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Boundary Value Problems for a Family of Domains in the Sierpinski Gasket

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Boundary Value Problems for a Family of Domains in the Sierpinski Gasket

Abstract

For a family of domains in the Sierpinski gasket, we study harmonic functions of finite energy, characterizing them in terms of their boundary values, and study their normal derivatives on the boundary. We characterize those domains for which there is an extension operator for functions of finite energy. We give an explicit construction of the Green's function for these domains.

Disciplines

Physical Sciences and Mathematics

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 6 **BOUNDARY VALUE PROBLEMS FOR A FAMILY OF**
 7 **DOMAINS IN THE SIERPINSKI GASKET**
 8

9 ZIJIAN GUO, RACHEL KOGAN, HUA QIU AND ROBERT S. STRICHARTZ
 10

11
 12 **ABSTRACT.** For a family of domains in the Sierpinski gasket, we
 13 study harmonic functions of finite energy, characterizing them in
 14 terms of their boundary values, and study their normal deriva-
 15 tives on the boundary. We characterize those domains for which
 16 there is an extension operator for functions of finite energy. We
 17 give an explicit construction of the Green’s function for these
 18 domains.
 19

20
 21 **1. Introduction**
 22

23 Consider the domain Ω_x in the Sierpinski Gasket (\mathcal{SG}) consisting of all
 24 points above the horizontal line L_x at the distance x from the top vertex q_0 ,
 25 for $0 < x \leq 1$.

26 Let $S(x) = \mathcal{SG} \cap L_x$. For x not a dyadic rational, this is a Cantor set.
 27 The boundary of Ω_x consists of $S(x)$ together with q_0 . By general principles,
 28 harmonic functions on Ω_x are determined by their boundary values, where
 29 harmonic functions are defined to be solutions of $\Delta h = 0$ on the interior of
 30 Ω_x , where Δ is the Kigami Laplacian on \mathcal{SG} . The study of such harmonic
 31 functions was initiated in [S1], and continued in [OS] for the special case
 32 $x = 1$. In this paper, we extend the results in [OS] to the general case. In
 33 Section 2, we give an explicit description of the analog of the Poisson kernel

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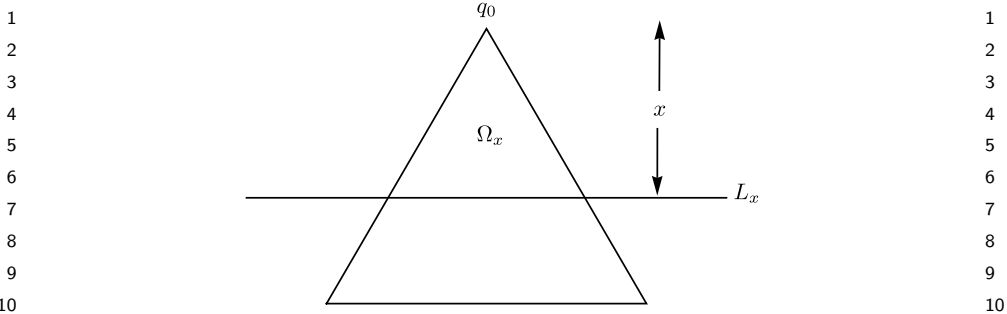


FIGURE 1

to recover the harmonic function from its boundary values, in terms of the Haar series expansion of the boundary values on $S(x)$, and we characterize the boundary values that correspond to harmonic functions of finite energy. In Section 3, we define normal derivatives on the boundary and give a description of the Dirichlet-to-Neumann map as a multiplier transform on the Haar series expansion.

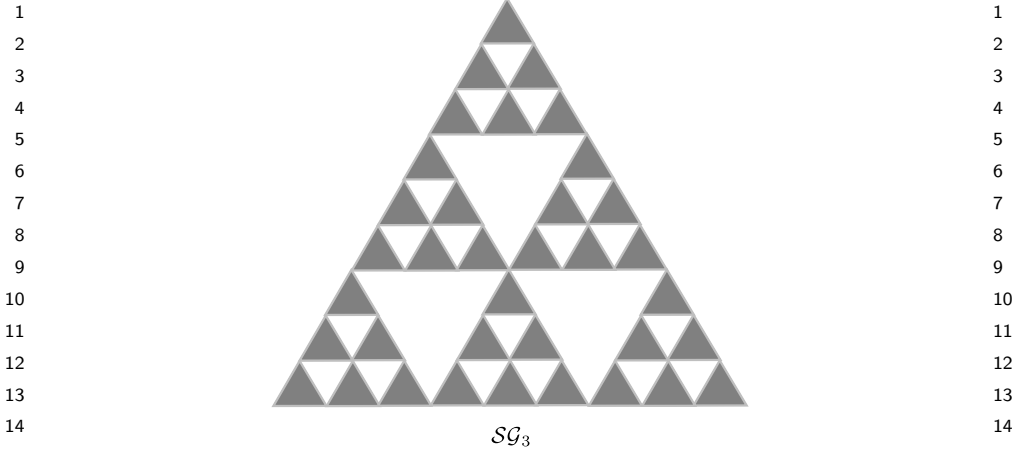
In Section 4, we study the extension problem for functions of finite energy on Ω_x to functions of finite energy on \mathcal{SG} . We are able to characterize the values of x for which such extensions are possible. In particular, the value $x = 1$ studied in [OS] does not admit such extensions. This may be regarded as the first of a family of Sobolev extension problems, based on Sobolev spaces on \mathcal{SG} discussed in [S2]. We leave these as open problems for future research. Related problems are studied in [LS] and [LRSU].

In Section 5, we give a construction of a Green's function on Ω_x to solve the Dirichlet problem $-\Delta u = F$ on Ω_x , $u|_{\partial\Omega_x} = 0$ via an integral transform of F . The construction of the Green's function is analogous to Kigami's construction on \mathcal{SG} .

The reader is referred to the books [Ki] and [S3] for a description of the theory of the Laplacian on \mathcal{SG} , and related fractals. It would be interesting to extend the results of this paper to other domains in \mathcal{SG} , and to domains in other fractals. In this regard, we offer the following cautionary tale. Consider the fractal \mathcal{SG}_3 , defined similarly to \mathcal{SG} but by subdivisions of the sides of triangles into three rather than two pieces (see Figure 2).

We may consider domains Ω_x defined as before, with the boundary $S(x)$ modeled as a Cantor set with divisions into three pieces. There is a natural analog of Haar functions on $S(x)$, with two generators as shown in Figure 3.

Because the second generator is symmetric rather than skew-symmetric, we cannot glue to zero at the top, so the analog of Lemma 2.3 does not hold. It is not clear how to overcome this difficulty.



SG_3

FIGURE 2

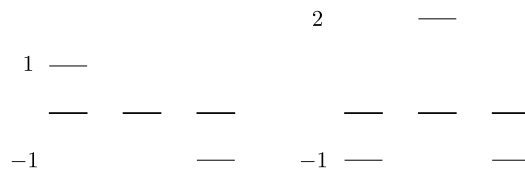


FIGURE 3. Haar generators.

2. Harmonic functions on Ω_x

For $0 < x \leq 1$, there is a unique representation

$$(2.1) \quad x = \sum_{k=1}^{\infty} 2^{-n_k}$$

for a sequence

$$(2.2) \quad 0 < n_1 < n_2 < \dots$$

of increasing positive integers. We will approximate Ω_x by the increasing sequence of domains $\Omega_x^{(m)}$ where each $\Omega_x^{(m)}$ is the closure of $\Omega_{x_{[m]}}$ where

$$(2.3) \quad x_{[m]} = \sum_{k=1}^m 2^{-n_k}$$

is the partial sum of (2.1). (Note that (2.3) is not the representation of $x_{[m]}$ of the form (2.1) since it is a finite binary representation.) The domain $\Omega_x^{(m)}$ is a finite union of cells, specifically 1 n_1 -cell, 2 n_2 -cells, 4 n_3 -cells, \dots , 2^{m-1}

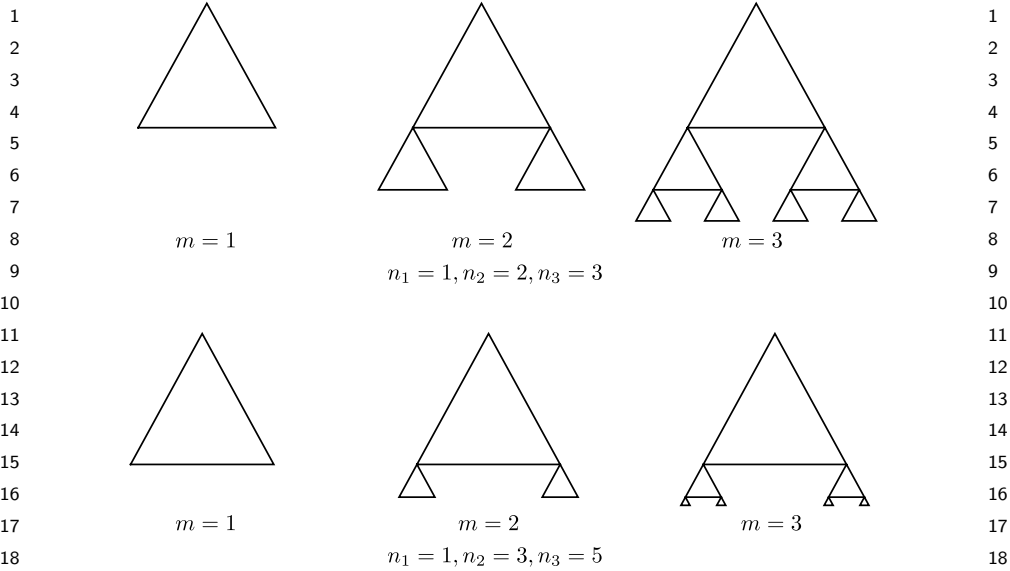


FIGURE 4. Some examples of $\Omega_x^{(m)}$ for $m = 1, 2, 3$.

n_m -cells. Figure 4 illustrates $\Omega_x^{(m)}$ for $m = 1, 2, 3$ for two choices of x . The boundary of $\Omega_x^{(m)}$ consists of the top vertex q_0 together with the 2^m bottom vertices of the n_m -cells.

Following [S1], we define

$$(2.4) \quad Rx = \sum_{k=2}^{\infty} 2^{-n_k} = x - 2^{-n_1}$$

and the function $\alpha_0(x)$ by the identity

$$(2.5) \quad \alpha_0(x) = \frac{1}{1 + 2(\frac{5}{3})^{n_2 - n_1}(1 - \alpha_0(Rx))}$$

which is easily solved to obtain a variant of a continued fraction representation

$$(2.6) \quad \alpha_0(x) = \lim_{k \rightarrow \infty} \alpha_0^{(k)}(x)$$

for

$$(2.7) \quad \alpha_0^{(k)} = \frac{1}{1 + 2(\frac{5}{3})^{n_2 - n_1} (1 - \frac{1}{1 + 2(\frac{5}{3})^{n_3 - n_2} (1 - \frac{1}{1 + 2(\frac{5}{3})^{n_k - n_{k-1}}})})}$$

See Figure 5 for the graph of $\alpha_0(x)$ on $(0, 1]$.

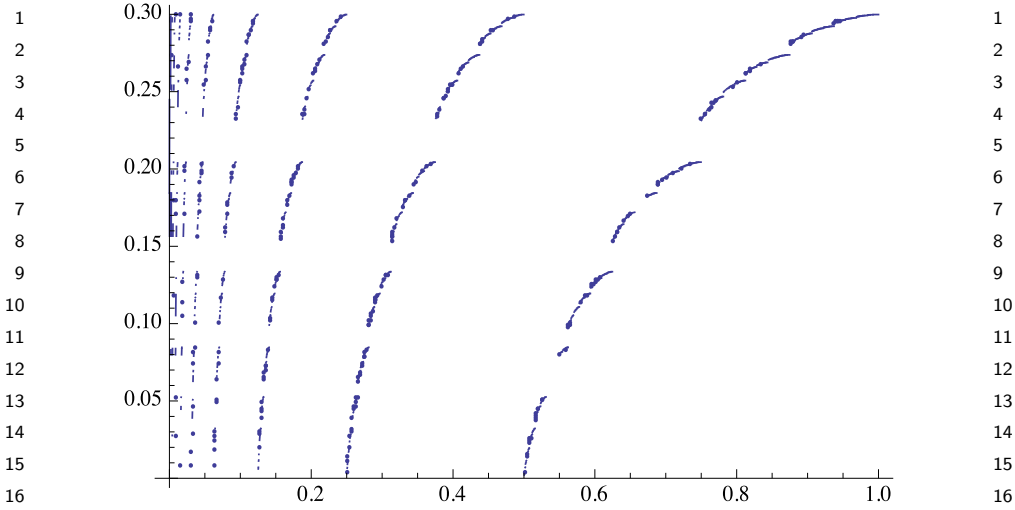


FIGURE 5. The graph of $\alpha_0(x)$.

We also define

$$(2.8) \quad \alpha_1(x) = \frac{1 - \alpha_0(x)^2}{2\alpha_0(x) + 1}, \quad \alpha_2(x) = \frac{\alpha_0(x) - \alpha_0(x)^2}{2\alpha_0(x) + 1}.$$

Note that

$$(2.9) \quad \alpha_0(x) + \alpha_1(x) + \alpha_2(x) = 1.$$

These functions enable us to describe harmonic functions in Ω_x . The boundary of Ω_x consists of the top vertex q_0 and $S(x) = L_x \cap \mathcal{SG}$. If x is not a dyadic rational, then $S(x)$ is a Cantor set. We will assume this holds. Then a harmonic function is determined by the value $h(q_0)$ and the expansion of $h|_{\mathcal{SG}}$ in a Haar basis.

DEFINITION 2.1. The harmonic function h_0 satisfies

$$(2.10) \quad h_0(q_0) = 1, \quad h_0|_{S(x)} = 0.$$

The harmonic function h_1 satisfies

$$(2.11) \quad h_1(q_0) = 0, \quad h_1|_{S(x) \cap F_0^{n_1-1} F_1(\mathcal{SG})} = 1, \\ h_1|_{S(x) \cap F_0^{n_1-1} F_2(\mathcal{SG})} = -1.$$

We write h_0^x and h_1^x when we need to explicitly show the dependence on x .

Note that $1 - h_0$ satisfies

$$(2.12) \quad (1 - h_0)(q_0) = 0, \quad (1 - h_0)|_{S(x)} = 1,$$

1 so that $1 - h_0$ and h_1 vanish at q_0 and give the first two Haar functions when 1
 2 restricted to $S(x)$. Also it is shown in [S1] that 2

$$3 \quad (2.13) \quad h_0(F_0^{n_1-1}F_1q_0) = h_0(F_0^{n_1-1}F_2q_0) = \alpha_0(x) \quad 3$$

4 and 4

$$5 \quad (2.14) \quad h_1(F_0^{n_1-1}F_1q_0) = -h_1(F_0^{n_1-1}F_2q_0) = \alpha_1(x) - \alpha_2(x). \quad 5$$

6 LEMMA 2.2. Let $y = 2^{n_1}Rx$. Then 6
 7

$$8 \quad (2.15) \quad h_0^x \circ (F_0^{n_1-1}F_1) = h_0^x \circ (F_0^{n_1-1}F_2) = \alpha_0(x)h_0^y \quad 8$$

9 and 9

$$10 \quad (2.16) \quad h_1^x \circ (F_0^{n_1-1}F_1) = -h_1^x \circ (F_0^{n_1-1}F_2) = 1 + (\alpha_1(x) - \alpha_2(x) - 1)h_0^y. \quad 10$$

11 *Proof.* The function $\alpha_0(x)h_0^y$ is a harmonic function on Ω_y with boundary 11
 12 values $\alpha_0(x)$ at q_0 and zero on $S(y)$. Note that $F_0^{n_1-1}F_1(S(y)) = S(x)$, so 12
 13 $h_0^x \circ (F_0^{n_1-1}F_1)$ is also a harmonic function on Ω_y vanishing on $S(y)$, and 13
 14 it assume the value $\alpha_0(x)$ at q_0 by (2.13). Thus, (2.15) holds. A similar 14
 15 argument shows that (2.14) implies (2.16). \square 15
 16 16
 17 17
 18 18

19 Next, we consider the general Haar basis functions on $L^2(S(x))$. Let $\omega =$ 19
 20 $(\omega_1, \dots, \omega_m)$ be a word of length $|\omega| = m$, with each $\omega_j = 1$ or 2. Then 20

$$21 \quad (2.17) \quad S_\omega(x) = S(x) \cap F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}(S\mathcal{G}) \quad 21$$

22 describe the dyadic pieces of $S(x)$. In particular, 22
 23

$$24 \quad (2.18) \quad S(x) = \bigcup_{|\omega|=m} S_\omega(x). \quad 24$$

25 The Cantor measure μ on $S(x)$ assigns measure 2^{-m} to each piece $S_\omega(x)$. 25
 26 The Haar function ψ_ω is supported on $S_\omega(x)$ and satisfies 26
 27

$$28 \quad (2.19) \quad \psi_\omega|_{S_{\omega_1}(x)} = 2^{m/2} \quad \text{and} \quad \psi_\omega|_{S_{\omega_2}(x)} = -2^{m/2}. \quad 28$$

29 Then $1 \cup \{\psi_\omega\}$ is an orthonormal basis for $L^2(S(x), d\mu)$. We define h_ω^x to be the 29
 30 harmonic function on Ω_x with boundary values $h_\omega^x(q_0) = 0$ and $h_\omega^x|_{S(x)} = \psi_\omega$. 30
 31 31
 32 32

33 LEMMA 2.3. Let $y_m = 2^{n_m}R^m x$. Then h_ω^x is supported in 33

$$34 \quad \Omega_x \cap F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}(S\mathcal{G}) \quad 34$$

35 and 35
 36

$$37 \quad (2.20) \quad h_\omega^x \circ (F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1}F_{\omega_m}) = 2^{m/2}h_1^{y_m}. \quad 37$$

38 *Proof.* The key observation is that, because of skew-symmetry, the function 38
 39 h_1 not only vanishes at q_0 but also has normal derivative vanishing at q_0 . 39
 40 Thus, we may glue the function defined by (2.20) to zero outside this cell 40
 41 and still have a harmonic function. This function clearly has the required 41
 42 boundary values for h_ω^x . \square 42
 43 43
 44 44

THEOREM 2.4. *The energies are given by*

$$(2.21) \quad \mathcal{E}(h_0^x) = (1 - \alpha_0(x))^2 \sum_{j=1}^{\infty} 2^{2-j} \left(\frac{5}{3}\right)^{2n_1-n_j},$$

$$(2.22) \quad \mathcal{E}(h_1^x) = 6 \left(\frac{1 - \alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} + 2 \left(\frac{3\alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} \mathcal{E}(h_0^y),$$

and

$$(2.23) \quad \mathcal{E}(h_\omega^x) = 2^m \left(\frac{5}{3}\right)^{n_m} \mathcal{E}(h_1^{y_m}),$$

where $m = |\omega|$. Moreover, there exist positive constants C_1 and C_2 , independent of x , such that

$$(2.24) \quad C_1 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \leq \mathcal{E}(h_\omega^x) \leq C_2 2^m \left(\frac{5}{3}\right)^{n_{m+1}}.$$

Proof. We compute the energy of h_0^x on the top cell $F_0^{n_1}(\mathcal{S}\mathcal{G})$ using (2.13) to be $(\frac{5}{3})^{n_1} 2(\alpha_0(x) - 1)^2$, since there are two edges where the difference of h_0^x is $\alpha_0(x) - 1$. On the remaining cells $F_0^{n_1-1}F_1(\mathcal{S}\mathcal{G})$ and $F_0^{n_1-1}F_2(\mathcal{S}\mathcal{G})$ the function h_0^x is equal to $\alpha_0(x)h_0^y \circ (F_0^{n_1-1}F_1)^{-1}$ and $\alpha_0(x)h_0^y \circ (F_0^{n_1-1}F_2)^{-1}$ by (2.15). These each have energy $\alpha_0(x)^2(\frac{5}{3})^{n_1}\mathcal{E}(h_0^y)$, so

$$(2.25) \quad \mathcal{E}(h_0^x) = 2 \left(\frac{5}{3}\right)^{n_1} ((\alpha_0(x) - 1)^2 + \alpha_0(x)^2 \mathcal{E}(h_0^y)).$$

Before iterating this identity, we observe that

$$(2.26) \quad (1 - \alpha_0(y))\alpha_0(x) = \frac{1}{2} \left(\frac{5}{3}\right)^{n_1-n_2} (1 - \alpha_0(x)).$$

This follows from (2.5) and the observation, from (2.7), that $\alpha_0(x)$ depends only on the sequence of differences $n_k - n_{k-1}$ and therefore $\alpha_0(y) = \alpha_0(2^{n_1}Rx) = \alpha_0(Rx)$. Thus,

$$\begin{aligned} \mathcal{E}(h_0^x) &= (1 - \alpha_0(x))^2 \left(2 \left(\frac{5}{3}\right)^{n_1} + \left(\frac{5}{3}\right)^{2n_1-n_2} \right) \\ &\quad + 4 \left(\frac{5}{3}\right)^{n_2} \alpha_0(x)^2 \alpha_0(y)^2 \mathcal{E}(h_0^{y_2}) \end{aligned}$$

and by iterating we obtain (2.21).

Similarly, we use (2.14) to compute the energy of h_1^x on the top cell $F_0^{n_1}(\mathcal{S}\mathcal{G})$ to be $(\frac{5}{3})^{n_1} 6(\alpha_2(x) - \alpha_1(x))^2 = (\frac{5}{3})^{n_1} 6(\frac{1-\alpha_0(x)}{2\alpha_0(x)+1})^2$ by (2.8). Then by using (2.16) we compute the energy in each of the other cells to be $(\frac{5}{3})^{n_1}(\alpha_1(x) - \alpha_2(x) - 1)^2 \mathcal{E}(h_0^y) = (\frac{5}{3})^{n_1}(\frac{3\alpha_0(x)}{2\alpha_0(x)+1})^2 \mathcal{E}(h_0^y)$, and by adding we obtain (2.22). Then (2.23) follows by Lemma 2.3.

To obtain the estimate (2.24), we observe that since $0 \leq \alpha_0(x) \leq \frac{3}{10}$ it follows from (2.7) that $\alpha_0(x)$ is bounded above and below by multiples of $(\frac{5}{3})^{n_1-n_2}$. It follows from (2.21) that $\mathcal{E}(h_0^x)$ is bounded above and below by multiples of $(\frac{5}{3})^{n_1}$ since the infinite series is dominated by its first term. We get the same estimate for $\mathcal{E}(h_1^x)$ using (2.22) since the second summand is bounded by a multiple of $(\frac{5}{3})^{2(n_1-n_2)}(\frac{5}{3})^{n_1}(\frac{5}{3})^{n_2-n_1}$. Then (2.24) follows from this estimate and (2.23). \square

COROLLARY 2.5. *Let h be the harmonic function on Ω_x with boundary values $h(q_0) = a$ and $h|_{S(x)} = f$, where*

$$(2.27) \quad f = b + \sum_{\omega} c_{\omega} \psi_{\omega}$$

for

$$(2.28) \quad c_{\omega} = \int_{S(x)} f \psi_{\omega} d\mu.$$

Then $\mathcal{E}(h)$ is bounded above and below by multiples of

$$(2.29) \quad \left(\frac{5}{3}\right)^{n_1} (a-b)^2 + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^m \left(\frac{5}{3}\right)^{n_{m+1}} |c_{\omega}|^2.$$

In particular, h has finite energy if and only if (2.29) is finite.

Proof. By subtracting a constant we may assume without loss of generality that $a = 0$ (this does not change c_{ω}). Then from (2.27) we have

$$(2.30) \quad h = b(1 - h_0) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} c_{\omega} h_{\omega},$$

and the functions $h_0 \cup \{h_{\omega}\}$ are orthogonal in energy by symmetry considerations. Thus,

$$(2.31) \quad \mathcal{E}(h) = b^2 \mathcal{E}(1 - h_0) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} |c_{\omega}|^2 \mathcal{E}(h_{\omega})$$

and the result follows by the estimates (2.24). \square

We are also interested in the corresponding result for the L^2 norm of h . Using similar reasoning, we can show that $\|h\|_2^2$ is bounded above and below by multiples of

$$(2.32) \quad \left(\frac{1}{3}\right)^{n_1} (a^2 + b^2) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^m \left(\frac{1}{3}\right)^{n_{m+1}} |c_{\omega}|^2.$$

Of course this allows the coefficients to grow so that $\sum_{\omega} |c_{\omega}|^2$ is infinite, meaning that the boundary values f on $S(x)$ may not be in $L^2(S(x))$.

3. Normal derivatives

We follow the general outline from [OS] to define a normal derivative on $S(x)$. We define

$$(3.1) \quad \partial_n u|_{S(x)} = \lim_{m \rightarrow \infty} 2^m \sum_{|\omega|=m} (-\partial_n u(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(S\mathcal{G})}$$

if the limit exists, where

$$(3.2) \quad \tilde{F}_\omega = F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \dots F_0^{n_m-n_{m-1}-1} F_{\omega_m}.$$

The cells $\tilde{F}_\omega(S\mathcal{G})$ for $|\omega| = m$ cover $S(x)$, and $\tilde{F}_\omega q_0$ is the top vertex. Since $\partial_n u(\tilde{F}_\omega q_0)$ is outer directed, upward, we insert the minus sign to get an outer directed normal across $S(x)$.

LEMMA 3.1. $\partial_n h_0^x$ is the constant function on $S(x)$ with value $-2(\frac{5}{3})^{n_1}(1 - \alpha_0(x))$.

Proof. We compute $\partial_n h_0^x(q_0) = 2(\frac{5}{3})^{n_1}(1 - \alpha_0(x))$ from the cell $F_0^{n_1}(S\mathcal{G})$. Next, consider the cell $F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1}(S\mathcal{G})$. The top vertex is $F_0^{n_1-1} F_{\omega_1} q_0$, and by symmetry (on the cell $F_0^{n_1}(S\mathcal{G})$), $\partial_n h_0^x(F_0^{n_1-1} F_{\omega_1} q_0) = \frac{1}{2} \partial_n h_0^x(q_0)$ for $\omega_1 = 1, 2$. Thus $2 \sum_{|\omega|=1} (-\partial_n h_0^x(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(S\mathcal{G})} = -\partial_n h_0^x(q_0) \chi_{S(x)}$. By similar reasoning, there is no change on the right side of (3.1) as m increases. \square

LEMMA 3.2. $\partial_n h_\omega^x = 6 \cdot 2^m (\frac{5}{3})^{n_{m+1}} (\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}) \psi_\omega$.

Proof. On the cell $F_0^{n_1}(S\mathcal{G})$ we compute (using (2.14))

$$\partial_n h_1^x(F_0^{n_1-1} F_1 q_0) = -\partial_n h_1^x(F_0^{n_1-1} F_2 q_0) = 3 \left(\frac{5}{3}\right)^{n_1} (\alpha_1(x) - \alpha_2(x)),$$

so we have

$$\sum_{|\omega|=1} 2(-\partial_n h_1^x(\tilde{F}_\omega q_0)) \chi_{S(x) \cap \tilde{F}_\omega(S\mathcal{G})} = 6 \left(\frac{5}{3}\right)^{n_1} (\alpha_1(x) - \alpha_2(x)) \psi_\emptyset,$$

and by the same reasoning as in Lemma 3.1, this does not change if we instead sum over $|\omega| = m$ for any $m \geq 2$. So this gives the correct result for $\omega = \emptyset$. We then use Lemma 2.3 to scale the result for general ω . \square

THEOREM 3.3. Suppose h and f are given as in Corollary 2.5. Then $\partial_n h$ is given by

$$(3.3) \quad 2(b-a) \left(\frac{5}{3}\right)^{n_1} (1 - \alpha_0(x)) + \sum_{m=0}^{\infty} \sum_{|\omega|=m} 6 \cdot 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \left(\frac{1 - \alpha_0(y_m)}{2\alpha_0(y_m) + 1}\right) c_\omega \psi_\omega$$

1 provided the series converges. In other words, the Dirichlet-to-Neumann map 1
 2 $f \rightarrow \partial_n h$ is a Haar series multiplier map with multiplier $6 \cdot 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \times$ 2
 3 $\left(\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}\right)$. 3
 4

5 COROLLARY 3.4. Suppose f satisfies 5

6 (3.4)
$$\sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^{2m} \left(\frac{5}{3}\right)^{2n_{m+1}} |c_{\omega}|^2 < \infty.$$
 6
 7
 8

9 Then $\partial_n h$ is well-defined in $L^2(S(x))$ and $\|\partial_n h\|_2^2$ is bounded above and below by 9
 10 a multiple of (3.4). 10

11 *Proof.* The theorem follows from Lemma 3.2, and the corollary follows from 11
 12 the fact that $\frac{1-\alpha_0(x)}{2\alpha_0(x)+1}$ is uniformly bounded above and below independent 12
 13 of x . \square 13
 14

15 Note that the finiteness of (3.4) is a stronger condition than the finiteness 15
 16 of (2.29), so harmonic functions of finite energy do not necessarily satisfy 16
 17 (3.4), but functions h satisfying the conditions of Corollary 3.4 automatically 17
 18 have finite energy. 18
 19

20 COROLLARY 3.5. Suppose h is as in Corollary 2.5 with coefficients that 20
 21 satisfy (3.4), and v is any function of finite energy of Ω_x , then the following 21
 22 Gauss–Green formula holds: 22

23 (3.5)
$$\mathcal{E}(h, v) = v(q_0)\partial_n h(q_0) + \int_{S(x)} v\partial_n h d\mu.$$
 23
 24
 25

26 *Proof.* v is continuous since v is of finite energy, hence has a well-defined 26
 27 restriction to $S(x)$ that is bounded and thus in $L^2(\mu)$. Apply the standard 27
 28 Gauss–Green formula on the domain $\bigcup_{|\omega|\leq m} \tilde{F}_{\omega}(S\mathcal{G})$ and take the limit as 28
 29 $m \rightarrow \infty$. \square 29
 30

31 **4. Extending functions of finite energy** 31

32 In this section, we will write Ω_x^+ for the region above $L(x)$ that was previ- 32
 33 ously denoted Ω_x , and Ω_x^- for the region below $L(x)$. Under the assumption 33
 34 that x is not a dyadic rational, $S(x)$ is the common boundary of Ω_x^+ and Ω_x^- . 34
 35 For functions u^{\pm} defined on Ω_x^{\pm} , we use $\mathcal{E}_{\Omega_x^{\pm}}(u^{\pm})$ to denote the energies of 35
 36 u^{\pm} , which are naturally defined by taking the graph energy sum with edges 36
 37 restricted to lie in Ω_x^{\pm} , and then computing the usual renormalized limit. 37
 38 Let $\text{dom } \mathcal{E}_{\Omega_x^{\pm}}$ to denote the collections of functions of finite energy on Ω_x^{\pm} , 38
 39 respectively. 39

40 The first issue that we address is under what conditions can we glue to- 40
 41 gether functions u^{\pm} of finite energy on Ω_x^{\pm} to obtain a function of finite energy 41
 42 on $S\mathcal{G}$. Since functions of finite energy are continuous, u^{\pm} must have bound- 42
 43 ary values on $S(x)$ that agree. It turns out that this is the only condition that 43
 44 44

1 we need to impose. This is not surprising since the same is true for gluing 1
 2 functions of finite energy on domains that intersect at a finite set of points. 2

3 THEOREM 4.1. Let $u^\pm \in \text{dom } \mathcal{E}_{\Omega_x^\pm}$, and suppose 3
 4

5 (4.1)
$$u^+|_{S(x)} = u^-|_{S(x)},$$
 5
 6

7 the values being defined by continuity. Then 7

8 (4.2)
$$u = \begin{cases} u^+ & \text{on } \overline{\Omega}_x^+, \\ u^- & \text{on } \overline{\Omega}_x^-, \end{cases}$$
 8
 9
 10

11 belongs to $\text{dom } \mathcal{E}$ in \mathcal{SG} and 11

12 (4.3)
$$\mathcal{E}(u) = \mathcal{E}_{\Omega_x^+}(u^+) + \mathcal{E}_{\Omega_x^-}(u^-).$$
 12
 13

14 *Proof.* Let S_m denote the strip of 2^m cells of order n_m containing $S(x)$, 14
 15 and let B_m^\pm denote the unions of the cells of order n_m contained in Ω_x^\pm . Then 15
 16

17
$$\mathcal{E}^{(n_m)}(u) = \mathcal{E}_{B_m^+}^{(n_m)}(u) + \mathcal{E}_{B_m^-}^{(n_m)}(u) + \mathcal{E}_{S_m}^{(n_m)}(u).$$
 17
 18

19 Since $\mathcal{E}_{B_m^\pm}^{(n_m)}(u) \rightarrow \mathcal{E}_{\Omega_x^\pm}(u^\pm)$ as $m \rightarrow \infty$, it suffices to show 19
 20

21 (4.4)
$$\mathcal{E}_{S_m}^{(n_m)}(u) \rightarrow 0.$$
 21
 22

23 Let C denote one of the n_m -cells in S_m with boundary points $x_m \in \Omega_x^+$ and 23
 24 $y_m, z_m \in \Omega_x^-$. We need to estimate 24

25 (4.5)
$$\left(\frac{5}{3}\right)^{n_m} [(u^+(x_m) - u^-(y_m))^2$$
 25
 26
$$+ (u^+(x_m) - u^-(z_m))^2 + (u^-(y_m) - u^-(z_m))^2].$$
 26
 27
 28

29 It suffices to estimate the first two terms in (4.5) since $u^-(y_m) - u^-(z_m) =$ 29
 30 $(u^+(x_m) - u^-(z_m)) - (u^+(x_m) - u^-(y_m))$, and by symmetry it suffices to esti- 30
 31 mate the first term. Let S_m^\pm be the portion of S_m above or below $S(x)$. There 31
 32 will be an infinite sequence of points $\{x_m, x_{m+1}, \dots\}$ in S_m^+ and $\{y_m, y_{m+1}, \dots\}$ 32
 33 in S_m^- , both converging to the same point $p \in S(x)$. Since $u^+(p) = u^-(p)$ by 33
 34 (4.1), we may write 34
 35

36 (4.6)
$$u^+(x_m) - u^-(y_m) = \sum_{j=m}^{\infty} (u^+(x_j) - u^+(x_{j+1}))$$
 36
 37
$$- \sum_{j=m}^{\infty} (u^-(y_j) - u^-(y_{j+1})).$$
 37
 38
 39
 40
 41

42 Now each pair (x_j, x_{j+1}) are vertices of a cell C_j of order n_{j+1} in Ω_x^+ . Note 42
 43 that all these cells are essentially disjoint, and $C = \bigcup_j C_j$. 43
 44

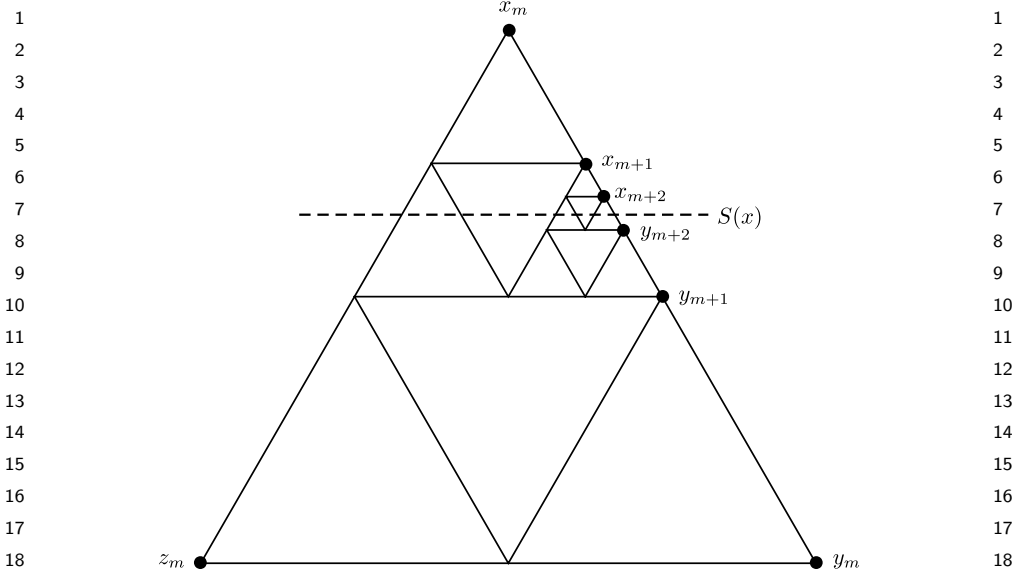


FIGURE 6

So we have the estimate

$$(4.7) \quad |u^+(x_j) - u^+(x_{j+1})| \leq \left(\frac{3}{5}\right)^{n_{j+1}/2} \mathcal{E}_{C_j}(u^+)^{1/2}.$$

By the Cauchy–Schwarz inequality, we obtain

$$(4.8) \quad \sum_{j=m}^{\infty} |u^+(x_j) - u^+(x_{j+1})| \leq \left(\sum_{j=m}^{\infty} \left(\frac{3}{5}\right)^{n_{j+1}}\right)^{1/2} \left(\sum_{j=m}^{\infty} \mathcal{E}_{C_j}(u^+)\right)^{1/2} \\ \leq c \left(\frac{3}{5}\right)^{n_m/2} \mathcal{E}_{C \cap S_m^+}(u^+)^{1/2}.$$

By similar reasoning, we obtain the same estimate with $|u^-(y_j) - u^-(y_{j+1})|$, so by (4.6) we have

$$(4.9) \quad \left(\frac{5}{3}\right)^{n_m} |u^+(x_m) - u^-(y_m)|^2 \leq c \mathcal{E}_{C \cap S_m^+}(u^+) + c \mathcal{E}_{C \cap S_m^-}(u^-).$$

Summing (4.9) over all the 2^m cells C yields

$$(4.10) \quad \mathcal{E}_{S_m}^{(n_m)}(u) \leq c \mathcal{E}_{S_m^+}(u^+) + c \mathcal{E}_{S_m^-}(u^-)$$

and $\mathcal{E}_{S_m^\pm}(u^\pm) \rightarrow 0$ because $\bigcap_m S_m^\pm = S(x)$ and $S(x)$ has measure zero in the Kusuoka measure (this follows easily from Theorem 5.1 of [AHS]). \square

1 It is easy to characterize the restrictions to $S(x)$ of functions of finite energy 1
 2 on Ω_x^+ . 2

3
 4 THEOREM 4.2. A function f on $S(x)$ is the restriction to $S(x)$ of a function 4
 5 u^+ of finite energy on Ω_x^+ if and only if f has a Haar series expansion (2.27) 5
 6 with (2.29) finite (here $a = 0$), and (2.29) is bounded by a multiple of $\mathcal{E}_{\Omega_x^+}(u^+)$. 6

7
 8 *Proof.* Let h be the harmonic function on Ω_x^+ with the same boundary 7
 9 values f . Since harmonic functions minimize energy, $\mathcal{E}_{\Omega_x^+}(h) \leq \mathcal{E}_{\Omega_x^+}(u^+)$, and 8
 10 the result follows from Corollary 2.5. \square 10

11
 12 However, there is no such simple result for Ω_x^- . We pose the following 12
 13 extension problem. 13

14
 15 PROBLEM 4.3. Does there exist a bounded linear extension operator (mean- 14
 16 ing $Tu|_{\Omega_x^+} = u$) $T : \text{dom}_{\Omega_x^+}(\mathcal{E}) \rightarrow \text{dom}_{SG}(\mathcal{E})$? 15

17
 18 There is a simple obstruction to solving this problem. 17

19
 20 DEFINITION 4.4. x satisfies the *nonconsecutive condition with bound N* if 19
 21 there are no N consecutive integers in the sequence $\{n_m\}$. If there is some N 20
 22 for which this holds then x is said to satisfy the *nonconsecutive condition*. 21

22
 23 Note that a generic value of x will not satisfy this condition. However, 22
 24 there are uncountably many (of Hausdorff dimension 1) values of x that do 23
 25 satisfy the condition. Perhaps the simplest choice has $n_m = 2m - 1$, with 24
 26 $N = 2$. 25

27
 28 THEOREM 4.5. Let E denote the collection of x satisfying the nonconsec- 27
 29 utive condition. Then the Hausdorff dimension of E is 1. 28

29
 30 *Proof.* Let E_N denote the collection of x satisfying the nonconsecutive 29
 31 condition with bound N . Then $E = \bigcup_{N \geq 2} E_N$ and 30

32
 33 (4.11)
$$E_2 \subset E_3 \subset \dots \subset E_N \subset \dots$$
 32

34
 35 We will first prove that the Hausdorff dimension of E_N is the unique positive 34
 36 root of the equation 35

37
 38 (4.12)
$$2 - 2^s - 2^{-Ns} = 0.$$
 37

39
 40 Consider the set E_N . We divide it into the disjoint union $E_N = \bigcup_{k \geq 1} E_{N,k}$ 39
 41 where $E_{N,k}$ is the set of x in E_N whose n_1 -digit is k . Obviously, for each 40
 42 k , $E_{N,k}$ is a similar copy of $E_{N,1}$ with contraction ratio 2^{1-k} . Since the 41
 43 Hausdorff dimension is stable for countable unions, we just need to compute 42
 44 the dimension of $E_{N,1}$. For this set, by the nonconsecutive condition, we can 43
 44 44

1 write

$$\begin{aligned}
 & 2 \\
 & 3 \quad (4.13) \quad E_{N,1} = \left(\bigcup_{j \geq 3} (2^{-1} + E_{N,j}) \right) \cup \dots \\
 & 4 \\
 & 5 \quad \quad \quad \cup \left(\bigcup_{j \geq N+1} (2^{-1} + \dots + 2^{-(N-1)} + E_{N,j}) \right). \\
 & 6 \\
 & 7
 \end{aligned}$$

8 Since $|E_{N,j}| \leq 1/2^j$, it is easy to check that the above union is disjoint. More-
 9 over, (4.13) is essentially a self-similar identity for the set $E_{N,1}$ with contrac-
 10 tion ratios,

$$11 \quad 2^{-2}, 2^{-3}, \dots; 2^{-3}, 2^{-4}, \dots; 2^{-N}, 2^{-(N+1)}, \dots,$$

12 satisfying the open set condition (with the open set $(2^{-1}, 1)$). (See [M] for the
 13 theory of infinitely generated self-similar sets.) Hence the Hausdorff dimension
 14 of $E_{N,1}$ is the solution of the equation

$$\begin{aligned}
 & 15 \\
 & 16 \quad (4.14) \quad 1 = \sum_{k=2}^N \sum_{j \geq k} (2^{-s})^j = \sum_{k=2}^N \frac{(2^{-s})^k}{1 - 2^{-s}} = \frac{2^{-2s} - 2^{-s(N+1)}}{(1 - 2^{-s})^2}, \\
 & 17 \\
 & 18
 \end{aligned}$$

19 which simplifies to (4.12). So we get the Hausdorff dimension of E_N .

20 Using (4.11), an easy calculation will show that the Hausdorff dimension
 21 of E is 1. □

22 If x fails to satisfy the nonconsecutive condition, then there are pairs of
 23 points in Ω_x^+ that are much closer to each other in \mathcal{SG} than in Ω_x^+ . For
 24 example, if $n_j = j$ for $j \leq N$ then the points $F_1 F_2^{N-1} q_0$ and $F_2 F_1^{N-1} q_0$ in Ω_x^+
 25 are distance on the order of $(\frac{3}{5})^N$ apart in the resistance metric on \mathcal{SG} , but are
 26 far apart in Ω_x^+ . Note that $h_1^x(F_1 F_2^{N-1} q_0) - h_1^x(F_2 F_1^{N-1} q_0) = 2h_1^x(F_1 F_0^{N-1} q_0)$
 27 and $\mathcal{E}(h_1^x)$ is bounded. The estimate analogous to (4.7) shows

$$\begin{aligned}
 & 28 \\
 & 29 \quad c \leq \left(\frac{3}{5} \right)^{N/2} \mathcal{E}(u)^{1/2} \\
 & 30
 \end{aligned}$$

31 for any extension u of h_1^x to \mathcal{SG} , hence $\mathcal{E}(u) \geq c(\frac{5}{3})^N$. This means that the
 32 bound on the operator T , if it exists, would be bounded below by a multiple
 33 of $(\frac{5}{3})^{N/2}$.

34 The same reasoning applies locally if $\{n_m\}$ has a consecutive string of N
 35 integers. Thus if such strings exist for all N then T cannot be bounded. On
 36 the other hand, it is easy to see that if the nonconsecutive condition holds
 37 for x then distances in Ω_x^+ and \mathcal{SG} are comparable. Note that this is very
 38 reminiscent of the type of condition that appears in the work of Peter Jones
 39 in the Sobolev extension problem in domains in Euclidean space ([J], [R]).

40
 41 **THEOREM 4.6.** *The extension Problem 4.3 has a positive solution if and*
 42 *only if x satisfies the nonconsecutive condition, in which case the bound on T*
 43 *is $O((\frac{10}{3})^{N/2})$.*

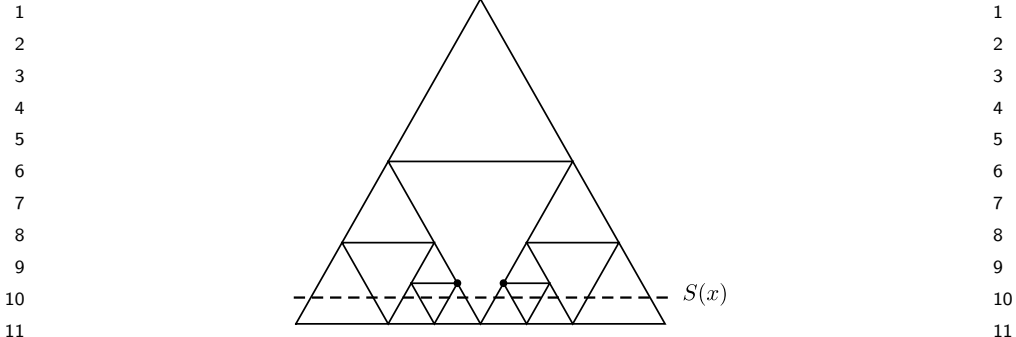


FIGURE 7

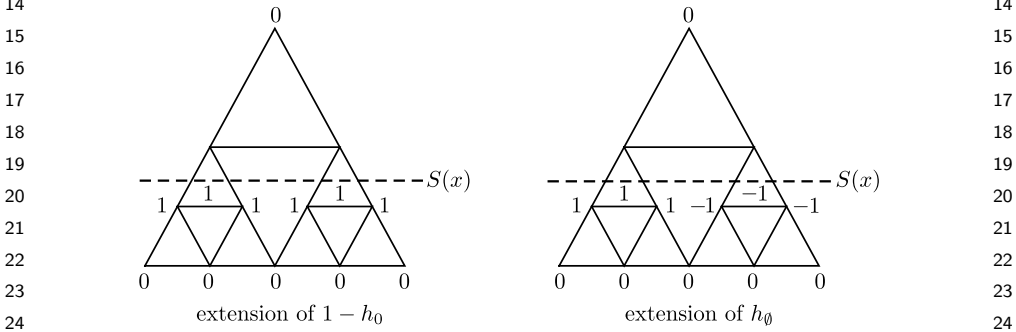


FIGURE 8

Proof. We need to construct an extension operator T under the assumption that x satisfies the nonconsecutive condition. In view of our previous results, it suffices to solve the extension problem for the functions h_ω (and also $1 - h_0$), say $Th_\omega = \tilde{h}_\omega$ where the functions \tilde{h}_ω are orthogonal in energy and

$$(4.15) \quad \mathcal{E}(\tilde{h}_\omega) \leq C(N)2^m \left(\frac{5}{3}\right)^{n_{m+1}}.$$

Suppose first that $N = 2$. Consider first $1 - h_0$ and h_0 . Assume for simplicity that $n_1 = 1$. Then $n_2 \geq 3$. Then $S(x)$ passes through the cells $F_1F_0(\mathcal{SG})$ and $F_2F_0(\mathcal{SG})$. We will extend $1 - h_0$ to be identically 1 on the bottom portions of these cells, and h_0 to be 1 on $F_1F_0(\mathcal{SG})$ and -1 on $F_2F_0(\mathcal{SG})$. On the remaining four cells of level 2, we make the extension harmonic with boundary values 0 on the bottom vertices (see Figure 8).

Note that the added energy of these extensions is exactly $8(\frac{5}{3})^2$. Also, since one extension is symmetric and one is skew-symmetric with respect to the vertical reflection, they are orthogonal in energy. If $n_1 > 1$, we may

1 repeat the same process on $F_0^{n_1-1}(\mathcal{SG})$ and then continue the extension to
 2 be identically zero on the complement of $F_0^{n_1-1}(\mathcal{SG})$. The added energy is
 3 exactly $8(\frac{5}{3})^{n_1+1}$, but the energy of the original functions was also a multiple
 4 of $(\frac{5}{3})^{n_1}$, so this is consistent with (4.15) with $m = 0$ and gives a uniform
 5 bound on the extension operator.

6 For the extension of h_ω , we just have to repeat the same procedure minia-
 7 turized. If $|\omega| = m$ then h_ω is supported on a cell of order $n_{m+1} - 1$ and since
 8 $n_{m+2} \geq n_{m+1} + 2$ the right side of Figure 8 describes h_ω and its extension
 9 (except for a factor of $2^{m/2}$) to that cell, and then we may glue this to zero
 10 in the complement of the cell. Thus, we get an extension with the same energy
 11 bound. For words ω with $|\omega| = m$, the extended function have disjoint
 12 support, so the energies are orthogonal. Comparing extensions for words of
 13 different length with overlapping support, we again have a symmetry/skew-
 14 symmetry dichotomy with respect to the local reflection in the vertical axis
 15 of the smaller supporting cell (this is the overlap of the supports) and so we
 16 again have energy orthogonality. This completes the proof for $N = 2$.

17 For general N , the argument is similar. In Figure 9, we show the extension
 18 of h_\emptyset when $N = 3$ and $n_1 = 1, n_2 = 2, n_3 \geq 4$.

19 Here we have 2^N cells of order N contributing to the energy, and this multi-
 20 plies the energy by $O((\frac{10}{3})^N)$. Since the norm of the extension is measured in
 21 terms of the square root of the energy, we obtain the $O((\frac{10}{3})^{N/2})$ bound. \square
 22

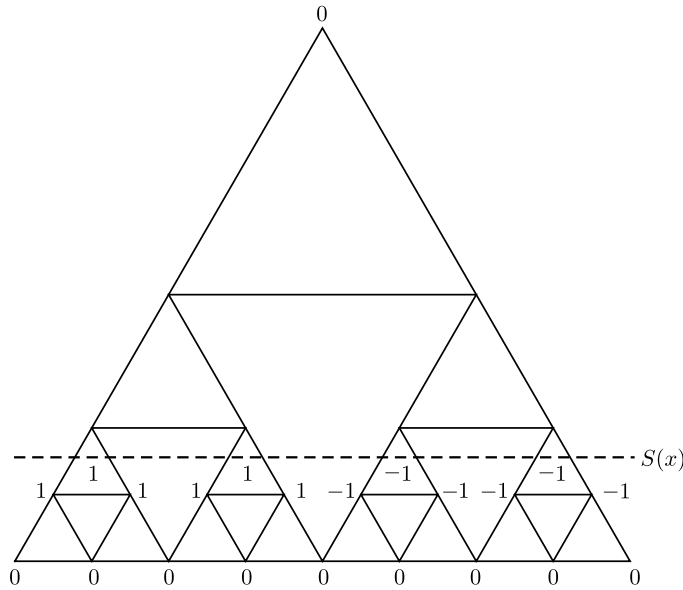


FIGURE 9

1 The optimal extension operator would produce functions that are harmonic 1
 2 on Ω_x^- . In particular, it would be interesting to have an explicit description 2
 3 of the functions h_ω^- that are harmonic on Ω_x^- and are equal to ψ_ω on $S(x)$, 3
 4 again under the nonconsecutive condition. 4

5 We may regard Theorem 4.2 as a trace theorem and Theorem 4.6 as an 5
 6 extension theorem for $\text{dom } \mathcal{E}$ regarded as a Sobolev space, and then we should 6
 7 ask if there are analogous results for other Sobolev spaces. In [S2], the 7
 8 spaces $\text{dom}_{L^2}(\Delta^k)$ on \mathcal{SG} are considered as Sobolev spaces ($\text{dom}_{L^2}(\Delta^k) =$ 8
 9 $\{u \in L^2(\mathcal{SG}) : \Delta^j u \in L^2(\mathcal{SG}) \text{ for all } j \leq k\}$). Similarly for the space $\{u \in$ 9
 10 $\text{dom}_{L^2}(\Delta^k) : \mathcal{E}(\Delta^k u) < \infty\}$. These spaces are easily characterized in terms 10
 11 of expansions in eigenfunctions of the Laplacian. A complete theory of the 11
 12 eigenspaces of the Laplacian on Ω_1 is given in [Q]. 12
 13

14 **PROBLEM 4.7.** For each of these Sobolev spaces, characterize the space 14
 15 of traces on $S(x)$ and restrictions to Ω_x^+ , for x satisfying the nonconsecutive 15
 16 condition. 16

17 It seems plausible that the trace problem may have a solution with a condi- 17
 18 tion similar to (2.29) for the Haar expansion (2.27) with different multiples of 18
 19 $|c_\omega|^2$ depending on the Sobolev space. The restriction problem is likely to be 19
 20 more challenging. It is clear that restrictions of functions in $\text{dom}_{L^2}(\Delta^k)$ must 20
 21 satisfy $\Delta^j u \in L^2(\Omega_x^+)$ for $j \leq k$, but that is not sufficient because all harmonic 21
 22 functions automatically have $\Delta^j u = 0$. It would seem that the characteriza- 22
 23 tion of restriction Sobolev spaces would also have to involve conditions on 23
 24 traces on $S(x)$. Related problems are discussed in [LS] and [LRSU]. 24
 25

26
 27 **5. Green's function** 27
 28

29 For a given k , let V_k denote the set of vertices on the k -level graph approx- 29
 30 imation of \mathcal{SG} . For a point $z \in V_k \setminus V_0$, let ϕ_z^k denote the piecewise harmonic 30
 31 spline of level k satisfying $\phi_z^k(t) = \delta_{zt}$ for $t \in V_k$ and extended harmonically on 31
 32 \mathcal{SG} . Notice that $\phi_z^k \in \text{dom}_0 \mathcal{E}$ because $z \notin V_0$, and it is supported in the two k - 32
 33 cells meeting at z . Recall that in the standard theory (see the books [Ki] and 33
 34 [S3]), the Green's function $G(s, t)$ to solve the Dirichlet problem $-\Delta u = F$ 34
 35 on \mathcal{SG} , subject to the boundary condition $u|_{V_0} = 0$ via an integral transform 35
 36 $\int_{\mathcal{SG}} G(s, t) F(t) dt$, has the following explicit formula, 36
 37

38 (5.1)
$$G(s, t) = \lim_{M \rightarrow \infty} G^M(s, t) \quad (\text{uniform limit})$$
 38
 39

40 with 40

41 (5.2)
$$G^M(s, t) = \sum_{k=1}^M \sum_{z, z' \in V_k \setminus V_{k-1}} g(z, z') \phi_z^k(s) \phi_{z'}^k(t),$$
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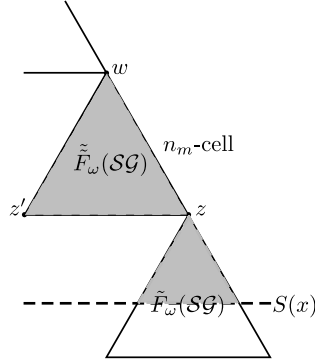


FIGURE 10. The support of $\phi_z^{n_m}$.

where

$$(5.3) \quad g(z, z') = \begin{cases} \frac{3}{10} \left(\frac{3}{5}\right)^k & \text{for } z = z' \in V_k \setminus V_{k-1}, \\ \frac{1}{10} \left(\frac{3}{5}\right)^k & \text{for } z \neq z' \in V_k \setminus V_{k-1}, \\ & \text{contained in the same } (k-1)\text{-cell,} \\ 0, & \text{otherwise.} \end{cases}$$

To get an analogous Green's function on Ω_x , we should first modify the definition of those piecewise harmonic splines ϕ_z^k whose support intersects the boundary $S(x)$ of the domain Ω_x . More specially, let ω be a word of symbols $\{1, 2\}$ with $|\omega| = m$ and $z = \tilde{F}_\omega(q_0)$. We redefine $\phi_z^{n_m}$ to be the piecewise harmonic spline with value 1 on z , 0 on $V_{n_m} \cap \Omega_x$ and $S(x)$, and extended harmonically on Ω_x . Obviously the support of $\phi_z^{n_m}$ is contained in two n_m -cells meeting at z , with $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$ on the cell $\tilde{F}_\omega(S\mathcal{G})$ and with values unchanged on the other cell, denoted by $\tilde{\tilde{F}}_\omega(S\mathcal{G})$, where

$$(5.4) \quad \tilde{\tilde{F}}_\omega = \begin{cases} F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \dots & \text{for } m \geq 2, \\ F_0^{n_m-1-n_{m-2}-1} F_{\omega_{m-1}} F_0^{n_m-n_{m-1}} & \text{for } m = 1. \end{cases}$$

LEMMA 5.1. Let $z = \tilde{F}_\omega(q_0)$, then

$$(5.5) \quad \mathcal{E}_{\Omega_x}(\phi_z^{n_m}, v) = \left(\frac{5}{3}\right)^{n_m} \left(\frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} v(z) - v(z') - v(w)\right)$$

for any $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$, where $z' = \tilde{F}_{\omega_1 \dots \omega_{m-1} (3-\omega_m)}(q_0)$ and $w = \tilde{F}_{\omega_1 \dots \omega_{m-1}}(q_0)$ are the two n_m -neighbors of z (see Figure 10).

1 *Proof.* On the cell $\tilde{F}_\omega(\mathcal{S}\mathcal{G})$, by using the localized Gauss–Green formula
 2 (see (3.5)),

$$\begin{aligned}
 \mathcal{E}_{\Omega_x \cap \tilde{F}_\omega(\mathcal{S}\mathcal{G})}(\phi_z^{n_m}, v) &= v(z) \partial_n \phi_z^{n_m}(z) \\
 &= 2 \left(\frac{5}{3}\right)^{n_m+1} (1 - \alpha_0(y_m)) v(z).
 \end{aligned}$$

3
 4
 5
 6
 7 The last equality follows from the same argument as the proof of Lemma 3.1
 8 with suitable scaling.

9 On the other cell $\tilde{\tilde{F}}_\omega(\mathcal{S}\mathcal{G})$, by using the standard theory,

$$\mathcal{E}_{\Omega_x \cap \tilde{\tilde{F}}_\omega(\mathcal{S}\mathcal{G})}(\phi_z^{n_m}, v) = \left(\frac{5}{3}\right)^{n_m} (2v(z) - v(z') - v(w)).$$

10
 11
 12
 13 Summing the energies on the two cells, we get the desired result by using
 14 (2.5). □

15
 16 Let T_x^m be the set of vertices in $V_{n_m} \cap \Omega_x$ which can be expressed as $\tilde{F}_\omega(q_0)$
 17 for some word $\omega = \omega_1, \dots, \omega_m$ of symbols $\{1, 2\}$, and $T_x = \bigcup_{m \geq 1} T_x^m$.

18
 19 DEFINITION 5.2. For fixed m , let

$$G_{\Omega_x}^m(s, t) = \sum_{k=1}^{n_m} \sum_{z, z' \in (V_k \setminus V_{k-1}) \cap \Omega_x} g_x(z, z') \phi_z^k(s) \phi_{z'}^k(t),$$

20
 21 with

$$\begin{aligned}
 (5.9) \quad g_x(z, z') &= \begin{cases} \frac{\alpha_0(y_{l-1}) + \alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1}) + 1} \left(\frac{3}{5}\right)^{n_l} & \text{for } z = z' \in T_x^l \text{ with } l \leq m, \\ \frac{\alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1}) + 1} \left(\frac{3}{5}\right)^{n_l} & \text{for } z \neq z' \in T_x^l, \text{ being } n_l\text{-neighbors,} \\ & \text{with } l \leq m, \\ g(z, z') & \text{for } z, z' \in V_k \setminus V_{k-1} \text{ contained in} \\ & \text{a } (k-1)\text{-cell in } \Omega_x, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

22
 23
 24
 25 Then it is obvious that $G_{\Omega_x}^m(s, t)$ converges uniformly to a function $G_{\Omega_x}(s, t)$
 26 as m goes to infinity.

27
 28 THEOREM 5.3. G_{Ω_x} is the Green’s function for Ω_x , namely

$$(5.10) \quad u(s) = \int_{\Omega_x} G_{\Omega_x}(s, t) F(t) dt$$

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 32 solves the Dirichlet problem $-\Delta u = F$ on Ω_x with $u|_{\partial\Omega_x} = 0$, for any contin-
 33 uous F .
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Proof. Similar to the \mathcal{SG} case, suppose we could prove

$$(5.11) \quad \mathcal{E}_{\Omega_x}(G_{\Omega_x}^m(\cdot, t), v) = \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t)$$

for any $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$.

Then just multiply (5.11) by $F(t)$ and integrate, using the standard arguments to interchange the energy and integral, to obtain

$$(5.12) \quad \mathcal{E}_{\Omega_x}(u_m, v) = \int_{\Omega_x} F(t) \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t) dt$$

for

$$(5.13) \quad u_m(s) = \int_{\Omega_x} G_{\Omega_x}^m(s, t) F(t) dt.$$

Since

$$(5.14) \quad \sum_{z \in V_{n_m} \cap \Omega_x} v(z) \phi_z^{n_m}(t) \rightarrow v(t)$$

uniformly as $m \rightarrow \infty$, the right side of (5.12) converges to $\int_{\Omega_x} F(t)v(t) dt$, and the left-hand side converges to $\mathcal{E}_{\Omega_x}(u, v)$ as m goes to ∞ . Thus, we have

$$(5.15) \quad \mathcal{E}_{\Omega_x}(u, v) = \int_{\Omega_x} Fv dt$$

for any $v \in \text{dom}_0 \mathcal{E}_{\Omega_x}$, which yields that $-\Delta u = F$ with $u|_{\partial\Omega_x} = 0$.

Hence, our goal is to prove (5.11). The function $G_{\Omega_x}^m(s, t)$, which we regard as a function of the single variable s , could be viewed as a linear combination of terms $\phi_z^k(s)$. Then it is clear that $\mathcal{E}_{\Omega_x}(G_{\Omega_x}^m(\cdot, t), v)$ is a linear combination of $v(z)$ for $z \in V_{n_m} \cap \Omega_x$. So we need to compute the combination coefficient of $v(z)$ for each z .

Let $z_0 \in V_{n_m} \cap \Omega_x$. If $z_0 \notin T_x$, it is easy to observe that there exists an n_m -cell containing z_0 as an interior point. The terms in $G_{\Omega_x}^m$ that contribute to the coefficient of $v(z_0)$ all have supports away from $S(x)$. Thus, the standard argument for the \mathcal{SG} case shows that the coefficient of $v(z_0)$ should be $\phi_{z_0}^{n_m}(t)$.

Hence, we only need to consider the case that $z_0 \in T_x$. We first do this when $z_0 \in T_x^m$. Let z'_0 denote the unique n_m -neighbor of z_0 in the same level. Then the only terms in $G_{\Omega_x}^m$ that contribute to the coefficient of $v(z_0)$ are

$$\begin{aligned} g_x(z_0, z_0) \phi_{z_0}^{n_m}(s) \phi_{z_0}^{n_m}(t), & \quad g_x(z_0, z'_0) \phi_{z_0}^{n_m}(s) \phi_{z'_0}^{n_m}(t), \\ g_x(z'_0, z_0) \phi_{z'_0}^{n_m}(s) \phi_{z_0}^{n_m}(t), & \quad g_x(z'_0, z'_0) \phi_{z'_0}^{n_m}(s) \phi_{z'_0}^{n_m}(t). \end{aligned}$$

By Lemma 5.1, the total contribution is

$$(5.16) \quad \left(\frac{5}{3}\right)^{n_m} \left(\frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z_0) - g_x(z'_0, z_0)\right) \phi_{z_0}^{n_m}(t) \\ + \left(\frac{5}{3}\right)^{n_m} \left(\frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z'_0) - g_x(z'_0, z'_0)\right) \phi_{z'_0}^{n_m}(t).$$

By substituting the value of $g_x(z_0, z_0) = g_x(z'_0, z'_0) = \frac{\alpha_0(y_{m-1}) + \alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1}) + 1} \left(\frac{3}{5}\right)^{n_m}$ and $g_x(z_0, z'_0) = g_x(z'_0, z_0) = \frac{\alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1}) + 1} \left(\frac{3}{5}\right)^{n_m}$ into (5.16), it is easy to verify that

$$(5.17) \quad \left(\frac{5}{3}\right)^{n_m} \left(\frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z_0) - g_x(z'_0, z_0)\right) = 1,$$

and

$$(5.18) \quad \left(\frac{5}{3}\right)^{n_m} \left(\frac{1 + \alpha_0(y_{m-1})}{\alpha_0(y_{m-1})} g_x(z_0, z'_0) - g_x(z'_0, z'_0)\right) = 0.$$

So the coefficient of $v(z_0)$ is $\phi_{z_0}^{n_m}(t)$.

Next, we consider the general case. Suppose $z_0 = \tilde{F}_\omega(q_0) \in T_x^l$ with $1 \leq l < m$. We need to compute the coefficient of $v(z_0)$. The previous discussion immediately shows that the contribution of terms in $G_{\Omega_x}^l$ to $v(z_0)$ is $\phi_{z_0}^{n_l}(t)$. Now we consider the terms in $G_{\Omega_x}^m - G_{\Omega_x}^l$. Let $z_1 = \tilde{F}_{\omega_1}(q_0)$ and $z_2 = \tilde{F}_{\omega_2}(q_0)$. Notice that in all the terms in $G_{\Omega_x}^m - G_{\Omega_x}^l$ that contribute to $v(z_0)$, only those which contain $\phi_{z_1}^{n_{l+1}}(s)$ or $\phi_{z_2}^{n_{l+1}}(s)$ have supports intersecting the boundary $S(x)$. Moreover, in calculating the energy $\mathcal{E}_{\Omega_x}(\phi_{z_i}^{n_{l+1}}, v)$, only the part $\phi_{z_i}^{n_{l+1}}|_{\tilde{F}_{\omega_i}(\mathcal{SG})}$ is involved in contributing to the coefficient of $v(z_0)$, for $i = 1, 2$. Comparing to the standard \mathcal{SG} case, the function $\phi_{z_i}^{n_{l+1}}(s)$ has been redefined, but the restriction of it to $\tilde{F}_{\omega_i}(\mathcal{SG})$ is unchanged. So the total contribution of $G_{\Omega_x}^m - G_{\Omega_x}^l$ to $v(z_0)$ is as same as the standard case, namely $\phi_{z_0}^{n_m}(t) - \phi_{z_0}^{n_l}(t)$. Thus, we get that in $\mathcal{E}_{\Omega_x}(G_{\Omega_x}^m, v)$, the coefficient of $v(z_0)$ is $\phi_{z_0}^{n_m}(t)$, as required.

Thus, we have proved (5.11). □

THEOREM 5.4. *For continuous F , the normal derivative of the solution u given by (5.10) is continuous on $S(x)$.*

Proof. From Theorem 5.3,

$$(5.19) \quad \partial_n u|_{S(x)} = \sum_{m \geq 1} \sum_{z, z' \in T_x^m} g_x(z, z') \partial_n \phi_z^{n_m}|_{S(x)} \int_{\Omega_x} \phi_{z'}^{n_m}(t) F(t) dt,$$

since only those terms containing ϕ_z^k whose supports intersect $S(x)$ contribute to the value of $\partial_n u|_{S(x)}$.

For fixed m , let $z = \tilde{F}_\omega(q_0) \in T_x^m$. Note that on the cell $\tilde{F}_\omega(\mathcal{SG})$, $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$. By Lemma 3.1, we have

$$(5.20) \quad \partial_n \phi_z^{n_m}|_{S(x)} = -2 \left(\frac{5}{3}\right)^{n_m+1} (1 - \alpha_0(y_m)) 2^m \chi_{S_\omega(x)}.$$

On the other hand, for $z, z' \in T_x^m$, $g_x(z, z')$ is bounded above by a multiple of $\alpha_0(y_{m-1}) \left(\frac{3}{5}\right)^{n_m}$, hence by a multiple of $\left(\frac{3}{5}\right)^{n_m+1}$ using (2.7). It is also easy to see that $\int_{\Omega_x} \phi_{z'}^{n_m}(t) F(t) dt$ is bounded above by a multiple of $\frac{1}{3^{n_m}} \|F\|_\infty$. Combing these estimates with (5.20), we conclude that $|\partial_n u|_{S(x)}$ is bounded above by a multiple of

$$(5.21) \quad \sum_{m \geq 1} \sum_{|\omega|=m} \frac{2^m}{3^{n_m}} \|F\|_\infty \chi_{S_\omega(x)}.$$

From (5.21), one can easily verify that $\partial_n u$ is continuous on $S(x)$. \square

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