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Abstract

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Comments

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Fermionic Strings in the Operator Formalism

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We extend the operator formulation of strings to the superconformal setting. We give a realization of the supersymmetry ghosts, both in components and in a manifestly supersymmetric formalism. We give a construction for the states associated with super Riemann surfaces of any genus with any number of punctures in the case of flat $10d$ spacetime, and from these states construct the measure. As an application we give a very simple, manifestly supersymmetric derivation of super holomorphic factorization. Finally we discuss the "integration ambiguity" from the operator point of view.

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1. Introduction

There are several ways of understanding string and superstring perturbation theory.² String amplitudes at the zero and one loop level can be efficiently dealt with using operator methods in the light cone gauge or in the covariant formalism (see [2] for details and references, and also [3]–[5]). For higher loop amplitudes it is usually desirable to maintain manifest all the world-sheet symmetries, and so string scattering amplitudes are usually computed using the functional integral formulation of the theory. Recently, however, a number of ambiguities first pointed out in [6] (see also [1] and references therein) in higher order superstring amplitudes have made the analysis of higher order string loops and of questions related to finiteness of string theory far more difficult than originally envisaged.

A second approach which has been gaining momentum during the past year involves an operator approach to conformal field theory and string theory in higher genus Riemann surfaces [7]–[20]. In this paper we extend the arguments of [16] to formulate superstrings in the operator formalism (see also [21]). In particular we construct the states associated to superstring theories on super Riemann surfaces (SRS) with arbitrary numbers of punctures³, both in components and in a manifestly world sheet supersymmetric formalism. One of the advantages of working in the operator formalism is that from the beginning we only deal with superconformally invariant quantities, and we do not need to construct slices in the space of frames for a super surface of genus g . It is also possible to extend the analysis of [26] and [27] to the moduli space of SRS. In this case it is the super stress tensor that plays the role of a connection in supermoduli space. Using the super stress tensor we can transport the states in supermoduli space.

In order to construct the integration measures describing scattering amplitudes, we give a thorough treatment of the superconformal ghost system. To do this, we introduce a gaussian representation for the oscillator algebra of the ghosts which provides a natural treatment of the operator δ -functions necessary to absorb the ghost charge for the supersymmetric ghost in the world sheet. As applications we derive the superstring measure and the holomorphic factorization theorem generalizing that of Belavin and Knizhnik [28].

The organization of this paper is as follows: In section 2 we extend the arguments of [16] to the matter supermultiplet of fermionic strings. The presentation is manifestly

² For a review of string perturbation theory with many references see [1].

³ For references to the literature of super Riemann surfaces and further details, see [22]–[25].

world sheet supersymmetric. We outline some of the basic properties of SRS that will be useful to us. In particular we identify the pieces of the matter states which break holomorphic factorization in supermoduli space. This is important in the derivation of the holomorphic factorization theorem. In section 3 we study the bosonic $\beta - \gamma$ systems [3], we introduce the gaussian representation for their oscillators, and construct the $\beta - \gamma$ states for arbitrary Riemann surfaces. In particular it will become clear that the use of the gaussian representation offers definite advantages in the treatment of δ -functions with operator arguments. In section 4 we analyze the full superconformal ghost system in a supersymmetric formalism, and compute their moduli and supermoduli dependence. The more involved computations appear in the construction of the state corresponding to the sphere with one Neveu-Schwarz and two Ramond punctures. The technical details are collected in appendix B. This section is to a large extent independent of the main body of this paper; its main purpose is to show that when enough explicit information about the superconformal structure of a SRS is known, one can compute the associated states in a manifestly world sheet supersymmetric fashion. Readers uninterested in these explicit details may safely go directly to section 5. In section 5, we study the measure on supermoduli space given by the operator formalism, we discuss the holomorphic factorization theorem, the behavior on the boundary of supermoduli space, and the measures for scattering amplitudes. We also derive the physical state conditions, and the way the operator formalism treats vertex operators in different pictures. We comment briefly on the implications for the measure of the possible non-splitness of supermoduli space. Section 6 presents the conclusions.

There are also two appendices. In appendix A we give a geometrical interpretation of the gaussian representation of section 3, and in appendix B we give the details on the computation of vertices for the ghost system.

2. The matter sector

We will first consider the matter sector on the super world sheet.⁴ For concreteness we will discuss only flat uncompactified spacetime; other cases can be treated analogously. Our treatment will closely follow the bosonic string case given in [16]. This procedure continues to work in superspace thanks to the existence of a super Riemann-Roch theorem, proved

⁴ General references for this section are [29][30][23][24][31].

in [24]. We will restrict attention to SRS with a nonsingular even spin structure in order to avoid the pathologies discussed in [24] [31], but our arguments seem to apply more generally.

As in the bosonic case the matter sector breaks holomorphic factorization, again as a consequence of a Weierstrass gap theorem. We will calculate this failure of factorization in a manifestly supersymmetric way and compare to the variation of the super period matrix. This result, combined with the factorization of the ghost sector and pole structure, will give the super version of the Belavin-Knizhnik theorem in sect. 5. Such results have been discussed in the language of path integrals and determinants in [32][33][34][1][35][36], and elsewhere.

In 2.1 we will consider split SRS only.⁵ In 2.2 the necessary arguments are developed for the general case; in 2.3 the variation of the state under moduli deformations is given. Finally in 2.4 we study holomorphic factorization.

2.1 Split case

First we recall the situation for the bosonic string. The field variable for the matter sector is a real function $y(z, \bar{z})$ on a Riemann surface Σ . To get a Hilbert space we must single out a parametrized curve on Σ , which we do by further specifying a point $P \in \Sigma$ and a choice of local complex parameter z near P . The triple (Σ, P, z) we will denote by $\tilde{\Sigma}$, and the space of such triples we call $\mathcal{P}$. (These constructions extend easily to any number of punctures.) Given this data, let $D = \{|z| < 1\} \subset \Sigma$, and $\Sigma_1 = \Sigma \setminus D$, and $D_1 = D \setminus P$, the punctured disk. Then we can write a field theory on Σ_1 with action

$$S = -i \int_{\Sigma_1} \partial y \wedge \bar{\partial} y \quad .$$

In canonical quantization on the curve $C = \{|z| = 1\}$, the field y and its momentum are canonically conjugate, so wave functions for this system are functionals of the boundary data $f \equiv y|_C$. Thus for example the "vacuum" state is

$$\Psi_0[f] = \int_{y|_C=f} [dy] e^{-S[y]} \quad . \quad (2.1)$$

Also we note that y has one oscillator for every integer. To fix a state, therefore, it suffices to require that a set of operators corresponding to the integers should annihilate Ψ_0 .

⁵ That is, families with all the odd moduli set to zero. See e.g. [22][23].

To characterize the vacuum (2.1), consider the operators

$$Q_0(h) = \oint_C [h(\partial - \bar{\partial})y - y(\partial - \bar{\partial})h] \quad (2.2)$$

Here h is an arbitrary real function on the curve C . The derivatives in the second term are defined by finding the unique harmonic extension of h to Σ_1 , a function we again call h . Eqn (2.2) is shorthand for $\oint_C (h\partial_z y - y\partial_z h)dz - \oint_C (h\partial_{\bar{z}} y - y\partial_{\bar{z}} h)d\bar{z}$. Since h and y are not analytic, these integrals depend on the specific curve C and not its homology class (which is zero).

We can write a harmonic h as the real part of $k + \tilde{k}$, where k is analytic on Σ_1 and $\tilde{k}$ antianalytic. Stokes' theorem then says that

$$Q_0(h) = 2 \oint_C [k\partial y + \tilde{k}\bar{\partial}y] + c.c.$$

To see that this annihilates Ψ_0 we need only show that ∂y is holomorphic inside correlation functions: then since C is homologically trivial we are done. But

$$\Psi_0[f] = \int_{y|_C=f} [dy] e^{-S[y+\delta]}$$

for any δ vanishing on C , by a change of variables. Again using Stokes' theorem this means that $\langle \bar{\partial}\partial y \rangle = 0$ and so ∂y is holomorphic.

Thus we find a conserved charge for every real function on C , or one for every integer. Imposing these charges sets the occupation number of every oscillator (eg. to zero if $\Sigma = \mathbb{P}^1$, the sphere) and so fixes Ψ_0 up to a constant.

Next we turn to a Majorana fermion $(\psi, \bar{\psi})$. In Minkowski signature ψ and $\bar{\psi}$ are real and independent. Each has one oscillator for every non-negative integer, so the pair has as many quantum degrees of freedom as y . Since ψ is its own canonical conjugate, we should not specify the boundary values by $\psi|_C = \gamma$ for some odd real function γ on C ; this fixes too much. Instead we need a way to specify just half of the degrees of freedom in ψ . This can be done via the nonlocal boundary conditions discussed in [15], leading to a wave functional $\Psi_1[W]$ depending holomorphically on a point W of the isotropic Grassmannian. However when a theory contains both ψ and $\bar{\psi}$ another possibility arises: one can now fix half the degrees of freedom by requiring

$$(\psi_\theta + \bar{\psi}_\theta)|_C = \gamma \quad (2.3)$$

for an odd function γ which is real, in the sense that $\gamma^* = \gamma$. This condition means the wave functional

$$\Psi_1[\gamma] = \int_{(\psi_\theta + \bar{\psi}_\theta)|_C = \gamma} [d\psi][d\bar{\psi}] e^{-S[\psi, \bar{\psi}]} \quad (2.4)$$

will not be holomorphic in γ , but this is consistent with the bosonic case (2.1).

The representation (2.4) requires several comments. First, we will provisionally take the action to be

$$S = i \int_{\Sigma} [\psi_\theta \partial_z \psi_\theta + \bar{\psi}_\theta \partial_z \bar{\psi}_\theta] dz \wedge d\bar{z} \quad (2.5)$$

and comment on this choice later. Note that ψ_θ denotes the component of a spinor field $\psi_\theta dz^{1/2}$ in the trivialization given by a local coordinate z . Thus ψ_θ is simply a function near the boundary, since by assumption a coordinate z has been supplied there, and so the boundary condition (2.3) makes sense.

Secondly we need to discuss the meaning of (2.4) in Euclidean space. There are no Majorana-Weyl fermions in Euclidean space, and so ψ can no longer be regarded as real. Instead we recall that in Euclidean path integrals $\int [d\psi] e^{-i \int \psi \bar{\partial} \psi} = \text{Pf } \bar{\partial} \equiv (\det \bar{\partial})^{1/2}$, the Pfaffian. Here $\int [d\psi]$ is a holomorphic integral, analogous to a contour integral in the complex plane: $\oint dz$ is a real integral, even though z is itself a complex variable.

When both ψ and $\bar{\psi}$ are present we have two independent holomorphic integrals: $\int [d\psi d\bar{\psi}]$. If the action S is real, however, as is the case for us, we can reformulate this integral in a way suitable for interpreting (2.4). Let a_1, a_2 be two complex Fourier coefficients of ψ making a contribution $-a_1 a_2 \lambda$ to the action S . Then the functional integral

$$\int da_1 da_2 d\bar{a}_1 d\bar{a}_2 e^{(a_1 a_2 \lambda + \bar{a}_1 \bar{a}_2 \lambda^*)} = \frac{1}{2} |\lambda|^2 \quad (2.6)$$

is a factor of $|\text{Pf } \bar{\partial}|^2$. Now let $a_i = b_i + ic_i$ where b_i, c_i are real Grassmann variables. Replace the two independent sets of holomorphic integrals in (2.6) by real integrals:

$$\int db_1 dc_1 db_2 dc_2 \exp [(b_1 + ic_1)(b_2 + ic_2)\lambda + c.c.] = |\lambda|^2 \quad .$$

Thus the free fermion integral can be regarded as a holomorphic integral over $[d\psi d\bar{\psi}]$, or as a single integral $[d^2\psi]$ with $\bar{\psi}$ the complex conjugate of ψ . In the second viewpoint the boundary condition $(\psi_\theta + \bar{\psi}_\theta)|_C = \gamma$ in (2.4) does fix half the degrees of freedom. We have belabored this point because eventually we wish to combine (2.4) with (2.1) into a superfield formulation.

Note that the odd real function γ appearing in (2.4) may be single- or double-valued depending on the kind of puncture being investigated. This will later follow naturally from the fact that there are just two real super-circles $\hat{C} = S^1|1$.

Finally we return to the question of the action (2.5). In [16] an additional boundary term was added to the action for Weyl fermions b, c , in order to account for the flux of the fermion current $j = bc$ off the boundary C . For Majorana fermions such a current $j = \psi\psi$ vanishes, and no such term is needed, in parallel with the bosonic case.

Again the state $\Psi_1[\gamma]$ is annihilated by a collection of charges. Suppose λ is a holomorphic spinor field on Σ_1 and $\tilde{\lambda}$ an antiholomorphic spinor. Then

$$Q_1(\lambda) = \oint \lambda \psi ; \quad \bar{Q}_1(\tilde{\lambda}) = \oint \tilde{\lambda} \bar{\psi}$$

are both conserved. The reasoning for this is entirely analogous to the bosonic case: ψ is holomorphic in correlations by a shift argument. We thus obtain two charges for each non-negative integer, since by the Riemann-Roch theorem one can find a basis of meromorphic spinors holomorphic off P of the form

$$\lambda_N = (z^{-N} + \text{regular}) dz^{1/2} \quad N > 0$$

near P . Here and in the sequel we will assume that Σ has a generic even spin structure, and hence no spinor zero modes. (This is the 'normal case' discussed in [24].) Thus we again get one condition for each oscillator, enough to characterize $\Psi_1[\gamma]$ up to a constant.

We have seen that for a given spin Riemann surface Σ with point P and local parameter z we get a unique state $\Psi_0[f] \otimes \Psi_1[\gamma] \equiv |\tilde{\Sigma}\rangle$ for the string matter sector, characterized by the vanishing of a set of charges determined by $\tilde{\Sigma} \in \mathcal{P}$.⁶ The charges $Q_0(h)$, $Q_1(\lambda)$, $\bar{Q}_1(\tilde{\lambda})$ can all be shown to commute using the canonical commutation relations and the fact that h , λ , $\tilde{\lambda}$ extend harmonically (resp. analytically) over the surface Σ_1 . This set of charges is sufficient to characterize the state $|\tilde{\Sigma}\rangle$ because every real h has a harmonic extension from C to Σ_1 , and because there is no Weierstrass gap for spinors $\lambda, \tilde{\lambda}$. Alternately one can simply construct enough $\{h_n\}$ by the method used in [14]. Since such a basis is useful in displaying the actual state obtained, we will generalize the latter approach to arbitrary SRS.

⁶ Again $\tilde{\Sigma}$ is the triple (Σ, P, z) , where now Σ includes a spin structure. As is well known, a spin Riemann surface suffices to specify a split SRS.

2.2 Arbitrary SRS

To extend the action, states, and charges to arbitrary SRS we now must define an integral over a SRS with boundary. More precisely we define an integral of $(\frac{1}{2}, \frac{1}{2})$ -differentials over $\hat{\Sigma}_1$ which depends not only on the boundary chosen but also on the superconformal structure.

A $(\frac{p}{2}, \frac{q}{2})$ -superdifferential on a SRS is a collection of smooth superfunctions transforming as $(D_\theta \theta')^p (\bar{D}_{\bar{\theta}} \bar{\theta}')^q$ under superconformal coordinate changes [29][30].⁷ On a compact SRS $\hat{\Sigma}$ there is an invariantly defined integral of $(\frac{1}{2}, \frac{1}{2})$ -superdifferentials. Let ϕ be a smooth $(\frac{1}{2}, \frac{1}{2})$ -superdifferential, and in local coordinates write $\phi = \phi_{\theta\bar{\theta}} d\theta d\bar{\theta}$. Here $d\sharp \equiv [dz|d\theta]$ is a half-volume form. Suppose ϕ has compact support in one coordinate neighborhood. Then we can let

$$\int_{\hat{\Sigma}} \phi \equiv \int_{\Sigma} i^*(D_\theta D_{\bar{\theta}} \phi_{\theta\bar{\theta}}) dz \wedge d\bar{z} \quad (2.7)$$

where $i : \Sigma \hookrightarrow \hat{\Sigma}$ is locally defined by $\{\theta = \bar{\theta} = 0\}$. The definition (2.7) is superconformally invariant, and using a partition of unity it defines an arbitrary integral [24].

Now suppose we are given a super Riemann surface $\hat{\Sigma}$, a super point $\hat{P}$, and a choice of superconformal coordinates z, θ near $\hat{P}$. (All of our considerations can be rephrased in the more precise language of families of SRS.) We denote the superspace of such triples by $\hat{\mathcal{P}}$, and we will again refer to such a triple as $\tilde{\Sigma} = (\hat{\Sigma}, \hat{P}; z, \theta)$. Embedded in $\hat{\mathcal{P}}$ is an ordinary space (no odd coordinates) consisting of the triples considered in the previous section; they are SRS built from a spin Riemann surface with local coordinate θ built from $dz^{1/2}$. This subspace we will call the “split locus” of $\hat{\mathcal{P}}$.

We can again define a super disk $\hat{D}$, super surface minus the disk $\hat{\Sigma}_1$, and punctured super disk $\hat{D}_1$. To define an integral with boundary, $\int_{\Sigma_1} \phi$, we simply use (2.7) again; on the right we use $\int_{\Sigma_1}$ and in the neighborhood of the boundary we use the specific superconformal coordinates supplied with Σ .

For our definition to be useful it must satisfy two conditions. First we should have

$$\int_{\hat{\Sigma}_1} \phi + \int_{\hat{D}} \phi = \int_{\hat{\Sigma}} \phi$$

⁷ Here $D_\theta = \partial_\theta + \theta \partial_z$; we reserve the symbol D for the invariant operator $dz \otimes D_\theta$. (The latter was called $\hat{\partial}$ in [23][31].) Also we reserve the term “ $\frac{p}{2}$ -superdifferential” for meromorphic $(\frac{p}{2}, 0)$ -superdifferentials.

for a global $(\frac{1}{2}, \frac{1}{2})$ superdifferential ϕ . This is easily verified. Secondly there should be a Stokes' theorem. To give this suppose first that $\phi = D\bar{\omega}$ for a smooth $(0, \frac{1}{2})$ -superdifferential $\bar{\omega}$ with support away from the boundary. Then $\int \phi = 0$. Now suppose $\phi = D\bar{\omega}$ has support only near the boundary. Then we can expand in the given coordinate to find

$$\int_{\hat{\Sigma}} D\bar{\omega} = \oint_C i^*(\bar{D}\bar{\omega})d\bar{z} \quad , \quad (2.8)$$

an integral over the ordinary S^1 . By the same argument as before (2.8) continues to hold for arbitrary $\bar{\omega}$; it is the desired Stokes formula. Eqn. (2.8) is really the best we can do, since there is no invariant definition of the loop integral of a smooth superdifferential; exactly as in the ordinary case we need the actual boundary, not its homology class.⁸

The $\frac{1}{2}$ -superdifferentials can be regarded as sections of a holomorphic bundle, the canonical bundle $\hat{\omega}$ of the SRS $\hat{\Sigma}$. The generator of this bundle is $d\mathbf{s} = [dz|d\theta]$, which is odd, so we say $\hat{\omega}$ is odd: its rank is $0|1$. If $\hat{\Sigma}$ is split we obtain g even sections of this bundle, namely $\hat{\omega}^i = \theta\omega_z^i d\mathbf{s}$ where $\{\omega_z^i dz\}$ are the ordinary abelian differentials on Σ . Also in this case we find no odd global holomorphic sections of $\hat{\omega}$ since we assume a non-degenerate even spin structure. Thus

$$h^0(\hat{\omega}) \equiv \dim H^0(\hat{\omega}) = (g|0) \quad .$$

Similarly there is one even holomorphic function (the constant) and no odd ones:

$$h^0(\hat{\mathcal{O}}) = (1|0)$$

These statements illustrate a special case of the general super Riemann-Roch theorem [24]: for a line bundle $\mathcal{F}$ of parity $|\mathcal{F}|$ we have

$$h^0(\mathcal{F}) - h^1(\mathcal{F}) = \Pi^{|\mathcal{F}|}(d+1-g|d) \quad (2.9)$$

Also one has Serre duality [37][38][24]:

$$H^1(\mathcal{F}) = H^0(\hat{\omega} \otimes \mathcal{F}^\vee)^\vee$$

(Our parity conventions differ slightly from [24].) Here $\Pi^{|\mathcal{F}|}$ exchanges the two numbers following it if $\mathcal{F}$ is odd and the checks mean dual. The result of Rosly, Schwarz, and

⁸ Also as in the classical case, however, we can integrate *meromorphic* $\frac{1}{2}$ -superdifferentials over homology classes, as we do for example in the definition of $\hat{\tau}$ below.

Voronov is that (2.9) continues to hold away from the split locus. Since the left side is an index, while all excursions in the odd moduli must be regarded as infinitesimal, this is to be expected. We will now use (2.9) to show the existence of enough conserved charges to characterize $|\tilde{\Sigma}\rangle$, the state associated to an arbitrary SRS, in a manifestly invariant way.

First we remark that the Weierstrass gap theorem continues to hold, with g even and no odd gaps as in the split case. The theorem says that for generic SRS we can choose a basis of meromorphic functions on $\hat{\Sigma}$, holomorphic off $\hat{P}$ of the form

$$\begin{aligned} u_M &= z^M + (\text{order } z^{-g}) & M < -g \\ \nu_M &= \theta z^M + (\text{regular}) & M < 0 \end{aligned} .$$

To prove this we first note that for any point $\hat{P}$ on $\hat{\Sigma}$ with coordinates (z_2, θ_2) we get a line bundle $\hat{\mathcal{O}}(N\hat{P})$ of rank $1|0$ and degree N . The holomorphic sections of $\hat{\mathcal{O}}(N\hat{P})$ can be represented as functions ϕ holomorphic away from $\hat{P}$ and with $(z_{12})^N \phi(z_1)$ holomorphic near $\hat{P}$. Here $z_{12} \equiv z_1 - z_2 - \theta_1 \theta_2$; in our case $z_2 = \theta_2 = 0$.

Then (2.9) implies that

$$h^0(\hat{\omega} \otimes \hat{\mathcal{O}}(N\hat{P})) - h^0(\hat{\mathcal{O}}(-N\hat{P})) = \Pi(N|g - 1 + N) .$$

Since the only global section of $\hat{\mathcal{O}}$ is the constant the second term on the left vanishes for $N > 0$:

$$h^0(\hat{\omega} \otimes \hat{\mathcal{O}}(N\hat{P})) = (g - 1 + N|N) , \quad N > 0 .$$

Thus there are two new sections of $\hat{\omega}$, one odd and one even, for every increment in N . Combining with the g even and no odd sections without poles we find as usual a basis of meromorphic $\frac{1}{2}$ -superdifferentials of the form

$$\begin{aligned} \tilde{u}_N &= [z^N + (\text{order } z^g)] d\mathbf{z} & -\infty < N < g \\ \tilde{\nu}_N &= [\theta z^N + (\text{regular})] d\mathbf{z} & -\infty < N < 0 \end{aligned} .$$

Following the usual argument⁹ using Serre duality we find the gap structure as claimed.

We can now return to the $y, \psi, \bar{\psi}$ system. Let X be a real superfield, i.e.

$$X = y(z, \bar{z}) + \theta \psi(z, \bar{z}) + \bar{\theta} \bar{\psi}(z, \bar{z}) + \bar{\theta} \theta F(z, \bar{z})$$

⁹ See e.g. the appendix to [16].

where y is real and $\bar{\psi}$ is the conjugate to ψ . We will suppress the auxiliary field F from subsequent formulas. A suitable action is now

$$S[X] = -i \int_{\hat{\Sigma}_1} DX \bar{D}X = i \int_{\hat{\Sigma}_1} D_\theta X \cdot D_{\bar{\theta}} X \cdot d\theta d\bar{\theta}$$

This is easily seen to reproduce the previous component version when $\hat{\Sigma}$ is split.

To define boundary conditions for X we embed a real super circle in Σ . Let $\hat{C}$ be the circle with commuting coordinate α and anticommuting real coordinate χ . χ lives either in the trivial or the Möbius bundle over S^1 . Then embed $\hat{C}$ in $\hat{\Sigma}$ via $z(\alpha, \chi) = e^{i\alpha}$, $\bar{z}(\alpha, \chi) = e^{-i\alpha}$, $\theta(\alpha, \chi) = \bar{\theta}(\alpha, \chi) = \chi$. Thus $\hat{C}$ is a "real axis" in both even and odd dimensions. If $F(\alpha, \chi)$ is now a real superfield on $\hat{C}$, we can define

$$\Psi[F] = \int_{X|_{\hat{C}}=F} [dX] e^{-S[X]} \quad (2.10)$$

Clearly at the split locus $\Psi[F] = \Psi_0[f] \otimes \Psi_1[\gamma]$ where $F = f(\alpha) + \chi\gamma(\alpha)$. Away from the split locus this factorization breaks down, but (2.10) still makes sense. We will also refer to the ray defined by (2.10) as $|\tilde{\Sigma}\rangle$, since clearly it is defined using only the data given in $\tilde{\Sigma}$.

What charges characterize the super vacuum? Suppose H is a superfield on $\hat{\Sigma}$ satisfying $D\bar{D}H = 0$; such an H will be called superharmonic. If we expand $H = h + \theta\lambda + \bar{\theta}\tilde{\lambda} + \bar{\theta}\theta K$ we find $K = 0$, h is harmonic, and $\bar{\partial}\lambda = \partial\tilde{\lambda} = 0$. Consider

$$Q(H) = \oint_{\hat{C}} [H(D - \bar{D})X - X(D - \bar{D})H] \quad (2.11)$$

In (2.11) the integrand is restricted to $\hat{C}$ by the previous embedding and then integrated over α and χ . At the split locus we at once find $Q(H) = Q_0(h) + 2Q_1(\lambda) + 2\bar{Q}_1(\tilde{\lambda})$, and hence $Q(H)\Psi = 0$. Moreover the same arguments as in the bosonic case show that $Q(H)$ is always conserved. For this we use the fact that DX is holomorphic in correlations, together with a simple lemma analogous to the identity $\oint df = 0$ for any single-valued ordinary function:

$$\oint_{\hat{C}} (D + \bar{D})F = 0$$

Here again F is defined near $\hat{C}$ and we restrict to $\hat{C}$ after differentiating.

To show that (2.11) characterize the vacuum we must now construct sufficiently many superharmonic functions H . The holomorphic superfunctions and their conjugates provide

examples, but we know from the super Weierstrass theorem that this choice misses g even functions, generically those with poles at $\hat{P}$ of order $1, \dots, g$. We now construct the g extra harmonic functions, in analogy with ref. [14].

Let $\hat{P}'$ be any point of $\hat{\Sigma}$, not necessarily the same as $\hat{P}$. Applying the super Riemann-Roch theorem to $\hat{\omega} \otimes \hat{\mathcal{O}}(2\hat{P}')$ we find $g + 1$ even sections. Thus there is just one $\frac{1}{2}$ -superdifferential meromorphic except for a double pole at $\hat{P}'$ and with vanishing a -periods. Call it $\eta_{\hat{P}'}$. To fix the normalization of η uniquely we wish to fix the "residue" of its double pole; this requires that we interpret it as a bidifferential $\eta(\hat{P}', \hat{P}'')$, a $\frac{1}{2}$ -superdifferential at each of P', P'' . Thus η plays the role of the unique bidifferential $\partial_{P'} \partial_{P''} \log E(P', P'')$ on an ordinary Riemann surface, where E is the prime form. In fact at the split locus one clearly has

$$\eta(z_1, z_2) = \theta_1 \theta_2 \partial_{z_1} \partial_{z_2} \log E(z_1, z_2) dz_1 dz_2 \quad .$$

We can now let

$$X_n(P') = \int^{P'} dz_1 (\partial_{z_1})^n D_{\theta_2} \eta_{\theta_1 \theta_2}(z_1, z_2) |_{z_2=0} \quad .$$

At the split locus these reduce to the harmonic, multivalued functions in [14]. As in the bosonic case, they depend on the coordinate choice at P but not at P' .

Finally one must correct the X_n by regular, non-holomorphic terms to correct the b -shifts. (The a -shifts vanish because the a -periods of η vanish.) Since we will not need the explicit form of the vacuum we leave this as an easy exercise, using some of the ideas in 2.4. This finally shows that the vacuum state $|\tilde{\Sigma}\rangle = \Psi[F]$, constructed via the path integral (2.10), can just as well be abstractly characterized as the state annihilated by two sets of commuting charges each indexed by the integers. The charges are intrinsically defined using only the given SRS, puncture, and local superconformal coordinate.

2.3 Variation in moduli

As in the bosonic case [26] one can define an action of the Neveu-Schwarz algebra $\text{Vect } S^{1|1}$ on the moduli space $\hat{\mathcal{P}}$ of SRS equipped with a marked point and a choice of local superconformal coordinate centered at the given point. Recall first that a superconformal vector field v^z on $\hat{\Sigma}$ is a (-1) -superdifferential; to it there corresponds a vector field on $\hat{\Sigma}$ preserving the superconformal structure:

$$V_v = v^z \partial_z + \frac{1}{2} (D_\theta v^z) D_\theta \quad . \quad (2.12)$$

The vector fields of the form (2.12) preserve the superconformal structure; they close under Lie brackets, and so one can define the bracket $[v, v']$.

Given a point $\tilde{\Sigma} = (\hat{\Sigma}, \hat{P}; z, \theta) \in \hat{\mathcal{P}}$ and a superconformal vector field v on a standard annulus $\{0 < |t| < 1\} \subset \mathbb{C}^{1|1}$ we get a tangent vector to $\hat{\mathcal{P}}$ at $\tilde{\Sigma}$ as follows: cut the disk $\hat{D}$ out of $\hat{\Sigma}$ and reglue it in place after performing the infinitesimal superconformal diffeomorphism generated by $V_v + \bar{V}_{\bar{v}}$. This gives $\tilde{\Sigma}' = (\hat{\Sigma}', \hat{P}'; z', \theta')$. Clearly $\tilde{\Sigma}'$ is close to $\tilde{\Sigma}$; we denote the real tangent vector to $\hat{\mathcal{P}}$ at $\tilde{\Sigma}$ obtained in this way by $(\delta_v + \bar{\delta}_{\bar{v}})|_{\tilde{\Sigma}}$. We will see that δ_v obtained in this way is in fact a holomorphic vector field. More generally we can let v be a function from $\hat{\mathcal{P}}$ to the fixed Neveu-Schwarz algebra; then δ_v is a complex vector field on $\hat{\mathcal{P}}$.

It now natural to ask whether the linear map

$$\delta : \hat{\mathcal{P}} \times \text{Vect}(S^{1|1}) \rightarrow T\hat{\mathcal{P}}$$

preserves the Lie algebra structures of both sides. The answer is clearly no, since the bracket on $\text{Vect}(S^{1|1})$ acts pointwise, instead of differentiating along $\hat{\mathcal{P}}$. Instead we will argue that

$$[\delta_v, \delta_{v'}] = \delta_{(\delta_v v' - \delta_{v'} v + [v, v'])} \quad (2.13)$$

On the left there appears the Lie bracket of two vector fields on $\hat{\mathcal{P}}$. On the right, δ_v differentiates the function v' along $\hat{\mathcal{P}}$; the bracket is the usual Neveu-Schwarz algebra on $\text{Vect}(S^{1|1})$. Thus for example if v_A are tangents corresponding to a good coordinate system m^A on $\mathcal{P}$, i.e. $\delta_{v_A} = \frac{\partial}{\partial m^A}$, then we must have

$$\frac{\partial v_A}{\partial m^B} - (-)^{AB} \frac{\partial v_B}{\partial m^A} = [v_A, v_B] \quad (2.14)$$

Before proving (2.13) we remark that the situation is analogous to that of a coset space of two Lie groups. In this case $\mathcal{P} = G/H$ carries a trivial bundle with fiber the Lie algebra $\mathfrak{g}$ and a map δ from $\mathfrak{g}$ to $T_g \mathcal{P} = \mathfrak{g}/\text{Ad}_g \mathfrak{h}$. A formula similar to (2.13) relates the Lie bracket in the fixed algebra $\mathfrak{g}$ to the one among tangents to $\mathcal{P}$.

To prove (2.13) we note that the differentials on $\tilde{\Sigma}'$ can be easily expressed in terms of those on $\tilde{\Sigma}$, just as in the bosonic case[16]. Namely, if s is a meromorphic superdifferential on $\tilde{\Sigma}$ we get s' on $\tilde{\Sigma}'$ by restricting s to $\hat{D}_1$ and setting

$$s' = (1 - \mathcal{L}_v)s \quad (2.15)$$

where $1 - \mathcal{L}_v$ is the infinitesimal superconformal diffeomorphism on $\hat{D}_1$ generated by the vector field V_v in (2.12). ($\mathcal{L}_v$ annihilates s .) Suppose momentarily that v, v' are constant over $\hat{\mathcal{P}}$, so that the derivative terms in (2.13) vanish. Then performing two successive deformations of $\tilde{\Sigma}$ of the form (2.15) we find that $[\delta_v, \delta_{v'}] = \delta_{[v, v']}$. Generalizing to arbitrary v, v' we easily obtain (2.13).

One can also define the trivial Hilbert bundle $\hat{\mathcal{P}} \times \mathcal{H}$ over $\hat{\mathcal{P}}$, where $\mathcal{H}$ is the Fock space of the conformal field theory we are studying. The ray $|\tilde{\Sigma}\rangle$ (eqn. (2.10)) can be regarded as a projective section of this bundle. We will now argue that this section obeys a system of differential equations, exactly like the "equation of motion" of the bosonic string:

$$\nabla_v |\tilde{\Sigma}\rangle \equiv (\delta_v - T_v) |\tilde{\Sigma}\rangle = a |\tilde{\Sigma}\rangle \quad , \quad (2.16)$$

where a is a scalar function which will later cancel when we include the ghosts and $T_v \equiv \oint_C v^z T_{z\theta} ds$.

To prove (2.16) we again use (2.15). A similar statement holds for superharmonic functions H : for a holomorphic deformation δ_v of $\tilde{\Sigma}$ we have

$$H' = (1 + V_v) H \quad . \quad (2.17)$$

This follows since the equation $D\bar{D}H = 0$ is superconformally invariant. Thus the state $|\tilde{\Sigma}'\rangle$ is characterized (up to normalization and phase) by the fact that the modified charges annihilate it:

$$Q((1 + V_v)H) |\tilde{\Sigma}'\rangle = 0 \quad .$$

Hence if we can show that

$$Q(V_v H) = [T_v, Q(H)] \quad (2.18)$$

then (2.16) will follow. For the matter system (2.18) can be shown directly using the canonical commutation relations, and similarly for the ghost system to be studied later.

The connection ∇ in (2.16) is projectively flat [39], i.e. the covariant derivatives obey the same Lie algebra under commutation as the vector fields on $\hat{\mathcal{P}}$. This follows since $[T_v, T_{v'}] = T_{[v, v']} + \text{c-number}$ implies that

$$[\nabla_v, \nabla_{v'}] = \nabla_{(\delta_{v,v'} - \delta_{v',v} + [v, v'])} + \text{c-number} \quad .$$

Thus eqn. (2.16) is consistent and, as in the bosonic case, we can use it to fix locally the normalization and phase of $|\tilde{\Sigma}\rangle$ once ghosts have been added to cancel the conformal anomaly.

2.4 Holomorphic factorization

We now examine the holomorphic properties of the state $|\tilde{\Sigma}\rangle$. Before doing so, however, we must first discuss the super period matrix [35].

By the super Riemann-Roch theorem there are g even holomorphic $\frac{1}{2}$ -superdifferentials. We can normalize them by demanding $\oint_{a_i} \hat{\omega}^j = \delta_i^j$. The super period matrix is then

$$\hat{\tau}^{ij} = \oint_{b^i} \hat{\omega}^j \quad (2.19)$$

Eqn. (2.19) manifestly depends only on a marking and on the superconformal structure, so there is no need to check that $\hat{\tau}$ is supersymmetric. Clearly at the split locus $\hat{\tau}^{ij}$ coincides with the ordinary τ^{ij} . We wish to compute the variation in $\hat{\tau}$ induced by a change in the superconformal structure of $\hat{\Sigma}$. To do this we follow the derivation given in [40] for ordinary Riemann surfaces. The key observation is that while the variation of $\hat{\tau}$ with respect to explicit holomorphic coordinates is quite messy, still the variation with respect to the natural vector fields δ_v defined earlier is extremely simple. These vector fields are not in general commuting, but one does have $[\delta_v, \bar{\delta}_v] = 0$, and this is enough.

We can see how the $\hat{\omega}^i$ vary by noting that the correlation $\langle B(P_1) \dots B(P_g) C(Q_1) \rangle$, for a BC system of weight $\frac{1}{2}$, varies via the insertion of the super stress tensor. Thus if we let $T_M = \int M_{\bar{\theta}}{}^z T_{\theta z} d\bar{z} d\bar{\theta}$, where M is a super Beltrami differential, then

$$\delta_M \langle B(P_1) \dots \rangle = \langle B(P_1) \dots T_M \rangle$$

where δ_M is the derivative in moduli space along the direction specified by M . Using the operator product expansion [29]

$$T(z_1)B(z_2) \sim \frac{1}{2} \frac{\theta_{12}}{z_{12}^2} B(z_2) + \frac{1}{2} \frac{1}{z_{12}} (D_{\theta_2} B(z_2)) + \frac{\theta_{12}}{z_{12}} \partial_{z_2} B(z_2) + \dots \quad (2.20)$$

we see that

$$\delta_M \hat{\omega}_{\theta_2}^i(z_2) = \int M_{\bar{\theta}_1}{}^{z_1} \eta_{z_1 \theta_1 \theta_2}^i(z_1, z_2) d\bar{z}_1 d\bar{\theta}_1$$

where η^i are bidifferentials with pole structure as in (2.20).

We also know that $\delta \hat{\omega}^i$ must have vanishing a -periods in order to preserve the normalization condition. As in the previous section this requirement together with (2.20) fixes the η^i completely. To compute $\delta_M \hat{\tau}^{ij}$ we now need the b -periods of η^i .

Cut the SRS into a polygon with boundary. The details of how this is done are immaterial since everything is holomorphic near the cut. Let $\hat{\omega}^j = Df^j$ for functions f^j well-defined on the cut surface. Then we can write

$$\oint_{b_j} d\mathbb{z}_1 \eta^i(\mathbb{z}_1, \mathbb{z}_2) = \oint_{\partial\hat{\Sigma}} f^j(\mathbb{z}_1) \eta^i(\mathbb{z}_1, \mathbb{z}_2) = \oint_{C'} f \eta \quad ,$$

where $\partial\hat{\Sigma}$ is the boundary of the cut surface and C' is a contour encircling $\mathbb{z}_2$. We now use the formulas (cf [29])

$$\begin{aligned} \frac{1}{2\pi i} \oint d\mathbb{z}_1 \theta_{12} z_{12}^{-2} f(\mathbb{z}_1) &= \partial_{\mathbb{z}_2} f(\mathbb{z}_2) \\ \frac{1}{2\pi i} \oint d\mathbb{z}_1 z_{12}^{-1} f(\mathbb{z}_1) &= D_{\theta_2} f(\mathbb{z}_2) \\ \frac{1}{2\pi i} \oint d\mathbb{z}_1 \theta_{12} z_{12}^{-1} f(\mathbb{z}_1) &= f(\mathbb{z}_2) \end{aligned}$$

to compute

$$\begin{aligned} \delta_M \hat{\tau}^{ij} &= \int d\mathbb{z}_2 d\bar{\mathbb{z}}_2 M_{\theta_2}^{\mathbb{z}_2}(\mathbb{z}_2) \oint_{b_j} d\mathbb{z}_1 \eta^i(\mathbb{z}_1, \mathbb{z}_2) \\ &= -i\pi \int M_{\theta}^{\mathbb{z}}(D_{\theta} \hat{\omega}_{\theta}^j \cdot \hat{\omega}_{\theta}^i + \hat{\omega}_{\theta}^j D_{\theta} \hat{\omega}_{\theta}^i) d\mathbb{z} d\bar{\mathbb{z}} \end{aligned}$$

Finally suppose that the super-Beltrami differential is of the form $M_{\theta}^{\mathbb{z}} = D_{\theta} v^{\mathbb{z}}$ where $v^{\mathbb{z}}$ is a (-1) -superdifferential which is holomorphic on the punctured disk $\hat{D}_1$. Parts integration then gives that

$$\delta_v \hat{\tau}^{ij} = -i\pi \oint_C v(D\hat{\omega}^j \cdot \hat{\omega}^i + \hat{\omega}^j D\hat{\omega}^i) \quad , \quad (2.21)$$

the analog of the Rauch formula [41].¹⁰ Note that (2.21) is manifestly superconformally invariant; it depends on the local coordinate only through the association of a (-1) -differential to each element of $\text{Vect } S^{1|1}$. This follows because while $D\hat{\omega}^i$ by itself is not a well-defined superdifferential, still the symmetrized form in (2.21) is, since ω_{θ}^i is an odd function. We also note that $\delta_v \hat{\tau}^{ij} = 0$, so δ_v really are the holomorphic vector fields on $\hat{\mathcal{P}}$.

Next consider the scalar Green function for the X system. To define $\mathcal{G}(\hat{P}, \hat{P}')$, and hence to define all the functional integrals written so far, we must endow $\hat{\Sigma}$ with a "super-Riemann structure" analogous to a metric [30][42]. For any choice, however, one finds

¹⁰ This is the covariant version of the formula given by Sonoda [35].

that $D_P \bar{D}_{P'} \mathcal{G}(\hat{P}, \hat{P}') = \langle DX \bar{D} X \rangle$ depends only on the superconformal structure, and that in fact

$$D_P \bar{D}_{P'} \mathcal{G}(\hat{P}, \hat{P}') = \pi \left[\delta(\mathbb{z}_P - \mathbb{z}_{P'}) - \hat{\omega}_i(\hat{P})(\hat{\tau}_2^{-1})^{ij} \hat{\omega}_j(\hat{P}') \right] . \quad (2.22)$$

Here δ is the reproducing kernel for $(\frac{1}{2}, 0)$ -superdifferentials, while $\hat{\tau}_2^{ij} = \frac{1}{2}(\hat{\tau} - \hat{\tau})^{ij} = \frac{1}{2} \int_{\hat{\Sigma}} \hat{\omega}^i \hat{\omega}^j$ by the super Riemann bilinear relation [24]. The latter expression shows that $\hat{\tau}_2$ is the correct normalization matrix for the zero modes of $DD^\dagger$, and so (2.22) follows since the nonzero eigenmodes of $DD^\dagger$ are all proportional to D acting on those of $D^\dagger D$.

Thus one finds that

$$\begin{aligned} \delta_M \bar{\delta}_{\bar{M}'} \log \det \hat{\tau}_2 &= -\frac{1}{4} \text{tr}[(\delta_M \hat{\tau}) \hat{\tau}_2^{-1} (\bar{\delta}_{\bar{M}'} \hat{\tau}) \hat{\tau}_2^{-1}] \\ \delta \bar{\delta} \log \det \hat{\tau}_2 &= -\frac{\pi^2}{4} ((D\hat{\omega}_1^i) \hat{\omega}_1^j + \hat{\omega}_1^i D\hat{\omega}_1^j) \hat{\tau}_{2jk}^{-1} ((\bar{D}\hat{\omega}_2^{\bar{k}}) \hat{\omega}_2^{\bar{l}} + \hat{\omega}_2^{\bar{k}} D\hat{\omega}_2^{\bar{l}}) \hat{\tau}_{2\bar{l}\bar{i}}^{-1} \\ &= -\frac{\pi^2}{2} [(\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \bar{D}\hat{\omega}_2)(\hat{\omega}_2 \cdot \hat{\tau}_2^{-1} \cdot D\hat{\omega}_1) \\ &\quad + (D\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \bar{D}\hat{\omega}_2)(\hat{\omega}_2 \cdot \hat{\tau}_2^{-1} \cdot \hat{\omega}_1)] . \end{aligned}$$

Recall that $D = d\mathbb{z} \otimes D_\theta$ is overall an even operator; the super Beltrami differential M_θ^* is odd.

We can now compare this formula to the variation of the matter sector vacuum. As in the bosonic string calculation [16] one simply finds the nonfactorizing terms of $\langle T(\mathbb{z}_1) \bar{T}(\mathbb{z}_2) \rangle$. For the matter sector $T = \frac{1}{2} : DX D^2 X :$, so

$$\begin{aligned} \langle T_1 \bar{T}_2 \rangle &= \frac{1}{4} \langle : DX_1 D^2 X_1 : : \bar{D} X_2 \bar{D}^2 X_2 : \rangle \\ &= \frac{\pi^2}{4} [(\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \hat{\omega})(D\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \bar{D}\hat{\omega}_2) + (\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \bar{D}\hat{\omega}_2)(D\hat{\omega}_1 \cdot \hat{\tau}_2^{-1} \cdot \hat{\omega}_2)] . \end{aligned}$$

We finally conclude that in the matter sector with d superfields X^μ one has

$$\delta \bar{\delta} \log \langle \dots \rangle = -\frac{d}{2} \delta \bar{\delta} \log \det \hat{\tau}_2 , \quad (2.23)$$

and so the matter amplitudes can be written as a holomorphically factorized piece times $(\det \hat{\tau}_2)^{-d/2}$.

One of the pleasant features of this proof is that the norm of the constant function has never explicitly entered. Since this factor is nilpotent and hence noninvertible for some perfectly good super-Riemann structures, its appearance in denominators forces some special considerations in the superfield determinant approach.

3. First order bosonic systems

3.1 The sewing state

In [16] there is a thorough treatment of the $b - c$ system [3] for spin two in the operator formalism. These fermionic fields provide the reparametrization ghosts in the first quantized formulation of string theory. In this section we want to extend the work in [16] to the superconformal ghost system of fermionic strings. The basic idea is to associate to every surface Σ with n punctures $P_1, \dots, P_n \in \Sigma$ and local coordinates z_i $i = 1, \dots, n$; $z_i(P_i) = 0$, a state in the Fock space of the $\beta - \gamma$ superconformal ghost system (we will follow the notation of [3]). There are a number of subtleties one encounters when extending the treatment for fermionic $b - c$ fields to β and γ . In particular, the only way of absorbing the vacuum charge for this system is through the introduction of operators of the form $\delta(\beta(z)), \delta(\gamma(z))$. These are rather unnatural in the standard oscillator approach to their quantization. One of the main aims of this section is to introduce a "gaussian" representation for the Fock space of the $\beta - \gamma$ system. In this formalism, δ -function operators appear naturally, and we can explicitly show how these operators interpolate between the different Bose seas. This is useful because for first order bosonic fields, the Fock spaces for different Bose sea levels provide inequivalent representations for the oscillator algebra. This also clarifies the role of the picture changing operator [3], as will be shown in later sections.

In this section we will consider first-order systems on a spin Riemann surface. Thus our constructions give the ghost states only at the split locus of $\widehat{\mathcal{M}}$, a restriction which we will rectify in sec. 4.

The simplest possible case is a sphere with a single puncture. The state associated in this case is the $SL_2(\mathbb{C})$ invariant vacuum. Let us briefly recall the operator treatment of an arbitrary $b - c$ fermionic system in this situation. The field $b(z)$ and $c(z)$ behave like j and $1 - j$ holomorphic differentials on the surface, $j \in \mathbb{Z}$ or $\mathbb{Z} + 1/2$. Since we are considering now a single puncture, the state obtained belongs to the Neveu-Schwarz Fock space $\mathcal{H}$ of the system i.e., both b and c are periodic around the puncture P . The oscillator expansions of b and c are:

$$(dz)^j b(z) = \sum_{n \in \mathbb{Z}} b_n z^{n-j} dz^j \quad (3.1a)$$

$$(dz)^{1-j}c(z) = - \sum_{n \in \mathbb{Z}} c_n z^{n-1+j} dz^{1-j} \quad (3.1b)$$

Using the method of conserved charges [14], we have to construct the space of meromorphic j and $1-j$ differentials with poles only at P . For $1-j$ differentials we get the basis

$$w_n(z) = z^n, \quad n \leq 2j-2 \quad (3.2)$$

and for j -differentials

$$\tilde{w}_n(z) = z^n, \quad n \leq -2j. \quad (3.3)$$

The associated conserved charges are:

$$\begin{aligned} Q_n &= \oint_0 b(z) z^n = b_n, \quad n > -j \\ \tilde{Q}_n &= \oint_0 c(z) z^n = c_n, \quad n > j-1 \end{aligned} \quad (3.4)$$

and the state is defined by:

$$Q_n |\phi\rangle = \tilde{Q}_n |\phi\rangle = 0. \quad (3.5)$$

This is indeed the standard $SL_2(\mathbb{C})$ invariant state [3]. Thus $L_0 |\phi\rangle = 0$. The charge of $|\phi\rangle$ is

$$Q |\phi\rangle = -(j-1/2) |\phi\rangle \quad (3.6)$$

$$Q = \oint c(z) b(z).$$

Similarly, the state associated to a surface with g handles and n punctures carries a charge $(2j-1)(g-1) + (j-1/2)n$. This assignment behaves additively under topological sums of orientable surfaces, and it can be determined mathematically by the Atiyah-Patodi-Singer index theorem for manifolds with boundaries. If we consider two surfaces Σ_1, Σ_2 with g_i handles and n_i punctures $i = 1, 2$, we can construct a surface with $g_1 + g_2$ handles and $n_1 + n_2 - 2$ punctures by choosing a puncture in each surface, cutting small discs around them, and gluing the boundaries. It is now readily verified that the charge assignment is additive under this operation. The state $|\phi\rangle$ is not a highest weight state for the oscillator algebra. Instead, when j is an integer we can introduce two states $|\pm\rangle$:

$$c_n |+\rangle = b_n |-\rangle = 0 \quad n \geq 0 \quad (3.7)$$

$$c_0 |-\rangle = |+\rangle$$

$$b_0|+\rangle = |-\rangle .$$

Then

$$|+\rangle = c_0 c_1 \dots c_{j-1} |\phi\rangle, \quad |\phi\rangle = b_{-j+1} \dots b_{-1} b_0 |+\rangle . \quad (3.8)$$

Since $[L_0, c_n] = -n c_n$, $[L_0, b_n] = -n b_n$, and $L_0 |\phi\rangle = 0$, we obtain:

$$L_0 |\pm\rangle = -j(j-1)/2 |\pm\rangle . \quad (3.9)$$

Let us now turn to the $\beta - \gamma$ system of spin 3/2. As we will see below, the extension of the oscillator formulation presented above has a number of serious drawbacks. Once these are explained, we will present a different representation of the $\beta - \gamma$ algebra which can easily be applied to the construction of states on surfaces with an arbitrary number of handles and punctures. As pointed out in [3] in the quantization of a first order bosonic system, we have to choose a particular representation of the oscillator algebra. This is equivalent to the choice of a "Bose sea level," and unlike the case for fermionic systems, different Bose sea levels give inequivalent representations of the oscillator algebra. Part of the motivation for the introduction of a "big" oscillator algebra in the $\beta - \gamma$ system in [3] (which includes the picture changing operators) centers upon the construction of operators which interpolate between the different representations. One of the basic problems in the analysis of the consistency of superstring perturbation theory is the picture independence of the scattering amplitudes for physical states.

On the sphere with one puncture P , the conserved charges for γ (resp. β) are obtained by constructing the holomorphic $-1/2$ (resp. $3/2$) differentials holomorphic off P . The $-1/2$ differentials are $\{z, 1, z^{-1}, z^{-2}, \dots\}$ times $dz^{-1/2}$, and the $3/2$ -differentials are $\{z^{-2}, z^{-3}, z^{-4}, \dots\}$ times $dz^{3/2}$. Since

$$\begin{aligned} \beta(z) &= \sum_{n \in \mathbb{Z} + 1/2} \beta_n z^{n-3/2} \\ \gamma(z) &= - \sum_{n \in \mathbb{Z} + 1/2} \gamma_n z^{n+1/2} \end{aligned} \quad (3.10)$$

the conserved charge conditions give

$$\begin{aligned} \beta_n |\phi_0^1\rangle &= 0 \quad n > -3/2 \\ \gamma_n |\phi_0^1\rangle &= 0 \quad n > 1/2 . \end{aligned} \quad (3.11)$$

This is the standard $SL_2(\mathbb{C})$ invariant vacuum. For notational convenience, the state $|\phi_g^n\rangle$ represents the state obtained for a surface of genus g and n punctures. For the moment we concentrate on Neveu-Schwarz (NS) punctures, so that the field is periodic. We will deal with Ramond punctures, and with the manifestly supersymmetric formalism in more detail in sect. 4. Notice that the state $|\phi_0^1\rangle$ is not a highest weight state of the oscillator algebra; to get such a state we need to construct a state $|gh\rangle$:

$$\beta_n|gh\rangle = \gamma_n|gh\rangle = 0 \quad n > 0 \quad . \quad (3.12)$$

For fermionic systems this is easy to do using local operators (3.7-8). Now however, the canonical commutation relations are

$$[\gamma_n, \beta_m] = \delta_{n+m,0} \quad (3.13)$$

and we can formally write $|gh\rangle$ as [6]

$$|gh\rangle = \delta(\gamma_{1/2})|\phi_0^1\rangle \quad . \quad (3.14)$$

It is easy to check that (3.12) is satisfied. However, $\delta(\gamma_{1/2})$ is a rather awkward operator in the $\beta - \gamma$ algebra. The ghost charge of $|\phi_0^1\rangle$ is taken to be -1. Using the Riemann-Roch theorem, the charge of the state $|\phi_g^n\rangle$ is $-(2g - 2 + n)$ (we will show this directly presently). Since we are taking $\gamma(z)$ to have charge +1, and $\beta(z)$ charge -1, it follows that

$$[Q_{gh}, \delta(\gamma_n)] = -\delta(\gamma_n); \quad [Q_{gh}, \delta(\beta_n)] = \delta(\beta_n) \quad (3.15)$$

Thus $|gh\rangle$ has zero ghost charge. Similarly, using $[L_0, \gamma_n] = -n\gamma_n$ we have $[L_0, \delta(\gamma_n)] = n\delta(\gamma_n)$. Hence

$$Q_{gh}|gh\rangle = 0 \quad L_0|gh\rangle = \frac{1}{2}|gh\rangle \quad . \quad (3.16)$$

Next, let us consider the state we would obtain for a surface with g handles and 1 puncture. The construction completely parallels the arguments in [16] and we will be brief. For $g > 1$ there are always $2g - 2$ holomorphic $3/2$ differentials. For generic Σ they can be written in the form:

$$S_n = (z^n + \sum_{m \geq 2g-2} B_{nm}^{(3/2)} z^m) dz^{3/2} \quad n = 0, 1, \dots, 2g-3 \quad , \quad (3.17)$$

and we can also construct $3/2$ differentials with arbitrary order poles at the point $P \in \Sigma$:

$$S_{-n} = (z^{-n} + \sum_{m \geq 2g-2} B_{-n,m}^{(3/2)} z^m) dz^{3/2}, \quad n = 1, 2, \dots \quad (3.18)$$

The $-1/2$ differentials meromorphic at P must have poles of order higher than $2g-2$ for a generic $P \in \Sigma$ as a consequence of the Weierstrass gap theorem. They can be written as:

$$\nu_{-n} = (z^{-n} + \sum_{m \geq 2-2g} B_{-n,m}^{(-1/2)} z^m) dz^{-1/2}; \quad n > 2g-2 \quad (3.19)$$

The conserved charges are the following:

$$Q(S_n) = \oint \gamma(z) S_n(z) = \gamma_{-n-3/2} + \sum_{m \geq 2g-2} B_{n,m}^{(3/2)} \gamma_{-m-3/2} \quad (3.20a)$$

$$Q(S_{-n}) = \oint \gamma(z) S_{-n}(z) = \gamma_{n-3/2} + \sum_{m \geq 2g-2} B_{-n,m}^{(3/2)} \gamma_{-m-3/2} \quad (3.20b)$$

$$\tilde{Q}(\nu_{-n}) = \oint \beta(z) \nu_{-n}(z) = \beta_{n+1/2} + \sum_{m > 2-2g} B_{-n,m}^{(-1/2)} \beta_{-m+1/2} \quad (3.21)$$

Notice that $Q(S_n)$ and $Q(S_{-1})$ contain only creation operators. Hence to implement these conservation laws, the state must contain δ -function operators. The charges (3.20a, b), (3.21) uniquely define the state $|\phi_g^1\rangle$:

$$\begin{aligned} |\phi_g^1\rangle &= \left(\prod_{n=0}^{2g-3} \delta(\gamma(S_n)) \right) \delta(\gamma(S_{-1})) \\ &\times \exp\left(- \sum_{n \geq 3} \sum_{m \geq 2g-2} B_{-n,m}^{(3/2)} \gamma_{-m-3/2} \beta_{-m+3/2}\right) |gh\rangle \end{aligned} \quad (3.22)$$

where

$$\gamma(w) = \oint \gamma(z) w(z) \quad (3.23)$$

The generalization to n punctures is straightforward. We have to construct the $3/2$ differentials holomorphic off $P_1, \dots, P_n \in \Sigma$. Again there are $(2g-2)$ holomorphic $3/2$ differentials, and there also exist $3/2$ differentials with arbitrary order poles, and in particular, with first order poles. Let $S_{-1}^{(i)}$ be the $3/2$ differential with a first order pole at puncture i and holomorphic elsewhere. At each puncture we have a set of oscillators $\gamma_n^{(i)}, \beta_n^{(i)}$, and we will get a state in the tensor product $\mathcal{H}^{\otimes n}$, with one copy per puncture. If now

$$\gamma(w) = \sum_{i=1}^n \oint_{P_i} \gamma^{(i)}(z_i) w(z_i) \quad (3.24)$$

then the state has the form:

$$|\phi_g^n\rangle = \prod_{n=0}^{2g-3} \delta(\gamma(S_n)) \prod_{i=1}^n \delta(\gamma(S_{-1}^{(i)})) g(B) |gh\rangle_1 \otimes \cdots \otimes |gh\rangle_n \quad (3.25a)$$

$$g(B) = \exp - \sum_{ij} \sum_{n,m} B_{-n,m}^{(3/2)ij} \gamma_{-m-3/2}^{(i)} \beta_{m+3/2}^{(j)} \quad (3.25b)$$

Thus $g(B)$ implements a Bogoliubov transformation on the tensor product of highest weight states $|gh\rangle_i$ for the oscillator algebra on the sphere. Since $\delta(\gamma)$ has ghost number -1, we immediately obtain that

$$Q_{gh} |\phi_g^n\rangle = -(2g - 2 + n) |\phi_g^n\rangle \quad (3.26)$$

The normalization of the state $|\phi_0^1\rangle$ is such that

$$\langle \phi_0^1 | \delta(\gamma_{-1/2}) \delta(\gamma_{1/2}) | \phi_0^1 \rangle = 1 \quad (3.27)$$

These manipulations are clearly formal and we would like to introduce a different formulation of the $\beta - \gamma$ system where the manipulations with δ -functions and inner products are well defined. Before we do this, we work out the sewing state. This is the state associated to the sphere with two punctures. By $SL_2(\mathbb{C})$ transformations, we can move the punctures to $z = 0$ and ∞ . Working for simplicity with NS punctures, if z is the local coordinate around 0 and w the local coordinate around ∞ , then $zw = 1$. The $3/2$ differentials holomorphic off $\{0, \infty\}$ are $z^n dz^{3/2}$ $n \in \mathbb{Z}$, and the $(-1/2)$ differentials are similarly $z^n (dz)^{-1/2}$. They yield the equations:

$$\left. \begin{aligned} (\beta_n^{(1)} - \beta_{-n}^{(2)}) |S_{12}\rangle &= 0 \\ (\gamma_n^{(1)} + \gamma_{-n}^{(2)}) |S_{12}\rangle &= 0 \end{aligned} \right\} \quad n \in \mathbb{Z} + 1/2 \quad (3.28)$$

with solution:

$$|S_{12}\rangle = \prod_{m=1}^{\infty} \exp (-\gamma_{-m+1/2}^{(1)} \beta_{-m+1/2}^{(2)} - \gamma_{-m+1/2}^{(2)} \beta_{-m+1/2}^{(1)}) |gh\rangle_1 \otimes |gh\rangle_2 \quad (3.29)$$

One has to be careful about the relative phases of $\beta^{(2)}$ and $\gamma^{(2)}$ in the derivation of (3.28).

Notice that the sewing state $|S_{12}\rangle$ has zero ghost charge. This operator is useful, because we can construct states on arbitrary surfaces, starting with states on the sphere with three punctures and using the sewing state to generate states for more complicated

surfaces. The ray obtained this way is uniquely defined by the surface the punctures and the local coordinates z , and not in how the particular surface was built up by sewing. This is the content of the charge transport argument (CTA) explained for the bosonic string and arbitrary fermionic $b - c$ system in [16]. The same argument applies in this case without any change. If Σ_i is a surface with g_i handles and n_i punctures, $i = 1, 2$, we can construct a surface $\Sigma_1 \cup \Sigma_2$ (using the notation of [18]) by identifying annuli around the punctures $P_i \in \Sigma_1, P_j \in \Sigma_2$ by the condition $z_i z_j = 1$. $\Sigma_3 = \Sigma_1 \cup \Sigma_2$ has $g_1 + g_2$ handles and $n_1 + n_2 - 2$ punctures. The state for Σ_3 is defined as

$$\langle S_{ij} | \Sigma_1 \rangle \otimes | \Sigma_2 \rangle = | \Sigma_3 \rangle . \quad (3.30)$$

Note first that the ghost charge behaves correctly since $Q_{gh} | S_{ij} \rangle = 0$. The only potential subtlety in the argument is whether for the $\beta - \gamma$ system the inner product is singular. Since $\langle gh | gh \rangle = 1$, and the states are constructed via Bogoliubov transformations of these states, generically, (3.30) is non-singular. The same argument works for the sewing of two punctures on the same surface. For $j \in \mathbb{Z} + 1/2$, the sewing only makes sense for pairs of either NS or R punctures. In sec. 4, we will explicitly construct the states associated with the three-punctured sphere in the two possible cases $|NNN\rangle$ or $|RRN\rangle$. It is also useful to remark that on the sphere with one puncture the conserved charges naturally lead to a vacuum state with zero Bose level. This is natural because one should expect to obtain an $SL_2(\mathbb{C})$ invariant state in this case. Ground states with different Bose sea level give a non-zero eigenvalue of L_0 . Notice also that the physical states in the NS sector of the superstring are constructed out of the state $|gh\rangle$ (for details and references see [2]).

3.2 The gaussian representation

We now study a “gaussian” representation of the $\beta - \gamma$ operator algebra which permits a justification of the δ -function manipulations. In the $\beta - \gamma$ system, one has

$$\gamma_n^\dagger = \gamma_{-n}, \quad \beta_n^\dagger = -\beta_{-n} . \quad (3.31)$$

We introduce the operators:

$$\begin{aligned} x_n &= \frac{1}{\sqrt{2}}(\gamma_n + i\beta_{-n}) \\ p_n &= \frac{1}{\sqrt{2}}(-\gamma_n + i\beta_{-n}) \end{aligned} \quad (3.32)$$

satisfying

$$\begin{aligned} [x_n, p_m] &= i\delta_{n,m} \\ x_n^\dagger &= x_{-n}, \quad p_n^\dagger = p_{-n} \end{aligned} \quad (3.33)$$

This Heisenberg algebra can be realized on a bosonic Fock space $\mathcal{F}^B$ of functionals $\phi\{x_n\}$, $n \in \mathbb{Z}$, where $p_n = -i\partial/\partial x_n$, and γ_n, β_n act according to

$$\begin{aligned} \gamma_n &= \frac{1}{\sqrt{2}}(x_n + i\frac{\partial}{\partial x_n}) \\ \beta_n &= \frac{-i}{\sqrt{2}}(x_{-n} - i\frac{\partial}{\partial x_{-n}}) \end{aligned} \quad (3.34)$$

The scalar product in $\mathcal{F}^B$ is defined by:

$$\langle \phi_1 | \phi_2 \rangle = \int \prod_n dx_n \phi_1^*(x) \phi_2(x) \quad ; \quad (3.35)$$

we will give precise integration rules momentarily. In this representation, all the states of the $\beta - \gamma$ system are given explicitly as gaussian functions of $\{x_n\}$. As a first example, we construct the Bose sea level introduced in [3], and defined by:

$$\begin{aligned} \beta_n |q\rangle &= 0 \quad n > -q - j \\ \gamma_n |q\rangle &= 0 \quad n \geq q + j \end{aligned} \quad (3.36)$$

where $q \in \mathbb{Z}$ in the NS sector, and $q \in \mathbb{Z} + 1/2$ in the R sector. These conditions translate into linear first order differential equations in the gaussian representation, with the result:

$$\Phi_q^{(j)}(x) = C \exp \frac{i}{2} \left(\sum_{n \geq q+j} x_n^2 - \sum_{n < q+j} x_n^2 \right) \quad (3.37)$$

Note that the Bose sea level determines the position of the plus and minus signs in the exponent of (3.37). If we act on $\Phi_q^{(j)}$ with polynomials of the γ_n, β_n oscillators, we will not move the Bose sea. Different Bose sea levels give inequivalent representations of the oscillator algebra, unlike what happens for fermionic systems, where the Fermi sea can be shifted by the action of fermionic oscillators. In the bosonic case, the change of level is achieved by the action of an infinite number of operators, and in particular with δ -function operators. Using the representation (3.34) and the integral representation of the δ -function, an easy computation shows that

$$\begin{aligned} \delta(\beta_n) e^{\frac{i}{2} x_n^2} &= \left(\frac{-1}{2\pi i} \right)^{1/2} e^{-\frac{i}{2} x_n^2} \\ \delta(\gamma_n) e^{-\frac{i}{2} x_n^2} &= \left(\frac{1}{2\pi i} \right)^{1/2} e^{\frac{i}{2} x_n^2} \end{aligned} \quad (3.38)$$

and therefore:

$$\begin{aligned}\delta(\beta_{-q-j})\Phi_q^{(j)}(x) &= \left(\frac{1}{2\pi i}\right)^{1/2}\Phi_{q+1}^{(j)}(x) \\ \delta(\gamma_{q+j-1})\Phi_q^{(j)}(x) &= \left(\frac{1}{2\pi i}\right)^{1/2}\Phi_{q-1}^{(j)}(x) \quad .\end{aligned}\tag{3.39}$$

Complex conjugation also modifies the Bose sea level:

$$(\Phi_q^{(j)})^* = \Phi_{-q-Q}^{(j)} \quad ; \quad Q = 2j - 1 \quad .\tag{3.40}$$

We can now briefly comment on the remarks made at the beginning of this section. In the representation (3.37) of $\Phi_q^{(j)}$, the Bose sea level is given by the division in the exponents between x_n^2 and $-x_n^2$. Changing $x_n^2 \rightarrow -x_n^2$ is equivalent to shifting the Bose sea by one unit. From (3.38-39), we immediately see that the operation of changing the sign of x_n^2 is accomplished by the δ -function operators, with a rather simple representation in the gaussian formulation.

Thus when we evaluate correlation functions, we have to make insertions of δ -function operators to absorb the vacuum charge Q . If $|\phi_{q=0}^{(j)}\rangle$ is the SL_2 invariant vacuum, then in the gaussian representation it is easy to show, using (3.38) that

$$\langle \phi_0^{(j)} | \delta(\gamma_{j-1}) \dots \delta(\gamma_{j-Q}) | \phi_0^{(j)} \rangle = 1\tag{3.41}$$

where the normalization constant has been chosen so that the right hand side is equal to 1. In particular, if $j = 3/2$:

$$\langle \phi_0^{(3/2)} | \delta(\gamma_{1/2}) \delta(\gamma_{-1/2}) | \phi_0^{(3/2)} \rangle = 1 \quad ,\tag{3.42}$$

thus verifying $\langle gh | gh \rangle = 1$. In this language, the sewing operator $|S_{12}\rangle$ takes an exceedingly simple form. Solving (3.28) in the gaussian representation, we obtain (in the NS sector for simplicity):

$$|S_{12}\rangle = \exp \left[i \sum_{n \in \mathbb{Z} + 1/2} x_n^{(1)} x_{-n}^{(2)} \right]\tag{3.43}$$

and sewing is equivalent to performing a Fourier transformation of the $x_n^{(1)}$ variables of one puncture with respect to the other. The integration over the x_n 's necessary to prove (3.41) involves integrating over oscillating gaussians. It can be defined by analytic continuation. We think of x_n and x_{-n} as independent real variables, and define the integration according to the standard rules of gaussian integration. That this prescription is correct, can be seen

Fig. 1: Sewing of the state $|S_{13}\rangle$, $|S_{24}\rangle$ to construct $|S_{12}\rangle$.

by explicitly computing the examples appearing in Fig. 1 and Fig. 2; the first corresponds to sewing two sewing operators: $\langle S_{34}|S_{13}\rangle \otimes |S_{24}\rangle = |S_{12}\rangle$; and the second corresponds to $\langle \phi_q^{(j)}|S_{12}\rangle = |\phi_q^{(j)}\rangle$. The sewing operator behaves like the insertion of the unit operator. As a final example, we can compute the one loop partition function for the $\beta - \gamma$ system in the NS sector using the gaussian representation. This is given by:

$$\langle S_{12}|q^{L_0^{(1)}}|S_{12}\rangle \quad . \quad (3.44)$$

In terms of oscillators

$$q^{L_0^{(1)}}|S_{12}\rangle = \prod_{n>0} \exp(-q^n \gamma_{-n}^{(1)} \beta_{-n}^{(2)} - q^n \gamma_{-n}^{(2)} \beta_{-n}^{(1)}) |gh\rangle_1 \otimes |gh\rangle_2$$

and

$$\langle S_{12}|q^{L_0^{(1)}}|S_{12}\rangle = q^{1/2} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{-2} \quad , \quad (3.45)$$

recalling that $L_0|gh\rangle = \frac{1}{2}|gh\rangle$. In the gaussian representation:

$$L_0 = \sum_{h>0} i \frac{k}{2} (x_{-h}^2 + \frac{\partial^2}{\partial x_{-h}^2} + i) - \sum_{h>0} i \frac{k}{2} (x_h^2 + \frac{\partial^2}{\partial x_h^2} - i) \quad .$$

Then q^{L_0} acting on (3.43) becomes:

$$\prod_n \frac{1}{(q^n + q^{-n})^{1/2}} \exp \left[-\frac{i}{2} \frac{q^n + q^{-n}}{q^n - q^{-n}} x_n^{(1)2} + \frac{i}{2} \left[\frac{2}{q^n - q^{-n}} x_n^{(1)} + q x_{-n}^{(2)} \right]^2 \frac{q^n - q^{-n}}{q^n + q^{-n}} \right] \quad (3.46)$$

after some simple algebra. Multiplying by $\langle S_{12}|$, and integrating, we obtain (3.45), providing the last check of the integration rules described above. ¹

¹ The derivation of (3.46) is substantially simplified if one used the propagator for the harmonic oscillator, with an appropriate analytic continuation.

Fig. 2: Sewing of the state $|S_{12}\rangle$ and $\langle\phi_q^{(j)}|$ to regenerate $|\phi_q^{(j)}\rangle$.

The definition of the states $|\pm\rangle$ for fermionic systems of spin j (3.7–8) can be easily generalized to bosonic $\beta - \gamma$ systems. The analog of $|gh\rangle$ is defined by:

$$\beta_n|gh\rangle = \gamma_n|gh\rangle = 0 \quad n \geq 1/2 \quad (3.47)$$

and it is obtained from the SL_2 invariant vacuum $|\phi_0^1\rangle$

$$\begin{aligned} |gh\rangle &= \delta(\gamma_{1/2}) \dots \delta(\gamma_{j-1}) |\phi_0^1\rangle \\ |\phi_0^1\rangle &= \delta(\beta_{1-j}) \dots \delta(\beta_{-1/2}) |gh\rangle \end{aligned} \quad (3.48)$$

Since

$$\begin{aligned} Q_{gh}|\phi_0^1\rangle &= (j - 1/2)|\phi_0^1\rangle \\ L_0|\phi_0^1\rangle &= 0 \end{aligned} \quad (3.49)$$

we obtain:

$$\begin{aligned} Q_{gh}|gh\rangle &= 0 \\ L_0|gh\rangle &= \frac{1}{2}(j - 1/2)^2|gh\rangle \end{aligned} \quad (3.50)$$

The δ -function operators can be identified with the scalar field ϕ that bosonizes the ghost number current $j(z) = -\gamma(z)\beta(z) \equiv -\partial_z\phi$ [3]. Using the operator product expansions of the energy momentum tensor $T(z)$ with $\gamma(w)$ and $\beta(w)$, one shows:

$$\begin{aligned} T(z)\delta(\gamma(w)) &= \frac{1}{2} \frac{\delta(\gamma(w))}{(z-w)^2} + \frac{1}{z-w} \partial_w \delta(\gamma(w)) \\ T(z)\delta(\beta(w)) &= -\frac{3}{2} \frac{\delta(\beta(w))}{(z-w)^2} + \frac{1}{z-w} \partial_w \delta(\beta(w)) \end{aligned} \quad (3.51)$$

The bosonization rule of [3] is:

$$\begin{aligned} \beta(z) &= e^{-\phi(z)} \partial \xi(z) \\ \gamma(z) &= e^{+\phi(z)} \eta(z) \end{aligned} \quad (3.52)$$

where $e^{-\phi(z)}(e^{\phi(z)})$ is a conformal fields of dimension $1/2(-3/2)$ and ξ, η is a first order fermionic system with spins 0 and 1 respectively. We can tentatively identify [6]

$$\begin{aligned}\delta(\gamma(z)) &= e^{-\phi(z)} \\ \delta(\beta(z)) &= e^{\phi(z)}\end{aligned}\quad (3.53)$$

To complete the identification, we need to compute the operator product expansion of $\delta(\gamma)$'s and $\delta(\beta)$'s. Since later we will use more general δ -function operators, let us explicitly construct the action of $\delta(\beta(f))$, $\delta(\gamma(f))$ on the vacuum. For simplicity, we take $j = 1/2$. The results are easily extended to other values of j . The operator product rules follow from the identities:

$$\begin{aligned}\delta(\beta(f))\Phi_0 &= N(f) \exp \frac{i}{2} \left(\sum_{n,m=1}^{\infty} A_{nm} x_{n-1/2} x_{m-1/2} - \sum_1^{\infty} x_{-n+1/2}^2 \right) \\ \beta(f) &= \sum \beta_{-n+1/2} f_{+n} = \oint \beta(z) f(z) \\ A_{nm} &= \delta_{nm} - 2 \frac{f_n f_m}{\sum f_n^2} \\ N(f) &= \left(\frac{i}{2\pi} \right)^{1/2} \left(\sum f_n^2 \right)^{-1/2}\end{aligned}\quad (3.54)$$

and

$$\begin{aligned}\delta(\gamma(f))\Phi_0 &= N(f) \exp \frac{i}{2} \left(\sum_1^{\infty} x_{n-1/2}^2 - \sum_{n,m=1}^{\infty} B_{nm} x_{-n+1/2} x_{-m+1/2} \right) \\ B_{nm} &= \delta_{n,m} - 2 \frac{f_n f_m}{\sum f_n^2} \\ N(f) &= \left(\frac{1}{2\pi i} \right)^{1/2} \left(\sum f_n^2 \right)^{-1/2}\end{aligned}\quad (3.55)$$

In particular we have

$$\begin{aligned}\delta(\beta(z))\delta(\gamma(w)) &\sim (z-w) \\ \delta(\beta(z))\delta(\beta(w)) &\sim (z-w)^{-1} \\ \delta(\gamma(z))\delta(\gamma(w)) &\sim (z-w)^{-1}\end{aligned}\quad (3.56)$$

These properties together with (3.51) allow us to compute the correlation functions with arbitrary numbers of $\delta(\beta(z))$ and $\delta(\gamma(w))$ operators on the sphere in the $|gh\rangle$ vacuum. This in particular implies that the identification (3.53) is correct. Furthermore, we can use the machinery developed so far to compute the correlation functions of $e^{\phi}, e^{-\phi}$ on arbitrary

Riemann surfaces. The argument is as follows: First, consider a $\beta - \gamma$ system of spin j on a surface Σ of genus g with a point $P \in \Sigma$ and a local coordinate $z(P) = 0$. If $w_n(z)$ (resp. $\tilde{w}_n(z)$) are the $1 - j$ (resp. j) differentials holomorphic off P , the conserved charges:

$$\begin{aligned} Q_n &= \oint_P \beta(z) w_n(z) \\ \tilde{Q}_n &= \oint_P \gamma(z) \tilde{w}_n(z) \end{aligned} \quad (3.57)$$

uniquely specify a ray $|\phi_g^1\rangle$ in the Fock space $\mathcal{F}^B$ at P :

$$Q_n |\phi_g^1\rangle = \tilde{Q}_n |\phi_g^1\rangle = 0 \quad (3.58)$$

If we solve (3.58) in the gaussian representation, we will obtain a gaussian function of the x_n 's without any of the δ -functions appearing in (3.22). By this explicit construction, the correlation functions of $e^\phi, e^{-\phi}$ on Σ can be represented in the form:

$$\langle 0 | \prod_{i=1}^N e^{\phi(z_i)} \prod_{j=1}^M e^{-\phi(w_j)} | \phi_g^1 \rangle_{\beta\gamma} \quad (3.59)$$

Since $[Q_{gh}, e^{\pm\phi}] = \pm e^{\pm\phi}$ and $|0\rangle, |\phi_g^1\rangle$ have respectively charges 1 and $-(2g - 2 + 1)$, (3.59) makes sense only when $N - M = (2j - 1)(g - 1)$. Any expectation value like (3.59) is a meromorphic function on Σ . The O.P.E.'s (3.56) give us some limited (although sufficient) information about the zeros and poles of (3.59) as a function of z_i, w_j . When $z_i \rightarrow z_j$ or $w_i \rightarrow w_j$, $i \neq j$ we get a first order pole, and a first order zero as $z_i \rightarrow w_j$. Fixing all variables but z_i (respectively w_i), (3.59) behaves like a $-j$ (resp. $j - 1$) differential. The divisor of a $-j$ (resp. $j - 1$) differential has degree $-j(2g - 2)$ (resp. $(j - 1)(2g - 2)$). From the O.P.E.'s (3.56) we find that the divisor of (3.59) as a function of z_i is $-\sum_{j \neq i}^N z_j + \sum_{k=1}^M w_k$ with degree $-N + 1 + M = -j(2g - 2) + g$. Thus, it differs from the divisor of a $-j$ differential by a negative divisor of degree g . Using now the Jacobi inversion theorem for generic points z_i, w_j , we know that these g points are uniquely determined [43][40]. Moreover, let b, c be a fermionic system with spin j ; in this case the full bosonization rule $b \sim e^{-\sigma}$ $c \sim e^{\sigma}$ has been proved by various methods [44][40]. Then the argument outlined before together with [43][44][40] shows that

$$\begin{aligned} &\langle 0 | \prod_{i=1}^N e^{\phi(z_i)} \prod_{j=1}^M e^{-\phi(w_j)} | \phi_g^1 \rangle_{\beta\gamma} = \\ &= [\langle 0 | \prod_{i=1}^N e^{\sigma(z_i)} \prod_{j=1}^M e^{-\sigma(w_j)} | \phi_g^1 \rangle_{bc}]^{-1} \end{aligned} \quad (3.60)$$

Since $b \sim e^{-\sigma}$, the field $e^{\sigma(z)}$ behaves like a j differential, so that in the right hand side of (3.60) it behaves like a $-j$ differential, and $e^{-\sigma}$ in the denominator behaves like a $j - 1$ differential. Since their divisors coincide using the Jacobi inversion theorem one proves the identity (3.60).

The complete bosonization rules for $\beta - \gamma$ on arbitrary Riemann surfaces requires more work. First we have to include the contribution of the ξ, η system. In the language of [3], we pass from the little algebra $\phi, \eta, \partial\xi$, to the big algebra ϕ, η, ξ ; the zero mode of ξ is the origin of picture changing when combined with the BRS operator Q_{BRS} . Since we will not need explicitly many of the particular features of the higher genus bosonization of $\beta - \gamma$, we have collected the results in the second part of appendix A.

Finally, in complete analogy with the bosonic string [16] and sec. 2, the variation of the ray $|\phi\rangle$ with moduli is given by the energy-momentum tensor. The argument is entirely similar. Let $\mathcal{P}(g, 1)$ be the moduli space of surfaces of genus g with one puncture P and a local coordinate $z(P) = 0$. The construction described in this section associates a state in the Fock space $\mathcal{H}$ at the puncture. If $\Sigma \in \mathcal{P}(g, 1)$ we can decompose the meromorphic vectors at P , $v(z)\partial/\partial z$ into three categories [26]: i) Those that extend holomorphically off P ii) Those that extend holomorphically to a neighborhood of P , iii) the rest. The first set (the Borel subset of $\text{Vect}(S^1)$ associated to Σ , or $B(\Sigma)$) describes conformal automorphisms of Σ . Since these do not change the data parametrized by $\mathcal{P}(g, 1)$, the action of $v \in B(\Sigma)$ will not change the ray $|\phi\rangle$. general, only for special coordinate For a conformal system with vanishing central extension, then one has $T(v)|\phi\rangle = 0$, $v \in B(\Sigma)$. This is the case for the full state associated to string theory. The vectors of class ii) are those which change the local parameter z (vectors vanishing at P), or move the puncture (vectors not vanishing at P); and finally, there are $3g - 3$ vectors in class iii). They represent the moduli deformations of the Riemann surface Σ . The arguments of [16] can easily be adapted to this case, to show that

$$\delta_v |\phi\rangle = T(v)|\phi\rangle \quad (3.61)$$

because for the $\beta - \gamma$ system, the state is also defined by a complete, maximal set of commuting conserved charges.

Using the gaussian representation of the $\beta - \gamma$ system, it is possible to give a simple geometrical interpretation of the states in the Fock space $\mathcal{F}^B$ as well as the δ -functions or $e^\phi, e^{-\phi}$ fields. This is the bosonic counterpart of the infinite grassmannian interpretation

of the states associated to a surface Σ , a puncture, and a local coordinate by a fermionic b, c system [10][12]. Since we will not make extensive use of this geometric interpretation, which is essentially independent of the main body of this paper, we have collected these results in appendix A. It is however important to point out that the correct definition of many of the manipulations carried out in this section can be achieved in the gaussian picture.

To summarize, in this section we have constructed the states associated to the $\beta - \gamma$ system on arbitrary genus spin Riemann surfaces with punctures and local coordinates. For simplicity of exposition, we chose all punctures to have NS periodicity conditions. We also introduced the gaussian representation which provides a satisfactory treatment of the various δ -functions appearing in the explicit construction of the $\beta - \gamma$ states. In section 4 we will extend the results of this section to Ramond punctures, and we will also present the manifestly supersymmetric construction of the states for the full B, C system $B = \beta + \theta b, C = c + \theta \gamma$, where b, c are the bosonic ghosts (spin 2, -1 resp.) and $\beta - \gamma$ the superconformal ghosts (spin 3/2, -1/2 resp.). This will get us away from the split locus to the full supermoduli space; in particular we will replace (3.61) by the full equation of motion for both even and odd directions on $\widehat{\mathcal{M}}$.

4. The ghost system: supersymmetric formulation

This section contains many of the technical details needed to complete the treatment of the $\beta - \gamma$ system in the previous section. We combine the $\beta - \gamma$ and the $b - c$ ghosts and provide a manifestly world-sheet supersymmetric construction of superconformal ghost states associated to an arbitrary SRS with any number of Neveu-Schwarz (NS) or Ramond (R) punctures.

4.1 Surfaces with one puncture

Throughout this section Σ is a complex 1|1 supermanifold of genus g with a given superconformal structure.¹¹ This means that on the tangent space we can find a one dimensional subspace of odd vector fields D such that D, D^2 provides a basis for the tangent space. Let P be a point in Σ and $s = (z, \theta)$ a local parameter vanishing at P .

¹¹ In this section we will not use the underlying Riemann surface so we can drop the hats without confusion.

This local parameter is defined in a super-disk D centered on P , and we can cover Σ with two coordinate patches: $U_0 = D$ $U_1 = \Sigma - P$. If $B^{(h)}$ is a chiral superconformal tensor field of dimension h , the conserved charges can be obtained by first constructing $(1-h)$ -tensor superfields extending holomorphically off P . If we denote by C the boundary of the ordinary disk, the conserved charges are:

$$Q_i = \oint_C d\mathbb{z} B^{(h)}(\mathbb{z}) w_i^{(1-h)}(\mathbb{z}) . \quad (4.1)$$

The count of meromorphic sections with prescribed poles for super line bundles is a consequence of the super Riemann-Roch theorem[24]. As in sect. 2, the arguments for the components fields [16] generalize straightforwardly to superfields. For the superconformal ghost system we have the superfields $B(\mathbb{z}) = \beta(z) + \theta b(z)$, $C(\mathbb{z}) = c(z) + \theta \gamma(z)$ of dimensions $3/2$ and -1 respectively. Their dynamics are described by the action [3]

$$S = \int_{\Sigma} B^{(3/2)}(\mathbb{z}) \bar{D} C^{(-1)}(\mathbb{z}) . \quad (4.2)$$

Using the conserved charges (4.1), we associate a ray to $\tilde{\Sigma}$ by the equations:

$$Q_i |W\rangle = \left[\oint_C B^{(3/2)}(\mathbb{z}) W_i^{(-1)}(\mathbb{z}) \right] |W\rangle = 0 \quad (4.3)$$

$$\tilde{Q}_j |W\rangle = \left[\oint_C C^{(-1)}(\mathbb{z}) \tilde{W}_j^{(3/2)}(\mathbb{z}) \right] |W\rangle = 0 . \quad (4.4)$$

As in the component analysis, these equations give a maximal set of conditions on the ray $|W\rangle$ which belongs to the tensor product of Fock spaces: $\mathcal{F}_{bc} \otimes \mathcal{F}_{\beta\gamma}^{(q)}$ ($q = 2g - 2 + 1$ is the ghost charge of the state for the $\beta - \gamma$ part; hence the state constructed lives in a definite Bose sea level for the surface Σ). In the one puncture case, we are forced to use NS boundary conditions. If $\hat{\mathcal{P}}(g, 1)$ denotes the moduli space of SRS of genus g with one NS puncture and a local parameter, we can interpret geometrically the construction of $|W\rangle$ as a projective section of a Fock bundle over $\hat{\mathcal{P}}(g, 1)$ with fiber $\mathcal{F}_{bc} \otimes \mathcal{F}_{\beta\gamma}^{(q)}$. By analogy with the component construction¹² we will also call $|W\rangle$ the super Krichever state for the $B - C$ system associated to the point $\Sigma \in \hat{\mathcal{P}}(g, 1)$.

As in sec. 2, it is easy to show that $|W\rangle$ satisfies the "equation of motion:"

$$\delta_v |W\rangle = T(v) |W\rangle \quad (4.5)$$

¹² This is closely related to the theory of the K-P hierarchy; see [12] and references therein.

for any tangent vector δ_v to $\hat{\mathcal{P}}(g, 1)$ at Σ , and

$$T(v) = \oint_C v(-C\partial B - \frac{3}{2}\partial CB + \frac{1}{2}DCDB) \quad (4.6)$$

with $T(\mathfrak{s})$ the super stress-energy tensor, a chiral superfield of dimension $3/2$. v is an element of $\text{Vect}(S^{1|1})$, the superconformal vector fields on $\hat{C} = S^{1|1}$; the charges change along v via the action of T :

$$Q_i \rightarrow Q_i + [T(v), Q_i] \quad (4.7)$$

Aside from some special cases (namely, the super-sphere with punctures and the super-torus), it is not yet possible to write very explicit representations of the state $|W\rangle$. This is mainly due to the fact that the superconformal structure of Σ depends on $2g - 2$ odd supermoduli parameters apart from the standard $3g - 3$ ordinary moduli, and many of the geometrical quantities appearing on ordinary Riemann surfaces (prime forms, ϑ -functions etc.) have not been constructed explicitly for SRS with non-vanishing odd moduli. Inside of supermoduli space $\widehat{\mathcal{M}}_g$, however, we have the split locus, and here things are fairly simple. In this submanifold, the state $|W\rangle$ can be written in terms of quantities related to prime forms and θ -functions. For arbitrary values of the odd moduli τ^a we can then compute the state $|W, \tau\rangle$ (where the τ are the supermoduli parameters) by choosing a basis of $-1/2$ differentials ν_α with poles of orders $1, \dots, 2g - 2$ at P , and writing $|W, \tau\rangle = \exp[\sum \tau^\alpha T_F(\nu_\alpha)]|W, \tau = 0\rangle$ where $T_F(z)$ is the world-sheet supercurrent. There are many subtleties involved with the global existence of coordinates of this form, but the ray $|W\rangle$ is always abstractly characterized by the changes and so is well-defined throughout $\hat{\mathcal{P}}$.

4.2 Two punctures

Given the technical difficulties in constructing directly the supermoduli dependence of the state $|W\rangle$, it is very convenient to develop a systematic procedure of constructing higher genus SRS starting with the three-punctured sphere and the sewing operator. We now proceed to construct explicitly the corresponding states. As will become apparent in the following, this sewing procedure gives a straightforward way of counting the number of moduli and supermoduli for a surface with arbitrary numbers of handles and punctures. We believe in fact that any $\tilde{\Sigma} \in \hat{\mathcal{P}}(g, p, q)$ (where $\hat{\mathcal{P}}(g, p, q)$ is the moduli space of SRS of genus g , p NS and q R punctures with local parameters) can be obtained by sewing. While this description of $\tilde{\Sigma}$ can be many-to-one, still the construction of the ray is independent of the way $\tilde{\Sigma}$ is obtained. It only depends on the superconformal properties of $\tilde{\Sigma}$.

Before proceeding with the construction we need to make more precise the notion of punctures. Even on an ordinary Riemann surface we do not mean that the “punctures” are literally deleted points; *e.g.* we do not admit functions with essential singularities at those points. Instead we view them simply as chosen, or marked points P_i and admit poles by tensoring in $\mathcal{O}(nP_i)$. Similarly super punctures are just marked super points, or more precisely maps from any parameter space into a family of SRS. They can be given as functions $(z_i(m, \tau), \theta_i(m, \tau))$; the even functions $(z - z_i)$ then define divisors Δ_i on the SRS just as in the bosonic case and we can talk about poles at the puncture. Also as in the bosonic case there is a normalization map from a SRS with super pinch to a nonsingular one with two super punctures; in particular normalization commutes with the D operator.

The case of spin punctures is more subtle.¹³ Unlike the super case something is actually singular at the spin puncture, but *not* the curve itself. Instead we define a smooth SRS with n spin punctures to be a smooth complex 1|1-supermanifold with a nonsingular divisor $\Delta = \Delta_1 + \dots + \Delta_n$ and a derivation D . D maps the super functions $\hat{\mathcal{O}}$ to sections of $\hat{\omega} \equiv \text{BER} \otimes \hat{\mathcal{O}}(\Delta)$. Away from Δ there must always exist good local coordinates z, θ such that $D = [dz|d\theta](\partial_\theta + \theta\partial_z)$ as usual. Near Δ_i , however, we require that there be good complex coordinates z, η such that $\Delta_i = \{z = 0\}$ and

$$D = [dz|d\eta](z^{-1}\partial_\eta + \eta\partial_z) \quad . \quad (4.8)$$

Thus D takes us to sections of the modified canonical bundle $\hat{\omega}$ above. There is nothing inherently ramified about the SRS at the punctures. Of course one can replace the good, non-superconformal coordinates z, η by “coordinates” $z, \theta = z^{1/2}\eta$, whereupon D regains its usual form, and we will do so later on. But the system of cuts usually taken to join pairs of spin punctures is not a part of the SRS proper.

Once again there is a good normalization map taking a spin-pinch curve to a SRS with spin punctures and commuting with D ; the two branches of the punctured curve are now glued along the 0|1-subvariety Δ_i [45].

As an example of a punctures SRS consider the smooth complex supermanifold with body the sphere and coordinate maps $z_2 = z_1^{-1}$, $\eta_2 = \eta_1$. Since η is not a spinor this is an example of a complex 1|1-supermanifold for which no nonsingular superconformal

¹³ The following definition seems to be a slight generalization of the one mentioned in [45], in that it works with odd parameters.

Fig. 3: The sphere with two punctures.

structure exists [23]. Now however let Δ_i be defined by z_i and D as (4.8). Then D is globally defined. To put this into a more traditional form we can now let

$$u_1^2 = -z_1, \quad u_2^2 = z_2, \quad \theta_i = \pm u_i \eta_i. \quad (4.9)$$

Consider now a super-sphere with two super punctures. It is clear by $OSp(2|1)$ invariance that the punctures (z_i, θ_i) $i = 1, 2$ can be chosen so that $z_1 = 0, z_2 = \infty$ and $\theta_1 = \theta_2 = 0$. Thus in this case there are no odd moduli and so the ray obtained is simply the tensor product of the $b - c$ and $\beta - \gamma$ states obtained in components, which we have already given.

For RR punctures the story is different. We parametrize this super-sphere using the construction of [46][45][25]. As mentioned above it is convenient to choose local parameters $u_1, u_2, \theta_1, \theta_2$ which are defined on double covers of the two coordinate patches (see Fig. 3): The superconformal transformation relating the two patches can be thought of as the "square root" of the NS-plumbing fixture. The changes of coordinates take the form:

$$\begin{aligned} u_2 &= tu_1^{-1} \pm \frac{1}{2} \theta_1 t \hat{t} u_1^{-2} \\ u_1 &= tu_2^{-1} + \frac{1}{2} \theta_2 t \hat{t} u_2^{-2} \\ \theta_2 &= t \hat{t} u_1^{-1} \pm \theta_1 t u_1^{-2} \\ \theta_1 &= \pm (t \hat{t} u_2^{-1} - \theta_2 t u_2^{-2}) \end{aligned} \quad (4.10)$$

These are in $OSp(2|1)$ for every value of $t, \hat{t}$. When we sew the twice-punctured spheres together using fixed vertices we will find that $t, \hat{t}$ become moduli parameters, analogous to the length and twist parameters of ordinary Riemann surfaces. In particular, $\hat{t}$ plays a role analogous to the twist, or phase of t : unlike $|t|$, neither is strictly speaking a modulus of the punctured sphere. (This explains why $\hat{t}$ is absent in the formulation of Deligne[45].) Nevertheless each tells something about the local coordinates chosen at the punctures, and so we do expect the state to depend on them.

As $t \rightarrow 0$ we obtain the so called spin node. The $\pm$ sign will characterize after sewing the two possible spin structures (whether we do or do not include $(-1)^F$ along the plumbing). This is important when one wants to sum over spin structures to implement the GSO projection. To construct the state $|RR\rangle$ we proceed in two steps.

First we build the superconformal vector fields that extend holomorphically outside of puncture (1). In the coordinates (4.9) these are $v = [dz_1|d\theta_1]^{-2}v^{z_1}$, where

$$v^{z_1} = A_n u_1^{2n} + B_n u_1^{2n+1} \theta_1 \quad (4.11)$$

and the same for puncture (2). Notice that the superconformal vector fields (4.11) define a single valued integrand for the conserved charges. This becomes clear when (4.11) is rewritten in terms of the good coordinates z, η given earlier.

Next, using the transformation rules of superconformal tensor fields of conformal weight -1 and the definition of the charges we construct the conserved charges. In components these are

$$Q_n^{[1]} = (-1)^n b_{n-1}^{(1)} + t^{2n-2} b_{-n+1}^{(2)} + (n-1) \beta_{-n+1}^{(2)} \hat{t} t^{2n-2} \quad (4.12)$$

$$Q_n^{[2]} = (-1)^{n-1/2} \beta_n^{(1)} \pm b_{-n}^{(2)} \hat{t} t^{2n} \mp t^{2n} \beta_{-n}^{(2)} \quad (4.13)$$

$$Q_n^{[3]} = b_{n-1}^{(2)} + (-1)^n b_{-n+1}^{(1)} t^{2n-2} \pm (n-1) (-1)^{-1/2+n} \beta_{-n+1}^{(1)} \hat{t} t^{2n-2} \quad (4.14)$$

$$Q_n^{[4]} = \beta_n^{(2)} \mp (-1)^{n-1/2} t^{2n} \beta_{-n}^{(1)} + (-1)^{n+1} b_{-n}^{(1)} \hat{t} t^{2n} \quad (4.15)$$

where the superscripts in the oscillators indicate the Hilbert space where they act. Notice that for $\hat{t} \neq 0$ (i.e. away from the split locus) we will get that the super Krichever state does not factorize into bc and $\beta\gamma$ parts, as expected. Solving (4.12) to (4.15) we obtain

$$\begin{aligned} |RR^{\hat{t}, t=1}\rangle &= \prod_{n=1}^{\infty} \exp[(nc_{-n}^{(1)} \beta_{-n}^{(2)} \mp inc_{-n}^{(2)} \beta_{-n}^{(1)} \pm i\gamma_{-n}^{(1)} b_{-n}^{(2)} + \gamma_{-n}^{(2)} b_{-n}^{(1)}) \hat{t}] \\ &\times \exp(\pm i\gamma_0^{(1)} b_0^{(2)} \hat{t}) |RR^{\hat{t}=0, t=1}\rangle \end{aligned} \quad (4.16)$$

where:

$$|RR^{\hat{t}=0, t=1}\rangle = |RR^{\hat{t}=0, t=1}\rangle_{bc} \otimes |RR^{\hat{t}=0, t=1}\rangle_{\beta\gamma} \quad (4.17)$$

$$|RR^{\hat{t}=0, t=1}\rangle_{bc} = \prod_{n=1}^{\infty} \exp(c_{-n}^{(1)} b_{-n}^{(2)} + c_{-n}^{(2)} b_{-n}^{(1)}) \exp(c_0^{(2)} b_0^{(1)}) |+\rangle_1 |-\rangle_2 \quad (4.18)$$

$$\begin{aligned}
|RR^{i=0,t=1}\rangle_{\beta\gamma} &= \prod_{n=1}^{\infty} \exp(\mp i\gamma_{-n}^{(1)}\beta_{-n}^{(2)} \pm i\gamma_{-n}^{(2)}\beta_{-n}^{(1)}) \exp(\mp i\gamma_0^{(1)}\beta_0^{(2)}) |-\frac{1}{2}\rangle_1 |-\frac{3}{2}\rangle_2 \\
\beta_0 | -1/2\rangle &= \gamma_0 | -3/2\rangle = 0 \quad , \\
\beta_n | -1/2\rangle &= \gamma_n | -1/2\rangle = \beta_n | -3/2\rangle = \gamma_n | -3/2\rangle = 0 \quad n > 0 \\
\delta(\beta_0) | -3/2\rangle &= | -1/2\rangle \quad , \delta(\gamma_0) | -1/2\rangle = | -3/2\rangle \quad .
\end{aligned} \tag{4.19}$$

From (4.16) it is a simple exercise to prove that:

$$|RR^{i,t}\rangle = \exp \left[\pm i G_0^{(1)} \hat{t} \right] |RR^{i=0,t}\rangle \quad . \tag{4.20}$$

Notice that the ghost number of $|RR\rangle$ is zero as expected from the Riemann-Roch theorem.

We can now address what happens to $|RR\rangle$ when we change the Bose sea charge i.e., when we choose a different representation of the $\beta - \gamma$ oscillators. Using the gaussian representation of sec. 3, we can show that $|RR\rangle$ is independent of the Bose sea level. The states annihilated by (4.12)–(4.15) but defined relative to different Bose seas are:

$$\begin{aligned}
|RR\rangle^{(q)} &= \exp \left(\sum_{n \geq -q} -i\gamma_{-n}^{(1)}\beta_{-n}^{(2)} + \sum_{n \geq q+1} i\gamma_{-n}^{(2)}\beta_{-n}^{(1)} \right) \\
&\times |q - 1/2\rangle_1 \otimes | -q - 3/2\rangle_2 \quad .
\end{aligned} \tag{4.21}$$

To prove that $|RR\rangle^{(q)} = |RR\rangle^{(q+1)} = |RR\rangle^{(q+2)} = \dots$, we rewrite (4.11) as follows:

$$\begin{aligned}
|RR\rangle^{(q)} &= \exp \left(\sum_{-q}^{\infty} -i\gamma_{-n}^{(1)}\beta_{-n}^{(2)} + \sum_{n=q+2}^{\infty} i\gamma_{-n}^{(2)}\beta_{-n}^{(1)} \right) \\
&\times \exp(i\gamma_{-q-1}^{(2)}\beta_{-q-1}^{(1)}) \delta(\gamma_{1+1}^{(1)}) \delta(\beta_{q+1}^{(2)}) |q + 3/2\rangle | -q - 5/2\rangle \quad .
\end{aligned} \tag{4.22}$$

Using the integral representation of $\delta(\beta), \delta(\gamma)$ we obtain after some simple algebra that

$$|RR\rangle^{(q)} = |RR\rangle^{(q+1)} \tag{4.23}$$

as was to be shown.

4.3 More punctures

Finally, we consider the state $|RRN\rangle$. The state $|NNN\rangle$ is straightforward to construct, and we leave it as an exercise to the reader. We only note that $|NNN\rangle$ depends on one supermodulus, a consequence of $OSp(2|1)$ invariance. We can always choose the commuting part of the positions of the punctures to be three specified points, say $0, 1, \infty$.

Using the fermionic transformations in $OSp(2|1)$, we can set two of $\theta_1, \theta_2, \theta_3$ to zero. Hence the supermoduli space of the sphere with three NS punctures has dimension $0|1$. For the $|RRN\rangle$ vertex, there are no moduli or supermoduli due to the antiperiodicity of the θ 's at the Ramond punctures. Since the algebra necessary to construct this vertex is somewhat involved, we have collected some of the details in appendix B. Here we simply outline the main features of the computation.

Since for $|RRN\rangle$ there are neither moduli nor supermoduli, the state will factorize into a tensor product of a $b - c$ and a $\beta - \gamma$ state. The main technical point is that the change of coordinates from the NS to the R puncture has to be constructed so that it is a) superconformal and b) changes periodic θ 's around the NS puncture to antiperiodic θ 's around the R puncture. For the R puncture we choose the meromorphic vector superfields (4.11), and for the NS :

$$v^{z_3} = A_n z_3^n + B_n \theta_3 z_3^n, \quad (4.24)$$

where (z_3, θ_3) are local parameters around puncture 3 (see Fig. 4). In appendix B we find the charges corresponding to (4.24), (4.11) and the state $|RRN\rangle$ which they define.

Fig. 4: Three-punctured sphere.

Having constructed the standard 2- and 3-point spheres, with choices for local coordinates at each puncture, we can now sew them together with standard gluing maps. These maps will not depend in any parameters, so that the final SRS we obtain depends on $1|0$ parameters for each $|NN\rangle$, $1|1$ for each $|RR\rangle$, $0|0$ for each $|RRN\rangle$, and $0|1$ for each $|NNN\rangle$. In order for this construction to provide good local coordinates for $\widehat{\mathcal{M}}$ we must now show that the number of parameters obtained in this way corresponds to the dimension of $\widehat{\mathcal{M}}$ in each genus and spin structure.¹⁴

¹⁴ It would be interesting to go further and see to what extent one obtains in this way global coordinates for super-Teichmüller space, a result analogous to the Fenchel-Nielsen theorem. This we will not attempt.

To check the counting, consider a state with one puncture (NS) on a genus 2 surface (see Fig. 5). The state will certainly depend on the spin structure on the surface, and the sewing prescription provides a definite procedure to obtain this dependence. There are several possibilities for sewing, for example:

- 1) Take three $|NN\rangle$ vertices. Since these vertices each depend on one supermodulus, and $|NN\rangle$ has no odd parameters, we obtain 3 supermoduli parameters. In each $|NN\rangle$, however, we have a parameter $q : z_i z_j = q$, and therefore the moduli count is 4. Notice that since $|NN^\pm\rangle$ has a $\pm$ sign (depending on the insertion of $(-1)^F$), we have on the cycles a_1, a_2 (see Fig. 5) NS boundary conditions, while the boundary conditions on the cycles b_1, b_2 depend on the $\pm$ sign in $|NN^\pm\rangle$.
- 2) We can instead consider two $|NRR\rangle$ vertices and one $|NN\rangle$. We will therefore need two $|RR^\pm\rangle$. Once again we obtain three supermoduli and four moduli.
- 3) Take three $|NRR\rangle$ vertices. Now we need three $|RR^\pm\rangle$ with three supermoduli, and one $|NN^\pm\rangle$ with no supermoduli.

Fig. 5: Sewing a complicated surface.

It is easy to generalize the previous count to an arbitrary surface with genus g, p NS and q R punctures. Then the supermoduli space has graded dimension

$$\dim \widehat{\mathcal{M}}(g, p; q) = (3g - 3 + p + q | 2g - 2 + p + q/2) \quad (4.25)$$

and the number of Ramond punctures must be even. Eqn. (4.25) indeed agrees with the well-known dimension when $p = q = 0$. With punctures we get a parameter space of dimension given by (4.25), a subspace of the infinite-dimensional $\widehat{\mathcal{P}}(g, p, q)$.

The sewing description is particularly useful when we want to analyze the behavior of string amplitudes at the boundary of supermoduli space. This boundary is described in terms of super plumbing fixtures [46][45][25] ($|NN\rangle, |RR\rangle$ states), in the limit when

$t \rightarrow 0$ (the even parameter vanishes), and therefore, as for the bosonic string, the tachyons and massless states of the string theory considered completely determine the structure of infinities for any order of string perturbation theory, as one would expect from naive physical arguments.

5. The measure

5.1 General remarks

From the previous sections we can now construct a state associated to a given superstring theory over the supermoduli space of SRS with genus g , n super-punctures, and local superconformal coordinates around them. We denote this space by $\widehat{\mathcal{P}}(g, n)$. In particular, if $n = 1$, we have a Fock space at the puncture:

$$\mathcal{F}_{bc} \otimes \mathcal{F}_{\bar{b}\bar{c}} \otimes \mathcal{F}_{\beta\gamma}^{(q)} \otimes \mathcal{F}_{\text{matter}} = \mathcal{F}^{TOT} \quad (5.1)$$

where for simplicity we are considering the heterotic string, so that we only need to include right moving superconformal ghosts. The space $\widehat{\mathcal{P}}(g, 1)$ contains the supermoduli space of Riemann surfaces $\widehat{\mathcal{M}}_g$, and we would like to construct a measure on it starting with the state given by the operator formalism in $\widehat{\mathcal{P}}(g, 1)$. A few remarks are in order before we proceed to the explicit construction: First, $\widehat{\mathcal{M}}_g$ is a space constructed over spinmoduli space $\text{Spin } \mathcal{M}_g$, i.e. the moduli space of Riemann surfaces with a chosen spin structure. One says that the body of $\widehat{\mathcal{M}}_g$ is $\text{Spin } \mathcal{M}_g$. Since there are 2^{2g} spin structures for a given surface, $\text{Spin } \mathcal{M}_g$ is a finite covering of $\mathcal{M}_g$, and the symplectic modular group $Sp(2g, \mathbb{Z})$ acts on the sheets of this covering. This discrete group acts transitively on the even and odd spin structures separately. The body has orbifold points at Riemann surfaces with conformal automorphisms which identify generically different spin structures. These orbifold singularities can be resolved as for ordinary moduli space $\mathcal{M}_g$ by introducing a point on the surface and a local parameter around it (see [18][47]).

The simplest supermanifolds are the split ones; they are constructed from ordinary vector bundles over an ordinary manifold. Let $\{U_\alpha, m_\alpha, \tau_\alpha\}$ be a coordinate cover of the supermanifold, where the U_α are open sets in the body and $\{m_\alpha\}$ are even and $\{\tau_\alpha\}$ are odd coordinates. Then the supermanifold is split wherever an atlas of coordinates can be found whose transition functions take the form:

$$m_\alpha = f_{\alpha\beta}(m_\beta), \quad \tau_\alpha = g_{\alpha\beta}(m_\beta) \cdot \tau_\beta \quad (5.2)$$

on the patch overlaps. It is not yet known whether $\widehat{\mathcal{M}}_g$ is split, although some progress has been made in this direction. It is clear that any supermanifold of dimension $n|1$ is split. Thus for example for genus one and with the odd spin structure, the supertori are described by a supermoduli space of dimension $1|1$, which is necessarily split. Also, the sphere with three NS punctures and local coordinates has a supermoduli space of dimension $0|1$, after $Osp(2|1)$ superconformal invariance has been fixed, and therefore it is split. As will be clear below, many of the complications in the construction of measures on moduli space coming from integrating the measure over the odd directions would naturally disappear if $\widehat{\mathcal{M}}_g$ were projected. Being projected is a property weaker than splitness. It means that the transition functions can be chosen to be:

$$\begin{aligned} m_\alpha &= f_{\alpha\beta}(m_\beta), \quad \tau_\alpha = f_{\alpha\beta}\tau_\beta + f_{\alpha\beta\gamma\delta}\tau_\beta\tau_\gamma\tau_\delta + \dots \\ &= P_\alpha(m_\beta, \tau_\beta) \end{aligned} \tag{5.3}$$

where P_α is an odd polynomial in $\{\tau_\beta\}$.

The issue of whether $\widehat{\mathcal{M}}_g$ is split or projected becomes relevant when one tries to integrate over $\widehat{\mathcal{M}}_g$. For physically acceptable superstring theories, it should be possible to integrate the measure unambiguously. This may give us a set of constraints on what superstring theories are physically acceptable, at least in perturbation theory [48].

A second useful remark is that the GSO projection, or equivalently, the sum over the various spin structures is achieved only after integration over $\widehat{\mathcal{M}}_g$, because the body of $\widehat{\mathcal{M}}_g$ is $\text{Spin } \mathcal{M}_g$.

Finally, we want to comment on the normalization of the state $|W\rangle$ which is necessary for computing string amplitudes. The strategy to fix the normalization of $|W\rangle$ parallels the one suggested for the bosonic strings in [18]. The motion of the state $|W\rangle$ as we change the data parametrized by $\widehat{\mathcal{P}}(g, 1)$ is given by the action of the energy-momentum tensor (the same as well as what will follow is true for the state associated to $\widehat{\mathcal{P}}(g, n)$). The energy-momentum tensor thus acts as a connection on the Fock bundle over $\widehat{\mathcal{P}}(g, 1)$. For each of the states tensoring into $|W\rangle$, the connection is projective, due to the central extension of the Virasoro algebra. For superstring theories, however, the total value of the central charge is zero. Thus the connection defined by the full energy-momentum tensor is flat. The state $|W\rangle$ is hence a flat line bundle over $\widehat{\mathcal{P}}(g, 1)$; locally there is no ambiguity in choosing the normalization of $|W\rangle$. We just start from a given normalization and fix the relative normalization of states at different points by parallel transporting with the super energy momentum tensor. The vanishing of central extension guarantees that when we

parallel transport about loops that are contractible we come back to the state we started with. In other words locally we can associate to each point in $\widehat{\mathcal{P}}(g, 1)$ a state, and not just a ray in $\mathcal{F}^{TOT}$.

To make sure that there is no problem in defining the phase globally it is clear that we should make sure that when we parallel transport the state around non-contractible loops we come back to the same state. If $H_1(\widehat{\mathcal{P}}(g, 1), \mathbb{Z}) = 0$, the relative normalizations of states will be well defined, because there are no non-trivial loops. For the bosonic string, $H_1(\mathcal{P}(g, n), \mathbb{Z}) = 0$ $g > 2$ [47]. The proof of the vanishing of H_1 relies heavily on specific properties of the mapping class group, and in particular on the fact that it is generated by Dehn twists around curves which do not split the surface.

How about the $H_1(\widehat{\mathcal{P}}(g, 1), \mathbb{Z})$? Do we expect that to be zero? $H_1(\widehat{\mathcal{P}}(g, 1), \mathbb{Z})$ is the same as $H_1(\text{Spin } \mathcal{P}(g, 1), \mathbb{Z})$, because we always use supermanifolds in the sense of Leites, and these are topologically identical to their body [49]. Then explicit computations have shown that the latter cohomology group is not zero; in particular we know there are global anomalies associated with fermions on the world sheet. From explicit computations this suggests that $H_1(\widehat{\mathcal{P}}(g, 1), \mathbb{Z})$ includes at least a $\mathbb{Z}_4$, which is probed by genus one surfaces (tori).¹⁵ Unfortunately $H_1(\text{Spin } \mathcal{P}(g, 1), \mathbb{Z})$ does not seem to have been calculated, though currently work is being done to this end [50]. For the case of heterotic string and superstring, the fact that there is some non-trivial H_1 means that there are extra conditions for the consistency of the theory. It would be interesting to see whether or not these conditions are completely probed at the one loop level. Of course for the ordinary uncompactified heterotic string or superstring the usual arguments for the absence of global anomalies implies that we have no global problems with fixing the normalization of the states. It would be nicer to know what the $H_1(\widehat{\mathcal{P}}(g, 1))$ (and its generalization to n punctures) is so that one would be able to phrase these as extra conditions for the consistency of the theory. We suspect that these conditions may be the same as the conditions required for one loop modular invariance.

Once these conditions are satisfied, there is no ambiguity in the relative normalization of $|W\rangle$ for different points on $\text{Spin } \mathcal{P}(g, 1)$. With a single puncture $\text{Spin } \mathcal{P}(g, 1)$ is a disconnected space, depending on whether the spin structure is even or odd. As in more

¹⁵ For example we could consider the bosonic string where one of the right-moving bosonic fields is replaced by a complex fermion; in such a case the state depends on $\text{Spin } \mathcal{P}$ and there is a fourth root of unity in modular transformations beginning with genus one.

standard treatments (see [51] [52] [53]), the relative contributions to the partition function or the scattering amplitudes of these disconnected pieces is constrained by the physical requirement of factorization of higher genus amplitudes, a necessary condition for unitarity in string theory. In addition, for the states associated to $\widehat{\mathcal{P}}(g, n)$, i.e., for Riemann surfaces with n punctures, depending on how many are in the NS sector and how many punctures in the Ramond sector, we get some states whose relative normalization needs to be fixed. This can be fixed if one notes that by sewing patches together we can get to a state with higher genus and fewer punctures, such that even if we start with different number of R and NS punctures, we end up with a fixed number of NS and R punctures; in such a case different number of relative punctures that we started with translates to different points on moduli of the higher genus surface, which can be reached by parallel transport of the higher genus surface (modulo the global anomalies discussed above). In this way, assuming that we have no global anomalies, we can fix the relative normalization of the states that we started with.

To construct a measure over $\widehat{\mathcal{M}}_g$ using the operator formalism, we proceed by analogy with the treatment of the bosonic string [16]. For a point¹⁶ $\tilde{\Sigma} \in \widehat{\mathcal{P}}(g, 1)$, the operator formalism gives a state $|W\rangle$ with ghost number $(3g - 3) + 3/2 - (2g - 2 + 1)$. Since the $OSp(2|1)$ invariant vacuum $|0\rangle$ has ghost number $-3/2|1$, to get a number $\langle 0 | \dots | W \rangle$ we have to insert B superfield operators to soak up the $|W\rangle$ charge. Notice that we are concentrating on the right moving ghost charge. The discussion of the left-moving ghosts runs parallel. In the heterotic string, for example, $|W\rangle$ has left moving ghost charge $3g - 3 + 3/2|0$. There is a projection $\pi : \widehat{\mathcal{P}}(g, 1) \rightarrow \widehat{\mathcal{M}}_g$ which is nothing but ignoring the puncture and the local coordinate. Let $\pi(\tilde{\Sigma}) \in \widehat{\mathcal{M}}_g$ have coordinates $(m_i, \tau_\alpha) \ i = 1, \dots, 3g - 3; \alpha = 1, \dots, 2g - 2$. The tangent space to $\widehat{\mathcal{M}}_g$ at $\pi(\tilde{\Sigma})$ is spanned by some superconformal vector fields $\tilde{v}_i, \tilde{v}_\alpha$, where $\tilde{v}_i$ (resp. $\tilde{v}_\alpha$) generate the even (resp. odd) deformations. We want to find representatives of $\tilde{v}_i, \tilde{v}_\alpha$ in the tangent space at $\tilde{\Sigma} \in \widehat{\mathcal{P}}(g, 1)$. As in the bosonic string, there is a good deal of arbitrariness in the choice of these representatives (v_i, v_α) , and one of the basic demands on the measure is its independence of the particular choice of (v_i, v_α) . Since we have the additional data of a local superconformal coordinate, we can find explicit representations of (v_i, v_α) in terms of meromorphic superconformal vector fields.

¹⁶ Again $\tilde{\Sigma}$ is the triple $(\Sigma, P; z, \theta)$.

Let Σ be a SRS and P the point selected on it. In terms of the local parameters we can divide the meromorphic superconformal vector fields $v(z, \theta) \equiv v(\mathbb{z})$ into three classes once again, based on the tangents δ_v which they induce on $\hat{\mathcal{P}}$ (sec. 2.3):

- i) Those extending holomorphically to a disc around P ; these correspond to superconformal reparametrizations of the local coordinates and infinitesimal motions of the marked point. Since for the time being we are analyzing $\hat{\mathcal{P}}(g, 1)$, the puncture has NS boundary condition, and the elements of $\text{Vect}(S^{1|1})$ (superconformal vector fields on the disc around P minus P itself) can be written as:

$$\begin{aligned} \ell_n &= -z^{n+1} \frac{\partial}{\partial z} - \frac{n+1}{2} z^n \theta \frac{\partial}{\partial \theta} & n \in \mathbb{Z} \\ g_m &= z^{m+1/2} \frac{\partial}{\partial \theta} - z^{m+1/2} \theta \frac{\partial}{\partial z} & n \in \mathbb{Z} + 1/2 \end{aligned} \quad (5.4)$$

Superconformal vector fields vanishing at P generate local reparametrizations of $\mathbb{z}$, and those not vanishing at P move the puncture. Recall that a superconformal vector field $v(\mathbb{z}) = v_0(z) + \theta v_1(z)$ acts by (2.12):

$$\begin{aligned} \delta z &= v_0 + \frac{1}{2} \theta v_1 \\ \delta \theta &= \frac{1}{2} (v_1 + \theta \partial v_0) \end{aligned} \quad (5.5)$$

Hence the generators $L_{+1}, G_{+1/2}$ corresponding to $\ell_{+1}, g_{+1/2}$ move the puncture, and L_n, G_n $n \leq 0, n < 0$ change the local superconformal coordinates.

- ii) Vector fields v extending holomorphically off P . They generate conserved charges:

$$\begin{aligned} T(v) &= \oint v T(\mathbb{z}) \equiv \oint_P v T(\mathbb{z}) \\ T(\mathbb{z}) &= T_{z\theta}^F(z) + \theta T_{zz}^B(z) \end{aligned} \quad (5.6)$$

which annihilate the superstring state because the central extension vanishes. The argument is identical to the one used for the bosonic string: such v have $\delta_v = 0$ and so by the equation of motion (2.16), (4.5) $T(v)$ annihilates the state.

- iii) Vector fields v which do not extend holomorphically to P nor off P . These correspond to δ_v which change the moduli of Σ . We can for example choose vectors:

$$\begin{aligned} v_i &= z^{-i} & i = 1, \dots, 3g-3 \\ v_\alpha &= \theta z^{-\alpha} & \alpha = 1, \dots, 2g-2 \end{aligned} \quad (5.7)$$

(for simplicity we are assuming that the point P is not a Weierstrass point so that there are no jumps in the poles representing supermoduli variations in (5.7). The meromorphic fields (5.7) in general have nonvanishing Lie brackets and so do not correspond to coordinates in supermoduli space; however, by adding vectors of types i), ii), to (5.7), we can construct vector fields v_A $A = (i, \alpha)$ with vanishing Lie brackets (see (2.14)). Such vectors represent true local coordinates on supermoduli space at $\tilde{\Sigma} \in \hat{\mathcal{P}}(g, 1)$.

In any case we have seen that the variation of the state $|W\rangle$ at $\tilde{\Sigma}$ with respect to supermoduli deformation is

$$\delta_A |W\rangle = T(v_A) |W\rangle \quad . \quad (5.8)$$

Notice that (5.8) is invariant under the change $v_A \rightarrow v_A + \eta_A$, where η_A is a vector of type (ii). These vectors generate the so called Borel subalgebra of $\text{Vect}(S^{1|1})$ associated to $\tilde{\Sigma}$ with puncture and coordinate. Eqn. (5.8) is not invariant, however, under $v_A \rightarrow v_A + \eta_A$ where η_A is now a vector of type (i). This is because the state $|W\rangle$ depends explicitly on the local parameter chosen at the puncture $P \in \Sigma$.

Unlike the case of ordinary Riemann surfaces, there are few explicit constructions of SRS. There are in principle two procedures to construct the state $|W\rangle$ associated to a generic point $\Sigma \in \mathcal{P}(g, 1)$. First, if we have enough explicit information about the complex structure of Σ (for instance, the analogs of prime forms, ϑ -functions and abelian differentials of the first, second, and third kinds), then an argument similar to the one presented in secs. 2–4 associates a unique ray to $\tilde{\Sigma}$. Alternatively, we can start on the split locus; there the state $|W\rangle$ can be constructed using standard information about Riemann surfaces. This gives a state $|W, m_i, \tau_\alpha = 0\rangle$, and next we exponentiate the generators v_α along the odd directions

$$|W, m_i, \tau_\alpha\rangle = \exp \left[\sum_\alpha \tau_\alpha \oint_P v_\alpha T(z) \right] |W, m_i; \tau = 0\rangle \quad . \quad (5.9)$$

In the examples constructed in section 4 both procedures agree. However those examples all correspond to split supermanifolds; globally there may be obstruction to finding appropriate vector fields v_α . In any case on open sets of $\hat{\mathcal{P}}(g, 1)$ we can use the parametrization (5.9), just as in the bosonic case [16]. This works because the Kodaira-Spencer map $v \mapsto \delta_v$ defined in sec. 2 is again locally onto in the super case [54] [55] .

We conclude this subsection with another example of how our construction works: The torus with one puncture and the odd spin structure is a SRS which can be constructed

explicitly. The standard torus is obtained by identifying the points in the super-plane (z, θ) as

$$\begin{aligned} z &\sim z + 1 & \theta &\sim \theta \\ z &\sim z + \tau & \theta &\sim \theta \end{aligned} \quad (5.10)$$

The identifications which exhibit the supermodulus of the torus with the odd spin structure are:

$$\begin{aligned} z &\sim z + 1 & \theta &\sim \theta \\ z &\sim z + \tau + \theta\delta & \theta &\sim \theta + \delta \end{aligned} \quad (5.11)$$

We can try to define new "coordinates" z', θ' taking the torus back to the form (5.10) as follows. Let

$$z' = z + \delta\theta \frac{\vartheta_1'(z)}{2\pi i \vartheta_1(z)} = z + \frac{\delta\theta}{2\pi i} \frac{d}{dz} \log \vartheta_1(z) \quad (5.12)$$

where $\vartheta_1(z|\tau) \equiv \vartheta[\frac{1}{2}](z|\tau)$ is the odd ϑ -function. Using

$$\vartheta_1(z+1) = -\vartheta_1(z) \quad , \quad \vartheta_1(z+\tau) = e^{i\pi\tau-2\pi iz} \vartheta_1(z) \quad (5.13)$$

we obtain

$$\begin{aligned} z' &\rightarrow z' + 1 & \text{if } z &\rightarrow z + 1 \\ z' &\rightarrow z' + \tau + \theta\delta & \text{if } z &\rightarrow z + \tau \end{aligned} \quad (5.14)$$

The new odd "coordinate" is:

$$\theta' = \theta - \delta \frac{1}{2\pi i} \frac{d}{dz} \log \vartheta_1(z) \quad , \quad (5.15)$$

which clearly satisfies $\theta' \rightarrow \theta'$ around each cycle. Of course z', θ' cannot be true coordinates, for then we would have removed δ by a coordinate transformation, and hence shown that δ isn't really a modulus. Indeed both (5.12) and (5.15) are singular at zero, and so are not good coordinates. Instead (5.12), (5.15) provide us with the explicit vector fields v needed for the Kodaira-Spencer construction. Using them the state $|\Phi, \tau, \delta\rangle$ can be obtained from $|\Phi, \tau, \delta = 0\rangle$ by exponentiating the world sheet supercurrent:

$$\exp \frac{\delta}{2\pi i} \oint_C \frac{\vartheta_1'(z)}{\vartheta_1(z)} T_F(z) |\Phi, \tau, \delta = 0\rangle = |\Phi, \tau, \delta\rangle \quad (5.16)$$

That this is the correct state can also be seen if we recall the super partition function [56]:

$$Z(\tau, \delta) = \text{tr } q^{L_0} e^{\delta G_0} \quad (5.17)$$

From the point $z'(P) = 0$, we can obtain δG_0 if we use a $-1/2$ differential with a single pole at $z = 0$ and a cut around the b-cycle with a shift of $2\pi i$ across it. This is exactly given by $\vartheta'_1(z)/\vartheta_1(z)$, hence [57]

$$G_0 = \frac{1}{2\pi i} \oint \frac{\vartheta'_1(z)}{\vartheta_1(z)} T_F(z) \quad . \quad (5.18)$$

The same answers can be obtained using the sewing operator on a surface with three punctures.

5.2 Construction of the measure

After this long digression, we are ready to write down the measure over supermoduli space in the operator language. Given a point $\tilde{\Sigma} \in \hat{\mathcal{P}}(g, 1)$, let v_i, v_α be the local meromorphic superconformal vectors representing a set of coordinate differentials $\frac{\partial}{\partial m^i}, \frac{\partial}{\partial \tau^\alpha}$ on the tangent space to supermoduli space. Since the projection $\pi : \hat{\mathcal{P}}(g, 1) \rightarrow \widehat{\mathcal{M}}_g$ is many to one, we first write down a candidate for the measure and then make sure that it has the desired properties: let

$$\mu(v_i, v_\alpha; m, \tau) \equiv \langle 0 | \prod_{\alpha=1}^{2g-2} \delta(B(v_\alpha)) \prod_{i=1}^{3g-3} B(v_i) | W \rangle \quad . \quad (5.19)$$

For a point $\tilde{\Sigma}$ away from the split locus, we have in general:

$$\begin{aligned} v_i(\mathbb{z}) &= w_i(z) + \theta \nu_i(z) \\ v_\alpha(\mathbb{z}) &= w_\alpha(z) + \theta \nu_\alpha(z) \end{aligned} \quad (5.20)$$

where ν_i, w_α are odd while w_i, ν_α are even. Then

$$\begin{aligned} B(v_i) &= b(w_i) + \beta(\nu_i) = \oint B(\mathbb{z}) v_i(\mathbb{z}) \\ B(v_\alpha) &= \beta(\nu_\alpha) + b(w_\alpha) = \oint B(\mathbb{z}) v_\alpha(\mathbb{z}) \quad . \end{aligned} \quad (5.21)$$

$|0\rangle$ is the $Osp(2|1)$ invariant vacuum, and $|W\rangle$ is the superstring state for $\tilde{\Sigma}$. We start by proving that (5.19) is a good measure in supermoduli space. The argument simplifies if we use graded δ -functions:

$$\delta^{3g-3|2g-2}(B(v_A)) \equiv \int \prod_{i=1}^{3g-3} d\eta^i \int \prod_{\alpha=1}^{2g-2} \frac{d\sigma^\alpha}{2\pi} \exp \left[i \sum_i \eta^i B(v_i) + i \sum_\alpha \sigma^\alpha B(v_\alpha) \right] \quad (5.22)$$

where the σ^α are even and the η^i are odd variables. For simplicity, we begin with a state $|W\rangle$ defined in $\widehat{\mathcal{P}}(g, 1)$. The first possible ambiguity in (5.19) is in the choice of v_i, v_α up to superconformal vector fields $v_i^\circ, v_\alpha^\circ$ which extend holomorphically off P . Then

$$\begin{aligned} B(v_i) &\rightarrow B(v_i) + B(v_i^\circ) \\ B(v_\alpha) &\rightarrow B(v_\alpha) + B(v_\alpha^\circ) \end{aligned} \quad (5.23)$$

Since $B(v_i^\circ), B(v_\alpha^\circ)$ are conserved charges used in the construction of $|W\rangle$, then $B(v_i^\circ)|W\rangle = B(v_\alpha^\circ)|W\rangle = 0$, and using the representation (5.22) of the δ -function, we find that

$$\mu(v_i, v_\alpha, m, \tau) = \langle 0 | \delta^{3g-3|2g-2}(B(v_A)) | W \rangle \quad (5.24)$$

is independent of (5.23). Next we want to make sure that (5.24) is independent of the local parameter z , and the position of the puncture $P \in \Sigma$. As for the bosonic string, this follows from the properties of the $OSp(2|1)$ invariant vacuum $|0\rangle : L_n|0\rangle = 0, n \geq -1, G_n|0\rangle = 0, n \geq -1/2$, and the equation of motion (5.12) for any v in the tangent space to $\widehat{\mathcal{P}}(g, 1)$ at $\tilde{\Sigma}$. Using (5.12) and the changes $v_i \rightarrow [v^\circ, v_i] + v_i$ where v° generates local reparametrizations and motions of P , we eventually obtain $T(v^\circ)$ acting on $\langle 0|$, using $[T(v), B(v')] = B([v, v'])$. Thus the variation vanishes. Finally, under a change of coordinates in $\widehat{\mathcal{M}}_g$:

$$\begin{aligned} m_i &\rightarrow m'_i = m'_i(m, \tau) \\ \tau_\alpha &\rightarrow \tau'_\alpha = \tau'_\alpha(m, \tau) \end{aligned} \quad (5.25)$$

Then:

$$\begin{aligned} \Pi dm_i \Pi d\tau_\alpha &= BER \begin{pmatrix} \frac{\partial m_i}{\partial m'_j} & \frac{\partial m_i}{\partial \tau'_b} \\ \frac{\partial \tau_\alpha}{\partial m'_j} & \frac{\partial \tau_\alpha}{\partial \tau'_b} \end{pmatrix} \Pi dm'_i \Pi d\tau'_\alpha \\ &\equiv BER(m, \tau / m', \tau') \Pi dm'_i \Pi d\tau'_\alpha \end{aligned} \quad (5.26)$$

Since under (5.25) v_i, v_α change by:

$$\begin{aligned} v'_i &= \frac{\partial m_j}{\partial m'_i} v_j + \frac{\partial \tau_b}{\partial m'_i} v_b \\ v'_\alpha &= \frac{\partial m_j}{\partial \tau'_\alpha} v_j + \frac{\partial \tau_b}{\partial \tau'_\alpha} v_b \end{aligned} \quad (5.27)$$

the standard properties of graded δ -functions imply that μ changes by the inverse Berezinian in (5.26). Hence the measure

$$\mu'(v'_i, v'_\alpha; m', \tau') \Pi dm' \Pi d\tau' = \mu(v_i, v_\alpha; m, \tau) \Pi dm \Pi d\tau \quad (5.28)$$

and this finally proves that (5.19) gives a good measure on $\widehat{\mathcal{M}}_g$ as was to be shown. If we have more punctures, i.e. if we start with $\tilde{\Sigma} \in \widehat{\mathcal{P}}(g, n)$, where the punctures are all NS , the state $|W\rangle$ has ghost number $3g - 3 + 3n/2 - (2g - 2 + n)$. Since $|0\rangle$ has ghost number $-3/2|1$, we can distribute the supermoduli generators among the different punctures; then the argument of the δ -function in (5.19) will involve operators in the Hilbert spaces of different punctures. We now project onto the $OSp(2|1)$ invariant vacua at each puncture: $\prod_{k=1}^n \langle 0|$. Arguments identical to those for the bosonic string [16] show that this measure is identical to (5.19); in other words, we can transfer the moduli deformations generated by meromorphic vectors at different punctures into a single puncture.

Next, we turn to the construction of superstring scattering measures. If we consider a scattering process with p NS and q R states, the scattering amplitude will be defined as a measure on $\widehat{\mathcal{M}}(g; p, q)$, the supermoduli space of SRS with genus g , p NS and q (even) R -punctures. This space has dimension $(3g - 3 + p + q|2g - 2 + p + q/2)$ (see sec. 4). The states are represented by kets in the Hilbert space of the punctures, and the natural candidate for the scattering measure is:

$$\mu_{sc} = {}^R\langle\psi_1|\dots{}^R\langle\psi_q|{}^N\langle\phi_{q+1}|\dots{}^N\langle\phi_{q+p}|\delta^{(3q-3+p+q|2g-2+\frac{q}{2}+p)}(B(v_A))|W, p, q\rangle \quad (5.29)$$

where $|W, p, q\rangle$ is constructed for each point $\tilde{\Sigma} \in \widehat{\mathcal{P}}(g; p, q)$ and v_A give a representation of the basis for the tangent space to $\widehat{\mathcal{M}}(g; p, q)$ at Σ . Once again, there is a projection $\pi : \widehat{\mathcal{P}}(g; p, q) \rightarrow \widehat{\mathcal{M}}(g; p, q)$ with a huge kernel, and our purpose is to show that μ_{sc} is independent of this kernel. The ghost number of $|W, p, q\rangle$ is

$$(3g - 3 + \frac{3}{2}(p + q)| - (2g - 2 + p + q)) \quad .$$

If we assume that the ghost number is the same for any pair of NS states, and also for any pair of R states, and use what we learned in pervious sections, the ghost number of $|\phi_{NS}\rangle$ is necessarily $(-1/2|0)$; and that of $|\psi_R\rangle$ is $(-1/2|-1/2)$. μ_{sc} will define a good measure on $\widehat{\mathcal{M}}(g; p, q)$ if it is independent of the local parameters around the punctures. This is equivalent to requiring the external states to be annihilated by those L_n^{total} and G_n^{total} which generate local reparametrizations. With the ghost numbers prescribed above, the scattering states take the form:

$$|\phi_{NS}\rangle = |\phi_{NS}\rangle_{\text{matter}} \otimes |-\rangle_{bc} \otimes |gh\rangle_{\beta\gamma} \quad (5.30)$$

with

$$\begin{aligned} L_0^{\text{matter}}|\phi_{NS}\rangle_{\text{matter}} &= \frac{1}{2}|\phi_{NS}\rangle_{\text{matter}} \\ L_n|\phi_{NS}\rangle_{\text{matter}} &= 0, \quad G_n|\phi_{NS}\rangle_{\text{matter}} = 0 \quad n > 0 \end{aligned} \quad (5.31)$$

For the Ramond states we obtain:

$$\begin{aligned} |\psi_R\rangle &= |\psi_R\rangle_{\text{matt}} \otimes |-\rangle_{bc} \otimes |-1/2\rangle_{\beta\gamma} \\ |-1/2\rangle_{\beta\gamma} &= e^{-\phi/2}|gh\rangle_{\beta\gamma} \\ L_0^{\text{matter}}|\psi_R\rangle_{\text{matter}} &= \frac{5}{8}|\psi_R\rangle_{\text{matter}} \end{aligned} \quad (5.32)$$

(recall: $L_0^{gh}|-1/2\rangle_{\beta\gamma} = \frac{3}{8}|-1/2\rangle_{\beta\gamma}$; $L_0^{gh}|-\rangle_{bc} = -|-\rangle_{bc}$). The states (5.30), (5.32) can be represented as being created out of the vacuum by vertex operators. It is important to realize that in the operator formalism one is naturally led to the simplest “picture” in the representation of the physical states: The zero picture for NS states, and the $-1/2$ picture for R states. It is usually argued that in the computation of n -point fermionic amplitudes, it is necessary to use different pictures [3]. Before integrating out the odd moduli this is not so, as a consequence of the explicit dependence of the scattering measure on the tangent vectors to $\widehat{\mathcal{M}}(g; p, q)$. Since there are $q/2$ supermoduli due to the R -punctures, we can choose the corresponding v_α ’s to be of the form $1/(z - z_a)$, where z_a $a = 1, \dots, q/2$ are the locations of half of the Ramond states. If we integrate over the corresponding odd directions, this is equivalent to inserting one $T_{z\theta}$ at each z_a . This together with $\delta(\beta(z_a))$ changes the Ramond operator to the $+1/2$ picture, and we end up with $q/2$ Ramond states on the $1/2$ picture, and the other $q/2$ in the $-1/2$ picture. This implies that the use of the formalism with different pictures is related to choosing a particular set of coordinates for $\widehat{\mathcal{M}}(g; p, q)$. This description is useful for example when we look at the factorization limit of scattering amplitudes. Factorization can be analyzed geometrically by pinching the surface Σ into two components. As for bosonic surfaces, pinching can be described in terms of the plumbing fixture joining Σ_1, Σ_2 (see [25]). The 2 and $3/2$ differentials representing the supermoduli of the plumbing construction have poles at the points $P_i \in \Sigma_i$ $i = 1, 2$ where the fixture is attached. The operators absorbing the vacuum charge can be divided into those distributed about Σ_i , and those at the plumbing fixture. The latter picture change the vertex operators (or states) appearing at P_1, P_2 in the factorization limit. To summarize, the operator formalism chooses the “minimal” pictures of the scattering states; and only after we integrate over the odd directions do we obtain scattering states in different pictures.

Using the results of section 2, it is easy now to prove the super holomorphic factorization. Recall that for the bosonic string this is equivalent to saying that

$$\delta\bar{\delta} \log \mu(v_1, \dots, v_{3g-3}, \bar{v}'_1, \dots, \bar{v}'_{3g-3}) = -13\delta\bar{\delta} \log \det \tau_2 \quad (5.33)$$

or equivalently that the measure takes the form:

$$\mu = \rho \wedge \bar{\rho} / (\det \tau_2)^{13} \quad (5.34)$$

Here $v_1, \dots, v_{3g-3}$ and $v'_1, \dots, v'_{3g-3}$ are two bases of holomorphic tangent vectors to $\mathcal{M}_g$ which vary holomorphically, τ_2 is imaginary part of the period matrix of the surface, and ρ is a holomorphic half-volume. The proof of (5.34) in the operator formalism is simply a consequence of the lack of holomorphic factorization (left-right separation) of the matter state for a generic $\tilde{\Sigma} \in \mathcal{P}(g, 1)$. In this case we only need to compute:

$$\frac{\langle 0|T(z)\overline{T(w)}|W\rangle}{\langle 0|W\rangle} - \frac{\langle 0|T(z)|W\rangle}{\langle 0|W\rangle} \frac{\langle 0|\overline{T(w)}|W\rangle}{\langle 0|W\rangle} \quad (5.35)$$

and then fold into the energy momentum tensors the meromorphic fields that generate moduli variations (see [16] for details). Since the matter states are constructed out of the two point function on the surface Σ , the whole contribution to (5.35) comes from its non-holomorphic pieces. These are easy to determine.

In the super case we once again observe that the failure of factorization comes only from the matter sector; all of our ghost manipulations have manifestly separated left- and right-movers.¹⁷ Thus the full contribution to the super version of (5.33) comes from (2.23) and one has

$$\mu = \rho \wedge \bar{\rho} \det(\hat{\tau}_2)^{-5} \quad (5.36)$$

where $\hat{\tau}_2$ is the imaginary part of the super period matrix of sect. 2.

Similarly for heterotic strings one can obtain [36]

$$\mu = \rho_L \wedge \bar{\rho}_R \cdot \det(\hat{\tau} - \bar{\tau}) \quad (5.37)$$

¹⁷ In [28][33][34][35] there are contributions to the holomorphic anomaly from every spin. These papers all use a coordinate invariant, Weyl-noninvariant regularization. Here in contrast we use a Weyl-invariant, coordinate-noninvariant regularization, following [40], [16]. The total result is the same, but the holomorphic anomaly is manifestly all contained in the matter sector.

where $\bar{\tau}$ is the conjugate of the ordinary period matrix.

To complete the analog of [28], we must describe the pole structure of the meromorphic form ρ at the boundary of $\widehat{\mathcal{M}}$. Since this boundary is described in terms of plumbing fixtures [46][25][45] as in the bosonic case, we can now use the sewing description of the superstring state (see sec. 4) to determine the pole structure of the measure. As in the bosonic string, the leading singularity comes from the possible tachyonic states appearing in the sewing operator $|S_{ij}^{\pm}\rangle$ representing the plumbing. In the absence of tachyons, we still have to worry about singularities coming from massless states. These conclusions are obvious from various naive physical arguments, and it is reassuring that in the operator formalism they are so simple to derive. The absence of divergences in superstring perturbation theory is therefore related to the GSO projection (integration on spin moduli space) and the absence of massless tadpoles. When we look at divergences in the partition function, there are two types of nodes that can be generated by pinching closed curves on Σ . We can either generate a splitting node (after pinching the surface falls into two disconnected pieces), or a non-splitting node (when we pinch a homologically non-trivial cycle). In the first case, and in the absence of external states, this will always be a super node (for notation and definitions see [25]), and the boundary behavior is determined by $|NN, \pm\rangle$. In the second case we can either have a super node (and again use $|NN, \pm\rangle$) or a spin node, in which case we use $|RR, \pm\rangle$ to determine the boundary behavior. Thus the divergences (if any) are determined by the tachyons and/or massless states in the spectrum of the superstring theory considered.

In connection with GSO projection, the procedure so far discussed in the literature has been to sum over the spin structures after integrating out the odd coordinates. Given the possible ambiguities in integrating over the odd variables first (see the next section), one could ask whether or not we can perform the GSO projection even *before* integration over the odd variables. It is not clear to us whether the geometry of super Riemann surfaces allows a notion of “change of spin structures.” This notion certainly makes sense for super Riemann surfaces which are split, but seems not to make sense in general. If, however, such a notion made sense, the more appropriate procedure to address the question of the vanishing of the cosmological constant in connection with the GSO projection may be to sum over the spin structure for the measure before integrating out the odd variables.

5.3 Ambiguities

Finally we would like to make some general remarks on the general properties of the measure in the operator formalism, and the connection with the total derivative ambiguities recently found [6][58] [59]. As we have shown, the measure $\mu(v_i, v_\alpha)$ transforms covariantly under general super coordinate transformations. The problem arises when we try to define integration over supermoduli space $\widehat{\mathcal{M}}_g$. If $\widehat{\mathcal{M}}_g$ is a split supermanifold, we can always choose split coordinate transformations, and the integration can be performed unambiguously in two steps: 1) Integrate over the odd directions. 2) Integrate the resulting measure over spin moduli space. For non-split supermanifolds, these two steps generate some ambiguities. To see this, let us compute what happens if we integrate over the odd directions before or after making an infinitesimal coordinate transformation. In particular, consider a change of the form:

$$\begin{aligned} m^i &\rightarrow m^i + \delta m^i = m^i + \sum_{p=0}^{g-1} \frac{1}{(2p)!} h_{a_1 \dots a_{2p}}^i(m) \tau_{a_1} \dots \tau_{a_{2p}} \\ \tau^a &\rightarrow \tau^a + \delta \tau^a = \tau^a + \sum_{p=0}^{g-1} \frac{1}{(2p-1)!} g_{a_1 \dots a_{2p-1}}^a(m) \tau_{a_1} \dots \tau_{a_{2p-1}} \end{aligned} \quad (5.38)$$

Even though (5.38) may seem very special, there is no loss of generality, because the transition functions in supermanifolds can always be written as the composition of a split transformation and a transformation of the form (5.38) [60]. Since the measure can be expanded locally as a polynomial in $\tau_1 \dots \tau_{2g-2}$:

$$\mu(m, \tau) = \sum_{p=0}^{g-1} \frac{1}{(2p)!} \mu_{a_1 \dots a_{2p}}(m) \tau_{a_1} \dots \tau_{a_{2p}} \quad (5.39)$$

(μ, h, g are antisymmetric in their lower indices). The berezinian (superjacobian) of (5.38) is:

$$\text{BER} \approx 1 + \frac{\partial}{\partial m^i} \delta m^i - \frac{\partial}{\partial \tau^a} \delta \tau^a \quad (5.40)$$

and:

$$\mu(m + \delta m, \tau + \delta \tau) = \mu(m, \tau) + \delta m^i \frac{\partial}{\partial m^i} \mu(m, \tau) + \delta \tau^a \frac{\partial}{\partial \tau^a} \mu(m, \tau) \quad (5.41)$$

Then:

$$\begin{aligned} \int dm d\tau \mu(m, \tau) &\rightarrow \int dm d\tau \text{BER} \mu(m + \delta m, \tau + \delta \tau) \\ &= \int dm d\tau [\mu(m, \tau) + \frac{\partial}{\partial m^i} (\delta m^i \mu(m, \tau)) - \frac{\partial}{\partial \tau^a} (\delta \tau^a \mu(m, \tau))]. \end{aligned} \quad (5.42)$$

Hence the change in the measure integrated over the odd direction is the top term of $\partial_i (\delta m^i \mu(m, \tau))$, or

$$\partial_i \left[\epsilon_{a_1 \dots a_{2g-2}} \sum_{k=0}^{g-1} \frac{\mu_{a_1 \dots a_{2k}} h^i_{a_{2k+1} \dots a_{2g-2}}}{(2k)!(2g-2-2k)} \right], \quad (5.43)$$

a total derivative. Note that in deriving (5.43) the last term on the right hand side of (5.42) does not contribute; this is because total derivatives in the odd directions trivially integrate to zero.

The total derivative (5.43) reflects the possible non-split character of the supermanifold considered. If supermoduli space were a projected supermanifold, the “boundary” terms in (5.43) could be made to vanish by an appropriate choice of coordinate transition functions. A particular example of (5.43) has been discussed in [6][58], where it is shown that a change in the choice of v_a is equivalent to a non-split change of coordinates similar to (5.38). Geometrically the $h^i_{a_1 \dots a_{2p}}$ represent the obstruction to splitness, and they can be characterized in terms of Čech cohomology groups. If $\text{Spin } \mathcal{M}_g$ is the spin moduli space for genus g , we can construct a vector bundle of rank $2g-2$ over it with fiber $H^1(K^{-1/2})$ (K is the canonical line bundle for the surface parametrized by a point in $\text{Spin } \mathcal{M}_g$). The tangent space to $\text{Spin } \mathcal{M}_g$ at m is isomorphic to $H^1(K^{-1})$. Since $h^i_{a_1 a_2 \dots a_{2p}}$ is defined in the overlap between two patches, then the obstruction $h^i_{a_1 \dots a_{2p}}$ is naturally an element of the group $H^1(\text{Spin } \mathcal{M}_g, H^1(K^{-1}) \otimes \wedge^{2p} H^1(K^{-1/2}))$ where $\wedge^{2p}$ is the $2p$ -th exterior power of the vector space $H^1(K^{-1/2})$. If these groups for $p = 1, \dots, g-1$ are trivial, all 1-cocycles can be written as coboundaries and we can write the transition functions in the form $m' = f(m) \quad \tau'_a = h_{ab}(m) \tau_b + \dots$. We should warn the reader that even if the ordinary supermoduli space were split there may be obstructions to splitness (or projectedness) for the compactified supermoduli space (adding to supermoduli space surfaces with nodes), and these are quite relevant in the analysis of string infinities, whose origin always lie in the boundary of moduli spaces of surfaces with punctures. The boundary contributions are related to tachyon poles and dilation tadpoles [48].

Some of the issues concerning the total derivatives (5.43) can be understood geometrically. In [60] the general theory of integration over a supermanifold $\widehat{\mathcal{M}}$ of discussion $p|q$ is analyzed. The most important result in [60] is establishing the exact sequence of sheaves

$$0 \rightarrow (\Omega^p \otimes \mathcal{D})^+ \hookrightarrow \Omega^p \otimes \mathcal{D} \rightarrow \text{BER} \rightarrow 0 \quad . \quad (5.44)$$

BER is the sheaf of Berezin forms (i.e. the volume forms on $\widehat{\mathcal{M}}$). If $\mathcal{M}$ is the body of $\widehat{\mathcal{M}}$, Ω^p are p -forms on $\mathcal{M}$. $\mathcal{D}$ is the sheaf of linear differential operators on $\widehat{\mathcal{M}}$. For sections of $\Omega^p \otimes \mathcal{D}$ the integration is well defined. We simply integrate first over the odd directions, and this gives a p -form which can be integrated over $\mathcal{M}$. The subsheaf $(\Omega^p \otimes \mathcal{D})^+$ consists of those elements of $\Omega^p \otimes \mathcal{D}$ whose integral over the odd directions produce an exact p -form. The measure gives an element in $H^0(\text{BER})$, and using the long exact sequence associated to the short sequence (5.44), we can identify the mismatch between the integrated measures $\tilde{\mu}_\alpha = \int d\tau_\alpha \mu_\alpha(m_\alpha, \tau_\alpha)$ $\alpha = 1, 2$ for two overlapping patches U_1, U_2 in terms of an element in $H^1((\Omega^p \otimes \mathcal{D})^+)$, the obstruction cocycle to splitting (5.44). Even when the latter obstruction vanishes, the value of the integral is ambiguous by a total divergence, again from the long exact sequence of (5.44).

There is a more physical way of understanding the total derivative ambiguities due to non-split coordinate transformations [6][58]. This is related to the cohomology of the BRST operator Q . For illustrative purposes, we consider the measure in the picture changing gauge [6]. Locally on $\widehat{\mathcal{M}}$ one can choose tangent vector fields such that

$$\frac{\partial}{\partial \tau^a} v_b = \frac{\partial}{\partial \tau^a} v_i = 0 \quad a, b = 1, \dots, 2g-2; \quad i = 1, \dots, 3g-3 \quad (5.45)$$

and so $\frac{\partial v^b}{\partial m^i} = 0$. The measure then takes the form:

$$\mu(m, \tau) = \langle 0 | \prod_{a=1}^{2g-2} \delta(B(v_a)) \prod_{i=1}^{3g-3} B(v_i) | W \rangle \quad . \quad (5.46)$$

Integrating over the odd directions, one obtains:

$$\tilde{\mu}(m) = \int d\tau \mu(m, \tau) = \langle 0 | \prod_{i=1}^{3g-3} B(v_i) \prod_{a=1}^{2g-2} Y(v_a) | W \rangle_{|\tau_a=0} \quad (5.47)$$

where $Y(v_a) = \delta(B(v_a))T(v_a)$ is the picture changing operator [3][6].¹⁸ In this gauge it is particularly easy to obtain the change in the measure. If we want to find the change of $\tilde{\mu}$ under a small change of $v_a \rightarrow v_a + \delta v_a$, a straightforward computation yields [6][58]:

$$\delta\tilde{\mu}(m) = \sum_j (-1)^j \frac{\partial}{\partial m^j} \langle 0 | \prod_{i \neq j} B(v_i) \sum_a \delta_a H(\beta(v_a)) \prod_{b \neq a} Y(v_b) | W \rangle_{|\tau=0} \quad (5.48)$$

where $H(x)$ is the Heaviside function, defined via contour integration, and δ_a stands for the variation $v_a \rightarrow v_a + \delta v_a$. The variation (5.48) can be rewritten as:

$$\delta\tilde{\mu}(m) = \langle 0 | \prod_i B(v_i) Q_{BRST} \sum_a \delta_a H(\beta(v_a)) \prod_{b \neq a} Y(v_b) | W \rangle_{|\tau=0} \quad (5.49)$$

The picture changing gauge can be realized for example by choosing $2g - 2$ points $z_a, a = 1 \dots 2g - 2$ and $-1/2$ differentials $1/(z - z_a)$. Since $\bar{\partial}(z - z_a)^{-1} = \pi\delta(z - z_a)$ in the functional integral language, this is equivalent to choosing δ -function gravitinos. The equality of (5.48) and (5.49) is a consequence of the defining properties (5.45) of the gauge used.

The total derivative (5.48) is closely related to another problem pointed out in [6]: the spurious poles in correlation functions involving supercurrents. The origin of this problem is the bosonic nature of the superconformal ghost. Bose statistics forces the $\beta - \gamma$ correlation functions to have a ϑ -function in the denominator, which according to the Riemann vanishing theorem will have g zeroes and thus g new poles in the correlation functions. These poles are called spurious because they are not predicted by the operator product expansion. In the picture changing gauge with $v_a = 1/(z - z_a)$, the possibility of avoiding these poles is equivalent to the independence of the scattering measures of the position of the z_a 's. Consider for instance the propagation of a NS state across a genus g surface. The measure is:

$${}_1\langle N | {}_2\langle N | \prod_{a=1}^{2g-2+2} \delta(B(v_a)) \prod_{i=1}^{3g-3+2} B(v_i) | W \rangle \quad (5.50)$$

In the picture changing gauge, $v_a = 1/(z - z_a), a = 1, \dots, 2g - 2 + 2$ and we can choose two of the insertion points to coincide with the positions x_1, x_2 of the external states. After integrating over the odd directions, in particular, the supermoduli associated to the two

¹⁸ Actually the Y which emerges from integrating (5.46) is not BRST-invariant due to normal ordering problems. This point is addressed in [61].

punctures x_1, x_2 , we change the external states from the picture 0 to the picture 1. The spurious poles in (5.50) are given by the position of the zeros of $\vartheta(x_1 - x_2 + \sum z_a - 2\Delta)$ (Δ is the Riemann class of the surface). To avoid these poles would require the full amplitude to be independent of the position of the z_n 's. However, changing the z_n 's is equivalent to changing the $-1/2$ differentials v_a , and this leads again to the total derivative (5.48). In [58] it was shown that changing $z_a \rightarrow z_a + \Delta z_a$ is equivalent to a non-split transformation. This can easily be shown using the Beilinson decomposition [26] of local meromorphic fields in the operator language. For completeness, we outline the argument. The reader should have little difficulty in filling in the algebraic details. For $g = 2$, and in the picture changing gauge, let $\tilde{v}_1, \tilde{v}_2$ be the supermoduli deformations, and v_1, v_2, v_3 the moduli deformations. If we only change $\tilde{v}_1 \rightarrow \tilde{v}_1 + \delta\tilde{v}_1$, (5.48) gives:

$$\begin{aligned} \delta\tilde{\mu} &= \sum_{i=1}^3 (-1)^i \langle 0 | \delta(B(\tilde{v}_2)) T(\tilde{v}_2) \delta(B(\tilde{v}_1)) B(\delta\tilde{v}_1) \prod_{j \neq i} B(v_j) | W \rangle_{\tau=0} \\ &= \frac{\partial}{\partial m^1} \langle 0 | \delta(B(\tilde{v}_2)) T(\tilde{v}_2) \delta(B(\tilde{v}_1)) B(\delta\tilde{v}_1) B(v_2) B(v_3) | W \rangle_{\tau=0} + \dots \end{aligned} \quad (5.51)$$

(we are ignoring throughout the antiholomorphic fields). The last term in (5.51) can be written as:

$$\begin{aligned} &\frac{\partial}{\partial m^1} \langle 0 | \delta(B(\tilde{v}_2)) B([\tilde{v}_2, \delta\tilde{v}_1]) \delta(B(\tilde{v}_1)) B(v_2) B(v_3) | W \rangle_{\tau=0} \\ &+ \frac{\partial}{\partial m^1} \langle 0 | \delta(B(\tilde{v}_2)) B(\delta\tilde{v}_1) T(\tilde{v}_2) \delta(B(\tilde{v}_1)) B(v_2) B(v_3) | W \rangle_{\tau=0} . \end{aligned} \quad (5.52)$$

The second term vanishes by the following argument. First select a basis for the meromorphic differentials $\tilde{v}_a$ (moduli deformations, reparametrizations and the Borel part: those extending holomorphically away from the points selected on the surface), and a dual basis of $3/2$ differentials. The pieces of $B(\delta\tilde{v}_1)$ projecting onto moduli deformations vanish due to the δ -functions, and the terms projecting into the Borel part or reparametrizations vanish as a consequence of the independence of the measure with respect to coordinate changes and choice of representatives for $\tilde{v}_a, v_i$. With respect to the first term in (5.52), we first note that $\tilde{v}_a(z) = \theta w_a(z)$ in this gauge, and that $[\tilde{v}, \tilde{v}'] = \tilde{v} \partial \tilde{v}' - \tilde{v}' \partial \tilde{v} + \frac{1}{2} D\tilde{v} D\tilde{v}' = ww'$ is an ordinary -1 differential. We now apply this to $[\tilde{v}_2, \delta\tilde{v}_1]$. Using the Beilinson decomposition again, and using the invariances of the measure, we find that only the projection onto even moduli deformations gives a non-vanishing contribution. In particular, since

(5.52) contains the insertion of $B(v_2), B(v_3)$, only the projection onto v_1 is relevant. Using a basis of quadratic differentials dual to the Beilinson decomposition, (5.52) becomes:

$$\frac{\partial}{\partial m^1} (h^1 \langle 0 | \delta(B(\tilde{v}_1)) \delta(B(\tilde{v}_2)) B(v_1) B(v_2) B(v_3) | W \rangle |_{\tau=0})$$

$$h^1 = \langle S_1, [\tilde{v}_2, \delta \tilde{v}_1] \rangle \quad . \quad (5.53)$$

S_1 is the differential dual to v_1 , and the bracket $\langle, \rangle$ is the standard contour integration around the point(s) selected on the surface. h is the tensor appearing in (5.38).

We now have several ways of understanding the change (5.48) of the measure integrated over the odd moduli with respect to changes in the super-Beltrami differentials: Non-split coordinate transformations, spurious poles, and BRST changes in the state $|W\rangle$. The last interpretation is perhaps the most appealing. The states $|W\rangle$ we have constructed for superstring theories are naturally BRST invariant. This can be shown either by building up $|W\rangle$ using the sewing operation, the charge transport argument and the BRST invariance of the 2 and 3 point vertices; or directly by writing

$$Q = \oint C(\mathbb{z}) (T_{\text{matt}}(\mathbb{z}) + \frac{1}{2} T_{gh}(\mathbb{z})) \quad (5.54)$$

and using the Beilinson decomposition for $T(\mathbb{z})$ and $C(\mathbb{z})$. For a basis of vector superfields v_N and its dual S_M :

$$\oint v_N S_M = \delta_{N,M} \quad (5.55)$$

we can write

$$\oint : C(\mathbb{z}) T(\mathbb{z}) := \sum_n : C(S_N) T(v_N) : \quad (5.56)$$

and using the conserved charges defining the state together with conformal invariance, it is easy to show that $Q|W\rangle = 0$. What the representation (5.49) of the total derivative ambiguity suggests is that if we want to have a measure free of ambiguities it should not matter whether we compute scattering amplitudes with the state $|W\rangle$, or any other state in the same BRST cohomology class $\{|W\rangle + Q|\chi\rangle\}$, where the state $|\chi\rangle$ apart from having the correct ghost number, should satisfy the "equation of motion" i.e. its variation under moduli and coordinate deformations should be generated by the stress-energy tensor. Since the measures for states $Q|\chi\rangle$ satisfying the equations of motion are total derivatives, the independence of the measure under motions of $|W\rangle$ in its BRST cohomology class is equivalent to the full decoupling of BRST spurious states, and in particular, the vanishing of tadpole diagrams, necessary for a consistent unitary string theory. This should also tie nicely the cohomology of Q with values in the Fock space (5.1) for states obtained at genus g , with the cohomology of the supermoduli space of genus g super Riemann surfaces. Preliminary results seem to support the picture just described [62].

6. Conclusions

We have seen how the operator formalism incorporates the geometry of superconformal field theories. In this language the measure of string theory is naturally defined in terms of arbitrary tangents to supermoduli space, and not just the specific coordinate differentials usually used in the path-integral approach. This and other simplifications permit very straightforward analysis of the main features of string amplitudes, including holomorphic factorization, sewing, and the decoupling of BRST spurious states, as well as the “ambiguity” in the integral over odd moduli. In particular it is clear that there is no ambiguity in the measure itself. On the other hand, even when supermoduli space splits it seems likely that the physical requirement of BRST decoupling will be crucial for string amplitudes and their factorization to be well defined [63].

Many interesting avenues remain to be explored. We expect that even if the exact sequence (5.44) does not split, still the obstruction to integrating the specific measures arising in string theory will vanish on shell by BRST invariance, consistent with [48]. There are also a number of interesting explicit calculations which can be efficiently performed using the analysis in this paper. More importantly, once states are characterized by algebras of charges one can abstract away from the path integral to more general superconformal field theories. From there one can hope that the formulation of superstrings given here will ultimately point to a good nonperturbative formulation of string theory.

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Note Added

After this work was completed we received some other papers dealing with sewing, integration over the supermoduli, and the matter state: [64] [65][66][67].

Appendix A.

In this appendix we want to interpret the results of sec. 3 in the framework of the infinite grassmannian of Segal and Wilson [68]. We follow the notation of [12][16]. This will also be useful to illustrate some of the properties of the gaussian representation of the $\beta - \gamma$ system. In the fermionic case the so-called Plücker map associates to every point in the grassmannian $Gr(H)$ an element in the fermionic Fock space $\mathcal{F}^F$ which is identified with the space of sections of the dual of the determinant line bundle, $DET^\vee$, over $Gr(H)$. Recall that given $H = L^2(S^1, \mathbb{C})$, we split $H = H_+ \oplus H_-$ where $H_+ = \{z^{-1}, z^{-2}, \dots\}$, $H_- = \{1, z, z^2, \dots\}$. H_+ represents the negative energy wave functions and H_- includes those with positive energy. A generic point $W \in Gr$ is a subspace $W \in H$ such that the orthogonal projections $pr_+ : W \rightarrow H_+$ is a Fredholm operator, and $pr_- : W \rightarrow H_-$ is a Hilbert-Schmidt operator. Physically this can be interpreted as being equivalent respectively to the requirement of finite charge and finite expectation value of the energy-momentum tensor in the state $|W\rangle$ associated to $W \in Gr(H)$. The grassmannian can be represented in a number of ways as an infinite dimensional homogeneous space; in particular $Gr(H) = U_{res}(H)/U(H_+) \times U(H_-)$, where $U_{res}(H)$ is the group of unitary operators on H preserving the conditions imposed on the subspaces $W \in Gr(H)$, and $U(H_\pm)$ are the unitary operators acting on $H_\pm$. Equivalently, if $GL_{res}(H)$ is the group of linear invertible operators preserving the conditions on W and U^+ is the subgroup preserving H_+ (operators of upper triangular form with respect to the splitting $H = H_+ \oplus H_-$), then $Gr(H) = GL_{res}(H)/U^+$. In the fermionic case, the SL_2 invariant vacuum is obtained by constructing the highest wedge product of a basis of H_+ . Given a point $w \in Gr(H)$ define its dual subspace $\tilde{W}$, via the bilinear pairing $\langle w, \tilde{w} \rangle = \oint w \tilde{w} = 0$. We can construct the associated conserved charges

$$\begin{aligned} Q_n &= \oint_0 dz \beta(z) w_n(z) \\ \tilde{Q}_n &= \oint_0 dz \gamma(z) \tilde{w}_n(z) \end{aligned} \quad (A.1)$$

In the fermionic case, the section of $DET^\vee$ associated to W is just $\prod_n Q_n \prod_n \tilde{Q}_n$ (this infinite determinant is defined only if the bases $w_n, \tilde{w}_n$ are admissible i.e. if we represent W as a map $w_+ \oplus w_- : H_+ \rightarrow H_+ \oplus H_-$, then w_+ should differ from the identity by a trace class operator). As in the fermionic case, the state Φ_w in the bosonic Fock space $\mathcal{F}^B$ is defined by the equations

$$Q_n \Phi_w = \tilde{Q}_n \Phi_w = 0 \quad (A.2)$$

Thus for example, the q -states (3.36), correspond to the following elements in $Gr(H)$:

$$\begin{aligned} W &= \{z^{Q+q}z^{-1}, z^{Q+q}z^{-2}, \dots\} \\ \widetilde{W} &= \{z^{-(Q+q)}z^{-2}, z^{-(Q+q)}z^{-3}, \dots\} \end{aligned} \quad (A.3)$$

We now show that the solutions to (A.2) are generalized gaussians. Since $U_{res}(H)$ acts transitively on $Gr(H)$, we can represent any point W as follows:

$$\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \rightarrow \begin{pmatrix} w \\ w^\perp \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \quad (A.4)$$

(A,D Fredholm, B,C Hilbert-Schmidt). In components:

$$\begin{aligned} w_n &= \sum_{m=1}^{\infty} (A_{nm}z^{-m} + B_{nm}z^{m-1}) \\ w_n^\perp &= \sum_{m=1}^{\infty} (C_{nm}z^{-m} + D_{nm}z^{m-1}) \end{aligned} \quad (A.5)$$

The orthogonality between $w, w^\perp$ implies:

$$AC^\dagger + BD^\dagger = 0 \quad (A.6)$$

Given the orthogonal basis $\{w_n^\perp\}$ of $W^\perp$, it is easy to check that the functions $\tilde{w}_n = z^{-1}(w_n^\perp)^*$ generate a basis of the dual subspace $\widetilde{W}$:

$$\tilde{w}_n = \sum_{m=1}^{\infty} (D_{nm}^*z^{-m} + C_{nm}^*z^{m-1}) \quad (A.7)$$

Now the charges associated to $W, \widetilde{W}$ are:

$$\begin{aligned} Q_n &= \sum_{m=1}^{\infty} (A_{nm}\beta_{m+j-1} + B_{nm}\beta_{-m+j}) \\ \tilde{Q}_n &= \sum_{m=1}^{\infty} (D_{nm}^*\gamma_{m-j} + C_{nm}^*\gamma_{-m-j+1}) \end{aligned} \quad (A.8)$$

and they commute because $\oint w_n \tilde{w}_n = 0$. In $\mathcal{F}^B$, the equations for Φ_W with half-odd integer spin j are

$$\left(-iJ\tilde{g}\Omega^{j-1/2}x + \tilde{g}\Omega^{j-1/2}\frac{\partial}{\partial x} \right) \Phi_{\tilde{g}} = 0 \quad (A.9)$$

where

$$\begin{aligned} \tilde{g} &= \begin{pmatrix} D^* & C^* \\ B & A \end{pmatrix} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Omega &= \begin{pmatrix} 0 & & & 1 & & & \\ 1 & 0 & & & 0 & & \\ & 1 & 0 & & & 0 & \\ & & 1 & \ddots & & & \ddots \\ 0 & & & & 0 & 1 & \\ & 0 & & & & 0 & 1 \\ & & 0 & & & & 0 & 1 \\ & & & 0 & & & & \ddots \end{pmatrix} \\ x &= \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \begin{bmatrix} x_{1/2} \\ x_{3/2} \\ \vdots \\ x_{-1/2} \\ x_{-3/2} \\ \vdots \end{bmatrix}, \quad \partial_x = \begin{pmatrix} \partial_+ \\ \partial_- \end{pmatrix}. \end{aligned} \quad (A.10)$$

The effect of Ω is to shift the "polarization" of the x 's; for example

$$\Omega \begin{bmatrix} x_{1/2} \\ x_{3/2} \\ \vdots \\ x_{-1/2} \\ x_{-3/2} \\ \vdots \end{bmatrix} = \begin{bmatrix} x_{-1/2} \\ x_{1/2} \\ \vdots \\ x_{-3/2} \\ x_{-5/2} \\ \vdots \end{bmatrix}. \quad (A.11)$$

If j is integral, we get an equation similar to (A.9), but with the subscript of x_n taking on integer values.

Since the matrix $\tilde{g}$ is invertible, it is possible to show that $\Omega^{\frac{1}{2}-j} \tilde{g}^{-1} J \tilde{g} \Omega^{j-\frac{1}{2}}$ is symmetric (a consequence of (A.6)), and then

$$\Phi_{\tilde{g}} = N \exp\left(\frac{i}{2} x^T \Omega^{\frac{1}{2}-j} \tilde{g}^{-1} J \tilde{g} \Omega^{j-\frac{1}{2}} x\right) \quad (A.12)$$

where N is a normalization constant. Finally, to establish that (A.12) is a ray associated to a point in $Gr(H)$, we have to show that $\Phi_{\tilde{g}}$ is invariant under the stability subgroup of H_+ , namely $U(H_+) \times U(H_-)$. This follows from:

$$g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \leftrightarrow \Phi_{\tilde{g}} = \Phi_1 \quad (A.13)$$

(because J commutes with $U(H_+) \times U(H_-)$), and if $\Phi_{g_1} = \Phi_{g_2}$ as rays, then one can easily show that $g_2 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} g_1$.

As an example, consider W to be a Bogoliubov transformation of H_+ and $j = 1/2$. Then:

$$\begin{aligned} Q_n &= B_{n-1/2} + \sum_{m=1}^{\infty} B_{nm} B_{-m+1/2} \\ \tilde{Q}_n &= \gamma_{n-1/2} - \sum_{m=1}^{\infty} B_{mn} \gamma_{-m+1/2} \end{aligned} \quad (A.14)$$

This corresponds to a $g \in GL_{res}(H)$ given by:

$$g_B = \begin{pmatrix} 1 & B \\ -B^\dagger & 1 \end{pmatrix} \quad (A.15)$$

so that

$$\tilde{g}_B = \begin{pmatrix} 1 & -B^T \\ B & 1 \end{pmatrix} \quad (A.16)$$

and if $\det \tilde{g}_B = \det(1 + BB^T) \neq 0$, its inverse is given by

$$\tilde{g}_B^{-1} = \begin{pmatrix} (1 + B^T B)^{-1} & (1 + B^T B)^{-1} B^T \\ -(1 + BB^T)^{-1} B & (1 + BB^T)^{-1} \end{pmatrix} \quad (A.17)$$

From (A.12) we obtain:

$$\Phi_B = \frac{1}{\det^{1/2}(1 + BB^T)} \exp \frac{i}{2} x^T \tilde{g}_B^{-1} J \tilde{g}_B x \quad (A.18)$$

and the normalization constant is chosen such that

$$\begin{aligned} \langle 0 | \Phi_B \rangle &= \int dx \Phi_0^* \Phi_B = 1 \\ \Phi_0^{(j=1/2)} &= \exp \frac{i}{2} \sum_{n=1}^{\infty} (x_{n-1/2}^2 - x_{-n+1/2}^2) \end{aligned} \quad (A.19)$$

The solution (A.18) can also be represented in the oscillator basis by:

$$\Phi_B = \exp \left(\sum_{n,m=1}^{\infty} B_{nm} \gamma_{-n+1/2} \beta_{-m+1/2} \right) |0\rangle \quad (A.20)$$

$$\beta_n |0\rangle = \gamma_n |0\rangle = 0 \quad n > 0$$

In the same way that the state Φ_W^F that one associates to the point W in the fermionic construction is a section of DET^* , it is easy to show that Φ_W^B is a section of the DET line

bundle over $Gr(H)$, albeit non-holomorphic. Similarly one can associate fermionic and bosonic states to those points on $Gr(H)$ which correspond to Dirac partitions (see [68] [12] for details). In the fermionic construction these states are obtained from $|H_+\rangle$ by the action of local operators (polynomials in the fermionic oscillators), whereas in the bosonic construction we have to use the δ -function operators introduced in sec. 3.

Similarly we have a bosonic Krichever map. It is known from the theory of the K-P hierarchy (see [68] for details) that to the geometrical data $(\Sigma, L, P, z, \sigma)$ we can associate a point in $Gr(H)$. The data consist of a Riemann surface Σ , a line bundle L over it, a point $P \in \Sigma$, a local parameter $z, z(P) = 0$, and a trivializing section σ of L over a neighborhood of P ; to it there corresponds $W = H^0(\Sigma - P, L) \equiv \{\text{sections of } L \text{ holomorphic off } P \text{ with poles only at } P\}$. By the Riemann-Roch Theorem it is easy to show that W is a point in $Gr(H)$ with virtual dimension $\deg L - g + 1$ i.e. index $pr_+ = \deg L - g + 1$. On top of the grassmannian we now consider the Fock spaces $\mathcal{F}^F = \Gamma(\text{DET}^-)$ and $\mathcal{F}^B = \Gamma(\text{DET}^+)$. The conserved charges associated to W and $\widetilde{W}$ now determine a unique ray on these Fock spaces for b, c or $\beta - \gamma$ systems respectively. The group $\text{Diff}(S^1)$ is naturally embedded in GL_{res} , and this action can be lifted projectively to $\mathcal{F}^F$ and $\mathcal{F}^B$, so that the central extension of the Virasoro algebra is obtained by restricting the extension of GL_{res} to $\text{Diff}(S^1)$. This provides an outline for a more geometrical proof of the basic "equation of motion."

$$\delta_v \Phi_{W(\Sigma, P, L, z)} = T(v) \Phi_{W(\Sigma, P, L, z)} \quad (A.21)$$

where v specifies a vector tangent to the moduli space at (Σ, P, z) , and the central extension of $T(v)$ obtained this way agrees with that obtained via ordinary oscillator methods.

Finally, the operators e^ϕ (or $\delta(\beta)$) introduced in section 3 can be understood in algebraic geometric terms. In the fermionic case, we start with a line bundle of spin j over the surface Σ . By Riemann-Roch has $(2j - 1)(g - 1)$ holomorphic sections ($j > 1$), hence when we compute correlation functions, we soak up the zero modes by inserting $N_j = (2j - 1)(g - 1)$ fields $b(z)$ at N_j points $P_1, \dots, P_{n_j}$. This can be thought of as computing first the Krichever state associated to the line bundle $\xi(-\sum_{i=1}^{n_j} P_i)$ with degree $g - 1$. If the points $\{P_i\}$ are suitably chosen, this new line bundle has no holomorphic sections, and we can now compute the correlation functions for the $b - c$ system. In the bosonic case we cannot change the line bundle $\xi \rightarrow \xi(-\sum_{i=1}^{n_j} P_i)$ by means of local operators. We can nevertheless define operators $e^{\phi(P_i)}$ such that:

$$\prod_{i=1}^{N_j} e^{\phi(P_i)} \Phi_{W(\Sigma, \xi)} \equiv \Phi_{W(\Sigma, \xi(-\sum P_i))} \quad (A.21)$$

If we now compute correlation functions using the grassmannian construction of the bosonic Fock state $\Phi_{W(\Sigma, \xi(-\sum P))}$, we obtain that the correlation functions of the e^ϕ operators agree with the results obtained in sec. 3 (see eqn. (3.60)).

Appendix B.

In sec. 4 we set up the calculation of the state $|RRN\rangle$. The calculation is straightforward if we choose a somewhat unorthodox set of local coordinates. Let u_i, θ_i be double-valued "coordinates" near spin punctures, $i = 1, 2$ and z_3, θ_3 at the super puncture. One can then take (see Fig. 4)

$$\begin{aligned} u_2 &= tu_1^{-1} \\ \theta_2 &= \epsilon \theta_1 tu_1^{-2} \\ z_3 &= -\log z_1 \\ \theta_3 &= \epsilon' \theta_1 u_1^{-1} \end{aligned} \tag{B.1}$$

where ϵ, ϵ' are relative $\pm$ signs which can be chosen at will, and t is a scale parameter which does not define any moduli in this case. In these coordinates the vector fields (4.24) do not extend to single-valued fields on the sphere. Had we started with more traditional coordinates $\tilde{z}_3 = z_1 - 1$, however, we would have gotten good vector fields; moreover the latter are all linear combinations of the ones considered here. Hence imposing that the charges of either set all vanish is equivalent to imposing the other, so we may as well use (B.1). We simply remember that P_1 and P_2 are logarithmic branch points, and the local coordinates $z_n (k = 1, 2)$ are given by

$$\begin{aligned} z_k &= r_k e^{i\theta_k} & \theta_k &\in (-\infty, +\infty) \\ & & r_k &\in (0, 1) \end{aligned} \tag{B.2}$$

Since $\theta_k \in (-\infty, +\infty)$, we have in fact an infinite number of sheets. A closed path C_k around P_k goes through all the sheets.

The conserved charges for the b, c part of the state are now:

$$b_n^{(1)} - b_{-n}^{(2)} - \sum_{m \geq -1} \frac{n^{m+1}}{(m+1)!} b_{-m}^{(3)} \quad n \geq 1 \tag{B.3}$$

$$b_n^{(2)} - b_{-n}^{(1)} + \sum_{m \geq -1} \frac{n^{m+1}}{(m+1)!} (-1)^{m+1} b_{-m}^{(3)} \quad n \geq 0 \tag{B.4}$$

$$b_n^{(3)} - \delta_{n,2} \left(\frac{1}{2} b_1^{(3)} + b_0^{(1)} \right) - \sum_{m \geq 1} b_{-m}^{(1)} \frac{m^{n-2}}{(n-2)!} - \sum_{m \geq 1} b_{-m}^{(2)} \frac{m^{n-2}}{(n-2)!} (-1)^n \quad (B.5)$$

These charges annihilate the state:

$$\begin{aligned} |RRN\rangle_{bc} = & \exp \left(\sum_{n \geq 1} c_{-n}^{(1)} b_{-n}^{(2)} + \sum_{n \geq 0} c_{-n}^{(2)} b_{-n}^{(1)} + \frac{1}{2} c_{-2}^{(3)} b_1^{(3)} + \right. \\ & + \sum_{\substack{m \geq 1 \\ n \geq 1}} c_{-n}^{(1)} b_{-m}^{(3)} \frac{n^{m+1}}{(m+1)!} + c_{-2}^{(3)} b_0^{(1)} + \sum_{\substack{m \geq 1 \\ n \geq 1}} c_{-m}^{(3)} b_{-n}^{(1)} \frac{m^{n-2}}{(n-2)!} + \\ & \left. + \sum_{\substack{m \geq 2 \\ n \geq 1}} c_{-n}^{(3)} b_{-m}^{(2)} \frac{m^{m-2}}{(n-2)!} (-1)^n \right) |+\rangle_1 |-\rangle_2 c_{-1}^{(3)} c_0^{(3)} |-\rangle_3 . \end{aligned} \quad (B.6)$$

For the $\beta - \gamma$ system, the conserved charges are:

$$\beta_n^{(1)} + \epsilon(-1)^{1/2} \beta_{-n}^{(2)} + \epsilon'(-1)^{1/2} \sum_{m \geq 0} \frac{n^m}{m!} \beta_{-m+1/2}^{(3)} \quad n \geq 0 \quad (B.7)$$

$$\beta_n^{(2)} + \epsilon(-1)^{-1/2} \beta_{-n}^{(1)} + \epsilon\epsilon' \sum_{n \geq 0} \frac{n^m}{m!} (-1)^m \beta_{-m+1/2}^{(3)} \quad n \geq 1 \quad (B.8)$$

$$\begin{aligned} \beta_{n+1/2}^{(3)} - \frac{1}{2} \delta_{n,1} \beta_{1/2}^{(3)} - \epsilon\epsilon' \delta_{n,1} \beta_0^{(1)} - \epsilon'(-1)^{-3/2} \sum_{m \geq 1} \beta_{-m}^{(1)} \frac{m^{n-1}}{(n-1)!} + \\ + \epsilon\epsilon'(-1)^n \sum_{m \geq 1} \beta_{-m}^{(2)} \frac{m^{n-1}}{(n-1)!} \quad n \geq 1 \end{aligned} \quad (B.9)$$

which annihilate the state:

$$\begin{aligned} |RRN\rangle_{\beta\gamma} = & \exp \left[\sum_{n \geq 0} \gamma_{-n}^{(1)} \beta_{-n}^{(2)} \epsilon(-1)^{1/2} + \sum_{n \geq 1} \gamma_{-n}^{(2)} \beta_n^{(1)} \epsilon(-1)^{-1/2} \right. \\ & - \frac{1}{2} \gamma_{-3/2}^{(3)} \beta_{1/2}^{(3)} + \sum_{\substack{n \geq 0 \\ m \geq 0}} \gamma_{-n}^{(1)} \beta_{-m+1/2}^{(3)} \epsilon'(-1)^{1/2} \frac{n^m}{m!} + \\ & + \sum_{\substack{n \geq 1 \\ m \geq 1}} \gamma_{-n-1/2}^{(3)} \beta_{-m}^{(1)} \epsilon'(-1)^{-1/2} \frac{m^{n-1}}{(n-1)!} + \\ & + \sum_{\substack{n \geq 1 \\ m \geq 0}} \gamma_{-n}^{(2)} \beta_{-m+1/2}^{(3)} \frac{1}{2} \epsilon\epsilon' \frac{n^m}{m!} (-1)^m - \epsilon\epsilon' \gamma_{-3/2}^{(3)} \beta_0^{(2)} + \\ & \left. + \sum_{\substack{n \geq 1 \\ m \geq 1}} \gamma_{-n-1/2}^{(3)} \beta_{-m}^{(2)} \epsilon\epsilon'(-1)^n \frac{m^{n-1}}{(n-1)!} \right] |-\frac{1}{2}\rangle_1 |-\frac{3}{2}\rangle_2 | -2\rangle_3 . \end{aligned} \quad (B.10)$$

The $b - c$ part of the state (B.6) is not the same as what one would obtain for the $|NNN\rangle$ vertex. This is because of our unorthodox choice of coordinate z_3 .

The basic integral used in (B.6), (B.10) is:

$$\oint_C \frac{dz}{2\pi i} z^{-m-1} (\log z)^{-n} = \begin{cases} (-1)^n \frac{m^{n-1}}{(n-1)!} & (m > 0) \\ 0 & (m < 0) \\ -1/2 & (m = 0, n = 1) \\ 0 & (m = 0, n > 1) \end{cases} \quad (\text{B.11})$$

where the contour surrounds a R -punctures; in other words, $z = re^{i\theta}$, $\theta \in (-\infty, +\infty)$. The formula for $m \neq 0$ in (B.11) follows directly from the Laplace identity:

$$\int_{-\infty}^{+\infty} (\beta + ix)^{-\nu} e^{-ipx} dx = \begin{cases} \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)} & p < 0 \\ 0 & p > 0 \end{cases} \quad (\text{B.12})$$

and $z = re^{i\theta}$. The case $m = 0$ can be obtained easily.

One may check the consistency of the procedure used by looking at the compatibility of the charges defining the state $|RRN\rangle$:

$$\sum_{k=1}^3 \oint_{C_k} dz_k d\theta_k V_n^{(k)} S_m^{(k)} = 0 \quad (\text{B.13})$$

where $V_n^{(k)}$, $S_m^{(k)}$ are the appropriate holomorphic section on the super-sphere minus three punctures. In our case this equation reduces to:

$$(-1)^{n+1} \oint_{C_1} dz_1 z_1^{-1-m} (\log z_1)^{-n} + \oint_{C_2} dz_2 z_2^{-1+m} (\log z_2)^{-n} + \oint_{C_3} z_3^{-n} e^{m z_3} = 0 \quad (\text{B.14})$$

which is shown to hold using (B.11).

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