12-1990

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Recommended Citation

La, H., Nelson, P. C., & Schwarz, A. S. (1990). Virasoro Model Space. *Communications in Mathematical Physics, 134* (3), 539-554. http://dx.doi.org/10.1007/BF02098446

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Abstract

The representations of a compact Lie group G can be studied via the construction of an associated "model space". This space has the property that when geometrically quantized its Hilbert space contains every irreducible representation of G just once. We construct an analogous space for the group Diff S1. It is naturally a complex manifold with a holomorphic, free action of Diff S1 preserving a family of pseudo-Kahler structures.All of the "good" coadjoint orbits are obtained from our space by Hamiltonian constraint reduction. We briefly discuss the connection to the work of Alekseev and Shatashvili.

Disciplines

Physical Sciences and Mathematics | Physics

UPR-0415T
IASSNS-HEP-90-25

Virasoro Model Space

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The representations of a compact Lie group G can be studied via the construction of an associated "model space". This space has the property that when geometrically quantized its Hilbert space contains every irreducible representation of G just once. We construct an analogous space for the group Diff S^1. It is naturally a complex manifold with a holomorphic, free action of Diff S^1 preserving a family of pseudo-Kahler structures. All of the "good" coadjoint orbits are obtained from our space by Hamiltonian constraint reduction. We briefly discuss the connection to the work of Alekseev and Shatashvili.

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1. Introduction

A geometrical understanding of the representation theory of the group of diffeomorphisms of the circle remains a desirable, and elusive, goal. Apart from its intrinsic interest a solution to this problem could shed light on a $2+1$-dimensional topological quantum field theory standing in the same relation to Virasoro as compact Chern-Simons-Witten theory does to Kac-Moody algebras [1] . Given the success of the method of orbits in understanding the representations of noncompact groups (see *e.g.* [2]), it is very natural to look to this method for help with Diff S^1 as well. Considerable progress has been made along these lines [3], but some problems stand out.

First, there are a variety of different types of orbit. Secondly, while every orbit has naturally the structure of a Hamiltonian dynamical system, there is in general no obvious choice of the additional structures needed to quantize these classical systems. Finally, once a quantization is chosen we find ourselves faced with a strongly-coupled system unless the central charge $c \gg 1$. In the latter case Witten has shown that indeed the familiar irreducible representations emerge.

Clearly it would be interesting to have an approach to this problem where all the representations come from quantizing a single space, with some natural choice of quantum data (*i.e.*, prequantization and polarization).

In the case of a compact, semisimple, finite-dimensional group G there is a well-known theorem with a similar flavor (see [4]). Such a group has a natural complexification G_c. Let N_+ be a maximal unipotent subgroup of G_c. For example, if $G = SU(n)$ then $G_c = SL(n, \mathbf{C})$ and N_+ consists of upper triangular matrices equal to 1 on the diagonal. Let $A = G_c/N_+$, a complex manifold of dimension $\frac{1}{2}(\dim G + \mathrm{rank} G)$. Then the space of holomorphic functions on A, subject to a certain square-integrability condition, is a representation of G, and moreover it is the sum of every irreducible representation with multiplicity one. We can thus refer to A as a *model space*, a space whose quantum mechanics yields a "model" for the representations of G.

Let us pause to sketch why this theorem is true. The Cartan torus $T \subset G$ commutes with N_+, and so acts on A from the *right*. It also commutes with *left* translations. Thus the space $\mathcal{H}_\lambda$ of eigenstates of the generators of T with eigenvalues given by some weight λ is a representation of G under left translation. But $\mathcal{H}_\lambda$ can also be regarded as the sections of a bundle over $(G_c/N_+)/T_c \cong G/T$; by the Borel-Weil-Bott theorem it is just the irreducible representation of weight λ. Letting λ range over the weight lattice we get each irreducible representation once.

We should contrast this result with two similar ones. First, the Peter-Weyl theorem tells us that the space of *all* L^2 functions on G (not necessarily holomorphic) also furnishes a representation of G. Now, however, each irreducible representation occurs with multiplicity equal to its dimension, and so the result is not so useful even if it remains true in infinite dimensions. Secondly, the generic orbit of G on its dual algebra $\mathbf{g}^\vee$ is a complex manifold of dimension $\dim_{\mathbf{c}} G/T = \frac{1}{2}(\dim G - \mathrm{rank}\ G)$, where T is a maximal torus of G. The Borel-Weil-Bott theorem tells us that the sections of a bundle over this orbit give one irreducible representation. Thus roughly speaking the difference between G/T and the model space A is that we have added in a complexified maximal torus (complex dimension rank G), and in so doing enriched the Hilbert space of states from one representation to all of them. It would be nice to have a corresponding result for Diff S^1.

The operation of taking all holomorphic functions on a space is reminiscent of geometric quantization. In the case of a single orbit of G it is well known that the above construction can be implemented by quantizing a certain classical dynamical system [2]. This approach seems bound to offer insights into infinite-dimensional systems, where a regularization is needed.

Recently Alekseev and Shatashvili have proposed to implement the above program for the group Diff $\equiv$ $\mathrm{Diff}_+ S^1$ of orientation preserving diffeomorphisms of the circle, in the hopes that a theorem similar to the one above will hold [5].[1] They have obtained some encouraging results to the effect that the quantization of A may contain the irreducible unitary representations of Diff, including the mysterious discrete series. Things did not quite work out, however. It seems clear that to make further progress one needs to be quite specific about the "model space" A and its global geometry. That is what we do here.

Specifically Alekseev and Shatashvili define their model space as a Hamiltonian dynamical system by writing down local canonical (or "Darboux") coordinates; they then obtain characters by path integration. To get the Hilbert space itself, however, one needs a precise global construction, and moreover a quantum structure on A. For individual orbits this has seemed problematical [3], but we will see that the model space has a very natural quantum structure.

[1] Indeed, some results of Chern-Simons-Witten gauge theory (see *e.g.* [6][7]) can be taken to support this for the case of loop groups. The recent work of H. Verlinde on the case of Vir is more subtle [8][1]; we can only hint at the connection to the present work.

In this paper we will construct a complex manifold $\mathcal{A}$ which is a suitable generalization of the model space A of a compact Lie group. Since the group Diff has no complexification, this is not quite straightforward. The appropriate method has already been used in a different context, however, by Kirillov and Yur'ev [9]. We will find on $\mathcal{A}$ a free, holomorphic action of Diff and a natural family of invariant pseudo-Kahler structures which implement the analog of the above prescription for compact groups. It seems rather remarkable and gratifying that this can be done at all. One feature of our approach is that all our constructions are complex-analytic, even for nonzero central charge. In principle $\mathcal{A}$ can then be quantized to get representations of Diff, but we will not be able to go this far. We will also explain the sense in which $\mathcal{A}$ decomposes into coadjoint orbits of Diff. Surprisingly the space $\mathcal{A}$, which has a very natural global definition, automatically excludes the pathological "unipotent" orbits Diff$/T$ (see [3]) while including the interesting ones Diff$/S^1$, Diff$/SL^{(n)}(2,\mathbf{R})$. That is, the latter orbits can be obtained from $\mathcal{A}$ by Hamiltonian constraint reduction.

Recently we received another paper [10], where a very different proposal is made for obtaining Virasoro representations from the diffeomorphism group.

2. Complex Structure

Our strategy will be as follows. While Diff admits no complexification, still we know that Diff$/S^1$ has a natural complex structure and invariant Kahler metric, indeed a two-parameter family of these [11][9][12]. Roughly we know we must take the maximal torus of Diff, namely the circle group of rigid rotations, complexify, and enlarge Diff$/S^1$ by that. Thus we take the space defined by Kirillov,

$$\mathcal{F} = \{f : f(0) = 0,\ f'(0) = 1\} \tag{2.1}$$

and enlarge it to

$$\mathcal{A} = \{f : f(0) = 0\}\,. \tag{2.2}$$

In both cases f is a holomorphic function on the unit disk $D = \{|z| < 1\}$, smooth and univalent up to the boundary. Kirillov showed that $\mathcal{F}$ can naturally be identified with Diff$/S^1$. In $\mathcal{A}$ we have added in the maximal torus (the angle $\arg f'(0)$), and complexified it (the magnitude $|f'(0)|$). Thus $\mathcal{A}$ is distinct from the space $\mathcal{T}_D$ appearing in [1], which was smaller than Diff$/S^1$.

Our plan is to identify $\mathcal{A}$ naturally with Diff$\times\mathbf{R}_+$.[2] The latter space has an obvious free action of Diff; we will show that on $\mathcal{A}$ this action is holomorphic. In later sections we will show that Diff$\times\mathbf{R}_+$ also has a natural invariant symplectic structure induced from the cotangent space $T^\vee$(Diff) (*cf.* [5]). We will see that on $\mathcal{A}$ this determines a Kahler structure. We will for illustration set the central charge to zero, then generalize in section seven. Finally the space Diff$\times\mathbf{R}_+$ projects to the dual algebra Vect$^\vee$, whereupon the action of Diff reduces to the usual coadjoint action.

To get started we must set up the identification $\mathcal{A} \simeq$ Diff$\times\mathbf{R}_+$. Begin with $f \in \mathcal{A}$. It takes the unit circle $\{|z|=1\}$ to a smooth non-self-intersecting contour K surrounding the origin. The *exterior* of K is thus topologically a disk in the Riemann sphere containing the point ∞. (See Fig. 1.) By the Riemann mapping theorem, we know that there is another function $G(u)$, holomorphic and single-valued everywhere *outside* the unit circle (*i.e.* for $|u|<1$, where $u=z^{-1}$), whose image is the exterior of K. Moreover there are many such maps, namely $G\circ M$ where M is any transformation in $SL(2,\mathbf{R})$. We can fix this freedom by imposing the additional conditions

$$G(0)=0, \quad G'(0) \text{ is real positive.} \tag{2.3}$$

We can rephrase these conditions in terms of

$$g(z) = 1/G(z^{-1}) \quad ;$$

then g has a simple pole at ∞ of real positive residue.

Fig. 1: Defining G from f.

[2] However $\mathcal{A}$ is *not* to be regarded as the central extension $\widehat{\text{Diff}}$; see section seven.

Having determined g we now let

$$(\gamma_f,\ s_f) = (f^{-1}\circ g,\ \ g'(\infty)^2) \quad . \tag{2.4}$$

What this means is that since f, g both take the unit circle to the contour K, we must have $g(\mathrm{e}^{i\theta}) = f(\mathrm{e}^{i\gamma(\theta)})$ for some diffeomorphism γ. The second entry just denotes the square of the real residue mentioned above; prime means $\frac{d}{dz}$. Clearly $s_f \in \mathbf{R}_+$. We note that this construction is independent of the actual centered complex coordinate z chosen on the disk D. Indeed if $\widetilde{z} = F(z)$ with $F(0) = 0$, then the map represented by $z \mapsto f(z)$ becomes $\tilde{z} \mapsto F\circ f\circ F^{-1}(\tilde{z})$, and $\frac{d}{d\tilde{z}}(F\circ f\circ F^{-1})(0) = \frac{d}{dz}f(0)$.

Thus we map $\Lambda : \mathcal{A} \to$Diff$\times\mathbf{R}_+$. Let us examine this map close to the base point, $f_0(z) = az$, where a is some real number. Thus let

$$f_\epsilon(z) = az + \epsilon\sum_{n>0}\varphi_n z^n \tag{2.5}$$

$$g_\epsilon(z) = az + \epsilon\sum_{n<2}\psi_n z^n \quad , \quad \psi_1 \ \text{ real} \tag{2.6}$$

$$s_\epsilon = a^2 + \epsilon\Delta \tag{2.7}$$

$$\gamma_\epsilon(\theta) = \theta + \epsilon\sum_{-\infty}^{\infty} v_n \mathrm{e}^{in\theta} \quad , \quad v_{-n} = \overline{v_n} \quad . \tag{2.8}$$

We have incorporated the conditions on f, g into these expansions. Expanding $g_\epsilon(\mathrm{e}^{i\theta}) = f_\epsilon(\mathrm{e}^{i\gamma_\epsilon(\theta)})$ we easily find

$$\begin{aligned} v_0 &= -\tfrac{1}{a}\mathrm{Im}\,\varphi_1, \qquad && \Delta = (2a)\mathrm{Re}\,\varphi_1 \quad , \\ v_n &= -\tfrac{i}{a}\varphi_{n+1}\ , && n>0 \quad , \end{aligned} \tag{2.9}$$

and so our map is invertible at the base point. In fact we can invert it everywhere, as follows (*cf.* [9]). Given $(\gamma, s) \in$ Diff $\times\ \mathbf{R}_+$ we construct a 2-sphere by gluing two standard disks $D_\pm$ using γ. The resulting space, with standard complex structure on each hemisphere, is isomorphic as a complex manifold to the usual sphere, by the uniformization theorem. Thus there is an invertible holomorphic function F from it to the Riemann sphere, or in other words holomorphic functions $F_\pm$ from the disk to the latter related by γ. F is well defined up to the automorphisms $SL(2,\mathbf{C})$ of the sphere. We use this freedom to set $F_+(0) = 0,\ F_-(\infty) = \infty,\ F'_-(\infty) = 1$. Finally we let $f(z) = \sqrt{s}F_+(z)$. This inverts the map Λ.

We now have that the map $\Lambda : \mathcal{A} \to$ Diff $\times\ \mathbf{R}_+$ is a bijection. It gives Diff$\times\mathbf{R}_+$ the desired complex structure.

3. Action of Diff

Recall [13] that a complex manifold M of dimension n is also a real manifold of dimension $2n$. We complexify the real tangent space to get $T_{\mathbf{C}}M$, a vector space of complex dimension $2n$, then split it into $T_{\mathbf{C}}M \simeq T^{(1,0)}M \oplus T^{(0,1)}M$, two complex pieces of dimension n. Every curve $P(\epsilon)$ in M has a tangent $\dot{P}(0)$ in the *real* tangent space of M; thus $\dot{P}(0) = V + \overline{V}$ where $V \in T^{(1,0)}M$ and $\overline{V}$ is its complex conjugate.

Consider the action of $U(1)$ on the complex plane: $P \to \theta \cdot P$ where $z_{\theta \cdot P} = \mathrm{e}^{i\theta} z_P$. For fixed θ we see that $z_{\theta \cdot P}$ depends holomorphically on z_P and we say the action is holomorphic. We can also formulate an infinitesimal criterion as follows. Fixing now P, the tangent $\frac{d}{d\theta}|_0(\theta \cdot P) = i(z_P \frac{\partial}{\partial z}|_P - \bar{z}_p \frac{\partial}{\partial \bar{z}}|_P)$. As noted above this has to be real, and it is. What we see is that its (1,0) bit is a holomorphic vector field on M. This is another criterion for the action of Diff to be holomorphic, and far more convenient for our purposes.

Fix any generator v for Diff. Thus $v \in$ Vect, the smooth vector fields on the circle, and we write $v = v(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}$. Letting v act from the left on any $\gamma_0 \in$ Diff gives us an action on Diff$\times \mathbf{R}_+$:

$$\gamma_\epsilon(\theta) = \gamma_0(\theta) + \epsilon v(\gamma_0(\theta)) \tag{3.1}$$

$$s_\epsilon \equiv s_0 \quad . \tag{3.2}$$

This action is of course globally well defined. Choose a base point f_0, not necessarily of the special form (2.5). Following [9] we will trivialize the tangent spaces $T_{f_0}\mathcal{A}$ by writing a tangent to f_0 as $\frac{d}{d\epsilon} f_\epsilon$ where

$$f_\epsilon(z) = f_0(z) + \epsilon\varphi(z) \quad , \tag{3.3}$$

where φ is holomorphic on the disk, $\varphi(0) = 0$, and similarly

$$g_\epsilon(z) = g_0(z) + \epsilon\psi(z) \quad , \tag{3.4}$$

where ψ is holomorphic off the disk. We don't permit any pole for ψ, even with real residue, because we are imposing (3.2).

We now want to find $\varphi_{v;f_0}$ corresponding to the fixed v and the chosen f_0. If it varies holomorphically as f_0 varies then the (1,0) part of the tangent to (3.3) will be a holomorphic vector field as desired. Again expanding $g_\epsilon(\mathrm{e}^{i\theta}) = f_\epsilon(\mathrm{e}^{i\gamma_\epsilon(\theta)})$ we find

$$\psi \circ g_0^{-1} = \varphi \circ f_0^{-1} + i[z f_0'(z) v(-i \log z)] \circ f_0^{-1} \text{ at points where } |g_0^{-1}| = 1. \tag{3.5}$$

This together with the boundary conditions on φ, ψ determines φ as follows.

Following [9], suppose we have a function F on the circle. Given a parametrized contour K in the plane we can regard F as a function on K and define its positive-frequency part as [3]

$$[F]^{>}_{K}(z) = \frac{z}{2\pi i} \oint_K \frac{F(w)dw}{w(w-z)} \tag{3.6}$$

for z a point inside K. Similarly define $[F]^{<}_{K}(z)$ by the same formula with z outside K. We then clearly have that on K, $F(z) = [F]^{>}_{K}(z) - [F]^{<}_{K}(z)$ for any contour K surrounding the origin, and $[F]^{>}_{K}$ is holomorphic inside K with $[F]^{>}_{K}(0) = 0$. Moreover the boundary condition on ψ clearly amounts to saying that $[\psi \circ g_0^{-1}]^{>}_{K} = 0$, since g_0^{-1} sends the exterior of K holomorphically to the exterior of the disk, and ψ is in turn holomorphic there. Similarly $[\varphi \circ f_0^{-1}]^{<}_{K} = 0$. We get

$$\varphi_{v;f_0} = i[(zf_0'(z)v(-i\log z)) \circ f_0^{-1}]^{>}_{K} \circ f_0. \tag{3.7}$$

Since v is fixed, everything in this formula depends holomorphically on f_0 and we are done.

We now have a holomorphic action of Diff on our space $\mathcal{A}$. In the next section we will proceed to investigate its symplectic structure. Before doing so, however, it is appropriate to ask how unique our construction is. The requirement that left actions of Diff on Diff$\times\mathbf{R}_+$ be holomorphic is a strong condition on our identification $\Lambda : \mathcal{A} \to \text{Diff} \times \mathbf{R}_+$, but suppose we replace (2.4) by

$$(\gamma_f,\ s_f) = (f^{-1} \circ g,\ \ \xi(g'(\infty)^2)) \quad , \tag{3.8}$$

where ξ is a real function. Then the induced action of Diff on $\mathcal{A}$, which doesn't change s at all, is completely unaffected. We will say more about this freedom shortly.

We close this section with an aside. While the group Diff has no complexification, still there is a complex semigroup, the "Neretin semigroup," which is the best substitute [14] [9] [15]. This complex semigroup can be shown to act holomorphically on $\mathcal{A}$; the action of Diff found in this section can be deduced from this action.

[3] This differs slightly from [9].

4. Symplectic Structure

We will begin by writing a symplectic form on $\mathcal{A}$ =Diff$\times\mathbf{R}_+$ and showing that it is indeed nondegenerate and left-invariant under Diff. This form is essentially the one proposed by Alekseev and Shatashvili; it is induced by a map from Diff$\times\mathbf{R}_+$ into the cotangent space $T^\vee$Diff. Finally we show that this 2-form is of type (1,1) in the complex structure of part two, and hence is the Kahler form of an invariant (pseudo-)Kahler metric on $\mathcal{A}$.

Trivialize $T^\vee\mathrm{Diff} \simeq \mathrm{Diff}\times\mathrm{Vect}^\vee$ by the map

$$(\gamma, b) \mapsto L^*_{\gamma^{-1}}(b) \in T^\vee_\gamma \mathrm{Diff} \quad , \tag{4.1}$$

where $L_{\gamma^{-1}}$ is left translation and b is a cotangent vector to Diff at the origin, *i.e.* a quadratic differential $b(\theta)(d\theta)^2$ on S^1. We include Diff$\times\mathbf{R}_+$ into $T^\vee$Diff by sending

$$(\gamma, s) \mapsto (\gamma, is\ell_0^*) = (\gamma, \tfrac{s}{2\pi}(\mathrm{d}\theta)^2) \quad . \tag{4.2}$$

It is traditional to use a complex basis for Vect in which $-i\ell_0$ corresponds to the middle element of the basis in (2.8). Hence $i\ell_0^*$ is the middle element of the dual basis, and $i\ell_0^* \leftrightarrow (2\pi)^{-1}(\mathrm{d}\theta)^2$.

We need a convenient description of two-forms on $T^\vee$Diff. Since these eat tangent vectors we introduce the natural trivialization $T(T^\vee\mathrm{Diff}) \simeq \mathrm{Diff}\times(\mathrm{Vect}^\vee\oplus\mathrm{Vect}\oplus\mathrm{Vect}^\vee)$ via

$$(\gamma,\ b;\ v,\ p) \to ((L_\gamma)_* v,\ p)|_{(\gamma,b)} \in T_{(\gamma,b)}(T^\vee\mathrm{Diff}) \quad . \tag{4.3}$$

Note that the tangent to a vector space, like $\mathrm{Vect}^\vee$, is naturally just that vector space.

The natural symplectic form on $T^\vee$Diff is now quite simple. Define a one-form α by the formula

$$\alpha(\gamma, b; v, p) \;\;\equiv\;\; \alpha(((L_\gamma)_* v,\ p)|_{(\gamma,b)}) = \langle (L_\gamma)_* v,\ L^*_{\gamma^{-1}} b\rangle = \langle v, b\rangle \quad , \tag{4.4}$$

the dual pairing of Vect with its dual. We will also let α denote the corresponding pulled-back one-form on $\mathcal{A} = \mathrm{Diff}\times\mathbf{R}_+$. Tangent vectors to $\mathcal{A}$ are given by $(\gamma, s; v, \Delta)$, where now Δ is a real number. Using the embedding (4.2) we get

$$\alpha(\gamma, s; v, \Delta) \;\;\equiv\;\; \alpha\big(((L_\gamma)_* v, \tfrac{\Delta}{2\pi}(\mathrm{d}\theta)^2)|_{(\gamma,(s/2\pi)(\mathrm{d}\theta)^2)}\big) = s v_0 \quad , \tag{4.5}$$

where v_0 is the middle expansion coefficient of v in (2.8).

The symplectic form Ω is now just the exterior derivative of α. For this we need the Lie bracket. With our trivialization of $T(T^\vee\mathrm{Diff})$ a vector field amounts to a pair of functions $(X(\gamma,b),\ \eta(\gamma,b))$ from Diff$\times$Vect$^\vee$ to Vect$\times$Vect$^\vee$. We will sometimes denote this vector field by $V_{X,\eta}$ to denote its dependence on these two functions. Considering the successive derivatives of a function f by two of these vector fields one gets

$$[V_{X,\eta}, V_{Y,\xi}] = V_{(V_{X,\eta}Y - V_{Y,\xi}X + [X,Y], V_{X,\eta}\xi - V_{Y,\xi}\eta)}. \tag{4.6}$$

Here $V_{X,\eta}Y$ denotes the derivative of the Vect-valued function Y, while $[X,Y]$ is taken pointwise and does not differentiate the functions X, Y with respect to $\gamma,\ b$. We thus have $\mathrm{d}\alpha = \Omega$, where at (γ, b)

$$\Omega(V_{X,\eta}, V_{Y,\xi}) = \langle \eta, Y\rangle - \langle \xi, X\rangle - \langle b, [X,Y]\rangle \quad , \tag{4.7}$$

a function on $T^\vee$Diff given two vector fields. Again let Ω denote also the corresponding pullback to $\mathcal{A}$. We then have that at (γ, s) (recall we set $b = \frac{s}{2\pi}(\mathrm{d}\theta)^2$)

$$\Omega((v,\Delta),\ (v',\Delta')) = \Delta v'_0 - \Delta' v_0 + 2is\sum_{-\infty}^{\infty} n v_n v'_{-n} \quad . \tag{4.8}$$

Here we have used the same expansion coefficients as in (2.8).

While we know that Ω is invariant as a differential form on $T^\vee$Diff, still one may worry that our choice of Diff$\times\mathbf{R}_+ \hookrightarrow T^\vee$Diff will spoil the invariance of Ω on $\mathcal{A}$. After all we did choose a basis, to define ℓ_0^*. We now check this invariance briefly. For any generator v_1 of Diff we get a vector field of the left action, which in our trivialization is seen to be $(\mathrm{Ad}_{\gamma^{-1}}v, 0)$ at the point (γ, s). Let us compute the Lie derivative of α along this vector:

$$(\mathcal{L}_{(\mathrm{Ad}_{\gamma^{-1}}v,0)}\alpha)(V_{X,\Delta}) = V_{X,\Delta}\alpha(V_{(\mathrm{Ad}_{\gamma^{-1}}v,0)}) + \Omega(V_{(\mathrm{Ad}_{\gamma^{-1}}v,0)}, V_{X,\Delta}) \quad . \tag{4.9}$$

The derivative in the first term substitutes $s \to s + \epsilon\Delta,\ \gamma \to \gamma\circ(1+\epsilon X)$ and takes the derivative of ϵ. Thus using (4.4), (4.7)

$$= \Delta(\mathrm{Ad}_{\gamma^{-1}}v)_0 - s([X, \mathrm{Ad}_{\gamma^{-1}}v])_0 - \Delta(\mathrm{Ad}_{\gamma^{-1}}v)_0 - s([\mathrm{Ad}_{\gamma^{-1}}v, X])_0 \tag{4.10}$$

and α is invariant, and hence Ω as well.[4]

[4] Compare the discussion of [5], where a residual global *right* invariance remains after "gauge-fixing."

It is clear from (4.8) that Ω is nondegenerate and hence an invariant symplectic form on Diff$\times\mathbf{R}_+$. In other words we have a Hamiltonian action of Diff on this space. In fact this action is strictly Hamiltonian, *i.e.* there is a globally defined moment map $\mu : \mathcal{A} \to \mathrm{Vect}^\vee$. One finds that, since Ω is exact, we have that

$$\mu(\gamma, s) = s\mathrm{Ad}^*_{\gamma^{-1}}(\ell_0^*) \tag{4.11}$$

generates the action (3.1)–(3.2). We also have a close relation to the usual Hamiltonian action of Diff on its coadjoint orbits. Since clearly $\mu(\gamma_1 \cdot \gamma, s) = \mathrm{Ad}^*_{\gamma_1^{-1}}\mu(\gamma, s)$, we see that one flow covers the other [16]. Furthermore, the map μ restricted to the inverse image of a generic coadjoint orbit is holomorphic. This is a meaningful statement, since such orbits are isomorphic to Diff$/S^1$, which has an invariant complex structure [11][9].

We note that the *single* form (4.8) corresponds to a *family* of forms on Diff$/S^1$ parameterized by s. Generalizing to arbitrary central charge gives the two-parameter family of Bowick and Rajeev, as we will see in section seven.

The space Diff $\times$ $\mathbf{R}_+$ has an important property: it is *multiplicity-free* in the sense of Guillemin and Sternberg [17]. In general a symplectic G-manifold is called multiplicity-free when every G-invariant function on it commutes with every other one. In our case the only Diff-invariant functions are clearly of the form $F(s)$, so this condition is satisfied. When this is so, the corresponding quantum state space will contain every representation at most once, essentially by Schur's lemma. This was shown in [17] for the case of real polarizations.

Finally we wish to stress that all our constructions so far are very natural. They do not depend on any choice of basis such as (2.5)–(2.8), as one sees from the main definitions (2.2), (2.4), (3.1)–(3.2), (4.4). Thus when it transpires that the pathological orbits Diff$/T$ are absent from $\mathcal{A}$, it will be clear that we have not simply taken them out by hand.

5. Pseudo-Kahler Structure

We have only to combine sections three and four. That is, an invariant closed nondegenerate two-form will be Kahler if it is of type (1,1) in the complex structure we found and the associated Hermitian structure is positive-definite. Since both the form Ω and the complex structure are Diff-invariant, we have only to check this assertion at $f(z) = az$ for a real constant a. We then have coordinates for the tangent space to $\mathcal{A}$ given by (2.9).

What we must now do is pass to the complexified tangent space. Thus we allow a_n, b_n Δ to be complex, or equivalently take $\bar{\varphi}_n$ independent of φ_n in (2.9), or v_{-n} independent of v_n in (4.8). We then extend Ω in (4.8) by linearity, obtaining $\varphi_1 = \frac{1}{2a}\Delta - iav_0$, $\bar{\varphi}_1 = \frac{1}{2a}\Delta + iav_0$, $\varphi_{n+1} = -iav_n$, $\bar{\varphi}_{n+1} = iav_{-n}$, and

$$\Omega = -i(\varphi_1\bar{\varphi}_1' - \bar{\varphi}_1\varphi_1') + 2i\sum_{>0} n(\varphi_{n+1}\bar{\varphi}_{n+1}' - \bar{\varphi}_{n+1}\varphi_{n+1}') \quad . \tag{5.1}$$

As claimed this has no (2,0) or (0,2) terms, *i.e.* none with $\varphi_n\varphi_m'$ or $\bar{\varphi}_n\bar{\varphi}_m'$. The corresponding Hermitian form [13] is indefinite:

$$H = -\varphi_1\bar{\varphi}_1' + 2\sum_{>0} n\varphi_{n+1}\bar{\varphi}_{n+1} \quad . \tag{5.2}$$

Hence we have a pseudo-Kahler structure.

We can now return to the question of how to fix the arbitrary real function ξ in (3.8). It is easy to see that regardless of ξ, Ω will always be a closed form of type (1,1):

$$\Omega = -i\xi'(a^2)(\varphi_1\bar{\varphi}_1' - \bar{\varphi}_1\varphi_1') + 2i\frac{\xi(a^2)}{a^2}\sum_{>0} n(\varphi_{n+1}\bar{\varphi}_{n+1}' - \bar{\varphi}_{n+1}\varphi_{n+1}') \quad .$$

As long as ξ takes $\mathbf{R}_+$ to $\mathbf{R}_+$, moreover, we will get a nondegenerate pseudo-Kahler form. In fact this freedom is completely expected. Consider the space $A = G_c/N_+$ analogous to our $\mathcal{A}$, where G is a compact Lie group (see section one). Clearly there is no natural choice of how to embed N_+ in G_c; any choice can be conjugated into an equivalent, different, choice by an element of G_c. Alternately if we fix a choice of N_+ we cannot expect to find any natural Kahler structure, by the same argument. In our case as we mentioned the nearest substitute for G_c is the Neretin semigroup. As a simple example of how it acts consider the transformation $\Xi : \mathcal{A} \to \mathcal{A}$, $\Xi(f)(z) = kf(z)$ for a real constant k. One easily shows that if we define Ω_ξ on $\mathcal{A}$ by the embedding (3.8), then an equally good choice is $\Xi^*\Omega_\xi = \Omega_{\xi'}$, where $\xi'(x) = k^2\xi(x)$. More complicated Ξ induce more complicated transformations of ξ. Thus the choice (2.4) is just one of many equivalent choices. It gives (5.1) a very simple form.

6. Quantum Mechanics

The quantization of an infinite-dimensional system like (5.1) is delicate and will no doubt require methods from the corresponding quantum field theory (see *e.g.* [5][8][7][18]). At the very least we will have to replace wavefunctions by cohomology classes [19]. Some geometrical remarks are in order first, however.

Suppose we have a dynamical system A on which a group G acts by symmetries. In general we cannot represent the Lie algebra g using Hamiltonian generating functions; the best we can do is to represent a central extension $\hat{\mathbf{g}}$ in such a way that the central generator is represented by a constant function [2]. It may seem surprising that a central extension can show up in *classical* mechanics, but it is already well known [20] that in the Virasoro geometrical action c enters as a classical parameter; see also section seven. In any case we have seen that in the present situation the moment map (4.11) affords a true representation of the algebra Vect in Poisson brackets.

Now suppose a G-invariant line bundle and connection (B, ∇) have been given with curvature of ∇ equal to Ω. Geometric quantization then tells us how to lift the action of g (respectively $\hat{\mathbf{g}}$) from A to B. Namely if μ_a is the moment of some generator and X_{μ_a} its Hamiltonian vector field, then

$$\hat{\mu}_a = -i\nabla_{X_{\mu_a}} + \mu_a \tag{6.1}$$

is the corresponding quantum operator on sections of B, and one shows that the $\hat{\mu}_a$ obey the same algebra under commutation as do the μ_a under Poisson bracket, namely g (respectively $\hat{\mathbf{g}}$). Finally if a G-invariant polarization is given then (6.1) acts on polarized sections of B.

In our case Diff acts on $\mathcal{A}$, (B, ∇) are defined in the usual way from the Kahler potential K of Ω, and the polarization is defined by the Diff-invariant complex structure on $\mathcal{A}$. Thus B is a holomorphic line bundle and $\nabla \equiv \mathrm{d} - i\partial K$ its Hermitian connection; since $\Omega = i\partial\bar{\partial}K$ this is a suitable choice. We finally get an action on wavesections defined by (6.1).

The point we wish to make is that (6.1) does not at first resemble the prescription we were trying to imitate. Recall from section one that in the compact case we wanted holomorphic *functions* on A (not sections of some bundle) with an action of G by left *translations*, *i.e.*

$$\hat{\mu}_a = -iX_{\mu_a} \quad , \tag{6.2}$$

where X_{μ_a} is the ordinary directional derivative.

To recover this prescription from geometric quantization, we must therefore verify two global properties. First, we must find that the bundle B is holomorphically trivial. This amounts to finding a single global Kahler potential. Second, to reduce (6.1) to (6.2) we need to show that K can be chosen such that everywhere

$$\mu_a = -i\langle \partial K, X_{\mu_a} \rangle = -iX^{(1,0)}_{\mu_a} K \quad . \tag{6.3}$$

Furthermore (6.3) amounts to requiring that the connection $-i\partial K$ be itself G-invariant [21]. A little manipulation reduces this condition to

$$X^{(1,0)}_{\mu_a}\mu_b + X^{(0,1)}_{\mu_b}\mu_a = 0 \quad , \tag{6.4}$$

where $X^{(1,0)}_{\mu_a}$ is the holomorphic part of X_{μ_a}. The real part of (6.4) merely says $\{\mu_a, \mu_b\} = -\{\mu_b, \mu_a\}$, but the imaginary part is new. We now briefly sketch why these two facts (triviality of B and (6.4)) are true for the case of $G =$Diff, $A = \mathcal{A}$.

First we notice that our manifold $\mathcal{A}$ has very little topology — it retracts to a circle epitomized by the phase of the first Taylor coefficient of f. So to study the triviality of B we can restrict attention to the submanifold $\mathcal{A}_0 = \{f_u\,,\ u \in \mathbf{C}^\times\}$, $f_u(z) = uz$. Here our formulas reduce to $\Omega = -i\mathrm{d}u\wedge\mathrm{d}\bar{u}$. We can therefore take $K = -|u|^2$, which is clearly global and invariant under the remnant symmetry $U(1) \subset$ Diff acting on $\mathcal{A}_0$. One easily shows that if we generalize to one of the other prescriptions (3.8), then $K(u) = -\int^{|u|^2} \frac{\mathrm{d}x}{x}\xi(x)$ is again global and invariant.

Next we want to verify the condition (6.4). It is enough to do so at the submanifold $\mathcal{A}_0$. Hence we need the moments μ_n of the Diff generators ℓ_n to first order near $\mathcal{A}_0$. Letting $u = \mathrm{e}^{i\alpha}a$ we find that at f_u the Hamiltonian vector field of ℓ_n is

$$\begin{aligned} X^{(1,0)}_{L_n} &= a\mathrm{e}^{in\alpha}\frac{\partial}{\partial\varphi_{n+1}} \qquad && n \geq 0 \\ &= 0 && n < 0 \quad , \end{aligned}$$

while the corresponding moments are

$$\begin{aligned} L_n &= \ \ 2ian\mathrm{e}^{in\alpha}\bar{\varphi}_{n+1} \qquad && n > 0 \\ &= -2ian\mathrm{e}^{in\alpha}\varphi_{-n+1} && n < 0 \\ &= \ \ a^2 - u\bar{\varphi}_1 - \bar{u}\varphi_1 && n = 0 \quad . \end{aligned}$$

Before verifying (6.4), we note that it was derived for the action of G, *i.e.* for *real* generators of Diff. Taking the linear combinations $L_n + L_{-n}$, $i(L_n - L_{-n})$, we easily verify it at $\mathcal{A}_0$. Again one can show that this works for any choice of ξ.

Away from $\mathcal{A}_0$ we extend the (1,0)-form $-i\partial K$ to an invariant (1,0)-form κ by the action of Diff, for which $\mathcal{A}_0$ is a slice. We can then integrate $\partial K = i\kappa$ to find K because $\mathcal{A}$ retracts onto $\mathcal{A}_0$.

We now know that our quantum mechanical system implements the analog of the theorem in section one for any choice of the real positive function ξ. We should however be careful to choose ξ so that the quantum mechanical operators satisfy $\hat{L}_n^\dagger = \hat{L}_{-n}$. Truncating to $\mathcal{A}_0$ we see that our choice $\xi(x) = x$ meets this condition, since then the metric e^{-K} makes our truncated system the same as the harmonic oscillator in an inverted potential, with $\hat{L}_0$ the Hamiltonian. This system can be quantized using 1-form wavefunctions.

7. Central extension

We now generalize the previous construction to find a *family* of invariant Kahler metrics on the space $\mathcal{A}$, parameterized by a real number t. That is, the space itself, its complex structure, and holomorphic action of Diff will always be the ones found in sections two and three. Our strategy is to invent a larger space $\widehat{\mathcal{A}}$ with just one symplectic structure $\widehat{\Omega}$, then find a Hamiltonian constraint reducing $\widehat{\Omega}$ to a family of symplectic manifolds all isomorphic to $\mathcal{A}$. In section eight we will then introduce further constraints to reduce $\mathcal{A}$ to individual coadjoint orbits.

We construct $\widehat{\mathcal{A}}$ by applying the previous recipe to $\widehat{\mathrm{Diff}}$, the central extension of Diff defined by the Lie algebra extension of $\widehat{\mathrm{Vect}} \cong \mathrm{Vect} \oplus \mathbf{R}$:

$$[(v_1, \nu_1), (v_2, \nu_2)] = \left([v_1, v_2], \frac{1}{24\pi} \oint v_1''' v_2\right) \quad .$$

Here and below prime means $\frac{\mathrm{d}}{\mathrm{d}\theta}$. Following [20] we have omitted the conventional factor of i to emphasize that $\widehat{\mathrm{Diff}}$ is a real manifold. In the usual way this extension defines a multiplication law

$$(\gamma_1, c_1) \cdot (\gamma_2, c_2) = (\gamma_1 \circ \gamma_2, c_1 + c_2 + c(\gamma_1, \gamma_2))$$

(see [22]), but we will not need the explicit form of the cocycle $c(\gamma_1, \gamma_2)$.

We will write $(\gamma, c) \in \widehat{\mathrm{Diff}}$, $(v, \nu) \in \widehat{\mathrm{Vect}}$, $(b, t) \in \widehat{\mathrm{Vect}}^\vee$. Thus a point in the cotangent is specified now by $(\gamma, c; b, t)$, and a vector by $V_{(v,\nu;\eta,\lambda)}|_{(\gamma,c;b,t)} \in T_{(\gamma,c;b,t)}(T^\vee\widehat{\mathrm{Diff}})$. Formulas like (4.6),(4.7) now have somewhat tedious generalizations. Analogous to (4.2) we include $\widehat{\mathrm{Diff}} \times \mathbf{R}_+ \times \mathbf{R}$ into $T^\vee\widehat{\mathrm{Diff}}$ by

$$(\gamma, c; s, t) \mapsto (\gamma, c; s\ell_0^*, t) \quad .$$

Then the natural symplectic form pulled back to $\widehat{\mathrm{Diff}} \times \mathbf{R}_+ \times \mathbf{R}$ is

$$\widehat{\Omega}(V_{(v_1,\nu_1;\Delta_1,\lambda_1)}, V_{(v_2,\nu_2;\Delta_2,\lambda_2)}) = \Delta_1(v_2)_0 + \lambda_1\nu_2 - \Delta_2(v_1)_0 - \lambda_2\nu_1 - s[v_1, v_2]_0 - \frac{t}{24\pi}\oint v_1'''v_2 \tag{7.1}$$

at $(\gamma, c; s, t)$, where now Δ_i are real numbers. As before this is closed.

We thus have a symplectic manifold $\widehat{\mathcal{A}} = \widehat{\mathrm{Diff}} \times \mathbf{R}_+ \times \mathbf{R}$ with an action of $\widehat{\mathrm{Diff}}$ by left translations. As before this action preserves the symplectic structure, and as before $\widehat{\mathcal{A}}$ is multiplicity-free. Eventually we want an action of Diff, not $\widehat{\mathrm{Diff}}$. For now, however, the Hamiltonian vector field corresponding to $(v, \nu) \in \widehat{\mathrm{Vect}}$ is

$$\widehat{U}_{(v,\nu)}|_{(\gamma,c;s,t)} = V_{(\mathrm{Ad}_{\gamma^{-1}}(v,\nu);0,0)}|_{(\gamma,c;s,t)} \quad . \tag{7.2}$$

One checks that again the corresponding moment map is

$$\hat{\mu}(\gamma, c; s, t) = \mathrm{Ad}^*_{\gamma^{-1}}(s\ell_0^*, t) \quad . \tag{7.3}$$

Note that since $\mathrm{Ad}_{(\gamma^{-1},c)}(v, \nu)$ is independent of c we abbreviate to $\mathrm{Ad}_{\gamma^{-1}}(v, \nu)$, and similarly Ad^*. Note also that $\hat{\mu}$ is therefore independent of c.

From (7.2) we see that the coordinate t is strictly first-class, *i.e.* it commutes with μ under Poisson bracket since $\widehat{U}_{(v,\nu)}t \equiv 0$. We may thus reduce $\widehat{\mathcal{A}}$ by a constraint setting $t - t_0 = 0$ for any constant t_0. Furthermore the flow generated by t is just $\frac{\partial}{\partial c}$, from (7.1). Hence for any t_0 we get a constraint reduction to $\widehat{\mathrm{Diff}} \times \mathbf{R}_+ \times \{t_0\}/\sim$, where $\sim$ identifies different values of c. But this space is just our $\mathcal{A} \cong \mathrm{Diff} \times \mathbf{R}_+$. Moreover the left action of $\widehat{\mathrm{Diff}}$ on $\widehat{\mathcal{A}}$ is seen to descend to the usual left action of Diff on $\mathcal{A}$. However, while the functions (7.3) descend to $\mathcal{A}$, they do not generate the Lie algebra of Vect — instead a central term remains, as desired.

We therefore get on $\mathcal{A}$ a *family* of symplectic forms

$$\Omega_{t_0}(V_{v_1,\Delta_1}, V_{v_2,\Delta_2}) = \Delta_1(v_2)_0 - \Delta_2(v_1)_0 - s[v_1, v_2]_0 - \frac{t_0}{24\pi}\oint v_1'''v_2 \quad .$$

Henceforth we regard t_0 as a parameter. We know that Ω_{t_0} is invariant since the constraint was first class. Using the same complex structure as in section two, invariant under the same action of Diff as in section three, we now see that every Ω_{t_0} is pseudo-Kahler by a calculation similar to (5.1):

$$\Omega_{t_0} = -i(\varphi_1 \bar{\varphi}_1' - \bar{\varphi}_1 \varphi_1') + 2i \sum_{>0} \left(n + \frac{t_0}{24a^2} n^3\right)\left(\varphi_{n+1} \bar{\varphi}_{n+1}' - \bar{\varphi}_{n+1} \varphi_{n+1}'\right) \quad . \tag{7.4}$$

This is the imaginary part of a Hermitian metric on $\mathcal{A}$ given by

$$H_{t_0}(\varphi, \bar{\varphi}') = -\varphi_1 \bar{\varphi}_1' + 2 \sum_{>} \left(n + \frac{t_0}{24a^2} n^3\right) \varphi_{n+1} \bar{\varphi}_{n+1}' \quad .$$

We have arranged for H, and hence also the Kahler potential, to be positive as $t_0 \to \infty$. Unlike the case of zero central charge, Ω_{t_0} and H_{t_0} are singular whenever $24a^2/t_0 = -n^2$ for integer n. (Recall that $b_0 = a^2/2\pi$.) Hence we should really define $\overline{\mathcal{A}}_{t_0}$ as a singular symplectic variety. Note that this problem was already present at the classical level (see (7.1)); it reflects our failure to find a slice in $T(T^\vee \widehat{\text{Diff}})$ suitable for "gauge-fixing" in the language of [5].[5] Far from being a pathology we expect the singularity of Ω to be the key to its correct quantization. For, as we cross the singularities the *signature* of H changes. For H of indefinite sign we know we should consider wavesections as Dolbeault cohomology classes [19], or equivalently introduce fermions. As noted by Alekseev and Shatashvili, such fermions are precisely what is needed to correct the signs in the character formula in [5]. (In that paper this phenomenon was not visible, however, because the complex structure was not available and hence Ω could not be converted into H.) We do not yet know how to make this conjecture precise.

The constraint formalism guarantees that each Ω_{t_0} will be closed. It does not follow, however, that $\Omega_{t_0} = \mathrm{d}\alpha_{t_0}$ for some invariant one-form α_{t_0}, even though $\widehat{\Omega} = \mathrm{d}\widehat{\alpha}$ for an invariant $\widehat{\alpha}$. In the language of section six this failure is responsible for the appearance of a central extension in the Poisson brackets of generators; upon quantization it gives us representations of $\widehat{\text{Diff}}$ as desired. Moreover the $\widehat{\text{Diff}}$-invariance condition for the geometric action is equivalent to the Virasoro Ward identity [24] , and the appropriate α_{t_0} is just the Virasoro geometric action with the base point s regarded as a dynamical variable as assumed in [5].

[5] The problem is indeed reminiscent of a similar singularity in Chern-Simons-Witten theory [23]. There the solution was to excise the bad subvariety by restricting attention to *stable* vector bundles on a fixed Riemann surface. This is not helpful in the present case, where the bad subvariety is in the interior of $\mathcal{A}$.

8. Reduction to orbits

Consider again the case with no central extension. Since the action of Diff in (3.1)–(3.2) does not affect s, we see that s is strictly first-class and may be set to any value s_0. Moreover the corresponding flow generated by the constraint $s - s_0$ is the *right* action of the rigid rotation group $S^1 \subset$ Diff. This follows because $\Omega(V_{(v,\Delta)}, V_{(\ell_0,0)}) = \Delta + s[\ell_0, v]_0 = \Delta$, while the derivative $V_{(v,\Delta)} s = \Delta$; from (4.3) $V_{(\ell_0,0)}$ generates the right action of S^1 on Diff.

Thus at $t_0 = 0$ $\mathcal{A}$ admits a Hamiltonian constraint reduction to a set of copies of Diff$/S^1$; under the moment map (4.11) these map to the usual coadjoint orbits as desired.

In the case of section seven, generically we have the same situation. At the special values of s, however, Ω_{t_0} becomes singular and we must reduce further to get a good dynamical system. We expect a new function ψ to become first-class at these special values. Since we are already constraining s, this means that the Poisson brackets satisfy

$$\{\psi, s\} = 0, \qquad \{\psi, \mu\} = 0 \quad .$$

The form of the centralizer in [3] suggests that we try

$$\psi(\gamma, s) = \langle \mathrm{Ad}_\gamma(\ell_0, 0), (\ell_n^*, t_0) \rangle \quad .$$

Thus the derivative $V_{(\ell_0,0)}\psi = 0$ and so $\{\psi, s\} = 0$. When $s = -\frac{t_0}{24} n^2$ we also find the derivative $V_{(\mathrm{Ad}_{\gamma^{-1}} v, 0)}\psi = 0$ for all v, and so $\{\psi, \mu\} = 0$ as desired. One can show that ψ generates right motions of a generator of $SL(2, \mathbf{R})$. (If such a generator exists then it must commute with s, since $[\ell_1, \ell_0]_0 = 0$, and also with μ, since right and left motions commute.)

9. Conclusion

Even though the diffeomorphism group has no complexification, we have found a space $\mathcal{A}$ which has all the attributes, save one, of the space G_c/N_+ for a compact group. The space $\mathcal{A}$ has a free action of Diff by holomorphic maps, and so the holomorphic functions on it furnish a large, reducible representation of Diff. $\mathcal{A}$ also carries a family of holomorphic line bundles with actions of $\widehat{\mathrm{Diff}}$, giving representations for various values of the central charge.

All this is nice, but we have seen much more. Since $\mathcal{A}$ is infinite-dimensional, the precise class of functions to allow is a delicate question. If $\mathcal{A}$ has the structure of a quantum-mechanical system, however, then we can imagine bringing to bear methods of 2d quantum field theory for its quantization. Remarkably we have seen that this is so. The holomorphic functions (or sections of a bundle), with the action of Diff (or an extension) above, actually arise from the quantization of a dynamical system. We constructed the classical and quantum data of this system. It turned out to be quite simple. For example the prequantum line bundle is just trivial — certainly not the case for the Borel-Weil-Bott theorem. Also we found that of the rather complicated catalog of Virasoro coadjoint orbits, only the interesting ones Diff/S^1 and $\mathrm{Diff}/SL^{(n)}(2,\mathbf{R})$ are present. The only missing attribute of the compact case is positivity of the Kahler metric on $\mathcal{A}$; we have suggested that this failure is not a pathology of our construction but instead a crucial feature for getting the representation theory right. In fact by taking the wavefunctions to be cohomology classes we expect to recover and interpret geometrically a form of the complex introduced by Felder [25] .

Quantization of the space $\mathcal{A}$ remains a somewhat daunting prospect, however. Even with some appropriate regularization replacing the condition of square-integrability, the fact that for $t < 0$ the pseudo-Kahler form degenerates will cause trouble. Similar difficulties appear in other approaches to quantizing $SL(2,\mathbf{R})$ gauge theory [26]. We think, however, that the present approach shows the issues in a particularly clear form.

We are grateful to R. Bott, A. Morozov, N. Reshetikhin, E. Verlinde, H. Verlinde, E. Witten, and especially S. Shatashvili for many discussions and suggestions. This work was supported in part by NSF grant PHY88-57200 and by the A. P. Sloan Foundation.

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