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Effective Field Equations for Fermionic Strings

HoSeong La

Philip C. Nelson

University of Pennsylvania, nelson@physics.upenn.edu

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Abstract

We show how to obtain loop-corrected effective field equations in tachyon-free heterotic string theories. The corrections are automatically gauge-invariant; they simultaneously secure Brst-decoupling and unambiguous loop amplitudes. We introduce a cutoff for two-dimensional field theory which is more general than the choice of a world-sheet metric, and using this we give an invariant description of factorization of fermionic amplitudes. As an example we work out the linearized corrections for the $O(16) \times O(16)$ heterotic string. They come from a finite corrected effective action.

Disciplines

Physical Sciences and Mathematics | Physics

Effective Field Equations for Fermionic Strings

HoSeong La[†]

Physics Department

Boston University

Boston, MA 02215 USA

Philip Nelson

Physics Department

University of Pennsylvania

Philadelphia, PA 19104 USA

We show how to obtain loop-corrected effective field equations in tachyon-free heterotic string theories. The corrections are automatically gauge-invariant; they simultaneously secure BRST-decoupling and unambiguous loop amplitudes. We introduce a cutoff for two-dimensional field theory which is more general than the choice of a world-sheet metric, and using it give an invariant description of factorization of fermionic amplitudes. As an example we work out the linearized corrections for the $O(16) \times O(16)$ heterotic string. They come from a finite corrected effective action.

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[†] Address after June 1st, 1989: Physics Department, University of Pennsylvania, Philadelphia, PA 19104.

1. Introduction

Perturbation theory for strings is superficially quite similar to its counterpart in quantum field theory. Each involves a prescription to separate a fixed background from fluctuations and then sum over all spacetime configurations of those fluctuations. In particular, field theory involves an integral over Feynman parameters describing the lengths of internal lines in a graph, while string theory involves an integral over an analogous “moduli space” of distinct 2-surfaces.

The structure of moduli space is complicated for several reasons. For one thing, given a surface it is impossible to say what precisely is the length of any internal tube. Not only have we no metric on the surface, but also we have no precise interaction points on a surface whose separation we could measure. Secondly, the various legs are attached to vertices with a twist, which has no analog in field theory. Finally, certain distinct-looking curves are actually conformally equivalent; to avoid overcounting we must make some identifications on the space of “Feynman parameters” before it becomes the desired moduli space.

Each of the above complications is vital to the finiteness and consistency of string theory. Each, however, contributes to the difficulty of carrying out a systematic perturbative analysis of string amplitudes. In particular, while modular invariance suffices to ensure ultraviolet finiteness, still many interesting string theories suffer from infrared divergences. The origin of these divergences is simple: dynamical effects destabilize the naive tree-level vacuum, making it unsuitable as a starting point for a perturbation expansion. As in any perturbative dynamical symmetry breaking problem, one needs to determine appropriate background shifts order by order in the coupling constants; they are fixed by the requirement that the corrected perturbation theory should be well-defined. Lovelace and Mansfield proposed that a similar mechanism might work for the string [1][2].

In string theory, however, any such procedure is complicated by at least two facts. First, as we already observed there is no natural IR cutoff procedure. Secondly, the required background shifts in general destroy world-sheet

conformal invariance. Each of these problems is without analog in field theory. The fundamental observation of Fischler and Susskind is that each cancels the other [3], leaving us with a well-defined prescription.

The mechanism proposed in [3] and refined in [4][5][6][7] (and elsewhere) works as follows. We must impose a single cutoff on both worldsheet field theory and on our moduli integrals. Then we determine counterterms in the $2d$ action by requiring a good limit as the cutoff is removed. These counterterms are then interpreted as changes in d -dimensional background fields, and we find modified field equations for which the modified backgrounds are solutions. The $2d$ cutoff requires us to choose some additional data not contained in the given Riemann surface. In [3][5][8] and elsewhere this choice was made implicitly, buried in a coordinate choice. Analogous constructions were given for fermionic strings in [9] [10] and elsewhere.

It is always desirable, however, to bring such choices out into plain view. Formalisms in which all chosen geometrical objects are explicitly visible are called “covariant”; thus for example special relativity is not covariant because the choice of a particular metric has been implicitly made. One normally works in special “inertial” coordinates in which this metric is a constant; by acknowledging the metric and assigning its appropriate transformation properties we can promote the formulas of special relativity (*e.g.* Maxwell’s equations) to coordinate-invariant status. Similarly it is desirable to disclose the geometrical object needed to cut off string amplitudes: we want a *superconformally covariant* cutoff.

Polchinski has shown how the choice of a worldsheet metric does the job for the bosonic string, simultaneously cutting off narrow necks and defining the normal ordering needed to insert background corrections off the usual mass shell [7]. In this paper we will make a different choice, one better suited to the fermionic string, but the principle is similar [11]. In any case, having brought this choice into view one can investigate the circumstances under which it drops out. This turns out to determine the background shifts, much as the tree-level condition of conformal invariance determines the tree-level effective action [12][1]. Remarkably one finds that the resulting *consistency* condition coincides

with the *stability* (no-tadpole) condition [3], a fact which was less surprising at tree level. The condition also arises from enforcing BRST decoupling, as conjectured by Mansfield [2] and shown for open strings in [9][10].

This paper has two main goals. First, we will show how the mechanism of [3] works for degenerations of any kind. In particular, a simple geometrical fact about the boundary of moduli space underpins the remarkable coincidence above; moreover exactly the same property continues to hold in the heterotic case. We explain this property and its consequences in sections 2 and 3. We also recall how the notion of stable punctured curves (and supercurves) lets one give a unified treatment of various degeneration phenomena [13].

Our other purpose is to show in some detail how all of our considerations extend to the super case [14] [15]. There has been some confusion in the literature on this point (see [15]), some of which can be traced to the use of explicit coordinates for supermoduli space. As discussed above, coordinate-dependent formalisms have hidden an implicit choice of cutoff prescription [16][17][18]. We have found that the operator formalism provides a suitably covariant framework to understand the issues, and in particular to implement the geometry discussed in section 3. While an operator approach was employed in [9][10], our method will have more in common with the bosonic analysis of [7], as it is not so closely tied to the one loop, open string case. Mandelstam has also given a prescription for cutting off amplitudes in the case of vanishing tadpoles which may turn out to be equivalent to the one presented here [19].

Specifically we will supply the general principle requested in the conclusion to [9] and verify the conjecture of [9], [18] that enforcing BRST invariance through the Fischler-Susskind mechanism suffices to guarantee well defined answers. We will also supply the “more geometrical notion” requested in the conclusion of [16] needed to generalize their genus-two cutoff prescription. The key technical points of our approach are that we work on supermoduli space instead of first attempting to integrate over odd directions¹ and we employ

¹ The dangers inherent in such attempts were pointed out by D. Kazhdan in July 1986.

the operator formalism described in ref. [20]. In particular our approach differs from some previous ones in that we use the superconformal description of super Riemann surfaces rather than the description with world sheet metrics. We replace the world-sheet metric, which spoils the manifest separation of left- and right-movers, by a family of local superconformal coordinates. The latter construction is holomorphic and may well play a role in taking the theory off shell; in any case it provides a more general cutoff for modular integrals and 2d field theory than does a world-sheet metric. Also we make no use of Beltrami differentials, delta-function gravitini, Weierstrass-point slices, nor projection or splitting of supermoduli space. Some of these latter techniques may well prove extremely useful for practical calculations, however. Throughout we will emphasize that the main physical issues arise already in the bosonic case, and are not inherently supergeometrical. Issues concerning the integral over the odd moduli have been dealt with in many ways [17][21][22][18][23][24][25], but for the present purposes we will not need detailed answers.

We will see how, at least to linearized order, a universal, finite correction to the effective equations of motion for massless physical fields in the $O(16) \times O(16)$ heterotic string [26] suffices to secure unambiguous vacuum amplitudes [14][15]. We also sketch how nonlinear corrections, for example mass renormalization [27][28][29][13], are in general both necessary and sufficient to remove ambiguities in higher-point amplitudes; they arise even in the supersymmetric heterotic theories. One can however conclude that in the latter theories flat spacetime is an exact (perturbative) solution, without BRST anomalies [18]. The corrected equations are automatically gauge-invariant, and they are the variational equations for a loop-corrected effective action. It seems likely that this situation persists to all orders, as proposed in the bosonic case by Tseytlin[6], but we will not pursue this idea here.

Sections 2 and 3 describe the relevant geometry and super geometry; an essential ingredient is the Beilinson-Kontsevich action of Virasoro on the extended moduli space [30]. We then review notation for superconformal field theory and some ideas of [31][32], as well as the “conformal normal ordering” prescription of Polchinski [33], and ideas from [34][35][20]. In sections 5 and 6

we discuss super factorization and the use of vacuum shifts to kill tadpoles, and in sections 7 and 8 we apply the technique to the $O(16) \times O(16)$ string.

The well-versed reader will find several sections of this paper to be very familiar. We have tried to present some existing results in a unified way, one which makes the super versions immediate once the bosonic versions are given. We have also tried to clarify some obscurities in the literature. Nevertheless we encourage the reader to peruse sections 2.3, 3.2, 3.4 and then pick up the story starting with 5.2. Some of the notation is reviewed in Appendix B.

2. Geometry

2.1. Coordinates

We recall the notion of augmented moduli space [30].² Let $\mathcal{M}_g$ denote the space of conformally distinct smooth Riemann surfaces with g handles. Let $\mathcal{M}_{g,n}$ denote conformally distinct smooth Riemann surfaces with g handles and n labeled points. The points may not coincide, and if we permute them we in general obtain a different pointed Riemann surface. We also denote by $\mathcal{P}_{g,n}$ the pointed surfaces on which we have chosen a local complex coordinate z_i near each of the marked points P_i . Each z_i is centered, $z_i(P_i) = 0$, and each is defined on some open neighborhood of P_i . Thus any other choice z'_i can be expressed as $z'_i = \sum_{n=0}^{\infty} A_n z_i^{n+1}$, in some suitable neighborhood of P_i .³

We will denote a triple (X, P, z) in $\mathcal{P}_{g,1}$ as $\tilde{X}$. Of course, the explicit mention of P is redundant; once z is given we have $P = \{z = 0\}$, so sometimes we will write $\tilde{X} = (X, z)$. Given $\tilde{X}$ we can always throw away z to obtain $(X, P) \in \mathcal{M}_{g,1}$. More generally, we always have a projection $\pi : \mathcal{P}_{g,n} \rightarrow \mathcal{M}_{g,n}$. Another useful way to think about π is to say that $\pi(X, z) = (X, [z])$, where $[z]$ is z taken modulo multiplication by any local holomorphic function f not

vanishing at P . Then $f \cdot z$ is another good coordinate again with a simple zero at the same point P .

Sometimes we will write $\mathcal{P}$ for $\mathcal{P}_{g,1}$ and $\mathcal{M}$ for $\mathcal{M}_{g,1}$ when the meaning is clear.

We can modify z in another way. Suppose G is some analytic map on the plane $\mathbb{C}$ which preserves the origin. Infinitesimally

$$G(w) = w + \sum_{n=0}^{\infty} \epsilon_n w^{n+1}. \quad (2.1)$$

Then the composition $z' = G \circ z$ is a new local coordinate centered at the same point P as before. Since we can compose the maps G , we see that $\pi : \mathcal{P}_{g,1} \rightarrow \mathcal{M}_{g,1}$ is a principal fiber bundle. The Lie algebra of its group we will call the positive Virasoro algebra, Vir_+ , and its generators ℓ_n , $n \geq 0$.

Even if G moves the origin we can still define the above action, but now it moves the point P . For example if $G(w) = w + \epsilon_{-1}$, then $z' = z + \epsilon_{-1}$ vanishes at $P' \neq P$, where P' is defined precisely by $z(P') = -\epsilon_{-1}$. In fact we can even define an action on $\mathcal{P}_{g,1}$ of any $G : \mathbb{C} \rightarrow \mathbb{C}$ which is holomorphic and single-valued on some neighborhood of the origin (excluding 0 itself) and meromorphic at 0. The Lie algebra of such G we call Vir . To define the action suppose first that z is a well-defined map to the unit disk of $\mathbb{C}$ and G is well-defined on this disk, except at 0. Excise the unit z -disk from X and reattach it using the map G restricted to an annulus (Fig. 1). This gives a new curve X' with a new local coordinate z' . If we like we can instead begin with a smaller disk and rescale everything to arrive at the same action; then if the original z or G is defined on some arbitrary domain, we just cut out a small enough disk around P and apply the above prescription⁴.

Our convention for generators for Vir is that ℓ_n generates the transformation $G \circ z = z - \epsilon z^{n+1}$. This diffeomorphism is generated by a vector field $v = v^z \frac{\partial}{\partial z}$ on $\mathbb{C}$:

$$\ell_n \leftrightarrow -z^{n+1} \frac{\partial}{\partial z}. \quad (2.2)$$

⁴ Strictly speaking what we get in this way is not an action of the group of G 's but rather of its Lie algebra Vir .

² The expert reader will want to skip to section 2.2.

³ Mathematically $\mathcal{P}_{g,n}$ involves the choice of the *germ* of a coordinate near each P_i ; the following discussion can be more precisely phrased in this language.

Fig. 1: (a) A surface with local coordinate; the shaded region is mapped to the disk; (b) Action of Vir_+ ; (c) Action of Vir ; the dotted regions are discarded.

If we allow reparameterizations G which are not holomorphic then we get another set of generators

$$\bar{\ell}_n \mapsto -\bar{z}^{n+1} \frac{\partial}{\partial \bar{z}}.$$

Strictly speaking only *real* vector fields generate diffeomorphisms; thus only generators of the form $\epsilon_n \ell_n + \bar{\epsilon}_n \bar{\ell}_n$ make sense, where the numbers $\bar{\epsilon}_n$ are complex conjugate to ϵ_n . But having said that we see the sense in which $\epsilon_n \ell_n$ by itself gives a vector in the complexified tangent of $\mathcal{P}$. In fact it is a holomorphic tangent.

We also note that Vir has an involution associated to the inversion $z \rightarrow w = z^{-1}$ of the plane. Since $v^w(w) = -z^{-2}v^z(z^{-1})$ we have that

$$(\ell_n)^T = -\ell_{-n}. \quad (2.3)$$

We will denote the action of Vir on $\mathcal{P}$ by $i_{\tilde{X}}(v)$, a tangent to $\mathcal{P}$ at $\tilde{X}$ corresponding to $v \in \text{Vir}$. We have seen that for $v \in \text{Vir}_+$ this deformation

of $\tilde{X}$ is vertical; it does not affect $\pi(\tilde{X}) = (X, P)$. More generally $i_{\tilde{X}}(v)$ will project to some tangent on $\mathcal{M}$, denoted by $\pi_* i_{\tilde{X}}(v)$. We stress that Vir does *not* act directly on $\mathcal{M}$, however, if $\tilde{X} = (X, P, z)$ and $\tilde{X}' = (X, P, z')$ then $i_{\tilde{X}}(v)$ and $i_{\tilde{X}'}(v)$ will in general project to *different* vectors on $\mathcal{M}$ (Fig. 2). Similarly $\pi : \mathcal{P} \rightarrow \mathcal{M}$ has no natural connection, or covariant derivative.

Fig. 2: The same element of Vir can generate different motions on $\mathcal{M}$ depending on z .

We will be interested in constructions which require us to *choose* a local coordinate z . More precisely for every (X, P) we want to choose a z which is a complex coordinate for X centered at P ; equivalently for every $(X, [z])$ we want a representative z of the class $[z]$. Let σ be such a choice: $\sigma(X, P) \equiv (X, P, z_{(X,P)})$. Note that $z_{(X,P)}(Q)$ for fixed X is a function of *two* points on X , with the property that $z_{(X,P)}(P) = 0$. Also we require that it is holomorphic in Q , but *not* necessarily in P or X .

We certainly cannot always choose σ globally. If we could then for X fixed $dz_{(X,P)}|_P$ would be a nowhere-vanishing cotangent field on X , but these do not exist unless X is a torus. If we allow σ to jump by a phase, though, then there is no further obstruction. That is, we take σ to be defined on patches; across patch boundaries we permit jumps of the form $z'_{(X,P)}(Q) = e^{i\alpha(X,P)} z_{(X,P)}(Q)$ where α is a real function independent of Q . Infinitesimally, then, z and z' are related by the action of a single generator, namely $\ell_0 - \bar{\ell}_0$. We will see that a coordinate family defined up to $U(1)$ will be all we need to cut off string amplitudes [7].

Given a point (X, P) of $\mathcal{M}$ we have no preferred $\tilde{X}$ in $\mathcal{P}$ which projects to (X, P) . Even if $\tilde{X}$ is given, we still don't know how to lift a tangent $V \in$

$T_{(X,P)}\mathcal{M}$ to $\tilde{V} \in T_{\tilde{X}}\mathcal{P}$; any $\tilde{V}$ with $\pi_*\tilde{V} = V$ will do. If we have a *slice* σ near (X, P) , though, then we can simply require that $\tilde{V}$ be parallel to the slice (Fig. 3) [11].⁵ Suppose for example that near Q our coordinate family obeys

$$z_P(\cdot) = z_Q(\cdot) - \Delta_{PQ}, \quad (2.4)$$

where $\Delta_{PQ} = z_Q(P)$ is a small number and X is fixed. Certainly $z_P(\cdot)$ is centered at P and holomorphic in both P and Q . Keeping X fixed, a coordinate for the modulus corresponding to the location of P is then just $u(X, P) = z_{Q_0}(P)$, the location of P in one *fixed* coordinate system. We can now read off the lifted vector $\tilde{V}$, which we will also denote by $\sigma_*(V)$. For example, we get

$$\sigma_* \left(\frac{\partial}{\partial u} \right) = i_{\tilde{X}}(\ell_{-1}) \quad , \quad \text{CNO family.} \quad (2.5)$$

The abbreviation stands for “conformal normal ordering”[33], for reasons to become clear later. See (2.2) for why (2.4) implies (2.5).

Fig. 3: Lifting a vector V using a slice σ .

The most general coordinate family is equally easy. To first order in Δ_{PQ} we must have

$$z_P(\cdot) = z_Q(\cdot) - \Delta_{PQ} + \sum_{n=0}^{\infty} (E_n \Delta_{PQ} + F_n \bar{\Delta}_{PQ}) z_Q(\cdot)^{n+1} + \mathcal{O}(\Delta_{PQ}^2). \quad (2.6)$$

⁵ Note that this use of the word “slice” has nothing to do with the slices of [17][18].

for some coefficients E_n, F_n depending on P . The extra terms are just a family of transformations (2.1), not necessarily holomorphic in P , reducing to the identity at $P = Q$. We then read off

$$\sigma_* \left(\frac{\partial}{\partial u} \right) = i_{\tilde{X}}(\ell_{-1} - E_n \ell_n - \bar{F}_n \bar{\ell}_n) \quad , \quad \text{general family.} \quad (2.7)$$

The last term comes from the $\mathcal{O}(\Delta_{PQ})$ term of $\tilde{z}_P(\cdot)$.

One way to obtain a coordinate family is described in [7]: we first choose a family of metrics for the family $\mathcal{M}_g$ of Riemann surfaces. Near each point Q of each X we then find a $z_{(X,Q)}$ in terms of which the given metric is as close as possible to the unit matrix. With this choice one finds that $F_1 = -\frac{1}{8}R$, $E_0 = E_1 = F_0 = 0$, where R is the scalar curvature of the chosen metric at Q . We will not make any use of this special family, however. For one thing, all of our formulas work with a general family. Moreover the analog in the heterotic string of a metric is a frame satisfying torsion constraints [36]; while this approach is equivalent to the superconformal route we will take [37], still it is easier to proceed without the constraints.

2.2. Sewing

We briefly review here some geometry associated to sewing, and then describe the key geometrical observation alluded to in the introduction. We will mainly be interested in separating pinches, returning later to the nonseparating ones.

We can turn $\mathcal{M}_g$ into a nice compact orbifold by admitting a special class of degenerate surfaces, the “stable curves”. The notion of stability is reviewed in [38]; essentially we allow only ordinary double points, and we allow only those curves which actually arise as limits of families of conformally-inequivalent surfaces. In particular attachment points never collide; a family where two such points approach each other is conformally equivalent to a family where they stay separate and a third pinch forms somewhere else.

The notion of stability extends to pointed curves [39]. Now the marked points never collide with each other, nor with attachment points. See Fig. 4.

Fig. 4: To prevent punctures from colliding we replace (a) by the conformally-equivalent (b).

In this way we see that the contact terms studied by Green and Seiberg, which arise from the collision of vertex operators, are really on the same footing as the other degeneration effects studied here [40][13].

Thus every pinched stable curve X can be decomposed into smooth pointed curves $X_1, \dots$ joined at certain of their punctures. Similarly given $X_1, \dots$ we can “sew” them to get X . The procedure is completely canonical once we say which points are to be joined. Counting dimensions we find that the locus Δ of once-pinched curves has one fewer complex dimension than does $\mathcal{M}_g$ itself: it is a “divisor” in $\mathcal{M}_g$.

We would like a way to exclude a tube around Δ , and hence cut off string integrals, which typically diverge there. We can do this locally as follows. Choose a coordinate q for $\mathcal{M}_g$ near Δ such that on Δ everywhere $q \equiv 0$ but $dq \neq 0$. Then let $\mathcal{M}_{g,\delta} = \{|q| > \delta\}$. The first condition means that dq annihilates the tangent to Δ , or in other words that dq is in the “conormal” bundle to Δ . The second condition says that dq “trivializes” the conormal bundle. Since in general this bundle is not trivial, q cannot be globally defined. Even if it were, however, we would still need to investigate the effects of different choices of cutoff (choices of transverse coordinate q) on our answers.

Given any local choice of q as above, and in particular a holomorphic choice, we can write any other choice as

$$q' = F(\vec{m}_{(1)}, \vec{m}_{(2)}, q) \cdot q \quad , \quad (2.8)$$

where F is a function on $\mathcal{M}_g$ which never vanishes on Δ . In general F is some arbitrary function. We now want to find a special class of transverse coordinates q which are always related to each other in a more specific way.

Consider a Riemann surface X_1 of genus g_1 with one puncture P_1 , and another X_2 of genus g_2 with puncture P_2 . We saw how to glue these and obtain a pinched curve. Given a complex number $q \neq 0$ we would like to construct a *nearly*-pinched surface X , but to do so we need more information. Choose local coordinates z_1, z_2 near P_1, P_2 . We can now excise the disks $\{|z_i| < |q|\}$ and glue $\{|q| < |z_1| < 1\}$ to $\{|q| < |z_2| < 1\}$ via the map $z_2 = q/z_1$; as $q \rightarrow 0$ we recover the pinched surface. What we have defined, then is not a unique isomorphism $\mathcal{M}_{g_1,1} \times \mathbb{C} \times \mathcal{M}_{g_2,1} \simeq \mathcal{M}_g$ but rather a map [41]

$$\infty : \mathcal{P}_{g_1,1} \times \mathbb{C} \times \mathcal{P}_{g_2,1} \rightarrow \mathcal{M}_g. \quad (2.9)$$

It is certainly many-to-one, since $\mathcal{P}$ is infinite-dimensional.⁶ Similarly one obtains

$$\infty : \mathcal{P}_{g_1,N_1+1} \times \mathbb{C} \times \mathcal{P}_{g_2,N_2+1} \rightarrow \mathcal{P}_{g_1+g_2,N_1+N_2} \quad . \quad (2.10)$$

To get *coordinates* for $\mathcal{M}_g$ we begin with a *family* of X_1 parameterized by $\vec{m}_{(1)}, i = 1, \dots, 3g_1 - 2$. (One modulus describes the location of P_1 .) Repeat for side two. Now choose families $\sigma_{1,2}$ of local coordinates. Composing $\sigma_{1,2}$ with the above map ∞ gives us the desired map $A : \mathcal{M}_{g_1,1} \times \mathbb{C} \times \mathcal{M}_{g_2,1} \rightarrow \mathcal{M}_g$, and with its coordinates $\vec{m}_{(1)}, \vec{m}_{(2)}, q$ for $\mathcal{M}_g$ near Δ , but now A depends on σ_1 and σ_2 . If we change σ_i to σ'_i then the *same* values of $\vec{m}_{(1)}, \vec{m}_{(2)}, q$ will correspond to a *different* surface X' . Phrased differently, changing σ_i to σ'_i induces a coordinate change

$$\vec{m}_{(i)} \mapsto \vec{m}'_{(i)}(\vec{m}_{(i)}, q) \quad (2.11)$$

$$q \mapsto q'(\vec{m}_{(i)}, q) \quad . \quad (2.12)$$

⁶ Some authors remove the redundant parameter q from (2.9), setting $q = 1$. We will remove the redundancy by choosing a coordinate family σ for $\mathcal{P}$, so we retain q .

It is of course the change in q which interests us most.

We claim that under this transformation (2.12) takes the form

$$q' = F_1(\vec{m}_{(1)})F_2(\vec{m}_{(2)})q + \mathcal{O}(q^2) \quad (2.13)$$

Furthermore,

$$F_h(\vec{m}_{(h)}) = \alpha_h, \quad \text{where} \quad z'_h = \alpha_h z_h + \mathcal{O}(z_h^2) \quad (2.14)$$

As promised, the transformation of q takes a very special form. For a general divisor the best we could have hoped for was a relation of the form (2.8). Before proving this, and finding the functions $F_h(\vec{m}_{(h)})$, let us remark on the significance of (2.13). The coefficient of q on the right side is the transition function for the conormal bundle on Δ , since q^2 can't contribute to $dq|_{q=0}$. We know that Δ has the product structure $\Delta \simeq \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$. What (2.13) says is that the conormal bundle *also* has a natural product structure.⁷ It is this "factorisation" of the first Taylor coefficient of q' which makes the Fischler-Susskind mechanism work; in the fermionic case it will again hold, and it will eliminate the integration problem [14][15].

To prove (2.13), first consider a special case. Let $z'_{(X_1, P_1)} = \alpha(\vec{m}_{(1)})z_{(X_1, P_1)}$ and $z'_{(X_2, P_2)} = z_{(X_2, P_2)}$, i.e. just rescale z_1 . Clearly for given $\vec{m}_{(1)}, \vec{m}_{(2)}$ we get precisely the same X if we glue z_2 to z_1 using q as when we glue z_2 to z'_1 using αq . Thus we have (2.13) with $F_1 = \alpha$, $F_2 \equiv 1$ and no $\mathcal{O}(q^2)$ terms, and similarly if we rescale $z_{(X_2, P_2)}$ as well.

Next consider the infinitesimal transformation $z'_{(X_1, P_1)} = z_{(X_1, P_1)} + \epsilon(z_{(X_1, P_1)})^{n+1}$, $n > 0$. This is generated by the vector field $v = z^{n+1} \frac{\partial}{\partial z}$ on the z -plane (Fig. 5). We need to find q' and distorted surfaces $(X'_1, P'_1), (X'_2, P'_2)$ such that when $(X'_1, P'_1, z'_{(X'_1, P'_1)})$ is joined to $(X'_2, P'_2, z_{(X'_2, P'_2)})$ using q' , then resulting X agrees with joining $(X_1, P_1, z_{(X_1, P_1)})$ to $(X_2, P_2, z_{(X_2, P_2)})$ using q .

It's easier than it sounds. Instead of letting v act on z_1 we can pass it through the inversion map and let it act on z_2 via $v^T = -\epsilon q^n z^{-n+1} \frac{\partial}{\partial z}$, $n > 0$. Note the factor of q . For large enough n the Beilinson-Kontsevich action of v^T

Fig. 5: Joining X_1 to X_2 to get a genus-3 surface. The transformation G is generated by $v = z^{n+1} \frac{\partial}{\partial z}$.

on (X_2, P_2, z_2) is *trivial*, and we get the same curve X by choosing $\vec{m}'_{(1)} = \vec{m}_{(1)}$, $q' = q$. For $n = 1$ we shift P_2 , while for intermediate values we distort X_2 itself to some $\tilde{X}_2$. The important point to note is that these shifts are already $\mathcal{O}(q)$.

Consider $n = 1$, so $v^T = -\epsilon q \frac{\partial}{\partial z}$. We have just seen that joining $(X_1, P_1, z_{(X_1, P_1)})$ to $(X_2, P_2, z_{(X_2, P_2)})$ via q is the same as joining $(X_1, P_1, z'_{(X_1, P_1)})$ to $(X_2, P'_2, \hat{z})$ via the same q if we take $\hat{z} = z_{(X_2, P_2)} - \epsilon q$ and P'_2 such that $\hat{z}(P'_2) = 0$. This is not precisely what we want, however. Even though $\hat{z}$ is centered on P'_2 , still we don't know that $\hat{z} = z_{(X_2, P'_2)}$. In general they will be related by an expansion similar to (2.6), where ϵq is substituted for Δ_{PQ} .

To replace $\hat{z}$ by $z_{(X_2, P'_2)}$ we now repeat the preceding two paragraphs, moving now from side two to side one. The coefficients $\epsilon q E_0, \epsilon q F_0$ from (2.6), if nonzero, will change q to q' , but now the difference will be $\mathcal{O}(q^2)$ since $q E_0, q F_0$ are already $\mathcal{O}(q)$. The higher $q E_n, q F_n$ will similarly distort (X_1, P_1) , by an amount of order q^2 once we pass them back through the inversion.

In short, when we change coordinate families σ_1 and σ_2 we induce a coordinate transformation of the form (2.13). The *lowest* Taylor coefficients of z' give rise to the factorized first term of (2.13). The higher Taylor coefficients

⁷ In fact it is the tensor product of the canonical bundles on $\mathcal{M}_{g_1}$ and $\mathcal{M}_{g_2}$.

give the messy, $\mathcal{O}(q^2)$ part. Eventually we will see that the latter is irrelevant for our purposes, as is the messy transformation (2.11).

2.3. Some comments

The previous subsection was a bit complicated, but the conclusion is simple:

While there is no standard way to excise a tube around Δ , still there is a coordinate-invariant sense in which a tube can respect the product structure near Δ . Such tubes are defined by $\{|q| < \delta\}$ where q is defined with the plumbing construction. They are all related by (2.13).

We will later show how cutting off integrals using such tubes can lead to a successful renormalization theory.

Let us compare the present situation to the one-loop open string case [5][8]. In that case the analog of q is a *real* number, since both the disk and crosscap are azimuthally symmetric. Moreover, both disk and crosscap are rigid; there are no moduli $\bar{m}_{(2)}$ for one side of the diagram (Fig. 6). For both these reasons, this case does not display the full dependence of the cutoff on choices. Similar remarks apply to the fermionic case [9][10]. Thus for a general degeneration we will not be able to summarize one side of a pinching diagram by one universal "boundary state". Nevertheless the BRST-cancellation approach of [9][10] will survive the generalization, just as in the bosonic string [7].

Fig. 6: Two conformally-equivalent views of a one-loop type-I divergence.

The divisor Δ can intersect itself. This happens for example when the dashed curves in Fig. 7a,b pinch off; the same curve Fig. 7c then arises in two different ways. We will not discuss this phenomenon in detail. Note however that it has a precise analog in field theory. Multiple degenerations lead to terms of order $(\log \delta)^2$, where δ is the cutoff. These terms in any given genus are automatically taken care of by lower-order renormalizations; the combinatorics which ensure this are the same as for ordinary φ^3 theory [6][42][43].

Fig. 7: Two different curves which approach the same twice-pinched curve.

More precisely we expect that one can systematically implement renormalization theory by induction on the number $r = 3g - 3 + n$ used by Sonoda [44]. Since for stable curves all pinches are independent and well-separated, one can associate counterterm diagrams to a multiply-pinched graph as easily as for one pinch. Since r decreases by one unit each time we pinch and normalize, one can consistently correct the equations of motion to each order in r before moving to the next. It would be worthwhile to work out the details of how this works.

3. Super Geometry

Many of the ideas used here are reviewed in [45] and references therein.

3.1. Super coordinates

Supermoduli space $\hat{\mathcal{M}}_g$ has a compactification by stable super curves [46]. Once again these are built from smooth super Riemann surfaces (SRS) joined

by separated, universal degenerations. Now however, there are *two* distinct degenerations [46][47], the super (or Neveu-Schwarz) pinches and the spin (or Ramond) pinches. In this paper we will deal only with super pinches, since in the heterotic string in flat spacetime only spacetime bosons can disappear into the (tree-level) vacuum.

There is also a straightforward notion of stable punctured SRS [46] (for an example see [45]). Once again punctures never collide with each other or with pinches. Once again there is a divisor Δ consisting of pinched SRS.

String theory supplies us with a well-defined, unambiguous integration density on $\hat{\mathcal{M}}_g$, or on $\hat{\mathcal{M}}_{g,n}$ for scattering amplitudes. This is particularly clear in the operator formulation [20][48]. Recall that a density on a supermanifold of dimension $p|q$ is a multilinear form which eats a basis of p even and q odd tangent vector fields and gives a function, much like an ordinary top form. If we take a linear transformation of the original basis, however, μ must transform by the superdeterminant, not the ordinary determinant, of the change.

A density of compact support has a well-defined integral (see *e.g.* [49][45]). Thus to integrate μ we can take any open cover $\{\mathcal{U}_\alpha\}$ of the compactified $\hat{\mathcal{M}}$ and any associated partition of unity $\{\rho_\alpha\}$. If a given $\mathcal{U}_\alpha$ does not intersect Δ then we simply integrate $\int_{\mathcal{U}_\alpha} \rho_\alpha \mu$ without further difficulty. In particular, on each $\mathcal{U}_\alpha$ we choose nonsingular coordinates, so that ‘spurious poles’ [50] due to degenerating coordinates never arise. Thus we see that in *principle* there is interesting physics only at the boundary Δ .⁸ In *practice* it will surely be helpful to employ the methods of refs. [17][21][22][18][23][24][25] to compute integrals on the interior of $\hat{\mathcal{M}}$, but these methods are not our present concern. Thus we will not worry about the existence of a modular-invariant slice of the universal curve, since strictly speaking it is not required. Nor indeed do we need global slices in the sense of [17] at all, meromorphic or otherwise. μ is globally defined, and we can integrate it in any atlas of good local coordinate systems.

We *do* need to understand the boundary. There is a well-defined integral over a supermanifold-with-boundary [49], so we now turn to the super version of the remarks in the previous section on how to exclude a tube surrounding Δ .

⁸ The preceding argument is due to E. Witten.

The space $\hat{\mathcal{P}}_{g,1}$ consists of SRS with a chosen “point” P and local superconformal coordinate $\mathfrak{z} = (z, \theta)$ centered at P . We put “point” in quotes because in general it is a problematic construction in superspace; in our case, however, P is again wholly redundant once $\mathfrak{z}$ is given. In fact we can again project $\hat{\mathcal{P}}$ to $\hat{\mathcal{M}}$ by sending (X, z, θ) to $(X, [z])$ where $[z]$ is z taken modulo multiplication by invertible local functions. Given such a z we recover P as “the point where $z = Dz = 0$,” and clearly $f \cdot z$ defines the same P for invertible f .

We now get an action of the Neveu-Schwarz algebra $\text{Diff } S^{1|1}$ on $\hat{\mathcal{P}}$ as before. The algebra consists of superconformal vector fields meromorphic near the origin of the z, θ plane; we take its generators to be

$$\begin{aligned} \ell_n &\leftrightarrow -z^{n+1} d\mathfrak{z}^{-2} \leftrightarrow -z^{n+1} \frac{\partial}{\partial z} \\ g_h &\leftrightarrow -\theta z^{h+1/2} d\mathfrak{z}^{-2} \leftrightarrow \frac{1}{2} z^{h+1/2} \left(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right) \end{aligned} \quad (3.1)$$

In the middle we list (-1) -superdifferentials; the corresponding superconformal vector fields appear on the right. For details see [51] [45]. Composing the corresponding transformations G of the plane with local coordinates z, θ gives an action on $\hat{\mathcal{P}}$ as before.

As in section 2.1, once given a fixed local coordinate z, θ we can obtain a coordinate family analogous to (2.4), which we will call the “superconformal normal ordering” (SCNO) family. Suppose that Q is a point located at $(z, \theta) = (u, \zeta)$. (More precisely, $Q = [z - u + \zeta\theta]$, where u, ζ are regarded as parameters, z, θ as coordinates on X .) Then to Q we associate

$$(z_Q, \theta_Q) = (z - u + \zeta\theta, \theta - \zeta), \quad (3.2)$$

a superconformal coordinate centered at Q . This association is not natural, as one finds by starting with another (z', θ') . For now, however, (z, θ) will be *fixed* while (z_Q, θ_Q) vary with Q .

The infinitesimal transformation $\epsilon \ell_{-1}$ takes $(z - u + \zeta\theta, \theta - \zeta)$ to $(z - u + \zeta\theta - \epsilon, \theta - \zeta)$. But the same thing can be accomplished by sending $u \mapsto u + \epsilon$. We thus have

$$\sigma_* \left(\frac{\partial}{\partial u} \Big|_Q \right) = i_{\sigma(Q)}(\ell_{-1}), \quad \text{SCNO family.} \quad (3.3)$$

A similar argument shows that

$$\sigma_* \left(\frac{\partial}{\partial \zeta} \Big|_Q + \zeta \frac{\partial}{\partial u} \Big|_Q \right) = -2i_{\sigma(Q)}(g_{-1/2}), \quad \text{SCNO family.} \quad (3.4)$$

It will be convenient to define a function on $\mathcal{M}$ with values in the algebra:

$$k_{-1/2} \equiv g_{-1/2} + \frac{1}{2}\zeta \ell_{-1}. \quad (3.5)$$

Then

$$\sigma_* \left(\frac{\partial}{\partial \zeta} \Big|_Q \right) = -2i_{\sigma(Q)}(k_{-1/2}). \quad (3.6)$$

Just as before, a more general coordinate family differs from (3.2) by a family of superconformal transformations; for such a σ the RHS of (3.3), (3.4), (3.6) receive corrections involving the derivatives of σ , similar to (2.7).

3.2. Super sewing

Again we first discuss only separating pinches. Again we get a transverse coordinate q to Δ only after choosing z_1, z_2 near the attachment points P_1, P_2 on lower-genus SRS. Actually q is not a good coordinate near Δ ; the good coordinate t is a square root of q [46][47][45], and we have sewing map⁹

$$z_2 = t^2/z_1, \quad \theta_2 = \pm it z_1^{-1} \theta_1.$$

Once again a modification of z_1, z_2 by rescaling (ℓ_0 transformation) rescales t , while other changes add only $\mathcal{O}(t^2)$ terms:

$$t' = F_1(\vec{m}_1, \vec{\tau}_1) F_2(\vec{m}_2, \vec{\tau}_2) t + \mathcal{O}(t^2), \quad (3.7)$$

analogous to (2.13). As in (2.14) the even functions F_k are just the lowest Taylor expansion of $s'_k(s_k)$, $k = 1, 2$. $\vec{\tau}_{(i)}$ are odd moduli of X_i . The proof of this assertion is a straightforward repetition of section 2.2. For example,

⁹ Actually the universal curve defined in this way is singular at Δ , unlike the bosonic case. Since we will always *exclude* a neighborhood of Δ , this is of no concern to us.

consider the action of $g_{1/2}$. Sending $v = z_1 \theta_1 d\bar{s}_1^{-2}$ through the neck gives $v^T \propto \frac{t^2}{z_2} \left(\frac{t}{z_2} \theta_2 \right) \left(\frac{z_2}{t} \right)^2 d\bar{s}_2^{-2}$, which, is indeed $\mathcal{O}(t)$. Henceforth we will write $q^{1/2}$ for t .

Thus we once again find a product structure both at Δ and *near* Δ , as expressed by the preferred family of pinching coordinates related by (3.7). In Appendix A we sketch a more concrete derivation of this fact in a particular case.

Eqn. (3.7) is our key result. It says that our cutoff is superconformally covariant, unlike the one employed in [40]. This in turn ensures that the counterterms we need will be well-behaved.

3.3. Spin structures

In this subsection we will investigate a subtlety involving the twice-punctured sphere. We will not need a detailed understanding of this point later, so the reader may wish to skip ahead.

As in the bosonic case [44], every SRS can be obtained by sewing together a few simple building blocks.¹⁰ One of these blocks is the sphere with two punctures. Regarded as a point of $\hat{\mathcal{M}}_{0,2}$ this sphere is rigid and unique. But before we can sew it we must choose (z_i, θ_i) at each of the two punctures, to get something in $\hat{\mathcal{P}}_{0,2}$. In the bosonic case it suffices to take z_1 to be the usual coordinate on the plane and $z_2 = q/z_1$. In the super case, however, there is an additional discrete choice we must make. This is because when z_i are specified we only know θ_i up to a sign. The *overall* sign of θ_1, θ_2 is immaterial¹¹, but the *relative* sign gives us *two* discrete families of sewing spheres, which we will call $\hat{S}_{12}^\sigma(q)$, $\sigma = \pm 1$. Neither of these choices is preferred, so neither can serve as the identity of the “algebra of SRS” (cf [41][53]).

¹⁰ More precisely, $\hat{\mathcal{M}}_g$ can be covered by patches; over each patch we can build the corresponding family of SRS by sewing together a fixed set of building blocks, each of which depends on some parameters. See [20].

¹¹ This is related to the $\mathbb{Z}_2$ orbifold structure of $\widehat{\mathcal{M}}$ noted in [52].

To see the need for σ , consider sewing $\hat{S}^2$ to itself to obtain a torus. Since we used super punctures we will obtain an even spin structure — but there is no preferred choice of even spin structure.

In any case $S_{12}^\sigma(q)$ has no odd moduli, so it is split. The above discussion boils down to saying that before we can sew we must choose not only local coordinates z_1, z_2 at P_1, P_2 , but also spinors $dz_1^{1/2}, dz_2^{1/2}$ at P_1, P_2 ; the relative sign of this choice determines σ . For concreteness we define the choices as follows. Let $q = 1$, so that $z_2 = z_1^{-1}$. A spinor field is $s = (A(z_1), B(z_2))$ where

$$A(z_2^{-1}) = -i\sigma z_2 B(z_2), \quad \sigma = \pm 1. \quad (3.8)$$

Then $dz_1^{1/2}, dz_2^{1/2}$ denote the respective basis sections of the spin bundle, i.e. $s = Adz_1^{1/2} = Bdz_2^{1/2}$, and

$$dz_1^{1/2} = i\sigma z_2^{-1} dz_2^{1/2}. \quad (3.9)$$

(The sign σ and the factor of i in (3.9) are the geometrical origin of the corresponding factors in the boundary conditions of ref. [54].) Clearly neither choice is preferred, as we can reverse σ just by exchanging P_1 and P_2 :

$$S_{12}^\sigma \cong S_{21}^{-\sigma}. \quad (3.10)$$

We can now describe sewing. To join two SRS X_1, X_2 we are to interpose $S_{12}^\sigma(q)$ between them and join P_1, P_2 on the latter to points P'_1, P'_2 on X_1, X_2 respectively. Let us consider the case where X_1, X_2 are themselves spheres (Fig. 8). Then the result of the sewing should be again a sphere. We need to know how σ behaves under this operation. A little algebra using (3.8) repeatedly shows that for $q_1 = q_2 = q_3 = 1$,

$$S_{12}^{\sigma_1} \cdot S_{23}^{\sigma_2} \cdot S_{34}^{\sigma_3} \cong S_{14}^{(-\sigma_1 \sigma_2 \sigma_3)}. \quad (3.11)$$

Fig. 8: Sewing spheres X_1, X_2 to get another sphere.

This is consistent with (3.10), and it shows once again that we cannot simply take $\sigma = +1$ always.

The choice of σ enters whenever we analytically continue a superdifferential of half-odd order from P_1 to P_2 . Thus for example if $v = z_1^n dz_1^{1/2}$ near P_1 , then $v = i\sigma z_2^{-n-1} dz_2^{1/2}$ near P_2 , a fact which enters into the construction of the sewing state in section 5.

3.4. Nonseparating pinches

We have already discussed how sewing $S_{12}^\sigma(q)$ to $S_{12}'(1)$ gives a torus with modular parameter q . Taking $q \rightarrow 0$ we get a pinched torus. More generally the locus Δ_0 of curves with a nonseparating pinch can be treated in just the same way as the separating case. A new issue arises here, however, related to the GSO projection. The holomorphic properties of amplitudes needed to make GSO work have been examined recently in [55]. Here we concentrate on the necessary geometry.

The measure μ on $\hat{\mathcal{M}}$ is well known to contain a term of order $q^{-1/2} \bar{q}^{-1}$. At one loop it is well known that this leading term cancels when the integral of μ is cut off and summed over spin structures. One may have qualms about the corresponding procedure at higher genus, however. The desired cancellation requires that we sum μ over distant regions of $\hat{\mathcal{M}}$ — but have we any canonical way to identify these regions?¹²

¹² Related questions have been raised for the type I case [56].

In fact there seems to be no canonical way to identify neighborhoods of Δ corresponding to all 4^g spin structures on a given SRS. This raises the spectre of an ambiguity in the lower-order remainder left after cancelling the tachyon. Fortunately, however, we need not sum all 4^g spin structures to effect the cancellation.¹³ When a nonseparating cycle pinches, all the spin structures fall into pairs differing by a sign as one crosses the pinching cycle. Summing any one pair suffices to remove the tachyon.

Moreover one *can* identify neighborhoods of Δ_0 near each of a pair of spin structures, once a choice of local coordinates at the pinching points has been chosen. One starts with a single family of curves of genus $g - 1$ with two punctures and local coordinates. One then sews in $S_{12}^\sigma(q)$ to obtain *two* families of pinching genus- g curves, for $\sigma = \pm 1$, and hence an identification between them. As we will see the sewing state when summed over σ has no tachyon; thus neither does μ when summed using this identification.

Thus we have only to worry about massless poles in vacuum amplitudes. But at Δ_0 these are suppressed by $(\log q)^{-5}$ and so cause no trouble [18]. Similarly in scattering amplitudes if all incoming states have the correct G -parity then again there will be no tachyon poles once spin structures are summed in pairs.

It is important to emphasize that this unambiguous cancellation mechanism is completely independent of the argument based on the product structure in Section 3.1. The latter deals admirably with massless poles, but it is completely helpless when faced with higher poles since the $\mathcal{O}(q^2)$ terms of (2.13) are *not* split nicely into two sides. Thus we have nothing to say about, say, the bosonic string. This is as it should be — perturbation theory should only work when one is close to a stable extremum of the effective action to begin with, and a tachyonic background is not of this form.

¹³ We thank E. Witten for emphasizing this.

4. Field Theory

The geometry studied so far is the arena for conformal and superconformal field theory, to which we now turn. In particular the constructions of section 2.3 provide the simplest language in which to study the factorization of bosonic [7] and fermionic [20] amplitudes. To see how this works we now briefly recall some ideas from [57][32] for the bosonic case.

4.1. Conformal field theory

The reader may wish to skip this section, which serves mainly to establish notation and link the viewpoint of [32][35][20] to that of [33][7] (and others).

A conformal field theory with $c = 0$ (for example string theory) associates to every Riemann surface with puncture and local coordinate a vector in a Fock space $\mathcal{H}$. We will denote the vector associated to $\tilde{X} \in \mathcal{P}_{g,1}$ by $|\tilde{X}\rangle$ or $|X, P, z\rangle$. Actually depending on the orientation chosen for a hole at P we can instead obtain a transposed vector $\langle \tilde{X}|$ in the dual space $\mathcal{H}^*$. It is important to note that in our notation the transpose is linear — not antilinear.

Similarly a point in $\mathcal{P}_{g,N}$ gives a vector in $\mathcal{H}^{\otimes N_1} \otimes (\mathcal{H}^*)^{\otimes N_2}$ where $N_1 + N_2 = N$, depending on chosen orientations.¹⁴

Thus given two vectors $\psi_1, \psi_2 \in \mathcal{H}$ we obtain a bilinear product¹⁵ (ψ_1, ψ_2) by pairing $\psi_1 \otimes \psi_2$ with $\langle S_{12}|$, the state associated to the standard $S_{12} \in \mathcal{P}_{0,2}$. More generally given $\tilde{X} \in \mathcal{P}_{g,N}$ and vectors $\psi_1, \dots, \psi_N$ we can pair $\langle \tilde{X}|$ with $\psi_1 \otimes \dots \otimes \psi_N$; the result is traditionally written as

$$\langle \psi_{1(P_1, z_1)} \dots \psi_{N(P_N, z_N)} \rangle_{\tilde{X}}. \quad (4.1)$$

¹⁴ When a path integral representation exists it can be used to define $|\tilde{X}\rangle$. We cut out disks near P_i and interpret $|\tilde{X}\rangle$ as a functional of boundary data on the holes. If unit disks collide or extend beyond the domain of z_i , we use smaller holes and then rescale $|\tilde{X}\rangle$ by the Virasoro action to be defined later. This suits the definition of $\tilde{X}$ in terms of *germs* of local coordinates mentioned earlier.

¹⁵ For strings in flat space there is also a *hermitian* product, but we will never use it; thus our factorization formulas will have analogs for any conformal field theory.

This notation encourages us to think of ψ_i as an “operator” inserted at P_i and “normal-ordered” with the help of z_i , ideas made more precise in [57][33].

Thus if we change z_i leaving ψ_i, P_i, X fixed, the expression (4.1) changes. One can think of the change as due to a re-normal ordering of $\psi_i(P_i)$, as suggested by (4.1), or dually as a change of the “Krichever” state $\langle X, \{P_i\}, \{z_i\} \rangle$.

The Virasoro algebra acts on $\mathcal{P}$, but it also acts on $\mathcal{H}$. Thus it makes sense to assert that $|\tilde{X}\rangle$ obeys a certain consistency, or “equivariance” condition (this point of view is expanded in [35][58]):¹⁶

$$\delta_v |\tilde{X}\rangle = T(v^T) |\tilde{X}\rangle. \quad (4.2)$$

Here δ_v is a derivative along the action of $v \in \text{Vir}$ on $\mathcal{P}_{g,1}$ at $\tilde{X}$, and v^T denotes the involution on Vir , eqn. (2.3). Eqn. (4.2) can easily be derived if $|\tilde{X}\rangle$ is constructed by the conserved-charge method [35], but it is more general than that [32]. Eqn. (4.2) is consistent with the basic commutator

$$[T(v), T(v')] = -T([v, v']) \quad ,$$

where on the right we have the Lie bracket of two vector fields on the annulus.

With the bilinear $(\ , \)$ we can define an adjoint: $(M\psi_1, \psi_2) \equiv (\psi_1, M^T \psi_2)$. We obtain [7]

$$(\alpha_n^\mu)^T = -\alpha_{-n}^\mu \quad , \quad c_n^T = -c_{-n} \quad , \quad b_n^T = b_{-n} \quad (4.3)$$

$$T(v^T) = -T(v)^T \quad . \quad (4.4)$$

To obtain (4.3) we proceed as follows. Let $w_n = z^n$, so that $\alpha_n^\mu = \oint w_n \partial X^\mu$, and consider $(\alpha_n \psi_1, \psi_2)$. Regarding w_n as a function of z_1 and analytically continuing to the neighborhood of P_2 we find an insertion of $-\alpha_{-n}$ at P_2 . (The sign comes from the fact that a positive contour around P_1 deforms to a *negative* contour around P_2 .) Similarly we obtain b_n^T by analytically continuing the *vector* field $v_n = z_1^{-n-2} \frac{\partial}{\partial z_1}$ to P_2 , etc.

¹⁶ Note that we define $T(v^T) \equiv \oint T_{zz}(v^T)z$; this operator is denoted by $T(v)$ in [35].

Using (4.4) we can translate (4.2) into the viewpoint of (4.1):

$$\delta_v |\tilde{X}| = -\langle \tilde{X} | T(v) \rangle \quad .$$

Thus in particular if we change normal ordering by $z \mapsto z' = z + v^2$, then we get

$$\langle \psi_{(P; z')} \rangle_X = \langle \psi_{(P; z)} \rangle_X - \langle (T(v)\psi)_{(P; z)} \rangle_X. \quad (4.5)$$

The Fock space contains a vector $|1\rangle$ associated to the insertion of the unit operator. Since the unit operator is the same in any normal ordering, and moreover amplitudes do not depend on where we insert it, we have that $|1\rangle$ is $SL(2, \mathbb{C})$ -invariant, and indeed

$$L_n |1\rangle = b_n |1\rangle = 0, \quad n \geq -1 \quad (4.6)$$

$$c_n |1\rangle = 0, \quad n \geq 2 \quad (4.7)$$

$$\alpha_n |1\rangle = 0, \quad n \geq 0 \quad .$$

Using this distinguished vector we can define a number, the string coupling K [7]:

$$(|1\rangle, c_1 \bar{c}_1 c_0 \bar{c}_0 c_{-1} \bar{c}_{-1} e^{ik \cdot s} |1\rangle) \equiv (2\pi)^{10} \delta^{10}(k^\mu) iK. \quad (4.8)$$

In a compactified spacetime the continuum delta-function gets replaced by a Kronecker delta. For brevity we will at times write (4.8) as

$$(|1\rangle, c_1 \bar{c}_1 c_0 \bar{c}_0 c_{-1} \bar{c}_{-1} |1\rangle) = iZ, \quad (4.9)$$

where Z is the “partition function,” but the explicit $e^{ik \cdot s}$ will be important in section 7.

The Polyakov measure now has an elegant formulation [59][35]. Given tangent vectors $V_1, \dots, \bar{V}_{3g-3}$ to $\mathcal{M}_g$ at some X we seek an alternating form $\mu(V_1, \dots, \bar{V}_{3g-3})$. Choose any P, z and compute $|\tilde{X}\rangle$. Next choose $v_i \in \text{Vir}$ such that $i_{\tilde{X}}(v_i)$ projects down to V_i . Thus each v_i is ambiguous by the addition of $\ell_n, n \geq -1$. Now let

$$\begin{aligned} \mu(V_1, \dots, \bar{V}_{3g-3}) &= (b(v_1^T) \cdots \bar{b}(\bar{v}_1^T) \cdots |\tilde{X}\rangle, |1\rangle) \\ &= \langle [\cdots b(v_2) b(v_1)]_{(P; z)} \rangle_X \quad . \end{aligned} \quad (4.10)$$

One shows that (4.10) is independent of P, z , and the chosen representatives v_i ; hence it indeed defines a differential form on $\mathcal{M}_g$ [35].

Eqn. (4.10) suggests a generalization. Replacing $|1\rangle$ by an arbitrary ψ of ghost charge $U = 1$, $\bar{U} = 1$, define a form of type $(3g - 2, 3g - 2)$ by

$$\tilde{\mu}_\psi(v_1, \dots, v_{3g-2}) \equiv \left(b(v_1^T) \dots \bar{b}(\bar{v}_{3g-2}^T) | \tilde{X} \right), \psi \Big).$$

In general $\tilde{\mu}$ must be regarded as a form on $\mathcal{P}$, since in general ψ will not obey (4.6). Now suppose ψ is BRST-exact, $\psi = (Q + \bar{Q})\lambda$. Dropping barred quantities for brevity we find

$$\begin{aligned} \tilde{\mu}_{Q\lambda}(v_1, \dots) &= (Qb(v_1^T) \dots | \tilde{X} \rangle, \lambda) \\ &= (T(v_1^T)b(v_2^T) \dots | \tilde{X} \rangle, \lambda) - (b(v_1^T)T(v_2^T) \dots | \tilde{X} \rangle, \lambda) + \dots \\ &= \delta_{v_1}(b(v_2^T) \dots | \tilde{X} \rangle, \lambda) - \delta_{v_2}(b(v_1^T) \dots | \tilde{X} \rangle, \lambda) \\ &\quad + ([T(v_1^T), b(v_2^T)] | \tilde{X} \rangle, \lambda) + \dots \\ &= \delta_{v_1}\tilde{\mu}_\lambda(v_2, \dots) - \delta_{v_2}\tilde{\mu}_\lambda(v_1, \dots) - \tilde{\mu}_\lambda([v_2, v_1]) + \dots \\ &= (\partial\tilde{\mu}_\lambda)(v_1, v_2, \dots) \end{aligned}$$

or [35]

$$\tilde{\mu}_{(Q+\bar{Q})\lambda} = d\tilde{\mu}_\lambda. \quad (4.11)$$

Similarly one finds

$$\tilde{\mu}_{b(v)}\psi = -i_v\tilde{\mu}_\psi \quad (4.12)$$

$$\tilde{\mu}_{T(v)}\psi = -\mathcal{L}_v\tilde{\mu}_\psi, \quad (4.13)$$

where i_v is the interior product with $i_{\tilde{X}}(v)$ and $\mathcal{L}_v$ is the corresponding Lie derivative.

Sometimes $\tilde{\mu}_\psi$ is actually the lift of a form μ_ψ on $\mathcal{M}_{g,1}$, just as with the Polyakov measure (4.10) [35][11]. This happens when ψ satisfies a weakened form of (4.6):

$$L_n\psi = b_n\psi = 0, \quad n \geq 0. \quad (4.14)$$

The conditions L_{-1}, b_{-1} have been dropped because in general $\langle \psi \rangle_X$ should depend on the insertion point P , unlike the Polyakov measure. In other cases

we can obtain a form μ_ψ by brute force, as follows: locally choose any section σ of $\mathcal{P}_{g,1} \rightarrow \mathcal{M}_{g,1}$, and let

$$\mu_{\psi,\sigma} \equiv \sigma^*\tilde{\mu}_\psi.$$

Thus at $(X, P) \in \mathcal{M}_{g,1}$

$$\mu_{\psi,\sigma}(V_1, \dots, V_{3g-2}) = \tilde{\mu}_\psi(\sigma_*V_1, \dots, \sigma_*V_{3g-2}), \quad (4.15)$$

where the RHS is evaluated at $\sigma(X, P)$. As explained in [11], (4.15) reproduces the “ $\hat{b}$ ” prescription of [7]; the corrections to b_{-1} are clearly visible in (2.7) and the following paragraph. On the other hand, for the conformal normal ordering (CNO) coordinate family (2.5) shows no such corrections, so we recover the formulas of [33]. The problem with the CNO family, however, is that it is not globally defined. When ψ satisfies the physical conditions (4.14) this does not matter since in any case $\mu_{\psi,\sigma}$ is independent of σ . As pointed out in [7], however (and as we will see in the heterotic case), states can enter into the factorization formula which violate (4.14). For these we will need a global family.

Actually no global choice of σ exists, as discussed in section 2.1. However, it will turn out that the relevant factorizing states do satisfy an even weaker form of (4.14), namely

$$(L_0 - \bar{L}_0)\psi = (b_0 - \bar{b}_0)\psi = 0. \quad (4.16)$$

In this case $\mu_{\psi,\sigma}$ is at least insensitive to $U(1)$ changes in σ , and we can choose a global coordinate family defined modulo $U(1)$. In general, however, this family will not be of the CNO form (2.4), nor indeed will it even be holomorphic. In this case we will need to use the general family and substitute (2.7) in (4.15). In particular an insertion of ψ will be accompanied by $\hat{b}_{-1}\hat{\bar{b}}_{-1}$ times $du \wedge d\bar{u}$, where again u is the parameter describing the location of the insertion point and

$$\hat{b}_{-1} = b_{-1} - E_n b_n - \bar{F}_n \bar{b}_n. \quad (4.17)$$

Substituting the special family with $\bar{F}_1 = -\frac{1}{8}R$, etc, we recover the prescription in [7].

4.2. Superconformal field theory

For simplicity we will consider only the heterotic string in flat spacetime. We follow the conventions of [51] (also see appendix B). The BRST operator for the supersymmetric sector is

$$Q = -\frac{1}{2} \oint \frac{ds}{2\pi i} C \left(\frac{1}{2} DCDB - \frac{3}{2} \partial CB - DX \cdot \partial X \right) .$$

In modes this is

$$\begin{aligned} Q = & -2c_{-1}c_1\bar{b}_0 - c_{-1}c_0\bar{b}_1 + c_1c_0\bar{b}_{-1} + \frac{1}{2}c_0(\gamma_{1/2}\beta_{-1/2} - \gamma_{-1/2}\beta_{1/2}) \\ & - c_1\gamma_{-1/2}\beta_{-1/2} + c_{-1}\gamma_{1/2}\beta_{1/2} - \frac{1}{4}\gamma_{-1/2}^2\bar{b}_1 - \frac{1}{4}\gamma_{1/2}^2\bar{b}_{-1} \\ & - \frac{1}{2}\gamma_{-1/2}\gamma_{1/2}\bar{b}_0 + \frac{1}{2}\gamma_{-1/2}d_{1/2} \cdot \alpha_0 + \frac{1}{2}\gamma_{-1/2}d_{-1/2} \cdot \alpha_1 \\ & + \frac{1}{2}\gamma_{1/2}d_{-1/2} \cdot \alpha_0 + \frac{1}{2}\gamma_{1/2}d_{1/2} \cdot \alpha_{-1} - c_{-1}\alpha_0 \cdot \alpha_1 - \frac{1}{2}c_0\alpha_0^2 \\ & - \frac{1}{2}c_0d_{-1/2} \cdot d_{1/2} - c_1\alpha_0 \cdot \alpha_{-1} - c_0\alpha_{-1} \cdot \alpha_1 . \end{aligned} \quad (4.18)$$

Similarly we get

$$\begin{aligned} \bar{Q} = & -2\bar{c}_{-1}\bar{c}_1\bar{b}_0 - \bar{c}_{-1}\bar{c}_0\bar{b}_1 + \bar{c}_1\bar{c}_0\bar{b}_{-1} \\ & - \bar{c}_{-1}\bar{\alpha}_0 \cdot \bar{\alpha}_1 - \frac{1}{2}\bar{c}_0\bar{\alpha}_0^2 - \frac{1}{2}\bar{c}_0\bar{d}_{-1/2} \cdot \bar{d}_{1/2} - \bar{c}_1\bar{\alpha}_0 \cdot \bar{\alpha}_{-1} - \bar{c}_0\bar{\alpha}_{-1} \cdot \bar{\alpha}_1 , \end{aligned} \quad (4.19)$$

where now there are 32 fermions $\bar{d}^A$ $A = 1, \dots, 32$. Thus

$$\{Q, b_n\} = L_n , \quad [Q, \beta_k] = -\frac{1}{2}G_k ,$$

where the stress tensor is

$$T \equiv \sum_z \theta z^{-n-2} L_n + \sum_{z=1/2} \frac{1}{2} z^{-k-3/2} G_k .$$

The machinery of the previous subsection generalizes to the superconformal case. Now however we need to bear in mind the sign $\sigma = \pm 1$ which arises when sewing world-sheet spinors (see sect. 3.3 and [54]). For example one has two bilinear forms $(\ , \)_\sigma$ corresponding to S_{12}^σ . Neither is symmetric: $(\psi_1, \psi_2)_\sigma = \pm (\psi_2, \psi_1)_{-\sigma}$ depending on the parities of $\psi_{1,2}$, since $S_{12}^\sigma \cong S_{21}^{-\sigma}$.

Again defining the transpose by $(M\psi_1, \psi_2)_\sigma = (\psi_1, M^T\psi_2)_\sigma$, we now have (see (4.3))

$$d_k^T = -i\sigma d_{-k} \quad \beta_k^T = i\sigma\beta_{-k} \quad \gamma_k^T = i\sigma\gamma_{-k} . \quad (4.20)$$

The Polyakov density now takes a form very parallel to the bosonic case [60][50][20][48]. If $V_1, \dots, V_{3g-3}, \Upsilon_1, \dots, \Upsilon_{2g-2}$ are a basis of tangents to $\hat{\mathcal{M}}_g$ with V_i even and Υ_α odd, define corresponding vectors $v_1, \dots, v_{2g-2}$ in the Neveu-Schwarz algebra as before and let

$$\mu(V_1, \dots, \bar{V}_1, \dots, \Upsilon_1, \dots) \equiv \left\langle [\bar{B}(\bar{v}_{3g-3}) \cdots \delta[B(v_{2g-2})] \cdots B(v_{3g-3}) \cdots]_{(P; a)} \right\rangle_{\tilde{X}} . \quad (4.21)$$

The operators $\delta[B(v)]$ are discussed extensively in [50][20][48]. With this definition μ transforms as a density. Similarly one can define $\bar{\mu}_\psi$ for an inserted state ψ , and relations (4.11)–(4.13) again hold. Given a family σ of superconformal coordinates one again defines a density $\mu_{\psi, \sigma}$ on $\hat{\mathcal{M}}_g$ using (4.15). For the superconformal normal-ordering (SCNO) coordinate family described in section 3.1 we see that an insertion of ψ should be accompanied by $B(\ell_{-1})\bar{B}(\bar{\ell}_{-1})\delta[B(-2k_{-1/2})]$, times $[dud\bar{u}|d\zeta]$, where $k_{-1/2}$ is defined in (3.5). The insertion equals $-\frac{1}{2}\bar{b}_{-1}\bar{b}_1\delta[\beta_{-1/2} + \frac{1}{2}\zeta\bar{b}_{-1}]$, which in turn is $-\frac{1}{2}\bar{b}_{-1}\bar{b}_1\delta(\beta_{-1/2})$. Performing the integral over ζ then differentiates $|\tilde{X}\rangle$, since ζ enters nowhere else; thus we recover the usual picture-changing formalism. We will see in section 7 that things work out very differently in the case of the dilaton, for which SCNO is not valid, essentially because of corrections analogous to (4.17).

Eqn. (4.21) implicitly contains the state $|1\rangle$ again. This time $|1\rangle$ is annihilated by the generators of $\text{Osp}(2|1)$, and indeed we have

$$G_k|1\rangle = \delta(\beta_k)|1\rangle = 0, \quad k \geq -1/2 \quad (4.22)$$

$$\delta(\gamma_k)|1\rangle = 0, \quad k \geq 3/2 \quad (4.23)$$

$$d_k|1\rangle = 0, \quad k \geq 1/2 ,$$

along with (4.14), (4.7). Any other state ψ satisfying (4.14) and (4.22) for $k \geq \frac{1}{2}$, and Q -closed, defines a good density μ_ψ . As we will see, however, in the

heterotic case an important state satisfies only $Q\psi = 0$ and (4.16). As in the bosonic case [7][11], this will be good enough.

We recall from [20] that there is no question of which picture ψ should be in. The insertions needed to change the odd moduli of X (including even the one describing where ψ is inserted) can all be made elsewhere on X , if desired, leaving ψ in its natural 0 picture required to give a good measure μ_ψ on $\widehat{\mathcal{M}}_{g,1}$.

5. Factorisation

5.1. Bosonic case

Again we briefly recapitulate the bosonic case and extend the discussion of [7][44]. When a separating pinch develops in a surface we get a map ∞ as in (2.9). A correlation function $\mathcal{A}$ on $\mathcal{M}_g$ thus gives, without any choices, a function $\tilde{\mathcal{A}}$ on $\mathcal{P}_{g,1} \times \mathbb{C} \times \mathcal{P}_{g,1}$:

$$\tilde{\mathcal{A}}(X_1, P_1, z_1; q; X_2, P_2, z_2) = \sum_a \langle \phi_a(P_1; z_1) \rangle_{X_1} q^{h_a} \bar{q}^{\bar{h}_a} \langle \phi_a^e(P_2; z_2) \rangle_{X_2} \quad (5.1)$$

Here $\{\phi_a\}$ is a basis of $\mathcal{H}$ in which $L_0, \bar{L}_0$ are diagonal, with eigenvalues $h_a, \bar{h}_a$, and $\phi^e = (\phi_a)^T$. For example if $\phi_a = |k\rangle$ then $\phi^e = c_1 \bar{c}_1 c_0 \bar{c}_0 c_{-1} \bar{c}_{-1} | -k \rangle$, where $|k\rangle = e^{ik \cdot x} |1\rangle$. As described in section 2.2, once we are given sections σ_i of $\mathcal{P}_{g,1} \rightarrow \mathcal{M}_{g,1}$ we can turn $\tilde{\mathcal{A}}$ into a function $\mathcal{A}_{\sigma_1 \sigma_2}$ on $\mathcal{M}_{g,1} \times \mathbb{C} \times \mathcal{M}_{g,1}$.

Our attitude towards (5.1) is that it is part of the axioms of conformal field theory. Before we proceed to use it, however, we should first pause to convince ourselves that it makes sense.

The momentum of the contributing states in (5.1) is fixed by momentum conservation; thus many terms are off shell and do not satisfy (4.14) (cf. [61]). Hence $\mathcal{A}_{\sigma_1 \sigma_2}$ depends on the coordinate families $\sigma_1 \sigma_2$ chosen:

$$\mathcal{A}_{\sigma_1 \sigma_2}(X_1, P_1; X_2, P_2) \neq \mathcal{A}_{\sigma'_1 \sigma'_2}(X_1, P_1; X_2, P_2). \quad (5.2)$$

How then can (5.1) be correct, since the original $\mathcal{A}$ was defined independently of any choices? The answer of course is that sewing (X_1, P_1) to (X_2, P_2) with

σ'_1, σ'_2 gives in general a surface X' different from X obtained from sewing with σ_1, σ_2 , so (5.2) *shouldn't* be an equality. Indeed we know how X' differs from X , so we can verify whether (5.1) behaves correctly under sewing.

If we allow additional punctures, then we sew $\tilde{X}_1, \tilde{X}_2$ to get $\tilde{X}$ or $\tilde{X}'$ as in (2.10). We again get a formula like (5.1), where on the LHS $\tilde{\mathcal{A}}$ is an n -point amplitude evaluated on the punctured sewn surface $\tilde{X}$ and on the RHS we have products of $(N_1 + 1)$ -point and $(N_2 + 1)$ -point functions where $N_1 + N_2 = N$. We want to check the consistency of (5.1) under changes of sewing coordinates z_i . By locality it suffices to check the case where X_1, X_2 are themselves 2-punctured spheres, but not necessarily the standard choice (Fig. 9). Let $\tilde{X}_1, \tilde{X}_2$ be the standard spheres, so that $w_i = z_i^{-1}, i = 1, 2$. Let $\tilde{X}'_1$ be a modified sphere with $z'_1 = z_1 + \epsilon_n z_1^{n+1}$. When we sew with $q = 1$, say, using z_1, z_2 we get $\tilde{X}$ with $w_2 = w_1^{-1}$, the standard sphere. Using z'_1, z_2 we instead get $\tilde{X}'$ with $w'_2 = w_1^{-1} + \epsilon_n w_1^{-n-1}$. Eqn. (5.1) applied to $\tilde{X}'$ then says that

$$\begin{aligned} & \langle \psi_{1(P_1; w_1)} \psi_{2(P_2; w_1^{-1} + \epsilon_n w_1^{-n-1})} \rangle_{\tilde{X}} = \\ & \sum_a \langle \psi_{1(P_1; w_1)} \phi_a(Q_1; w_1^{-1} + \epsilon_n w_1^{-n-1}) \rangle_{X_1} \langle \phi_a^e(Q_2; w_2^{-1}) \psi_{2(P_2; w_2)} \rangle_{X_2} \end{aligned} \quad (5.3)$$

Again, the RHS involves correlation functions on $\tilde{X}'_1, \tilde{X}_2$ while the LHS is a correlation function on the surface $\tilde{X}'$ built from them with $q = 1$. Let us rewrite both sides of (5.3) using (4.5), (4.4):

$$\begin{aligned} LHS &= \langle \psi_{1(P_1; w_1)} [(1 - \epsilon_n L_n) \psi_2]_{(P_2; w_1^{-1})} \rangle_{\tilde{X}} \\ &= \langle [(1 + \epsilon_n L_{-n}) \psi_1]_{(P_1; w_1)} \psi_{2(P_2; w_2^{-1})} \rangle_{\tilde{X}} \end{aligned} \quad (5.4)$$

$$\begin{aligned} RHS &= \langle \psi_{1(P_1; w_1)} [(1 - \epsilon_n L_n) \phi_a]_{(P_2; w_1^{-1})} \rangle_{X_1} \langle \phi_a^e(Q_2; w_2^{-1}) \psi_{2(P_2; w_2)} \rangle_{X_2} \\ &= \langle [(1 + \epsilon_n L_{-n}) \psi_1]_{(P_1; w_1)} \phi_a(P_2; w_2^{-1}) \rangle_{X_1} \langle \phi_a^e(Q_2; w_2^{-1}) \psi_{2(P_2; w_2)} \rangle_{X_2} \end{aligned} \quad (5.5)$$

The equality of the right-hand sides of (5.4)–(5.5), however, is nothing but (5.1) applied to the original $\tilde{X}$! Thus (5.1) is indeed compatible with the equivariance

Fig. 9: Sewing the same spheres two different ways.

conditions (4.2) on $|X_1\rangle$, $|X_2\rangle$, and $|X\rangle$. The heart of the argument was essentially the contour deformation which moved L_n from one puncture to another in (5.4)–(5.5).

Eqn. (5.1) is still not quite what we need since amplitudes come from volume forms μ on $\mathcal{M}$, not functions $\mathcal{A}$. But from the above discussion and section 4.1 it is easy to show that the Polyakov measure obeys ¹⁷

$$(\infty^* \mu)(\tilde{V}_1^{(1)}, \dots, \frac{\partial}{\partial q}; \tilde{V}_1^{(2)}, \dots) = \sum_a q^{h_a-1} \langle b \dots b [b_0 \phi_a]_{(P_1, z_1)} \rangle_{X_1} \langle \phi_{(P_2, z_2)}^* b \dots b \rangle_{X_2} \quad (5.6)$$

We have suppressed all barred quantities. On the left $\infty^* \mu$ is a form on $\mathcal{P}_{g,1} \times \mathbb{C} \times \mathcal{P}_{g,1}$ and $\tilde{V}_i^{(j)}$ are tangents to $\mathcal{P}_{g,1}$. On the right $b \dots b$ denotes ghost insertions on X_j associated to $\tilde{V}_i^{(j)}$. The insertion of $b_0 = -b(\ell_0)$ is dual to $\partial(\log q)$; hence the new factor of q^{-1} . Eqn. (5.6) is compatible with (4.2) by an argument like the one above.

5.2. Super case

The explicit form of the bosonic sewing state $|S_{12}\rangle = b_0^{(1)} \bar{b}_0^{(1)} |\phi_a\rangle^{(1)} \otimes |\phi_a^*\rangle^{(2)}$ appears in [35][7]. We now turn to its super analog. Our notation differs slightly from [20] due to the sign $\sigma = \pm 1$ discussed in section 3.2. We need a state

¹⁷ In [58] it was conjectured that the following condition should be considered as an “equivariance” under the “algebra of Riemann surfaces,” on the same footing as section 4.1.

annihilated by $Y^{(1)} - (Y^{(2)})^T$ where $Y^{(i)}$ is any mode of any field inserted at P_i . Using (4.3), (4.20) one finds the super part of $|S_{12}^\sigma\rangle$ to be, at $q = 1$,

$$\sigma \exp \left[\sum_k i\sigma (-d_{-k}^{(1)} \cdot d_{-k}^{(2)} + \beta_{-k}^{(1)} \gamma_{-k}^{(2)} - \gamma_{-k}^{(1)} \beta_{-k}^{(2)}) \right] \delta(\gamma_{1/2}^{(1)}) \delta(\gamma_{1/2}^{(2)}) |1\rangle^{(1)} \otimes |1\rangle^{(2)} \quad (5.7)$$

Here $\delta(\gamma_{1/2})|1\rangle$ is annihilated by $\beta_k, \gamma_k, k \geq \frac{1}{2}$. We will justify the overall sign σ in a moment.

In order to use (5.7) to join $|\bar{X}_1\rangle$ to $|\bar{X}_2\rangle$ we must first convert $|S_{12}^\sigma\rangle$ to its adjoint $\langle S_{12}^\sigma|$, then pair it with $|\bar{X}_1\rangle \otimes |\bar{X}_2\rangle$. But which of $(\ , \)_{\pm 1}$ are we to use to take the adjoint? σ does not belong to either puncture P_1 or P_2 ; it sits between them. Our answer, justified below, is to use *either* sign: to obtain $\langle S_{12}^\sigma|$ we take the $(\ , \)_{\pm}$ adjoint on Fock space $\mathcal{H}_1$ and the *same* $(\ , \)_{\pm}$ adjoint on $\mathcal{H}_2$. This is our super sewing prescription.

To justify our prescription we must first verify that it behaves correctly with respect to change of coordinates s_1, s_2 . This is a straightforward repetition of section 5.1. We must also check, however, that the measure behaves correctly under sewing. That is, under sewing $|S_{12}^\sigma\rangle$ should obey a condition based on (3.11). In the bosonic case, where $\sigma = \pm 1$ does not enter, this condition is not hard to verify. In the super case when we pair $\langle S_{23}^{\sigma_2}|$ with $|S_{12}^{\sigma_1}\rangle \otimes |S_{34}^{\sigma_3}\rangle$ we must obtain $|S_{14}^{-(\sigma_1 \sigma_2 \sigma_3)}\rangle$. To see how this works drop all the oscillators and consider only the conformal Killing spinors. Since $\gamma_{1/2}^T = i\sigma \gamma_{-1/2}$ we get $\langle S_{23}^{\sigma_2}| = -\sigma_2 \langle 1| \delta(\gamma_{-1/2}^{(2)}) \otimes \langle 1| \delta(\gamma_{-1/2}^{(3)})$ and so the required pairing is $-\sigma_1 \sigma_2 \sigma_3 \delta(\gamma_{1/2}^{(1)}) |1\rangle^{(1)} \otimes \delta(\gamma_{1/2}^{(4)}) |1\rangle^{(4)} = |S_{14}^{-\sigma_1 \sigma_2 \sigma_3}\rangle$. We omit the details for the oscillators.

As another check one can sew two spheres to get a torus amplitude. For the matter fermions one gets (for general q) $\prod_k (1 - q^k \sigma_1 \sigma_2)$; the strange factors of i in (5.7) cancel out.

As in the bosonic case (section 5.1) the full sewing insertion is $|q|^{-2}dq \wedge d\bar{q}|S_{12}^\sigma\rangle \equiv \sum_a |q|^{-2}dq \wedge d\bar{q} q^{L_0} \bar{q}^{L_0} b_0^{(1)} \bar{b}_0^{(1)} \phi_a^{(1)} \otimes \phi^{a(1)}$ or¹⁸

$$|S_{12}^\sigma\rangle = \int \frac{d^{10}k}{(2\pi)^{10}iK} q^{k^2/2-1/2} \bar{q}^{k^2/2-1} \times \sigma \exp \left[\sum_{z_+} \left\{ q^n \left(-\alpha_{-n}^{(1)} \cdot \alpha_{-n}^{(2)} + c_{-n}^{(1)} b_{-n}^{(2)} - b_{-n}^{(1)} c_{-n}^{(2)} \right) + \text{conj.} \right\} \right. \\ \left. + \sum_{z_+ - 1/2} \sigma \left\{ i q^k \left(\beta_{-k}^{(1)} \gamma_{-k}^{(2)} - \gamma_{-k}^{(1)} \beta_{-k}^{(2)} - d_{-k}^{(1)} \cdot d_{-k}^{(2)} \right) - i \bar{q}^k \bar{d}_{-k}^{(1)} \cdot \bar{d}_{-k}^{(2)} \right\} \right] \\ \times c_1^{(1)} \bar{c}_1^{(1)} c_1^{(2)} \bar{c}_1^{(2)} \delta(\gamma_{1/2}^{(1)}) \delta(\gamma_{1/2}^{(2)}) |k\rangle^{(1)} \otimes |k\rangle^{(2)}. \quad (5.8)$$

Here $|k\rangle = e^{ik \cdot \sigma} |1\rangle$. The string coupling constant K (see (4.8)) is needed to get correct sewing [7].

We can now extract the relevant states from (5.8). Suppose X degenerates into X_1, X_2 where X_2 has no external lines. Then the relevant states of (5.8) have ghost charge appropriate to disappear into the vacuum. That is, $\hat{b}_{-1} \hat{\bar{b}}_{-1} \delta(\beta_{-1/2}) \phi^a$ must have the same *total* ghost charge as $|1\rangle$. Only the total charge ($\hat{b} + \hat{\bar{b}} + \beta$ charge) matters, since we have seen how the choice of a coordinate family can mix then (see for example (4.17)). The leading term of the ghost part of (5.8) thus involves $\phi^a = q^{-1/2} \bar{q}^{-1} c_1 \bar{c}_1 \delta(\gamma_{1/2}) |1\rangle^{gh}$. The next term is then $|q|^{-2} i (\beta_{-1/2}^{(1)} \gamma_{-1/2}^{(2)} - \gamma_{-1/2}^{(1)} \beta_{-1/2}^{(2)})$, which involves the states $c_1 \bar{c}_1 |1\rangle^{gh}$, $\gamma_{-1/2} c_1 \bar{c}_1 \delta(\gamma_{1/2}) |1\rangle^{gh}$. Both have the wrong total ghost number, so we discard them.

Any term with more powers of q gives no pole and so is not dangerous near Δ . On the other hand one more power of $\bar{q}$ is interesting; the states with appropriate ghost charge are

$$|D_1\rangle \equiv -\frac{1}{2} c_1 \gamma_{-1/2} \delta(\gamma_{1/2}) |1\rangle \\ |D_2\rangle \equiv -2 c_1 \bar{c}_1 \bar{c}_{-1} \beta_{-1/2} \delta(\gamma_{1/2}) |1\rangle. \quad (5.9)$$

The factors will prove convenient later. We note that $|D_{1,2}\rangle$ both fail the restrictive physical state conditions (4.14)(4.22), and so cannot be inserted using

¹⁸ In [15] we set $\sigma = 1$ for simplicity.

SCNO, i.e. with a holomorphic coordinate family like (3.2). Both however satisfy (4.16) and are BRST-closed, so they can be inserted with a family defined modulo $U(1)$.

Turning now to the matter part of (5.8), we need only consider states which are gauge- and Lorentz scalars. This gives the tachyon and the graviton trace,

$$|\text{grav}\rangle \equiv d_{-1/2} \cdot \bar{a}_{-1} c_1 \bar{c}_1 \delta(\gamma_{1/2}) |1\rangle. \quad (5.10)$$

This state satisfies the strong conditions (4.14)(4.22) and is BRST-closed.

We now see how the tachyon decouples. As discussed in sect. 3.4, when a nonseparating cycle pinches we can sum the spin structures in pairs. The sum illustrated in Fig. 10 involves a complicated state $|\tilde{X}\rangle$ independent of the pinching parameter q , and a *simple* state $|S_{12}^+ + S_{12}^- \rangle$ which does depend on q . A look at the sewing state, however, reveals that this sum projects onto even fermion number relative to $|1\rangle$. We thus see that the elimination of the tachyon is really *geometrically necessary*¹⁹ and globally well-defined.

Fig. 10: Geometry of GSO.

We are thus left with

$$|S_{12}^\sigma\rangle = i \left\{ \frac{1}{10} |\text{grav}\rangle^{(1)} \otimes |\text{grav}\rangle^{(2)} + |D_2\rangle^{(1)} \otimes |D_1\rangle^{(2)} + |D_1\rangle^{(1)} \otimes |D_2\rangle^{(2)} \right\} + \dots$$

Again the ellipsis denotes terms with vanishing vev in flat spacetime or with higher powers of $|q|$. It will be helpful to rediagonalize this formula. Let

$$|\text{dil}_\pm\rangle = |D_1\rangle \pm |D_2\rangle. \quad (5.11)$$

¹⁹ Since we do not consider Ramond pinches we do not see the difference between $SO(32)$ and $O(16) \times O(16)$ projections at this point.

Then

$$|S_{12}^c\rangle = i \left\{ \frac{1}{10} |\text{grav}\rangle^{(1)} \otimes |\text{grav}\rangle^{(2)} + \frac{1}{2} |\text{dil}_+\rangle^{(1)} |\text{dil}_+\rangle^{(2)} \right\} + \dots;$$

we will see that $|\text{dil}_-\rangle$ decouples.

6. Shifts

In the picture-changing formalism one computes integrals in two steps. First one sets up coordinates for $\widehat{\mathcal{M}}$ using a choice of world-sheet gravitino fields χ_a , $a = 1, \dots, 2g - 2$ and integrates the density μ over the odd variables. One then integrates the resulting density μ_0 on $\mathcal{M}$. The procedure has an exact analogy in ordinary geometry: suppose we wish to integrate a density μ over an manifold N with a given submanifold N_0 . We can if we choose proceed in two steps: first choose a fibration $\pi : N \rightarrow N_0$ and integrate μ over the fibers of π to get μ_0 . If N is compact then the integral $\int_{N_0} \mu_0$ is independent of the projection π chosen, by Fubini's theorem. If we exclude some open set U_0 from N_0 , however, then of course our answer will depend on the chosen π simply because $U = \pi^{-1}(U_0)$ depends on π (Fig. 11).

Fig. 11: Dashed lines indicate two different projections from N to N_0 . The shaded regions lie over the same excluded set U_0 .

This dependence on projection arises in string theory as follows. If we change the basis χ_a to $\chi_a + \Delta\chi_a$, then the partition function Z changes by a term of the form $\delta_{\Delta\chi} Z = \int_{\mathcal{M}} \delta_{\Delta\chi} \mu_0 = \int_{\mathcal{M}} \frac{\partial W^i}{\partial m^i}$ for some W^i [50]. One can show that this ‘ambiguity’ is just the result of a change of integration region, just as depicted in Fig. 11 [16][20][48]. In the language of Fig. 11, clearly what we must do is to specify the excluded set U directly on N . We now have the tools to do this, and to study the effect of varying U within a good class of cutoffs.

It has always been clear that the ‘ambiguity’ is closely related to a failure of BRST decoupling, via the observation that [50] $\delta_{\Delta\chi} Y$ is BRST-exact, where Y is the picture-changing operator. This suggests that the variation $\delta_{\Delta\chi} Z$ is just due to a failure to enforce BRST. Our goal in the next sections is to show that this variation is not really anomalous in the heterotic string, i.e. it can be cured by modifications to the equations of motion, and that these modifications also suffice to remove the ‘ambiguity’. Actually, we will only discuss the *linearized* corrections to the equations of motion in detail. Even at this level our statement has nontrivial content; for example in the type-I string it is sometimes false [10].

6.1. BRST cancellation

If $|\text{dil}_\pm\rangle$ or $|\text{grav}\rangle$ have tadpoles, then the measure μ will diverge at Δ and we will have to cut off the moduli integrals. We will use the prescriptions of sections 2.2, 3.2 to obtain the cutoff. While these prescriptions require choices of extra local data (local coordinates z_i near the attachment points), we will see that the resulting BRST anomaly, and the condition for it to cancel, do not depend on this choice. Moreover, once the anomaly is cancelled the amplitudes themselves become choice-independent and hence unambiguous [14][15].

Let us insert into X several BRST-invariant states ψ_α , of which one is exact: $\psi_0 = Q\lambda$. To begin with we suppose that all of the ψ_α are on the same side, X_1 (Fig. 12a). We have reviewed how the resulting scattering measure μ is a total derivative on (super) moduli space; thus the amplitude reduces to a boundary integral around the tube $\{|q| = \delta\}$ in $\mathcal{M}_{g_1, N+1} \times \mathbb{C} \times \mathcal{M}_{g_2, 1}$. Of the many terms in the total derivative only the one which omits dq is relevant; all other terms

when integrated around the boundary involve (dq, V) , where V is tangent to the boundary, and this vanishes. The interesting term is thus the one which differentiates with respect to q ; it arises due to the commutator $\{Q, b_0\}$. All other commutators of Q with B insertions give derivatives with respect to the other moduli $\tilde{m}_{(1)}, \tilde{\tau}_{(1)}, \tilde{m}_{(2)}, \tilde{\tau}_{(2)}$.

Fig. 12: Origin of background shifts.

We now specialize to the bosonic case for brevity; the fermionic case is the same. The measure is now

$$\mu = \sum_a q^{h_a} \bar{q}^{\bar{h}_a} \left\langle b \cdots b \lambda_{(Q_0; w_0)} \psi_{1(Q_1; w_1)} \cdots [L_0 \bar{b}_0 \phi_a]_{(P_1; z_1)} \right\rangle_{X_1} \times \left\langle \phi_{(P_2; z_2)}^e b \cdots b \right\rangle_{X_2} |q|^{-2} dq \wedge d\bar{q} \quad (6.1)$$

Integrating $\tilde{m}_{(2)}$ we get

$$\sum_a d(q^{h_a} \bar{q}^{\bar{h}_a} \frac{d\bar{q}}{\bar{q}}) \left\langle b \cdots b \lambda \psi_1 \cdots \psi_N [\bar{b}_0 \phi_a]_{(P_1; z_1)} \right\rangle_{X_1} \cdot iZ_{(g_2)}^e \quad (6.2)$$

where $iZ_{(g_2)}^e$ is the vev of ϕ^e on X_2 appearing in (6.1). A priori $Z_{(g_2)}^e$ seems to depend on the family of local coordinates z_2 . But we have already seen that the only relevant factorizing states, while they do not satisfy the strong physical conditions (4.14)(4.22) (see [35][33][20]), do at least satisfy (4.16) and are BRST-closed. Thus they have unambiguous expectation values [11]. Integrating (6.2)

over $\{|q| > \delta\}$, we see that as $\delta \rightarrow 0$ only $h_a = \bar{h}_a = 0$ survives.²⁰

$$\text{anomaly} = \sum_{h_a = \bar{h}_a = 0} \left\langle b \cdots b \lambda \psi_1 \cdots \psi_N [\bar{b}_0 \phi_a]_{(P_1; z_1)} \right\rangle_{X_1} \cdot 2\pi i \cdot iZ_{(g_2)}^e \quad (6.3)$$

The remarkable thing about the anomaly, then, is that it is homological — the factor of $2\pi i$ is independent of the shape of the cutoff tube. Thus the expression (6.3) is totally independent of z_2 , and depends on z_1 only through the normal ordering of ϕ_a .

We now seek a small change in the background $\delta^{(g_2)} \phi_{b.g.}$ such that the diagram Fig. 12b cancels Fig. 12a. That is, we want

$$\left\langle b \cdots b \lambda \psi_1 \cdots \psi_N [Q \delta \phi_{b.g.}]_{(P_1; \tilde{z}_1)} \right\rangle_{X_1} = -\text{anomaly} \quad ,$$

where $\tilde{z}_1 = z_1/\delta$, so that $\tilde{z}_1 = 1$ is the same as $z_1 = \delta$. That this is possible at all is due to the homological character of the anomaly mentioned above, and in particular to the fact that all tadpoles are physical, a point stressed in [18]. Thus we require a shift of the physical massless backgrounds satisfying

$$Q \delta^{(g_2)} \phi_{b.g.} = + \sum_{h_a = \bar{h}_a = 0} [\delta]^{-L_0 - \bar{L}_0} \bar{b}_0 \phi_a \cdot 2\pi Z_{(g_2)}^e \quad (6.4)$$

The new factor on the right arises because we need to compare a state inserted with z_1 to a state inserted with $\tilde{z}_1$. Since the relevant ϕ_a are all annihilated by L_0 , however, this factor is unity.

As promised, (6.4) is a universal, choice-independent equation of motion for the background shift. Eqn. (6.4) shows how to take the $2d$ CFT slightly off shell, by an amount of order g_2 in the string coupling, to cancel boundary effects. If all the tadpoles vanish, e.g. in the usual heterotic string, then the above argument shows that BRST-spurious states decouple for 1-point functions, since for these Fig. 12 is the only dangerous diagram. In such diagrams arguments based on contour deformation (e.g. [60]) are applicable.

²⁰ Note that naively (6.1) seems to say that $h = \bar{h} = 0$ terms vanish! Actually however (6.2) shows there is a “contact term”, or delta-function, at $q = 0$. (cf. [40]).

To linearized order we consider at most one insertion, and the cancellation condition (6.4) is the whole story. To get higher-order corrections we must allow the case where some of the ψ_α are on X_1 , while others are on X_2 . Repeating the same argument for Fig. 13, we find that in addition to shifts mass renormalization [27][28] is also needed to cancel BRST anomalies [29][13]. More generally higher nonlinear corrections can be found to do the job [6].

Fig. 13: A divergence requiring mass renormalization.

6.2. Choice-independence

We can now see how (6.4) secures choice independence for the vacuum amplitude itself, which is *not* homological. We begin with a fallacious argument. Near Δ we can represent the measure as a form on $\mathcal{M}_{g_1,1} \times \mathbb{C} \times \mathcal{M}_{g_2,1}$:

$$\mu_{\sigma_1, \sigma_2} = \sum_a q^{h_a} \bar{q}^{\bar{h}_a} \mu_{[b_0 \bar{b}_0 \phi_a], \sigma_1}^{(1)} \wedge \frac{dq \wedge d\bar{q}}{|q|^2} \wedge \mu_{\phi_a, \sigma_2}^{(2)} ,$$

and similarly for scattering amplitudes. Why not simply let

$$\begin{aligned} i\mathcal{Z} &\equiv \int \mu \\ &= \sum_a \left[\int_{\mathcal{M}_{g_1,1}} \mu^{(1)} \right] \cdot \left[\int_{|q|>\delta} q^{h_a-1} \bar{q}^{\bar{h}_a-1} dq \wedge d\bar{q} \right] \cdot \left[\int_{\mathcal{M}_{g_2,1}} \mu^{(2)} \right] ? \end{aligned} \quad (6.5)$$

Since the relevant states ϕ_a all have choice-independent vevs, it seems that (6.5) is completely independent of the choices of coordinate families σ_1, σ_2 !

The problem with this argument is that of course (6.5) *does* secretly depend on σ_1, σ_2 : the fixed region $\{|q| > \delta\}$ of $\mathcal{M}_{g_1,1} \times \mathbb{C} \times \mathcal{M}_{g_2,1}$ corresponds to a

region of $\mathcal{M}_g$ which *does* depend on σ_1, σ_2 via (2.13), (3.7). Eqn. (6.5) looks superficially invariant because as the region of integration gets dilated so does the *integrand*. But this merely shoves the ambiguity problem into the interior of $\mathcal{M}$; we then see it in the shaded regions of Fig. 14.

Fig. 14: As we dilate $\mathcal{U}$ for fixed interior patch $\mathcal{U}_1$ the ambiguity shifts to the shaded region.

The lesson to be learned is that we must be careful about changes in coordinate family σ_i — but not *very* careful. From (2.13)–(2.14) we see that the only source of concern is ℓ_0 transformations; all others give effects of $\mathcal{O}(q^2)$, which vanish as the cutoff is removed, so long as the theory has no tachyons. Suppose we leave z_2 alone. In order to preserve the region of integration when z_1 changes to $z_1 + \epsilon_n(\bar{m}_1)z_1^{n+1}$, we just modify the limit in (6.5) from δ to $\delta(1 + \epsilon_0) + \mathcal{O}(\delta^2)$. Since the divergence is logarithmic we get the variation

$$i\delta\mathcal{Z} = \sum_a \int_{\mathcal{M}_{g_1,1}} \epsilon_0(\bar{m}_1) \mu_a^{(1)} \cdot i\mathcal{Z}_{(g_2)}^a \cdot 2\pi i + \mathcal{O}(\delta) , \quad (6.6)$$

where the sum is over $h_a = \bar{h}_a = 0$ only. Note that in the bosonic string our argument is invalidated by the tachyon pole, since the $\mathcal{O}(q^2)$ terms of (2.13) do *not* factorize into factors corresponding to sides 1 and 2. In the fermionic string, however, we have already dealt with the tachyon and (6.6) is correct.

Now we add the counterterm diagrams to (6.6). As the normal ordering is rescaled we get

$$i\delta\mathcal{Z}^{c.t.} = \int_{\mathcal{M}_{g_1,1}} \sum_n \epsilon_n(\bar{m}_1) \left\langle b \dots b [\delta^{L_0 + \bar{L}_0} L_n \delta\phi_{b.g.}]_{(P_1; z_1)} \right\rangle_{X_1} . \quad (6.7)$$

Again we have inserted a rescaling factor so that we can insert $\delta\phi_{b,g}$ on the circle $z_1 = 1$, not $z_1 = \delta$. Only the first term of the sum survives the limit $\delta \rightarrow 0$. Noting again that $h_a = \bar{h}_a = 0$, we see that (6.7) cancels (6.6) if $\delta^{(g_2)}\phi_{b,g}$ is slightly off-shell:

$$L_0\delta^{(g_2)}\phi_{b,g} = -\sum_a b_0\bar{b}_0\phi_a iZ_{(g_2)}^a \cdot 2\pi i . \quad (6.8)$$

We now see that (6.4) is sufficient to ensure (6.8): just act on (6.4) by $b_0 - \bar{b}_0$ and use $(b_0 - \bar{b}_0)\delta\phi_{b,g} = 0$. Thus the BRST-cancellation condition guarantees not only independence of the cutoff δ [7], but also independence of the cutoff *prescription*, or choice of z_1 . Similar arguments apply when z_2 is varied.

All of the preceding arguments apply without modification to the super case, using Eqn. (3.7).

6.3. Comments

We will not discuss the higher-order corrections mentioned at the end of sect. 6.1 in much detail. But consider *e.g.* Fig. 13. We remarked that when a spurious state $Q\lambda$ was inserted on X_1 BRST decoupling in general required the other state ψ_1 to be slightly off its naive mass shell. Now consider the discussion of sect. 6.2. Again we find an ambiguity as we vary the choice of z_1 defining the cutoff. This time, however, the ambiguity is proportional to a two-point amplitude. In fact it cancels precisely when ψ_1 is taken off its naive mass-shell in the way required by BRST. Similar remarks apply to the higher nonlinear corrections. These have been discussed in the bosonic string by Ooguri and Sakai [43].

7. Two VEVs

So far our arguments have been general, relying only on properties of conformal field theory. For illustration we will now specialize to a starting vacuum which is flat $10d$ space. We have found that in order to compute the required background shifts we need to know the tadpole amplitudes $\langle\langle\text{grav}\rangle\rangle, \langle\langle\text{dil}_\pm\rangle\rangle$, where the double brackets imply an integral over moduli space.

7.1. Dilaton

We first note that using (4.18)–(4.19), (5.9)(5.11) we get

$$|\text{dil}_\pm\rangle = (Q + \bar{Q})(c_0 \mp \bar{c}_0)c_1\beta_{-1/2}\delta(\gamma_{1/2})|1\rangle. \quad (7.1)$$

Naively these states seem to decouple, since we have already seen that a BRST-exact state gives a total derivative; the same can be said for the analogous bosonic states $(Q + \bar{Q})(c_0 \pm \bar{c}_0)|1\rangle$. In fact only one state decouples in this way. The point is that even though $\tilde{\mu} = d\tilde{\nu}$ is globally a total derivative on $\mathcal{P}$, and even though $\tilde{\mu}$ descends to a well-defined μ_σ on $\mathcal{M}$, still $\tilde{\nu}$ may *not* descend to a globally-defined form.²¹ This can happen because while $(b_0 - \bar{b}_0)$ annihilates $|\text{dil}_\pm\rangle$, still it does *not* annihilate $(c_0 - \bar{c}_0)c_1\beta_{-1/2}\delta(\gamma_{1/2})|1\rangle$. As discussed in section 2.1 this wouldn't necessarily be a problem if we could find a global section σ of $\mathcal{P}$ — but we cannot. We must instead use a section defined up to $U(1)$ and see what happens to $|\text{dil}_\pm\rangle$. On the other hand, for $|\text{dil}_-\rangle$ no such reservations apply and we at once find $\langle\langle\text{dil}_-\rangle\rangle = 0$.²²

Another possible confusion concerns superconformal normal ordering (SCNO). In this prescription $|\text{dil}_+\rangle$ gets multiplied by $b_{-1}\bar{b}_{-1}\delta(\beta_{-1/2})$, giving zero (see section 4.2). As discussed at the end of section 4.1, however, SCNO is not always applicable. Requiring σ to be defined up to $U(1)$ means that σ cannot be holomorphic, and we need to use general normal ordering.

Let us consider a coordinate family differing from the SCNO family of section 3.1 (Eqn. (3.2)). If Q is a “point” with $z(Q) = u$, $\theta(Q) = \zeta$ (see section 3.1), let

$$z_Q = M(Q)^2(z - u + \zeta\theta) , \quad \theta_Q = M(Q)(\theta - \zeta) \quad (7.2)$$

²¹ The situation is precisely analogous to a point particle moving in the field of a magnetic monopole. The sphere plays the role of $\mathcal{M}$; the Hopf bundle that of $\mathcal{P}$.

²² In [7] an argument for decoupling of $|\text{dil}_-\rangle$ was given using twist conservation. There is no twist in the heterotic case.

or

$$\begin{aligned} z_Q(\cdot) &= M(Q)^2(z(\cdot) - z(Q) + \theta(Q)\theta(\cdot)) \\ \theta_Q(\cdot) &= M(Q)(\theta(\cdot) - \theta(Q)) \end{aligned}$$

Here M is an arbitrary nonvanishing complex function. (More general departures from (3.2) lead to modifications to $\hat{b}$ which do not contribute to $\langle\langle \text{dil}_+ \rangle\rangle$.)

We now make a simplifying choice. We will see that $\langle\langle \text{dil}_+ \rangle\rangle$ is proportional to a Cech cohomology class, as suggested by the fact that it is "almost" decoupled. To calculate the constant of proportionality we can restrict attention to split SRS X , since the Euler number is a topological invariant. While inessential, this restriction simplifies the algebra. For example, when X is split it is meaningful (and possible) to require $\frac{\partial M}{\partial \theta} = 0$ throughout X , where θ belongs to any split superconformal coordinate (z, θ) ; in this case we also know what the conjugate $\bar{M}$ means. Let M denote $M(Q)$ and ∂M denote $\frac{\partial M}{\partial z}|_Q$. We can Taylor expand M at any P near Q :

$$M(P) = M + \Delta \partial M + \bar{\Delta} \bar{\partial} M + \mathcal{O}(\Delta^2) \quad ,$$

where

$$\Delta = z(P) - z(Q).$$

Also let

$$\begin{aligned} \delta &= \theta(P) - \theta(Q) \\ A &= 2M^{-1}\partial M \quad B = 2M^{-1}\bar{\partial} M \end{aligned}$$

Expanding $z_P(\cdot), \theta_P(\cdot)$ in terms of $z_Q(\cdot), \theta_Q(\cdot)$ then shows analogous to (3.2) that

$$\begin{aligned} \sigma_* \left(\frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial u} \right) &= -2M i(g_{-1/2}) - A \zeta i(\ell_0) - \bar{B} \zeta i(\bar{\ell}_0). \\ \sigma_* \left(\frac{\partial}{\partial u} \right) &= M^2 i(\ell_{-1}) - i(A \ell_0 + \bar{B} \bar{\ell}_0). \end{aligned} \quad (7.3)$$

For $M \equiv 1$ we have $A = B = 0$, recovering (3.2).

Thus to evaluate the pulled-back density μ_σ with our coordinate family we must use the modified insertions

$$\begin{aligned} \hat{b}_{-1} &= b_{-1} - M^{-2}(A b_0 + \bar{B} \bar{b}_0) \\ \hat{\bar{b}}_{-1} &= \bar{b}_{-1} - \bar{M}^{-2}(\bar{A} \bar{b}_0 + B b_0) \\ \hat{\beta}_{-1/2} &= \beta_{-1/2} + \zeta M^{-1}(A b_0 + \bar{B} \bar{b}_0) \end{aligned} \quad (7.4)$$

These appear in the combinations $du \cdot M^2 \hat{b}_{-1}$, $d\zeta \cdot 2M(\hat{\beta}_{-1/2} - \frac{1}{2}M\zeta \hat{b}_{-1})$. Note that $d\zeta \cdot 2M\hat{\beta}_{-1/2}$ is not correct, since by (7.3) $\beta_{-1/2}$ corresponds not to $\partial/\partial \zeta$ but to the nonintegrable differential $\frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial u}$. As in sect. 3.1, we give a name to the generator corresponding to $\frac{\partial}{\partial \zeta}$:

$$k_{-1/2} = g_{-1/2} + \frac{1}{2}M\zeta \ell_{-1} = \hat{k}_{-1/2},$$

a function on $\hat{\mathcal{M}}_{g,1}$ with values in the NS algebra. As indicated the $b_0, \bar{b}_0$ terms of (7.4) cancel from $\hat{k}_{-1/2}$.

Next we suppose that on another, overlapping patch we have a different coordinate family, also obtained from the SCNO family via (7.2) but differing by $U(1)$:

$$\widetilde{M} = e^{i\pi f} M \quad , \quad (7.5)$$

where $f = \bar{f}$ is a function on X , independent of θ . Thus on this patch

$$\tilde{A} = A + 2\pi i \partial f, \quad \tilde{B} = B + 2\pi i \bar{\partial} f \quad . \quad (7.6)$$

We can now write

$$\begin{aligned} \langle\langle \text{dil}_+ \rangle\rangle &= \int_{\hat{\mathcal{M}}} \int_X [d^2 u | d\zeta] \left\langle B(-M^2 \hat{\ell}_{-1}) B(-\bar{M}^2 \hat{\bar{\ell}}_{-1}) \delta \left[B(2M k_{-1/2}) \right] [Q\psi]_{(P;z)} \right\rangle_X \\ &= \frac{1}{2} \int [d^2 u | d\zeta] M \bar{M}^2 \left\langle \left\{ \hat{b}_{-1} \hat{\bar{b}}_{-1} \left[\delta(\tilde{\beta}), Q \right] + \hat{b}_{-1} \hat{\bar{L}}_{-1} \delta(\tilde{\beta}) \right. \right. \\ &\quad \left. \left. - \hat{L}_{-1} \hat{\bar{b}}_{-1} \delta(\tilde{\beta}) \right\} \psi_{(P;z)} \right\rangle_X \end{aligned}$$

Here $|\psi\rangle \equiv (c_0 - \bar{c}_0)c_1\beta_{-1/2}\delta(\gamma_{1/2})|1\rangle$ from (7.1) and $\tilde{\beta} \equiv B(k_{-1/2}) = \beta_{-1/2} - \frac{1}{2}\zeta M b_{-1}$. Also we suppress the ghost insertions needed to get a measure on $\hat{\mathcal{M}}$, keeping only those associated to moving the insertion point P . After some arithmetic the first term is seen to be the integral of a total derivative in ζ , and hence vanishes. Writing the state $\langle X|$ as $\langle X_0|(1 - \zeta M 2T(k_{-1/2}))$ where $\langle X_0|$ is independent of ζ we find the remaining terms to be

$$\begin{aligned} &\frac{1}{2} \int [d^2 u | d\zeta] M \bar{M}^2 \left\langle (1 - \zeta M G_{-1/2}) \right. \\ &\quad \times \left[-\frac{\overleftarrow{\partial}}{\partial u} \bar{M}^{-2} \hat{b}_{-1} \left(\delta(\beta_{-1/2}) - \frac{1}{2}M\zeta b_{-1} \delta'(\beta_{-1/2}) \right) \right. \\ &\quad \left. \left. + \frac{\overleftarrow{\partial}}{\partial u} M^{-2} \hat{\bar{b}}_{-1} \left(\delta(\beta_{-1/2}) - \frac{1}{2}M\zeta b_{-1} \delta'(\beta_{-1/2}) \right) \right] \psi_{(P;z)} \right\rangle_{X_0} \end{aligned}$$

The derivatives differentiate $\langle X_0 |$. The terms $\delta(\beta_{-1/2})$ annihilate $|\psi\rangle$. Thus the term $\zeta M G_{-1/2}$, normally responsible for supplying the picture-changing operator [50], *cancels*. Integrating by parts, we then easily get

$$\begin{aligned} \langle\langle \text{dil}_+ \rangle\rangle &= -\frac{1}{4} \int d^2u \langle (\partial_u \hat{b}_{-1} M^2 - \partial_u \hat{\bar{b}}_{-1} \bar{M}^2) (c_0 - \bar{c}_0) \rangle_{X_0} \\ &= -\frac{1}{4} iZ \int_X d[du(A - \bar{B}) - d\bar{u}(\bar{A} - B)] \quad , \end{aligned}$$

where iZ is the usual vacuum amplitude. As expected the density μ is a total derivative: $\mu = d\nu_\alpha$ on each patch $\mathcal{U}_\alpha$ where the family σ is defined. Moreover across patch boundaries we get from (7.6)

$$\nu_\alpha - \nu_\beta = [du(-4\pi i)\partial_u f - d\bar{u}4\pi i\partial_{\bar{u}} f] \quad . \quad (7.7)$$

Eqn. (7.7) is actually valid even if $\sigma_\alpha, \sigma_\beta$ are not of the special form (7.2); in general we let

$$f_{\alpha\beta} = \frac{1}{2\pi i} \log \frac{dz_\beta}{dz_\alpha} \quad . \quad (7.8)$$

Adding up the contributions we now get

$$\begin{aligned} \langle\langle \text{dil}_+ \rangle\rangle &= -\frac{1}{4} iZ \sum_{(\alpha\beta)} \int_{W_{\alpha\beta}} (\nu_\alpha - \nu_\beta) \\ &= -i\pi\chi iZ \quad . \end{aligned} \quad (7.9)$$

Here $W_{\alpha\beta}$ is an edge between $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ and we have used the standard definition of the Euler number χ .

7.2. Graviton

We seek

$$\langle\langle \text{grav} \rangle\rangle = \langle\langle d_{-1/2} \cdot \bar{\alpha}_{-1} c_1 \bar{c}_1 \delta(\gamma_{1/2}) \rangle\rangle \quad .$$

As mentioned it suffices to use SCNO. Thus

$$\begin{aligned} \langle\langle \text{grav} \rangle\rangle &= \int_{\hat{\mathcal{M}}} \int_X [d^2u | d\zeta] \langle b_{-1} \bar{b}_{-1} \delta[\beta_{-1/2}] [c_1 \bar{c}_1 d_{-1/2} \cdot \bar{\alpha}_{-1} \delta(\gamma_{1/2})]_{(P;z)} \rangle_X \\ &= -\frac{1}{2} \int_{\hat{\mathcal{M}}} \int_X [d^2u | d\zeta] \langle (1 - \zeta G_{-1/2}) d_{-1/2} \cdot \bar{\alpha}_{-1} \rangle_{X_0} \\ &= \frac{1}{2} \int_{\hat{\mathcal{M}}} \int_{X_0} d^2u \langle \alpha_{-1} \cdot \bar{\alpha}_{-1} \rangle_{X_0} \quad . \end{aligned}$$

Using the bosonic string result [7] in our notation gives

$$= -\frac{i\pi}{2} \cdot 10 \cdot (\chi - 2) iZ \quad .$$

The factor of $\chi - 2$ arises as follows. We can write $\partial x \bar{\partial} x$ as $\partial(x \cdot \bar{\partial} x) - x \cdot \partial \bar{\partial} x$. The first term yields a total derivative $d\nu_\alpha$ on each CNO patch α , but $\nu_\alpha \neq \nu_\beta$ since $L_1 : x \cdot \partial x : |1\rangle = -10|1\rangle \neq 0$. Thus we again build up the Euler number just as in (7.9). The second term seems to vanish by the equations of motion, but a contact contribution arises instead. As mentioned before, all our correlations really contain $e^{ik \cdot x_0}$ from the sewing state (5.8); a quantity like $\langle\langle \text{grav} \rangle\rangle$ is really the coefficient of the resulting factor of $(2\pi)^{10} \delta^{10}(k^\mu)$. When $\partial_x \bar{\partial}_{\bar{x}} x$ contracts with $e^{ik \cdot x}$ we get a singular contribution which persists when k is set to zero, and this changes χ to $\chi - 2$ [7].

8. Equations of motion

In the $O(16) \times O(16)$ string the vacuum amplitude Z is nonzero, and hence so are the two vevs just computed. They imply the BRST anomaly (6.3). As in sect. 6.1, we now introduce a BRST-*non*invariant $\delta^{(g_2)} \phi_{b.g.}$ to eliminate the problem:

$$\begin{aligned} |\delta^{(g_2)} \phi_{b.g.}\rangle &= \frac{1}{4\pi} \left[(B_{\mu\nu} + h_{\mu\nu}) d_{-1/2}^\mu \bar{\alpha}_{-1/2}^\nu c_1 \bar{c}_1 - \tilde{\Phi} \left(\frac{1}{2} c_1 \gamma_{-1/2} + 2c_1 \bar{c}_1 \bar{\alpha}_{-1} \beta_{-1/2} \right) \right. \\ &\quad \left. + (c_0 + \bar{c}_0) \left(\zeta_\mu c_1 d_{-1/2}^\mu + \eta_\mu c_1 \bar{c}_1 \beta_{-1/2} \bar{\alpha}_{-1}^\mu \right) \right] \delta(\gamma_{1/2}) |1\rangle. \end{aligned} \quad (8.1)$$

Here $B_{\mu\nu}$, $h_{\mu\nu}$, $\tilde{\Phi}$, ζ_μ , and η_μ are all functions of the zero mode of x^μ . Eqn. (8.1) is the most general background with quantum numbers appropriate to cancel the boundary term; for example we have not included any gauge fields, as they will not be induced to lowest order in tadpoles. Note that $(\bar{b}_0 - b_0) |\delta \phi_{b.g.}\rangle = 0$ is chosen according to (4.16) and each term in eq.(8.1) has the total (ghost+antighost) charge 1.

Applying the BRST operator (the relevant terms are given in eq.(4.18)) to $|\delta^{(g_2)}\phi_{b,g}\rangle$, we have

$$\begin{aligned}
& (Q + \bar{Q})|\delta^{(g_2)}\phi_{b,g}\rangle \\
&= \left[-\frac{1}{2}p^2 h_{\mu\nu} - \frac{1}{2}(ip_\mu \zeta_\nu + ip_\nu \zeta_\mu) - \frac{1}{4}(ip_\mu \eta_\nu + ip_\nu \zeta_\mu) \right] d_{-1/2}^{[\mu} \bar{\alpha}_{-1}^{\nu]} (c_0 + \bar{c}_0) c_1 \bar{c}_1 \delta(\gamma_{1/2}) |1\rangle \\
&+ \left[-\frac{1}{2}p^2 B_{\mu\nu} + \frac{1}{2}(ip_\mu \zeta_\nu - ip_\nu \zeta_\mu) - \frac{1}{4}(ip_\mu \eta_\nu - ip_\nu \zeta_\mu) \right] d_{-1/2}^{[\mu} \bar{\alpha}_{-1}^{\nu]} (c_0 + \bar{c}_0) c_1 \bar{c}_1 \delta(\gamma_{1/2}) |1\rangle \\
&+ \left[\frac{i}{2}p^\nu h_{\mu\nu} - \frac{i}{2}p^\nu B_{\mu\nu} + \frac{i}{2}p_\mu \tilde{\Phi} - \frac{1}{2}\eta_\mu \right] \bar{\alpha}_{-1}^\mu c_1 \bar{c}_1 \gamma_{-1/2} \delta(\gamma_{1/2}) |1\rangle \\
&+ \left[\frac{i}{2}p^\nu h_{\mu\nu} + \frac{i}{2}p^\nu B_{\mu\nu} + ip_\mu \tilde{\Phi} - 2\zeta_\mu \right] d_{-1/2}^\mu c_1 \bar{c}_1 \bar{c}_{-1} \delta(\gamma_{1/2}) |1\rangle \\
&+ \left[-\frac{1}{4}p^2 \tilde{\Phi} + \frac{i}{2}p^\mu \zeta_\mu \right] (c_0 + \bar{c}_0) c_1 \gamma_{-1/2} \delta(\gamma_{1/2}) |1\rangle \\
&+ \left[-p^2 \tilde{\Phi} + ip^\mu \eta_\mu \right] (c_0 + \bar{c}_0) c_1 \bar{c}_1 \bar{c}_{-1} \beta_{-1/2} \delta(\gamma_{1/2}) |1\rangle .
\end{aligned} \tag{8.2}$$

Thus by imposing similar condition to (6.4) such that

$$(Q + \bar{Q})|\delta^{(g_2)}\phi_{b,g}\rangle = \sum_{h_a = \bar{h}_a = 0} (\bar{b}_0 - b_0) \phi_a \cdot 2\pi Z_{(g_2)}^a , \tag{8.3}$$

we get the condition for overall BRST decoupling:

$$\frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_\mu \left(\zeta_\nu + \frac{1}{2}\eta_\nu \right) - \frac{1}{2}\partial_\nu \left(\zeta_\mu + \frac{1}{2}\eta_\mu \right) = \eta_{\mu\nu} \frac{8\pi^2}{K} \frac{1}{10} \langle\langle \text{grav} \rangle\rangle_{g_2} \tag{8.4a}$$

$$\frac{1}{2}\partial^2 B_{\mu\nu} + \frac{1}{2}\partial_\mu \left(\zeta_\nu - \frac{1}{2}\eta_\nu \right) - \frac{1}{2}\partial_\nu \left(\zeta_\mu - \frac{1}{2}\eta_\mu \right) = 0 \tag{8.4b}$$

$$\frac{1}{2}\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial^\nu B_{\mu\nu} + \frac{1}{2}\partial_\mu \tilde{\Phi} - \frac{1}{2}\eta_\mu = 0 \tag{8.4c}$$

$$\partial^\nu h_{\mu\nu} + \partial^\nu B_{\mu\nu} + \partial_\mu \tilde{\Phi} - 2\zeta_\mu = 0 \tag{8.4d}$$

$$\frac{1}{4}\partial^2 \tilde{\Phi} + \frac{1}{2}\partial^\mu \zeta_\mu = \frac{8\pi^2}{K} \frac{1}{4} \langle\langle \text{dil}_+ \rangle\rangle_{g_2} \tag{8.4e}$$

$$\partial^2 \tilde{\Phi} + \partial^\mu \eta_\mu = \frac{8\pi^2}{K} \langle\langle \text{dil}_+ \rangle\rangle_{g_2} . \tag{8.4f}$$

These are gauge invariant equations of motion. By a proper choice of gauge, we can recover the usual gauge fixed ones easily. Furthermore, if the two vevs

vanish, then (8.4) reduce to the equations of motion for superstrings. Recall that the two dilaton states $|\text{dil}_\pm\rangle$ played different roles; e.g. one of them had no vev even when $Z_{g_2} \neq 0$. The RHS of eqs. (8.4e, f) are consistent with this asymmetry. In fact one can see that this is the only way to give a consistent set of equations of motion after a simple analysis. The source terms of (8.4e, f) are originally

$$E_1 \equiv \frac{8\pi^2}{K} \left(\frac{1}{2} \right) \langle\langle D_2 \rangle\rangle \quad \text{for eq.(8.4e)} \tag{8.5}$$

$$E_2 \equiv \frac{8\pi^2}{K} (2) \langle\langle D_1 \rangle\rangle \quad \text{for eq.(8.4f)}$$

respectively, where $|D_1\rangle$, $|D_2\rangle$ are defined in eq.(5.9). Due to the antisymmetry of $B_{\mu\nu}$ $4 \times (8.4c) - (8.4d)$ should be

$$\partial^\mu \left(\zeta_\mu - \frac{1}{2}\eta_\mu \right) = \partial^\mu \partial^\nu B_{\mu\nu} = 0.$$

On the other hand, $4 \times (8.4e) - (8.4f)$ says that

$$2\partial^\mu \left(\zeta_\mu - \frac{1}{2}\eta_\mu \right) = 4E_1 - E_2.$$

Thus we have $4E_1 = E_2$ from the consistency of the equations. This in turn implies

$$\langle\langle D_1 \rangle\rangle - \langle\langle D_2 \rangle\rangle = \langle\langle \text{dil}_- \rangle\rangle = 0. \tag{8.6}$$

In other words, one combination of dilatons decouples as we expected.

Since $\langle\langle D_1 + D_2 \rangle\rangle = \langle\langle \text{dil}_+ \rangle\rangle$, we get the source terms in eq.(8.5)

$$\langle\langle D_1 \rangle\rangle = \frac{1}{2} \langle\langle \text{dil}_+ \rangle\rangle$$

$$\langle\langle D_2 \rangle\rangle = \frac{1}{2} \langle\langle \text{dil}_+ \rangle\rangle$$

which lead to the correct source terms in eq.(8.4e, f).

After eliminating the auxiliary fields ζ and η , we get a homogeneous, decoupled equation for B

$$\partial^2 B_{\mu\nu} + \partial_\mu \partial^\alpha B_{\nu\alpha} - \partial_\nu \partial^\alpha B_{\mu\alpha} = 0 \tag{8.7}$$

and

$$\partial^2(\tilde{\Phi} + \frac{1}{2}h^\alpha_\alpha) = \frac{8\pi^2}{K} (\langle\langle\text{grav}\rangle\rangle_{g_2} + \langle\langle\text{dil}_+\rangle\rangle_{g_2}) \quad (8.8)$$

$$2\partial^2\tilde{\Phi} + \partial^\mu\partial^\nu h_{\mu\nu} = \frac{8\pi^2}{K} \langle\langle\text{dil}_+\rangle\rangle_{g_2} \quad (8.9)$$

Requiring that eqn. (8.8) should become the equation of motion for the dilaton alone, we can define $\tilde{\Phi} = \Phi - \frac{1}{2}h^\alpha_\alpha$. Thus we have

$$\begin{aligned} \partial^2\Phi &= \frac{8\pi^2}{K} (\langle\langle\text{grav}\rangle\rangle_{g_2} + \langle\langle\text{dil}_+\rangle\rangle_{g_2}) \\ \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h^\alpha_\alpha + 2\partial^2\Phi &= \frac{8\pi^2}{K} \langle\langle\text{dil}_+\rangle\rangle_{g_2}. \end{aligned} \quad (8.10)$$

Using the vevs computed before we can now recognize (8.10) as the same equations of motion as those arising in the bosonic string [7], namely the equations of linearized gravity plus dilaton with a cosmological constant given by the vacuum amplitude $i\mathcal{Z}$.

Thus we see that the linearized, loop-corrected equations of motion for physical background fields are well-defined and consistent at $g_1 + g_2$ loops if they are well-defined at fewer loops. We therefore obtain inductively finite, unambiguous string dynamics, as in the bosonic case[29][42] and as in the no-tadpole case [18].

While in this section we worked about flat space, clearly the method is applicable to any exact tree-level vacuum. The discussion of sect. 6, for example, employed only general facts about CFT like factorization and normal ordering. Thus even if we stop short of a full nonlinear treatment, where we use multiple insertions of $\delta\phi$ in (8.3) and so get nonlinear equations instead of (8.4), still we can get the analogs of (8.4) linearized about any nontrivial solution and then deduce the full form. This method was used in [62][54] to deduce the Green-Schwarz anomaly-cancelling term.

9. Conclusions

Most of our conclusions were summarized in the introduction and in [15]. We have seen how to start with an approximate solution to the string equations of motion and perturbatively find an exact solution. If the original background is not too far from the true solution (*e.g.* if it gives no tachyons and no unphysical tadpoles), then the procedure succeeds: we get finite, well-defined corrections to the equations of motion. When the background obeys the corrected equations of motion then amplitudes are well-defined without any remaining ambiguities.

The whole procedure suggests a number of generalizations. We have seen how the corrections to scattering amplitudes at $g_1 + g_2$ loops involve integrals over g_1 -loop surfaces of shifts which are themselves generated by g_2 -loop tadpole integrals. Thus the background shifts can be thought of as contributions from the interior of the space of *disconnected* curves $\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$, introduced to cancel trouble at the *boundary* of $\mathcal{M}_{g_1+g_2}$. This suggests that we construct a universal moduli space along the lines of [31], in which these two regions are adjacent. Rather than having a dependence on choices which cancels between two distant regions, we could then expect a single corrected measure which is everywhere independent of choices. It was speculated in [58] that the right construction would also make every g -loop surface very close to other surfaces with many loops (see also [31]), so that the sewing conditions (3.11) become the finite form of some infinitesimal equivariance condition like (4.2). If this proves correct then the results of this paper may be a step towards verifying Friedan and Shenker's suggestion that nonperturbatively many BRST classes could become identical, giving a unique string ground state [31].

In a different direction, we have seen how the notion of a *family* of local coordinates $z_Q(P)$ arises naturally as the most general world-sheet cutoff; it also preserves the holomorphic structure of the theory even as we go off shell. Such a family contains a lot of information: at each point Q we have an infinite number of degrees of freedom, not just one as suggested by the scale factor of a metric. We already know that such a general cutoff permits us to get gauge-invariant equations of motion for background fields [11], while merely using

CNO does not [33]. The renormalization group associated with such a general cutoff may then lead to generalized string field equations (cf. [1][63]) with more gauge invariance than the one associated to changes of CNO cutoff. In any case introducing this cutoff allowed us to dispense with the world-sheet metric and so clarifies issues involving world-sheet curvature in the beta functions; for example, the coupling of the dilaton is no different from that of other fields[7].

Finally, it would be also very interesting to explore the notion of 'loop-corrected conformal field theory' in general. The present approach, which puts loop corrections on the same footing as ordinary world-sheet renormalization and respects the underlying conformal structure, is a beginning.

On a more prosaic level, we have given the constructions needed to extend the unitarity analysis of [7] to fermionic strings. We have also given the necessary ingredients for a simple, intuitive inductive proof of finiteness in string theories *with tadpoles*. This is important since we want susy to break; when eventually a realistic nonperturbative breaking mechanism is found it can be substituted for the rather crude modification of GSO used here for illustrative purposes.

Numerous questions still remain. We have not given a rigorous discussion of multiple divergences. We have also not sought the nonlinear corrections to the quantum equations of motion, which arise from multiple insertions of $\delta\phi_{b,g}$. Most important, we have not given any general proof that the corrected equations of motion are always consistent in modular-invariant closed fermionic string theories, or equivalently that the loop-corrected RG flow is always integrable. This is a problem whose solution could shed light on the correct closed superstring field theory.

Note Added

We have recently learned that the construction of sect. 2.3, and the result (2.13), have been worked out by S. Wolpert [64].

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Appendix A.

As a check on the discussion of sect. 3.2 we will give an explicit example of the phenomenon described in (3.7), the existence of a family of 'good' coordinates near Δ . We will show that coordinates built from the picture-changing formalism are all related in the special way given by (3.7), to lowest nontrivial order in the odd moduli ζ .

Suppose we are given a local family of ordinary spin Riemann surfaces $\Psi_0 : \mathbb{C}^{3g-3} \rightarrow \mathcal{M}_g^{\text{spin}}$, and $2g-2$ families of marked points with local coordinates, i.e. sections $\sigma_a : \mathcal{M}_g^{\text{spin}} \rightarrow \mathcal{P}_g^{\text{spin}}$. We can then deform Ψ_0 into

$$\Psi : \mathbb{C}^{3g-3|2g-2} \rightarrow \widehat{\mathcal{M}}_g$$

by the agency of Beltrami superdifferentials concentrated on the points P_a :

$$\mu_a = \theta \bar{\theta} \chi_{a\bar{z}}^\theta d\bar{s}_a \otimes ds_a^{-2}, \quad \chi_{a\bar{z}}^\theta = \delta^{(2)}(z_a),$$

where z_a is the local coordinate centered at the point P_a and $ds_a = [dz_a | d\theta_a]$. The inverse Ψ^{-1} then supplies the desired coordinate system on $\widehat{\mathcal{M}}$.

Now repeat the process with new points $\tilde{P}_a$ to get $\tilde{\Psi}$. Suppose the $\tilde{P}_a$ are each close to the original P_a . The composition $\tilde{\Psi}^{-1} \circ \Psi : \mathbb{C}^{3g-3|2g-2} \rightarrow \mathbb{C}^{3g-3|2g-2}$ is then generated by an infinitesimal vector field W . In fact one has [48] $W = W^i \frac{\partial}{\partial m^i} + W^a \frac{\partial}{\partial \zeta^a}$ with

$$W^i = \zeta^a \langle \Delta \mu_a, \Phi^i \rangle, \quad (A.1)$$

where Φ^i, Φ^a are a basis of 3/2-superdifferentials dual to $\partial/\partial m^i, \partial/\partial \zeta^a$. This relation has nothing to do with supergeometry; it can be derived for coordinate changes in an ordinary vector bundle. It simply says that if the new tangents $\tilde{\Psi}_*(\partial/\partial \zeta^a)$ have components along the old $\Psi_*(\partial/\partial m^i)$ then this means that the new coordinates $\tilde{m}^i$ involve ζ^a . We would like to evaluate (A.1) and see whether (3.7) is satisfied. We will work only to $\mathcal{O}(\zeta^2)$, neglecting ζ^4 and higher terms in $\tilde{m}^i$.

To verify (3.7), begin with a pinching family Ψ_0 constructed by the bosonic plumbing construction. Then the coordinates m^i divide into $\tilde{m}_{(1)}, \tilde{m}_{(2)}, t$, where $t = q^{1/2}$ is the pinching coordinate. Choose sections $\sigma_a^{(k)} = (P_a^{(k)}, z_a^{(k)})$, $k = 1, 2$ of $\mathcal{P}_{g_h,1} \rightarrow \mathcal{M}_{g_h,1}$. Thus the $P_a^{(1)}$ are independent of $\tilde{m}_{(2)}$, and vice versa. We want first to make sure that W^t , the infinitesimal transformation of t , contains no term of the form

$$W^t = \gamma_{ab}^t(\tilde{m}_{(1)}, \tilde{m}_{(2)}) \zeta_{(1)}^a \zeta_{(2)}^b t, \quad (\text{A.2})$$

since no such term can be written in the form (3.7). To evaluate (A.1) we now note that Φ^t is a 3/2-superdifferential satisfying $\langle \mu_a, \Phi^t \rangle = 0$, $\langle \mu_i^{(k)}, \Phi^t \rangle = 0$, $k = 1, 2$, and $\langle \mu_t, \Phi^t \rangle = 1$. Thus

$$\Phi^t(\#) = \frac{\left\langle \prod^{2g_1-1} \delta[\beta(z_a^{(1)})] \prod^{2g_2-1} \delta[\beta(z_a^{(2)})] \prod^{3g_1-2} b(\mu_i^{(1)}) \prod^{3g_2-2} b(\mu_i^{(2)}) B(\#) \right\rangle}{\left\langle \prod^{2g_1-1} \delta[\beta(z_a^{(1)})] \prod^{2g_2-1} \delta[\beta(z_a^{(2)})] \prod^{3g_1-2} b(\mu_i^{(1)}) \prod^{3g_2-2} b(\mu_i^{(2)}) t^{-1} b_0 \right\rangle}.$$

In the numerator we have included insertions for all of the tangents except $\partial/\partial t$. The b_0 in the normalization factor is the contour integral of $b(z)$ around the neck, which is $t \int d^2 z b \mu_t$; hence the extra factor of t^{-1} . Since $\langle \mu_a^{(k)}, \Phi^t \rangle = 0$ we have

$$\langle \Delta \mu_a, \Phi^t \rangle = -\langle \tilde{\mu}_a, \Phi^t \rangle = \frac{\langle \dots \beta(\tilde{z}_a) \rangle}{\langle \dots t^{-1} b_0 \rangle}.$$

Evidently this vanishes when $z_a = \tilde{z}_a$, or when all of the odd moduli are set to zero. To lowest order in ζ we have

$$\langle \Delta \mu_a, \Phi^t \rangle = \frac{t \langle \dots \zeta^b T_F(z_b) \beta(\tilde{z}_a) \rangle}{\langle \dots b_0 \rangle}. \quad (\text{A.3})$$

To check (A.2) we now suppose that $z_b^{(1)}, \tilde{z}_a^{(2)}$ lie on opposite sides of the pinch. Examination of the sewing state (5.8) then shows that by ghost-charge conservation the numerator has an extra factor of t compared to the denominator, so that the term (A.2) is actually $\mathcal{O}(t^2)$, consistent with (3.7).

When $z_b^{(1)}, \tilde{z}_a^{(1)}$ are on the *same* side of the pinch then the $\zeta_{(1)}^a \zeta_{(1)}^b$ term analogous to (A.2) is $\mathcal{O}(t)$. In this case, however, factorization shows that to leading order in t the correlations on side 2 cancel from the numerator and denominator of (A.3), so that the function γ_{ab}^t in (A.2) is independent of $\tilde{m}_{(2)}$, again consistent with (3.7).

Appendix B.

In this appendix we will summarize our convention for the relevant ingredients used in the text.

We use Euclidean world sheet metric γ^{ab} with the signature $(++)$ for the coordinate (τ, σ) . The complex coordinate can be chosen as $z = \tau + i\sigma$. The bosonic string space-time coordinate is decomposed as

$$X^\mu(z, \bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z})$$

with the mode expansions

$$\begin{aligned} \partial X^\mu(z) &= \sum \alpha_n^\mu z^{-n-1} \\ X^\mu(z) &= \frac{1}{2} q^\mu + \alpha_0^\mu \ln z + \sum_{n \neq 0} \frac{\alpha_n^\mu}{-n} z^{-n} \end{aligned} \quad (\text{B.1})$$

and the mode coefficients satisfy

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= -n \eta^{\mu\nu} \delta_{n+m} \\ [\alpha_0^\mu, q^\nu] &= -\eta^{\mu\nu}, \quad \alpha_0^\mu = -ip^\mu = -\theta^\mu. \end{aligned} \quad (\text{B.2})$$

The stress energy tensor is given by

$$T^{(X)}(z) = -\frac{1}{2} \partial X^\mu(z) \partial X_\mu(z) = \sum L_n^{(X)} z^{-n-2}$$

$$L_n^{(X)} = -\frac{1}{2}\alpha_0^2 - \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m . \quad (B.3)$$

The fermionic matter field satisfies $\psi^\mu(z, \bar{z}) = \psi^\mu(z) + \bar{\psi}^\mu(\bar{z})$ and

$$\psi^\mu(z) = \sum d_n^\mu z^{-n-1/2}, \quad \{d_n^\mu, d_m^\nu\} = -\eta^{\mu\nu} \delta_{n+m} . \quad (B.4)$$

The conformal and superconformal ghosts are

$$\begin{aligned} b(z) &= \sum b_n z^{-n-2}, \quad b_n|1\rangle = 0, \quad n > -2 \\ c(z) &= \sum c_n z^{-n+1}, \quad c_n|1\rangle = 0, \quad n > 1 \\ \{b_n, c_m\} &= \delta_{n+m}, \quad \{b_n, b_m\} = 0 = \{c_n, c_m\} \\ \beta(z) &= \sum \beta_n z^{-n-3/2}, \quad \beta_n|1\rangle = 0, \quad n > -3/2 \\ \gamma(z) &= \sum \gamma_n z^{-n+1/2}, \quad \gamma_n|1\rangle = 0, \quad n > 1/2 \\ [\gamma_n, \beta_m] &= \delta_{n+m}, \quad [\gamma_n, \gamma_m] = 0 = [\beta_n, \beta_m]. \end{aligned}$$

We can construct the BRST-operator

$$Q = -\oint \frac{dz}{2\pi i} C \left(T^X + \frac{1}{2} T^{BC} \right) \quad (B.5)$$

$$\begin{aligned} T^X &= -\frac{1}{2} D X^\mu D^2 X_\mu \\ T^{BC} &= \frac{1}{2} D C D B - \frac{3}{2} \partial C B - C \partial B \\ X^\mu &= X^\mu + \theta \psi^\mu, \quad D = \partial_\theta + \theta \partial \\ B &= \beta + \theta b, \quad C = c + \theta \gamma . \end{aligned}$$

After integrating over θ , we have

$$\begin{aligned} Q &= \oint \frac{dz}{2\pi i} \left[-\frac{1}{2} c \partial X^\mu \partial X_\mu + \frac{1}{2} c \psi^\mu \partial \psi_\mu + \frac{1}{2} \gamma \psi^\mu \partial X_\mu \right. \\ &\quad \left. + c \partial c b - \frac{1}{4} \gamma^2 b - \frac{1}{2} c \partial \beta \gamma - \frac{3}{2} c \beta \partial \gamma \right], \end{aligned} \quad (B.6)$$

which can be written in terms of modes as

$$\begin{aligned} Q &= \sum c_{-n} \left(L_n^{(\alpha)} + L_n^{(d)} \right) + \frac{1}{2} \sum \gamma_{-k} G_k^{(\alpha, d)} \\ &\quad - \frac{1}{2} \sum (m-n) : c_{-m} c_{-n} b_{n+m} : + \sum \left(\frac{3m}{2} + k \right) : c_{-m} \beta_{-k} \gamma_{m+k} : \\ &\quad - \frac{1}{4} \sum : \gamma_{-k} \gamma_{-l} b_{k+l} : . \end{aligned} \quad (B.7)$$

$$\begin{aligned} L_n^{(\alpha)} &= -\frac{1}{2} \sum : \alpha_{-m} \cdot \alpha_{n+m} : \\ L_n^{(d)} &= -\frac{1}{2} \sum \left(m + \frac{n}{2} \right) : d_{-m} \cdot d_{n+m} : \\ G_k^{(\alpha, d)} &= \sum : \alpha_{-m} \cdot d_{k+m} : , \end{aligned}$$

where $k, l \in \mathbb{Z} + 1/2$ for NS sector and $k, l \in \mathbb{Z}$ for R sector, and the normal ordering $: \cdot :$ is with respect to $SL(2, \mathbb{C})$ invariant vacuum $|1\rangle$. For superstring $\bar{Q}$ can be obtained by taking the conjugate, while for heterotic strings we need to drop the antiholomorphic superconformal ghosts.

The total ghost charge U is determined by

$$\begin{aligned} [U, c] &= c, \quad [U, \gamma] = \gamma, \quad [U, \delta(\beta)] = \delta(\beta) \\ [U, b] &= -b, \quad [U, \beta] = -\beta, \quad [U, \delta(\gamma)] = -\delta(\gamma) . \end{aligned} \quad (B.8)$$

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