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Monotone Subsequences in the Sequence of Fractional Parts of Multiples of an Irrational

Abstract

Hammersley [7] showed that if $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables whose common distribution is continuous, and if $l_n^+(l_n^-)$ denotes the length of the longest increasing (decreasing) subsequence of $X_1, X_2, \dots, X_n$, then there is a constant c such that $l_n^-/n^{1/2} \rightarrow c$ and $l_n^+/n^{1/2} \rightarrow c$ in probability, as $n \rightarrow \infty$. Kesten [8] showed that in fact there is almost sure convergence. Logan and Shepp [11] proved that $c \geq 2$, and recently Versik and Kerov [13] have announced that $c = 2$.

Disciplines

Physical Sciences and Mathematics

Comments

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Monotone subsequences in the sequence of fractional parts of multiples of an irrational*)

By *David W. Boyd* and *J. Michael Steele* at Vancouver

1. Introduction

Hammersley [7] showed that if $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables whose common distribution is continuous, and if $l_n^+ (l_n^-)$ denotes the length of the longest increasing (decreasing) subsequence of $X_1, X_2, \dots, X_n$, then there is a constant c such that $\frac{l_n^+}{n^{\frac{1}{2}}} \rightarrow c$ and $\frac{l_n^-}{n^{\frac{1}{2}}} \rightarrow c$ in probability, as $n \rightarrow \infty$.

Kesten [8] showed that in fact there is almost sure convergence. Logan and Shepp [11] proved that $c \geq 2$, and recently Veršik and Kerov [13] have announced that $c = 2$.

If α is an irrational then the sequence $\{\alpha\}, \{2\alpha\}, \dots$ of fractional parts of multiples of α is uniformly distributed in the unit interval. Franklin [5] calls such a sequence a “Weyl sequence” and applies various tests to determine its quality as a pseudo-random sequence. In this spirit, it is reasonable to investigate $l_n^+(\alpha)$ and $l_n^-(\alpha)$, the lengths of the longest increasing and decreasing subsequences of $\{\alpha\}, \dots, \{n\alpha\}$. We will be particularly interested in the behaviour of $\frac{l_n^+}{n^{\frac{1}{2}}}$ and $\frac{l_n^-}{n^{\frac{1}{2}}}$.

Some work along these lines was done by del Junco and Steele in [2]. Using discrepancy estimates, they were able to show that $\frac{\log l_n^+}{\log n} \rightarrow \frac{1}{2}$ and $\frac{\log l_n^-}{\log n} \rightarrow \frac{1}{2}$ for almost all α , and in particular for algebraic irrationals.

Here we shall be able to obtain more precise results by establishing the exact connection between $l_n^+ (l_n^-)$ and the continued fraction expansion of α . We will find piecewise linear functions of n , λ_n^+ and λ_n^- whose vertices are explicitly determined by α , which satisfy $\lambda_n^+ - 2 < l_n^+ \leq \lambda_n^+$ and $\lambda_n^- - 2 < l_n^- \leq \lambda_n^-$. The results are precise enough to show, for example,

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that there is no α for which $\frac{l_n^+}{n^{\frac{1}{2}}}$ or $\frac{l_n^-}{n^{\frac{1}{2}}}$ tends to a limit. In fact $n^{\frac{1}{2}}$ is the correct order of magnitude of l_n^+ and l_n^- precisely when α has bounded partial quotients. For example, if $\alpha = \frac{5^{\frac{1}{2}} + 1}{2}$, then

$$\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = \frac{2}{5^{\frac{1}{4}}} = 1.337481 \dots$$

$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = 5^{\frac{1}{4}} = 1.495349 \dots$$

This example gives the minimum value attainable for the difference $\limsup \frac{l_n^+}{n^{\frac{1}{2}}} - \liminf \frac{l_n^+}{n^{\frac{1}{2}}}$.

This last result invites comparison with a result of del Junco and Steele concerning the van der Corput sequence, to the effect that

$$\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = 2^{\frac{1}{2}}$$

$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = \frac{3}{2}.$$

An interesting contrast to the result for random sequences is that

$$(1) \quad n \leq l_n^+(\alpha) \quad l_n^-(\alpha) \leq n, \text{ for all } \alpha.$$

In fact $\limsup \frac{l_n^+ l_n^-}{n} = 2$ for all α , and $\liminf \frac{l_n^+ l_n^-}{n} = 1$ if α has unbounded partial quotients. The lower inequality is essentially the familiar result of Erdős and Szekeres [4], but the upper inequality is peculiar to the sequence $\{n\alpha\}$. Although it is an easy consequence of our formulas for λ_n^+ and λ_n^- , a more direct proof would be desirable. Note that, by contrast with (1), for random sequences $\frac{l_n^+ l_n^-}{n} \rightarrow 4$.

The result (1), as well as aspects of the structure of the longest monotone subsequences, was suggested by a computation of l_n^+ and l_n^- for $\alpha = 2^{\frac{1}{2}}, \frac{5^{\frac{1}{2}} + 1}{2}$ and e for $n \leq 100,000$, using the algorithm of Fredman [6]. Only the values of n for which $l_n^+ > l_{n-1}^+$ or $l_n^- > l_{n-1}^-$ were printed, and these values suggested the connection with continued fractions.

The pattern of the proof is as follows. We first show that l_n^+ is the solution to a certain integer programming problem. We then define λ_n^+ to be the solution of the corresponding linear programming problem, so that obviously $\lambda_n^+ \geq l_n^+$. Since there are only two more constraints than variables, we are able to explicitly find λ_n^+ , and the structure of the extremum shows that $\lambda_n^+ - 2 < l_n^+$. Finally, we analyse the asymptotic behaviour of $\frac{\lambda_n^+}{n^{\frac{1}{2}}}$. Since obviously

$l_n^-(\alpha) = l_n^+(1 - \alpha)$, the results for l_n^- follow automatically.

2. One-sided diophantine approximation

If α is an irrational, we will denote its continued fraction expansion by $\alpha = [a_0; a_1, a_2, \dots]$, so that if $\alpha = a_0 + \alpha_0$ with $a_0 = [\alpha]$, $\alpha_0 = \{\alpha\}$, then for $n = 1, 2, \dots$

$$a_n = \left[\frac{1}{\alpha_{n-1}} \right] \text{ and } \alpha_n = \left\{ \frac{1}{\alpha_{n-1}} \right\}.$$

If we write $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ for the principal convergents, then [10], pp. 1–8,

$$(2) \quad a_n + \alpha_n = \frac{1}{\alpha_{n-1}},$$

$$(3) \quad q_n \alpha - p_n = (-1)^n (q_{n+1} + \alpha_{n+1} q_n)^{-1}, \quad q_{-1} = 0, \quad q_0 = 1,$$

$$(4) \quad q_{n+1} = a_{n+1} q_n + q_{n-1}, \text{ for } n = 0, 1, \dots$$

We shall write $\sigma_n = |q_n \alpha - p_n|$, so that $\sigma_n = ||q_n \alpha||$ for $n > 1$, where, as usual, $||x|| = \min(\{x\}, 1 - \{x\})$. Then (2), (3) and (4) imply

$$(5) \quad \sigma_n = (q_{n+1} + \alpha_{n+1} q_n)^{-1},$$

$$(6) \quad \sigma_{n+1} = \alpha_{n+1} \sigma_n, \quad n = 0, 1, \dots$$

We define the intermediate convergents $\frac{p_{n,k}}{q_{n,k}}$ by

$$(7) \quad p_{n,k} = k p_{n+1} + p_n, \quad 0 < k < a_{n+2},$$

$$(8) \quad q_{n,k} = k q_{n+1} + q_n, \quad 0 < k < a_{n+2}.$$

Note that $q_{n,0} = q_n$ and $q_{n,a_{n+2}} = q_{n+2}$. Defining $\sigma_{n,k} = |q_{n,k} \alpha - p_{n,k}|$, (3) and (6) imply that

$$(9) \quad \sigma_{n,k} = \sigma_n - k \sigma_{n+1}, \text{ for } 0 \leq k \leq a_{n+2}.$$

The sequence q_n is characterized by the following well-known result:

Lemma 1. For $n \geq 2$, q_n is the smallest integer $q > q_{n-1}$ such that $||q\alpha|| < ||q_{n-1}\alpha||$.

Proof. See [10], p. 10 or [1], p. 2.

We define $\{q_n^+\}$ to be the sequence of even-ordered denominators q_{2m} and $q_{2m,k}$ ($1 \leq k \leq a_{2m+2} - 1$) arranged in increasing order $q_0 < q_{0,1} < \dots < q_2 < \dots$, and $\{q_n^-\}$ to be the corresponding sequence of odd-ordered denominators. The following is an analogue of Lemma 1 which we have been unable to find explicitly stated in the standard literature:

Lemma 2. (i) For $n \geq 1$, q_n^+ is the smallest integer $q > q_{n-1}^+$ such that $\{q\alpha\} < \{q_{n-1}^+\alpha\}$.

(ii) For $n \geq 1$, q_n^- is the smallest integer $q > q_{n-1}^-$ such that $\{q_{n-1}^-\alpha\} < \{q\alpha\}$.

Proof. By (9), $\{q_n^+\alpha\}$ forms a decreasing sequence. Let q satisfy $q > q_{n-1}^+ = q_{2m,k}$ and $\{q\alpha\} < \{q_{n-1}^+\alpha\}$. Write $q = q_{n-1}^+ + r$. Then, in order for $\{q_{n-1}^+\alpha + r\alpha\} < \{q_{n-1}^+\alpha\}$, we must have

$$(10) \quad \{q\alpha\} = \{q_{n-1}^+\alpha\} + \{r\alpha\} - 1 = \{q_{n-1}^+\alpha\} - ||r\alpha||.$$

Thus $||r\alpha|| < \{q_{n-1}^+\alpha\} = \{q_{2m,k}\alpha\} \leq \{q_{2m}\alpha\} = ||q_{2m}\alpha||$. By Lemma 1, this means that $r \geq q_{2m+1}$ so that $q \geq q_{2m,k} + q_{2m+1} = q_{2m,k+1} = q_n^+$.

To prove (ii), apply (i) to $1 - \alpha$.

Remark. The q_n^+ are the denominators in the semi-regular continued fraction defined by $\alpha = b_0 \frac{+1}{b_1} \frac{-1}{b_2} \frac{-1}{b_3} \dots$, with $b_n \geq 2$ for all $n \geq 1$.

The easiest way to see this is to show that these denominators satisfy Lemma 2. It also follows from a result of Tietze [12], p. 163.

3. An integer programming problem

It is clear that $l_n^-(\alpha) = l_n^+(1 - \alpha)$ so we will limit ourselves in this section to l_n^+ .

Lemma 3. Let α and n be fixed. Let q_m^+ be defined as in the previous section, and let M satisfy $q_M^+ \leq n < q_{M+1}^+$. Then $l_n^+(\alpha)$ is the solution to the following:

$$(11) \quad l_n^+ = \max \sum_{m=1}^M r_m,$$

where the r_i are integers for which

$$(12a) \quad \sum_{m=1}^M r_m q_m^+ \leq n,$$

$$(12b) \quad \sum_{m=1}^M r_m \{q_m^+ \alpha\} \leq 1,$$

$$(12c) \quad r_1 \geq 0, \dots, r_M \geq 0.$$

Proof. Suppose $0 < \{n_1 \alpha\} < \dots < \{n_l \alpha\} < 1$ is an increasing sequence of length l with $1 \leq n_1 < \dots < n_l \leq n$. If we write $d_1 = n_1$, $d_k = n_k - n_{k-1}$ for $k = 2, \dots, l$, then we obtain $\{d_k \alpha\} = \{n_k \alpha\} - \{n_{k-1} \alpha\}$, and we have

$$(13) \quad \sum_{k=1}^l \{d_k \alpha\} \leq 1 \quad \text{and} \quad \sum_{k=1}^l d_k \leq n.$$

By definition, l_n^+ is the maximum value of l under these constraints.

We claim that, in (13), it is no restriction to choose d_k from among the q_m^+ . For, if $q_m^+ < d_k < q_{m+1}^+$, then $\{d_k \alpha\} > \{q_m^+ \alpha\}$, by Lemma 2, so if d_k is replaced by q_m^+ , both inequalities in (13) continue to hold. But then, collecting terms in q_m^+ and $\{q_m^+ \alpha\}$, (13) reduces to (12).

4. A linear programming problem

Let $\lambda_n^+(\alpha)$ denote the solution to the problem defined by (11) and (12), where r_m and n are no longer required to be integers. Define $\lambda_n^-(\alpha) = \lambda_n^+(1 - \alpha)$. Then obviously $\lambda_n^+ \geq l_n^+$ and $\lambda_n^- \geq l_n^-$. Now that real coefficients are to be allowed, we find that the intermediate denominators $q_{2m,k}$ which appear in q_m^+ are no longer needed:

Lemma 4. Let K satisfy $q_{2K} \leq n < q_{2K+2}$. Then

$$\lambda_n^+ = \max \sum_{k=1}^K x_k,$$

where $(x_1, \dots, x_K)$ satisfies

$$(14a) \quad \sum_{k=1}^K x_k q_{2k} \leq n,$$

$$(14b) \quad \sum_{k=1}^K x_k \sigma_{2k} \leq 1,$$

$$(14c) \quad x_1 \geq 0, \dots, x_K \geq 0.$$

Proof. By (8) and (9), $q_{2m,k} = (1-c)q_{2m} + cq_{2m+2}$ and $\sigma_{2m,k} = (1-c)\sigma_{2m} + c\sigma_{2m+2}$, where $c = \frac{k}{a_{2m+2}}$. Thus, in (12a), we may replace any q_i^+ of the form $q_{2m,k}$ by a convex combination of q_{2m} and q_{2m+2} , and in (12b), replace $\{q_{2m,k}\alpha\} = \sigma_{2m,k}$ by the same combination of σ_{2m} and σ_{2m+2} , without affecting the inequalities.

Theorem 1. Define $\beta_{-1} = (1 - \{\alpha\})^{-1}$ and $\beta_m = q_m \sigma_m^{-1}$, $m = 0, 1, \dots$. Then, for any given α , λ_n^+ and λ_n^- are the following piecewise linear functions of n :

$$(15) \quad \lambda_n^+ = \begin{cases} q_{2k+1} + n\sigma_{2k+1}, & \text{if } \beta_{2k} \leq n < \beta_{2k+2}, k = 0, 1, \dots, \\ n, & \text{if } 0 \leq n < \beta_0, \end{cases}$$

$$(16) \quad \lambda_n^- = \begin{cases} q_{2k} + n\sigma_{2k}, & \text{if } \beta_{2k-1} \leq n < \beta_{2k+1}, k = 0, 1, \dots, \\ n, & \text{if } 0 \leq n < \beta_{-1}. \end{cases}$$

Proof. We begin with (15). The constraint region (14) has $\binom{K+2}{K}$ possible vertices.

Apart from $(0, \dots, 0)$, either $K-1$ or $K-2$ of the x_k must be 0. If all x_k but x_i are 0, which we call a vertex of type I, then (14a), (14b) give

$$(17) \quad x_i = \min(\sigma_{2i}^{-1}, nq_{2i}^{-1}).$$

If all x_k but x_i and x_j are zero, called a vertex of type II, then x_i, x_j must solve the equations

$$(18a) \quad x_i \sigma_{2i} + x_j \sigma_{2j} = 1,$$

$$(18b) \quad x_i q_{2i} + x_j q_{2j} = n$$

so that

$$(19a) \quad x_i = \sigma_{2i}^{-1} \frac{\beta_{2j} - n}{\beta_{2j} - \beta_{2i}},$$

$$(19b) \quad x_j = \sigma_{2j}^{-1} \frac{n - \beta_{2i}}{\beta_{2j} - \beta_{2i}}.$$

Assuming $i < j$, then $\sigma_{2i}^{-1} < \sigma_{2j}^{-1}$ and $\beta_{2i} < \beta_{2j}$, so the condition for $x_i \geq 0, x_j \geq 0$ is seen to be $\beta_{2i} \leq n \leq \beta_{2j}$. Among the vertices of types I and II, we seek to maximize $\lambda = x_1 + \dots + x_K$. Let us denote the right member of (17) by $f_i(n)$ and by $g_{i,j}(n)$ the value of $\lambda = x_i + x_j$ given by (19). Then

$$(20) \quad \lambda_n^+ = \max(\max_i f_i(n), \max_{i,j} g_{i,j}(n)),$$

where the maximum over i, j is restricted to those which satisfy $\beta_{2i} \leq n \leq \beta_{2j}$.

Define v_i to be the point $(\beta_{2i}, \sigma_{2i}^{-1})$, $i = 0, 1, \dots$ and $v_{-1} = (0, 0)$. Then the graph of $f_i(n)$ is the line segment $v_{-1}v_i$ followed by the horizontal line from v_i to ∞ . The graph of $g_{i,j}(n)$ is the line segment from v_i to v_j . Let us denote the right member of (15) by $f(n)$. Then, in fact, the graph of $f(n)$ is the polygonal line $v_{-1}v_0v_1\dots$. To see this, observe that

$$f(\beta_{2k}) = q_{2k+1} + \beta_{2k}\sigma_{2k+1} = q_{2k+1} + q_{2k}\sigma_{2k}^{-1}\sigma_{2k+1} = q_{2k+1} + q_{2k}\alpha_{2k+1} = \sigma_{2k}^{-1}, \text{ by (5) and (6).}$$

Also $f(\beta_{2k+2}) = q_{2k+1} + \beta_{2k+2}\sigma_{2k+1} = \sigma_{2k+2}^{-1}$, by a similar calculation. The function f is thus continuous, increasing and concave, since the slope σ_{2k+1} decreases with k . Thus, the graphs of $f_i(n)$ and $g_{i,j}(n)$ lie strictly below the graph of $f(n)$, except for $f_0(n)$ and $g_{i,i+1}(n)$, $i = 0, 1, \dots$ which coincide with $f(n)$ on the intervals $[0, \beta_0]$, $[\beta_{2i}, \beta_{2i+2}]$ respectively. This shows that $\lambda_n^+ = f(n)$.

To prove (16), one uses $\lambda_n^-(\alpha) = \lambda_n^+(1 - \alpha)$. The two cases $\alpha > \frac{1}{2}$ and $\alpha < \frac{1}{2}$ need to be distinguished. If $\alpha > \frac{1}{2}$, note that $\beta_{-1} = \beta_1$.

Corollary 1. For all α , $\lambda_n^+ - 2 < l_n^+ \leq \lambda_n^+$ and $\lambda_n^- - 2 < l_n^- \leq \lambda_n^-$.

Proof. Clearly $\lambda_n^+ \geq l_n^+$. On the other hand, by the proof of Theorem 1, if $(x_1, \dots, x_K)$ satisfies (14) and has $\lambda_n^+ = x_1 + \dots + x_K$, then $(x_1, \dots, x_K)$ need only have two non-zero components. Thus

$$l_n^+ \geq \sum_{k=1}^K [x_k] > \left(\sum_{k=1}^K x_k \right) - 2 = \lambda_n^+ - 2.$$

The proof for λ_n^- follows by the standard symmetry.

5. The asymptotic behaviour of l_n^+ and l_n^-

Theorem 2. The sequence $\frac{\lambda_n^+}{n^{\frac{1}{2}}}$ oscillates between local maxima of size $\frac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}}$ attained at $n = \beta_{2k}$ and local minima of size $2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}$ attained at $n = \beta_{2k+1}$. In a self explanatory notation,

$$(21a) \quad \text{loc max } \frac{\lambda_n^+}{n^{\frac{1}{2}}} = \frac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}}$$

$$(21b) \quad \text{loc min } \frac{\lambda_n^+}{n^{\frac{1}{2}}} = 2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}.$$

In the same notation,

$$(22a) \quad \text{loc max } \frac{\lambda_n^-}{n^{\frac{1}{2}}} = \frac{1}{(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}}$$

$$(22b) \quad \text{loc min } \frac{\lambda_n^-}{n^{\frac{1}{2}}} = 2(q_{2k}\sigma_{2k})^{\frac{1}{2}}.$$

Proof. By (15), if $\beta_{2k} \leq n \leq \beta_{2k+2}$ we have

$$\frac{\lambda_n^+}{n^{\frac{1}{2}}} = q_{2k+1} n^{-\frac{1}{2}} + \sigma_{2k+1} n^{\frac{1}{2}}$$

which, as a function of n , decreases to a minimum of $2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}$ at $n = \beta_{2k+1}$, and attains a local maximum at $n = \beta_{2k}$ equal to

$$q_{2k+1} q_{2k}^{-\frac{1}{2}} \sigma_{2k}^{\frac{1}{2}} + \sigma_{2k+1} q_{2k}^{\frac{1}{2}} \sigma_{2k}^{-\frac{1}{2}} = \sigma_{2k}^{\frac{1}{2}} q_{2k}^{-\frac{1}{2}} (q_{2k+1} + \alpha_{2k+1} q_{2k}) = \frac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}},$$

using (5) and (6). This proves (21), and (22) follows as usual.

Corollary 2. Let $A = \liminf q_{2k}\sigma_{2k}$ and $B = \liminf q_{2k+1}\sigma_{2k+1}$. Then

$$\begin{aligned} \limsup \frac{l_n^+}{n^{\frac{1}{2}}} &= A^{-\frac{1}{2}}, & \liminf \frac{l_n^+}{n^{\frac{1}{2}}} &= 2B^{\frac{1}{2}}, \\ \limsup \frac{l_n^-}{n^{\frac{1}{2}}} &= B^{-\frac{1}{2}}, & \liminf \frac{l_n^-}{n^{\frac{1}{2}}} &= 2A^{\frac{1}{2}}. \end{aligned}$$

Proof. A direct consequence of (21) and (22).

Remarks. The quantity $q_k\sigma_k = q_k\|q_k\alpha\|$ appears naturally in the theory of rational approximation, as does the quantity $v(\alpha) = \liminf q_k\|q_k\alpha\|$ [1], p. 11. The set of values $v(\alpha)$ is called the Lagrange spectrum. By a theorem of Hurwitz, the largest value of $v(\alpha)$ is $5^{-\frac{1}{2}}$. The proof of Corollary 4, below, is modelled on Davenport's proof of the theorem of Hurwitz, as given in [1], p. 11.

Two real numbers α, α' are said to be equivalent if $\alpha = \frac{a\alpha' + b}{c\alpha' + d}$, where a, b, c, d are integers with $ad - bc = \pm 1$. A necessary and sufficient condition for α and α' to be equivalent is that their continued fraction expansions can be shifted so as to coincide beyond some point [1], p. 9.

Using (5), we have the following convenient representation:

$$(23) \quad (q_k\sigma_k)^{-1} = a_{k+1} + \alpha_{k+1} + \frac{q_{k-1}}{q_k} = [a_{k+1}; a_{k+2}, \dots] + [0; a_k, a_{k-1}, \dots, a_1].$$

Thus, for example, if $\alpha = \tau = \frac{5^{\frac{1}{2}} + 1}{2} = [1; 1, 1, \dots]$, then

$$(q_k\sigma_k)^{-1} = [1; 1, 1, \dots] + [0; 1, \dots, 1] \rightarrow \frac{5^{\frac{1}{2}} + 1}{2} + \frac{5^{\frac{1}{2}} - 1}{2} = 5^{\frac{1}{2}}.$$

Thus $\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = \limsup \frac{l_n^-}{n^{\frac{1}{2}}} = 5^{\frac{1}{4}} = 1.495349\dots$, while $\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = \liminf \frac{l_n^-}{n^{\frac{1}{2}}} = 1.337481\dots$

This is an extreme example as the following shows:

Corollary 3. Let $a = \limsup \frac{l_n^+}{n^{\frac{1}{2}}}$ and $b = \liminf \frac{l_n^+}{n^{\frac{1}{2}}}$. Then $0 < b$ and $a < \infty$ if and only if $\{a_n\}$ is a bounded sequence. There is no α for which $a = b$, and in fact the minimum values of $\frac{a}{b}$ and $a - b$ are $\frac{5^{\frac{1}{2}}}{2}$ and $5^{\frac{1}{4}} - \frac{2}{5^{\frac{1}{4}}}$ attained only for $\alpha = \frac{5^{\frac{1}{2}} + 1}{2}$ and numbers equivalent to it.

Proof. The first statement is obvious since, by (23)

$$(24) \quad a_{k+1} < (q_k \sigma_k)^{-1} < a_{k+1} + 2.$$

To prove the second statement, let $A_k = q_k \sigma_k$. Then the following inequality may be derived from (2), (4) and (5), [1], p. 12:

$$(25) \quad a_{k+1}^2 A_k^2 + 2 A_k (A_{k-1} + A_{k+1}) < 1.$$

Suppose that an infinite set of a_m satisfy $a_m \geq 2$. To be specific, consider the case where $a_{2k} \geq 2$ for an infinite set of k . Then (25) implies, for these k , that

$$4 A_{2k-1}^2 + 2 A_{2k-1} (A_{2k-2} + A_{2k}) < 1,$$

and

$$A_{2k}^2 + 2 A_{2k} (A_{2k+1} + A_{2k-1}) < 1.$$

Thus

$$(26) \quad 4 B^2 + 4 A B \leq 1,$$

$$(27) \quad A^2 + 4 A B \leq 1.$$

We may rewrite (26) and (27) as

$$(28) \quad \frac{B}{A} \leq (4 A B)^{-1} - 1,$$

$$(29) \quad \frac{A}{4 B} \leq (4 A B)^{-1} - 1.$$

Taking the geometric mean of (28) and (29), we find that $(4 A B)^{-1} - 1 \geq \frac{1}{2}$, so $(4 A B)^{-1} \geq \frac{3}{2}$

which implies that $\frac{a}{b} = (4 A B)^{-\frac{1}{2}} \geq \left(\frac{3}{2}\right)^{\frac{1}{2}} > \frac{5^{\frac{1}{2}}}{2}$. A similar proof applies if an infinite number of a_{2k+1} are ≥ 2 . Thus, only if all but a finite number of a_m are equal to 1 we will have

$\frac{a}{b} = \frac{5^{\frac{1}{2}}}{2}$, and this is in fact attained by such α as we saw above.

$$a-b=a\left(1-\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)+\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}a-b\right)\geq a\left(1-\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)\geq 1-\left(\frac{2}{3}\right)^{\frac{1}{2}}=.1835\dots,$$
$$a - b = 5^{\frac{1}{4}} - \frac{2}{5^{\frac{1}{4}}} = .1578 \dots$$

Example. We see from (21), (22) and (24) that $\text{loc max} \frac{\lambda_n^+}{\frac{1}{n^2}} \cong a_{2k+1}^{\frac{1}{2}}$ and $\text{loc max} \frac{\lambda_n^-}{\frac{1}{n^2}} \cong a_{2k}^{\frac{1}{2}}$

$$\alpha = \tan\left(\frac{1}{2}\right) = [0; 1, 1, 4, 1, 8, 1, 12, 1, 16, \dots],$$
$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = \infty, \quad \liminf \frac{l_n^+}{n^{\frac{1}{2}}} = 2,$$

$$\limsup \frac{l_n^-}{n^{\frac{1}{2}}} = 1, \quad \liminf \frac{l_n^-}{n^{\frac{1}{2}}} = 0.$$

In fact, one can be more precise about $\frac{l_n^+}{n^{\frac{1}{2}}}$. By making the obvious estimates in

$$q_{2k+1} = 4k q_{2k} + q_{2k-1}, \quad q_{2k} = q_{2k-1} + q_{2k-2},$$

one has $4^k k! < q_{2k+1} < 8^k (k+1)!$, so $q_{2k+2} \cong q_{2k+1} \cong (ck)^k$ (compare the discussion for e given in [10], p. 78). If $\beta_{2k} \leq n < \beta_{2k+2}$, then

$$q_{2k}(q_{2k+1} + \alpha_{2k+1} q_{2k}) \leq n < q_{2k+2}(q_{2k+3} + \alpha_{2k+1} q_{2k+2}),$$

so $n \cong (ck)^k$ and hence $k \cong \frac{\log n}{\log \log n}$. Hence

$$\text{loc max} \frac{l_n^+}{n^{\frac{1}{2}}} \cong (4k)^{\frac{1}{2}} \cong \left(\frac{\log n}{\log \log n} \right)^{\frac{1}{2}}.$$

The metric theory of continued fractions gives us similarly precise information for almost all α . The following result is sufficient for our purposes:

Lemma 5 (E. Borel and F. Bernstein [9], p. 63). *Let $\phi(k)$ be an increasing function of k . If $\sum_{k=1}^{\infty} \phi(k)^{-1} = \infty$ then for almost all α , $a_k \geq \phi(k)$ for infinitely many k . On the other hand, if $\sum_{k=1}^{\infty} \phi(k)^{-1} < \infty$, then for almost all α , $a_k \leq \phi(k)$ except for a finite set of k .*

Corollary 4: *Let $l_n^{\pm}$ denote either l_n^+ or l_n^- . For almost all α , and any $\varepsilon > 0$, there are constants $C > 0$ and $D < \infty$ so that for all n ,*

$$(30) \quad Cn^{\frac{1}{2}}(\log n)^{-\frac{1}{2}-\varepsilon} \leq l_n^{\pm} \leq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}.$$

On the other hand

$$l_n^{\pm} \leq Cn^{\frac{1}{2}}(\log n)^{-\frac{1}{2}+\varepsilon} \quad \text{and} \quad l_n^{\pm} \geq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}-\varepsilon}$$

hold for infinitely many n .

Proof. For almost all α , $q_k \cong c^k$ for a universal constant c [9], p. 66. Thus $\beta_{2k} \leq n$ implies that $n \geq c_1^k$ so $k \leq c_2 \log n$. By Lemma 5, for almost all α , $a_k \leq c_3 k^{1+2\varepsilon}$ and thus, by (21 a) and (24),

$$l_n^+ \leq \lambda_n^+ \leq n^{\frac{1}{2}}(q_{2k}\sigma_{2k})^{-\frac{1}{2}} \leq n^{\frac{1}{2}}(a_{2k+1} + 2)^{\frac{1}{2}} \leq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}.$$

The remainder of the proof is similar.

Remark. Of course $(\log n)^{\frac{1}{2}+\varepsilon}$ can be replaced by $\psi(\log n)$ for any increasing ψ for which $\sum \psi(k)^{-2} < \infty$.

Corollary 5. *For all α , $n \leq l_n^+ l_n^- \leq 2n$, and $\limsup \frac{l_n^+ l_n^-}{n} = 2$. If α has unbounded partial quotients, then $\liminf \frac{l_n^+ l_n^-}{n} = 1$. For all α ,*

$$(31) \quad \liminf \frac{l_n^+ l_n^-}{n} \leq 1 + \frac{2}{5^{\frac{1}{2}}} = 1.894427\dots,$$

which is attained only for $\alpha = \frac{5^{\frac{1}{2}} + 1}{2}$ and equivalent numbers.

Proof. The fact that $l_n^+ l_n^- \geq n$ is a result of Erdős and Szekeres [4]. By Theorem 1, if $\beta_m \leq n \leq \beta_{m+1}$, then

$$\lambda_n^+ \lambda_n^- = g(n) = (q_m + n\sigma_m)(q_{m+1} + n\sigma_{m+1}).$$

By (5) and (6), $g(\beta_m) = 2\beta_m$ and $g(\beta_{m+1}) = 2\beta_{m+1}$. Thus $g(n) - 2n$ is quadratic in n and vanishes at $n = \beta_m, \beta_{m+1}$ and hence satisfies $g(n) < 2n$ for $\beta_m < n < \beta_{m+1}$. Hence

$$l_n^+ l_n^- \leq \lambda_n^+ \lambda_n^- \leq 2n.$$

Since $\lambda_n^+ \lambda_n^- = 2n$ for $n = \beta_m$, it follows that $\limsup \frac{l_n^+ l_n^-}{n} = \limsup \frac{\lambda_n^+ \lambda_n^-}{n} = 2$.

On the other hand $\frac{g(n)}{n}$ has a minimum at $n = (\beta_m \beta_{m+1})^{\frac{1}{2}}$. Using $q_{m+1} \sigma_m + q_m \sigma_{m+1} = 1$, we have

$$\text{loc min } \frac{\lambda_n^+ \lambda_n^-}{n} = 1 + 2(A_m A_{m+1})^{\frac{1}{2}},$$

where $A_m = q_m \sigma_m$. Thus

$$(32) \quad \liminf \frac{l_n^+ l_n^-}{n} = 1 + \liminf 2(A_m A_{m+1})^{\frac{1}{2}},$$

which is 1 if $\{a_m\}$ is unbounded. Let $c_m = A_m A_{m+1}$. Then (25) becomes

$$(33) \quad a_{m+1}^2 A_m^2 < 1 - 2c_{m-1} - 2c_m.$$

Thus

$$(34) \quad a_{m+1}^2 a_{m+2}^2 c_m^2 < (1 - 2c_{m-1} - 2c_m)(1 - 2c_m - 2c_{m+1}).$$

Let $c = \liminf c_m$, which is clearly at most $\frac{1}{4}$ by (33). If $a_m \geq 2$ for infinitely many m , then

(34) shows that $4c^2 \leq (1 - 4c)^2$, so $c \leq \frac{1}{6}$. Thus $\liminf \frac{l_n^+ l_n^-}{n} \leq 1 + \frac{2}{6^{\frac{1}{2}}} < 1 + \frac{2}{5^{\frac{1}{2}}}$, from

(32). It is easy to check that equality holds in (31) for any α equivalent to τ .

Added in proof. With respect to Lemma 2 we remark that J. L. Nicolas states that result without proof in: Répartition Modulo 1, Lecture notes in Mathematics **475**, New York 1975, p. 115.

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