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Limit Cycles of Planar Quadratic Differential Equations

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Abstract
Since Hilbert posed the problem of systematically counting and locating lhe limit cycle of polynomial systems on the plane in 1900, much efTort has been expended in its investigation. A large body of literature - chiefly by Chinese and Soviet authors - has addressed this question in the context of differential equations whose field is specified by quadratic polynomials, In this paper we consider the class of quadratic differential equations which admit a unique equilibrium state, and establish sufficient conditions, algebraic in system coefficients, for the existence and uniqueness of a limit cycles. The work is based upon insights and techniques developed in an earlier analysis of such systems [1] motivated by questions from mathematical control theory.

Comments
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Limit Cycles of Planar Quadratic Differential Equations

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INTRODUCTION

Since Hilbert posed the problem of systematically counting and locating the limit cycle of polynomial systems on the plane in 1900, much effort has been expended in its investigation. A large body of literature—chiefly by Chinese and Soviet authors—has addressed this question in the context of differential equations whose field is specified by quadratic polynomials. In this paper we consider the class of quadratic differential equations which admit a unique equilibrium state, and establish sufficient conditions, algebraic in system coefficients, for the existence and uniqueness of a limit cycles. The work is based upon insights and techniques developed in an earlier analysis of such systems [1] motivated by questions from mathematical control theory.

Until the fifties, work on quadratic systems chiefly concerned the existence of a center. In 1952, Bautin [2] showed that a given equilibrium state can support as many as but no more than three limit cycles under a quadratic field. Three years later, a paper by Petrovskii and Landis [3] purported to show that a quadratic system could support no more than three cycles on the entire plane. Although this result was called into question by several researchers (and the authors later acknowledged an error in the proof [4]) it apparently inspired a number of attempts to complete the Hilbert program for quadratic differential equations [5–7]. A useful survey of the general literature was given by Coppel [8] in 1966, and Ye Yanquian [15] has recently summarized the last decade's contributions to the quadratic limit cycle problem. Notably, Shi Songling [10] has presented a quadratic differential equation with four limit cycles, finally demonstrating the invalidity of the result in [3]. Thus, the Sixteenth Hilbert Problem remains unsolved, even for quadratic systems.

By a "quadratic system" we mean the differential equation

$$\dot{x} = Ax + \begin{bmatrix} x^T G x \\ x^T H x \end{bmatrix}, \tag{1}$$

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where $A, G, H \in \mathbb{R}^{2\times 2}$ (and x^TGx denotes the scalar product of the vectors x and $Gx \in \mathbb{R}^2$). We adopt the convention

$$B(x) \triangleq \begin{bmatrix} x^TGx \\ x^THx \end{bmatrix}; \qquad \text{and} \qquad f(x) \triangleq Ax + B(x),$$

and will assume, throughout the paper, that neither A nor B is identically zero. The presentation is organized as follows. In Section 2 we state the central result and include an example to illustrate the conditions listed. Section 3 provides a brief review of some algebraic results in $\mathbb{R}^2$ which will be very helpful throughout this investigation. Section 4 establishes the existence of limit cycles as a geometric interpretation of the techniques from the preceding section. Finally, uniqueness is proven in Section 5, and a brief conclusion follows in Section 6.

2. Statement of the Main Result

For ease of exposition, it is helpful to introduce some notation and terminological conventions along with the central theorem. To begin with, we will show (Lemma 2, in Section 3) that "almost all" quadratic systems which admit a unique equilibrium state at the origin may be written in the form[1]

$$\dot{x} = Ax + c^T x\, Dx, \tag{2}$$

where $c \in \mathbb{R}^2$ and $D \in \mathbb{R}^{2\times 2}$. Thus, we will find it often necessary to refer to the affine line $\{A + \mu D \mid \mu \in \mathbb{R}\}$, which we will call the *pencil* (A, D) [16]. A linear transformation of the plane is *nodal* if it has two real eigenvectors, *critical* if it has a unique eigenspace, and *focal* if its eigenvalues are complex conjugates. We may now state the main result.

Theorem 1. *System* (2) *has one and only one limit cycle if*

(i) *A is focal, with non-zero trace;*

(ii) *the pencil (A, D) includes bounded nodal values whose eigenvalues have opposite sign to the real part of the eigenvalues of A, and no other nodal values.*

To convince the reader that these conditions are algebraic and easily computed, we require some more notation. Denote the skew symmetric matrix $J \triangleq \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, and the symmetric part of any matrix, A, by

[1] It will be seen that no quadratic transformation with a unique singularity which may not be written as (2) can give rise to limit cycles.

$A_s \triangleq \frac{1}{2}[A + A^T]$. We will say that two symmetric matrices *agree in sign* if both are either positive definite or negative definite; they *oppose in sign* if one is positive definite and the other is negative definite.

Theorem 2. *The conditions of Theorem* 1 *are equivalent to the following:*

(i) *$[JA]_s$ is sign definite,* $\operatorname{tr}\{A\} \neq 0$; *and either*

(iia) $\operatorname{tr}\{A\} > 0$, *and $[JD]_s$ and $[D^TJA]_s$ agree in sign, or*

(iib) $\operatorname{tr}\{A\} < 0$, *and $[JD]_s$ and $[D^TJA]_s$ oppose in sign.*

A few remarks are now in order. It is clear that systems of the form (2) are not generic within the class of general quadratic differential equations, since the factorization of $B(x)$ depends upon the resultant of its coefficients vanishing. There are problems in control theory—e.g., adaptive control [18] and bilinear systems [17]—wherein such systems arise naturally. However, Eq. (2) deserves attention from a purely mathematical point of view. Earlier work [1, 11, 12] has established that homogeneous quadratic systems which may not be written in the form $c^Tx\,Dx$ must be unstable. It may be shown [1, 13], in consequence, that system (1) must have unbounded solutions if it cannot be written in the form (2). Imposing the added condition that the linear part of the field not be unstable and adjusting for special cases permits the following characterization of any globally asymptotically stable quadratic differential equation.

Theorem 3 (Koditschek and Narendra [1]). *System* (1) *is globally asymptotically stable if and only if*

(i) *the eigenvalues of A have non-positive real part;*

(ii) *there exist a $c \in \mathbb{R}^2$ and $D \in \mathbb{R}^{2\times 2}$ such that $B(x) = c^Tx\,Dx$;*

(iii) *the pencil (A, D) includes stable nodal values with bounded eigenvalues, and no other nodal values; any singular value of the pencil has a kernel in $\langle c_\perp \rangle$ if and only if A is non-singular.*

In fact, according to [13], conditions (ii) and (iii) of this theorem are essentially necessary for the boundedness of solutions to any quadratic system (1) as well.[2]

In the sequel, we will confine our attention to quadratic systems of the form (2), and specifically to those shown below (Corollary 3.2) to have a

[2] The qualification "essentially" is required since there are some special cases of bounded but not g.a.s. behavior, excluded by the theorem, e.g., where D is critical and singular, A is singular, and $AD = \alpha D$.

single equilibrium state. It should be noted that the conditions of Theorems 1, 2, and 3 specify open sets in the space of coefficients of system (2).

We conclude this overview of the main result with an example. Consider the system

$$\dot{x}_1 = \sigma x_1 - x_2 + x_1(x_1 - x_2)$$
$$\dot{x}_2 = x_1 + \sigma x_2 + x_1(x_1 + x_2),$$

which may be written as (2) with $A = \sigma I + J$, $D = I + J$, and $c = [{}^1_0]$. We note that A is focal for all values of σ, $\text{tr}\{A\} = 2\sigma$, $[JD]_s = -I$, and $[D^TJA]_s = (\sigma - 1)I$. When $\sigma < 0$ then the system satisfies the conditions of Theorem 3, and, hence, is globally asymptotically stable. When $\sigma = 0$ the system still satisfies the conditions of Theorem 3, even though the linear part of the field has pure imaginary eigenvalues. Systems of this nature, whose linearized equations are critical, necessitated a separate proof in [1, Lemma 4.9] precluding the possibility of a limit cycle. When $0 < \sigma < 1$, this system satisfies the conditions of Theorem 2: the origin is unstable; all trajectories may be shown to be bounded; there are no other equilibrium states—there exists one and only one stable limit cycle. When $\sigma = 1$ the field vanishes on the line $x_1 = -1$: all trajectories are bounded, but there is no more limit cycle. Finally, for $\sigma > 1$ it can be shown that all trajectories (excepting 0) tend to infinity.

3. Some Algebra of the Plane

We will use the following notation throughout the paper. If $x, y \in \mathbb{R}^2$, then x^Ty denotes the scalar product of x and y, $|x, y|$ denotes the determinant of the array formed by the coordinates of x and y, $x_- \triangleq Jx$ denotes the orthogonal complement of x, and $\langle x \rangle \triangleq \{\alpha x \mid \alpha \in R\}$ denotes the line through the origin containing x. The following relation between inner products, determinants, and quadratic forms in $\mathbb{R}^2$ will be used extensively:

$$|x, y| = y^T x_- = y^T Jx.$$

Lemma 1. *The linear transformation of the plane, A, is focal, critical, or nodal if and only if $[JA]_s$ is sign definite, semi-definite, or indefinite, respectively.*

Proof. Since x is an eigenvector of A if and only if $|Ax, x| = 0$, A has no eigenvectors if x^TJAx never vanishes for $x \neq 0$, a unique eigenspace if x^TJAx vanishes on a unique line, and two eigenvectors if the quadratic form vanishes on two lines. These are equivalent to the conditions that $[JA]_s$ be definite, semi-definite, or indefinite, respectively. ∎

We may now show how Eq. (2) arises from the consideration of quadratic systems with a unique equilibrium state.

Lemma 2. *If A is bijective then $f(x)$ vanishes at a non-zero point in $\mathbb{R}^2$ unless there exist a $c \in \mathbb{R}^2$ and $D \in \mathbb{R}^{2\times 2}$ such that*

$$B(x) = c^T x\, Dx.$$

Proof. If for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$, $\lambda A x_0 = B(x_0) = 0$ then $-1/\lambda x_0$ is an equilibrium state of system (1). Hence, we require that $|Ax, B(x)| = 0$ implies $B(x) = 0$ for any $x \neq 0$. Since $|Ax, B(x)| = x_1^3 q(x_2/x_1)$, where q is a cubic polynomial in x_2/x_1, there exists at least one real zero of q (say, v_0). The system has a unique equilibrium state only if $B = 0$ on $\langle [{}^{1}_{v_0}] \rangle$. This, in turn, implies that both quadratic forms in B share a common zero line, or $G = [cd_1^T]_s$, $H = [cd_2^T]_s$ where $c = [{}^{1}_{v_0}]$ and $d_1, d_2 \in \mathbb{R}^2$. Defining $D = [{}^{d_1^T}_{d_2^T}]$, we have the desired result. ∎

Since it has been shown [8] that no limit cycles may occur in (1) unless A is focal, Lemma 2 implies that (2) is the only quadratic differential equation with a unique equilibrium state capable of supporting a limit cycle.

The following result establishes the connection between nodal values of the pencil (A, D) and sign agreement or opposition of $[JD]_s$ and $[D^TJA]_s$.

Lemma 3. *If x is not an eigenvector of D then it is an eigenvector of $A + \mu D$ with corresponding eigenvalue, λ, if and only if*

$$\mu \triangleq -\frac{|Ax, x|}{|Dx, x|} \qquad \textit{in which case} \qquad \lambda \triangleq -\frac{|Ax, Dx|}{|Dx, x|}.$$

Proof. Define $\alpha(x) \triangleq |Ax, x|$ and $\delta(x) \triangleq |Dx, x|$. Since $|[\delta A - \alpha D]x, x| = \delta|Ax, x| - \alpha|Dx, x| = 0$ for all $x \in \mathbb{R}^2$, it follows that $[\delta A - \alpha D]x = \eta(x)x$ for some real-valued function η. But

$$x^T x\eta = x^T[\delta A - \alpha D]x = \begin{vmatrix} x^TAx & x^TDx \\ x^TJAx & x^TJ\,Dx \end{vmatrix}$$
$$= |[x, J^Tx][Ax, Dx]| = -x^Tx\,|Ax, Dx|.$$

Hence $\eta(x) = -|Ax, Dx|$ and the result follows. ∎

Corollary 3.1. *The conditions of Theorem 1 and Theorem 2 are equivalent.*

Proof. According to Lemma 1 the conditions labelled (i) in each theorem are equivalent. Condition (iia) of Theorem 2 gurantees that $\lambda(x)^3$ is bounded and always negative. This implies that the pencil (A, D) has bounded and stable and only bounded and stable nodal values according to Lemma 3. Since the eigenvalues of A have positive real part, this satisfies condition (ii) of Theorem 1. Similarly (iib) implies that the pencil has bounded and unstable and only bounded and unstable nodal values whose eigenvalues have opposite sign to be real part of the eigenvalues of A. Thus (iia) and (iib) both imply (ii) of Theorem 1.

Conversely, if $[JD]_s$ is not sign definite then the pencil (A, D) has arbitrarily large nodal values violating (ii) of Theorem 2,[4] while if $[D^TJA]_s$ is not definite, a nodal value of the pencil has a zero eigenvalue, violating that condition as well. The necessity of the sign agreement and opposition condition is now evident. ∎

COROLLARY 3.2. *The conditions of Theorem* 1 *or* 2 *guarantee that* (2) *has a unique equilibrium state at the origin.*

Proof. f cannot vanish at $y \neq 0$ unless $|Ax, B(x)| = 0$ on the line $\langle y \rangle$. Since $|Ax, B(x)| = c^T x\, |Ax, Dx| = c^T x x^T D^T J A x$, and the quadratic form is sign definite under the hypothesis, f could only vanish on $\langle c_\perp \rangle$. However, $B(c_\perp) = 0$ while $Ac_\perp \neq 0$, so this is impossible. ∎

4. Existence of Limit Cycles

We now put the algebra of the preceding section to good geometric use. As shown in the proof of Corollary 3.1, the assumption that $\mu(x)$ is bounded (and that A is focal) immediately implies that D is focal. The sign agreement condition may be interpreted to show that the spiral curve defined by a single loop of the linear trajectory, $e^{tD}y$, defines a positive-invariant region in the phase plane for arbitrarily large values of y.

LEMMA 4. *Condition* (iia) *of Theorem* 2 *implies that all trajectories of system* (2) *remain bounded.*

Proof. Choose a point, y, on $\langle c \rangle$ whose sign is opposite to the sign of the real part of the eigenvalues of D, say, on the positive ray. Let $\Delta \triangleq \{e^{tD}y \mid t \in [0, t^*]; \; e^{t^*D}y = \gamma y; \; 0 < \gamma < 1\}$ be a complete spiral loop and let $\Lambda \triangleq \{\zeta y \mid \zeta \in [\gamma, 1]\}$ join its end-points as depicted in Fig. 1.

[3] In the sequel, we will denote the quadratic ratios in x defined by Lemma 3 as $\mu(x)$ and $\lambda(x)$.

[4] Note that there can be no cancellation of factors in μ since A is focal.

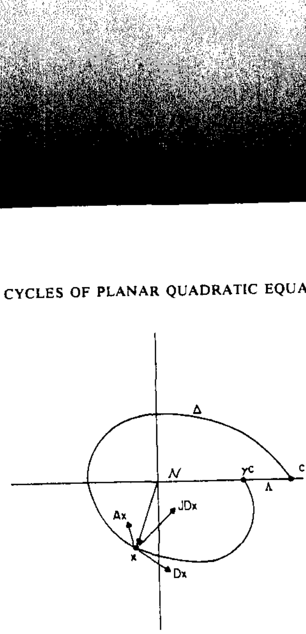

FIGURE 1

The normal to the curve at any point $x \in \Delta$ lies in $\langle JDx \rangle$ and since $x^T JDx = |Dx, x|$, JDx is either interior directed or exterior directed, depending upon whether $|Dx, x|$ is negative or positive, respectively. With no loss of generality, we assume $|Dx, x| < 0$, hence JDx is the interior directed normal to Δ at x. Similarly, Jy is the interior directed normal to Λ for any $y \in \Lambda$. We must now show that $f^T(x)\, JDx > 0$ for $x \in \Delta$, and $f^T(y)\, Jy > 0$ for $y \in \Lambda$. This will imply that any trajectory originating inside the spiral bounded region must remain within that region for all time. Since the region may be constructed arbitrarily far from the origin, that demonstration concludes the proof.

Expanding the first inequality, we have

$$\begin{aligned} f^T JDx &= x^T [A^T + c^T x D^T]\, JDx = x^T A^T JDx \\ &= |Dx, Ax| = -|Ax, Dx| > 0 \end{aligned}$$

for all $x \in \mathbb{R}^2$. Expanding the second inequality, we have

$$\begin{aligned} f^T Jy &= -y^T Jf = -y^T JAy - c^T y y^T JDy \\ &= -|Ay, y| - c^T y\, |Dy, y| \end{aligned}$$

hence, because $c^T y > 0$ for $y \in \Lambda$, and $|Dy, y| < 0$, the desired inequality holds when the second term dominates the first far enough away from the origin. ∎

LEMMA 5. *Condition* (iib) *of Theorem* 2 *implies that system* (2) *has unbounded solutions for every initial condition outside a compact neighborhood of he origin.*

Proof. Let y be a point on $\langle c \rangle$ whose sign is the same as the sign of the real part of the eigenvalues of D, say, on the negative ray. Let Δ and Λ be as in the proof of Corollary 2.2, depicted in Fig. 1. Assume again with no loss of generality that JDx is the interior directed normal to Δ at x and Jy the interior normal to Λ for $y \in \Lambda$. We need to show that $f^TDx < 0$ for $x \in \Delta$ and $f^TJy < 0$ for $y \in \Lambda$. Since $[Ax, Dx]$ has opposite sign to $[Dx, x]$ under the assumption that the pencil has positive real eigenvalues, the first inequality follows for every spiral loop Δ. The second inequality holds on Λ outside of the last loop for which $|yc^Ty|$ is less than the constant $|y^TJAy/y^TJDy|$. ∎

Having elucidated the geometric implications of the apparatus developed in Section 3, we are now able to show that a limit cycle must exist under the conditions of Theorem 1 or 2. According to the results of Lyapunov, the local stability behavior of system (2) is entirely determined by the spectrum of A. According to Lemmas 4 and 5, and Corollary 3.1, the global boundedness of system (2) is determined by the spectrum of the pencil (A, D) in its nodal range. The following result depends crucially on the special nature of limit sets of planar dynamical systems established by the Poincaré-Bendixson Theorem.

PROPOSITION 1. *The conditions* (i) *and* (iia) *of Theorem* 2 *guarantee the existence of a stable limit cycle of system* (2). *The conditions* (i) *and* (iib) *guarantee that an unstable limit cycle exists.*

Proof. Assume that (i) holds, and the eigenvalues of A have positive real parts. Then the origin is totally unstable, hence for some positive definite symmetric matrix, P, $\mathbb{R}^2 - \{x \mid x^TPx < \gamma\}$ for any $\gamma > 0$ is a positive invariant set of system (2). If either version of (ii) holds, then the origin is the sole critical point of system (2), according to Corollary 3.2. By Lemma 4, if condition (iia) of Theorem 2 holds, then all solutions of (2) are bounded: in particular, the Jordan Curve $\Delta \cup \Lambda$ bounds a positive-invariant set, $\mathcal{T}$, containing the origin. Thus $\mathcal{T} - \{x^TPx < \gamma\}$ is a compact positive-invariant set, free of critical points. In consequence of the Poincaré-Bendixson Theorem, the positive limit set of a trajectory in $\mathcal{T} - \{x^TPx < \gamma\}$ must be a limit cycle [14].

If the eigenvalues of A have negative real part and condition (iib) holds, then an identical argument concerning negative limit sets using Lemma 5 will establish the existence of a limit cycle. ∎

While the question of necesssity is not formally addressed in this paper, it is useful to remark upon the existence of limit cycles of (2) when the conditions of Theorem 2 are not met. Assuming (ii), condition (i) of Theorem 1 or 2 is certainly necessary according to the results of Coppel [8]. Note that when A has purely imaginary eigenvalues and nodal values of the pencil (A, D) have negative eigenvalues, Theorem 3 guarantees global asymptotic stability, while a similar argument establishes that all non-zero solutions of (2) grow without bound when the pencil (A, D) has positive eigenvalues in this case (see the example). If (i) holds and $A + \mu(x)D$ has a zero eigenvalue for some $x \in \mathbb{R}^2$ then system (2) has at least one critical point distinct from the origin. On the other hand, given condition (i), there is a case where (ii) is violated due to a nodal matrix D, while the plane is left free of additional equilibrium states, and the possibility of a limit cycle remains. As will be seen below, there is good reason to suspect that system (2) cannot support a limit cycle unless D has complex conjugate eigenvalues. If true, this would imply that the conditions of Theorem 2 are both necessary and sufficient for a quadratic system (1) with a single critical point to support a limit cycle.

5. UNIQUENESS

We finally show that the limit cycle established by Theorems 1 and 2 is indeed unique. Along the way we will restate the conditions of that theorem (Lemma 6, below) and provide a better intuitive sense of the mechanism underlying the isolated periodic solution. This is achieved by a transformation to polar coordinates.

Assuming A has complex conjugate eigenvalues we may always find a coordinate system (under linear transformation of the state) such that $A = \sigma I + \omega J$—where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\sigma, \omega \in \mathbb{R}$—and $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, defining the polar coordinate transformation $\rho \triangleq [x_1^2 + x_2^2]^{1/2}$, $\theta \triangleq \arctan x_2/x_1$, Eq. (2) may be written as

$$\begin{aligned} \dot{\rho} &= (1/\rho) f^T(x)x = \rho[\sigma + \rho d(\theta)] \\ \dot{\theta} &= \frac{f^T(x)Jx}{x^Tx} = \omega + \rho\, \bar{d}(\theta), \end{aligned} \tag{3}$$

where d and $\bar{d}$ are functions of θ only and are defined by

$$d(\theta) \triangleq \cos\theta \frac{x^T Dx}{x^Tx}; \qquad \bar{d}(\theta) \triangleq \cos\theta \frac{x^TD^TJx}{x^Tx}.$$

Under the assumptions of Theorem 2, $\bar{d}$ is sign definite for $\theta \in [-\pi/2, \pi/2]$ and we assume, with no loss of generality, that $\operatorname{sgn} \omega = \operatorname{sgn} \bar{d} > 0$. We define $\eta(\theta) \triangleq \sigma\bar{d}(\theta) - \omega d(\theta)$ and assert the following.

LEMMA 6. *The following conditions are equivalent to those stated in*

Theorem 2, *and hence are sufficient for the existence of a limit cycle of system* (3): *either*

(a) $\sigma > 0$ *and* $\eta < 0$ *or*,
(b) $\sigma < 0$ *and* $\eta > 0$ *for* $\theta \in [-\pi/2, \pi/2]$.

Proof. Since $A = \sigma I + \omega J$, condition (i) of Theorem 2 is equivalent to one of the sign conditions on σ. From Lemma 2, the eigenvalues of $A + \mu(x)D$ are given by

$$\lambda(x) = \frac{-[Ax, Dx]}{[Dx, x]} = \frac{-1}{[Dx, x]}(\sigma[x, Dx] + \omega[Jx, Dx]) = \frac{\eta}{\bar{d}}.$$

Thus, for $\theta \in [-\pi/2, \pi/2]$, the sign conditions on $\bar{d}$ and η are equivalent to condition (ii) of Theorem 2. ∎

As reported in [8], limit cycles of quadratic differential equations enclose convex regions, hence, any periodic solution of (3) must have an angular derivative, $\dot{\theta}$, of constant sign: no limit cycle may leave the region $\mathscr{C} \triangleq \{x \in \mathbb{R}^2 \mid \omega + \rho\bar{d} > 0\}$. Consider $x(t; p_0)$, a trajectory in $\mathscr{C}$ originating at p_0, a point on the negative x_2-axis. For some $t_2 > t_1 > 0$ we must have $x(t_1; p_0) = p_1$, a point on the positive x_2-axis, and $x(t_2; p_0) = p_2$, a point on the negative x_2-axis, as depicted in Fig. 2. Denote the resulting curve in the right half plane over the interval $[0, t_1]$ as Γ_1, and the left half plane curve, over the interval $[t_1, t_2]$, as Γ_2. Evidently, Γ_2 may be expressed as $x(s; p_1)$,

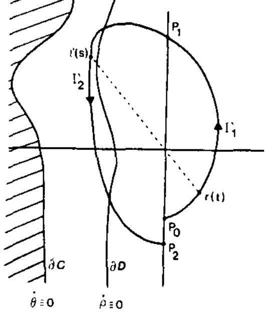

FIGURE 2

where $s \in [0, t_2 - t_1]$. We will map Γ_1 into Γ_2 by reflecting a point in the right-hand curve onto the unique point in the left-hand curve which intersects the line through the origin containing the original point. This may be justified as follows: since $\dot{\theta}$ is sign definite, for every $t \in [0, t_1]$ there exists a unique $s \in [0, t_2 - t_1]$ and $\zeta > 0$ such that

$$x(s; p_1) = -\zeta x(t; p_0).$$

For convenience we shall denote points on the right-hand curve, Γ_1, by $r(t)$, and on the left-hand curve, Γ_2, by $l(s)$, letting $\rho \triangleq \|r(t)\|$ and $\lambda \triangleq \|l(s)\| = \zeta\rho$. The chief advantage of this map is the induced functional dependence of s on t, hence the ability to write a differential equation for ρ and λ using the same angular interval. From (3) and the above, we have, for fixed initial conditions,

$$\begin{aligned} \frac{d}{d\theta}\ln\rho &= \frac{\sigma + \rho d}{\omega + \rho\bar{d}} \\ \frac{d}{d\theta}\ln\lambda &= \frac{\sigma - \lambda d}{\omega - \lambda\bar{d}} \end{aligned} \qquad \theta \in [-\pi/2, \pi/2]. \tag{4}$$

The restatement of Theorem 2 in Lemma 3 lends added insight into the mechanism by which $x(t; p_0)$ grows and decays on $\Gamma_1 \cup \Gamma_2$. Considering case (i) of Lemma 3, since $\sigma\bar{d} > 0$ on $[-\pi/2, \pi/2]$, the condition $\eta < 0$ necessitates $d > 0$ on that interval. Hence, from (4), while ρ must increase on Γ_1, λ becomes negative when Γ_2 enters the region $\mathscr{D} \triangleq \{x \in R^2 \mid x_1 < -\sigma(x^T Dx/x^T x)\}$ in the left half plane. Moreover, $\mathscr{C}$ has a boundary, $\partial\mathscr{C}$, in the left half plane and $\eta < 0$ implies $\partial\mathscr{C} \subseteq \mathscr{D}$—i.e., that certain trajectories contained in $\mathscr{C}$ must enter $\mathscr{D}$. Since $(d/d\theta)\lambda \to -\infty$ as $l(s) \to \partial\mathscr{C}$, the growth of a trajectory on Γ_1 is countered with increasing effect on a portion of Γ_2, resulting in a limit cycle. Notice that if D has real eigenvalues then $\bar{d}$ is no longer sign definite, hence d may not be sign definite, and these remarks are no longer valid, underscoring the importance of the requirement that D be focal.

The differential equations in (4) define two families of functions, $\rho(\theta; \rho_0)$ and $\lambda(\theta; \lambda_0)$, parametrized by initial condition on the negative and positive x_2-axes, respectively. Observing that $\lambda_0 = \rho(\pi/2; \rho_0)$—i.e., that λ_0 depends upon ρ_0—and that the vector fields in (4) are smooth when $x \in \mathscr{C}$, we may explicitly regard ρ and λ as functions of θ and ρ_0, continuously differentiable in both arguments. Since distinct integral curves of autonomous systems defined by smooth fields remain distinct over all time, we have $(\partial/\partial\rho_0)\rho > 0$ and $(\partial/\partial\rho_0)\lambda > 0$ for all $\theta \in [-\pi/2, \pi/2]$. Hence, the function

$$\psi(\rho_0) \triangleq \frac{\lambda(\pi/2, \rho_0)}{\rho(-\pi/2, \rho_0)} = \frac{\lambda(\pi/2, \rho_0)}{\rho_0},$$

which reresents the ratio of the magnitudes of the end-points of the curve $\Gamma_1 \cup \Gamma_2$ (both on the negative x_2-axis), is a continuously differentiable function of ρ_0. Evidently, $\Gamma_1 \cup \Gamma_2$ is the integral curve of a limit cycle if and only if $\psi = 1$. The proof of uniqueness involves a demonstration that ψ is monotone in ρ_0 over an interval of interest, and hence may pass through 1 at most once. That demonstration depends upon the following computation.

LEMMA 7. *Conditions* (a) *and* (b) *of Lemma* 6, *respectively, imply*

$$\text{(a)} \quad \frac{\partial}{\partial\theta} \ln\left(\frac{\partial\lambda}{\partial\rho_0}\frac{\partial\rho}{\partial\rho_0}\right) < 2\left[\frac{\partial}{\partial\theta}(\ln\lambda\rho) - \sigma/\omega\right],$$

$$\text{(b)} \quad \frac{\partial}{\partial\theta} \ln\left(\frac{\partial\lambda}{\partial\rho_0}\frac{\partial\rho}{\partial\rho_0}\right) > 2\left[\frac{\partial}{\partial\theta}(\ln\lambda\rho) - \sigma/\omega\right].$$

Proof.

$$\frac{\partial}{\partial\theta}\ln\left(\frac{\partial\lambda}{\partial\rho_0}\frac{\partial\rho}{\partial\rho_0}\right) = \frac{\partial^2\lambda}{\partial\theta\,\partial\rho_0}\bigg/\frac{\partial\lambda}{\partial\rho_0} + \frac{\partial^2\rho}{\partial\theta\,\partial\rho_0}\bigg/\frac{\partial\rho}{\partial\rho_0}.$$

From (4) we have

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\lambda}{\partial\theta}\right) = \left[\frac{\partial}{\partial\theta}\ln\lambda + \lambda\frac{\eta}{(\omega-\lambda\bar{d})^2}\right]\frac{\partial\lambda}{\partial\rho_0}$$

and

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\rho}{\partial\theta}\right) = \left[\frac{\partial}{\partial\theta}\ln\rho + \rho\frac{\eta}{(\omega+\rho\bar{d})^2}\right]\frac{\partial\rho}{\partial\rho_0}.$$

Since $\bar{d} > 0$, we have $\omega/(\omega-\lambda\bar{d}) > 1$, and $\omega/(\omega+\rho\bar{d}) < 1$, for all $\theta \in [-\pi/2, \pi/2]$. Hence, if condition (i) of Lemma 3 holds then

$$\lambda\frac{\eta}{(\omega-\lambda\bar{d})^2} < \frac{\lambda}{\omega}\frac{\eta}{\omega-\lambda\bar{d}} = \frac{\partial}{\partial\theta}\ln\lambda - \sigma/\omega$$

and

$$\rho\frac{\eta}{(\omega+\rho\bar{d})^2} < \frac{\rho}{\omega}\frac{\eta}{\omega+\rho\bar{d}} = \frac{\partial}{\partial\theta}\ln\rho - \sigma/\omega,$$

since $\eta < 0$ on $[-\pi/2, \pi/2]$ and substituting from (4).

Since $\partial\lambda/\partial\rho_0$, $\partial\rho/\partial\rho_0 > 0$, this implies

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\lambda}{\partial\theta}\right) < \left(2\frac{\partial}{\partial\theta}\ln\lambda - \sigma/\omega\right)\frac{\partial\lambda}{\partial\rho_0}$$

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\rho}{\partial\theta}\right) < \left(2\frac{\partial}{\partial\theta}\ln\rho - \sigma/\omega\right)\frac{\partial\rho}{\partial\rho_0}$$

yielding (a), above. The identical argument holds for (b) with signs reversed, since $\eta > 0$. ∎

We may now state the second principal result of this paper.

PROPOSITION 2. *Under the conditions of Theorem* 2, *system* (2) *has only one limit cycle.*

Proof. $x(t; p_0)$ is a limit cycle of (2) if and only if $\psi(\rho_0) = 1$ in system (3). According to Theorem 2, $\mathscr{S} \triangleq \{\rho_0 > 0 \mid \psi(\rho_0) = 1\}$ is non-empty, and bounded away from the origin, hence $\rho_0^* \triangleq \inf \mathscr{S}$ exists and $\rho_0^* > 0$. We will show that $(d/d\rho_0)\psi$ is sign definite for all $\rho_0 > \rho_0^*$, hence $x(t; p_0^*)$ is the only limit cycle of (2).

Note that

$$\frac{d}{d\rho_0}\psi = \frac{1}{\rho_0}\left[\frac{\partial}{\partial\rho_0}\lambda(\pi/2, \rho_0) - \psi\right].$$

We will show below that under condition (a) of Lemma 6,

$$\frac{\partial}{\partial\rho_0}\lambda(\pi/2, \rho_0) < \psi^2 e^{-2\pi\sigma/\omega},$$

and hence

$$\frac{d}{d\rho_0}\psi < \frac{\psi}{\rho_0}\left[\psi e^{-2\pi\sigma/\omega} - 1\right].$$

Since $\sigma/\omega > 0$ and $\psi(\rho_0^*) = 1$, this is clearly negative for $\rho_0 > \rho_0^*$. Similarly, under condition (b) of Lemma 6 the inequalities are reversed, and $\sigma/\omega < 0$ so that $(d/d\rho_0)\psi > 0$ for all $\rho_0 > \rho_0^*$.

To obtain the bound on $(\partial/\partial\rho_0)\lambda(\pi/2, \rho_0)$ we recall that $\lambda(-\pi/2, \rho_0) = \rho(\pi/2, \rho_0)$ and $\rho(-\pi/2, \rho_0) = \rho_0$, hence

$$\ln\frac{\partial}{\partial\rho_0}\lambda(\pi/2, \rho_0) = \ln\left[\frac{\partial}{\partial\rho_0}\lambda(\pi/2, \rho_0)\frac{\partial}{\partial\rho_0}\rho(\pi/2, \rho_0)\right]$$

$$- \ln\left[\frac{\partial}{\partial\rho_0}\lambda(-\pi/2, \rho_0)\frac{\partial}{\partial\rho_0}\rho(-\pi/2, \rho_0)\right]$$

$$= \int_{-\pi/2}^{\pi/2}\frac{\partial}{\partial\theta}\ln\left[\frac{\partial}{\partial\rho_0}\lambda(\theta, \rho_0)\frac{\partial}{\partial\rho_0}\rho(\theta, \rho_0)\right]d\theta.$$

Applying Lemma 7 to case (a) yields

$$\ln \frac{\partial}{\partial \rho_0} \lambda(\pi/2, \rho_0) < \int_{-\pi/2}^{\pi/2} 2 \left(\frac{\partial}{\partial \theta} \ln \rho\lambda - \sigma/\omega \right) d\theta$$

$$= \ln \left(\frac{\lambda(\pi/2, \rho_0)}{\rho(-\pi/2, \rho_0)} \right)^2 - 2\pi\sigma/\omega$$

$$= \ln \psi^2 - 2\pi\sigma/\omega,$$

hence $(\partial/\partial\rho_0)\lambda(\pi/2, \rho_0) < \psi^2 e^{(-2\pi\sigma/\omega)}$ as claimed. Case (b) proceeds identically with the signs reversed. ∎

6. Conclusions

This paper presents sufficient conditions for the existence of limit cycles of quadratic systems with a unique equilibrium state. The conditions guarantee that the limit cycle is unique. The results are based upon insights and techniques developed during an earlier investigation of the global stability properties of (1) [1], facilitated by the expression of that system in the form (2). They strongly suggest that these conditions are necessary as well, hence, that no quadratic system with a unique equilibrium state can support more than one limit cycle. That result, the uniqueness of a limit cycle around any equilibrium state of (2), the relation of limit cycles of (2) to those supported by general quadratic systems (1), all remain to be rigorously established.

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