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Operator Formalism and Holomorphic Factorization in Supermoduli

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In this talk I'd like to describe some joint work with Alvarez-Gaumé, Gomez, Sierra and Vafa [1] (for background see also [2][3]). One of the things we did was to use the SRS theory described in Giddings' lectures to make contact with the operator formalism in a manifestly supersymmetric way. To begin with I should say what the previous sentence means. When we say that the Polyakov action is manifestly coordinate-invariant we mean that it depends only on geometrical objects, and not on their explicit components. These are 26 scalars and one new object, namely a conformal structure on the world sheet Σ. Invariance is "manifest" because the action is built from bits which have previously been shown to be individually invariant, namely the Dolbeault operators $\partial, \bar{\partial}$ and integration of $(1,1)$-forms.

In the string case scalars get replaced by scalar superfunctions X^{μ}. And we have now seen that a ***superconformal structure*** replaces the conformal structure. Again there are primitive operations which one can check to be intrinsically-defined and then assemble into complicated objects, which will automatically be invariant. Such constructions include:

- A canonical bundle $\widehat{\omega}$ of superdifferentials of weight $1/2$.
- D. It takes scalars to sections of the canonical bundle $\widehat{\omega}$.
- Integration. It takes $\left(\frac{1}{2}, \frac{1}{2}\right)$ - superdifferentials to numbers, or more precisely to functions on any parameter space.
- Contour integration. It takes *holomorphic* $\frac{1}{2}$-superdifferentials (or antiholomorphic ones) to numbers, given a homology cycle in the underlying Riemann surface.

We will add to this list, but recall that the first two are enough to get an intrinsic action. Note that a $(\frac{1}{2}, \frac{1}{2})$-superdifferential can never be holomorphic. Also for contour integration to work it's essential that the integrand be holomorphic near the contour, just as in the ordinary case: an arbitrary $(\frac{1}{2}, 0)$ - superdifferential has an integral depending on the *curve*, not simply its homology class, because it's not a closed form. We'll later see how to specify a curve, and hence the integral of an arbitrary form.

The operator formalism for string theory has a long history, but its extension to multiloop amplitudes is relatively recent. The problem is that globally on a compact Riemann surface there is no vector field we can call Euclidean "time", and hence no consistent Hilbert space formulation of amplitudes. We can however find local patches where a radial quantization exists. The decisive step comes with the realization that the rest of the surface, however complicated, can be summarized by a state $|\widetilde{\Sigma}\rangle$ in the corresponding Hilbert space:

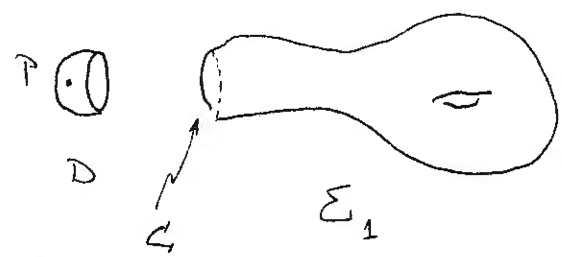

Fig. 1

Let's make this more precise. To associate a state to the full surface Σ we must choose a circle C somewhere on Σ and a normal to C. We do this by supposing Σ to be equipped with a chosen point P and a *local coordinate* z, such that $z_P = 0$. Then $C = \{|z| = 1\}$ and the normal, "time", is given by the conformal structure. Similarly we build a state $\langle 0|$ from the interior of C, a disk D, with the standard local coordinate.

To construct $\langle 0|$ and $|\Sigma\rangle$, consider a real scalar field $y(z, \bar{z})$, with conformally-invariant action $S = \int_\Sigma \partial y \wedge \bar{\partial} y$. Such a field has one oscillator for every Fourier mode of y. The quantum wave functional for y is thus a function of each Fourier coefficient, or in other words a functional $\Psi[f]$ of the boundary value f of y at time 0, *i.e.* on C. We can build such a functional:

$$\Psi[f] = \int_{y|_C = f} [dy] e^{-S[y]} \quad .$$

The corresponding state is either $|0\rangle$ or $|\widetilde{\Sigma}\rangle$ depending on whether y and the action are taken to be defined on the disk D or on $\Sigma_1 \equiv \Sigma \backslash D$.

Pretty clearly the full partition function is

$$Z = \int_\Sigma [dy] e^{-S} = \int [df] \Psi_{\Sigma_1}[f] \Psi_D[f] \equiv \langle 0|\widetilde{\Sigma}\rangle \quad .$$

Similarly if we glue two cylinders together we get the same propagator as on one long cylinder — the "semigroup property" of path integrals.

Why is this interesting? Why not simply do path integrals? Several reasons were sketched in the papers [2], [1]. First of all, the state $|\Sigma\rangle$ contains much more than just the partition function. We can instead compute $\langle 0|y(P_1)\cdots|\widetilde{\Sigma}\rangle$ where the y are evaluated on C; since y is harmonic this gives correlations everywhere. Secondly the operator formalism is intimately tied into the constructions of string field theory and may help make them more natural. It may also serve as a bridge to nonperturbative string theory via the universal Grassmannian. In the super case the operator formalism may provide a better understanding of GSO projection than just as a sum over spin structures. Finally it may help us to rephrase, and resolve, the integration ambiguity, by connecting it to BRST.

That was a long list of maybe's. My point today is that the operator formalism sometimes holds substantial *practical* advantages over path integral methods. We will see how it affords us a tremendous simplification in the proof of holomorphic factorization. Also in [2], [1] other practical advantages are discussed involving modular invariance and the construction of vertex and spin operators.

Recall that in the bosonic string moduli space has $3g-3$ even coordinates t^i. The string measure is a top form on moduli space; that is it eats $3g-3$ holomorphic tangent vectors and $3g-3$ antiholomorphic and yields a number. Let $V_1, \ldots V_{3g-3}$ be holomorphic tangent vector fields which vary holomorphically, and similarly $\overline{V}'_1, \ldots \overline{V}'_{3g-3}$. Then $\mu(V_1, \ldots \overline{V}'_{3g-3})$ is a function on moduli space. If factorization of left- and right-movers were perfect we would expect $\mu(\cdots)$ to be of the form $\mu(\cdots) = f(t)g(\bar{t})$ locally. This property is clearly unaffected if we replace $V_1, \ldots V_{3g-3}$ by another such basis. The theorem of Belavin-Knizhnik however says that instead we have

$$\delta\bar{\delta} \log \mu(V_1, \ldots \bar{V}'_{3g-3}) = -13\delta\bar{\delta}\ \log \det \tau_2 \tag{1}$$

where τ_2 is the imaginary part of the period matrix. Thus μ itself is of the form

$$\mu = (\det \tau_2)^{-13} \psi \wedge \bar{\psi}$$

where ψ is a holomorphic $(3g-3)$-form. All kinds of goodies come from this formula. (See *e.g.* [4].) We would like the super analog, and indeed one exists [5]–[10]. In the past however it's turned out to be a bit messy.

Let's begin with the RHS of (1). We can define super Abelian differentials on a SRS as global holomorphic sections of $\widehat{\omega}$; call them $\widehat{\omega}^i$. How many are there? In the usual case we can answer using the Riemann-Roch theorem: there are always g of them. Similarly in the super case a super Riemann-Roch theorem exists :

$$h^0(\mathcal{F}) - h^0(\widehat{\omega}\otimes\mathcal{F}) = \Pi^{|\mathcal{F}|}(d+1-g|d) \tag{2}$$

where $d =$ degree of $\mathcal{F}$, $|\mathcal{F}|$ is the parity of the bundle $\mathcal{F}$, and Π exchanges the two following numbers. Taking $\mathcal{F}$ trivial,

$$h^0(\widehat{\mathcal{O}}) - h^0(\widehat{\omega}) = (1-g|0) \quad .$$

Suppose from now on that we work at an even nonsingular spin structure. (Otherwise the partition function vanishes!) Then the only holomorphic function is the constant and so $h^0(\widehat{\omega}) = (g|0)$; there are g even super Abelian differentials. We can therefore normalize them by $\oint_{a_i}\widehat{\omega}^j = \delta_i^j$ as usual.

Since this exhausts our freedom we find a matrix [11][8][10]

$$\widehat{\tau}^{ij} = \oint_{b_i}\widehat{\omega}^j \quad ,$$

the super period matrix. It depends on a canonical homology basis as always. Note however that one need not check that $\widehat{\tau}$ is otherwise invariant; this is manifest since it is built from invariant objects. Unlike the bosonic case $\widehat{\tau}$ does *not* contain enough information to reconstruct all the moduli t^i and supermoduli ζ^a — after all, every entry of $\widehat{\tau}$ is commuting. Nevertheless it is what we want.

We are supposed to find the variation of $\widehat{\tau}$ as the SRS is changed. We could attempt to write an explicit coordinate system for moduli space, for example as in [12] or [9], then compute $\partial\widehat{\tau}/\partial t^i$, $\partial\widehat{\tau}/\partial\zeta^a$, but this is quite messy. Instead we will use a very nice collection of tangents to super moduli space provided by [13]. These tangents aren't integrable — but for the computation of $\delta\bar{\delta}$ it doesn't matter, so long as we put the same thing on both sides of (1).

Recall that we deal not with a Riemann surface Σ but a triple $(\Sigma, P, z) \equiv \widetilde{\Sigma}$. We will refer to the moduli space of such triples as $\mathcal{P}$ in the bosonic case. (In the super case $\widehat{\mathcal{P}}$

consists of $(\widehat{\Sigma}, \widehat{P}; z, \theta)$ where $\widehat{P}$ is a marked point and z, θ are superconformal coordinates centered at $\widehat{P}$.) How can we deform $\widetilde{\Sigma}$? Referring to Fig. 1, suppose one has a vector field $v^z(z)$ defined and holomorphic on the unit disk, except possibly for poles at the center. We can take the clutching function which glues D onto Σ_1 and modify it by composing with the infinitesimal diffeomorphism $1+v$. Using this new clutching function we glue D back onto Σ_1 to obtain a new surface Σ', close to Σ (Fig. 2). The marked point still corresponds to the center of the disk; the local coordinate still corresponds to the standard z on D. Thus v deforms $\widetilde{\Sigma}$ to $\widetilde{\Sigma}'$ and hence defines a tangent δ_v to every point of $\mathcal{P}$. These tangents do not commute; instead the Lie brackets are

$$[\delta_v, \delta_{v'}] = \delta_{[v,v']} \quad ,$$

where on the left appears the Lie bracket of vector fields on $\mathcal{P}$; on the right is a bracket of vector fields on the Riemann surface itself.

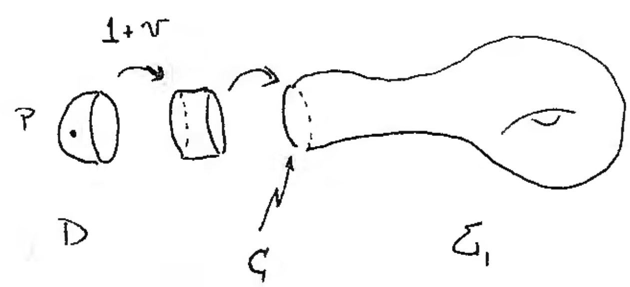

Fig. 2

For every generator $v \in$ Vect S^1, the Virasoro algebra, we get δ_v. What is the kernel of the map δ? Suppose first that v extends holomorphically to the entire disk D. Then we have changed clutching functions by a coboundary so $\Sigma' = \Sigma$. However, $\widetilde{\Sigma}' \neq \widetilde{\Sigma}_1$, because the local parameter z gets replaced by $z + v^z$. Such v therefore modify the coordinate and (if $v(0) \neq 0$) the location of the marked point P. On the other hand if v^z extends not to D but to Σ_1 then again $\Sigma' = \Sigma$. Since no extra information on Σ_1 enters $\widetilde{\Sigma}$, such v give trivial deformations, $\delta_v = 0$. The subalgebra of such v is called Borel $(\widetilde{\Sigma})$, in analogy to Lie group theory.

For SRS we first must find the analog of v. Not every holomorphic diffeomorphism is permitted now, since some spoil the superconformal structure. Indeed to be permissible a vector field X on $\widehat{\Sigma}$ must satisfy $[X, D_\theta] \propto D_\theta$, where D_θ is the vector field $\partial_\theta + \theta\partial_z$. These can all be given in terms of a (-1)-superdifferential $v = v^z(D_\theta)^{\otimes 2}$: in superconformal coordinates

$$X_v = v^z \frac{\partial}{\partial z} + \frac{1}{2}\left(D_\theta v^z\right) D_\theta \quad . \tag{3}$$

The map from v to X_v is easily shown to be superconformally invariant. The fields X preserving D form a closed subalgebra by the Jacobi identity, so we induce a Lie structure on the (-1)-superdifferentials:

$$[v,v'] = (v^z\partial_z v'^z - v'^z\partial_z v^z + \tfrac{1}{2}D_\theta v^z D_\theta v'^z)\cdot(D_\theta)^{\otimes 2} \quad .$$

Since $v^z = v_0^z + \theta v_1^\theta$ is a holomorphic superfield, it has enough degrees of freedom to correspond to the Neveu-Schwarz algebra Vect $S^{1|1}$. In exactly the same way as before we thus get a map δ from Vect $S^{1|1}$ to tangent vector fields δ_v on $\widehat{\mathcal{P}}$. Again there is a "Borel" subalgebra at each point of $\widehat{\mathcal{P}}$, Borel $(\widetilde{\Sigma})$, containing those v extending to all of $\hat{\Sigma}_1$. Again these form the kernel of δ. Again the Riemann-Roch theorem can be used to show that δ is onto: every tangent to $\widehat{\mathcal{P}}$ arises this way. Of course not every tangent vector *field* arises this way unless we allow v to depend on the chosen $\widetilde{\Sigma}$. In this case we find

$$[\delta_v, \delta_{v'}] = \delta_{(\delta_v v' - \delta_{v'} v + [v,v'])} \quad . \tag{4}$$

Thus *e.g.* if v_A are chosen to correspond to coordinate differentials $\partial/\partial t^A$ we must have

$$\frac{\partial v_A}{\partial t^B} - (-)^{AB}\frac{\partial v_B}{\partial t^A} = [v_A, v_B] \quad .$$

The set of tangents obtained in this way is very pleasant; unlike the case with super Beltrami differentials everything is holomorphic. (Other subtleties are eliminated as well.) For example it is obvious what happens to the various holomorphic differentials on $\widehat{\Sigma}$. If s is such a differential (*e.g.* a super abelian differential), use the given local coordinate to get a differential on the standard disk. Then the new differential D given by

$$s' = (1 - \pounds_v)s$$

clearly extends holomorphically to $\widehat{\Sigma}'$. Here $\pounds_v$ is the Lie derivative along X_v given in (3). From this one can easily prove (4). Instead we will now use it to find $\delta_v \widehat{\tau}^{ij}$.

Consider a BC system of weight $\frac{1}{2}$. We will shortly see that its correlations obey

$$\delta_v\langle B(\widehat{P}_1)\cdots B(\widehat{P}_q)C(\widehat{Q})\rangle = \langle B(\widehat{P}_1)\cdots T_v\rangle \tag{5}$$

where $T_v = \oint_C T_{z\theta}(\mathbf{z})v^z(\mathbf{z})d\mathbf{z}$, and T is the stress tensor. The correlation function $\langle B(\widehat{P}_1)\cdots C(\hat{Q})\rangle$ is a super abelian differential on each of $\widehat{P}_i$. Writing the infinitesimal variation of $\widehat{\omega}^i$ as v folded with some kernel η,

$$\delta_v\widehat{\omega}^i(\mathbf{z}_2) = \left[\oint_C d\mathbf{z}_1 v^{z_1}(\mathbf{z}_1)\,\eta_{z_1\theta_1\theta_2}(\mathbf{z}_1,\mathbf{z}_2)\right] d\mathbf{z}_2$$

we see from (5) that η is a $\frac{3}{2}$-superdifferential at $\mathbf{z}_1$, and a $\frac{1}{2}$-superdifferential at $\mathbf{z}_2$ with pole structure dictated by the operator product expansion of T with B. For B of weight $\frac{1}{2}$:

$$T(\mathbf{z}_1)\,B(\mathbf{z}_2) \sim \frac{1}{2}\frac{\theta_{12}}{(z_{12})^2}B(\mathbf{z}_2) + \frac{1}{2}\frac{1}{z_{12}}\left(D_{\theta_2}B(\mathbf{z}_2)\right) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}B(\mathbf{z}_2)\cdots$$

Now the super RRT (2) implies that for fixed $\mathbf{z}_2$ there are just $g+1$ even $\frac{1}{2}$-superdifferentials with a given second-order pole (take $\mathcal{F} = \widehat{\mathcal{O}}(2\widehat{P}')$, degree $d = 2$). But η must satisfy g conditions: the a-periods of $\delta\widehat{\omega}^i$ must all vanish. Furthermore the normalization of η is fixed by the pole structure above — so η is unique.

Now cut $\widetilde{\Sigma}$ into a polygon; as usual we multiply by 1 and add zero:

$$\oint_{b_k} \mathrm{d}\mathbf{z}_1\eta^i(\mathbf{z}_1,\mathbf{z}_2) = \oint_{a_k}\widehat{\omega}^i\oint_{b_k}\eta^i - \oint_{b_k}\widehat{\omega}^j\oint_{a_k}\eta^i$$

using the known periods of $\widehat{\omega}$ and η. Now let $\widehat{\omega}^i = Df^i$ for functions f defined on the cut surface. As usual our expression becomes

$$= \oint_{\partial\widehat{\Sigma}} f^i\eta^i = \oint_{C'} f^j\eta^i$$

where C' is a small contour surrounding the pole.

The poles in η give, by the super Cauchy formula, terms with $f^j(\mathbf{z}_2)$ and $Df^j(\mathbf{z}_2)$. The former cancel while the latter is $\widehat{\omega}^j$; one obtains

$$\delta_v\widehat{\tau}^{ij} = -i\pi\oint_C v\left(D\widehat{\omega}^j\cdot\widehat{\omega}^i + \widehat{\omega}^j D\widehat{\omega}^i\right) \quad ,$$

a super Rauch formula. Note that τ varies holomorphically with moduli: $\delta_v\widehat{\overline{\tau}} = 0$, so $\overline{\delta}_{\overline{v}}\widehat{\tau} = 0$. Therefore the double variation is quite simple:

$$\begin{aligned}
\delta_v\overline{\delta_{\bar{v}'}}\log\det\hat{\tau}_2 &= \tfrac{1}{4}\mathrm{Tr}\big((\delta_v\hat{\tau})\hat{\tau}_2^{-1}(\overline{\delta_{\bar{v}'}\hat{\tau}})\hat{\tau}_2^{-1}\big)\\
\delta\bar{\delta}\,\log\,\det\hat{\tau}_2 &= -\frac{\pi^2}{4}((D\hat{\omega}_1^i)\hat{\omega}_1^j + \hat{\omega}_1^i D\hat{\omega}_1^j)\hat{\tau}_{2jk}^{-1}((\bar{D}\hat{\bar{\omega}}_2^k)\hat{\bar{\omega}}_2^\ell + \hat{\bar{\omega}}_2^k(\bar{D}\hat{\bar{\omega}}_2^\ell))\hat{\tau}_{2\ell i}^{-1}\\
&= -\frac{\pi^2}{2}[(\hat{\omega}_1\cdot\hat{\tau}_2^{-1}\cdot\bar{D}\hat{\bar{\omega}}_2)(\hat{\bar{\omega}}_2\cdot\hat{\tau}_2^{-1}\cdot D\hat{\omega}_1)\\
&\quad + (D\hat{\omega}_1\cdot\hat{\tau}_2^{-1}\cdot\bar{D}\hat{\bar{\omega}}_2)(\hat{\bar{\omega}}_2\cdot\hat{\tau}_2^{-1}\cdot\hat{\omega}_1) \quad .
\end{aligned} \tag{6}$$

Next we turn to the left side of (1). To compute it we need a superfield Green function $\mathcal{G}(\widehat{P},\widehat{P}')$. As usual we define it by finding a basis of eigenfunctions of $\bar{D}^\dagger\bar{D}$. To define the dagger we must endow the surface with not just a superconformal structure but also a "super Riemann structure" [14], analogous to a Riemannian metric on an ordinary Riemann surface. In both cases a frame E^A defined up to $U(1)$ gives a good volume form, here

$$\mathrm{vol} = \mathrm{d}\mathbf{z}\mathrm{d}\bar{\mathbf{z}}\ \mathrm{Ber}\left({E_M}^A\right)$$

and from this we get the adjoint. A new twist in the super case is that the metric on field space is not positive: for example the volume of the SRS, $\|1\|^2 = \int_{\widehat{\Sigma}} \mathrm{vol}$ may be zero! We can always arrange by a suitable super Weyl transformation to avoid this problem; since ultimately SW transformations are symmetries we lose no generality by making this intermediate step. Similarly the indefiniteness of the metric means that the kernel of $\bar{D}^\dagger \bar{D}$ is not automatically equal to that of $\overline{D}$, and similarly $\bar{D}\bar{D}^\dagger$. One can easily check by hand, however, that these kernels do coincide.

Then we can let $\mathcal{G}(\widehat{P}_1, \widehat{P}_2) = \Sigma' \lambda_n^{-1} \phi_n(\widehat{P}_1)\phi_n(\widehat{P}_2)$ as usual, where ϕ_n are normalized modes of $\bar{D}^\dagger \bar{D}$ and λ_n depend on the even and odd moduli. The usual manipulations give $D_1 \bar{D}_1 \mathcal{G}$, a Weyl-dependent quantity. However

$$D_1 \bar{D}_2 \mathcal{G}(1,2) = \pi \left[\delta(1-2) - \widehat{\omega}_i(1)\left({\widehat{\tau}_2}^{\,-1}\right)^{ij} \widehat{\overline{\omega}}_j(2)\right] \tag{7}$$

is Weyl-invariant. The super-period matrix enters as the normalization matrix of the zero modes of $\bar{D}^\dagger$, *i.e.* $\left(\widehat{\omega}^i, \widehat{\omega}^j\right) = \int_{\widehat{\Sigma}} \widehat{\overline{\omega}}^i \widehat{\omega}^j = \oint_{a_k} \widehat{\overline{\omega}}^i \oint_{b_k} \widehat{\omega}^i - \oint_{b_k} \widehat{\overline{\omega}}^i \oint_{a_k} \widehat{\omega}^j = {\widehat{\tau}_2}^{\,ij}$ by the super Riemann bilinear relations [11].

We can now vary the matter sector state and compare the answer to (6). (In the Weyl-invariant regularization we are using, the ghosts will not contribute to the holomorphic anomaly; this is obvious since they have completely separate Fock spaces.) To do this we again vary the state (see below):

$$\delta_v \delta_{\bar{v}} \log\langle\cdots\rangle = \langle \cdots T_v \overline{T}_{\bar{v}'} \rangle \quad . \tag{8}$$

Note that δ_v commutes with $\bar{\delta}_{\bar{v}'}$. Using the known stress tensor $T = -\frac{1}{2} : DXD^2X:$ we can compute the contractions which contribute to (8). From the non-factorizing second term of (7) these are easily seen to reproduce the structure of terms in (7), the variation of $\det \widehat{\tau}_2$. Doing it carefully one finds

$$\delta\bar{\delta} \log\langle\cdots\rangle = -\frac{d}{2} \delta\bar{\delta} \log\det\widehat{\tau}_2$$

or for $d = 10$,

$$\langle\cdots\rangle = (\det\widehat{\tau}_2)^{-5} |F|^2$$

where F is holomorphic on supermoduli space. This is the desired result — the super Belavin-Knizhnik theorem.

We have twice used the variation of a state with respect to moduli. Let me sketch briefly where this formula comes from. Parallel to the bosonic case let

$$S[X] = -i \int_{\widehat{\Sigma}} DX\bar{D}X$$
$$\Psi[F] = \int_{X|_{\widehat{C}}=F} [dX] e^{-S[X]} \quad .$$

At the split locus Ψ splits into $\Psi_{\mathrm{Bose}} \cdot \Psi_{\mathrm{Fermi}}$. Here $\widehat{C} = S^{1|1}$, a real super circle embedded in $\widehat{\Sigma}$. $\widehat{C}$ is to be regarded as a "real axis":

$$z = e^{i\alpha} \quad \bar{z} = e^{-i\alpha} \quad \theta = \bar{\theta} = \chi \quad .$$

F is a function on $\widehat{C}$: $F = f_0(\alpha) + \chi f_1(\alpha)$. This is the correct amount of boundary data for one scalar and one Weyl fermion. $\widehat{C}$ sits in $\widehat{\Sigma}$ in a way dictated by the given superconformal coordinates.

We now wish to characterize the state Ψ, which we will again write as $|\widetilde{\Sigma}\rangle$. It turns out that

$$Q(H) = \oint_{\widehat{C}} \{H(D-\bar{D})X - X(D-\bar{D})H\}$$

all annihilate the state, for any real super function H on the circle $\widehat{C}$. At the split locus these charges include the usual bosonic $Q_{\mathrm{Bose}}(h_0) = \oint_C [h_0(\partial\text{-}\bar{\partial})y\text{-}y(\partial\text{-}\bar{\partial})h_0]$. In both formulas the normal derivatives are defined by taking H (resp. h_0) and extending it harmonically: $D\bar{D}H = 0$ (resp. $\partial\bar{\partial}h_0 = 0$). This can always be done in the ordinary case via the existence theorem for harmonic functions. Alternately we can display enough harmonic functions (one for each fourier mode of h_0), as a consequence of the Weierstrass gap theorem. Similarly a super analog of this theorem, proved using the super RRT (2), gives the existence of H.

We therefore have one "conserved" change (a charge annihilating the state with one boundary) for each super function on $S^{1|1}$. These suffice to determine $|\widetilde{\Sigma}\rangle$ up to a constant. This fact is very useful because now we can investigate the variation of $|\widetilde{\Sigma}\rangle$ by studying that of $Q(H)$. If we can show

$$Q\left((1+X_v)\,H\right) = Q(H) + [T_v, Q(H)] \tag{9}$$

where X_v is as in (3) then

$$Q\left(\left(1+X_v\right)H\right)|\widetilde{\Sigma}'\rangle = 0$$

will imply that $|\Sigma'\rangle = (1+T_v)|\Sigma\rangle$ up to a constant normalization. But (9) is not hard to verify explicitly; using the operator product expansion T_v differentiates the fields in Q, and by parts integration this is the same as differentiating H.

We can rephrase this situation in a geometrical way. If we let $\nabla_v = \delta_v + T_v$ then it defines a covariant derivative, for which [1]

$$\nabla_v|\widetilde{\Sigma}\rangle = 0 \quad . \tag{10}$$

This makes sense, since from the operator product expansion of T with itself we find that ∇ is flat, *i.e.* $[\nabla_v, \nabla_{v'}] = \nabla_{[v,v']}$ for constant v, v'. The formula (10) expresses the variations needed in (5) and (8).

In conclusion we again want to stress that the operator formalism has the potential to solve various outstanding problems in string theory. But even at the practical level a number of constructions become quite simple in this framework, and that seems reason enough to study it for now.

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