1987

A New Topological Term in 2d Field Theory

Philip C. Nelson
University of Pennsylvania, nelson@physics.upenn.edu

A New Topological Term in 2d Field Theory

Abstract

This talk describes some work done jointly with L. Alvarez-Gaume, J.B. Bost, G. Moore, and C. Vafa [1][2], building on [3] and [4).

We were concerned with establishing bosonization results on two-dimensional surfaces with complicated topology. Far from being a mere curiosity, bosonization is of great interest in string theory. For example, bosonization has been used in light-cone gauge to prove the equivalence of the Green-Schwarz and NSR superstring [5][6]. Bosonization also plays a key role in understanding the gauge and super-symmetry of the heterotic string [7] and in formulating the covariant fermion emission vertex [8][9].

The papers [1], [2] generalize existing results on Fermi-Bose equivalence for Fermi fields of any spin on the sphere [10]-[13]. In this talk I will only discuss a subproblem, that of bosonizing spin 1/2 on the torus. It turns out that this problem is only slightly more difficult than the sphere case. One needs a way to "tell" the bosonic theory which of the various spin structures it is to mimic; this is accomplished by adding to the bosonic action a new global term. The new term is already familiar to mathematicians as the parity of a spin structure; it has an immediate generalization to any genus surface.

Disciplines

Physical Sciences and Mathematics | Physics

Comments

At the time of publication, author Philip C. Nelson was affiliated with Harvard University. Currently, he is a faculty member in the Physics & Astronomy Department at the University of Pennsylvania.

5I

HUTP-87/A012

A New Topological Term in 2d Field Theory

Philip Nelson
Lyman Laboratory of Physics
Harvard University
Cambridge MA 02138

This talk describes some work done jointly with L. Alvarez-Gaumé, J.-B. Bost, G. Moore, and C. Vafa [1][2], building on [3] and [4].

We were concerned with establishing bosonization results on two-dimensional surfaces with complicated topology. Far from being a mere curiosity, bosonization is of great interest in string theory. For example, bosonization has been used in light-cone gauge to prove the equivalence of the Green-Schwarz and NSR superstring [5][6]. Bosonization also plays a key role in understanding the gauge- and super-symmetry of the heterotic string [7] and in formulating the covariant fermion emission vertex [8][9].

The papers [1], [2] generalize existing results on Fermi-Bose equivalence for Fermi fields of any spin on the sphere [10]–[13]. In this talk I will only discuss a subproblem, that of bosonizing spin 1/2 on the torus. It turns out that this problem is only slightly more difficult than the sphere case. One needs a way to "tell" the bosonic theory which of the various spin structures it is to mimic; this is accomplished by adding to the bosonic action a new global term. The new term is already familiar to mathematicians as the parity of a spin structure; it has an immediate generalization to any genus surface.

1. Bosonic action

We will let Σ denote any compact Riemann surface, a 2d "spacetime" in the euclidean formalism. We begin by reviewing the situation when Σ is the sphere.

We will consider only free fermions, that is, fermions interacting with background gauge and gravitational fields but without self-interactions such as mass or quartic terms. This is the case of interest for the NSR superstring in flat spacetime.

The prototypical fermionic system one might wish to bosonize has action

$$S_f = \int_\Sigma \bar{\psi}\, i\partial\!\!\!/ \psi \quad .$$

In euclidean path integrals ψ and $\bar{\psi}$ are independent, and it has become traditional to rename the fields, with $\psi_1 \mapsto c \quad \psi_2 \mapsto \bar{b} \quad \bar{\psi}_1 \mapsto b$ and $\bar{\psi}_2 \mapsto \bar{c}$. Using complex notation and rescaling fields we then have

$$S_f = \frac{i}{2\pi} \int_\Sigma (b\bar{\partial}c + \bar{b}\partial\bar{c}) \quad , \tag{1.1}$$

where b and c are sections of a spin bundle L, and $\bar{b}$ and $\bar{c}$ are sections of $\bar{L}$. $\bar{\partial}$ is the Cauchy-Riemann operator. The integrand is a (1,1)-form and so can be integrated over Σ without the use of any metric. That is, (1.1) is classically conformally invariant. It is also invariant under the global chiral transformation $b \mapsto e^{i\alpha}b,\ c \mapsto e^{-i\alpha}c$.

Similarly the prototypical scalar action for a single real field φ has the form

$$\begin{aligned} S_1 &= \int_\Sigma 2\pi \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi \; d(\mathrm{vol}) \\ &= \int_\Sigma 4\pi i\, \partial\varphi \wedge \bar{\partial}\varphi \quad . \end{aligned} \tag{1.2}$$

The second form makes it clear that S_1 is also classically conformally invariant. It also has an invariance under shifts $\varphi \mapsto \varphi +$ constant.

The operator analysis of bosonization on $2d$ Minkowski spaces teaches us two important physical lessons. (See for example [14]–[18],[10].) First of all, the correspondence should be between the fermionic bilinears and *exponentials* of φ, properly normal-ordered. Secondly, the bosonic field φ is properly to be regarded as a *periodic*, or circle-valued, field. This fact is compatible with the first if the normalization of φ is chosen such that its ambiguity does not affect its exponential. We can see the periodicity of φ either in the periodic sine-Gordon potential of [14] or in the case where spacetime is the Minkowski cylinder [10]. In either case the crucial physical basis of bosonization is that fermions in (1.1) correspond to *solitons*, or states where φ is multiple-valued, in (1.2). In particular, when spacetime has noncontractible loops the partition function of the bosonic system gets important contributions from soliton sectors.

On the sphere, of course, the second observation above is immaterial, since there are no noncontractible loops on Σ and hence no solitons. To make the first observation concrete in our present notation we will assume that we have

$$\begin{aligned} b\bar{b} &\propto e^{q\varphi} \\ c\bar{c} &\propto e^{-q\varphi} \end{aligned} \tag{1.3}$$

and find q, starting with the case of spin-1/2. First note that (1.2) gives a two-point function with singularity

$$\langle \varphi(z)\varphi(w)\rangle \sim -\frac{1}{16\pi^2}\log|z-w|^2 \ .$$

In proving this we have used the fact that

$$\bar{\partial}_P\partial_P \log|z_P - z_Q|^2 = 2\pi i\,\delta_Q(P) \tag{1.4}$$

where $\delta_Q(P)$, the delta function, is a (1,1)-form at P. (1.4) is easily shown by integrating on a small disk and using deRham's theorem.

The classical stress tensor of (1.2) is

$$T = -8\pi^2\partial\varphi\partial\varphi \ , \tag{1.5}$$

a $(2,0)$-tensor. Its quantum version looks the same but with normal ordering to remove self-contractions. T is defined so that in operator products [12]

$$T(z)\psi(w) \sim \frac{h}{(z-w)^2}\,\psi + \text{less singular terms} \quad .$$

where h is the spin of ψ. One can check the normalization of (1.5) by showing that $\partial\varphi$ has spin one. Then

$$T(z)e^{\pm q\varphi(w)} \sim \left(\frac{1}{z-w}\right)^2 \frac{q^2}{16\pi^2}\cdot\left(-\frac{1}{2}\right) + \cdots \quad .$$

Thus choosing $q = 4\pi i$ gives $b\,,c$ of spin one half.

It is now simple to show the equivalence of (1.1) and (1.2), still in spin $1/2$. First, the zero-point functions agree up to an overall multiplicative constant:

$$Z_b[g] = \text{const.} \cdot Z_f[g] \ . \tag{1.6}$$

Each side of (1.6) is a functional of a metric chosen on Σ to regularize the theories. (1.6) holds because both bosons and nonchiral fermions are free of gravitational anomalies, and both have the *same* anomalous variation under Weyl transformations. Since on the sphere every metric g is related to any reference g_0 by coordinate and Weyl transformations [19], we see that (1.6) really does hold up to a constant. As for the higher correlation functions, they follow from the fact that

$$\begin{aligned} \langle e^{4\pi i\varphi(z)} e^{-4\pi i\varphi(w)} \rangle = \exp[16\pi^2 \langle \varphi(z)\varphi(w) \rangle] \sim \frac{1}{|z-w|^2} \\ \sim \langle b\bar{b}(z) c\bar{c}(w) \rangle \quad . \end{aligned} \tag{1.7}$$

For spins other than one half, the bosonic action (1.2) must be supplemented by a term S_2 explicitly breaking invariance under shifts of φ, since the corresponding transformation of b, c is anomalous. We cannot discuss this term here; see [1],[2].

2. The Torus

When we move up in complexity from the sphere to the torus we at once encounter two novel features. First, it is no longer true that every metric is related to every other by coordinate and Weyl symmetries: a residual "moduli space" of inequivalent metrics remains [19]. Secondly, there is now a variety of inequivalent bundles over Σ in which b and c could take values. Thus we need some way to tell the bosons what spin structure they are to mimic. Fortunately on the torus we still have a canonical formalism, which we can use to address the problem.

A spin bundle L is a bundle whose square is the cotangent to Σ. On the torus the cotangent is trivial, so that we can parametrize the four possible choices for L by measuring the difference between L and one particular spin bundle, the trivial one. If we take the torus defined by the unit square in $\mathbf{C}$, we then have

$$\begin{aligned} b(1) &= e^{2\pi i\theta}\, b(0) \\ b(i) &= e^{-2\pi i\phi}\, b(0) \end{aligned} \quad , \tag{2.1}$$

and similarly for c. In this talk I will restrict to untwisted spin bundles, i.e. θ, $\phi = 0$ or $\frac{1}{2}$. The field c then lives in the bundle $K \otimes L^{-1} \cong L$. Also we will not consider $\theta = \phi = 0$ since with this choice $Z_f = 0$ due to the fermion zero mode; that is, we consider only the three "even" spin structures.

Certainly (1.6) cannot hold as it stands on the torus, since one side depends on θ, ϕ while the other does not. Instead one expects that the bosonic theory with action S_1 should give the *sum* over all spin structures of the corresponding fermionic theories. Detailed calculation affirms this expectation [3]. To bosonize just one spin structure one must add to S_1 a new term S_3 depending on θ, ϕ. We will find S_3 by canonically quantizing and applying the physical lesson that fermions correspond to solitons of the field φ.

Since moduli will not play an important role in this talk we will continue to take the torus to be the unit square in C, with identifications and (2.1). We will quantize with euclidean time running up the imaginary axis. Then the partition functions $Z_f(\theta, \phi)$ are traces over the Ramond and Neveu-Schwarz Hilbert spaces, for $\theta = 0$, $\frac{1}{2}$ respectively. ϕ on the other hand denotes the boundary conditions in time. We then have

$$Z_f(\tfrac{1}{2}, \tfrac{1}{2}) = Tr_{NS} e^{-H}$$
$$Z_f(\tfrac{1}{2}, 0) = Tr_{NS} (-)^F e^{-H}$$

Hence $Z_f(\frac{1}{2}, \frac{1}{2}) \pm Z_f(\frac{1}{2}, 0)$ is a trace over the even (resp. odd) fermion-number space. It must therefore in the bosonic language receive contributions only from states of even (resp. odd) soliton number.

A functional integral over φ includes a sum over *all* soliton sectors. Thus our modification to the action S_3 must have the effect of weighting the various winding sectors in such a way as to cancel the odd-soliton contributions to $Z_b(\frac{1}{2}, 0) + Z_b(\frac{1}{2}, \frac{1}{2})$, and so on. A possible set of weighting factors is shown in Fig. 1. In the left column we have shown the spin structures for the fermionic system. On the right the boxes represent the contributions to the bosonic path integral from the winding sectors with $(n, m) = (\frac{1}{2}\cdot\text{even}\ ,\ \frac{1}{2}\cdot\text{even}\ )$, $(\frac{1}{2}\cdot\text{even}, \frac{1}{2}\cdot\text{odd})$, and so on. Each box on the right thus represents a sum $Z_{b,i}^{\text{partial}}$, $i = 1, \ldots 4$ over

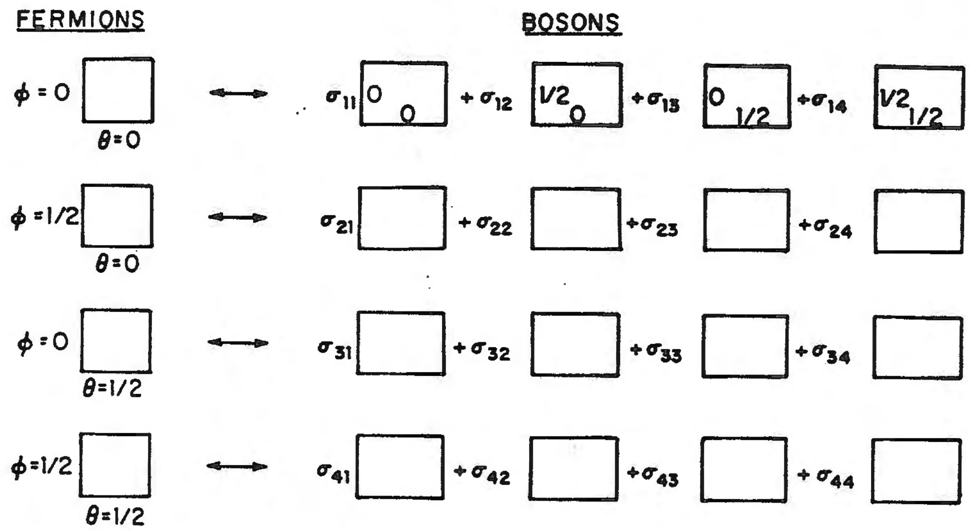

Fig. 1: Weighting the soliton sectors.

an infinite subclass of field configurations. The effects of S_3 are in the phases σ_{ij}: we have $Z_b(\frac{1}{2},0) = \sum_{i=1}^{4} \sigma_{3i} Z_{b,i}^{\text{partial}}$, etc.

The conditions that $Z_b(\frac{1}{2},0) \pm Z_b(\frac{1}{2},\frac{1}{2})$ have no odd (resp. even) soliton contribution now imply that in these combinations $Z_{b,3}^{\text{partial}}$ and $Z_{b,4}^{\text{partial}}$ must cancel (resp. $Z_{b,1}^{\text{partial}}$ and $Z_{b,2}^{\text{partial}}$), i.e.

$$\sigma_{33} = -\sigma_{43}\,, \qquad \sigma_{34} = -\sigma_{44}$$

$$\sigma_{31} = \sigma_{41}\,, \qquad \sigma_{32} = \sigma_{42}$$

We can get further conditions by letting the torus degenerate with very long time. Then only the ground state contributes to the fermionic partition function. With zeta-function regularization the split ground state of the Ramond sector has nonzero energy, so that $Z_f(0,\phi) \to 0$. Take $\phi = \frac{1}{2}$. On the bosonic side, only the zero-soliton sectors contribute, but they do so independently of the time winding $2m$, in the limit. Thus the two contributing partial sums $Z_{b,1}^{\text{partial}}$ and $Z_{b,2}^{\text{partial}}$ must cancel from $Z_b(0,\frac{1}{2})$, so that $\sigma_{21} = -\sigma_{22}$.

We must also impose the condition of modular invariance on the σ_{ij}. Requiring for example that our prescription be unchanged when we exchange the roles of space and time gives relations like $\sigma_{21} = \sigma_{31}$, $\sigma_{22} = \sigma_{33}$, etc. Requiring that the torus with corners 0, 1, $i+2$, $i+1$ give the same answers as the unit square gives $\sigma_{31} = \sigma_{41}$, $\sigma_{33} = \sigma_{44}$, etc. These conditions fix σ_{ij}, $i \neq 1$ up to an overall constant, which we take to be unity:[1]

$$\sigma_{ij} = \begin{cases} -1 \,, & i = j \\ +1 \,, & i \neq j \end{cases} \quad . \tag{2.2}$$

We can restate (2.2) in a way which makes its modular invariance obvious. Note again that the four spin structures split into one "odd" one (the trivial bundle) and three "even" ones. The names indicate that the number of zero modes of $\bar{\partial}_L$ is odd (= 1) or even (= 0) in the respective cases [20]. Here $\bar{\partial}_L$ is the Cauchy-Riemann operator coupled to the holomorphic bundle L [19]. Note also that the 1-form $d\varphi$ corresponds to a flat bundle $F(d\varphi)$, the one whose holonomy around a loop γ is given by $d\varphi$. The prescription (2.2) simply says that we must add to S_1 the topological term

$$S_3 = i\pi\sigma(L \otimes F(d\varphi)) \quad , \tag{2.3}$$

where $\sigma(L')$ is 0 or 1, depending on whether L' is even or odd. Note that $L \otimes F(d\varphi)$ really is a spin bundle, since with our normalization $d\varphi$ defines a half-integral cohomology class.

The proof that $S_1 + S_3$ is the correct bosonic action, as well as the generalization to arbitrary twists θ, ϕ, comes after we generalize everything to genus $g \geq 1$ [1],[2]. We emphasize, however that (1.2) and (2.3) are by now a very plausible prescription on the torus, and that in higher genus essentially no new physics will be needed.

3. Higher Genus

Various problems beset us when we try to generalize this discussion to world-sheets with more than one handle. For one thing, the anomaly term S_2 alluded to above becomes problematical since it involves not $d\varphi$ but φ itself, and so is ill-defined for multiple-valued field configurations φ. Secondly, when the cotangent to Σ is no longer trivial, then the trivial bundle is not a spin bundle and we

---

[1] We can use the same reasoning to fix the σ_{1j}. However, to fix the relative sign of σ_{1j} relative to the others we must interpolate between the spin structures, as we do in [1],[2].

cannot refer all spin bundles to it as we did in (2.1). Instead we need to choose a reference spin bundle — but any such choice automatically breaks coordinate invariance, since a diffeomorphism of Σ in general transforms a chosen spin structure into another, different one. Remarkably, these two problems cancel each other. The extra term in S_2 needed to make it well-defined gives it a coordinate transformation which cancels that of S_3 when a suitable reference spin bundle is chosen.

Also on higher genus surfaces there is no obvious operator formalism. Questions about bosonization then turn into path-integral questions about functional determinants. To prove that bosonization works we had to have recourse to methods of algebraic geometry. These are described in an introductory way in [19], and the full details of their application are in [2].

References

[1] L. Alvarez-Gaumé, J. B. Bost, G. Moore, P. Nelson, and C. Vafa, "Bosonization in arbitrary genus," Phys. Lett. **B178** (1986) 41.

[2] L. Alvarez-Gaumé, J. B. Bost, G. Moore, P. Nelson, and C. Vafa, "Bosonization on higher genus Riemann surfaces," Harvard Preprint HUTP-86/A062.

[3] L. Alvarez-Gaumé, G. Moore, and C. Vafa, "Theta functions, modular invariance, and strings," Comm. Math. Phys. **106**(1986)1.

[4] J. Bost and P. Nelson, "Spin-1/2 bosonization on compact surfaces," Phys. Rev. Lett. **57** (1986) 795.

[5] R. Shankar, Phys. Lett. **92B**, 333 (1980); E. Witten, in *Fourth workshop on grand unification, ed. H. Weldon, P. Langacker, and P. Steinhardt*, (Birkhauser, 1983).

[6] R. Nepomechie, "Nonabelian bosonization, triality, and superstring theory," Phys. Lett. **178B** (1986) 207; **180B** (1986) 423; L. Brown and R. Nepomechie, Non-abelian bosonization: current correlation functions," Washington preprint.

[7] D. Gross, J. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **54**(1985)502; Nucl. Phys. **B256**(1985)253; **B267**(1986)75.

[8] D. Friedan, E. Martinec, and S. Shenker, Phys. Lett.**B160**(1985)55.

[9] V. Knizhnik, "Covariant superstring amplitudes from the sum over fermionic surfaces," Phys. Lett. **178B** (1986) 21.

[10] For a history, see D. Wolf and J. Zittartz, Zeit. Phys. **B51** (1983) 65.

[11] R. Marnelius, Nucl. Phys. **B211** (1983) 14.

[12] D. Friedan, E. Martinec, and S. Shenker, "Conformal invariance, supersymmetry, and string theory," Nucl. Phys. **B271** (1986)93.

[13] W. Siegel and B. Zwiebach, Nucl. Phys. **B263** (1986) 105.

[14] S. Coleman, Phys. Rev. **D11** (1975) 2088.

[15] S. Mandelstam, Phys. Rev. **D11** (1975) 3026.

[16] T. Banks *et al.*, Nucl. Phys. **B108** (1976) 119.

[17] I. Frenkel, J. Funct. Anal. **44** (1981) 259.

[18] P. Goddard and D. Olive, in *Vertex operators in mathematics and physics*, ed. J. Lepowsky, S. Mandelstam, and I. M. Singer (Springer, 1985).

[19] P. Nelson, "Lectures on strings and moduli space" HUTP-86/A047, to appear in Phys. Reports.

[20] M. Atiyah, "Riemann surfaces and spin structures," Ann. Sci. Ec. Norm. Sup. 4^e série **4** (1971) 47.