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Abstract

We show how to construct topological N = 1 and 2 supergravity theories from appropriately constrained N = 3 and 4 ghost plus matter systems. In particular N = 4 susy is not needed to obtain the N = 1 topological theory. We give a description of the relevant supermoduli spaces which must be integrated to obtain the amplitudes of these theories, as well as explicit formulas for the ingredients entering the integrands, in particular the various supercurrents with their inhomogeneous terms.

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Semirigid Construction of Topological Supergravities

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We show how to construct topological $N = 1$ and 2 supergravity theories from appropriately constrained $N = 3$ and 4 ghost plus matter systems. In particular $N = 4$ susy is not needed to obtain the $N = 1$ topological theory. We give a description of the relevant supermoduli spaces which must be integrated to obtain the amplitudes of these theories, as well as explicit formulas for the ingredients entering the integrands, in particular the various supercurrents with their inhomogeneous terms.

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1. Introduction

Two-dimensional topological gravity remains a fascinating and enigmatic subject, due to its conjectured link with ordinary Liouville gravity and matrix models [1][2][1]. Since two-dimensional supergravity is the basis for more interesting string theories than ordinary gravity, it is of some interest to see whether a similar link exists between topological supergravity and super Liouville theory, or super-KDV flows [4]. It has not been entirely clear how to construct the required topological theory, however. For example the nature of the moduli space in question has been elusive [4][5]. Moreover, the case of ordinary gravity has also proved more subtle than originally appeared; certain inhomogeneous terms in the supercurrent used to integrate odd moduli, as well as modifications to the operator insertions themselves, are needed [6][7]. Clearly what is needed is a fundamental definition of the theory starting from some principle.

Such a principle was given in ref. [8]. One begins with the observation that the matter systems coupled to topological gravity should be coordinate-invariant even prior to introducing gravity. In particular [9] they should have an algebra of symmetries with a stress tensor T and a nilpotent charge Q_s such that $T = \{Q_s, G\}$ for some operator G. Moreover T must be anomaly-free in order to generate true symmetries. In two dimensions we can satisfy these requirements by beginning with local $N = 2$ susy, because the $N = 2$ superconformal algebra contains a subalgebra of the type requested above. In particular this "twisted" subalgebra is always *anomaly-free*, regardless of the central extension of the full $N = 2$ algebra on the matter system [10][11].

One can now simply gauge the twisted subalgebra. In the process the full (matter plus ghost) stress tensor becomes a BRST-commutator. This ensures that T decouples and the theory becomes truly topological. Other total commutators decouple as well, provided they obey a certain "equivariance" condition [1][12]. Rather than gauging the subalgebra one can gauge the full $N = 2$ symmetry and impose a constraint to reduce the symmetry; this was the approach used in [8]. One ends up with a completely explicit prescription for computing amplitudes as integrals of ordinary CFT correlations over a supermoduli space. The latter is then a reduction of the $N = 2$ supermoduli space consisting of those $N = 2$ SRS whose patching functions respect the topological constraint. Such surfaces are called semirigid SRS, or "SSRS." We will also write "$TN = 0$" to denote these surfaces, since they arise in describing ordinary topological gravity.

[1] For the status of this conjecture see ref. [3].

It is not hard to guess a generalization of this program to get topological supergravities. We must find anomaly-free subalgebras of the $N > 2$ superconformal algebras [13] and gauge them. By identical reasoning to the above, we then obtain topological theories with $TN > 0$ susy. We will carry out this program below. Since the construction follows from a very explicit principle we will find directly the various ingredients needed in the path integral, including the full inhomogeneous supercurrents. Moreover the resulting moduli space for $N = 3$, say, has a natural projection $\pi : \widehat{\mathcal{M}}_{TN=1} \to \widehat{\mathcal{M}}_{N=1}$. Integrating over the fibers of π one can in principle reduce the amplitudes of topological $N = 1$ supergravity to integrals over $\widehat{\mathcal{M}}_{N=1}$, just as in the $TN = 0$ case [6][8]. To describe a full $N = 1$ topological supergravity theory, one couples a topological matter system such as the $N = 1$ topological superconformal field theory recently constructed [14].

Other approaches to topological supergravity exist [4][15][16][5]. Some of these begin with a different principle, namely the quantization of flat $OSp(2|1,\mathbf{R})$ gauge connections [15][5] or with BRST gauge-fixing of diffeomorphisms [16] along the lines of [17]. As mentioned in [8], we find the semirigid approach gives a clearer understanding of why the final CFT is free, and generally simplifies the construction. Finally it will become clear in the sequel that to end up with $N = 1$ we only need to start with $N = 3$ (not $N = 4$ [5]). We will simply verify that our constrained symmetry closes without extension to $N = 4$. One could of course begin with $N = 4$ symmetry. In this case we will show that the resulting theory is *extended*, or $TN = 2$, topological supergravity.

2. Unbroken geometry

We first recall some facts about extended superconformal geometry [18][19][20] [21][22].

An N-superconformal surface may be regarded as a complex supermanifold Σ of dimension $1|N$ equipped with an extra structure. The structure can be conveniently specified by describing those coordinate transformations which leave it fixed. Take $\Sigma = \mathbf{C}^{1|N}$ and define

$$D_i = \frac{\partial}{\partial\theta^i} + g_{ij}\theta^j \, \frac{\partial}{\partial z}$$

where $g_{ij} = \delta_{ij}$. Then the allowed coordinate transformations will take $\mathbf{z} \equiv (z, \vec{\theta})$ to $\mathbf{z}'$ with

$$\begin{pmatrix} \partial/\partial z \\ \vec{D} \end{pmatrix} = \left(\begin{array}{c|c} a^2 & \vec{\omega} \\ \hline \vec{0} & a\overset{\leftrightarrow}{M} \end{array}\right) \begin{pmatrix} \partial/\partial z' \\ \vec{D}' \end{pmatrix} \quad . \tag{2.1}$$

Here a is a nowhere-vanishing even function, $\vec{\omega}$ are odd functions, and $\overset{\leftrightarrow}{M}$ is a matrix of even functions obeying $M^t g M \equiv g$. Since matrix functions of the form (2.1) form a group, the corresponding set of coordinate transformations forms a group: the "N-superconformal transformations." We will sometimes write

$$D_i = F_i{}^j D'_j \quad ; \qquad F_i{}^j = a M_i{}^j \quad . \tag{2.2}$$

An N-superconformal surface is then just a supermanifold patched together from pieces of $\mathbf{C}^{1|N}$ by N-superconformal transition functions. There is a more intrinsic description of such a space as possessing an integrable reduction of its structure group [22][23], but we will not need this level of refinement here.

We should note an important difference between $N = 1, 2$ and $N > 2$. For $N = 1$ the sole operator D never mixes with ∂_z, by (2.1). For $N = 2$ the operators $D_{1,2}$ do mix with each other, but when $M \in SO(2, \mathbf{C})$ the combinations $D_\pm \equiv \frac{1}{\sqrt{2}}(D_1 \pm D_2)$ do not. This is simply because in this basis $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the matrix M must be diagonal. For $N > 2$ however, the D_i will in general mix across patch boundaries; only the $O(N, \mathbf{C}) \times \mathbf{C}^\times$ structure is global. In the case of $N = 2$ it was essential to forbid "twisting," *i.e.* one must in fact require [8] $M \in SO(2, \mathbf{C})$. We will see that for $N > 2$ twisting will be possible and desirable.

As on ordinary SRS one can easily translate (2.1) into a condition on $\mathbf{z}'$:

$$D_i z' = g_{jk} \theta'^j D_i \theta'^k \quad .$$

Moreover, as in $N = 1$ the most general infinitesimal superconformal transformation can be specified by a single even function: if $v = v(z, \vec{\theta})$ then we let

$$V_v \equiv v\, \partial_z + \tfrac{1}{2}\, g^{ij} (D_i v) D_j \quad , \tag{2.3}$$

which gives for (2.1), (2.2)

$$F_i{}^j = \delta_i^j + \tfrac{1}{2}\, D_i D^j v \ , \qquad a = 1 + \tfrac{1}{2}\, \partial_z v \quad . \tag{2.4}$$

The function v is not a scalar. Writing (2.3) out in another coordinate system $(z', \vec{\theta}')$ shows that $V_v = v' \partial_{z'} + \frac{1}{2}\, g^{ij} (D'_i v') D'_j$ with

$$v'(z', \vec{\theta}') = a^2(z, \vec{\theta}) v(z, \vec{\theta}) \quad . \tag{2.5}$$

Since the basis vector $\frac{\partial}{\partial z'} = a^{-2}\frac{\partial}{\partial z}$ (modulo $\mathcal{E}$), we see that v should be regarded as a section of the line bundle $\mathcal{T} = T\Sigma/\mathcal{E}$. We will remind ourselves of this by writing $v = v^z(z, \vec{\theta})$, but note that this does *not* mean v is the z component of a true tangent vector; eqn. (2.5) shows that v transforms homogeneously under superconformal transformations.

We will need the formula for the Lie bracket of two infinitesimal transformations. One finds $[V_v, V_w] = V_{[v,w]}$ where

$$[v, w]^z \equiv v^z \partial_z w^z - w^z \partial_z v^z + \tfrac{1}{2}(D_i v^z)(D^i w^z) \equiv \pounds_v w^z \quad . \tag{2.6}$$

The reader can show using (2.4) that (2.6) is the infinitesimal version of (2.5):

$$v'(\mathbf{z}) = v(\mathbf{z}) + [v, w](\mathbf{z}) \equiv (1 - \pounds_w)v \quad .$$

Taking the superdeterminant of (2.4) shows that the Berezin differential $\mathrm{d}\mathbf{z} = [\mathrm{d}z|\mathrm{d}\theta^1 \ldots \mathrm{d}\theta^N]$ transforms as a section of $\mathcal{T}^{(\frac{N}{2}-1)}$, or in other words as a $(1 - \frac{N}{2})$-differential. Thus we get a natural dual pairing between p-differentials and $(1 - \frac{N}{2} - p)$-differentials, the analog of Serre duality (which is the case $N = 0$). For example the stress tensor in CFT is paired with the generator v, a (-1)-differential:

$$\mathbf{T}[v] = \int \mathrm{d}\mathbf{z}\; \mathbf{T}(\mathbf{z})v(\mathbf{z}) \quad .$$

Thus $\mathbf{T}$ is a $(2 - \frac{N}{2})$-differential. We can define the Lie derivative $\pounds_w \mathbf{T}$ by requiring $\mathbf{T}[v]$ to be invariant and using (2.6). In fact for any p-differential one has

$$\pounds_w \mathbf{T} = w\partial\mathbf{T} + p\mathbf{T}\partial w + \tfrac{1}{2}\, D_i w D^i \mathbf{T} \quad . \tag{2.7}$$

We close this section by recalling central extensions to the various superconformal algebras. A central extension is defined by a cocycle, a bilinear form $C(v_1, v_2) = -C(v_2, v_1)$. It was shown that the unique generators for $N = 3$ and for untwisted and twisted $N = 4$ are [20][21]

$$C_3(v_1, v_2) = \oint \mathrm{d}\mathbf{z}\; v_1^z \epsilon^{ijk} D_i D_j D_k v_2^z \tag{2.8}$$

$$\begin{aligned} C_4(v_1, v_2) &= \oint \mathrm{d}\mathbf{z}\; v_1^z \epsilon^{ijkl} D_i D_j D_k D_l^{-1} v_2^z \\ \widetilde{C}_4(v_1, v_2) &= \oint \mathrm{d}\mathbf{z}\; v_1^z \partial_z v_2^z \quad . \end{aligned} \tag{2.9}$$

3. Broken $N = 3$

We now specialize to $N = 3$ geometry. To break the full symmetry we must supply Σ with a choice of some further additional structure. Again we specify this structure by giving the group of allowed coordinate transformations preserving it. Namely, in (2.1) we require M to live in a subgroup $G \subseteq O(3, \mathbf{C})$. Choose the standard basis $\{D_+, D_3, D_-\}$ for $\mathcal{E}$, in which $g_{ij} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Then we require that M gives $D_- \propto D'_-$, without mixing into D'_+, D'_3. Since M must still be orthogonal we find

$$\begin{pmatrix} \partial_z \\ \\ \vec{D} \end{pmatrix} = \left(\begin{array}{c|ccc} a^2 & & \cdots & \\ \hline 0 & 1 & x & -x^2/2 \\ 0 & & a & -xa \\ 0 & & & a^2 \end{array}\right) \begin{pmatrix} \partial_{z'} \\ \\ \vec{D}' \end{pmatrix} \quad . \tag{3.1}$$

The ellipsis denotes odd functions; x, a are even functions. Since matrices of the form (3.1) form a group, we again get a group of coordinate transformations, smaller than the 3-superconformal transformations.

Eqn. (3.1) is a natural generalization of the corresponding $N = 2$ reduction [8]:

$$\begin{pmatrix} \partial_z \\ D_+ \\ D_- \end{pmatrix} = \left(\begin{array}{c|cc} a^2 & \cdots & \\ \hline 0 & 1 & 0 \\ 0 & 0 & a^2 \end{array}\right) \begin{pmatrix} \partial_{z'} \\ D'_+ \\ D'_- \end{pmatrix} \quad . \tag{3.2}$$

Both are more transparent when examined infinitesimally. From (2.4) we see that (3.2) requires $D_+ D_- v \equiv 0$, which in turns means $D_- v = \epsilon \equiv$ const. But the $N = 2$ cocycle [24] vanishes identically on such generators:

$$\begin{aligned} C(v_1, v_2) &= \oint v_1^z \partial_z (D_+ D_- - D_- D_+) v_2^z \\ &= \oint (v_1^z \partial_z D_+ \epsilon_2 - \epsilon_1 \partial_z D_+ v_2^z) \\ &= 0 \quad , \end{aligned}$$

and so we have found the desired anomaly-free subalgebra. In exactly the same way, for $N = 3$ the requirement (3.1) implies $D_3 D_- v = D_+ D_- v = 0$, or $D_- v =$ const. One can readily verify using (2.6) that these indeed form a closed algebra, and using (2.8) that this 3-semirigid algebra is anomaly-free. This means in particular that in any matter system with $N = 3$ local susy we can gauge the semirigid symmetries regardless of the central charge — a hallmark of topological theories.

The generator $v^z(\mathbf{z})$ of an unbroken $N = 3$ superconformal transformation has an expansion in θ^i with eight terms, four even functions corresponding to Virasoro and $SO(3)$ current algebra plus four odd functions corresponding to an isotriplet of supercurrents plus one abelian generator. The restricted generator contains only four independent currents, plus one additional charge. We will write the solution to $D\_v = \text{const}$ as

$$v = (v_0 + \theta^3\lambda) + \theta^+(\nu + \theta^3\ell) + \theta^+\theta^-\partial_z(v_0 + \theta^3\lambda) + \theta^-\epsilon \tag{3.3}$$

where v_0, ℓ are even functions of z while ν, λ are odd functions; ϵ is an odd constant.

We have grouped the generators in pairs in (3.3) to emphasize that our semirigid algebra contains the $N = 1$ susy algebras, with generators corresponding to v_0, λ, together with same-spin partners corresponding to ν, ℓ. Thus in particular we have $N = 1$ susy as desired. The isolated parameter ϵ corresponds to a single generator Q_s mixing the usual generators with their topological partners.

There are various ways to "twist" the $N = 3$ algebra in the sense of imposing boundary conditions between $z = 1$ and $z = e^{i2\pi}$. The original $N = 3$ superconformal transformation group has an $O(3)$ global automorphism group that acts on $\theta^\pm, \theta^3$. Its global conjugacy classes are known [13] to be $\mathbf{Z}_2 \times U(1)$, in which the $U(1)$ may be taken to rotate $\theta^\pm$ into each other. However, the symmetry breaking $D\_v = \text{const}$ requires θ^+ to be global and so explicitly breaks the above $U(1)$. (The local automorphism group works similarly.) Thus, only $\mathbf{Z}_2$ survives, leaving only Neveu-Schwarz and Ramond sectors corresponding to those of $N = 1$ susy as desired.

Clearly we would like to identify θ^3 with the usual spin-$\frac{1}{2}$ coordinate on $N = 1$ superspace and θ^+ with the usual scalar odd coordinate found in ordinary topological gravity [8], but this does not quite make sense yet; (2.3) shows that under (3.3) we have

$$\begin{aligned} &\theta^+ \to \theta^+ + \epsilon \\ &\theta^3 \to \theta^3 + \lambda + \theta^3\partial_z v_0 - \theta^+(\ell + \theta^3\partial_z\nu) + \theta^+\theta^-\partial_z(\lambda + \theta^3\partial_z v_0) \quad . \end{aligned} \tag{3.4}$$

To deal with the first of these we simply require that patching functions lie in the subgroup with $\epsilon = 0$.[2] A tedious but straightforward verification then shows that the general allowed

[2] Later we will find another reason for this restriction.

transition function is $\mathbf{z} \mapsto \mathbf{z}'$

$$\begin{aligned}
z' &= f_0 + \theta^+\rho_- + \theta^3 n_0\nu_0 - \theta^+\theta^-\nu_0\partial_z\nu_0 - \theta^3\theta^+\nu_0\alpha_- + \theta^+\theta^-\theta^3((\partial_z n_0)\nu_0 - n_0\partial_z\nu_0)\\
\theta^{+\prime} &= \theta^+\\
\theta^{3\prime} &= \nu_0 + \theta^+ n_- + \theta^3 n_0 + \theta^+\theta^-\partial_z\nu_0 + \theta^3\theta^+\alpha_- + \theta^+\theta^-\theta^3\,\partial_z n_0\\
\theta^{-\prime} &= (\rho_- - n_-\nu_0) - \tfrac{1}{2}\,\theta^+(n_-)^2 + \theta^-(n_0)^2 - \theta^3 n_0 n_-\\
&\quad + \theta^+\theta^-(-\partial_z\rho_- + \partial_z n_-\nu_0 - n_-\partial_z\nu_0) + 2\theta^-\theta^3 n_0\partial_z\nu_0\\
&\quad - \theta^3\theta^+ n_-\alpha_- + \theta^+\theta^-\theta^3(-2(\partial_z\nu_0)\alpha_- + n_0\partial_z n_- - (\partial_z n_0)n_-)\ .
\end{aligned} \tag{3.5}$$

Here $f_0,\ \rho_-,\ n_0,\ \nu_0,\ \alpha_-,\ n_-$ are six functions of z subject to two conditions:

$$\begin{aligned}
\partial_z f_0 &= (n_0)^2 - \nu_0\partial_z\nu_0\\
\partial_z\rho_- &= n_0\alpha_- + \partial_z n_-\nu_0 \quad .
\end{aligned} \tag{3.6}$$

Eqn. (3.5) is the general local solution to the semirigid condition (3.1). Expanding about $f_0 = z$, $n_0 = 1$, $n_- = \nu_0 = \rho_- = \alpha_- = 0$, we recover the transformations generated by (2.3) with $D_v = 0$. We can now deal with the inhomogeneous transformation law (3.4). As in the $TN = 0$ case we can simplify (3.5) considerably, and at the same time deal with the inhomogeneous transformation. Note that according to (3.1), D_- does transform homogeneously. Hence the line bundle $\mathcal{D} \subseteq \mathcal{E} \subseteq T\Sigma$ which it spans is the same in every coordinate patch of an $N = 3$ SSRS. Since $[D_-, D_-] \equiv 0$ this line bundle is integrable and hence defines a *flow*. We can thus define a reduced supermanifold $\overline{\Sigma}$ of dimension $1|2$ as the quotient of Σ by this flow [22][8].[3] Furthermore, since $[D_3, D_-] = 0$ we see that D_3 projects to a well-defined vector field $\overline{D}_3$ on $\overline{\Sigma}$, one which moreover transforms homogeneously.

We can make this prescription very concrete. On the $z|\theta^+\theta^3$ plane $\mathbf{C}^{1|2}$ consider the transformations

$$\begin{aligned}
z' &= \mathcal{F} + \theta^3\mathcal{G}\mathcal{N}\\
\theta^{+\prime} &= \theta^+\\
\theta^{3\prime} &= \mathcal{G} + \theta^3\mathcal{N}
\end{aligned} \tag{3.7}$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{N}$ are functions of z, θ^+ subject to the condition

$$\partial_z\mathcal{F} = \mathcal{N}^2 - \mathcal{G}\partial_z\mathcal{G} \quad . \tag{3.8}$$

[3] In the $N = 2$ case we had the alternative option of quotient by the flow of D_+. In the present case this does not work, since by (3.1) D_+ mixes into D_3, D_- and so does not define a line bundle.

Eqns (3.7)–(3.8) define exactly the usual $N=1$ superconformal transformations with an extra nonparticipating odd variable θ^+. Thus we call them "*augmented* $N=1$ transformations."

We promised that the augmented $N=1$ surface $\overline{\Sigma}$ would have a distinguished line bundle spanned by $\overline{D}_3$. This is clearly the case, since by construction (3.7)–(3.8) preserve $\overline{D}_3 = \frac{\partial}{\partial\theta^3} + \theta^3 \frac{\partial}{\partial z}$ up to a multiplier. What is not so clear is what such surfaces have to do with the desired $TN=1$ semirigid surfaces.

What we claim is that the group of transformations (3.7)–(3.8) is in fact isomorphic to the group of $TN=1$ semirigid transformations defined by (3.5)–(3.6). This is exactly the same situation as in ordinary topological gravity [8], where we found that the $TN=0$ transformation group was the same as the augmented $N=0$ conformal transformations.

To prove the isomorphism we need only examine the corresponding Lie algebras. Substituting (3.3) with $\epsilon=0$ into (2.6) immediately yields the same algebra as the infinitesimal form of (3.7)–(3.8). Either way we get a doubled form of the Ramond or Neveu-Schwarz algebra, with each generator paired with a same-spin partner.

We can also display the finite form of this correspondence. The augmented transformation (3.7) corresponds to the semirigid transformation (3.5) when

$$\begin{aligned} \mathcal{F} &= f_0 + \theta^+(2\rho_- - n_-\nu_0) \\ \mathcal{G} &= \nu_0 + \theta^+ n_- \\ \mathcal{N} &= n_0 + \theta^+\alpha_- \quad . \end{aligned} \tag{3.9}$$

One checks that $\mathcal{F}$, $\mathcal{G}$, $\mathcal{N}$ obey (3.8) whenever $f_0, \ldots, n_-$ obey (3.6), and that (3.9) is a homomorphism.

Just as in $TN=0$, we thus see that any family of $N=1$ SRS can be promoted to a family of $TN=1$ surfaces as follows. We first write the surface by giving patching functions

$$\begin{aligned} z_\alpha &= F_{\alpha\beta}(z_\beta;\vec{m},\vec{\zeta}) + \theta^3_\beta G_{\alpha\beta}(z_\beta;\vec{m},\vec{\zeta}) N_{\alpha\beta}(z_\beta;\vec{m},\vec{\zeta}) \\ \theta^3_\alpha &= G_{\alpha\beta}(z_\beta;\vec{m},\vec{\zeta}) + \theta^3_\beta N_{\alpha\beta}(z_\beta;\vec{m},\vec{\zeta}) \end{aligned} \tag{3.10}$$

where $\vec{m}$ are $3g-3$ even moduli and $\vec{\zeta}$ are $2g-2$ odd moduli. Next we promote (3.10) to a family of augmented $N=1$ SRS (3.7) by introducing $3g-3$ *odd* $\hat{m}^a$ and $2g-2$ *even* $\hat{\zeta}^\mu$ and substituting

$$\begin{aligned} m^a &\mapsto m^a + \theta^+\hat{m}^a \\ \zeta^\mu &\mapsto \zeta^\mu + \theta^+\hat{\zeta}^\mu \end{aligned}$$

into (3.10), along with

$$\theta^+_\alpha = \theta^+_\beta \quad .$$

Finally we solve (3.9) for $(f_0)_{\alpha\beta}, \dots (n_-)_{\alpha\beta}$ to define a family of $TN = 1$ semirigid surfaces depending on $5g-5|5g-5$ moduli. Since (3.9) is an isomorphism, we see at once that replacing (3.10) by an equivalent presentation of the same family of $N = 1$ SRS we get an equivalent family of SSRS.

Using the above construction thus yields not only the moduli space of $TN = 1$ surfaces, but also the *projection* down to the usual $N = 1$ supermoduli space: we simply project the point with coordinates $(\vec{m}, \hat{\vec{\zeta}}, \vec{\zeta}, \hat{\vec{m}})$ to the point with coordinates $(\vec{m}, \vec{\zeta})$. Just as in [8], ordinary conformal field theory techniques will yield a density on $\widehat{\mathcal{M}}_{TN=1}$, once we have studied the BC system in the next section. This density can then be integrated over $\hat{\vec{m}}, \hat{\vec{\zeta}}$ to get a density on $\widehat{\mathcal{M}}_{N=1}$. The rest of the integration then parallels the fermionic string.

Before closing this long section we need some physical mechanism for breaking the full $N = 3$ symmetry of section two down to the present $TN = 1$ subgroup. As in [8] we will propose a *constraint* on the fields which does not respect (2.1) but does respect (3.1). An immediate candidate is thus

$$D_- \mathbf{C}^z = q \quad . \tag{3.11}$$

Here q is a constant and $\mathbf{C}^z$ is the BRST ghost field present whenever we gauge superconformal symmetry. We will eventually adopt the convention that $q = -2$, but we do not expect the final results to depend on the choice of q. The ghost $\mathbf{C}^z$ is always a tensor of the same type as the generator v^z, eqn. (2.5). Since the lhs of (3.11) is not a scalar, it certainly breaks the symmetry. In fact it breaks $N = 3$ susy down to the subgroup preserving the line bundle spanned by D_-. But this was precisely our definition of the $TN = 1$ subgroup at the start of this section. We then saw (eqn. (3.1)) that under these transformations the line bundle $\mathcal{D}_- \otimes \mathcal{T}$ was *trivial*, as required by (3.11). (Recall that $\mathbf{C}$ is a section of $\mathcal{T} \equiv T\Sigma/\mathcal{E}$.)

One may object that the lhs of (3.11) is not tensorial, since we have no covariant derivative. But consider an infinitesimal semirigid transformation. We find using (3.1), (2.4), (2.5) that

$$D'_-(\mathbf{C}^{z\prime}) = (1 - \partial_z v) D_-((1 + \partial_z v)\mathbf{C}^z) = D_- \mathbf{C}^z$$

since $D_- v = 0$. Hence (3.11) really does leave unbroken the desired semirigid group of transformations.

4. Ghost System

We have just seen that the constraint $D_{\_}\mathbf{C}^z = q$, where q is a constant, reduces the symmetry of the $N = 3$ ghost system down to an anomaly-free group of transformations. But this constraint by itself is inconsistent with the canonical commutation relations for the components of $\mathbf{B}$, $\mathbf{C}$. As usual in constrained systems we must also require that this inconsistency not matter, by restricting the observables of the theory to be independent of the components of $\mathbf{B}$ conjugate to the constrained components of $\mathbf{C}$. In fact we must ask this not only of the observables (physical states) but also of the other ingredients entering the string measure, namely the ghost insertions $\mathbf{B}[v]$ corresponding to moduli and the unbroken components of the stress tensor. We will make this decoupling clear by expanding $\mathbf{B}$ in an unconventional manner.

First consider the unbroken $N = 3$ ghost system. The ghost $\mathbf{C}^z$ is related to the generator of superconformal transformations v^z and is a (-1)-differential. The antighost $\mathbf{B}_\theta$ and the stress tensor $\mathbf{T}_\theta$ are always dual to v^z and are $(\frac{1}{2})$-differentials for N=3. We have written $\mathbf{T}_\theta$ and $\mathbf{B}_\theta$ with the subscript θ to remind us of their weights. The ghost propagator on the plane is

$$\mathbf{B}_\theta(\mathbf{z}_1)\mathbf{C}^z(\mathbf{z}_2) = \frac{\theta^+_{12}\theta^-_{12}\theta^3_{12}}{\mathbf{z}_{12}} = \mathbf{C}^z(\mathbf{z}_2)\mathbf{B}_\theta(\mathbf{z}_1)$$

where

$$\mathbf{z}_{12} \equiv z_1 - z_2 - \theta^i_1\theta^j_2 g_{ij} \quad ,$$
$$\theta^i_{12} \equiv \theta^i_1 - \theta^i_2 \quad .$$

The ghost stress tensor is obtained by the method of ref. [25]. We demand that $\mathbf{T}_\theta$ consists of bilinears in $\mathbf{B}_\theta$ and $\mathbf{C}^z$ with weight $(\frac{1}{2})$ and is neutral with respect to $SO(3)$. Further, we require that $[\mathbf{T}(v), \phi] = \pounds_v\phi$ for $\phi = \mathbf{B}_\theta, \mathbf{C}^z$ and $\pounds_v$ as defined in (2.7). This determines the stress tensor uniquely to be

$$\mathbf{T}_\theta = -\mathbf{C}^z\partial_z\mathbf{B}_\theta - \tfrac{1}{2}\partial_z\mathbf{C}^z\mathbf{B}_\theta + \tfrac{1}{2}D_i\mathbf{C}^z D_j\mathbf{B}_\theta g^{ij} \quad . \tag{4.1}$$

We expand the fields in components as follows

$$\begin{aligned}
\mathbf{C}^z &\equiv C + \theta^+\Gamma^- + \theta^-\Gamma^+ + \theta^+\theta^-\Xi \\
\mathbf{B}_\theta &\equiv \Lambda + \theta^+\Omega^- + \theta^-\Omega^+ + \theta^+\theta^-(B + \partial_z\Lambda) \\
\mathbf{T}_\theta &\equiv \tfrac{1}{2}\Psi + \theta^+J^- + \theta^-J^+ + \theta^+\theta^-(G^B + \tfrac{1}{2}\partial_z\Psi) \\
&= (\tfrac{1}{2}\psi + \theta^3\tfrac{1}{2}j^3) + \theta^+(\tfrac{1}{2}j^- + \theta^3G^-) + \theta^-(\tfrac{1}{2}j^+ + \theta^3G^+) \\
&\quad + \theta^+\theta^-\{(G^3 + \theta^3T^B) + \tfrac{1}{2}\partial_z(\psi + \theta^3j^3)\} \quad .
\end{aligned} \tag{4.2}$$

In the first three lines of (4.2) the fields C, Γ, *etc.* are functions of z, θ^3. In the last line we have expanded Ψ, $J^\pm$, and G^B in an evident notation. Thus j^i are the $SO(3)$ generators, G^i the supersymmetry generators, T^B the Virasoro generator and ψ an abelian, spin 1/2 generator.

The constraint $D_-\mathbf{C}^z = q$ implies that

$$\Gamma^+ = q; \qquad \Xi = \partial_z C \quad . \tag{4.3}$$

This constraint breaks the $N = 3$ superconformal transformations down to the semirigid transformation given in (3.5). After this breaking is done one can attribute spin 0 to θ^+ and spin 1 to θ^- ; θ^3 continues to have spin $\frac{1}{2}$, as we can see from (3.1). The virtue of the unusual expansion for $\mathbf{B}$ in (4.2) is that to be compatible with the constraints (4.3) an observable need only be independent of Λ and Ω^-. In particular $\mathbf{B}[v]$ will be of this form when $D_- v = 0$. The virtue of the expansion for $\mathbf{T}$ in (4.2) is that the generators of the semirigid transformations are simply T^B, G^3, G^+ and j^+.

The unbroken generators are given by the following expressions which are obtained by substituting (4.3) into (4.1) and using (4.2) to define them.

$$\begin{aligned} G^B &= -C\partial_z B + \tfrac{1}{2}D_3 C D_3 B - \tfrac{3}{2}\partial_z C B - \Gamma^-\partial_z\Omega^+ - \tfrac{1}{2}D_3\Gamma^- D_3\Omega^+ - \tfrac{3}{2}\partial_z\Gamma^-\Omega^+ \\ J^+ &= C\partial_z\Omega^+ - \tfrac{1}{2}D_3 C D_3\Omega^+ + \tfrac{3}{2}C\partial_z\Omega^+ + \frac{q}{2}B \\ Q_s &= -q\oint_{N=1} J^- = \frac{q}{2}\oint_{N=1}\Gamma^- B \quad . \end{aligned} \tag{4.4}$$

In the last line the measure is $[\mathrm{d}\mathbf{z}]_{N=1} \equiv [\mathrm{d}z|\mathrm{d}\theta^3]$. Notice that the generators do not contain any terms involving Λ and Ω^-. This satisfies the requirement dual to $D_-\mathbf{C}^z = q$. As promised, inhomogeneous terms in G^+ and j^+ occur naturally in the semirigid formalism. We choose $q = -2$ just as in the case of $TN = 0$ [8]. This gives the following commutation relations.

$$\{Q_s, Q_s\} = 0; \qquad [Q_s, J^+] = G^B.$$

However, Q_s does not define a global nilpotent charge, since it has a non-zero commutation relations with some of the unbroken generators as we have shown. The global charge is obtained from the BRST generator of the complete $N = 3$ system after substituting the constraint. The BRST charge is given by

$$Q_{brst} = \tfrac{1}{2}\oint_{N=3}\mathbf{C}^z\mathbf{T}_\theta \quad .$$

Here the measure is $[\mathrm{d}\mathbf{z}]_{N=3} \equiv [\mathrm{d}z|\mathrm{d}\theta^3\mathrm{d}\theta^+\mathrm{d}\theta^-]$. On imposing the constraint, we obtain

$$\begin{aligned} Q_T &\equiv Q_{brst}|_{D_-\mathbf{C}=-2} \\ &\equiv Q_s + Q_v \end{aligned} \tag{4.5}$$

where

$$Q_v = \frac{1}{2}\oint_{N=1}[CG^B + \Gamma^-\hat{J}^+]$$

and $\hat{J}^+$ is the generator J^+ without the inhomogeneous term. Further, Q_v corresponds to the BRST generator of the unbroken algebra. Now we have

$$\{Q_T, Q_T\} = 0; \qquad \{Q_T, G^B\} = 0; \qquad [Q_T, J^+] = 0 \quad .$$

Thus Q_T defines a global nilpotent charge and hence the theory is topological. We can assign ghost numbers to all the fields in the theory by means of the following ghost number charge

$$\begin{aligned} U_T &= \oint_{N=3} -\mathbf{B}_\theta\mathbf{C}^z = U_v + U_s \\ U_v &= \oint_{N=1} -BC \\ U_s &= \oint_{N=1} -\Psi = \oint_{N=1} \Omega^+\Gamma^- \end{aligned} \quad . \tag{4.6}$$

The ghost number assignments for the fields $(C, B, \Gamma^-, \Omega^+)$ are then $(1, -1, 2, -2)$. We would like to compare our results with those obtained by Hughes and Li [5]. We find on making the required identifications[4] that their expressions for G^+ and T^B agree with the ones we obtain while those for G^3 and j^+ do not agree. Of course, they do not obtain the inhomogeneous terms. In addition, certain terms in their expressions disagree with ours by numerical factors which are important for the closure of the algebra. Since our terms were derived directly from the $N = 3$ system, closure is automatic. This non-closure led Hughes and Li [5] to conclude that one needs to introduce more currents. This led to the conclusion that $TN = 1$ is obtained by twisting the $N = 4$ superconformal algebra. As we shall show, we instead obtain $TN = 2$ by starting from $N = 4$.

[4] We make the following identifications: $(G^B, J^+) \to (t_{gh}+\theta^3\Gamma, B-\theta^3\beta)$ and $(C, B, \Gamma^-, \Omega^+) \to (c + \theta^3\mathcal{C}, -b + \theta^3\mathcal{B}, -\gamma - \theta^3\Gamma, B - \theta^3\beta)$.

To make the discussion more complete, we give the unbroken subalgebra by expanding the unbroken generators into their modes

$$
\begin{aligned}
T^B &= \sum_m z^{-m-2} L_m \quad , & G^+ &= \sum_m \tfrac{1}{2} z^{-m-2} G^+_m \quad , \\
G^3 &= \sum_r \tfrac{1}{2} z^{-r-\frac{3}{2}} G^3_r \quad , & j^+ &= \sum_r z^{-r-\frac{3}{2}} j^+_r \quad .
\end{aligned}
\tag{4.7}
$$

The unbroken subalgebra is

$$
\begin{aligned}
&[L_m, L_n] = (m-n) L_{m+n} & & \\
&[L_m, G^3_r] = (\tfrac{1}{2}m - r) G^3_{m+r} & &\{G^3_r, G^3_s\} = 2L_{r+s} \\
&\{G^+_m, G^+_n\} = 0 & &[L_m, j^3_r] = (\tfrac{1}{2}m - r) j^3_{m+r} \\
&\{G^+_m, G^3_r\} = 2(\tfrac{1}{2}m - r) j^3_{m+r} & &[j^+_r, j^+_s] = 0 \\
&[G^+_m, j^+_s] = 0 & &[G^3_r, j^+_s] = G^+_{s+r}
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
[Q_s, j^+_r] &= -\, G^3_r & \{Q_s, G^+_m\} &= 2L_m \quad , \\
\{Q_s, G^3_r\} &= \quad 0 & [Q_s, L_m] &= 0 \quad .
\end{aligned}
\tag{4.9}
$$

Eqns. (4.8)–(4.9) are indeed seen to be anomaly-free. This completes the derivation of a topological ghost sector directly from the $N = 3$ system using the semirigid construction introduced in [8]. It is anomaly free by construction and the algebra closes naturally. The physical observables are cohomology classes defined by Q_T, subject to some "equivariance" conditions. The theory is free from the beginning and so free CFT can be used in calculations. As in string theory and topological gravity , the ghost system can be used to obtain a well defined measure on supermoduli space. The observables in this theory presumably measure the topology of the supermoduli space of $N = 1$ SRS.

5. Broken N=4

We now extend our construction to $N = 4$ geometry. Again, we require M to live in a subgroup $G \subseteq O(4, \mathbf{C})$. We find it convenient to choose complex odd coordinates $(\xi, \tilde{\theta}, \theta, \tilde{\xi})$ and the basis as $\{D_+, \widetilde{D}_+, \widetilde{D}_-, D_-\}$ for $\mathcal{E}$ with metric $g_{ij} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. Here,

$$
D_+ = \partial_\xi + \tilde{\xi}\partial_z, \;\; D_- = \partial_{\tilde{\xi}} + \xi\partial_z, \;\; \widetilde{D}_+ = \partial_\theta + \tilde{\theta}\partial_z, \;\; \widetilde{D}_- = \partial_{\tilde{\theta}} + \theta\partial_z. \tag{5.1}
$$

Demanding that M preserves $D_- \propto D'_-$ and is orthogonal, we get

$$\begin{pmatrix} \partial_z \\ D_+ \\ \widetilde{D}_+ \\ \widetilde{D}_- \\ D_- \end{pmatrix} = \left(\begin{array}{c|cccc} a^2 & & \cdots & & \\ \hline 0 & 1 & x & y & -xy \\ 0 & 0 & ab & 0 & -aby \\ 0 & 0 & 0 & a/b & -ax/b \\ 0 & 0 & 0 & 0 & a^2 \end{array}\right) \begin{pmatrix} \partial_{z'} \\ D'_+ \\ \widetilde{D}'_+ \\ \widetilde{D}'_- \\ D'_- \end{pmatrix} \quad . \tag{5.2}$$

The entries a, b, x, y are even functions. Using (2.4), we find that the above structure (5.2) of restricted 4-superconformal transformation implies that $\widetilde{D}_+ D_- v = \widetilde{D}_- D_- v = 0$ and $\partial_z v = \frac{1}{2} D_- D_+ v$. These are satisfied if and only if $D_- v =$ constant. We see again that these form a closed algebra, and from (2.9) that the algebra is anomaly-free.

The original unbroken $N = 4$ superconformal transformation is generated by a super-diffeomorphism vector $v(\mathbf{z})$. It consists of eight even components and eight odd components: even components corresponding to a spin two, Virasoro algebra, two $SU_l(2) \times SU_r(2)$ isovectors plus one $U(1)$ spin one current and odd components corresponding to two $SU_l(2) \times SU_r(2)$ isospinors of spin 3/2 plus two $SU_l(2) \times SU_r(2)$ isospinors of spin 1/2.

Once restricted to $D_- v =$ constant, the super-diffeomorphism vector is reduced to four even and four odd components in addition to one odd, constant component. It is

$$v(\mathbf{z}) = (v_0 + \theta\alpha + \tilde{\theta}\tilde{\alpha} + \theta\tilde{\theta} w_0) + \xi(\beta + \theta u + \tilde{\theta}\tilde{u} + \theta\tilde{\theta}\gamma) + \xi\tilde{\xi}\partial_z(v_0 + \theta\alpha + \tilde{\theta}\tilde{\alpha} + \theta\tilde{\theta} w_0) + \tilde{\xi}\epsilon. \tag{5.3}$$

Here, $v_0, w_0, u, \tilde{u}$ are even functions of z, $\alpha, \tilde{\alpha}, \beta, \gamma$ odd functions of z and ϵ is an odd constant. The first and the third terms generate an $N = 2$ susy algebra consisting of Virasoro, two supercurrents and one $U(1)$ current. The second term generates the same-spin partners of this multiplet. The global automorphism group of the original $N = 4$ superconformal transformation is $O(4)$ or $SO(4)$, depending on whether we have a $\mathbf{Z}_2$ discrete parity symmetry $SU_l(2) \leftrightarrow SU_r(2)$ or not. We find its local automorphism group to be $SO(4) = SU_l(2) \times SU_r(2)$ in which the first $SU(2)$ acts on $(\theta, \tilde{\theta})$ and the second on $(\xi, \tilde{\xi})$. However, the symmetry breaking condition $D_- v = q$ breaks the latter *explicitly.* Thus, there remains either a $\mathbf{Z}_2$ outer automorphism group for the $l \leftrightarrow r$ symmetric case or none for the $l \leftrightarrow r$ asymmetric case. In both cases, Neveu-Schwarz and the Ramond algebras are continuously connected by a spectral flow. Finally, the last term in (5.3) corresponds to a global generator $Q_s \equiv \frac{\partial}{\partial \tilde{\xi}}$, mixing between the above two sets of generators of opposite statistics.

As in the $N = 3$ case, the identification of $\theta, \tilde{\theta}$ as the two odd coordinates of $N = 2$ superspace and ξ as the usual scalar odd coordinate in topological gravity is not quite consistent yet. We find that (5.3) generates the following odd-coordinate transformations

$$
\begin{aligned}
\xi' &= \xi + \epsilon, \\
\theta' &= \theta + (\tilde{\alpha} + \theta(\partial_z v_0 - w_0) + \theta\tilde{\theta}\partial_z \tilde{\alpha}) \\
&\quad + \xi(-\tilde{u} + \theta(\gamma - \partial_z \beta) - \theta\tilde{\theta}\partial_z \tilde{u}) \\
&\quad + \xi\tilde{\xi}\partial_z(\tilde{\alpha} + \theta(\partial_z v_0 - w_0) + \theta\tilde{\theta}\partial_z \tilde{\alpha}) \\
\tilde{\theta}' &= \tilde{\theta} + (\alpha + \tilde{\theta}(w_0 - \partial_z v_0) + \theta\tilde{\theta}\partial_z \alpha) \\
&\quad + \xi(-u + \tilde{\theta}(\partial_z \beta - \gamma) - \theta\tilde{\theta}\partial_z u) \\
&\quad + \xi\tilde{\xi}\partial_z(\alpha + \tilde{\theta}(w_0 - \partial_z v_0) + \theta\tilde{\theta}\partial_z \alpha) \\
\tilde{\xi}' &= \tilde{\xi} + (\beta + \theta u + \tilde{\theta}\tilde{u} + \theta\tilde{\theta}\gamma) \\
&\quad + 2\tilde{\xi}(v_0 + \theta\alpha + \tilde{\theta}\tilde{\alpha} + \theta\tilde{\theta}w_0) \\
&\quad + \xi\tilde{\xi}\partial_z(\beta + \theta u + \tilde{\theta}\tilde{u} + \theta\tilde{\theta}\gamma) \quad .
\end{aligned}
\tag{5.4}
$$

As in the previous sections the patching functions for SSRS will be of this form with $\epsilon = 0$, so that ξ is global. Also (5.2) has been again constructed so that D_- transforms homogeneously. Therefore, the line bundle $\mathcal{D} \subseteq \mathcal{E} \subseteq T\Sigma$ which it spans remains the same in every coordinate patch of an $N = 4$ SSRS. Again, $[D_-, D_-] \equiv 0$ implies the line bundle is integrable, and thus gives a well-defined flow. Hence, we can define a reduced $1|3$ supermanifold $\overline{\Sigma}$ as Σ modded out by the flow. The $\widetilde{D}_\pm$ project to well-defined vector fields $\overline{\widetilde{D}}_\pm$ on $\overline{\Sigma}$, which transform into each other.

Indeed, let us consider the following transformations on the $z|\theta\tilde{\theta}\xi$ plane $\mathbf{C}^{1|3}$

$$
\begin{aligned}
z' &= \mathcal{F} + \theta\mathcal{L}\mathcal{J} + \tilde{\theta}\mathcal{I}\mathcal{M} + \theta\tilde{\theta}\partial_z(\mathcal{I}\mathcal{L}) \\
\theta' &= \mathcal{I} + \theta\mathcal{J} + \theta\tilde{\theta}\partial_z \mathcal{I} \\
\tilde{\theta}' &= \mathcal{L} + \tilde{\theta}\mathcal{M} - \theta\tilde{\theta}\partial_z \mathcal{L} \\
\xi' &= \xi \quad .
\end{aligned}
\tag{5.5}
$$

Here, $\mathcal{F}, \mathcal{J}, \mathcal{M}$ and $\mathcal{I}, \mathcal{L}$, are even and odd functions of z, ξ subject to the condition

$$
\mathcal{J}\mathcal{M} = \partial_z \mathcal{F} + \mathcal{L}\partial_z \mathcal{I} + \mathcal{I}\partial_z \mathcal{L} \quad . \tag{5.6}
$$

Eqns (5.5)–(5.6) define the usual $N = 2$ superconformal transformations with an extra odd variable ξ, and form the $N = 2$ version of "augmented transformations". Indeed, the two covariant derivatives $\overline{\overline{D}}_+ = \frac{\partial}{\partial\theta} + \tilde{\theta}\partial_z$ and $\overline{\overline{D}}_- = \frac{\partial}{\partial\tilde{\theta}} + \theta\partial_z$ are preserved up to multiplicative factors.

The group transformations (5.5)–(5.6) are isomorphic to the group of $TN = 2$ semirigid transformations defined infinitesimally by (5.4), the very same situation as in $TN = 0, 1$. The isomorphism can be proven by examining the Lie algebras. From this way, we find a $N = 2$ algebra, paired by its BRST partner of the same spin contents. In fact, by choosing

$$
\begin{aligned}
\mathcal{I} &= \tilde{\alpha} - \xi\tilde{u} + \xi\tilde{\xi}\partial_z\tilde{\alpha} \\
\mathcal{L} &= \alpha - \xi u + \xi\tilde{\xi}\partial_z\alpha \\
\mathcal{J} &= 1 + (\partial_z v_0 - w_0) + (\partial_z\beta - \gamma) + \partial_z(\partial_z v_0 - w_0) \\
\mathcal{M} &= 1 - (\partial_z v_0 - w_0) - (\partial_z\beta - \gamma) - \partial_z(\partial_z v_0 - w_0)
\end{aligned}
\tag{5.7}
$$

we find (5.5) corresponds to (5.4). This identification (5.7) certainly obeys (5.6). Therefore, (5.7) is a Lie algebra isomorphism. It is indeed possible to promote $N = 2$ SRS to a family of $TN = 2$ geometries in the same way as in $TN = 0, 1$.

We now briefly indicate the generalization of sect. four to the present case. We find it convenient to write the $\mathbf{B}, \mathbf{C}$ ghost fields in terms of $N = 2$ superfields of $z|\theta\tilde{\theta}$ $\mathbf{C}^{1|2}$ superspace. They are

$$
\begin{aligned}
\mathbf{B} &= \Lambda + \xi\tilde{\Omega} + \tilde{\xi}\Omega + \xi\tilde{\xi}(B + \partial_z\Lambda) \\
\mathbf{C} &= C + \xi\tilde{\Gamma} + \tilde{\xi}\Gamma + \xi\,\tilde{\xi}\,\Xi \quad .
\end{aligned}
\tag{5.8}
$$

Here, Λ, B, C, Ξ are even $N = 2$ superfields, while $\Omega, \tilde{\Omega}, \Gamma, \tilde{\Gamma}$ are odd superfields. Similarly, the stress tensor is

$$
\mathbf{T} = J + \xi\tilde{G} + \tilde{\xi}G + \xi\tilde{\xi}(T^B + \partial_z J). \tag{5.9}
$$

consisting of $U(1)$ current, two supercurrents and the ordinary bosonic stress tensor.

We expect again that the SSRS symmetry breaking is induced by

$$
D_-\mathbf{C} = q \quad . \tag{5.10}
$$

Imposing this condition to (5.8), we find $\Xi = \partial_z C$ and $\Gamma = q$, $\partial_z q = 0$. In fact as before q is a constant. In $N = 4$ local susy the ghost C and its antighost B carry conformal dimensions minus one and zero respectively. This fixes the stress tensor of the $\mathbf{B}, \mathbf{C}$ ghost system uniquely. We find

$$
\mathbf{T} = -\mathbf{C}\partial_z\mathbf{B} + \tfrac{1}{2}[D_-\mathbf{C}\cdot D_+\mathbf{B} + D_+\mathbf{C}\cdot D_-\mathbf{B} + D_\alpha\mathbf{C}\cdot D^\alpha\mathbf{B}] \quad . \tag{5.11}
$$

The indices α run over $N=2$ $(\theta,\tilde{\theta})$ odd-coordinates. Expanding Eqn. (5.11) using (5.8),

$$\begin{aligned} J &= \tfrac{1}{2}D_\alpha(CD^\alpha\Lambda)+\tfrac{1}{2}\tilde{\Gamma}\Omega+\tfrac{1}{2}q\tilde{\Omega} \\ G &= \partial_z(C\Omega)-\tfrac{1}{2}D_\alpha CD^\alpha\Omega+\tfrac{1}{2}qB \\ \tilde{G} &= -\tfrac{1}{2}D_\alpha(CD^\alpha\tilde{\Omega}+\tilde{\Gamma}D^\alpha\Lambda)-\tfrac{1}{2}\tilde{\Gamma}B \\ T^B &= -\partial_z(CB)+\tfrac{1}{2}D_\alpha CD^\alpha B-\partial_z(\tilde{\Gamma}\Omega)+\tfrac{1}{2}D_\alpha\Omega D^\alpha\tilde{\Gamma} \end{aligned} \tag{5.12}$$

Then, we find that T^B and G survive the symmetry breaking (5.10) as unbroken generators, while J and $\tilde{G}$ are broken due to their linear terms proportional to q.

In the original $N=4$ superconformal geometry, the BRST charge Q_{brst} is defined as

$$\begin{aligned} Q_{brst} &= \tfrac{1}{2}\oint_{N=4}\mathbf{C}^z\mathbf{T} \\ &= \tfrac{1}{2}\oint_{N=2}[CT^B+\tilde{\Gamma}G-q\tilde{G}] \end{aligned} \tag{5.13}$$

where we have dropped surface terms. Here, the integration measures are denoted compactly as $[\mathrm{d}\mathbf{z}]_{N=4}\equiv \mathrm{d}z\mathrm{d}^2\theta\mathrm{d}^2\xi$ and $[\mathrm{d}\mathbf{z}]_{N=2}\equiv \mathrm{d}z\mathrm{d}^2\theta$. It is straightforward to see that the constrained BRST charge consists of two pieces $Q_T\equiv Q_{brst}|_{D_-\mathbf{C}=q}\equiv Q_v+Q_s$ in which

$$\begin{aligned} Q_s &\equiv \oint_{N=4}2\tilde{G} \;=\; \oint_{N=2}\tfrac{1}{2}\tilde{\Gamma}B \\ Q_v &\equiv \tfrac{1}{2}\oint_{N=2}[CT^B+\tilde{\Gamma}\hat{G}] \end{aligned} \tag{5.14}$$

after fixing $q=-2$. We denoted $\hat{G}$ as the q-independent portion of G in (5.12).

Similarly, the total chiral ghost charge operator U_T is calculated

$$\begin{aligned} U_T &\equiv \oint_{N=4}-\mathbf{BC}=U_v+U_s \\ U_v &= \oint_{N=2}-BC \\ U_s &= \oint_{N=4}-2U \;=\; \oint_{N=2}\tilde{\Gamma}\Omega \quad . \end{aligned} \tag{5.15}$$

Again, we have dropped the surface term and a harmless constant term in U_s.

Both Q_s and U_s are not globally well-defined since they do not commute with the unbroken generators G or T^B. However, the total sum operators Q_T and U_T are easily seen to commute with G and T^B, and thus are globally well-defined. Indeed, we find they form an anomaly-free, closed $N=2$ superalgebra in which

$$\begin{aligned} [\,U_0, C\,] &= +C, & [\,U_T, \tilde{\Gamma}\,] &= +\tilde{\Gamma}, \\ [\,U_T, B\,] &= -B, & [\,U_0, \Omega\,] &= -\Omega, \\ \{Q_T, C\} &= \;\;0\;\;, & [Q_T, \tilde{\Gamma}] &= \;0, \\ \{Q_T, B\} &= T^B, & [Q_T, \Omega] &= \;0\quad. \end{aligned} \tag{5.16}$$

Therefore, the BRST commutators of the unbroken generators T^B and G vanish identically due to the nilpotence of Q_{brst}. This completes the semirigid construction of topological $N=2$ supergravity.

6. Conclusions

The approach to topological gravity advocated in this paper and in [8] seems quite different from the traditional approach of starting from a lagrangian which is zero or some other topological invariant. Instead we obtain topological theories as reductions, or truncations, of larger ordinary field theories. This fits with the view of topological matter systems advocated in [26] and elsewhere. Furthermore, while the approach to topological gravity through $SL(2,\mathbf{R})$ gauge theory suggested that higher matrix models could correspond to $SL(n,\mathbf{R})$ gauge theory, the present approach makes it seem more natural that they correspond to semirigid geometry coupled to higher $N=2$ minimal matter, as indeed seems to be the case [27][28]. It will be very interesting to see whether the present framework, possibly in conjunction with matter systems like the one in [14], will similarly reproduce the amplitudes of Liouville supergravity. Even more interesting would be to find a simple dynamical origin of the symmetry-breaking reduction which makes these theories topological in the first place.

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