December 1991

The Control of Natural Motion in Mechanical Systems

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Abstract

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This paper offers a unified exposition of some recent results [13, 12, 15] heretofore scattered throughout a more mathematically oriented literature that strengthen our original suggestion [8, 9] concerning the utility of controlling natural motion as a means of simultaneously encoding, planning and effecting tasks in mechanical systems. The chief theoretical insight, Theorem 2, is a global global version of Lord Kelvin's century old result on the dissipation of total energy. Establishing this extension yields a rather general design principlethe notion of a navigation function-that seems to have useful application in a variety of settings. Roughly speaking, it offers a checklist of criteria for achieving the strongest possible convergence properties allowed on a configuration space by a smooth and bounded force/torque control strategy. Some simple examples introduced here may aid the exposition of these ideas. A sequel [10] to this paper illustrates how the ideas may be applied in more realistic settings.

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The Control of Natural Motion in Mechanical Systems[1]

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1 Introduction

This paper offers a unified exposition of some recent results [13, 12, 15] heretofore scattered throughout a more mathematically oriented literature that strengthen our original suggestion [8, 9] concerning the utility of controlling natural motion as a means of simultaneously encoding, planning and effecting tasks in mechanical systems. The chief theoretical insight, Theorem 2, is a global global version of Lord Kelvin's century old result on the dissipation of total energy. Establishing this extension yields a rather general design principle—the notion of a *navigation function*—that seems to have useful application in a variety of settings. Roughly speaking, it offers a checklist of criteria for achieving the strongest possible convergence properties allowed on a configuration space by a smooth and bounded force/torque control strategy. Some simple examples introduced here may aid the exposition of these ideas. A sequel [10] to this paper illustrates how the ideas may be applied in more realistic settings.

By "natural motion" is meant the unforced response of a closed loop dynamical system resulting from the introduction of a suitable feedback law to some dynamical "plant" with inputs and outputs. Roughly speaking, the appeal to natural motion attempts to place all of the "intelligence" required for proper functioning in the analog computer comprised of the plant's intrinsic dynamics. In the case of a mechanical system—a plant such as a robot or a satellite governed primarily by the interchange of power between various sources of potential and kinetic energy—the intrinsic analog computer is a set of double integrators (one for each degree of freedom) that manifest Newton's Second Law. This motivating principle owes much to fundamental work of Hogan [6], and the idea of a navigation function in particular represents a refinement of Khatib's original work in artificial potential fields [7]. The utility of total energy as a basis for controller design in mechanical systems has been championed for some time by Arimoto [16].

The class of tasks amenable to the methods introduced in this paper includes merely setpoint problems—albeit reasonably complicated ones as the sequel [10] will hopefully demonstrate. Thus, in its appearance here, this design philosophy merely represents an extension to nonlinear mechanical systems of the class of "PD controllers." In a parallel program of research [4, 3] we have begun to show how certain tasks requiring periodic steady-state behavior may be amenable to similar treatment. Of course, the design principle can no longer be as simple as a navigation function in such cases. It remains to be seen how general a class of tasks the natural control point of view can address.

The use of total energy for control applications has been rediscovered many times in the engineering community—for example, consult the historical sketch in [11]. In contrast, the extent to which global conclusions about the phase portrait of a mechanical system may be drawn from analysis of the total energy function appears not to have been addressed in the previous literature. Although nonlinear dynamical systems give rise to extremely complicated behavior in general, most engineers understand intuitively that a "dissipative" system is in some sense very simple. Thus, the formal confirmation in the present case, Theorem 2, should not seem very surprising. Hopefully, the design principles embodied in the notion of a navigation function which emerge from this theorem will justify the effort involved in its presentation.

2 Dissipative Mechanical Systems

This section introduces the setting for these ideas. A more elaborate exposition is given in [12], and all of this work is strongly influenced by the excellent text of Abraham and Marsden [1].

2.1 Σ, The Mechanical Control System. Given a configuration space, $\mathcal{Q}$, the *phase space*, $\mathcal{P}$—the union of the vector space of all infinitesimal motions permissible at each configuration—models all possible velocities a mechanical system may experience. The *wrench space*, $\mathcal{U}$—the union of the dual vector space to the infinitesimal motions permissible at each

[1]This work was supported in part by the National Science Foundation under grant no. DMC-8505160.

Contributed by the Dynamic Systems and Control Division for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received by the Dynamic Systems and Control Division May 11, 1987; revised manuscript received March 11, 1991. Associate Editor: M. Tomizuka.

configuration—models all possible forces to which a mechanical system may be subjected.

We will narrow the scope of the presentation to consider a *mechanical control system*, as determined by the ordered pair, consisting of an n-dimensional configuration space and a generalized inertia tensor, M

$$\Sigma \triangleq (\mathcal{Q}, M), \tag{1}$$

that gives rise to the second order vector field f_Σ: $\mathcal{P} \times \mathcal{U} \to \mathbb{R}^{2n}$ derived from Lagrange's equations [12, 1]

$$\left(\frac{d}{dt} D_{\dot{q}} - D_q\right) \dot{q}^T M(q) \dot{q} = u.$$

Example 2.1.1. *One Degree of Freedom Prismatic Robot.* The configuration space is some closed real interval, say $\mathcal{Q}_P \triangleq [-\epsilon, 1]$. Its boundary consists of the endpoints, $\partial\mathcal{Q}_P = \{-\epsilon, 1\}$. The phase space arising from $\mathcal{Q}_P$ is the closed vertical strip in $\mathbb{R}^2$, $\mathcal{P}_P = [-\epsilon, 1] \times \mathbb{R}$. Its boundary is formed by the two vertical lines through the endpoints of the configuration space

$$\partial\mathcal{P}_P = (-\epsilon \times \mathbb{R}) \cup (1 \times \mathbb{R}).$$

An ideal actuator applies arbitrary and instantaneous forces, u. Supposing that the robot has mass M, the mechanical control system,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f_{\Sigma_P}((x,y),u) = \begin{bmatrix} y \\ M^{-1}u \end{bmatrix}, \tag{2}$$

is a double integrator.

Example 2.1.2. *One Degree of Freedom Revolute Robot.* If there are joint limits then this is the same situation as in Example 2.1.1. Otherwise, the configuration space consists of the planar rotations,

$$\mathcal{Q}_R = \mathrm{SO}(2) \triangleq \{R \in \mathbb{R}^{2\times 2}: R^T R = \mathrm{I} \text{ and } |R| = 1\},$$

that can be put into smooth one-to-one correspondence with the circle, $\mathbb{S}^1$, via $R(\theta) = \exp\{\theta J\}$ where J is the unit skew symmetric matrix of $\mathbb{R}^{2\times 2}$. There is evidently no boundary, $\partial\mathcal{Q}_R = \emptyset$, and the phase space is the cylinder, $\mathcal{P}_R = \mathbb{S}^1 \times \mathbb{R}$. An ideal actuator applies arbitrary and instantaneous torques, u. Supposing that the robot has moment of inertia M, the Lagrangian dynamics

$$\begin{bmatrix} \dot{R} \\ \dot{\omega} \end{bmatrix} = f_{\Sigma_R}((R,\omega),u) \triangleq \begin{bmatrix} \omega J R \\ M^{-1}u \end{bmatrix}, \tag{3}$$

give rise to the same double integrator, f_{Σ_R} as in (2), with the exception that the setting is a cylinder rather than a rectangle.

Thus, we limit our consideration of mechanical systems to fully actuated holonomically constrained physical mechanisms governed by Lagrangian dynamics. The absence of external disturbance forces such as the gravitational potential presumes the availability of a model permitting their exact cancellation via the actuators. This, of course, excludes a great variety of important situations.[2] The assumption of an ideal wrench source—full actuation—excludes vehicles with controlled surface angles, or indeed, any nonholonomically constrained mechanism. It completely ignores the likely circumstance of unactuated dynamics arising from transmissions or imperfect actuators. Similarly, the restriction to tensorial inertia precludes consideration of many mechanisms that interact physically with their environments such as underwater vehicles whose effective inertia is governed by the surrounding fluid and may vary with velocity. Assessing the extent to which these ideas may apply to such situations falls beyond the scope of this paper.

2.2 Γ, The Gradient "Planning" System. Consider the class of twice differentiable real valued functions $\varphi \in C^2[\mathcal{Q}, \mathbb{R}]$ on the configuration space, $\mathcal{Q}$, of the mechanical system, Σ. The associated wrench or *covector field*, $D\varphi$, is related to the *gradient vector field*, gradφ, via the inertia,

$$\mathrm{grad}\varphi \triangleq M^{-1} D\varphi.$$

Thus, a *gradient system* (which in this paper will always mean the dynamics of the negative gradient vector field) is determined by the triple consisting of a configuration space, an inertia tensor, and a scalar valued function,

$$\Gamma \triangleq (\mathcal{Q}, M, \varphi). \tag{4}$$

One calls φ a *Morse function* if its hessian (matrix of second derivatives) is non-singular at every critical point [5]. A vector field is *transverse* to a surface if it is never tangent [5]: in this paper "transverse" will specifically mean "pointing away from the interior on the boundary surface." Morse functions with transverse gradients give rise to systems, Γ, whose trajectories have very simple behavior which may be summarized as follows [12, Proposition 2.1]: (*i*) trajetories originating in $\mathcal{Q}$ remain there for all future time; (*ii*) every trajectory approaches an extremum of φ in steady state; (*iii*) there is a dense open set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ from which all trajectories tend asymptotically toward the local minima of φ.

So easily characterized a positive limit set is most unusual in the dynamical systems literature and we will take the attributes of Γ described above as a model for desired asymptotic behavior of some the closed loop mechanical system. Note that application of the wrench, $D\varphi$, as an input to the mechanical system, results in a "lift" of the gradient vector field over the phase space. The gradient system, Γ, thus achieves the character of a "planning" system whose limit behavior we wish to carry over automatically to the eventual physical dynamics through the controller.

Example 2.2.1. Γ_H, *A Hook's Law Gradient System.* Consider the end-point task of moving from anywhere in the configuration space of Example 2.1.1 to a desired position, $q_d = 0$. Any Hook's Law spring potential, $\varphi_H \triangleq 1/2\, K_1 q^2$, (where $K_1 > 0$) will result in a valid gradient system. In particular, adopting the inertia of Example 2.1.1, the gradient system, $\Gamma_H = (\mathcal{Q}_P, M, \varphi_H)$ associated with this setup is

$$\dot{q} = -\mathrm{grad}\varphi_H = -M^{-1} K_1 q.$$

Clearly, gradφ_H is transverse since grad$\varphi_H(1) > 0$ and grad$\varphi_H(-\epsilon) < 0$. Moreover, φ_H always has a positive definite hessian, $D^2\varphi_H = K_1 > 0$. Thus, we are guaranteed that all solutions of Γ_H tend to some extremum of φ_H (a minimum) on $\mathcal{Q}_P$, hence, every solution of Γ_H tends to q_d as desired.

Example 2.2.2. *An Induced Gradient System.*[3] Suppose in Example 2.1.2 that a gripper frame origin, $g(\theta) = R(\theta)\,[{}^1_0]$, has been established, and that it is desired to move the gripper to coincide with the workspace goal location, $w_d = [{}^1_0] = R(0)\,[{}^1_0]$. A two degree of freedom Hook's Law potential, $\varphi_H = 1/2\,[w - w_d]^T [w - w_d]$, may be composed with the "kinematic map," g, to obtain an induced potential,

$$\varphi_T \triangleq \varphi_H(g(\theta))$$

$$= \frac{1}{2}\,[1, 0][R(\theta) - R(0)]^T\,[R(\theta) - R(0)]\,[{}^1_0]$$

[2]The author is indebted to one of the reviewers for helping clarify the physical limitations imposed by this setup.

[3]This somewhat ungainly derivation of φ_T is undertaken as a means of motivating both the satellite tracking section and the induced kinematics section of the sequel paper [10].

$$= \frac{1}{2}\, tr\ \{(2I - R(\theta)^T R(0) - R(0)^T R(\theta))\ [1,\ 0]\ [{}^{1}_{0}]\}$$

$$= 1 - tr\ \{R(\theta)^T\ [1,\ 0]\ [{}^{1}_{0}]\} = 1 - \cos\,\theta.$$

The reader may readily verify that φ_T varies smoothly between 0 (attained when $R = R(0)$) and 2 (attained when R is 180 deg away from $R(0)$), taking intermediate values at all other points. Adopting the inertia of Example 2.1.2, the gradient system, $\Gamma_T = (\mathcal{Q}_R, M, \varphi_T)$ associated with this setup is

$$\dot{\theta} = -\,\mathrm{grad}\varphi_T = -M^{-1}\sin\theta$$

The critical points of φ_T occur at $\theta = n\pi$, $n = \ldots, -1, 0, 1, \ldots$ Its hessian is unity at $\theta = 2n\pi$ and -1 at $\theta = (2n+1)\pi$. All trajectories of Γ_T tend to the minima of φ_T corresponding to the rotation $R(0)$ except the (unstable) equilibrium solutions at its maxima corresponding to the rotation 180° degrees away from $R(0)$.

2.3 Δ, The Dissipative Mechanical System. The chief object of study in this paper, the dissipative mechanical system, Δ, arises from superimposing a dissipative term, $d(p)$, with the property $p_2^T d(p_1, p_2) < 0$, upon the wrench associated with φ.

$$u \triangleq d(q,\dot{q}) + [D\varphi]^T(q), \tag{5}$$

A dissipative mechanical system,

$$\Delta \triangleq (\mathcal{Q}, M, \varphi,\ d), \tag{6}$$

is defined by the vector field,

$$f_\Delta(p) \triangleq f_\Sigma(p, d + D\varphi). \tag{7}$$

Example 2.3.1 Σ_{HR}, *A Hook-Rayleigh Dissipative System.* Consider a Rayleigh damping law, $d_R(q,\dot{q}) \triangleq -K_2\dot{q}$, where $K_2 > 0$. Applying this to Γ_H of Example 2.2.1 yields a "Hook-Rayleigh" dissipative mechanical system, $\Delta_{HR} = (\mathcal{Q}_P, M, \varphi_H, d_R)$, given by

$$\dot{p} = f_{\Delta_{HR}}(p) = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix} p.$$

Interpreting the total mechanical energy as a Lyapunov function yields:

Theorem 1 (Lord Kelvin (1886) [17, §345]). *If q_0 is a local minimum of φ in $\mathcal{Q}$, then $(q_0,\ 0)$ is a stable equilibrium state of the dissipative mechanical system, Δ (6), in $\mathcal{P}$.*

3 Controlling the Global Steady State Properties of Dissipative Mechanical Systems

Theorem 1 reveals certain analogies between the local state behavior of Δ in $\mathcal{P}$ and that of Γ in $\mathcal{Q}$. We must now address the extent to which this correspondence may be made global.

3.1 Problems With the Global Extension. There are two technical obstacles to extending Lord Kelvin's observation. As it turns out, neither of these will have serious practical implications, but both must be addressed in order to refine the criteria to be placed upon φ.

3.1.1 Finite Escape. Although Theorem 1 precludes unbounded trajectories, finite escape might still occur across the boundary of the phase space. Appealing to intuition, consider a mass rolling around on a terrain under the influence of the earth's gravitational potential and viscous friction due to air drag. Tranversality simply implies that the terrain slopes away from any forbidden region (the boundary of configuration space). Yet a mass traveling with sufficient kinetic energy could roll uphill and crash through the boundary.

Example 2.1.1. (continued). Although grad φ_H, from Example 2.2.1, is directed toward the interior of $\mathcal{Q}_P$, on the boundary, its "lift" in $f_{\Delta_{HR}}$ of Example 2.3.1 is directed *away* from the interior of $\mathcal{P}_P$ on the upper half of the line through the point $[1,\ 0]$ and the lower half of the line through $[-\epsilon, 0]$. Consequently, it may be observed that the trajectory of $f_{\Delta_{HR}}$ through every initial condition in a neighborhood of these open half line segments must escape from $\mathcal{P}_P$ in finite time.

It is intuitively clear and may be readily proven in complete generality [12, Lemma 3.1] that solutions of any dissipative mechanical system originating on a boundary point of the configuration space (i.e., actually touching an obstacle) will have finite escape (i.e., crash into the obstacle) if their nonzero initial velocity vector points in the wrong direction. However, the situation is much worse in general.

Example 2.1.1. (continued). If $\delta \triangleq \sqrt{K_2^2 - 4K_1} < 0$, then the system is underdamped. Now all solutions of Δ_{HR} originating at zero velocity from any configuration in the subinterval of $\mathcal{Q}_P$,

$$\epsilon\ \exp\ \{K_2/\delta\pi\} \le q \le 1,$$

will escape from $\mathcal{P}_P$ in finite time—they will crash into the left hand boundary at the configuration $q = -\epsilon$.

Thus, while initial conditions of the gradient system, Γ_H, remain within $\mathcal{Q}_P$ for all time and eventually reach q_d, solutions of Δ_{HR} originating well within the interior of the configuration space may nevertheless crash into a configuration space obstacle *even if they start with zero initial velocity*!

Transversality was sufficient to avoid finite escape in the planning system, Γ, but, quite apparently, more is required to achieve the same property in Δ.

3.1.2 Spurious Equilibria. Theorem 1, like any local result, applies to some neighborhood of $(q_0, 0)$ which we may take to be a ball. In fact, a rather general result in dynamical systems theory dictates that the global domain of attraction (that is, the set of initial conditions whose trajectories asymptotically approach the stable equilibrium state) be topologically equivalent to a ball [2]. But since our configuration spaces will almost never be topologically equivalent to a ball (Example 2.1.1 represents the rare positive case), this implies that global asymptotic stability is impossible in general.

Example 2.1.2. (continued). It is immediately clear that no gradient system on SO(2), can have a globally asymptotically stable equilibrium state. For every continuous function on a compact set attains both a minimum and a maximum on that set. Thus, Γ_T, with one stable and one unstable equilibrium state (considered as rotation matrices rather than angles) possesses the strongest possible convergence properties that any smooth dynamical system on SO(2) may have. In fact, this is good enough from any practical point of view since the maximum is the only initial condition whose trajectory fails to converge as desired, and the probability of starting exactly on that configuration is zero.

The properties of gradient systems reviewed in Section 2.2 imply that the absence of global asymptotic stability presents no real problem in general since the initial conditions whose trajectories do not converge to a minimum form a closed and nowhere dense set in $\mathcal{Q}$. In practice, this means that we will never encounter a trajectory of Γ that does not converge to a minimum. Under what conditions might the same be said of Δ?

3.2 Resolving the Problems.

3.2.1 Admissible Functions Preclude Finite Escape. Let b denote the lowest value φ attains on the configuration space boundary (if there is no boundary, then take $b = +\infty$). It is intutitively clear and can be readily demonstrated [12, Proposition 3.2] that the "lowest boundary total energy set,"

$$\mathcal{E}_b \triangleq \{q,\dot{q})\epsilon\mathcal{P}: \frac{1}{2}\,\dot{q}^T M\dot{q}+\varphi(q)\leq b\},$$

is positive invariant under Δ—that is trajectories originating in $\mathcal{E}_b$ never leave it. The contrary would incur a net increase in total energy. $\mathcal{E}_b$ is in some sense the largest subset of $\mathcal{P}$ that can be bounded entirely by a total energy surface and thus be invariant. Taking any larger value, $b' > b$, gives rise to a set, $\mathcal{E}_{b'}$, whose boundary includes some of $\partial\mathcal{P}$ from which finite escape is unavoidable as seen in Section 3.1.1. Unfortunately, $\mathcal{E}_b$, in general, is too small, since it will not include many perfectly valid configurations.

Example 2.3.1. (continued). The total energy function for Δ_{HR} is

$$\eta=\frac{1}{2}\,K_1 p_1^2+\frac{1}{2}\,p_2^2.$$

The bounded energy set for energy level $\eta \equiv 1$ is a truncated ellipse just touching (1, 0) and tangent to the vertical line $1 \times \mathbb{R}$ comprising the right hand boundary of $\mathcal{P}_P$. This ellipse is truncated on the left hand side of the plane by the a line segment contained in the left-hand boundary, $-\epsilon \times \mathbb{R}$. Thus $\mathcal{E}_1$ is bounded only partially by a total energy surface. Trajectories originating in this set, cannot escape through the ellipsoidal portion of the boundary but, as we have seen, certainly can escape through the left hand truncating line. On the other hand, φ_H takes its lowest value, b, on $\partial\mathcal{Q}_P$, at $-\epsilon$, that is $b = 1/2\ \epsilon^2 K_1 = \varphi_H(-\epsilon)$. Thus, $\mathcal{E}_b$, the largest subset of $\mathcal{P}_P$ completely enclosed by a total energy surface, is positive invariant. But no configurations beyond $q = +\epsilon$ are included in this "safe" set.

We can surely design potential functions φ that explode to an infinite height on the boundaries of the configuration space and thus leave $\mathcal{P}$ invariant.[4] The obvious objection is that grad φ must also explode in magnitude as one approaches the boundaries of $\mathcal{Q}$ and our feedback control law, (5) could never be achieved by a physically realizable actuator with hard torque bounds. The requirement that φ be bounded and smooth seems a very important practical constraint. Smoothness (continuity of at least the second derivatives) affords a reliance on standard analytical tools from the theoretical point of view, and avoids "chattering" and other unrealistic demands on actuators from the practical point of view. For example, smooth functions on a compact set (in this case, $\mathcal{E}_b$) are bounded—thus the input function in (5) is guaranteed to be bounded.

The notion of admissibility reconciles the smoothness constraints with the imperative of incorporating all legal configurations within the "safe" set. Requiring φ to take the same "lowest" value, b, on all boundary points includes all of $\partial\mathcal{Q} \times 0$—that is, all boundary configurations at zero velocity—in $\mathcal{E}_b$. Requiring φ to take strictly lower values than b in the interior of $\mathcal{Q}$ includes each level surface of φ together with sufficiently small velocities in $\mathcal{E}_b$ as well. We will follow Hirsch [5], and say that a smooth Morse function on $\mathcal{Q}$ is *admissible* if it achieves its maximal value uniformly on $\partial\mathcal{Q}$ and nowhere else.

Example 3.2.1. *An Admissible Potential.* The boundary of $\mathcal{Q}_P$ in Example 2.1.1 may be represented as the zero level set of two appropriately chosen scalar valued functions, say

$$\beta_L(q) \triangleq |q+\epsilon|^2; \quad \beta_R(q) \triangleq |q-1|^2.$$

[4]In point of fact, this is exactly the method proposed by almost all previous proponents of "artificial potential fields." A notable exception is provided by Newman and Hogan [14] who obtained bounded solutions by recourse to discontinuous ("logical") gradients that we seek to avoid.

Retaining the Hook's law potential from Example 2.2.1 that properly encodes the desired destination, consider the new cost function,

$$\varphi_A=\frac{\varphi_H}{\varphi_H+\beta_L\beta_R}.$$

It attains its lowest value (zero) on the good configuration, q_d, and its highest value (one) on the bad configurations, $\partial\mathcal{Q}$.

3.2.2 Morse Functions Assure Convergence. Section 3.1.1 demonstrated that the conditions on φ which preclude finite escape behavior in Γ are not sufficient to do so in Δ. In contrast, once finite escape behavior has been ruled out, the condition which guarantees convergence in Γ is readily seen work equal effectively in Δ. The equilibrium states of Δ are exactly the critical points of φ at zero velocity, and they constitute the entire positive limit set of $\mathcal{E}_b$ under f_Δ [12, Proposition 3.3]. Moreover, their local stability properties are inherited directly from those of Γ—a minimum of φ corresponds to an asymptotically stable equilibrium state of f_Δ; maxima and saddles correspond to unstable equilibrium states [12, Lemmas 3.4, 3.5]. Finally, if φ is a Morse function, then f_Δ has analogous nondegeneracy properties admitting the conclusion all initial conditions, apart from a closed and nowhere dense subset of $\mathcal{E}_b$ have trajectories that approach an equilibrium state corresponding to a minimum [12, Proposition 3.6]. Thus questions of global convergence of Δ on $\mathcal{E}^{b_1}$ are reduced to questions of global convergence of Γ on $\mathcal{Q}$.

4 Conclusion: Navigation Functions Effect a Global Plan and Its Control

Say that φ, a twice differentiable function of $\mathcal{Q}$ taking values in the unit interval is a *navigation function* if it: (*i*) is a Morse function; (*ii*) is admissible, taking the value zero at some distinguished interior point, $q_0 \in \mathcal{Q}$, and the value unity on the boundary, $\partial\mathcal{Q}$ (or at some finite number of interior maxima if there is no boundary); (*iii*) has a unique minimum at q_0. The following summary of the reasoning above shows that Δ has global steady state behavior on $\mathcal{E}_1$ analogous to the global steady state behavior of Γ on $\mathcal{Q}$. Namely, at every configuration there may be found sufficiently small velocities from which initial conditions the trajectory of f_Δ will never intersect an obstacle and will tend asymptotically toward the distinguished point, $(q_0, 0)$.

Theorem 2 ([12]). *Let Δ be a dissipative mechanical system (6), and suppose that φ is a navigation function on $\mathcal{Q}$. Then (i) almost every initial condition whose total energy is less than unity has a trajectory which asymtotically approaches the minimum of φ (at zero velocity) and (ii) for all valid configurations there can be found a velocity such that the resulting point in $\mathcal{P}$ has total energy less than unity.*

4.1 Theoretical Remarks. Will there be situations where in fact no navigation function can be devised? In a recent paper [13] we have been able to answer this question quite unequivocally: we show that smooth navigation functions exist on any smooth manifold for any desired interior point, q_d. Thus, the examples Example 4.1.1 and Example 4.1.2 are not anomalous simple cases but represent the situation in general.

Example 4.1.1. *A Navigation Function for SO(2).* It is clear that 1/2 φ_T in Example 2.2.2 is a navigation function. Thus, almost all trajectories of $\Delta_T R = (\mathcal{Q}_R, M, \varphi_T d_R)$ approach the desired orientation, R_d.

Example 4.1.2. *A Navigation Function for the Interval* $[-\epsilon, 1]$. It can be shown [13] that φ_A in Example 3.2.1 has a single minimum at 0. Since it is clearly a Morse function, it now follows that almost all trajectories of $\Delta_A R = (\mathcal{Q}_P, M, \varphi_A, d_R)$ approach this desired configuration.

How hard is it to actually construct navigation functions? The sequel [10] will present more useful extensions of the special examples, φ_T and φ_A, above. While there are no general constructive results presently available, the invariance of the navigation properties under change of coordinates provides a useful hint of how to extend solutions in simple cases to obtain navigation functions in more complex settings [15].

4.2 Practical Remarks. It seems worth sketching how to obtain feedback controllers, (5), that respect torque limits from a navigation function, $\tilde{\varphi}$. Suppose that the actuators are capable of delivering torques whose magnitude does not exceed the bound $U < \infty$. Let G denote the least upper bound of $\|\text{grad } \tilde{\varphi}\|$ over $\mathcal{Q}$. Assume for the sake of concreteness that a Rayleigh dissipative field Example 2.3.1 will be used and denote its (induced operator) norm K. Let the inverse inertia tensor, $M^{-1}(q)$, have an (induced operator) norm that is bounded by the constant $1/M_0$ over all the configuration space. Now take in (5) the wrench associated with $\tilde{\varphi} = c\varphi$, where c is a constant scalar "gain" to be determined as follows. Assuming all initial conditions are chosen within $\mathcal{E}_c$[5] it follows that all trajectories have total energy less than c, hence, $\|\dot{q}\|^2 \leq c/M_0$, thus we require that $U > cG + K(c/M_0)^2$, guaranteed by

$$c \leq \frac{(K^2 + 4M_0GU)^{1/2} - K}{2GM_0^{1/2}}.$$

This represents a very conservative strategy, of course, and it would be useful to develop more theoretical insight into the nature of the transient response of Δ.

Now suppose that no contact with the configuration space boundaries is required. In particular, suppose we require that the clearance from the boundary be greater than some distance which it has been ascertained holds at all configurations, q, with the property $\varphi(q) < \varphi_0 < 1$. Then choosing a "speed limit," κ_0—the maximal allowable initial kinetic energy—such that $c = \kappa_0/(1 - \varphi_0)$ satisfies the previous inequality guarantees that all trajectories originating in $\mathcal{E}_c$ will remain the specified distance away from the boundaries and will not require excessive torques to do so. If contact with the boundaries is required then choose $\kappa_0 = 0$ and repeat the same argument.

4.3 Programmatic Remarks. Presumably, one important attribute of an "intelligent machine" is its *autonomy*—the ability to perform a task successfully without intervention from some higher level supervisor. Having once encoded a goal in terms of the steady state behavior of some dynamical system, a reasonable measure of the machine's autonomy is the relative size in state space of initial conditions that result in the desired steady state. From this point of view, the navigation function represents a design principle that bestows the most autonomy possible upon a mechanical system commanded to perform a task that can be encoded as a setpoint problem in the face of configuration constraints.

[5]It should be understood in this discussion that $\mathcal{E}_c$ and all further references to total energy involve the scalar function $1/2\ \dot{q}^T M(q)\dot{q} + \tilde{\varphi}(q)$.

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Transactions of the ASME®

Published Quarterly by The American Society of Mechanical Engineers

Transactions of the ASME, Journal of Dynamic Systems, Measurement, and Control (ISSN 0022-0434) is published quarterly (Mar., June, Sept., Dec.) for $125.00 per year by The American Society of Mechanical Engineers, 345 East 47th Street, New York, NY 10017. Second class postage paid at New York, NY and additional mailing offices.
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