have for a.e. ω an $N(\omega)$ such that

$$c_n \sup_{f \in \mathscr{F}} |\int f(x)\mu(dx) - \int f(x)\mu(n, \omega, dx)| \geqq c_n h_n/2 \tag{1.6}$$

for all $n \geqq N(\omega)$. Finally, since the only property of h_n required in the construction is that $h_n \to 0$, the h_n can be selected such that $c_n h_n \to \infty$. Such a choice completes the proof of the theorem.

3. Applications and extentions. One should note that the class constructed in Theorem 1 certainly satisfies the Sethuraman conditions S_1 and S_2. Consequently, the question posed by N. Glick is completely answered. In the style of that question one now has that

$$\sup_{f \in \mathscr{F}} |\int f(x)\mu(dx) - \int f(x)\mu(n, \omega, dx)| = o_p(1)$$

is best possible under the conditions S_1 and S_2.

The construction used in Theorem 1 obviously can be applied to yield more general results. In particular one can prove

THEOREM 2. *Let X_i, $i = 1, 2, \cdots$ be* i.i.d. *random variables which take values in R^d and which have distribution μ. Suppose also that μ is not concentrated on any countable set. Then for any $c_n \to \infty$ there is a uniformly bounded class $\mathscr{F}$ which is compact under uniform convergence such that* (1.4) *holds with probability one.*

The proof of Theorem 2 requires only mild modifications of the technique used previously so it will not be given.

4. A conjecture. The two previous results should not make one overly doubtful of relationships such as suggested by (1.3). In particular, if one considers a class G of functions on $[0, 1]$ which are uniformly bounded and have uniformly bounded derivatives, the techniques of the present note do nothing to discredit the conjecture that

$$n^{\frac{1}{2}} \sup_{f \in G} |\int f(x)\mu(dx) - \int f(x)\mu(n, \omega, dx)| = O_p(1) . \tag{4.1}$$

There are reasons to believe that (4.1) is in fact true. If (4.1) is true this fact would have several uses, and if (4.1) is false this too should be learned.

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