2017

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Recommended Citation

Bhattacharya, B. B., & Mukherjee, S. (2017). Degree Sequence of Random Permutation Graphs. *The Annals of Applied Probability, 27* (1), 439-484. http://dx.doi.org/10.1214/16-AAP1207

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Abstract

In this paper, we study the asymptotics of the degree sequence of permutation graphs associated with a sequence of random permutations. The limiting finite-dimensional distributions of the degree proportions are established using results from graph and permutation limit theories. In particular, we show that for a uniform random permutation, the joint distribution of the degree proportions of the vertices labeled $\lceil nr_1 \rceil, \lceil nr_2 \rceil, \ldots, \lceil nr_s \rceil$ in the associated permutation graph converges to independent random variables $D(r_1)$, $D(r_2), \ldots, D(r_s)$, where $D(r_i) \sim \mathrm{Unif}(r_i, 1-r_i)$, for $r_i \in [0,1]$ and $i \in \{1,2,\ldots,s\}$. Moreover, the degree proportion of the mid-vertex (the vertex labeled n/2) has a central limit theorem, and the minimum degree converges to a Rayleigh distribution after an appropriate scaling. Finally, the asymptotic finite-dimensional distributions of the permutation graph associated with a Mallows random permutation is determined, and interesting phase transitions are observed. Our results extend to other nonuniform measures on permutations as well.

Keywords

Combinatorial probability, graph limit, limit theorems, Mallow's model, permutation limit

Disciplines

Physical Sciences and Mathematics

The Annals of Applied Probability
2017, Vol. 27, No. 1, 439–484
DOI: 10.1214/16-AAP1207

DEGREE SEQUENCE OF RANDOM PERMUTATION GRAPHS

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In this paper, we study the asymptotics of the degree sequence of permutation graphs associated with a sequence of random permutations. The limiting finite-dimensional distributions of the degree proportions are established using results from graph and permutation limit theories. In particular, we show that for a uniform random permutation, the joint distribution of the degree proportions of the vertices labeled $\lceil nr_1 \rceil, \lceil nr_2 \rceil, \ldots, \lceil nr_s \rceil$ in the associated permutation graph converges to independent random variables $D(r_1), D(r_2), \ldots, D(r_s)$, where $D(r_i) \sim \text{Unif}(r_i, 1 - r_i)$, for $r_i \in [0, 1]$ and $i \in \{1, 2, \ldots, s\}$. Moreover, the degree proportion of the mid-vertex (the vertex labeled $n/2$) has a central limit theorem, and the minimum degree converges to a Rayleigh distribution after an appropriate scaling. Finally, the asymptotic finite-dimensional distributions of the permutation graph associated with a Mallows random permutation is determined, and interesting phase transitions are observed. Our results extend to other nonuniform measures on permutations as well.

1. Introduction. Let $[n] := \{1, 2, \ldots, n\}$, and S_n denote the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. For any permutation $\pi_n \in S_n$ associate a *permutation graph* $G_{\pi_n} = (V(G_{\pi_n}), E(G_{\pi_n}))$, where $V(G_{\pi_n}) = [n]$ and there exists an edge (i, j) if $(i - j)(\pi_n(i) - \pi_n(j)) < 0$, that is, whenever i, j determines an *inversion* in the permutation π_n. The permutation graphs associated with π_n and π_n^{-1} are isomorphic, and the adjacency matrix $\boldsymbol{Q}_n = ((q_n(i, j)))$ associated with the permutation graph G_{π_n} is

$$q_n(i, j) := \begin{cases} 1, & \text{if } (i - j)\big(\pi_n(i) - \pi_n(j)\big) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the permutation graph G_{π_n} denote by $d_n(i) := \sum_{j=1}^n q_n(i, j)$ the *degree* of the vertex labeled $i \in [n]$. Note that $d_n(i)$ counts the number of $j \in [n]$ such that i, j is an inversion in the permutation π_n, and $\frac{1}{2} \sum_{i=1}^n d_n(i) = |E(G_{\pi_n})|$, is the number of inversions in π_n. Thus, the degree sequence of a permutation graph is the building block of the number of inversions, which has found versatile applications in integer sorting [30] and combinatorial searching [31], and has been widely

Received April 2015; revised March 2016.
MSC2010 subject classifications. Primary 05A05, 60F05; secondary 60C05.
Key words and phrases. Combinatorial probability, graph limit, limit theorems, Mallow's model, permutation limit.

studied for random permutations [18, 22, 30]. Other applications of permutation graphs are discussed later in Section 1.2.

In this paper, using results from the emerging literature on graph and permutation limit theories [26, 37], we study the asymptotic distribution of the degree sequence of random permutation graphs.

1.1. *Summary of results.* Given a sequence $\{\pi_n\}_{n\geq 1}$, with $\pi_n \in S_n$, of random permutations, the *permutation process* is the stochastic process on (0, 1] defined by [1]

$$\pi_n(t) := \frac{\pi_n(\lceil nt \rceil)}{n}. \tag{1.1}$$

Recall that $d_n(i) := \sum_{j=1}^n q_n(i, j)$ is the degree of the vertex labeled $i \in [n]$ in the permutation graph G_{π_n}. The quantity $d_n(i)/n$ will be referred to as the *degree proportion* of the vertex $i \in [n]$. The *degree process* is a stochastic process on (0, 1] obtained from the degree proportion

$$d_n(t) := \frac{d_n(\lceil nt \rceil)}{n}. \tag{1.2}$$

The following is an informal summary of the results obtained in the paper. The formal statements are given in Section 3.

(1) We show that the finite-dimensional convergence of the permutation process $\pi_n(\cdot)$, associated with a sequence $\{\pi_n\}_{n\geq 1}$ of random permutations, implies the convergence in distribution of permutation sequence $\{\pi_n\}_{n\geq 1}$ in the sense of Hoppen et al. [26], and the finite-dimensional convergence of the degree process $d_n(\cdot)$ (Theorem 3.1).

(2) As a consequence of the above, we derive the finite-dimensional convergence of the degree process of a uniformly random permutation graph, that is, the permutation graph associated with a permutation chosen uniformly at random from S_n (Corollary 3.3). Figure 1 shows the degree proportion of the vertices in the permutation graph associated with a uniformly random permutation of length $n = 10^5$.

(3) It follows from Corollary 3.3 that the degree proportion of the mid-vertex, that is, the vertex labeled $\lceil n/2 \rceil$, converges to 1/2 in probability (This is illustrated in the fan-like structure in Figure 1 around the point 1/2.) We show that $d_n(\lceil n/2 \rceil)/n$ has a CLT around 1/2 after an appropriate rescaling (Theorem 3.4).

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[1] Throughout the paper, π_n will be used interchangeably to denote both the permutation and the permutation process depending on the context. In particular, for $a \in [n]$ $\pi_n(a)$ will denote the image of a under the permutation π_n. On the other hand, for $t \in [0, 1]$ $\pi_n(t) = \pi_n(\lceil nt \rceil)/n$ will denote the permutation process evaluated at t. Similarly, d_n will be used to denote both the degree of a vertex and the degree process.

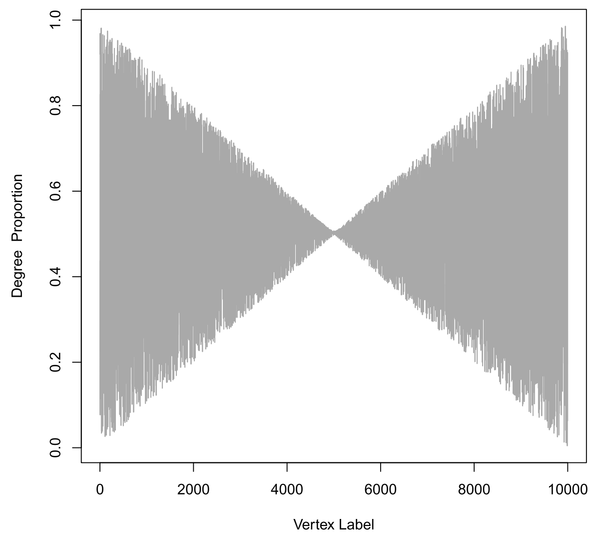

FIG. 1. *Degree proportion of the labeled vertices of the permutation graph associated with a random permutation of length* $n = 10^5$.

(4) In Theorem 3.5, we derive the asymptotics of the minimum degree $\delta(G_{\pi_n})$ in a uniformly random permutation graph G_{π_n}. We show that $\delta(G_{\pi_n})/\sqrt{n}$ converges to a Rayleigh distribution with parameter $\frac{1}{\sqrt{2}}$.

(5) Finally, we derive sufficient conditions for verifying whether the finite-dimensional distributions of a permutation process converge. These conditions can be easily verified for many nonuniform (exponential) measures on permutations. These conditions together with the recent work of Starr [47], can be used to explicitly determine the limiting distribution of the degree process for a Mallows random permutation, for all $\beta \in \mathbb{R}$ (Theorem 3.6). For each $a \in (0, 1]$, the limiting density of $d_n(\lceil na \rceil)/n$ has a interesting phase transition depending on the value of β: there exists a critical value $\beta_c(a)$ such that for $\beta \in [0, \beta_c(a)]$ the limiting density is a continuous function supported on $[a, 1 - a]$. However, for $\beta > \beta_c(a)$ the density breaks into two piecewise continuous parts.

1.2. *Related work.* Permutation statistics such as the number of inversions, the number of descents, and the length of the longest increasing subsequence are encoded in the permutation graph. For example, the number of edges $|E(G_{\pi_n})|$ in the permutation graph is the number of inversions in the permutation π_n. Similarly, the number of edges $(i, i + 1) \in E(G_{\pi_n})$, for $i \in [n]$, is the number of descents in π_n. The largest clique in a permutation graph corresponds to the longest decreasing subsequence in the permutation. Similarly, an increasing subsequence in a permutation corresponds to an independent set of the same size in the corresponding permutation graph.

Central limit theorems for the number of inversions and the number of descents in a uniform random permutation have been extensively studied (the interested reader may refer to Knuth [30] and Fulman [22] and the references therein for further details). Asymptotics for the maximum clique and the independent set in the permutation graph associated with a uniform random permutation follow from the seminal work of Baik et al. [4] on the length of the longest increasing subsequence. Recently, Mueller and Starr [42] and later Bhatnagar and Peled [6] studied the length of the longest increasing subsequence in a Mallows random permutation.

Acan and Pittel [1] studied when $\sigma(n,m)$, a permutation chosen uniformly at random among all permutations of $[n]$ with m inversions, is indecomposable (refer to [8, 21] and the references therein for more on indecomposable permutations). The probability $p(n,m)$ that $\sigma(n,m)$ is indecomposable, is same as the probability that the random permutation graph $G_{\sigma(n,m)}$ is connected. Acan and Pittel [1] showed that $p(n,m)$ has a phase transition from 0 to 1 at $m_n := (6/\pi^2)n\log n$. They also studied the behavior of $G_{\sigma(n,m)}$ at the threshold.

Permutation graphs were introduced by Pnueli et al. [44] and Even et al. [19], and have found applications in many applied problems, such as VLSI channel routing, scheduling, memory allocation [25], and genome rearrangement [3, 5]. In applications, the following pictorial description of permutation graphs is often useful: For $\pi_n \in S_n$ and $a \in [n]$, denote by $\ell_n(a)$ the line segment with endpoints $(a,0)$ and $(\pi_n(a),1)$. The endpoints of these segments lie on the two parallel lines $y=0$ and $y=1$, and two segments have a nonempty intersection if and only if they correspond to an inversion in the permutation. The permutation graph G_{π_n} is the intersection graph[2] of the segments $\{\ell_n(a)\}_{a=1}^n$. This representation of permutation graphs are useful in wire-routing problems in very large-scale integrated (VLSI) circuits [45]. For example, to represent the wiring between two circuit modules, an intersection between two line-segments denotes that the corresponding wires will cross each other in a planar layout. Note that the degree of a node in a permutation graph denotes how many times the segment intersects with other segments, and thus, determines the wire-routing complexity.

The limit theory of geometric intersection graphs was initiated by Diaconis et al. [16, 17]. Using the framework of graph limit theory introduced by Borgs et al. [10, 11, 37], they characterized the limits of interval graphs (intersection graph of a family of intervals on the real line), threshold graphs,[3] and permutation graphs. As a consequence, the asymptotic empirical degree proportion, that is, the degree

[2]The *intersection graph* of a family of sets $\{S_1, S_2, \dots, S_n\}$ is the graph obtained by creating one vertex v_i for each set S_i, and connecting two vertices v_i, v_j by an edge whenever $S_i \cap S_j \neq \varnothing$.

[3]A graph $G = (V(G), E(G))$ is a *threshold graph* if there is a real number t, and weight function $w : V(G) \to \mathbb{R}$ such that $(u,v) \in E(G)$ if and only if $w(u) + w(v) > t$. Threshold graphs are a special case of interval graphs.

proportion of a uniformly random vertex, can be derived (see Section 2.5). However, the results of Diaconis et al. [16, 17] say nothing about the degrees of specific vertices, and hence the degree process. In general, deriving the limiting distribution of the degree sequence in labeled geometric intersection graphs is a more delicate question. In this paper, we explore some elegant connections between the graph and permutation limit theories, and used them to derive new results on the asymptotic distribution of the degree sequence in random permutation graphs.

1.3. *Organization.* The rest of the paper is organized as follows: Section 2 contains preliminaries about weak convergence of measures and the basics of graph and permutation limit theories and their connections. The results are formally stated in Section 3. The proof of Theorem 3.1 and the connections between the permutation process and the degree process are discussed in Section 4. The degree process for the uniformly random permutation and the CLT for the mid-vertex are proved in Section 5. The conditions needed to verify the convergence of the permutation process are discussed in Section 6. These conditions are also verified for a general class of exponential measures on permutations. Applications of these results to derive the limiting degree process of a Mallows random permutation are in Section 7. The limiting distribution of the minimum degree of the permutation graph associated with a uniformly random permutation is proved in Section 8. Appendix A discusses weak convergence of random probability measures, and Appendix B proves the asymptotics of the empirical degree proportion of random permutation graphs.

2. Graph and permutation limit theories. In this section, we discuss the basic definitions regarding the convergence of graph and permutation sequences and their connections. We begin by recalling preliminaries about weak convergence of probability measures.

2.1. *Preliminaries.* Let $\mathscr{B}([0,1]^2)$ be the Borel sigma-algebra of $[0,1]^2$, and $\mathcal{P}([0,1]^2)$ be the space of all probability measures on $([0,1], \mathscr{B}([0,1]^2))$. The law and the distribution function of a random variable $Z \sim \mu \in \mathcal{P}([0,1]^2)$ will be denoted by $\mathscr{L}(Z)$ and F_μ, respectively.

The space $\mathcal{P}([0,1]^2)$ equipped with the Lévy–Prokhorov metric [7] is a Polish space, which induces the topology of weak convergence: a sequence of measures $\mu_n \in \mathcal{P}([0,1]^2)$ converges *weakly* to a measure $\mu \in \mathcal{M}$ (denoted by $\mu_n \to \mu$), if

$$\mu_n(f) := \int_{[0,1]^2} f \, \mathrm{d}\mu_n \to \int_{[0,1]^2} f \, \mathrm{d}\mu := \mu(f), \tag{2.1}$$

for all continuous function $f : [0,1]^2 \to \mathbb{R}$. A *random* measure is a probability measure on $(\mathcal{P}([0,1]^2), \mathscr{B}(\mathcal{P}([0,1]^2)))$, where $\mathscr{B}(\mathcal{P}([0,1]^2))$ is the Borel sigma-algebra of space $\mathcal{P}([0,1]^2)$ equipped with the Lévy–Prokhorov metric. The space

$\mathcal{P}([0,1]^2)$ is compact, and, hence, the collection of all probability measures on $\mathcal{P}([0,1]^2)$ is tight.

A sequence $\{\mu_n\}_{n\geq 1}$ of random measures in $\mathcal{P}([0,1]^2)$, converges *weakly in distribution* to a random measure μ (denoted by $\mu_n \stackrel{D}{\to} \mu$) if for any continuous map $\omega : \mathcal{P}([0,1]^2) \mapsto \mathbb{R}$, $\mathbb{E}\omega(\mu_n) \to \mathbb{E}\omega(\mu)$. A simple sufficient criterion for weak convergence in distribution is

$$\mu_n(f) \stackrel{D}{\to} \mu(f) \tag{2.2}$$

for any continuous function $f : [0,1]^2 \to \mathbb{R}$ (refer to Proposition A.1 in Appendix A for the proof).

Let $\{Z_n(t)\}_{t\in(0,1]}$ be a sequence of stochastic processes with the sample paths $Z_n(\cdot) \in [0,1]$, for all $n \geq 1$ [this includes the degree process (1.2) and the permutation process (1.1)]. To define convergence of such stochastic processes, equip the space $[0,1]^{(0,1]}$ with the product topology. It follows from Proposition A.1 that $\{Z_n(t)\}_{t\in(0,1]}$ *converges in law/distribution* to a process $\{Z(t)\}_{t\in(0,1]}$ [denoted by $Z_n(\cdot) \stackrel{w}{\Rightarrow} Z(\cdot)$] if for any fixed $s \geq 1$ and $0 < t_1 < t_2 < \cdots < t_s \leq 1$,

$$\big(Z_n(t_1), Z_n(t_2), \dots, Z_n(t_s)\big) \stackrel{D}{\to} \big(Z(t_1), Z(t_2), \dots, Z(t_s)\big). \tag{2.3}$$

In this paper, the limiting process $Z(\cdot)$ will often have mutually independent finite-dimensional marginals. In this case, the sample paths $Z(\cdot)$ are nonmeasurable almost surely. For this reason, we consider $[0,1]^{(0,1)}$ equipped with the product topology, instead of nicer spaces such as the space of continuous functions $\mathbb{C}(0,1]$, or the space of cadlag functions $\mathbb{D}(0,1]$.

Finally, recall that the Kolmogorov–Smirnov distance between $\mu, \nu \in \mathcal{P}([0,1]^2)$ is defined as

$$\|\mu-\nu\|_{\mathrm{KS}} := \sup_{0\leq x,y\leq 1} \big|F_\mu(x,y) - F_\nu(x,y)\big|.$$

Convergence in Kolmogorov–Smirnov distance implies weak convergence, but the converse does not hold in general. On $\mathcal{P}([0,1]^2)$ however these two notions turn out to be equivalent, as shown in [26].

2.2. *Graph limit theory.* The theory of graph limits was developed by Lovász and coauthors [10, 11, 37], and has received phenomenal attention over the last few years. Here we mention the basic definitions about the convergence of graph sequences. If F and G are two graphs, then define the homomorphism density of F into G by

$$t(F,G) := \frac{|\hom(F,G)|}{|V(G)|^{|V(F)|}},$$

where $|\hom(F,G)|$ denotes the number of homomorphisms of F into G. In fact, $t(F,G)$ is the probability that a random mapping $\phi : V(F) \to V(G)$ defines a graph homomorphism.

Let $\mathscr{W}$ be the space of all measurable functions from $[0,1]^2$ into $[0,1]$ that satisfy $W(x,y) = W(y,x)$, for all x, y. For a simple graph H with $V(H) = \{1, 2, \ldots, |V(H)|\}$, let

$$t(H, W) = \int_{[0,1]^{|V(H)|}} \prod_{(i,j)\in E(H)} W(x_i, x_j)\, \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{|V(H)|}.$$

DEFINITION 2.1 [10, 11, 37]. A sequence of graphs $\{G_n\}_{n\geq 1}$ is said to *converge to* W if for every finite simple graph H,

$$\lim_{n\to\infty} t(H, G_n) = t(H, W). \tag{2.4}$$

Moreover, in light of equation (2.3), a sequence of random graphs $\{G_n\}_{n\geq 1}$ is said to *converge in distribution* to a random graphon W, if for any $s \geq 1$ and all finite collection $\{H_1, H_2, \ldots, H_s\}$ of finite simple graphs,

$$\{t(H_1, G_n), \ldots, t(H_s, G_n)\} \xrightarrow{D} \{t(H_1, W), \ldots, t(H_s, W)\}. \tag{2.5}$$

The limit objects, that is, the elements of $\mathscr{W}$, are called *graph limits* or *graphons*. A finite simple graph G on $[n]$ can also be represented as a graphon in a natural way: Define $f^G(x,y) = \mathbf{1}\{(\lceil nx\rceil, \lceil ny\rceil) \in E(G)\}$, that is, partition $[0,1]^2$ into n^2 squares of side length $1/n$, and let $f^G(x,y) = 1$ in the (i,j)th square if $(i,j) \in E(G)$, and 0 otherwise. Observe that $t(H, f^G) = t(H, G)$ for every simple graph H and therefore the constant sequence G converges to the graph limit f^G.

The notion of convergence in terms of subgraph densities outlined above can be captured by the *cut-distance* defined as

$$d_\square(f, g) := \sup_{S,T\subset[0,1]} \left| \int_{S\times T} [f(x,y) - g(x,y)]\, \mathrm{d}x\, \mathrm{d}y \right|,$$

for $f, g \in \mathscr{W}$. Define an equivalence relation on $\mathscr{W}$ as follows: $f \sim g$ whenever $f(x,y) = g_\sigma(x,y) := g(\sigma x, \sigma y)$, for some measure preserving bijection $\sigma : [0,1] \mapsto [0,1]$. The orbit of $g \in \mathscr{W}$ is the set of all functions g_σ, as σ varies over all measure preserving bijections from $[0,1] \mapsto [0,1]$. Denote by $\widetilde{g}$ the closure of the orbit of g in $(\mathscr{W}, d_\square)$. The space $\{\widetilde{g} : g \in \mathscr{W}\}$ of closed equivalence classes is denoted by $\widetilde{\mathscr{W}}$ and is associated with the following natural metric:

$$\delta_\square(\widetilde{f}, \widetilde{g}) := \inf_\sigma d_\square(f, g_\sigma) = \inf_\sigma d_\square(f_\sigma, g) = \inf_{\sigma_1,\sigma_2} d(f_{\sigma_1}, g_{\sigma_2}).$$

The space $(\widetilde{\mathscr{W}}, \delta_\square)$ is compact [10], and the metric $\delta_\square$ is commonly referred to as the *cut-metric*.

The main result in graph limit theory is that a sequence of graphs $\{G_n\}_{n\geq 1}$ converges to a limit $W \in \mathscr{W}$ in the sense defined in (2.4) if and only if $\delta_\square(\widetilde{f^{G_n}}, \widetilde{W}) \to 0$ [10], Theorem 3.8. More generally, a sequence $\{\widetilde{W}_n\}_{n\geq 1}$ converges to $\widetilde{W} \in \widetilde{\mathscr{W}}$ if and only if $\delta_\square(\widetilde{W}_n, \widetilde{W}) \to 0$.

2.3. *Permutation limits*. Hoppen et al. [26, 27] developed a theory of permutation limits, which is analogous to the theory of graph limits. For $\pi_n \in S_n$ and $\sigma \in S_a$, σ is a sub-permutation of π_n if there exists $1 \le i_1 < \cdots < i_a \le n$ such that such that $\sigma(x) < \sigma(y)$ if and only if $\pi_n(i_x) < \pi_n(i_y)$. For example, 132 is a sub-permutation of 7,126,354 induced by $i_1 = 3, i_2 = 4, i_3 = 6$.[4] The density of a permutation $\sigma \in S_a$ in a permutation $\pi_n \in S_n$ is

$$\rho(\sigma, \pi_n) = \begin{cases} \binom{n}{a}^{-1} \left|\{\sigma \in S_a : \sigma \text{is sub-permutation of } \pi_n\}\right|, & \text{if } a \le n \\ 0, & \text{if } a > n. \end{cases}$$

Let $\mathcal{M} \subset \mathcal{P}([0,1]^2)$ be the set of all probability measures with uniform marginals on $([0,1], \mathscr{B}[0,1]^2)$. For $a \ge 1$ and $\nu \in \mathcal{M}$, sample a independent points $(x_1, y_1), (x_2, y_2), \dots, (x_a, y_a)$ in $[0,1]^2$ randomly from the measure ν. Let σ_x and σ_y be the permutations of order a such that $x_{\sigma_x(1)} < x_{\sigma_x(2)} < \cdots < x_{\sigma_x(a)}$ and $y_{\sigma_y(1)} < y_{\sigma_y(2)} < \cdots < y_{\sigma_y(a)}$, respectively (since the marginals of ν are uniform, ties do not occur with probability 1). Define $\sigma_y^{-1} \circ \sigma_x$ as the ν-*random permutation* of order a, which gives by the relative order of the vertical coordinates of the points $(x_1, y_1), (x_2, y_2), \dots, (x_a, y_a)$ with respect to their horizontal coordinates [26], Definition 1.4. Denote by $\rho(\sigma, \nu)$ the probability that a ν-random permutation of order a is σ.

DEFINITION 2.2. An infinite sequence $\{\pi_n\}_{n\ge1}$ of permutations is said to *converge to a measure* $\nu \in \mathcal{M}$ if

$$\lim_{n\to\infty} \rho(\sigma, \pi_n) = \rho(\sigma, \nu), \tag{2.6}$$

for every finite permutation σ. Moreover, a sequence of random permutations $\{\pi_n\}_{n\ge1}$ *converges in distribution to random measure* $\nu \in \mathcal{M}$ if

$$\big(\rho(\sigma_1, \pi_n), \rho(\sigma_2, \pi_n), \dots, \rho(\sigma_s, \pi_n)\big) \xrightarrow{D} \big(\rho(\sigma_1, \nu), \rho(\sigma_2, \nu), \dots, \rho(\sigma_s, \nu)\big),$$

for all $s \ge 1$ and all collections of finite permutations $\{\sigma_1, \sigma_2, \dots, \sigma_s\}$.

Drawing parallel from graph limit theory, the permutation limit objects, that is, the elements of $\mathcal{M}$ are referred to as *permutons*. Any permutation $\pi_n \in S_n$ can be represented as probability measure with uniform marginals in a natural way. Define $\nu_{\pi_n} \in \mathcal{M}$ as follows:

$$\mathrm{d}\nu_{\pi_n} := f_n(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.7}$$

[4] Sub-permutations are often referred to as *patterns* and their combinatorial properties are well studied (the interested reader may refer to Bona [9], the recent paper of Janson et al. [29] and the references therein).

where $f_n(x, y) = n\mathbf{1}\{(x, y) : \pi_n(\lfloor nx \rfloor) = \lfloor ny \rfloor\}$. As in graph limit theory, ν_{π_n} has the following interpretation: partition $[0, 1]^2$ into n^2 squares of side length $1/n$, and define $f_n(x, y) = n$ for all (x, y) in the (i, j)th square if $\pi_n(i) = j$ and 0 otherwise. The measure ν_{π_n} will be referred to as the *permuton associated with* π_n.

The space $\mathcal{M}$ is a closed subset of $\mathcal{P}([0, 1]^2)$, and, hence, Polish and compact. The convergence defined in Definition 2.2 can be metrized by embedding all finite permutations in $\mathcal{M}$ [as in (2.7)], equipped with any metric which induces the topology of weak convergence. Analogous to graph limit theory, Hoppen et al. ([26], Lemma 5.3) showed that a sequence of (random) permutations π_n convergences in distribution to a permuton ν [in the sense of (2.2)] if and only if the corresponding sequence of measures ν_{π_n} converges weakly in distribution to ν. More generally, any sequence of (random) measures $\{\nu_n\}_{n\geq 1}$ in $\mathcal{M}$ converges weakly in distribution to a (possibly random) measure ν if and only if $(\rho(\sigma_1, \nu_n), \rho(\sigma_2, \nu_n), \dots, \rho(\sigma_s, \nu_n)) \xrightarrow{D} (\rho(\sigma_1, \nu), \rho(\sigma_2, \nu), \dots, \rho(\sigma_s, \nu))$, for all $s \geq 1$ and all collections of finite permutations $\{\sigma_1, \sigma_2, \dots, \sigma_s\}$.

2.4. *Limit of permutation graphs*. Diaconis et al. [16, 17] studied the limits of threshold graphs and interval graphs. Their methods also apply to other geometric intersection graphs, which include permutation graphs [17]. However, instead of describing the limit object as symmetric function from $[0, 1]^2$ to $[0, 1]$, they represented the graph limit as a measure on $[0, 1]^2$ with uniform marginals.

For every measure ν on $[0, 1]^2$ with uniform marginals, Diaconis et al. [16, 17] defined a unique graph limit object $\widetilde{W}_\nu \in \widetilde{\mathscr{W}}$ by specifying $t(F, W_\nu)$ for all graphs F as follows:

$$\begin{aligned} t(F, W_\nu) &:= \mathbb{E} \prod_{(i,j)\in E(F)} K(X_i, X_j) \\ &= \int_{[0,1]^{2|V(F)|}} \prod_{(i,j)\in E(F)} K(x_i, x_j)\, \mathrm{d}\nu(x_1)\, \mathrm{d}\nu(x_2) \cdots \mathrm{d}\nu(x_{|V(F)|}), \end{aligned} \tag{2.8}$$

where $X_1, X_2, \dots, X_n$ are independent and identically distributed from ν and $K : [0, 1]^2 \times [0, 1]^2 \to [0, 1]$ is given by $K((a_1, b_1), (a_2, b_2)) = \mathbf{1}\{(a_1 - a_2)(b_1 - b_2) < 0\}$.

In the graph limit literature it is usually convenient to represent a graphon by a functional $W : [0, 1]^2 \to [0, 1]$. However, Diaconis et al. [17] described permutation graph limits in terms of a permuton ν. Thus, every probability measure ν on $[0, 1]^2$ defines a graph limit W_ν by (2.8). Diaconis et al. [17] showed that every permutation graph limit may be represented in terms of a measure ν on $([0, 1]^2, \mathscr{B}([0, 1]^2))$ via (2.8) with the two marginal distributions of ν both being uniform on $[0, 1]$.

Glebov et al. [23] pointed out that if $\{\pi_n\}_{n\geq 1}$ is a convergent sequence of permutations (as in Definition 2.2), then the sequence of permutation graphs $\{G_{\pi_n}\}_{n\geq 1}$ is also convergent. Therefore, each permuton ν can be associated with an equivalence class $\widetilde{W}_\nu \in \widetilde{\mathscr{W}}$. However, the map $\nu \to \widetilde{W}_\nu$ is not one-to-one, and convergence of a sequence of permutation graphs $\{G_{\pi_n}\}_{n\geq 1}$ does not necessarily imply convergence of the permutons associated with the sequence $\{\pi_n\}_{n\geq 1}$:

REMARK 2.1. Suppose $\nu \in \mathcal{M}$ be a permuton which is not exchangeable, that is, if (X, Y) has distribution ν then (Y, X) has distribution $\mu \neq \nu$. Let $\{\pi_n\}_{n\geq 1}$ be a sequence of permutations such the associated permutons converge to ν. Then $\{\pi_n^{-1}\}_{n\geq 1}$ converge to $\mu \neq \nu$. However, since the graphs G_{π_n} and $G_{\pi_n^{-1}}$ are isomorphic, $\widetilde{W}_\nu$ are $\widetilde{W}_\mu$ are identical. Thus, if $\{\sigma_n\}_{n\geq 1}$ is a sequence of permutations defined by $\sigma_n = \pi_n$ for n odd, and $\sigma_n = \pi_n^{-1}$ for n even, then the sequence of graphs $\{G_{\sigma_n}\}_{n\geq 1}$ converges to $\widetilde{W}_\nu = \widetilde{W}_\mu$. However, the sequence of permutons associated with $\{\sigma_n\}_{n\geq 1}$ converges to ν along the odd subsequence, and μ along the even subsequence.

For other interesting connections between graph and permutation limit theories, refer to recent papers by Král' and Pikhurko [33], Glebov et al. [23, 24].

2.5. *Empirical distribution of the degree proportion.* Given a sequence of permutation graphs $(G_{\pi_n})_{n\geq 1}$, the empirical distribution of the degree proportion, that is, the degree proportion of uniformly randomly chosen vertex in G_{π_n} is

$$\kappa(G_{\pi_n}) := \frac{1}{n}\sum_{i=1}^{n} \delta_{\frac{d_n(i)}{n}}. \tag{2.9}$$

Diaconis et al. [17] pointed out that if G_{π_n} converges to a graphon W_ν, then $\kappa(G_{\pi_n})$ converges weakly to the distribution of the random variable

$$W_1(\mathbf{X}) := \int_{[0,1]^2} K(\mathbf{X}, \mathbf{z})\, \mathrm{d}\nu(\mathbf{z}), \tag{2.10}$$

where $\mathbf{X} = (X_1, Y_1)$ is a random element in $[0, 1]^2$ with distribution ν, and $K((x_1, y_1), (x_2, y_2)) = \mathbf{1}\{(x_1 - x_2)(y_1 - y_2) < 0\}$.

The following proposition is immediate from the above discussion. We include a proof in Appendix B.1 for the sake of completeness.

PROPOSITION 2.1. *Let $\{\pi_n\}_{n\geq 1}$ be a sequence of permutations such that the corresponding permuton sequence $\{\nu_{\pi_n}\}_{\geq 1}$ converges to a permuton ν. Then the empirical distribution of the degree proportion* (2.9)

$$\kappa(G_{\pi_n}) := \frac{1}{n}\sum_{i=1}^{n} \delta_{\frac{d_n(i)}{n}} \xrightarrow{D} X_1 + Y_1 - 2F_\nu(X_1, Y_1), \tag{2.11}$$

where $(X_1, Y_1) \sim \nu$ and F_ν is the distribution function of ν.

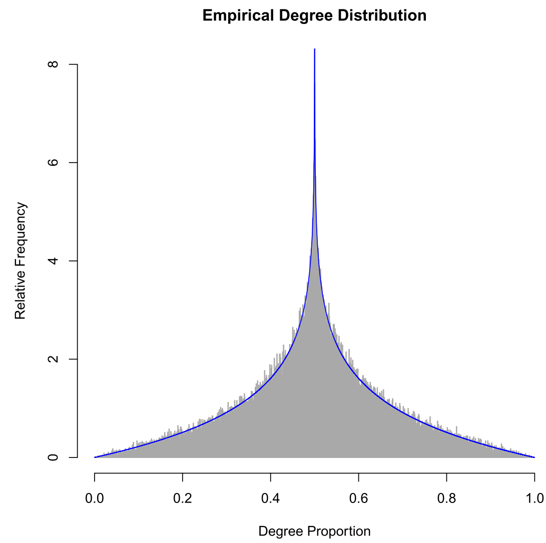

FIG. 2. *The empirical distribution of the degree proportion of the permutation graph associated with a random permutation of length $n = 10^5$. The blue curve represents the density of the limiting distribution $f(z) = -\log|2z - 1|$.*

The following corollary is an application of the above result for the uniform random permutation. The proof is given in Appendix B.2 (see also Figure 2).

COROLLARY 2.2. *Let $\pi_n \in S_n$ be a uniformly random permutation and G_{π_n} the associated permutation graph. Then the empirical distribution of the degree proportion*

$$\kappa(G_{\pi_n}) \xrightarrow{D} Z := (1-U)V + U(1-V), \tag{2.12}$$

where U, V are independent $\mathrm{Unif}(0,1)$. *Equivalently, Z has the same distribution as* $\mathrm{Unif}(V, 1-V)$, *where $V \sim$* $\mathrm{Unif}(0,1)$, *and has a density with respect to Lebesgue measure given by $f_Z(z) = -\log|1-2z|$, for* $0 \le z \le 1$.

3. Statement of the results. In this section, we formally state the results obtained in the paper. In Section 3.1, we discuss the general result which relates the convergence of the permutation process with the convergence of the degree process. The application of this general result for the uniform random permutation, and the asymptotics of the mid-vertex and the minimum degree are given in Section 3.2. The degree process of the Mallows random permutation is discussed in Section 3.3.

3.1. *Convergence of the degree process.* Given a sequence $\{\pi_n\}_{n\ge1}$ of random permutations, the convergence of the associated permuton sequence $\{\nu_{\pi_n}\}_{n\ge1}$ is not enough for the degree sequence to converge (see Example 4.1 in Section 4). Therefore, the convergence of the degree sequence does not follow from the results of Diaconis et al. [17], unlike the empirical distribution of the degree proportion (Proposition 2.1). Thus, additional assumptions are needed on the permutation process to ensure the convergence of the degree process, which makes the problem more challenging.

The following theorem shows that the convergence of the permutation process (1.1) implies that both the corresponding permutons and the degree process converge. To the best of our knowledge, this connection between the convergence of the permutation process and permutons is new, and might be of independent interest. In fact, under regularity conditions (discussed in Section 6) the two notions of convergence are equivalent.

THEOREM 3.1. *Let $\pi_n \in S_n$ be a sequence of random permutations such that*

$$\pi_n(\cdot) \stackrel{w}{\Rightarrow} Z(\cdot), \tag{3.1}$$

where $Z(\cdot)$ is a stochastic process in $(0, 1]$. *Then there exists a* (*random*) *measure $\mu \in \mathcal{M}$ such that the permuton $\nu_{\pi_n} \stackrel{D}{\to} \mu$, and the degree process*

$$d_n(\cdot) \stackrel{w}{\Rightarrow} D(\cdot), \tag{3.2}$$

where

$$D(t) = t + Z(t) - 2F_\mu\big(t, Z(t)\big),$$

and F_μ is the distribution function of the measure μ.

The above theorem, which is proved in Section 4.2.1, will be used to determine the limiting degree process for various random permutations. Note that the limiting measure μ might be random. In that case, the finite-dimensional distributions degree process can be dependent (see Example 4.2). However, for most of the examples considered in this paper the limiting measure is nonrandom, and the corresponding degree process has independent finite-dimensional distributions. This is summarized in the following corollary and proved in Section 4.2.2.

COROLLARY 3.2. *Suppose the permutation process $\pi_n(\cdot) \stackrel{w}{\Rightarrow} Z(\cdot)$, and the finite-dimensional marginals of $Z(\cdot)$ are independent. Then the following hold*:

- *ν_{π_n} converges to a nonrandom measure $\mu \in \mathcal{M}$ with law $\mathscr{L}(X, Y) \sim \mu$ defined as follows*:

$$X \sim \mathrm{Unif}[0, 1], \quad \textit{and} \quad \mathscr{L}(Y | X = x) \sim \mathscr{L}\big(Z(x)\big).$$

- *The finite-dimensional distributions of the limiting degree process $D(\cdot)$* [*defined in* (3.2)] *are independent.*

The above results imply that in order to determine the convergence of the degree process, it suffices to verify the convergence of the permutation process. This requires the permutation process to have some regularity, which is formalized in Section 6. The regularity conditions ensure that the finite-dimensional distributions of the permutation process are "equicontinuous", which can be verified easily for exponential models on permutations, like the Mallow's model [38] and Spearman's rank correlation models [14].

3.2. *Uniform random permutation.* A *uniformly random permutation graph* G_{π_n} is the permutation graph associated with a uniformly random permutation $\pi_n \in S_n$. In this section we state the results on the asymptotic degree proportion of a uniformly random permutation graph. To this end, we need some notation. For $a, b \in [0, 1]$, denote by $[a, b]$ the interval $[a \wedge b, a \vee b]$, where $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$, and let $\mathrm{Unif}(a, b)$ denote the uniform distribution over $[a, b]$.

3.2.1. *Convergence of the degree process.* The convergence of the finite-dimensional distributions of the degree process for a uniformly random permutation graph is an easy consequence Theorem 3.1 (see Section 5.1 for the proof).

COROLLARY 3.3. *Let $\pi_n \in S_n$ be a uniform random permutation, and G_{π_n} the associated permutation graph. Then the degree process* (1.2),

$$d_n(t) \overset{w}{\Rightarrow} D(\cdot),$$

where $D(r) \sim \mathrm{Unif}(r, 1 - r)$, and the coordinates of $D(\cdot)$ are mutually independent.

Figure 1 shows the degree proportion of the labeled vertices in the permutation graph associated with a random permutation of length $n = 10^5$. The symmetry in the figure around the *mid-vertex*, that is the vertex labeled $n/2$, is because the distribution of $d_n(i)$ and $d_n(n+1-i)$ are the same for every $i \in [n]$. This is confirmed by the above corollary, which shows that for $r \in [0, 1]$, the degree proportion of the vertices labeled $\lceil nr \rceil$ and $\lceil n(1-r) \rceil$ both converge to the uniform distribution over the interval $[r, 1 - r]$. The shrinking length of this interval as r approaches 1/2, explains the fan-like structure in Figure 1. Moreover, when $r = 1/2$, this implies that $d_n(\lceil n/2 \rceil)/n$ converges in probability to 1/2, as can be seen from Figure 1.

3.2.2. *CLT of the mid-vertex.* It follows from Corollary 3.3 that the degree proportion of the mid-vertex, that is, the vertex labeled $\lceil n/2 \rceil$, converges to 1/2 in probability. Therefore, it is reasonable to expect a central limit theorem for $d_n(\lceil n/2 \rceil)/n$ around 1/2 after an appropriate rescaling. This is detailed in the following theorem and illustrated in Figure 3. The proof is given in Section 5.2.2.

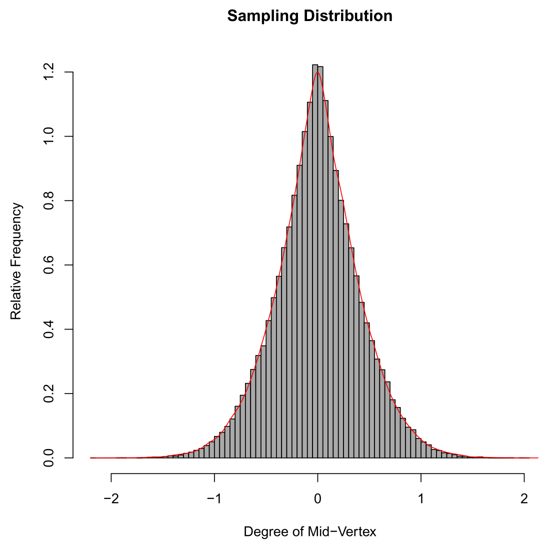

FIG. 3. *The sampling distribution of the degree of the mid-vertex* (*the vertex labeled* $n/2$), *for the permutation graph associated with a uniform random permutation of length* $n = 10^4$, *repeated over* 10^5 *samples. The red curve is the density of the limiting distribution given by Theorem* 3.4.

THEOREM 3.4. *Let* $\pi_n \in S_n$ *be a uniform random permutation, and* G_{π_n} *the associated permutation graph. Then*

$$\sqrt{n}\left(\frac{d_n(\lceil n/2\rceil)}{n} - \frac{1}{2}\right) \xrightarrow{D} N\big(0, U(1-U)\big), \tag{3.3}$$

where $U \sim \mathrm{Unif}[0, 1]$.

3.2.3. *Asymptotics for the minimum degree.* In this section, we obtain the limiting distribution of the minimum degree of a uniformly random permutation graph. The proof is given in Section 8.

THEOREM 3.5. *Let* $\delta(G_{\pi_n})$ *be the minimum degree of a uniformly random permutation graph* G_{π_n}. *Then*

$$\frac{\delta(G_{\pi_n})}{\sqrt{n}} \xrightarrow{D} \Gamma,$$

where Γ *is the Rayleigh distribution with parameter* $\frac{1}{\sqrt{2}}$, *that is*, $\mathbb{P}(\Gamma > \gamma) = e^{-\gamma^2}$ *for all* $\gamma > 0$.

The degree proportion of a uniformly random permutation graph has different variability depending on the label of the vertex, as shown in Corollary 2.2 and the above theorem: the minimum degree scales as $\sqrt{n}$, whereas the degree of a typical vertex scales as n (Proposition 2.1).

3.3. *Mallows random permutation.* The Mallows model [38] is a popular non-uniform model on permutations. For $\beta \in \mathbb{R}$, denote by $\pi_n \sim M_{\beta,n}$ the Mallows random permutation over S_n with probability mass function

$$m_{\beta,n}(\sigma) := \frac{e^{-\beta\cdot\frac{\lambda(\sigma)}{n}}}{\sum_{\sigma\in S_n} e^{-\beta\cdot\frac{\lambda(\sigma)}{n}}}, \tag{3.4}$$

where $\lambda(\sigma) = |\{(i, j) : (i - j)(\pi_n(i) - \pi_n(j)) < 0\}|$ is the number of inversions of the permutation σ. The uniform random permutation corresponds to the case $\beta = 0$. Diaconis and Ram [18] studied a Markov chain on S_n for which the Mallows model gives the limiting distribution. Tail bounds for the displacement of an element in a Mallows permutation was studied by Braverman and Mossel [12]. Recently, Mueller and Starr [42] and later Bhatnagar and Peled [6] studied the length of the longest increasing subsequence in a Mallows permutation.

Recently, permutation modeling has found applications in statistics and machine learning. Consistent estimation of parameters in exponential models on permutations has been studied by Mukherjee [43]. Location and scale mixtures of Mallows model have been considered in [2, 35]. A generalized version of Mallows model was studied in [13, 41], which was extended to infinite permutations in [39, 40]. Huang et al. [28] and Kondor et al. [32] study inference on permutations via Fourier analysis of representation of finite groups with the focus of reducing computational complexity. Modeling of partially ranked data using Mallows models and its extensions were considered in [36].

Starr [47] derived the limit of empirical permutation measure associated with a sequence of Mallows random permutations. Using this and Theorem 3.1, we compute the limiting density of the degree proportion in a Mallows random permutation. The limiting density exhibits interesting phase transitions depending on the value of β. This is summarized in the following theorem and proved later in Section 7.

THEOREM 3.6. *Fix $\beta \in \mathbb{R}$. Let $\pi_n \sim M_{\beta,n}$ be a Mallows random permutation with parameter β, and G_{π_n} the associated permutation graph. Then the degree process* (1.2)

$$d_n(\cdot) \overset{w}{\Rightarrow} D_\beta(\cdot),$$

where the coordinates of $D_\beta(\cdot)$ are mutually independent, and the distribution of $D_\beta(a)$ is defined as follows: Let

$$R(a, \beta) = \frac{4(e^\beta - e^{a\beta})(e^{a\beta} - 1)}{(e^\beta - 1)^2}, \quad \textit{and} \quad a_c(\beta) = \frac{1}{2} - \frac{\log\cosh(\beta/2)}{\beta}, \tag{3.5}$$

and the function $h_{a,\beta} : [0, 1] \to \mathbb{R}^+ \cup \{0\}$,

$$h_{a,\beta}(z) = \frac{\beta e^{\frac{1}{2}\beta(a-z)}}{(1 - e^{-\beta})(e^{\beta(a+z)} - e^\beta R(a, \beta))^{\frac{1}{2}}}. \tag{3.6}$$

Then:

1. *if* $a \notin [a_c(\beta), 1 - a_c(\beta)]$, $D_\beta(a)$ *has density*

$$g_{a,\beta}(z) = h_{a,\beta}(z), \qquad z \in [a, 1-a]; \tag{3.7}$$

2. *if* $a \in (a_c(\beta), 1 - a_c(\beta))$, $D_\beta(a)$ *has density*

$$g_{a,\beta}(z) = \begin{cases} h_{a,\beta}(z), & \text{for } z \in [a, 1-a], \\ 2h_{a,\beta}(z), & \text{for } z \in \left[1 - a + \frac{1}{\beta} \log R(a,\beta), a \wedge 1 - a\right). \end{cases} \tag{3.8}$$

The above theorem gives the limiting distribution of the degree process of the permutation graph G_{π_n} associated with a Mallows random permutation $\pi_n \sim M_{\beta,n}$. For $\beta \in \mathbb{R}$ fixed, the limiting distribution of $d_n(\lceil na \rceil)/n$ has a phase transition depending on the value of $a \in [0, 1]$. There exist two *critical points* $a_c(\beta)$ and $1 - a_c(\beta)$, such that for $a \notin [a_c(\beta), 1 - a_c(\beta)]$, the limiting density of $D_\beta(a)$ is a continuous function supported on $[a, 1-a]$. However, if a is in the *critical interval* $(a_c(\beta), 1 - a_c(\beta))$, the density of $D_\beta(a)$ breaks into two piecewise continuous parts on the intervals

$$\left[1 - a + \frac{1}{\beta} \log R(a,\beta), a \wedge 1 - a\right), \quad \text{and} \quad (a \wedge 1 - a, a \vee 1 - a],$$

with a discontinuity at the point $a \wedge 1 - a$. The plots of the limiting density of $D_\beta(a)$ are shown in Figure 4 and Figure 5, for $\beta = 2$ and $a = 0.1$ and $a = 0.55$. The changes in the support of $D_\beta(a)$ for values of a in the critical interval are depicted in Figure 6.

The *critical curves* $\beta \mapsto a_c(\beta)$ and $\beta \mapsto 1 - a_c(\beta)$ are shown in Figure 7. For a fixed $\beta_0 \in \mathbb{R}$ the critical interval $(a_c(\beta_0), 1 - a_c(\beta_0))$ is the interval between the

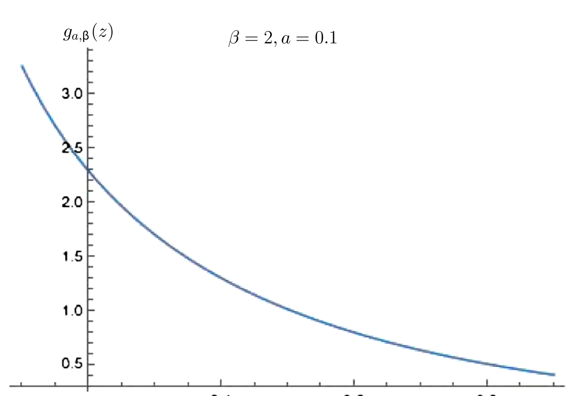

FIG. 4. *Density of* $D_\beta(a)$ *for* $\beta = 2$ *and* $a = 0.1$. *For* $\beta = 2$, $a_c(\beta) = 0.28311$. *Since* $0.1 \notin (0.28311, 0.71689)$, *the density of* $D_2(0.1)$ *is a continuous function supported on* $[0.1, 0.9]$.

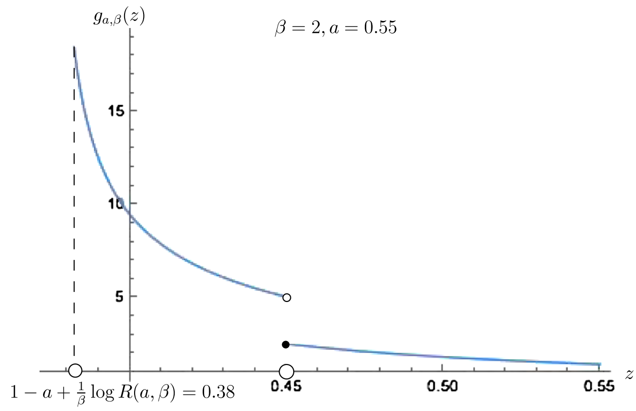

FIG. 5. *Density of* $D_\beta(a)$ *for* $\beta = 2$ *and* $a = 0.55$. *For* $\beta = 2$, $a_c(\beta) = 0.28311$. *As* $0.55 \in (0.28311, 0.71689)$, *the density of* $D_2(0.55)$ *is a piecewise continuous function with the pieces supported on* $[0.38, 0.45)$ *and* $[0.45, 0.55]$, *with a discontinuity at* $1 - a = 0.45$.

two curves intercepted by the vertical line at β_0. Note that for $\beta = 0$, the $a_c(\beta) = 1 - a_c(\beta) = 1/2$, that is, the critical interval is empty. Therefore, for a uniform random permutation, the limiting density has no phase transition, as elaborated in Corollary 3.3.

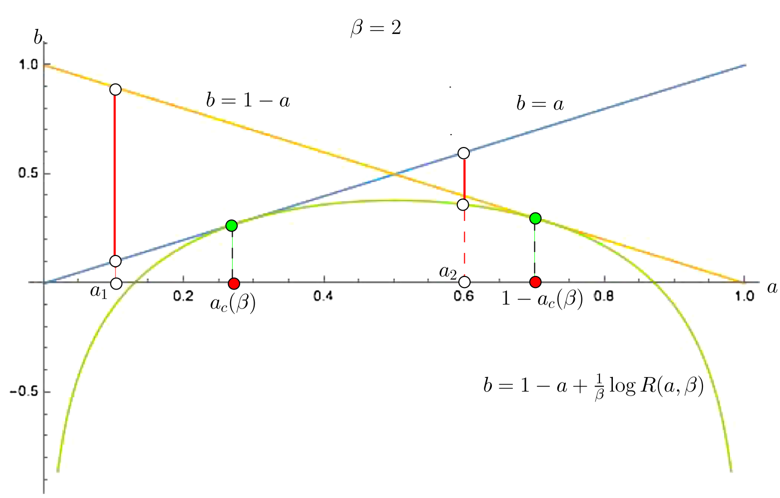

FIG. 6. *The support of* $D_\beta(a)$, *for* $\beta = 2$ *and* $a \in [0, 1]$: *The points of tangency of the curve* $b = 1 - a + \frac{1}{\beta}\log R(a, \beta)$ *with the two straight lines* $b = a$ *and* $b = 1 - a$ *are colored green. The corresponding red points on the* x*-axis are the critical points* $a_c(\beta)$ *and* $1 - a_c(\beta)$. *For a fixed* $a \in [0, 1]$, *the support of* $D_\beta(a)$ *is the interval intercepted by the vertical line at* a *either between the two straight lines* [*if* $a \notin (a_c(\beta), 1 - a_c(\beta))$, *e.g., when* $a = a_1$], *or between the curve and one of the two straight lines* [*if* $a \in (a_c(\beta), 1 - a_c(\beta))$, *e.g., when* $a = a_2$].

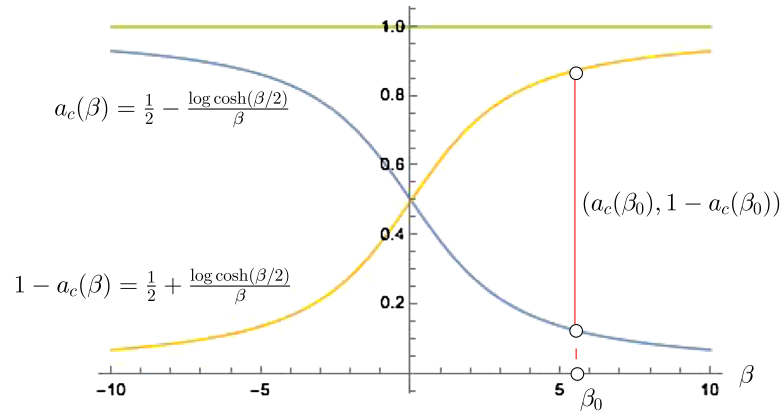

FIG. 7. *The critical transition curves $a_c(\beta)$ and $1 - a_c(\beta)$ for $\beta \in [-10, 10]$. For a fixed $\beta_0 \in \mathbb{R}$, the critical interval $(a_c(\beta_0), 1 - a_c(\beta_0))$ is the interval between the two curves intercepted by the vertical line at β_0.*

The phase transition in the density of $D_\beta(a)$ can be reinterpreted by fixing $a \in [0, 1]$ and varying β: Theorem 3.6 shows that for a fixed $a \in [0, 1]$, there exists a critical point $\beta_c(a)$ [obtained by solving for β in $a_c(\beta) = a$] such that for $\beta \in [0, \beta_c(a)]$, the density of $D_\beta(a)$ is a continuous function supported on $[a, 1 - a]$. However, for $\beta > \beta_c(a)$ the density of $D_\beta(a)$ breaks into two piecewise continuous parts with a discontinuity at the point $a \wedge 1 - a$. If $\beta = 1/T$ denotes the inverse temperature, then this phenomenon is the effect of *replica symmetry breaking* in statistical physics as one moves from the high temperature to the low temperature regime.

4. Limiting degree proportion of random permutations. In this section, we derive the limiting degree process for a general sequence of random permutations (proofs of Theorem 3.1 and Corollary 3.2). We begin with a simple example which shows that the convergence of the permuton sequence does not necessarily imply the convergence of the degree process.

EXAMPLE 4.1. Let e_n be the identity permutation on S_n and σ_n the permutation that takes 1 to n and n to 1, and keeps all the remaining indices fixed. If π_n is a sequence of permutations such that $\pi_n = e_n$ for n even, and $\pi_n = \sigma_n$ for n odd, then the permuton ν_{π_n} converges to the limiting measure which is uniform on the diagonal of the unit square (since $\pi_n(i) = i$ for all $i \in [n]/\{1, n\}$). However, for $t = 1$ the degree process $d_n(1)$ converges to 0 along n even, and to 1 along n odd.

The rest of the section is organized as follows: In Section 4.1, we show that the convergence of the finite-dimensional distributions of the permutation process (1.1) implies the convergence of the associated permutons. This is then used to show the convergence of the degree process (proofs of Theorem 3.1 and Corollary 3.2) in Section 4.2. In Section 4.3, we construct a sequence of random permutations, where the finite-dimensional distributions of the limiting degree process are dependent.

4.1. *Permutation process and permutons.* For $\pi_n \in S_n$, define the *empirical permutation measure*

$$\bar{\nu}_{\pi_n} = \frac{1}{n} \sum_{i \in [n]} \delta_{(\frac{i}{n}, \frac{\pi_n(i)}{n})}. \tag{4.1}$$

It is easy to check that the permuton ν_{π_n} associated with the permutation π_n satisfies $\|\nu_{\pi_n} - \bar{\nu}_{\pi_n}\|_{\mathrm{KS}} \xrightarrow{P} 0$, and so a sequence of permutations $\pi_n \in S_n$ converges to a permuton ν if and only if $\bar{\nu}_{\pi_n}$ converges weakly to ν.

With these definitions, we now show that the convergence of the permutation process implies the convergence of the permutons.

THEOREM 4.1. *Let $\pi_n \in S_n$ be a sequence of random permutations such that*

$$\pi_n(\cdot) \overset{w}{\Rightarrow} Z(\cdot). \tag{4.2}$$

Then $(\bar{\nu}_{\pi_n}, \pi_n(\cdot))$ converges jointly weakly in distribution. In particular, there exists a (possibly random) measure $\mu \in \mathcal{M}$, such that the permuton sequence $\{\nu_{\pi_n}\}_{n\geq 1}$ converges in distribution to $\mu \in \mathcal{M}$.

PROOF. By Proposition A.1, it suffices to show that the vector

$$\big(\nu_{\pi_n}(f), \pi_n(s_1), \dots, \pi_n(s_b)\big)$$

converges in distribution, for any continuous function $f : [0, 1]^2 \mapsto [0, 1]$ and real numbers $s_1, \dots, s_b \in (0, 1]$. Therefore, it suffices to show that the limit

$$\lim_{n\to\infty} \mathbb{E}\left(\bar{\nu}_{\pi_n}(f)^a \prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\right), \tag{4.3}$$

exists for all positive integers $a, b \geq 1$ and continuous functions $g_1, g_2, \dots, g_b : [0, 1] \mapsto \mathbb{R}$. (Note that for any random measure $\mu \in \mathcal{P}([0, 1]^2)$, $\mathbb{E}\mu(f)^a = \mathbb{E}(\int f \,\mathrm{d}\mu)^a$ is the ath moment of $\mu(f)$.)

For verifying the existence of the limit in (4.3), first note that

$$\begin{aligned}
&\mathbb{E}\left(\bar{\nu}_{\pi_n}(f)^a \prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\right) \\
&\quad = \mathbb{E}\left(\frac{1}{n^a} \sum_{t_1, t_2, \dots, t_a \in [n]} \prod_{i=1}^{a} f\left(\frac{t_j}{n}, \frac{\pi_n(t_j)}{n}\right) \prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\right) \qquad (4.4)\\
&\quad = \mathbb{E}\left(\prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big) \int_{[0,1]^a} \prod_{i=1}^{a} f\big(x_i, \pi_n(x_i)\big)\,\mathrm{d}x_i\right) + o(1),
\end{aligned}$$

where the last equality follows from the uniform continuity of f.

Now, by assumption (4.2), for fixed $\{x_i\}_{i\in[a]}$,

$$\prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\prod_{i=1}^{a} f\big(x_i,\pi_n(x_i)\big)\xrightarrow{D}\prod_{j=1}^{b} g_j\big(Z(s_j)\big)\prod_{i=1}^{a} f_i\big(x_i,Z(x_i)\big).$$

Therefore, by the dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}&\left(\prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\prod_{i=1}^{a} f_i\big(x_i,\pi_n(x_i)\big)\right)\\ &=\mathbb{E}\left(\prod_{j=1}^{b} g_j\big(Z(s_j)\big)\prod_{i=1}^{a} f_i\big(x_i,Z(x_i)\big)\right).\end{aligned} \tag{4.5}$$

The RHS above is measurable in $\{x_i\}_{i\in[a]}$, as it is the limit of measurable functions. Another application of dominated convergence theorem gives

$$\begin{aligned}\mathbb{E}&\left(\bar{\nu}_{\pi_n}(f)^a\prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\right)\\ &=\int_{[0,1]^a}\mathbb{E}\left(\prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big)\prod_{i=1}^{a} f\big(x_i,\pi_n(x_i)\big)\right)\prod_{i\in[a]}\mathrm{d}x_i+o(1)\\ &=\int_{[0,1]^a}\mathbb{E}\left(\prod_{j=1}^{b} g_j\big(Z(s_j)\big)\prod_{i=1}^{a} f\big(x_i,Z(x_i)\big)\right)\prod_{i\in[a]}\mathrm{d}x_i+o(1).\end{aligned}$$

This implies $(\bar{\nu}_{\pi_n},\pi_n(\cdot))$ converges jointly weakly in distribution.

Finally, since $\|\nu_{\pi_n}-\bar{\nu}_{\pi_n}\|_{\mathrm{KS}}\to 0$ in probability, $\{\nu_{\pi_n}\}_{n\geq 1}$ converges in distribution to the permuton $\mu\in\mathcal{M}$, since the space $\mathcal{M}\subset\mathcal{P}([0,1]^2)$ of measures with uniform marginals is closed. □

4.2. *Proofs of Theorem* 3.1 *and Corollary* 3.2. Theorem 4.1 can now be used to prove the convergence of the degree process. To this end, we need the following definitions: For $\pi_n\in S_n$, define the random variable $a_n(i):=\sum_{j=1}^{i-1}q_n(i,j)$, for $i\in[n]$. Note that $a_n(i)\in[0,i-1]$ represents the number of edges in G_{π_n} connecting the vertex i to vertices $j\in[i-1]$. The quantity $a_n(i)$ will be referred to as the *backward-degree* of the vertex i. Similarly, one can define the *forward-degree* of the vertex i as $b_n(i):=\sum_{j=i+1}^{n}q_n(i,j)$. Note that $b_n(i)\in[0,n-i]$ and $d_n(i)=a_n(i)+b_n(i)$, where $d_n(i)$ is the degree of the vertex i. The *degree process* $d_n(t)=d_n(\lceil nt\rceil)/n=a_n(t)+b_n(t)$, where $a_n(t)=a_n(\lceil nt\rceil)/n$ and $b_n(t)=b_n(\lceil nt\rceil)/n$, are the *backward-degree process* and the *forward-degree process*, respectively. Note that $\sum_{i=1}^{n}a_n(i)=\sum_{i=1}^{n}b_n(i)$ is the number of inversions of π_n.

4.2.1. *Proof of Theorem* 3.1. By Theorem 4.1, we have

$$\big(\bar{\nu}_{\pi_n}, \pi_n(\cdot)\big) \overset{D}{\to} \big(\mu, Z(.)\big),$$

where $\mu \in \mathcal{M}$ is a random measure such that $\bar{\nu}_{\pi_n} \overset{D}{\to} \mu$, and $Z(\cdot)$ is a stochastic process on $(0, 1]$ such that $\pi_n(\cdot) \overset{w}{\Rightarrow} Z(\cdot)$. Thus, fixing $k \geq 1$ and $0 < t_1 < t_2 < \cdots < t_k \leq 1$ we have

$$\big(\bar{\nu}_{\pi_n}, \pi_n(t_1), \dots, \pi_n(t_k)\big) \overset{D}{\to} \big(\mu, Z(t_1), \dots, Z(t_k)\big).$$

Applying Skorohod's representation theorem on the separable metric space $\mathcal{M} \times [0, 1]^k$ (see Billingsley [7], Theorem 6.7), without loss of generality assume that the above convergence happens almost surely.

Now for any $t \in (0, 1]$,

$$\begin{aligned} a_n(t) &= \frac{1}{n}\sum_{a=1}^{\lceil nt\rceil} \mathbf{1}\big\{\pi_n(a) > \pi_n\big(\lceil nt\rceil\big)\big\} \\ &= \bar{\nu}_{\pi_n}\big(\big[0, \lceil nt\rceil/n\big] \times \big(\pi_n(t), 1\big]\big) = \bar{\nu}_{\pi_n}\big([0, t] \times \big(\pi_n(t), 1\big]\big) + o(1) \\ &= t - F_{\nu_{\pi_n}}\big(t, \pi_n(t)\big) + o(1), \end{aligned} \tag{4.6}$$

where the last step uses $\|\bar{\nu}_{\pi_n} - \nu_{\pi_n}\|_{\mathrm{KS}} = o(1)$. By a similar argument,

$$b_n(t) = \pi_n(t) - F_{\nu_{\pi_n}}\big(t, \pi_n(t)\big) + o(1). \tag{4.7}$$

Combining (4.6) and (4.7), for any $1 \leq i \leq k$ we have

$$\begin{aligned} d_n(t_i) &= a_n(t_i) + b_n(t_i) = t_i + \pi_n(t_i) - 2F_{\nu_{\pi_n}}\big(t_i, \pi_n(t_i)\big) + o(1) \\ &= t_i + \pi_n(t_i) - 2F_\mu\big(t_i, \pi_n(t_i)\big) + \|\nu_{\pi_n} - \mu\|_{\mathrm{KS}} + o(1) \\ &\overset{\text{a.s.}}{\to} t_i + Z(t_i) - 2F_\mu\big(t_i, Z(t_i)\big), \end{aligned}$$

where the last step uses $\|\nu_{\pi_n} - \mu\|_{\mathrm{KS}} \overset{\text{a.s.}}{\to} 0$ (see Hoppen et al. [26], Lemma 2.1), and the fact that the function F_μ is continuous in each coordinate when the other coordinate is held fixed. Indeed, this follows from the observation that any $\mu \in \mathcal{M}$ has continuous marginals. Thus, we have

$$\begin{aligned} &\big(d_n(t_1), \dots, d_n(t_k)\big) \\ &\quad \overset{\text{a.s.}}{\to} \big(t_1 + Z(t_1) - 2F_\mu\big(t_1, Z(t_1)\big), \dots, t_k + Z(t_k) - 2F_\mu\big(t_k, Z(t_k)\big)\big), \end{aligned} \tag{4.8}$$

from which finite-dimensional convergence of $d_n(\cdot)$ follows.

4.2.2. *Proof of Corollary* 3.2. For the first part, by Proposition A.1 it suffices to check that for every continuous function $f : [0,1]^2 \mapsto [0,1]$ we have

$$\bar{\nu}_{\pi_n}(f) \xrightarrow{D} f(X,Y),$$

where (X,Y) is as defined in the statement of the corollary. Since $\nu_{\pi_n}(f)$ is a bounded random variable, it suffices to check that for every positive integer $a \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\bar{\nu}_{\pi_n}(f)^a = \mathbb{E}f(X,Y)^a.$$

To this effect, using (4.4) and (4.12) it follows

$$\mathbb{E}f(X,Y)^a = \int_{[0,1]^a} \mathbb{E}\left(\prod_{j=1}^a f\bigl(x_j, Z(x_j)\bigr)\right) \prod_{j=1}^a \mathrm{d}x_j.$$

The RHS above equals to $(\int_0^1 \mathbb{E}f(x, Z(x))\,\mathrm{d}x)^a$, under the assumption of independence. Therefore, $\bar{\nu}_{\pi_n}(f)$ converges in probability to the nonrandom quantity

$$\int_0^1 \mathbb{E}f\bigl(x, Z(x)\bigr)\,\mathrm{d}x = \mathbb{E}_\mu f(X,Y),$$

thus completing the proof of the first part.

The independence of the finite-dimensional marginals of $Z(\cdot)$, implies the same for the degree process $D(\cdot)$ by (3.2).

4.3. *A dependent degree process*. Even though Theorem 3.1 allows for μ to be random, in most examples in this paper μ turns out to be nonrandom and the corresponding degree process has independent finite-dimensional distributions. In this section, we construct a sequence of random permutations where the limiting permuton is random and the finite-dimensional distributions of the degree process are not independent:

EXAMPLE 4.2. Suppose W_n is a uniform random variable on $[n] := \{1, 2, \dots, n\}$, and $\pi_n \in S_n$ defined by

$$\pi_n(i) := (i + W_n - 1 \mod n) + 1. \tag{4.9}$$

Note that π_n is a cyclic shift of the identity permutation, where the length of the shift is chosen uniformly random.

PROPOSITION 4.2. *Let* $\{\pi_n\}_{n\geq 1}$ *be the sequence of random permutations as defined in* (4.9). *Then the degree process*

$$d_n(t) \overset{w}{\Rightarrow} D(t) := W \cdot \mathbf{1}\{W + t < 1\} + (1 - W) \cdot \mathbf{1}\{W + t \geq 1\}, \tag{4.10}$$

where $W \sim \mathrm{Unif}[0,1]$.

PROOF. We will first show that the permutation process $\pi_n(\cdot)$ converges weakly in distribution. For $s \geq 1$ and let $g_1, g_2, \dots, g_b$ be continuous functions on [0, 1]. Then, it is easy to see that, for $s_1, s_2, \dots, s_b \in (0, 1]$

$$\mathbb{E} \prod_{j=1}^{b} g_j\big(\pi_n(s_j)\big) \to \int_0^1 \prod_{j=1}^{b} g_j(s_j + u \mod 1)\,\mathrm{d}u,$$

where $x \mod 1$ denotes the fractional part of x, for $x \in \mathbb{R}$. Hence, $\pi_n(\cdot) \overset{w}{\Rightarrow} Z(\cdot)$, where $Z(\cdot)$ is a stochastic process defined by

$$Z(t) = W + t \mod 1, \qquad \text{with } W \sim \mathrm{Unif}[0, 1]. \tag{4.11}$$

This implies, by Theorem 3.1, that $\nu_{\pi_n} \overset{D}{\to} \mu$, for some random measure $\mu \in \mathcal{M}$. Moreover, by (4.12), for any $a \geq 1$, and continuous function $f : [0, 1]^2 \to [0, 1]$ and positive integer a,

$$\begin{aligned} \mathbb{E}\big(\mu(f)^a\big) &= \int_{[0,1]^a} \mathbb{E}\Bigg(\prod_{i=1}^{a} f\big(x_i, Z(x_i)\big)\Bigg) \prod_{i\in[a]} \mathrm{d}x_i \\ &= \int_{[0,1]^a} \prod_{i=1}^{a} f(x_i, \omega + x_i \mod 1) \prod_{i\in[a]} \mathrm{d}x_i\,\mathrm{d}\omega \\ &= \int_0^1 \bigg(\int_0^1 f(x, x + \omega \mod 1)\,\mathrm{d}x\bigg)^a \mathrm{d}\omega \\ &= \int_0^1 \big(\kappa_w(f)^a\big)\,\mathrm{d}\omega, \end{aligned} \tag{4.12}$$

where κ_s is the joint law of $(V, s + V \mod 1)$, where $V \sim \mathrm{Unif}[0, 1]$, for $s \in [0, 1]$. Therefore, limiting random measure $\mu \sim \kappa_W$, with $W \sim [0, 1]$.

To compute the limit of the degree process, we compute the distribution function of κ_s. In this case, with $U \sim \mathrm{Unif}[0, 1]$ and $0 \leq a, b \leq 1$

$$\begin{aligned} F_{\kappa_s}(a, b) &= \mathbb{P}(U \leq a, U + s \mod 1 \leq b) \\ &= \mathbb{P}(U \leq a, U + s \leq b) + \mathbb{P}(U \leq a, 1 \leq U + s \leq b + 1) \\ &= \min(a, b - s)_+ + \min(a + s - 1, b)_+, \end{aligned} \tag{4.13}$$

which implies

$$\begin{aligned} &F_{\kappa_W}\big(t, Z(t)\big) \\ &\quad = \min\big(t, (W + t \mod 1) - W\big)_+ + \min(t + W - 1, W + t \mod 1)_+ \\ &\quad = t \cdot \mathbf{1}\{W + t < 1\} + (t + W - 1)\mathbf{1}\{W + t \geq 1\}. \end{aligned} \tag{4.14}$$

Therefore, by Theorem (3.1) $d_n(\cdot) \overset{w}{\Rightarrow} D(\cdot)$ where: $W \sim U[0, 1]$, and

$$\begin{aligned} D(t) &:= t + (W + t \mod 1) - 2F_{\kappa_W}(t, W + t \mod 1) \\ &= W \cdot \mathbf{1}\{W + t < 1\} + (1 - W) \cdot \mathbf{1}\{W + t \geq 1\}. \end{aligned} \tag{4.15}$$

In this case, the finite-dimensional distributions of the permutation process (4.11) and the degree process (4.15) are not independent, and the limiting permuton is random. □

5. Uniformly random permutation graph. This section is organized as follows: In Section 5.1, we derive the finite-dimensional convergence of the degree process of a uniformly random permutation graph, as a direct consequence of Theorem 3.1. The CLT of the degree proportion of the mid-vertex (Theorem 3.4) is proved in Section 5.2.

5.1. *Proof of Corollary* 3.3. For a uniformly random permutation, the permutation process $\pi_n(\cdot) \overset{w}{\Rightarrow} Z(\cdot)$, where $Z(t)$ is independent Unif[0, 1], for all $t \in [0,1]$. Then by Corollary 3.2, the permuton ν_{π_n} converges to the Lebsegue measure on $[0,1]^2$. Therefore, for $a \in [0,1]$, by Theorem 3.1 the finite-dimensional distributions of the degree process converges to the finite-dimensional distributions of the process $\{D(a)\}_{a\in[0,1]}$, where $D(a)$ is independent for every $a \in [0,1]$ and

$$D(a) = a + U_a - 2aU_a = (1-a)U_a + a(1-U_a) \sim \mathrm{Unif}(a, 1-a),$$

where $\{U_a\}_{a\in[0,1]}$ are independent Unif(0, 1), for every $a \in [0,1]$.

5.2. *CLT for the mid-vertex.* In this section, we prove the CLT of the degree proportion of the mid-vertex $d_n(\lceil n/2\rceil)/n$ (Theorem 3.4). We begin with a technical estimate about the hypergeometric distribution (Section 5.2.1). Using this, the proof of Theorem 3.4 is given in Section 5.2.2.

5.2.1. *A hypergeometric estimate.* Recall the hypergeometric distribution: A nonnegative integer valued random variable X is said to follow the *hypergeometric distribution* with parameters (N, M, r) if

$$\mathbb{P}(X = x) = \frac{\binom{M}{x}\binom{N-M}{r-x}}{\binom{N}{r}} \qquad \text{for } x \in \big[\max\{0, r+M-N\}, \min\{M, r\}\big],$$

where $N \geq \max\{M, r\}$.

Recall the forward and backward degree proportions $a_n(i)$ and $b_n(i)$ defined in Section 3, respectively. Note that $(i-1) - a_n(i) + b_n(i) = \pi_n(i) - 1$, thus giving the simple relation $b_n(i) - a_n(i) = \pi_n(i) - i$. Using this relation, the following proposition gives a concentration result for $d_n(i)$ around its conditional mean given $\pi_n(i)$.

PROPOSITION 5.1. *The conditional distribution of $a_n(i)|\{\pi_n(i) = j\}$ is hypergeometric with parameters $(n-1, i-1, n-j)$. Consequently, for $R > 0$*

$$\mathbb{P}\big(\big|d_n(i) - m_n(i,j)\big| > R|\pi_n(i) = j\big) \leq 2e^{-\frac{R^2}{2n}},$$

where $m_n(i,j) := \frac{(i-1)(n-\pi_n(i))-(\pi_n(i)-1)(n-i)}{n-1}$.

PROOF. Given $\pi_n(i) = j$ and $a_n(i) = a$, $b_n(i) = a + j - i =: b$. Hence, to count the number of permutations with $\pi_n(i) = j$ and $a_n(i) = a$, it suffices to choose a indices less than i and b indices greater than i, which are inverted with i, and then arranging them in $(j-1)!(n-j)!$ ways. This gives

$$\begin{aligned}\mathbb{P}\big(a_n(i) = a|\pi_n(i) = j\big) &= \frac{1}{(n-1)!}\binom{i-1}{a}\binom{n-i}{b}(j-1)!(n-j)! \\ &= \frac{(i-1)!(j-1)!(n-i)!(n-j)!}{a!(i-1-a)!b!(n-i-b)!(n-1)!} \\ &= \frac{\binom{i-1}{a}\binom{(n-1)-(i-1)}{(n-j)-a}}{\binom{n-1}{n-j}},\end{aligned}$$

and so $a_n(i)|\pi_n(i) = j$ follows the hypergeometric distribution with aforementioned parameters.

Therefore, $\mathbb{E}(d_n(i)|\pi_n(i) = j) = \mathbb{E}(a_n(i)|\pi_n(i) = j) + \mathbb{E}(b_n(i)|\pi_n(i) = j) = \frac{(i-1)(n-j)+(j-1)(n-i)}{n-1}$. To prove the second conclusion note that

$$\begin{aligned}&\left|d_n(i) - \frac{(i-1)(n-j)+(j-1)(n-i)}{n-1}\right| > R \\ \Leftrightarrow\quad &\left|a_n(i) - \frac{(i-1)(n-j)}{n-1}\right| > \frac{R}{2}.\end{aligned}$$

An application of the bound in [46] now gives the desired conclusion. $\square$

5.2.2. *Proof of Theorem* 3.4. Let $Z_n = \sqrt{n}(\frac{d_n(\lceil n/2\rceil)}{n} - \frac{1}{2})$ and $m_n(j) = \frac{(\lceil n/2\rceil-1)(n-j)}{n-1}$. Now, fixing $\delta > 0$,

$$\begin{aligned}&\mathbb{P}(Z_n \le x) \\ &\quad= \frac{1}{n}\sum_{j=1}^{n}\mathbb{P}\big(Z_n \le x|\pi_n(\lceil n/2\rceil) = j\big) \\ &\quad= \frac{1}{n}\sum_{n\delta\le j\le n(1-\delta)}\mathbb{P}\big(Z_n \le x|\pi_n(\lceil n/2\rceil) = j\big) + O(\delta) \qquad (5.1)\\ &\quad= \frac{1}{n}\sum_{n\delta\le j\le n(1-\delta)}\mathbb{P}\bigg(\frac{a_n(\lceil n/2\rceil) - m_n(j)}{n^{\frac{1}{2}}} \le \lambda_n(x,j)\bigg|\pi_n(\lceil n/2\rceil) = j\bigg) \\ &\qquad+ O(\delta),\end{aligned}$$

where $\lambda_n(x, j)$ satisfies $\lim_{n\to\infty}\max_{n\delta\le j\le n(1-\delta)}|\lambda_n(x, j) - x/2| = 0$, for all $x \in \mathbb{R}$.

By Proposition 5.1,

$$
\begin{aligned}
\sigma_n^2(\lceil n/2\rceil), j) &:= \operatorname{Var}(a_n(\lceil n/2\rceil)|\pi_n(\lceil n/2\rceil) = j) \\
&= \frac{(\lceil n/2\rceil - 1)(j-1)(n-\lceil n/2\rceil)(n-j)}{(n-1)^2(n-2)} \geq C(\delta)n,
\end{aligned} \tag{5.2}
$$

for some $C(\delta) > 0$, and for all $j \in [n\delta, n(1-\delta)]$. Using the Berry–Esseen theorem for hypergeometric distribution [34], Theorem 2.2, there exists a universal constant C such that with $C'(\delta) := C/\sqrt{C(\delta)} < \infty$,

$$
\begin{aligned}
&\left|\mathbb{P}\left(\frac{a_n(\lceil n/2\rceil) - \mu_n(j)}{n^{\frac{1}{2}}} \leq \lambda_n(x,j)\middle|\pi_n(i) = j\right) - \Phi\left(\sqrt{n}\cdot\frac{\lambda_n(x,j)}{\sigma_n(\lceil n/2\rceil, j)}\right)\right| \\
&\quad \leq \frac{C'(\delta)}{n^{\frac{1}{2}}}.
\end{aligned} \tag{5.3}
$$

Finally, note that

$$
\max_{n\delta \leq j \leq n(1-\delta)} \left|\frac{\sigma_n^2(\lceil n/2\rceil, j)}{n} - \frac{j(n-j)}{4n^2}\right| = o(1),
$$

where the $o(1)$ term goes to zero as $n \to \infty$. Moreover, since the function Φ is uniformly continuous on $\mathbb{R}$,

$$
\max_{n\delta \leq j \leq n(1-\delta)} \left|\Phi\left(\sqrt{n}\cdot\frac{\lambda_n(x,j)}{\sigma_n(\lceil n/2\rceil, j)}\right) - \Phi\left(\frac{x}{\sqrt{(j/n)(1-j/n)}}\right)\right| = o(1). \tag{5.4}
$$

Combining (5.1), (5.3) and (5.4), we have

$$
\mathbb{P}(Z_n \leq x) = \frac{1}{n} \sum_{n\delta \leq j \leq n(1-\delta)} \Phi\left(\frac{x}{\sqrt{(j/n)(1-j/n)}}\right) + o(1) + O(\delta).
$$

On taking limits as $n \to \infty$ followed by $\delta \to 0$, we have

$$
\lim_{n\to\infty} \mathbb{P}(Z_n \leq x) = \int_0^1 \Phi\left(\frac{x}{\sqrt{u(1-u)}}\right) \mathrm{d}u,
$$

which completes the proof of the theorem.

6. Convergence of the permutation process.

The convergence of the permutation process requires some regularity assumptions. In this section, we introduce the notion of equicontinuity for a sequence of random permutations, and verify this for most standard exponential models on permutations.

6.1. *Equicontinuous permutations.* Let $\mathscr{F}$ be a family of functions from $[0, 1]$ to $\mathbb{R}$. The family $\mathscr{F}$ is *equicontinuous at a point* $x_0 \in [0, 1]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|f(x_0) - f(x)\right| < \varepsilon \qquad \text{for all } f \in \mathscr{F} \text{ and all } x \text{ such that } |x - x_0| < \delta.$$

The family $\mathscr{F}$ is *equicontinuous* if it is equicontinuous at each point in $[0, 1]$.

To introduce the notion of equicontinuous permutations, we need some definitions: For $s \geq 1$ fixed, define

$$\mathcal{P}_n(s) := \left\{(j_1, j_2, \dots, j_s) \in [n]^s : j_a \neq j_b, \text{for } a \neq b \in [s]\right\}. \tag{6.1}$$

For $\mathbf{j} = (j_1, j_2, \dots, j_s) \in \mathcal{P}_n(s)$, define the function $r_{\pi_n}(\cdot|\mathbf{j}) : (0, 1]^s \to [0, 1]$ as follows:

$$\begin{aligned} r_{\pi_n}(\mathbf{x}|\mathbf{j}) &= r_{\pi_n}(x_1, x_2, \dots, x_s|\mathbf{j}) \\ &:= \mathbb{P}\big(\pi_n(\lceil nx_1\rceil) = j_1, \dots, \pi_n(\lceil nx_s\rceil) = j_s\big). \end{aligned} \tag{6.2}$$

DEFINITION 6.1. A sequence $\{\pi_n\}_{n\geq 1}$ of random permutations is said to be *equicontinuous* if for all $s \geq 1$, the following holds:

$$\lim_{\delta\to 0}\lim_{n\to\infty} \sup_{\substack{(\mathbf{x},\mathbf{y})\in B_s(\delta)\\ \mathbf{j}\in\mathcal{P}_n(s)}} \left|\frac{r_{\pi_n}(\mathbf{x}|\mathbf{j})}{r_{\pi_n}(\mathbf{y}|\mathbf{j})} - 1\right| = 0, \tag{6.3}$$

where $B_s(\delta) = \{x_1, \dots, x_s, y_1, \dots, y_s \in (0, 1] : \max_{i\in[s]} |x_i - y_i| \leq \delta\}$.

Informally, the above definition says that a sequence of random permutations $\{\pi_n\}_{n\geq 1}$ is *equicontinuous* if for all $s \geq 1$, the collection of functions $\{r_{\pi_n}(\cdot|\mathbf{j})\}_{\mathbf{j}\in\mathcal{P}_n(s)}$ is uniformly equicontinuous. Next, we show that if a sequence of permutons converges and the permutations are equicontinuous, then the permutation process also converges.

PROPOSITION 6.1. *Let $\pi_n \in S_n$ be a sequence of random equicontinuous permutations such that the permuton sequence $\nu_{\pi_n} \stackrel{D}{\to} \mu$. Then the permutation process*

$$\pi_n(\cdot) \stackrel{w}{\Rightarrow} Z(\cdot),$$

where the finite-dimensional distributions of $Z(\cdot)$ are as follows: *Let (X_1, Y_1), $(X_2, Y_2), \dots, (X_a, Y_a)$ be independent draws from the random measure $\mu \in \mathcal{M}$. Then*

$$\mathscr{L}\big(Z(x_1), \dots, Z(x_s)\big) \sim \mathscr{L}\big(Y_1|X_1 = x_1, \dots, Y_a|X_a = x_a\big), \tag{6.4}$$

for $0 < x_1 < x_2 < \cdots < x_a \leq 1$.

PROOF. Let $s \geq 1$ and $g_1, g_2, \dots, g_s$ be continuous functions from $(0, 1] \mapsto [0, 1]$. For any sequence $\pi_n \in S_n$ of random permutations define function $G^{(n)}_{g_1,g_2,\dots,g_s} : (0, 1]^a \to [0, 1]$ by

$$G^{(n)}_{g_1,g_2,\dots,g_s}(x_1, x_2, \dots, x_s) = \mathbb{E}\left(\prod_{i=1}^{s} g_i\big(\pi_n(x_i)\big)\right). \tag{6.5}$$

To begin with, we show that the sequence of functions $\{G^{(n)}_{g_1,g_2,\dots,g_s}(\cdot)\}_{n\geq 1}$ is uniformly equicontinuous on $(0, 1]^s$: Indeed, recalling (6.2) we have

$$\begin{aligned}
&\sup_{\mathbf{x},\mathbf{y}\in B_s(\delta)} \big|G^{(n)}_{g_1,g_2,\dots,g_s}(\mathbf{x}) - G^{(n)}_{g_1,g_2,\dots,g_s}(\mathbf{y})\big| \\
&\quad \leq \sup_{\mathbf{x},\mathbf{y}\in B_s(\delta)} \sum_{\mathbf{j}\in\mathcal{P}_n(s)} \left|\prod_{k=1}^{s} g_k\left(\frac{j_k}{n}\right)\right| \cdot \big|r_{\pi_n}(\mathbf{x}|\mathbf{j}) - r_{\pi_n}(\mathbf{y}|\mathbf{j})\big| \\
&\quad \leq \sup_{\mathbf{x},\mathbf{y}\in B_s(\delta)} \sum_{\mathbf{j}\in\mathcal{P}_n(s)} \big|r_{\pi_n}(\mathbf{x}|\mathbf{j}) - r_{\pi_n}(\mathbf{y}|\mathbf{j})\big| \\
&\quad \leq \sup_{\substack{(\mathbf{x},\mathbf{y})\in B_s(\delta) \\ \mathbf{j}\in\mathcal{P}_n(s)}} \left|\frac{r_{\pi_n}(\mathbf{x}|\mathbf{j})}{r_{\pi_n}(\mathbf{y}|\mathbf{j})} - 1\right|,
\end{aligned} \tag{6.6}$$

which converges to 0 as δ converges to 0, uniformly in n. This proves the equicontinuity of $\{G^{(n)}_{g_1,g_2,\dots,g_s}(\cdot)\}_{n\geq 1}$.

Proceeding to prove (6.4), by Proposition A.1 it suffices to prove convergence of finite-dimensional distributions. Thus, fixing $s \geq 1$ and continuous functions $g_1, g_2, \dots, g_s : (0, 1] \mapsto [0, 1]$ and $x_1, \dots, x_s \in (0, 1]$ it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\prod_{i=1}^{s} g_i\big(\pi_n(x_i)\big) = \mathbb{E}\left(\prod_{i=1}^{a} g_i(Y_i)|X_1 = x_1, \dots, X_s = x_s\right). \tag{6.7}$$

To show this, define a finite collection of random variables $\{(U_{j,n}, V_{j,n})\}_{j=1}^{s}$, where $\{U_{j,n}\}_{j=1}^{s}$ are i.i.d. Unif[0, 1] independent of π_n, and $V_{j,n} := \pi_n(U_{j,n})$. Then, as in (4.4) we have

$$\begin{aligned}
\mathbb{E}\prod_{j=1}^{s} g_j(U_{j,n}, V_{j,n}) &= \int_{[0,1]^s} \mathbb{E}\left(\prod_{j=1}^{s} g_j\big(x_j, \pi_n(x_j)\big)\right) \prod_{j=1}^{s} \mathrm{d}x_j \\
&= \mathbb{E}\prod_{j=1}^{s} \bar{\nu}_{\pi_n}(g_j) + o(1).
\end{aligned} \tag{6.8}$$

Since $\bar{\nu}_{\pi_n} \xrightarrow{D} \mu$, the RHS of (6.8) converges to the quantity $\mathbb{E}\prod_{j=1}^{s}\mu(g_j) = \mathbb{E}\prod_{j=1}^{s} g_j(X_j, Y_j)$. Thus, the joint law of $\{(U_{j,n}, V_{j,n})\}_{j=1}^{a}$ converges to the joint law of $\{(X_j, Y_j)\}_{j=1}^a$. Now, as

$$\mathbb{E}\left(\prod_{j=1}^{s} g_j(V_{j,n})|U_{1,n} = x_1, \dots, U_{s,n} = x_s\right) = \mathbb{E}\prod_{j=1}^{s} g_j\big(\pi_n(x_s)\big), \tag{6.9}$$

to prove (6.7) it suffices to show that the corresponding conditional distributions converge:

$$\mathscr{L}\big(\{V_{j,n}\}_{j=1}^s|\{U_{j,n}\}_{j=1}^s\big) \xrightarrow{D} \mathscr{L}\big(\{Y_j\}_{j=1}^s|\{X_j\}_{j=1}^s\big).$$

This follows by the equicontinuity of $\{G_{g_1,g_2,\dots,g_s}^{(n)}\}_{n\geq 1}$ and an application of [49], Theorem 4. $\square$

6.2. *Exponential models on permutations*. Let $\theta \in \mathbb{R}$, and $T_n : S_n \to \mathbb{R}$ be any function. Suppose $\pi_n \in S_n$ is a sequence of random permutations with probability mass function

$$Q_{n,\theta}(\sigma) := \frac{e^{\theta T_n(\sigma)}}{\sum_{\sigma\in S_n} e^{\theta T_n(\sigma)}}. \tag{6.10}$$

One of the most common exponential models on permutations, is the Mallows model, where $T_n(\sigma)$ is the number of inversions of σ scaled by n. The limiting degree process of the Mallows random permutation is explicitly computed in Section 7. Here, we determine a simple criterion for the convergence of the degree process of a sequence of random permutations distributed as (6.10). We begin with the following technical definition.

DEFINITION 6.2. Fix $s \geq 1$ and $\mathbf{j} \in \mathcal{P}_n(s)$ [defined in (6.1)]. For a fixed vector $\mathbf{x} = (x_1, x_2, \dots, x_s) \in (0, 1]^s$, denote by

$$\Gamma(\mathbf{j}, \mathbf{x}) = \big\{\pi_n \in S_n : \pi_n\big(\lceil nx_1\rceil\big) = j_1, \dots, \pi_n\big(\lceil nx_s\rceil\big) = j_s\big\}.$$

For any two fixed vectors $\mathbf{x} = (x_1, x_2, \dots, x_s) \in (0, 1]^s$ and $\mathbf{y} = (y_1, y_2, \dots, y_s) \in (0, 1]^s$, define the bijection

$$\Phi_{\mathbf{x},\mathbf{y}} : \Gamma(\mathbf{j}, \mathbf{x}) \to \Gamma(\mathbf{j}, \mathbf{y})$$

as follows: for each $\pi_n \in \Gamma(\mathbf{j}, \mathbf{x})$, define its image $\widetilde{\pi}_n$ as

$$\widetilde{\pi}_n(k) = \begin{cases} j_t = \pi_n\big(\lceil nx_t\rceil\big), & \text{if } k = \lceil ny_t\rceil \text{ for } t \in [s], \\ \pi_n\big(\lceil ny_t\rceil\big), & \text{if } k = \lceil nx_t\rceil \text{ for } t \in [s], \\ \pi_n(k), & \text{otherwise}. \end{cases} \tag{6.11}$$

Informally, $\Phi_{\mathbf{x},\mathbf{y}}$ takes a permutation π_n and interchanges the coordinates $\lceil nx_1\rceil, \lceil nx_2\rceil, \dots, \lceil nx_s\rceil$ to the coordinates $\lceil ny_1\rceil, \lceil ny_2\rceil, \dots, \lceil ny_s\rceil$ to get $\widetilde{\pi}_n$. Using this bijection it is easy to get a sufficient condition for the convergence of the permutation process for exponential models.

COROLLARY 6.2. *A sequence of random permutations* $\{\pi_n\}_{n\geq 1}$ *from* (6.10) *is equicontinuous whenever for all* $s \geq 1$ *the following holds*:

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{\substack{(\mathbf{x},\mathbf{y})\in B(\delta)\\ \mathbf{j}\in\mathcal{P}_n(s)}}\sup_{\pi_n\in\Gamma(\mathbf{j},\mathbf{x})}\big|T_n(\pi_n)-T_n\big(\Phi_{\mathbf{x},\mathbf{y}}(\pi_n)\big)\big|=0, \tag{6.12}$$

where $\Phi_{\mathbf{x},\mathbf{y}}$ *is the bijection defined in* (6.11).

PROOF. Fix $s \geq 1$, $j \in \mathcal{P}_n(s)$, and $\mathbf{x}, \mathbf{y} \in B(\delta)$. Let $\widetilde{\pi}_n$ be the image of π_n defined by the bijection $\Phi_{\mathbf{x},\mathbf{y}}$ in (6.11). Then using (6.12), for $\varepsilon > 0$ arbitrary there exists $\delta = \delta(\varepsilon)$ and $N = N(\varepsilon, \delta)$ such that for all $\delta < \delta(\varepsilon)$ and $n \geq N(\varepsilon, \delta)$ we have

$$\sup_{\substack{(\mathbf{x},\mathbf{y})\in B(\delta)\\ \mathbf{j}\in\mathcal{P}_n(s)}}\sup_{\pi_n\in\Gamma(\mathbf{j},\mathbf{x})}\big|T_n(\pi_n)-T_n\big(\Phi_{\mathbf{x},\mathbf{y}}(\pi_n)\big)\big|\leq\varepsilon.$$

Along with (6.10), for any $\pi_n \in \Gamma(\mathbf{j}, \mathbf{x})$ this gives

$$\left|\frac{Q_{n,\theta}(\pi_n)}{Q_{n,\theta}(\Phi_{\mathbf{x},\mathbf{y}}(\pi_n))}-1\right|=\left|e^{\theta T_n(\pi_n)-\theta T_n(\Phi_{\mathbf{x},\mathbf{y}}(\pi_n))}-1\right|\leq e^{|\theta|\varepsilon}-1,$$

and so

$$\left|\frac{\mathbb{P}(\pi_n\in\Gamma(\mathbf{j},\mathbf{x}))}{\mathbb{P}(\pi_n\in\Gamma(\mathbf{j},\mathbf{y}))}-1\right|=\left|\frac{\sum_{\pi_n\in\Gamma(\mathbf{j},\mathbf{x})}Q_{n,\theta}(\pi_n)}{\sum_{\widetilde{\pi}_n\in\Gamma(\mathbf{j},\mathbf{y})}Q_{n,\theta}(\widetilde{\pi}_n)}-1\right|\leq e^{|\theta|\varepsilon}-1.$$

The conclusion follows as $\varepsilon > 0$ is arbitrary. □

Consider the following general class of 1-parameter exponential family on the space of permutations S_n, with probability mass function

$$Q_{n,f,\theta}(\sigma)=\frac{e^{\theta\sum_{i=1}^n f(\frac{i}{n},\frac{\sigma(i)}{n})}}{\sum_{\sigma\in S_n}e^{\theta\sum_{i=1}^n f(\frac{i}{n},\frac{\sigma(i)}{n})}}, \tag{6.13}$$

where $f : [0, 1]^2 \to [0, 1]$ is any continuous function. This is a special case of the model (6.10) with $T_n(\sigma) = \sum_{i=1}^n f(\frac{i}{n}, \frac{\sigma(i)}{n})$. Popular choices of the function f includes the *Spearman's Rank Correlation Model*: $f(x, y) = -(x - y)^2$ and the *Spearman's Footrule Model*: $f(x, y) = -|x - y|$. These models find applications in statistics for analyzing ranked data [14, 15]. Feigin and Cohen [20] gave a nice application of such models for analyzing agreement between several judges in a contest. For other choices of f and their various properties, refer to Diaconis [14]. Consistent estimation of parameters in such models has been studied recently by Mukherjee [43].

Using Corollary 6.2, it can easily shown that any sequence of random permutations $\{\pi_n\}_{n\geq 1}$ distributed as (6.13) is equicontinuous, that is, their corresponding degree process converges.

COROLLARY 6.3. *Fix $\theta \in \mathbb{R}$ and $f : [0,1]^2 \to [0,1]$ be any continuous function. Let $\pi_n \in S_n$ be a sequence of random permutations distributed as* (6.13). *Then $\{\pi_n\}_{n\geq 1}$ is equicontinuous and the degree process $d_n(\cdot)$ converges.*

PROOF. The convergence of the degree process follows from the equicontinuity (Proposition 6.1), as convergence of ν_{π_n} was already verified in [43], Theorem 1.5. To prove equicontinuity, let $T_n(\sigma) = \sum_{i=1}^n f(i/n, \sigma(i)/n)$. Fix $\delta > 0$, $s \geq 1$, $\mathbf{j} = (j_1, j_2, \dots, j_s) \in \mathcal{P}_n(s)$, and let $(\mathbf{x}, \mathbf{y}) \in B(\delta)$. For $\pi_n \in \Gamma(\mathbf{j}, \mathbf{x})$ and $\widetilde{\pi}_n \in \Gamma(\mathbf{j}, \mathbf{y})$, and using the bijection (6.11) we get

$$\begin{aligned} \left|T_n(\pi_n) - T_n(\widetilde{\pi}_n)\right| &\leq \sum_{t=1}^{s}\left| f\left(\frac{\lceil nx_t\rceil}{n}, \frac{\pi_n(\lceil nx_t\rceil)}{n}\right) - f\left(\frac{\lceil ny_t\rceil}{n}, \frac{\pi_n(\lceil nx_t\rceil)}{n}\right)\right| \\ &\leq s \sup_{\substack{|x_1-x_2|\leq\delta+\frac{1}{n},\\ y\in[0,1]}} \left|f(x_1,y) - f(x_2,y)\right|, \end{aligned} \tag{6.14}$$

which goes to 0, after taking $n \to \infty$ and $\delta \to 0$, by the continuity of f. □

7. Degree process of the mallows random permutation: Proof of Theorem 3.6.

Recall the Mallows probability mass function $m_{\beta,n}(\cdot)$ from (3.4). Starr [47] derived the limiting density of the empirical permutation measure associated with a sequence of Mallows random permutations:

THEOREM 7.1 (Starr [47]). *Let $\pi_n \in S_n$ be a Mallows random permutation with parameter β. Then the empirical permutation measure $\bar{\nu}_{\pi_n}$ converges weakly in distribution to a random variable which has density in $[0,1]^2$ given by*

$$m_\beta(x,y) = \frac{\frac{\beta}{2}\sinh(\frac{\beta}{2})}{(\exp(\frac{\beta}{4})\cosh(\frac{1}{2}\beta(x-y)) - \exp(-\frac{\beta}{4})\cosh(\frac{1}{2}\beta(x+y-1)))^2}, \tag{7.1}$$

and distribution function

$$M_\beta(a,b) = -\frac{1}{\beta}\log\left(1 - \frac{2\exp(-\frac{1}{2}\beta(a+b-1))(\sinh(\frac{a\beta}{2})\sinh(\frac{\beta b}{2}))}{\sinh(\frac{\beta}{2})}\right). \tag{7.2}$$

PROOF. The proof of (7.1) can be found in Starr [47]. The expression for the distribution function (7.2) follows by directly integrating the density (7.1) m_β (see also Starr and Walters [48], Theorem 2.4). □

The above theorem together with Theorem 3.1 can be used to derive the limiting densities of the degree proportions in a Mallows random permutation.

7.1. *Proof of Theorem* 3.6. The proof of Theorem 3.6 has two parts: to show the existence of limit of the degree process $d_n(\cdot)$ by verifying (6.12) in Corollary 6.2, and the explicit computation of the density of the limiting distribution using Theorem 7.1.

7.1.1. *Existence of the limit.* In this section, we show that the degree process of a Mallows random permutation converges. In light of Proposition 6.1, it suffices to verify that the Mallows random permutation is equicontinuous:

LEMMA 7.1. *Let $\beta \in \mathbb{R}$ and $\pi_n \sim M_{\beta,n}$ be a sequence Mallows random permutations. Then $\{\pi_n\}_{n\geq 1}$ is equicontinuous.*

PROOF. Fix $\varepsilon > 0$ and $1 \leq i_1 < i_2 < \cdots < i_s \leq n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_s) \in (0,1]^s$ and $\mathbf{y} = (y_1, y_2, \ldots, y_s) \in (0,1]^s$ and consider the bijection (6.11) between $\Gamma(i_1, i_2, \ldots, i_s, \mathbf{x})$ and $\Gamma(i_1, i_2, \ldots, i_s, \mathbf{y})$. If $\widetilde{\pi}_n$ denotes the image of π_n under this bijection, then

$$\frac{1}{n}\bigl|\lambda(\pi_n) - \lambda(\widetilde{\pi}_n)\bigr| \leq \frac{1}{n}\sum_{a=1}^{s}\bigl|\lceil nx_a\rceil - \lceil ny_a\rceil\bigr| \leq s\delta + \frac{2s}{n},$$

which goes to zero after taking limits as $n \to \infty$ and $\delta \to 0$. Equicontinuity of $\{\pi_n\}_{n\geq 1}$ now follows from Corollary 6.2. □

The above result and Proposition 6.1 implies that the permutation process $\pi_n(\cdot) \stackrel{w}{\Rightarrow} W_\beta(\cdot)$, such that for every $t \geq 0$, $W_\beta(t)$ is independent and distributed according to conditional law of $Q_2|Q_1 = t$, where $(Q_1, Q_2) \sim M_\beta$. Since the distribution of M_β of (Q_1, Q_2) has uniform marginals,

$$\mathbb{P}\bigl(W_\beta(t) \leq w\bigr) = \int_0^w m_\beta(t, y)\,\mathrm{d}y, \tag{7.3}$$

and $W_\beta(t)$ has density $m_\beta(t,\cdot)$. Theorem 3.1 then implies that $d_n(\cdot) \stackrel{w}{\Rightarrow} D_\beta(\cdot)$, where

$$D_\beta(t) = t + W_\beta(t) - 2M_\beta\bigl(t, W_\beta(t)\bigr),$$

and $D_\beta(t)$ is independent for all $t \geq 0$. Therefore, for indices $0 \leq r_1 < r_2 < \cdots < r_s \leq 1$,

$$\begin{aligned}&\left(\frac{d_n(\lceil nr_1\rceil)}{n}, \frac{d_n(\lceil nr_2\rceil)}{n}, \ldots, \frac{d_n(\lceil nr_s\rceil)}{n}\right)\\ &\qquad\stackrel{D}{\to} \bigl(D_\beta(r_1), D_\beta(r_2), \ldots D_\beta(r_s)\bigr),\end{aligned}$$

as desired.

7.1.2. *Calculating the limiting density.* Fix $\beta \in \mathbb{R}$ and $a \in [0,1]$ and suppose $W \sim W_\beta(a)$ be distributed as in (7.3). To find the density of $D_{a,\beta}$ for $a \in [0,1]$, we have to find the density of the random variable

$$
\begin{aligned}
J_{a,\beta}(W) & \\
&:= a + W - 2M_\beta(a, W) \\
&= a + W + \frac{2}{\beta}\log\left(1 - \frac{2\exp(-\frac{1}{2}\beta(a+W-1))(\sinh(\frac{\beta a}{2})\sinh(\frac{\beta W}{2}))}{\sinh(\frac{\beta}{2})}\right) \\
&= a + W + \frac{2}{\beta}\log\big(1 - \varphi_\beta(a)\big(1 - e^{-\beta W}\big)\big),
\end{aligned} \tag{7.4}
$$

where

$$\varphi_\beta(a) := e^{\frac{1}{2}(\beta - \beta a)}\operatorname{csch}\left(\frac{\beta}{2}\right)\sinh\left(\frac{a\beta}{2}\right).$$

We begin by establishing properties of the function $J_{a,\beta} : \mathbb{R} \to \mathbb{R}$ defined as $J_{a,\beta}(w) = a + w - 2M_\beta(a, w)$, for $a \in [0, 1]$. Recall that an interval $[a, b]$ is always interpreted as $[a \vee b, a \wedge b]$.

LEMMA 7.2. *Let $\beta > 0$, $a \in [0, 1]$, and $a_c(\beta)$ be as defined in Theorem 3.6. Then for $J_{a,\beta}$ as defined above, the following hold*:

(a) *The function $J_{a,\beta}$ is strictly convex in $\mathbb{R}$.*

(b) *For $a \in [0, a_c(\beta)]$ the function $J_{a,\beta}$ is strictly increasing and for $a \in [1 - a_c(\beta), 1]$, the function $J_{a,\beta}$ is strictly decreasing in* $[0, 1]$.

(c) *For $a \in (a_c(\beta), 1 - a_c(\beta))$, the function $J_{a,\beta}$ has a minimum at $z_0 \in (0, 1)$, and is strictly decreasing in $[0, z_0)$ and strictly increasing in $(z_0, 1]$.*

PROOF. The derivatives of the function $J_{a,\beta}$ are

$$J'_{a,\beta}(z) = \frac{\mathrm{d}}{\mathrm{d}z}J_{a,\beta}(z) = 1 - \frac{2\varphi_\beta(a)e^{-\beta z}}{1 - \varphi_\beta(a)(1 - e^{-\beta z})}$$

and

$$
\begin{aligned}
J''_{a,\beta}(z) &= -2\varphi_\beta(a) \cdot \frac{-\beta e^{-\beta z}(1 - \varphi_\beta(a)(1 - e^{-\beta z})) + \beta e^{-2\beta z}\varphi_\beta(a)}{(1 - \varphi_\beta(a)(1 - e^{-\beta z}))^2} \\
&= \frac{\beta e^{-\beta z}(1 - \varphi_\beta(a))}{(1 - \varphi_\beta(a)(1 - e^{-\beta z}))^2}.
\end{aligned}
$$

Note that $\varphi_a(\beta) = \frac{1-e^{-a\beta}}{1-e^{-\beta}} \le 1$, for all $\beta > 0$, and so $J_{a,\beta}$ is a convex function.

The convexity of $J_{a,\beta}$ implies that $J'_{a,\beta}$ is increasing, and $J'_{a,\beta}(z) = 0$ has at most one solution z_0 in $[0, 1]$:

$$z_0 := \frac{1}{\beta}\log\left(\frac{\varphi_\beta(a)}{1 - \varphi_\beta(a)}\right) \in (0, 1) \quad \Leftrightarrow \quad a \in \big[a_c(\beta), 1 - a_c(\beta)\big],$$

where $a_c(\beta)$ is defined in Theorem 3.6. Therefore, for $a \notin [a_c(\beta), 1 - a_c(\beta)]$, the function $J_{a,\beta}$ is strictly monotone: for $a \in [0, a_c(\beta)]$ the function $J_{a,\beta}$ is strictly increasing, and for $a \in [1 - a_c(\beta), 1]$, the function $J_{a,\beta}$ is strictly decreasing in $[0, 1]$.

For $a \in [a_c(\beta), 1 - a_c(\beta)]$, the function $J_{a,\beta}$ has a minimum at z_0, and is strictly decreasing in $[0, z_0)$ and strictly increasing in $(z_0, 1]$. $\square$

The above lemma shows that for fixed β, depending on the value of a, the range of the function $J_{a,\beta}$, and hence the support of $D_\beta(a)$, undergoes a phase transition. Calculating the density of $D_\beta(a)$ involves some tedious calculations with Jacobian transformations. The main steps of the calculations are given below. For $\beta > 0$ and $a \in [0, 1]$, recall the definition $R(a, \beta)$ from (3.5) and the function $h_{a,\beta}$ from (3.6):

$$\begin{aligned} h_{a,\beta}(x) &= \frac{\beta e^{\frac{1}{2}\beta(a-x)}}{(1-e^{-\beta})\sqrt{e^{\beta(a+x)} - e^{\beta} R(a,\beta)}} \\ &= \frac{\beta e^{\frac{1}{2}\beta(a-x+2)}}{\sqrt{4e^{\beta}(e^{\beta a}-1)(e^{\beta a}-e^{\beta}) + e^{\beta(a+x)}(1-e^{\beta})^2}}. \end{aligned} \tag{7.5}$$

Assume that $\beta > 0$, and consider the two cases [recall the definition of $J_{a,\beta}$ from (7.4)]:

1. Suppose $a \notin [a_c(\beta), 1 - a_c(\beta)]$. In this case, $J_{a,\beta}$ is monotone in $[0, 1]$ (Lemma 7.2) and the equation $J_{a,\beta}(z) = w$ has a unique solution $J^{-1}_{a,\beta}(w) \in [0, 1]$ (Figure 8). Then by the Jacobian transformation and direct calculations, the density of $J_{a,\beta}(W)$ simplifies to
$$g_{a,\beta}(w) = \left|\frac{\mathrm{d}}{\mathrm{d}w} J^{-1}_{a,\beta}(w)\right| m_\beta\big(a, J^{-1}_{a,\beta}(w)\big) = h_{a,\beta}(w).$$
The support of $J_{a,\beta}(W)$ is $[J_{a,\beta}(0), J_{a,\beta}(1)] = [a, 1 - a]$.
2. Suppose $a \in [a_c(\beta), 1 - a_c(\beta)]$. In this case, for $w \in [0, 1]$ the equation $J_{a,\beta}(z) = w$ has at most two solutions in $[0, 1]$ depending on the value of w (Lemma 7.2).
   - 2.1. $w \in [J_{a,\beta}(0), J_{a,\beta}(1)] = [a, 1 - a]$. This situation is same as the previous case, that is, $J_{a,\beta}(z) = w$ has a unique solution $J^{-1}_{a,\beta}(w) \in [0, 1]$ (Figure 9), and the density of $J_{a,\beta}(W)$ simplifies to
   $$g_{a,\beta}(w) = \left|\frac{\mathrm{d}}{\mathrm{d}w} J^{-1}_{a,\beta}(w)\right| m_\beta\big(a, J^{-1}_{a,\beta}(w)\big) = h_{a,\beta}(w),$$
   for $w \in [a, 1 - a]$.
   - 2.2. $w \notin [J_{a,\beta}(0), J_{a,\beta}(1)] = [a, 1 - a]$ (refer to Figure 9). In this case, $J_{a,\beta}(z) = w$ has two solutions given by $J^{-1}_{a,\beta,1}(w) = -\frac{2}{\beta}\log\gamma_1(w)$ and

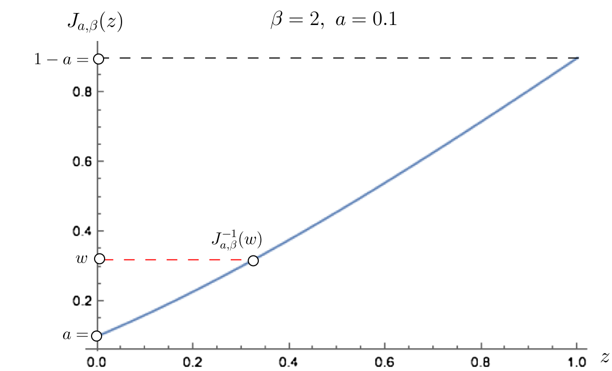

FIG. 8. *Plot of $J_{a,\beta}$ for $\beta = 2$ and $a = 0.1$. For $\beta = 2$, $a_c(\beta) = 0.28311$. Since $0.1 \notin (0.28311, 0.71689)$, the function $J_{0.1,2}$ is monotone with range $[0.1, 0.9]$, and for $w \in [0.1, 0.9]$ the equation $J_{0.1,2}(z) = w$ has a unique solution $J^{-1}_{0.1,2}(w) \in [0, 1]$.*

$J^{-1}_{a,\beta,2}(w) = -\frac{2}{\beta} \log \gamma_2(w)$, where $\gamma_1(w)$ and $\gamma_2(w)$ are roots of the quadratic

$$\gamma e^{\frac{\beta}{2}(w-a)} + 1 = \varphi_\beta(a)\big(1 - \gamma^2\big)$$
$$\Rightarrow \quad \varphi_\beta(a)\gamma^2 + \gamma e^{\frac{\beta}{2}(w-a)} + 1 - \varphi_\beta(a) = 0.$$

The above quadratic equation is obtained by simplifying the equation $J_{a,\beta}(z) = w$ and substituting $\gamma = e^{-\beta z/2}$. Then by the Jacobian transfor-

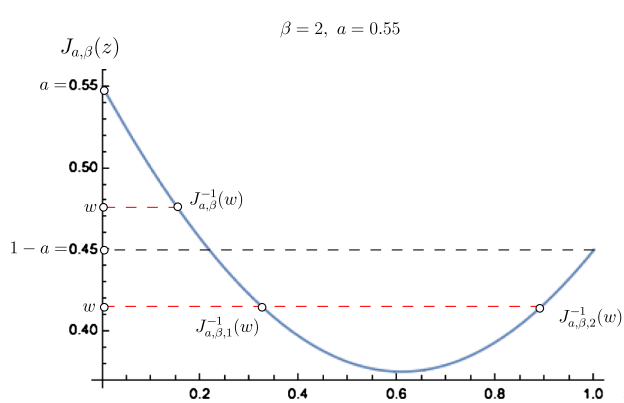

FIG. 9. *Plot of $J_{a,\beta}$ for $\beta = 2$ and $a = 0.55$. For $\beta = 2$, $a_c(\beta) = 0.28311$, and $0.55 \in (0.28311, 0.71689)$. Then, the function $J_{0.55,2}$ is convex with minimum at 0.375. Therefore, for $w \in [0.45, 0.55]$ the equation $J_{0.55,2}(z) = w$ has a unique solution $J^{-1}_{0.55,2}(w) \in [0, 1]$. However, for $w \in [0.375, 0.45]$, the equation $J_{0.55,2}(z) = w$ has two solutions $J^{-1}_{0.55,2,1}(w)$ and $J^{-1}_{0.55,2,2}(w)$.*

mation, the density of $J_{a,\beta}(Z)$ is

$$g_{a,\beta}(w) = \sum_{s=1}^{2} \left| \frac{\mathrm{d}}{\mathrm{d}w} J_{a,\beta,s}^{-1}(w) \right| m_\beta\big(a, J_{a,\beta,s}^{-1}(w)\big). \tag{7.6}$$

Substituting $J_{a,\beta,1}^{-1}(w)$, $J_{a,\beta,2}^{-1}(w)$, and the density $m_\beta(a, \cdot)$ (7.1), and simplifying (7.6) gives $g_{a,\beta}(w) = 2h_{a,\beta}(w)$. Since the function $J_{a,\beta}$ has a minimum at z_0, and is strictly decreasing in $[0, z_0)$ and strictly increasing in $(z_0, 1]$, the support of $J_{a,\beta}(W)$ is

$$\big[J_{a,\beta}(z_0), J_{a,\beta}(0) \vee J_{a,\beta}(1)\big] = \left[1 - a + \frac{1}{\beta} \log R(a, \beta), a \wedge 1 - a\right].$$

For $\beta < 0$ the result follows from the observation: if $\pi_n \sim M_{\beta,n}$, then $\sigma_n(i) = \pi_n(n+1-i)$ is distributed as $M_{-\beta,n}$ (see [6], Lemma 2.2). Note that every interval $[a, b]$ should be interpreted as $[a \wedge b, a \vee b]$.

8. Asymptotics for the minimum degree: Proof of Theorem 3.5. This section gives the proof of the limiting Rayleigh distribution of the minimum degree in a uniformly random permutation graph.

For $i \in [n]$ define

$$c_n(i) = \begin{cases} i + \pi_n(i), & \text{for } 1 \le i < \dfrac{n+1}{2}, \\ 2(n+1) - i - \pi_n(i), & \text{for } \dfrac{n+1}{2} < i \le n. \end{cases} \tag{8.1}$$

The following lemma shows that the degrees $d_n(i)$ can be small (order $\sqrt{n}$) only if $c_n(i)$ is small, which can happen only if i is such that either i or $n + 1 - i$ is small (order $\sqrt{n}$).

LEMMA 8.1. *For any $\gamma \in (0, \infty)$:*

$$\lim_{M\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}, c_n(i) > M\sqrt{n}\big) = 0.$$

PROOF. By symmetry, it suffices to show that

$$\lim_{M\to\infty} \lim_{n\to\infty} \sum_{1 \le i \le \frac{n+1}{2}} \mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}, i + \pi_n(i) > M\sqrt{n}\big) = 0,$$

which follows if we can show the following:

$$\lim_{n\to\infty} \sum_{\frac{n+1}{4} \le i \le \frac{n+1}{2}} \mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}\big) = 0, \tag{8.2}$$

$$\lim_{M\to\infty}\lim_{n\to\infty}\sum_{\frac{M\sqrt{n}}{2}\le i\le\frac{n+1}{4}}\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n}\big)=0, \tag{8.3}$$

$$\lim_{M\to\infty}\lim_{n\to\infty}\sum_{1\le i\le\frac{M\sqrt{n}}{2}}\sum_{\frac{M\sqrt{n}}{2}\le j\le n}\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n},\pi_n(i)=j\big)=0. \tag{8.4}$$

Recall by Proposition 5.1, $a_n(i)|\{\pi_n(i)=j\}$ is hypergeometric with parameters $(n-1,i-1,n-j)$. Therefore, since $b_n(i)|\{\pi_n(i)=j\}=a_n(i)+(j-i)$, we have

$$\begin{aligned}\mathbb{E}\big(d_n(i)|\pi_n(i)=j\big)=\mathbb{E}\big(a_n(i)+b_n(i)\big)&=\frac{2(i-1)(n-j)+(j-i)(n-1)}{n-1}\\&=\frac{j(n-2i+1)+(n+1)i-2n}{n-1}.\end{aligned}$$

Therefore, for $\frac{n+1}{4}\le i\le\frac{n+1}{2}$,

$$\begin{aligned}\mathbb{E}\big(d_n(i)|\pi_n(i)=j\big)-\gamma\sqrt{n}&\ge j\left(\frac{n+1}{2(n-1)}\right)+i\left(\frac{n+1}{n-1}\right)-(\gamma\sqrt{n}+2)\\&\ge\frac{n+1}{8},\end{aligned}$$

for all n large enough. An application of Proposition 5.1 now gives

$$\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n}|\pi_n(i)=j\big)\le 2e^{-\frac{n}{128}}.$$

On adding over i and j gives

$$\begin{aligned}\sum_{\frac{n+1}{4}\le i\le\frac{n+1}{2}}\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n},i+\pi_n(i)>M\sqrt{n}\big)&\le\sum_{\frac{n+1}{4}\le i\le\frac{n+1}{2}}\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n}\big)\\&\le ne^{-\frac{n}{128}},\end{aligned}$$

from which (8.2) follows.

Proceeding to prove (8.3), for $\frac{n+1}{4}\le i\le\frac{n+1}{2}$,

$$\mathbb{E}\big(d_n(i)|\pi_n(i)=j\big)-\gamma\sqrt{n}\ge j\left(\frac{n+1}{2(n-1)}\right)+i-(\gamma\sqrt{n}+2)\ge\frac{i+j}{2}$$

for all $M\ge 4\gamma+8$. Lemma 5.1 gives

$$\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n}|\pi_n(i)=j\big)\le 2\sum_{j\ge1}e^{-(i^2+j^2)/8},$$

which on summing over i and j gives

$$\sum_{\frac{M\sqrt{n}}{2}\le i\le\frac{n+1}{4}}\mathbb{P}\big(d_n(i)\le\gamma\sqrt{n}\big)\le 2\int_{M/2}^{\infty}e^{-x^2/8}\,\mathrm{d}x\int_0^{\infty}e^{-y^2/8}\,\mathrm{d}y.$$

Since the RHS of the above equation goes to 0 on letting $M \to \infty$, (8.3) follows.

Finally, to show (8.4), for $1 \le i \le \frac{M\sqrt{n}}{2}$, and $\frac{M\sqrt{n}}{2} \le j \le n$, note that

$$\mathbb{E}\big(d_n(i)|\pi_n(i) = j\big) - \gamma\sqrt{n} \ge j/2 - \gamma\sqrt{n} - 2 + i \ge \frac{i+j}{4},$$

for $M \ge 4\gamma + 8$. Thus by a similar argument as before, we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{1\le i\le \frac{M\sqrt{n}}{2}} \sum_{\frac{M\sqrt{n}}{2}\le j\le n} & \mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}, \pi_n(i) = j\big) \\ & \le 2\int_0^{M/2} e^{-x^2/32}\,\mathrm{d}x \int_{M/2}^{\infty} e^{-y^2/32}\,\mathrm{d}y,\end{aligned}$$

which goes to 0 as $M \to \infty$ as before. This completes the proof of the lemma. □

The following lemma now strengthens the above result to show that $d_n(i)$ and $c_n(i)$ are close for those indices i where either i or $n + 1 - i$ is small.

LEMMA 8.2. *For any* $\gamma > \varepsilon > 0$:

(a) $\lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}(d_n(i) \le \gamma\sqrt{n}, c_n(i) > (\gamma + \varepsilon)\sqrt{n}) = 0$,
(b) $\lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}(d_n(i) > \gamma\sqrt{n}, c_n(i) \le (\gamma - \varepsilon)\sqrt{n}) = 0$.

PROOF. We claim that for every fixed $M < \infty$, $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{1\le i\le n} \mathbb{P}\big(|d_n(i) - c_n(i)| > \varepsilon\sqrt{n}, c_n(i) \le M\sqrt{n}\big) = 0. \tag{8.5}$$

Proceeding to complete the proof of the lemma using (8.5), note that

$$\begin{aligned}&\mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}, c_n(i) > (\gamma + \varepsilon)\sqrt{n}\big) \\ &\quad\le \mathbb{P}\big(d_n(i) \le \gamma\sqrt{n}, c_n(i) > M\sqrt{n}\big) \\ &\qquad + \mathbb{P}\big(|d_n(i) - c_n(i)| > \varepsilon\sqrt{n}, c_n(i) \le M\sqrt{n}\big).\end{aligned}$$

Summing over i and letting $n \to \infty$ followed by $M \to \infty$, the second term goes to 0 by (8.5), and the first term goes to 0 by Lemma 8.1. This completes the proof of part (a). The proof of part (b) follows by similar calculations.

Turning to the proof of (8.5), note that by symmetry it suffices to show that

$$\lim_{n\to\infty} \sum_{1\le i\le \frac{n+1}{2}} \mathbb{P}\big(|d_n(i) - c_n(i)| > \varepsilon\sqrt{n}, c_n(i) \le M\sqrt{n}\big) = 0,$$

which follows if we can show

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le M\sqrt{n}} \sum_{1\le j\le M\sqrt{n}} \mathbb{P}\big(|d_n(i) - i - j| > \varepsilon\sqrt{n}|\pi_n(i) = j\big) = 0.$$

To this end, for $1 \le i, j \le M\sqrt{n}$,

$$\begin{aligned}\left|\mathbb{E}\big(d_n(i)|\pi_n(i)=j\big)-i-j\right| &= \left|\frac{(i-1)(n-j)+(j-1)(n-i)}{n-1}-i-j\right| \\ &\le 2M,\end{aligned} \tag{8.6}$$

and

$$\mathrm{Var}\big(d_n(i)|\pi_n(i)=j\big)=\frac{(i-1)(j-1)(n-i)(n-j)}{(n-1)^2(n-2)} \le 2M.$$

Therefore, by Chebyshev's inequality,

$$\mathbb{P}\big(\left|d_n(i)-i-j\right| \ge \varepsilon\sqrt{n}|\pi_n(i)=j\big) \le \frac{4M^2}{(\varepsilon\sqrt{n}-2M)^2}.$$

This readily gives

$$\frac{1}{n}\sum_{1\le i\le M\sqrt{n}}\sum_{1\le j\le M\sqrt{n}}\mathbb{P}\big(\left|d_n(i)-i-j\right| > \varepsilon\sqrt{n}|\pi_n(i)=j\big) \le \frac{4M^4}{(\varepsilon\sqrt{n}-2M)^2},$$

which goes to 0 on letting $n \to \infty$, for every $M < \infty$ and $\varepsilon > 0$. □

8.1. *Completing the proof of Lemma* 3.5. Using the above lemmas, we can now complete the proof of the theorem. To this end, it suffices to show that for any $\gamma > 0$

$$\lim_{n\to\infty}\mathbb{P}\big(d_n(i) > \gamma\sqrt{n}, \text{for all } 1 \le i \le n\big) = e^{-\gamma^2/2}. \tag{8.7}$$

Note that

$$\begin{aligned}&\mathbb{P}\big(d_n(i) > \gamma\sqrt{n}, 1 \le i \le n\big) \\ &\quad \le \sum_{i=1}^n \mathbb{P}\big(d_n(i) > \gamma\sqrt{n}, c_n(i) \le (\gamma-\varepsilon)\sqrt{n}\big) \\ &\qquad + \mathbb{P}\big(c_n(i) > (\gamma-\varepsilon)\sqrt{n}, i \in [n]\big),\end{aligned}$$

and so by Lemma 8.2

$$\limsup_{n\to\infty}\mathbb{P}\big(d_n(i) > \gamma\sqrt{n}, 1 \le i \le n\big) \le \limsup_{n\to\infty}\mathbb{P}\big(c_n(i) > (\gamma-\varepsilon)\sqrt{n}, 1 \le i \le n\big).$$

A similar argument gives

$$\limsup_{n\to\infty}\mathbb{P}\big(c_n(i) > \gamma\sqrt{n}, 1 \le i \le n\big)\limsup_{n\to\infty}\mathbb{P}\big(d_n(i) > (\gamma-\varepsilon)\sqrt{n}, 1 \le i \le n\big),$$

and so to prove (8.7) and hence the theorem, it suffices to prove the following lemma.

LEMMA 8.3. *Let $c_n(\cdot)$ be as defined in* (8.1). *Then for any $\gamma > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}\Big) = e^{-\gamma^2}. \tag{8.8}$$

PROOF. Note that

$$\begin{aligned}\mathbb{P}\Big(\min_{1\le i\le \gamma\sqrt{n}} c_n(i) > \gamma\sqrt{n}\Big) &= \mathbb{P}\big(\pi_n(j) > \gamma\sqrt{n} - j, 1 \le j \le \gamma\sqrt{n}\big)\\ &= \frac{(n - \lfloor\gamma\sqrt{n}\rfloor + 1)^{\lfloor\gamma n\rfloor}}{n(n-1)\cdots(n - \lfloor\gamma n\rfloor + 1)} \\ &= \frac{(n - \lfloor\gamma\sqrt{n}\rfloor + 1)^{\lfloor\gamma n\rfloor}(n - \lfloor\gamma n\rfloor)!}{n!}.\end{aligned} \tag{8.9}$$

Moreover,

$$\begin{aligned}&\mathbb{P}\Big(\min_{n+1-\gamma\sqrt{n}\le j\le n} c_n(j) > \gamma\sqrt{n}|\pi_n(i), 1 \le i \le \gamma\sqrt{n}\Big)\\ &\quad= \mathbb{P}\big(\pi_n(j) < 2(n+1) - j - \gamma\sqrt{n}, n + 1 - \gamma\sqrt{n} \le j \le n|\pi_n(i),\\ &\qquad 1 \le i \le \gamma\sqrt{n}\big)\\ &\quad\ge \frac{(n - 1 - 2\lfloor\gamma\sqrt{n}\rfloor)^{\lfloor\gamma\sqrt{n}\rfloor}(n - 2\lfloor\gamma\sqrt{n}\rfloor)!}{(n - \lfloor\gamma\sqrt{n}\rfloor)!},\end{aligned} \tag{8.10}$$

where the lower bound uses the following argument: The probability of the event is minimized when all the $\pi_n(i)$, for $i \in [\lfloor\gamma\sqrt{n}\rfloor]$, are at most $n + 1 - \gamma\sqrt{n}$. This minimizes the choices of $\pi_n(j)$, for $n+1-\gamma\sqrt{n} \le j \le n$. In this case, each $\pi_n(j)$ has $(n - 1 - 2\lfloor\gamma\sqrt{n}\rfloor)$ choices, and the bound follows.

Denote $A_{n,\gamma} = \{\min_{1\le i\le\gamma\sqrt{n}} c_n(i) > \gamma\sqrt{n}\}$. Now, combining (8.10) and (8.9) and taking limits as $n \to \infty$ gives the lower bound

$$\begin{aligned}&\mathbb{P}\Big(\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}\Big)\\ &\quad= \mathbb{E}\Big(\mathbb{P}\Big(\min_{n+1-\gamma\sqrt{n}\le j\le n} c_n(j) > \gamma\sqrt{n}|\pi_n(i), 1 \le i \le \gamma\sqrt{n}\Big)\mathbf{1}\{A_{n,\gamma}\}\Big)\\ &\quad\ge \frac{(n - \lfloor\gamma\sqrt{n}\rfloor + 1)^{\lfloor\gamma n\rfloor}(n - 1 - 2\lfloor\gamma\sqrt{n}\rfloor)^{\lfloor\gamma\sqrt{n}\rfloor}(n - \lfloor\gamma n\rfloor)!(n - 2\lfloor\gamma\sqrt{n}\rfloor)!}{n!(n - \lfloor\gamma\sqrt{n}\rfloor)!}\\ &\quad\to e^{-\gamma^2}.\end{aligned} \tag{8.11}$$

For the upper bound, setting $N_n := |\{1 \le i \le \gamma\sqrt{n} : \pi_n(i) \ge n + 1 - \gamma\sqrt{n}\}|$ and fixing a large integer M we have

$$\begin{aligned}\mathbb{P}\Big(\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}\Big) &\le \mathbb{P}\Big(\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}, N_n \le M\Big) + \mathbb{P}(N_n > M)\\ &\le \mathbb{P}\Big(\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}, N_n \le M\Big) + \frac{\mathbb{E}N_n}{M},\end{aligned} \tag{8.12}$$

by Markov's inequality. Now, since

$$\mathbb{E}N_n = \sum_{i=1}^{\lfloor \gamma\sqrt{n}\rfloor} \frac{\lfloor \gamma\sqrt{n}\rfloor}{n} \le \gamma^2,$$

the second term in the RHS of (8.12) to 0 after taking limits as $n \to \infty$ followed by $M \to \infty$. Again, by a similar argument as the lower bound, on the set $\{N_n \le M\}$ we have

$$\begin{aligned}\mathbb{P}\Big(&\min_{n+1-\gamma\sqrt{n}\le j\le n} c_n(j) > \gamma\sqrt{n} | \pi_n(i), 1 \le i \le \gamma\sqrt{n}\Big)\\ &= \mathbb{P}\big(\pi_n(j) < 2(n+1) - j - \gamma\sqrt{n}, n+1-\gamma\sqrt{n} \le j \le n | \pi_n(i),\\ &\quad 1 \le i \le \gamma\sqrt{n}\big)\\ &\le \frac{(n-1-2\lfloor\gamma\sqrt{n}\rfloor + M)^{\lfloor\gamma\sqrt{n}\rfloor}(n-2\lfloor\gamma\sqrt{n}\rfloor+M)!}{(n-\lfloor\gamma\sqrt{n}\rfloor)!}.\end{aligned} \tag{8.13}$$

Therefore, using (8.9), (8.12) and (8.13),

$$\begin{aligned}\mathbb{P}\Big(&\min_{1\le i\le n} c_n(i) > \gamma\sqrt{n}, N_n \le M\Big)\\ &\le \frac{(n-1-2\lfloor\gamma\sqrt{n}\rfloor + M)^{\lfloor\gamma\sqrt{n}\rfloor}(n-2\lfloor\gamma\sqrt{n}\rfloor+M)!}{(n-\lfloor\gamma\sqrt{n}\rfloor)!}\mathbb{P}(A_{n,\gamma})\\ &\to e^{-\gamma^2},\end{aligned} \tag{8.14}$$

by taking limits as $n \to \infty$ and $M \to \infty$. This completes the proof of the upper bound, which combined with the lower bound (8.11) gives the result. □

APPENDIX A: WEAK CONVERGENCE IN DISTRIBUTION

Let $\{\mu_n\}_{n\ge1}$ be a sequence of random measures in $\mathcal{P}([0,1]^2)$, and let $\{Z_n(\cdot)\}_{n\ge1}$ be a sequence of random functions in $[0,1]^{(0,1]}$. Recall that, equipped with the Lévy–Prokhorov metric, the space $\mathcal{P}([0,1]^2)$ is compact. Consider the product topology on $\mathcal{P}([0,1]^2) \times [0,1]^{(0,1]}$. Now, we have the following proposition:

PROPOSITION A.1. *Let $\{\mu_n, Z_n(\cdot)\}_{n\ge1}$ be a random sequence defined in $\mathcal{P}([0,1]^2) \times [0,1]^{(0,1]}$, equipped with the product topology, such that for every continuous function $f : [0,1]^2 \mapsto \mathbb{R}$, every $b \ge 1$ fixed, and reals $s_1, \dots, s_b \in (0,1]$, the sequence*

$$\big(\mu_n(f), Z_n(s_1), Z_n(s_2), \dots, Z_n(s_b)\big) \tag{A.1}$$

converges in distribution. Then there exists $(\mu, Z(\cdot)) \in \mathcal{P}([0,1]^2) \times [0,1]^{(0,1]}$ such that

$$\big(\mu_n, Z_n(\cdot)\big) \xrightarrow{D} \big(\mu, Z(\cdot)\big). \tag{A.2}$$

PROOF. To begin, note that the space $\mathcal{P}([0,1]^2) \times [0,1]^{(0,1]}$ is compact under product topology, by Tychonoff's theorem. Hence, the sequence is $\{\mu_n, Z_n(\cdot)\}_{n\geq 1}$ is tight in $\mathcal{P}([0,1]^2) \times [0,1]^{(0,1]}$, and it suffices to show that all sub-sequential limits in distribution of $\{\mu_n, Z_n(\cdot)\}_{n\geq 1}$ are the same. Assume, on the contrary, that $(\mu, Z(\cdot))$ and $(\nu, Y(\cdot))$ are two different sub-sequential limits in distribution.

Note that any $\nu \in \mathcal{P}([0,1]^2)$ is characterized by $\{\nu(f_s)\}_{s\geq 1}$ for a countable collection of continuous functions $\{f_s\}_{s\geq 1}$, where $f_s : [0,1]^2 \mapsto \mathbb{R}$. Indeed, for a particular choice take the sequence $\{f_{a,b}(x,y) = x^a y^b\}_{a,b\geq 0}$ which evaluates the mixed (a,b)th moment $\int x^a y^b \, d\nu$ of ν, and hence, characterizes the measure ν (as ν is supported on a bounded set). Thus, there is a one-one correspondence between $\nu \in \mathcal{P}([0,1]^2)$ and the infinite sequence $\{\sigma(f_s)\}_{s\geq 1} \in [0,1]^{\mathbb{N}}$. Then endowing $[0,1]^{\mathbb{N}}$ with product topology, weak convergence of measures in $\mathcal{P}([0,1]^2)$ is equivalent to convergence in product topology of $[0,1]^{\mathbb{N}}$. Thus, we can view $(\mu_n, Z_n(\cdot)) \in \mathcal{P}[0,1]^2 \times [0,1]^{(0,1]}$ as a random variable in $[0,1]^{(0,1]+\mathbb{N}}$. Now, since finite-dimensional distributions determine joint distribution in $[0,1]^{(0,1]+\mathbb{N}}$, [7] there must exist $a, b \geq 1$ and continuous functions $(f_1, \ldots, f_a)$ from $[0,1]^2 \to \mathbb{R}$ and real numbers $s_1, s_2, \ldots, s_b \in (0,1]$ such that

$$
\begin{aligned}
&\big(\mu(f_1), \ldots, \mu(f_a), Z(s_1), \ldots, Z(s_b)\big) \\
&\quad \overset{D}{\neq} \big(\nu(f_1), \ldots, \nu(f_a), Y(s_1), \ldots, Y(s_b)\big).
\end{aligned} \tag{A.3}
$$

However, $f := \sum_{i=1}^{a} \alpha_i f_i$ is a bounded continuous function on $[0,1]^2$, and by the given assumption (A.1) we have $(\mu_n(f), Z_n(s_1), \ldots, Z_n(s_b))$ converges in distribution. Thus,

$$
\left(\sum_{i=1}^{a} \alpha_i \mu(f_i), Z(s_1), \ldots, Z(s_b)\right) \overset{D}{=} \left(\sum_{i=1}^{a} \alpha_i \nu(f_i), Y(s_1), \ldots, Y(s_b)\right),
$$

and since this holds for all $\alpha_1, \ldots, \alpha_a \in \mathbb{R}$, by the Cramér–Wold theorem, we have

$$
\big(\mu(f_1), \ldots, \mu(f_a), Z(s_1), \ldots, Z(s_b)\big) \overset{D}{=} \big(\nu(f_1), \ldots, \nu(f_a), Y(s_1), \ldots, Y(s_b)\big),
$$

which is a contradiction to (A.3). This completes the proof of the proposition. □

APPENDIX B: EMPIRICAL DEGREE PROPORTION

In this section we derive the limiting distribution of the empirical degree proportion (2.9). This follows easily from the results of Diaconis et al. [17]. We include the proofs here for the sake of completeness.

B.1. Proof of Proposition 2.1. Let $K : [0,1]^2 \times [0,1]^2 \mapsto [0,1]$ be $K((x_1, y_1), (x_2, y_2)) = \mathbf{1}\{(x_1 - x_2)(y_1 - y_2) < 0\}$. Suppose $(X_1, Y_1) \sim \nu$ and let F_ν be the distribution function of ν. From Diaconis et al. [17] and (2.10), it follows that the empirical degree proportion $\kappa(G_{\pi_n})$ converges weakly to the law of

$$\begin{aligned} W_1(X_1, Y_1) &= \int_{[0,1]^2} \mathbf{1}\big\{(X_1 - x_2)(Y_1 - y_2) < 0\big\} \, d\nu(x_2, y_2) \\ &= \nu\big([0, X_1] \times [Y_1, 1]\big) + \nu\big([0, Y_1] \times [X_1, 1]\big). \end{aligned}$$

The limiting degree distribution in (2.11) now follows from the fact that ν has uniform marginals.

B.2. Proof of Corollary 2.2. When π_n is a uniform random permutation, the limiting permuton is $\nu = \mathrm{Unif}(0,1) \times \mathrm{Unif}(0,1)$ and (2.12) follows from Proposition 2.1 by direct substitution.

To get the density of the limiting random variable, let $Z := (1-U)V + U(1-V)$ where U, V are independent Unif[0, 1]. For $z \le 1/2$, conditioning on U the distribution function of Z can be calculated as

$$\mathbb{P}(Z \le z) = \int_0^z \frac{z-u}{1-2u} \, du + \int_{1-z}^1 \frac{z-(1-u)}{2u-1} \, du.$$

Simplifying and differentiating the above expression with respect to z gives the desired density for $z \le 1/2$. For $z > 1/2$, the density can be derived similarly. The density vanishes at the end points, and blows up to infinity at $z = 1/2$.

Acknowledgments. The authors are indebted to their advisor Persi Diaconis for valuable discussions. The authors thank Shannon Starr for useful comments and Hera Y. He for her help with the simulations. The authors also thank an anonymous referee for carefully reading the manuscript and for providing many helpful comments, which improved the presentation of the paper.

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