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Introduction

The operator formalism for string theory provides a generalization of the notions of Hilbert space of states, BRST cohomology, and the rest of conformal field theory from string tree amplitudes (CFT on a sphere) to more complicated Riemann surfaces. One basic inspiration for this approach was the work of Krichever and Novikov, Sato and Sato, and others on the KP hierarchy. One can build an infinite-dimensional Grassmannian containing, among other things, the moduli spaces of Riemann surfaces of every genus. The details of this construction are not so important to the present discussion, but what is important is that over this Grassmannian there is a bundle with a natural section. The bundle consists of semi-infinite wedge products of first-quantized state vectors, and so it is natural to interpret it as the Fock space of a very simple conformal quantum field theory: a single fermion. Thus the existence of a natural section amounts to a naturally-defined *state* of the given field theory associated to any given point of the Grassmannian. In particular to every Riemann surface with a chosen point and local coordinate about that point one has a ray in the Hilbert space summarizing that surface.

The operator formalism seeks to generalize the above situation in a number of ways. One begins with a *general* conformal field theory and again seeks to associate to each underlying Riemann surface with puncture and local coordinate a state in whatever Fock space was appropriate for the sphere. For certain theories the principle guiding the choice turns out to be very simple: to each Riemann surface we find certain geometrical data, namely a collection of diffferentials which extend nicely from a unit circle around the chosen point to the rest of the surface; we then form corresponding charge operators on the Fock space H and require that they all annihilate the desired state [1][2]. For example, such charge conditions suffice to determine the appropriate state for spin-0 bosons and spin-1 fermions, and hence for the bosonic string.

The utility of the above construction comes when we note that the partition function of a given conformal theory on a given surface is just the inner product of the above state with the standard $SL(2,\mathbf{C})$-invariant vacuum. Correlation functions can similarly be computed as easily as on the sphere, once the appropriate master state is known. Finally, insertions of external states can also be easily computed by taking the product not with the vacuum but with the desired state.

In the talk I described the generalization of the above setup from conformal theories to superconformal geometry, as worked out in detail in [3]. The text of this talk can be found in the review article [4]. Here instead I would like to describe some other features of the operator formalism which seem to be applicable to a wider class of CFT's than just free bosons and fermions. Specifically I will focus on the 'equation of motion' satisfied by the basic state [5]. There turns out to be a remarkable parallel between the operator formalism and the action of a Lie algebra in a differential complex [6]. In this light the 'equation of motion' becomes an *equivariance* condition, a fact which I expect will survive (and guide) the generalization from individual moduli spaces to the correct universal moduli space.

The Ingredients

We have noted above that the basic state associated to a Riemann surface is not in general well defined given only the surface and a puncture. Instead one needs to make a choice of a local coordinate z near the puncture; changing z then modifies the special state by introducing a stress tensor. To get a well defined state we accordingly work on the space $\mathcal{P}$ of moduli of Riemann surfaces with a puncture and local coordinate centered on the puncture. (One can just as easily work with n punctures.) Over $\mathcal{P}$ we build the trivial bundle $\mathcal{H} = H \otimes \mathcal{P}$ where H

is the Fock space of the original CFT. Thus the basic state becomes a *section* of $\mathcal{H}$.

$\mathcal{P}$ is itself a principal bundle over the desired moduli space $\mathcal{M}$ of Riemann surfaces with puncture; the group is simply $\mathrm{Diff}_+ S^1$ of changes of local coordinate z. Moreover we also know that this group acts on the bundle $\mathcal{H}$, via the action of the stress tensor as mentioned above. But we can do even better than this. The *full* diffeomorphism group (or rather the Virasoro algebra) acts on $\mathcal{P}$ [7]. The construction is described *e.g.* in [5][4]. Moreover, when a state is uniquely defined by charge conditions such as those described above one can show that as we move in any direction in $\mathcal{P}$ (not just in the vertical directions) that the state changes by the corresponding insertion of the stress tensor:

$$\delta_v |\Sigma\rangle = T(v)|\Sigma\rangle \quad . \tag{1}$$

Here $v \in \mathrm{Vir}$ is any vector field on the circle and δ_v is the corresponding vector on $\mathcal{P}$, regarded as a directional derivative. $|\Sigma\rangle$ is the section of $\mathcal{H}$ evaluated at $\Sigma \in \mathcal{P}$, and $T(v) = \oint T_{zz}(z) v^z(z) \mathrm{d}z$ is the mode of the stress tensor corresponding to v.

Eqn. (1) is very general. Even if the conserved-charge method is not sufficient to fix a state for a general CFT, we always expect to have (1). Accordingly we would like to take (1) as a fundamental property and find a suitable mathematical home for it.

Operations

Consider as an analogy a principal G-bundle P over some manifold M, where G is a Lie group. Then G acts on P from the right; similarly G acts on all the functions, forms, *etc.* defined on P. Furthermore, to every element v of the Lie algebra $\mathbf{g}$ there is a vector field R_v on P.

We can cast the situation in algebraic terms as follows [6]: we again note that $\mathbf{g}$ acts on the complex of differential forms (Ω, d) on P. In fact Ω serves as a representation of $\mathbf{g}$, and d commutes with the action. More generally we will say that an "operation" of $\mathbf{g}$ consists of a representation θ of $\mathbf{g}$ in a differential complex (Ω, D). For every $v \in \mathbf{g}$ we thus get $\theta(v)$ linear in v; $\theta(v)$ is a linear map of Ω of degree 0 satisfying

$$\theta([v, v']) = [\theta(v), \theta(v')] \quad . \tag{2a}$$

We further require that for every $v \in \mathbf{g}$ there be given another operator on Ω called $i(v)$. $i(v)$ is again linear in v, again a linear operator on Ω, but it reduces the degree of a form by one. It must satisfy three axioms:

$$i(v)^2 = 0 \tag{2b}$$

$$[\theta(v), i(v')] = i([v, v']) \tag{2c}$$

$$[i(v), D]_+ = \theta(v) \quad . \tag{2d}$$

In our example, we can satisfy conditions (2) by letting

$$\theta(v) \equiv \pounds_{(R_v)} \quad , \quad \text{Lie derivative} \tag{3a}$$

$$i(v) \equiv i_{(R_v)} \quad , \quad \text{Interior product.} \tag{3b}$$

Here R_v is the vector field on P associated to $v \in \mathbf{g}$.

Note that one has at once from (2) that

$$[D, \theta(v)] = 0 \quad , \tag{4}$$

so that θ in (3) really does commute with D. (Remember that in the example D is the exterior derivative.)

To summarize, an operation of $\mathbf{g}$ in a complex (Ω, D) consists of a choice of $\theta(\cdot)$, $i(\cdot)$ satisfying the axioms (2). The differential forms on any principal G-bundle provides an example of an operation, and in this context the notion is very powerful. For example, Chevalley's theorem on the cohomology of symmetric spaces is naturally phrased in terms of just such an operation [6]. Now however we will discard the example and focus on another, more interesting one: conformal field theory.

String Theory

Strictly speaking we will not be interested in an arbitrary conformal field theory, but only in string theories; that is, we ask for a BRST operator Q and associated ghost field b to be singled out. We will construct an operation of the Virasoro algebra, Vir, on a certain complex.

Again let $\mathcal{P} = \mathcal{P}_{g,1}$ be the moduli space of curves of genus g with one puncture and local coordinate. In order to get a differential complex, one might try taking the differential forms on $\mathcal{P}$ and tensoring them in with the bundle $\mathcal{H}$ of states. This doesn't quite work. Instead consider

$$\Omega \ \equiv \ \{\text{sections of } \mathcal{H}\} \otimes \{\text{skew forms on Vir}\} \quad . \tag{5}$$

Certainly given any $\mathcal{H}$-valued differential form $\widetilde{\omega}$ we get an $\omega \in \Omega$, just by letting

$$\omega(v_1, \ldots, v_p) \ \equiv \ \widetilde{\omega}(V_{v_1}, \ldots, V_{v_p}) \quad .$$

Here V_v is the tangent to $\mathcal{P}$ given by $v \in \text{Vir}$. The point of using (4) is that given a tangent to $\mathcal{P}$ we cannot go backwards and obtain a corresponding v; in general there is an ambiguity due to the "Borel" subspace of Vir [7].

The space Ω has two gradings: $\Omega^{p,n}$ is the subspace of p-forms of ghost number n. We can define a diagonal operator β as follows: for a section $s \in \Omega$,

$$\text{“ } \beta s = b(\cdot) \wedge s \text{ ”} \quad ,$$

or more precisely,

$$(\beta s)(v_1, \ldots, v_p) \ \equiv \ \frac{1}{p} \sum (-)^j b(v_j) s(v_1, \ldots, \widehat{v_j}, \ldots) \quad . \tag{6}$$

Then β leaves the combined grading $p + n$ unchanged.

To make Ω into a complex we must supply a differential D which raises p by one unit. First we can define a d on skew forms on Vir as follows: Roughly speaking,

$$\text{“}\mathrm{d} = \sum v^A \otimes \delta_{v_A}\text{”} \quad ,$$

where v_A is a basis of Vir and v^A is the dual basis. More precisely if s is a 0-form then $\mathrm{d}s$ is the 1-form given by $(\mathrm{d}s)(v) \ \equiv \ \delta_v s$, the derivative. Acting on 1-forms,

$$(\mathrm{d}s)(v, v') = \delta_v(s(v')) - \delta_{v'}(s(v)) - s([v, v']) \quad ,$$

and so on. Thus $\mathrm{d}^2 = 0$. Now we can define the desired D on $\mathcal{H}$-valued forms:

$$D \ \equiv \ \mathrm{d} - Q\beta \quad . \tag{7}$$

One can readily verify that $D^2 = 0$ when acting on, say, 0-forms:

$$\begin{aligned}(D^2 s)(u,v) &= (\mathrm{d}Q\beta s)(u,v) + (Q\beta \mathrm{d}s)(u,v) + (Q\beta Q\beta)(u,v)\\ &= \big([\delta_u(Qb(v)a) - Qb(v)\delta_u s + Qt(u)b(v)s] + (u \leftrightarrow v)\big) - Qb([u,v])s\\ &= 0 \quad .\end{aligned}$$

We can now identify the family of states $\Sigma\rangle$ over $\mathcal{P}$ associated to any string theory as a *closed 0-form*, that is, as a representative of a cohomology class for the complex (Ω, D). This follows since this state satisfies $Q\Sigma\rangle = 0$ [5]; one thus has

$$(D\Sigma\rangle)(v) = \delta_v\Sigma\rangle - Qb(v)\Sigma\rangle = (\delta_v - T(v))\Sigma\rangle \quad ,$$

and this vanishes because of the 'equation of motion' satisfied by $\Sigma\rangle$.

Next we need to find an appropriate $i(\cdot)$. This is simply the interior product of a skew form on Vir with a vector $v \in$Vir; it does not affect the Fock space part of Ω at all. Certainly one has that $(2b)$ is satisfied. Moreover the basic state $\Sigma\rangle$ is a 0-form and so automatically satisfies $i(v)\Sigma\rangle = 0$ for any v.

Using i we now use $(2d)$ to *define* θ and see whether $(2a,c)$ are satisfied. Thus

$$\begin{aligned}\theta(v) &\equiv Di(v) + i(v)D\\ &= \pounds_v + Q(\beta i_v + i_v\beta)\\ &= \pounds_v + Qb(v) \quad ,\end{aligned} \tag{8}$$

where $\pounds_v = [\mathrm{d}, i(v)]_+$ satisfies the desired $(2a)$. Then one easily shows that θ also satisfies $(2a)$ and so furnishes the desired representation of Vir on Ω. Finally it is trivial to check that

$$[\theta(v), i(u)] = [\pounds_v, i(u)] = i([v,u]) \quad .$$

Thus we have indeed found an operation of Vir on (Ω, D); the required axioms are satisfied on arbitrary sections of Ω, not just on the basic section $\Sigma\rangle$. Applied to $\Sigma\rangle$, however, we find that any string theory background supplies us with an *equivariant cohomology class* over $\mathcal{P}$: since

$$i(v)\Sigma\rangle = 0, \quad \theta(v)\Sigma\rangle = 0 \quad , v \in \mathrm{Vir} \quad , \tag{9}$$

we have a class in $H^0_{i=\theta=0}(\Omega)$.

We can say the same thing in another way which makes contact with the analysis in [5]. Let $\widetilde{\mu} = \langle\psi\beta^{2g-2}\Sigma\rangle$ correspond to the differential form on moduli space (actually on $\mathcal{P}$) for the insertion of the state ψ. If $Q\psi\rangle = 0$ then $\widetilde{\mu}$ is a closed differential form [5]. From the present viewpoint this follows at once from the fact that since $\Sigma\rangle$ is equivariant, then both $\Sigma\rangle$ and $\beta^{2g-2}\Sigma\rangle$ are closed under D.

Conclusion

It remains to be seen what fundamental significance, if any, this equivariant structure has for string theory proper. One can speculate that the correct generalization of string perturbation theory to a universal moduli space must retain this structure, and that any equivalence of perturbatively distinct string backgrounds will be due to the vanishing of *equivariant* cohomology classes on the universal space [8].

In any case the framework described here is certainly incomplete. No mention has been made of the *ring* structure of Ω, corresponding in the classical analogy to wedge product of differential forms. Undoubtedly the correct way to define a product on our Ω is via *sewing* of surfaces. Since universal moduli space, whatever it is, will contain all of the different genera moduli spaces in close proximity (see for example [9]), one may then wonder whether the equivariance condition (9) admits a generalization. Parallel to the locus of any given genus, (9) as usual describes the variation of a state as the moduli are changed. Transverse to the locus, perhaps a generalization of (9) will exist joining different genera. This relation could be the infinitesimal analog of sewing; its flatness conditions could be infinitesimal generators of the sewing consistency relations.

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