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## Risk vs. Profit-Potential: A Model for Corporate Strategy

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## Risk vs. Profit-Potential: A Model for Corporate Strategy

### Abstract

A firm whose net earnings are uncertain, and that is subject to the risk of bankruptcy, must choose between paying dividends and retaining earnings in a liquid reserve. Also, different operating strategies imply different combinations of expected return and variance. We model the firm's cash reserve as the difference between the cumulative net earnings and the cumulative dividends. The first is a diffusion (additive), whose drift/volatility pair is chosen dynamically from a finite set,  $A$ . The second is an arbitrary nondecreasing process, chosen by the firm. The firm's strategy must be *nonclairvoyant*. The firm is bankrupt at the first time,  $T$ , at which the cash reserve falls to zero ( $T$  may be infinite), and the firm's objective is to maximize the expected total discounted dividends from 0 to  $T$ , given an initial reserve,  $x$ ; denote this maximum by  $V(x)$ . We calculate  $V$  explicitly, as a function of the set  $A$  and the discount rate. The optimal policy has the form: (1) pay no dividends if the reserve is less than some critical level,  $a$ , and pay out all of the excess above  $a$ ; (2) choose the drift/volatility pairs from the upper extreme points of the convex hull of  $A$ , between the pair that minimizes the ratio of volatility to drift and the pair that maximizes the drift; furthermore, the firm switches to successively higher volatility/drift ratios as the reserve increases to  $a$ . Finally, for the optimal policy, the firm is bankrupt in finite time, with probability one.

### Keywords

corporate policy, wiener processes, economics modelling

### Disciplines

Economics | Statistics and Probability

### Comments

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# Risk vs. Profit-Potential; A Model for Corporate Strategy

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## ABSTRACT

A firm whose net earnings are uncertain, and that is subject to the risk of bankruptcy, must choose between paying dividends and retaining earnings in a liquid reserve. Also, different operating strategies imply different combinations of expected return and variance. We model the firm's cash reserve as the difference between the cumulative net earnings and the cumulative dividends. The first is a diffusion (additive), whose drift/volatility pair is chosen dynamically from a finite set,  $A$ . The second is an arbitrary nondecreasing process, chosen by the firm. The firm's strategy must be *nonclairvoyant*. The firm is bankrupt at the first time,  $T$ , at which the cash reserve falls to zero ( $T$  may be infinite), and the firm's objective is to maximize the expected total discounted dividends from 0 to  $T$ , given an initial reserve,  $x$ ; denote this maximum by  $V(x)$ . We calculate  $V$  explicitly, as a function of the set  $A$  and the discount rate. The optimal policy has the form: (1) pay no dividends if the reserve is less than some critical level,  $a$ , and pay out all of the excess above  $a$ ; (2) choose the drift/volatility pairs from the upper extreme points of the convex hull of  $A$ , between the pair that minimizes the ratio of volatility to drift and the pair that maximizes the drift; furthermore, the firm switches to successively higher volatility/drift ratios as the reserve increases to  $a$ . Finally, for the optimal policy, the firm is bankrupt in finite time, with probability one.

## 1. A Theory of Corporate Decision Making

Consider a firm with fixed plant and equipment that produces an uncertain stream of net revenues. (Net revenue equals gross revenue less all costs except the cost of capital.) Part of this revenue may be paid out in dividends, with the remainder accumulated in a *cash reserve*. Since the net revenue can be negative as well as positive, the cash reserve can fluctuate up and down. We suppose that the firm starts with a positive cash reserve, say  $x$ , and that it becomes bankrupt the first time, if ever, that the cash reserve falls to zero. When the firm becomes bankrupt, it ceases to exist.

The firm's manager controls the dividends, which must be nonnegative, and can also influence the stochastic process of net revenues. In other words, the manager's policy has two parts, a dividend policy and an operating policy. (We shall give a more precise description of this in a moment.) We define the *profit* of the firm to be the expected total dividends paid out during

the life of the firm, discounted at some fixed rate, say  $r > 0$ . We shall say that the manager's policy is optimal if it maximizes the firm's profit. For a Bachelier model of the firm's net revenue process, we characterize the firm's optimal policy and the resulting maximum profit,  $V(x)$ , that the firm can attain starting with a cash reserve  $x$ .

In this (simplified) model of the firm, the assets are in two parts: (1) fixed, or illiquid, assets, which produce the stream of net revenues, and (2) the cash reserve, which produces no net revenues, but is completely liquid and provides insurance against bankruptcy. There is no reinvestment of net revenues in the illiquid assets. Furthermore, the cash reserve can be increased only from retained earnings, but not by an infusion of outside capital. We have adopted the convention that the firm becomes bankrupt when the cash reserve falls to zero, but any other critical level (including negative ones) could be postulated. For example, if the firm could not obtain a line of credit that exceeds  $b$ , then the critical level for the cash reserve would be  $-b$ . However, once the cash reserve fell to this critical level, the firm could not be revived with more investment. We have also implicitly assumed that the illiquid assets of the firm have no salvage at the time of bankruptcy, but this assumption can easily be relaxed (see Section 3).

Roughly speaking, we may interpret our model as describing a firm that has reached its optimal scale of operation. A more general model would allow for investment of retained earnings in both new productive (illiquid) capital assets, and the cash reserve. However, such investment would eventually encounter diminishing returns to scale, until the optimal scale is reached. On the other hand, firms can sometimes sell parts of their productive assets with little if any loss in the value of the sold assets, i.e., the productive assets may be more or less liquid, rather than completely illiquid as in our present model. In particular, bankruptcy may result in only a partial loss to creditors and investors, followed by a "reorganization" with the addition of new capital. The analysis of a more general model embodying these features would be desirable, but is beyond the scope of the present paper.

We turn now to a formal statement of the problem. Let  $X_t$  denote the cash reserve at time  $t$ , and let  $R_t$  denote the *accumulated* net revenues up through time  $t$  (here time is continuous and nonnegative). In the light of the preceding discussion it is natural to assume that  $R_t$  evolves according to the Bachelier additive model,

$$(1.1) \quad dR_t = \begin{cases} \mu dt + \sigma dW_t, & 0 < t < \tau_0, \\ 0, & t > \tau_0, \end{cases}$$

where  $\tau_0$  is the first  $t$  for which  $X_t \leq 0$ , i.e., when the firm becomes bankrupt. Here  $\mu$  is the expected (instantaneous) rate of net revenue, and  $\sigma dW_t$  represents the contribution to net revenue due to random fluctuations beyond the firm's control, with  $W_t$  a standard Brownian motion or Wiener process. The effect of the manager's operating policy is to control  $\mu$  and  $\sigma$  as a function of time, as described below.

We note in passing that, in models of individual investors in securities markets, it has become fashionable to describe fluctuations in asset values with the *multiplicative* model [?, ?, ?],

$$(1.2) \quad dR_t = R_t(\mu dt + \sigma dW_t) .$$

Such a model embodies an assumption of *constant returns to scale*, which, as we have noted above, is inappropriate for an individual firm, although it might provide a good approximation in the case of a single investor in the securities market whose assets are small relative to the market as a whole.

Returning to our model, we assume that the firm has available  $n$  alternate business plans — plan  $i$  is identified by the parameters  $(\mu_i, \sigma_i^2)$ ,  $i = 1, \dots, n$ , as in (??) — and that the firm is able to switch continuously between plans.<sup>1</sup> If the firm uses plan  $(\mu_t, \sigma_t^2) \in \mathcal{A} = \{(\mu_i, \sigma_i^2), i = 1, \dots, n\}$  at time  $t$ , then the accumulated net revenue,  $R_t$ , of the firm evolves according to the model

$$(1.3) \quad dR_t = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq \tau_0 .$$

Unless the firm distributes some dividends at some time, there is not much point to being in business. We denote the total dividends distributed up to time  $t$  by  $Z_t$ , where  $dZ_t \geq 0$ . Hence cash reserve  $X_t$  of the firm evolves according to

$$(1.4) \quad dX_t = \mu_t dt + \sigma_t dW_t - dZ_t, \quad 0 \leq t \leq \tau_0 ,$$

where  $\tau_0$  (as before) is the first  $t \geq 0$  for which  $X_t = 0$ , and

$$(1.5) \quad dX_t = dZ_t = 0 \quad \text{for } t \geq \tau_0 .$$

Note that  $X_0 = x \geq 0$ , given. The firm wants to choose its corporate policy  $[(\mu_t, \sigma_t^2), Z_t]$  in

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<sup>1</sup>This is a serious assumption which is not appropriate in reality since management cannot appear to be too capricious, but if  $n$  is large and if  $(\mu_i, \sigma_i^2)$  varies slowly with  $i$  then it does not seem unreasonable when the actual switching takes place only between adjacent plans,  $i$  and  $i \pm 1$ , which will be proven to always take place under *optimal* corporate management.

such a way that the total *profit*, *i.e.*, the expected total discounted dividends,

$$E_x \int_0^\infty e^{-rt} dZ_t ,$$

is maximized, where  $r > 0$  is a given rate of discount. Accordingly, define

$$(1.6) \quad V(x) = \sup E_x \int_0^\infty e^{-rt} dZ_t .$$

Here the sup or max is taken over all policies  $[(\mu_t, \sigma_t^2), Z_t]$  which do not see into the future, *i.e.*  $(\mu_t, \sigma_t^2)$  and  $Z_t$  are independent of the future increments  $dW_s$ ,  $s > t$  of the noise process,  $W$ . More precisely,  $\mu_t$ ,  $\sigma_t^2$ , and  $Z_t$  should be “nonclairvoyant” in the sense of adaptedness to the standard filtration; of course it is necessary to assume that  $\mu_t$ ,  $\sigma_t^2$ ,  $Z_t$  should be jointly measurable in  $w$  and  $t$ .

We will find  $V(x)$  in §2 as well as the optimal policy for the firm, including both the optimal choice of  $(\mu_t, \sigma_t^2)$  and the optimal dividend stream,  $Z_t$ ,  $t \geq 0$ . In each case the solution will be stationary in time and will only be a function of the current cash reserve,  $X_t$ . We will see that there is a threshold,  $a$ , such that dividends are paid only when  $X_t \geq a$ , and then are paid at an infinite rate so that, in fact,  $0 \leq X_t \leq a$  for all  $t > 0$ , under optimal management.<sup>2</sup> The optimal choice of plan at each time  $t$ ,  $(\mu_t, \sigma_t^2)$  is also only a function of  $X_t$ . This is a simple consequence of the formulation of the problem; if we change the problem, even slightly, say using a finite horizon instead of exponential discounting, the optimal solution no longer is of stationary form and appears impossible to find in closed, explicit, form. This is familiar in problems of this type [?, ?, ?, ?, ?]. We will show in §2 that the firm will go bankrupt at time  $\tau_0 < \infty$  with probability one under optimal management. This seems somewhat paradoxical, since one believes optimally managed firms prefer to remain solvent, but in fact it is a theorem, and might explain why profit oriented firms actually operate on the edge of bankruptcy and often wind up on the wrong side of it. If instead, one redefines the criterion (??) of optimality to include a term involving  $P\{\tau_0 = \infty\}$  in the maximization, then the solution exhibits nonstationarity and apparently cannot be found explicitly, as discussed in §3.

We find (in §2) the exact condition on  $\mathcal{A} = \{(\mu_i, \sigma_i^2), i = 1, \dots, n\}$  under which each of the  $n$  (distinct) plans,  $(\mu_i, \sigma_i^2)$ , has a chance to actually become part of the corporate policy (only if  $X_t$  ever enters the set where  $(\mu_i, \sigma_i^2)$  is the optimal choice). If  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$  is the

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<sup>2</sup>Policies of this type, called *barrier policies* by M. Harrison, arise in a variety of problems of “regulated Brownian motion;” see [?].

ordering of the drift parameters, without loss of generality, then one also needs that the plans satisfy:

$$(1.7) \quad \frac{\sigma_1^2}{\mu_1} \leq \frac{\sigma_2^2}{\mu_2} \leq \dots \leq \frac{\sigma_n^2}{\mu_n}$$

or else one of the plans will completely dominate another one and any time the one plan say  $(\mu_1, \sigma_1^2)$  would be used,  $(\mu_2, \sigma_2^2)$  would be preferable — if (??) failed and say  $\sigma_2^2/\mu_2 \leq \sigma_1^2/\mu_1$  — because plan 2 would be more profitable ( $\mu_2 \geq \mu_1$ ) and its risk would be sufficiently small. There is however one more condition needed for each  $(\mu_i, \sigma_i^2) \in \mathcal{A}$  to be needed. We must have that each  $(\mu_i, \sigma_i^2)$  is actually an extreme point of  $\mathcal{A}^*$ , the convex hull of  $\mathcal{A}$ . This is because, through rapid switching between the plans in  $\mathcal{A}$ , the firm can actually obtain any plan in  $\mathcal{A}^*$ , and if a plan in  $\mathcal{A}^*$  dominates one of  $(\mu_i, \sigma_i^2)$  in the above sense, then  $(\mu_i, \sigma_i^2)$  can be eliminated from it. One can restate this last condition as:

$$(1.8) \quad \frac{\sigma_i^2 - \sigma_{i-1}^2}{\mu_i - \mu_{i-1}} \leq \frac{\sigma_{i+1}^2 - \sigma_i^2}{\mu_{i+1} - \mu_i}, \quad i = 2, 3, \dots, n-1,$$

so that (??) and (??) will be shown to be necessary and sufficient for each plan in  $\mathcal{A}$  to be needed, *i.e.*, so that  $V(\mathcal{A}, x) > V(\mathcal{A}', x)$  for  $\mathcal{A}' \subset \mathcal{A}$ . Note that the essential condition here is (??), which for  $n = 2$  indicates the way the two different plans,  $(\mu_1, \sigma_1^2)$  and  $(\mu_2, \sigma_2^2)$  should be compared, and thus allows an intrinsic comparison of drift and volatility, two parameters which appeared difficult to quantitatively compare until now. It is particularly reassuring that the conditions (??) and (??) do not involve the (interest or discount) rate  $r$ , *i.e.*, (??) gives a universal way to compare drift and volatility, and is even independent of other plans in  $\mathcal{A}$  for  $n > 2$ .

In Section 3 we consider the somewhat more general problem of determining

$$(1.9) \quad V_c(x) = \sup E_x \left[ \int_0^\infty e^{-rt} dZ_t + ce^{-r\tau_0} \right],$$

where  $c$  is a given parameter. This problem formulation arises in two situations: (1) there is a positive salvage value of the firm's assets at the time of bankruptcy ( $c > 0$ ); and (2) the manager is paid a constant salary throughout the life of the firm, and the goal is to maximize some weighted average of the expected discounted totals of dividends and the manager's salary ( $c < 0$ ).<sup>3</sup> Because of the form of the criterion, the optimal solution is again stationary, and also  $\tau_0 < \infty$  w.p.1. Furthermore,  $\tau_0$  is stochastically decreasing in  $c$ .

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<sup>3</sup>For a related class of problems, see [?].

Finally in §4 we consider a more technical variant of the theory where  $\mathcal{A}$  consists of a continuum of plans (*i.e.*  $\mathcal{A}^*$  has infinitely many extreme points), Here we give a technique based on the smooth-fit principle [?, ?, ?, ?] for guessing and proving the optimal strategy and obtaining  $V(\mathcal{A}, x)$ . In particular we consider the case where for each  $0 \leq u \leq \beta < \infty$ , the company has a plan with  $\mu(u) = u$ , and  $\sigma(u) = \theta u$ , where  $\theta$  is fixed. We give in §4 the exact solution in this case by way of illustrating the general method in the continuous case. It is interesting that as  $\beta \rightarrow \infty$ ,  $V(x; \beta, \theta) \rightarrow \infty$ , so if the firm has a choice of unboundedly aggressive policies (with  $\frac{\sigma}{\mu}$  fixed, but  $\frac{\sigma^2}{\mu}$  unbounded) then it can obtain an infinite profit.<sup>4</sup>

## 2. Finitely many plans; which one to choose?

In order to solve (??) for  $V$  we must first guess it — for which we use the heuristic “principle of smooth fit” as in [?, ?, ?]. Suppose we can find a function  $\bar{V} = \bar{V}(x)$ ,  $x \geq 0$ , which satisfies:

$$(2.1) \quad \begin{aligned} (a) \quad & \bar{V}(0) = 0; \\ (b) \quad & 0 \leq \bar{V}(x) \leq \text{const. } x, 0 \leq x < \infty ; \\ (c) \quad & \bar{V} \in C^2[0, \infty) . \end{aligned}$$

$$(2.2) \quad \bar{V}'(x) \geq 1, \quad x \geq 0 .$$

$$(2.3) \quad -r\bar{V}(x) + \mu_i \bar{V}'(x) + \frac{1}{2} \sigma_i^2 \bar{V}''(x) \leq 0 \text{ for all } x \geq 0 \text{ and all } i = 1, \dots, n .$$

Then it is easy to see that  $V(\mathcal{A}, x) \leq \bar{V}(x)$ . To see this let  $(\mu_t, \sigma_t^2)$  be any choice in  $\mathcal{A}$  for each  $t$ , and  $Z_t$ , an arbitrary dividend scheme. Define the process  $Y_t$ ,  $t \geq 0$  by

$$(2.4) \quad Y_t = \bar{V}(X_t) e^{-rt} + \int_0^t e^{-rs} dZ_s$$

where  $X_t$  is the fortune stream of the firm employing the plan  $(\mu_t, \sigma_t^2)$  and  $Z_t$ , *i.e.*,  $X_0 = x$ , and, as in (??),

$$(2.5) \quad dX_t = \mu_t dt + \sigma_t dW_t - dZ_t, \quad 0 \leq t \leq \tau_0 .$$

We see that  $Y$  has the Ito differential

$$(2.6) \quad dY_t = e^{-rt} \left\{ -r\bar{V}(X_t) dt + \bar{V}'(X_t)(\mu_t dt + \sigma_t dW_t - dZ_t) + \frac{1}{2} \bar{V}''(X_t) \sigma_t^2 dt + dZ_t \right\} .$$

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<sup>4</sup>Dutta and Radner [?] consider the situation in which the set  $\mathcal{A}$  is compact and strictly convex. They first study the case in which the “dividend rate” is bounded, and then let the bound increase without limit. Of course, without further specification of the set  $\mathcal{A}$  they obtain only qualitative properties of the optimal policy and the value function. The methods of this paper can be used for arbitrary  $\mathcal{A}$ .



Thus the expected increment of  $dY_t$  is (since  $E\bar{V}'(X_t)\sigma_t dW_t = 0$  because  $X_t$  is independent of  $dW_t$  under Ito calculus, which applies because of (??c))

$$(2.7) \quad E dY_t = e^{-rt} dt \left\{ -r\bar{V}(X_t) + \bar{V}'(X_t)r\mu_t + \frac{1}{2}\bar{V}''(X_t)\sigma_t^2 + (1 - \bar{V}'(X_t))dZ_t \right\} .$$

The latter gives  $E dY_t \leq 0$  because  $dZ_t \geq 0$  and (??) and (??) hold. Thus  $Y_t$  is an expectation decreasing (submartingale) process and so  $EY_\infty \leq EY_0$ , which gives from (??), and an (easy) argument based on (??a,b)

$$(2.8) \quad E_x Y_\infty = E_x \int_0^\infty e^{-rs} dZ_s \leq E_x Y_0 = \bar{V}(x) .$$

Since  $(\mu_t, \sigma_t^2)$  and  $Z_t$  are arbitrary, we obtain  $V(\mathcal{A}, x) \leq \bar{V}(x)$ . We now need to construct  $\bar{V}$  satisfying (??)–(??) in such a way that equality will hold in the above argument.

After a little experimentation, it is easy to guess that  $\bar{V}$  and  $V$  must be of the following form. If we assume that

$$(2.9) \quad 0 < \mu_1 < \mu_2 < \dots < \mu_n$$

then we must try to find thresholds

$$(2.10) \quad a_0 = 0 < a_1 < \dots < a_n$$

for which for  $i = 1, \dots, n$

$$(2.11) \quad -r\bar{V}(x) + \mu_i\bar{V}'(x) + \frac{\sigma_i^2}{2}\bar{V}''(x) = 0 \quad \text{for } a_{i-1} \leq x \leq a_i$$

$$(2.12) \quad \bar{V}(x) = \xi + x \quad \text{for } a_n \leq x$$

$$(2.13) \quad \bar{V}(0) = 0, \quad \bar{V} \in C^2 .$$

Now equality will hold throughout (??) provided we set

$$(2.14) \quad (\mu_t, \sigma_t^2) = (\mu_i, \sigma_i^2) \quad \text{when } X_t \in (a_{i-1}, a_i)$$

and

$$(2.15) \quad dZ_t = 0 \quad \text{for } X_t < a_n; \quad dZ_t = \infty \quad \text{for } X_t > a_n .$$

We need only check the details. In order to do this we need to simultaneously determine when each of the plans  $(\mu_i, \sigma_i^2)$  are actually needed and correspond to a nonzero interval  $(a_{i-1}, a_i)$ .

We will show for “neededness” the plans must also satisfy

$$(2.16) \quad \frac{\sigma_{i-1}^2}{\mu_{i-1}} < \frac{\sigma_i^2}{\mu_i}, \quad i = 1, \dots, n$$

as well as

$$(2.17) \quad \frac{\sigma_i^2 - \sigma_{i-1}^2}{\mu_i - \mu_{i-1}} < \frac{\sigma_{i+1}^2 - \sigma_i^2}{\mu_{i+1} - \mu_i}, \quad i = 1, 2, \dots, n-1$$

equivalently, (??) says that each plan  $(\mu_i, \sigma_i^2)$  must be an extreme point of  $\mathcal{A}^*$ , the convex hull of  $\mathcal{A}$ .

So suppose (??), (??), and (??) hold for  $\mathcal{A} = \{(\mu_i, \sigma_i^2); i = 1, \dots, n\}$  and define  $\alpha_i$  and  $\beta_i$  the roots of  $\frac{1}{2}\gamma^2\sigma_i^2 + \gamma\mu_i - r = 0$ , *i.e.*

$$(2.18) \quad \alpha_i = \frac{-\mu_i + \sqrt{\mu_i^2 + 2r\sigma_i^2}}{\sigma_i^2} \quad \beta_i = \frac{-\mu_i - \sqrt{\mu_i^2 + 2r\sigma_i^2}}{\sigma_i^2}.$$

Now set  $a_0 = 0$  and for  $i = 1, 2, \dots, n-1$  define  $a_i$  by

$$(2.19) \quad a_i - a_{i-1} = \frac{1}{\alpha_i - \beta_i} \log \left[ \frac{P_{i+1}(\beta_i)}{P_{i+1}(\alpha_i)} \frac{P_{i-1}(\alpha_i)}{P_{i-1}(\beta_i)} \right]$$

where the quadratic  $r$  polynomials  $P_i$  are defined for  $i = 0, 1, \dots, n$  by

$$(2.20) \quad P_i(x) = \frac{1}{2}x^2\sigma_i^2 + x\mu_i - r$$

where  $\sigma_0 = \mu_0 = 0$  and so  $P_0(x) \equiv -r$ , a degenerate quadratic. We will check that (??) holds and then we set

$$(2.21) \quad \bar{V}(x) = \begin{cases} A_i e^{\alpha_i x} - B_i e^{\beta_i x} & \text{for } a_{i-1} \leq x \leq a_i, \quad i = 1, \dots, n \\ x + \xi & a_n \leq x. \end{cases}$$

We will now show that  $A_i, B_i, i = 1, \dots, n$  and  $\xi$  can be chosen so that  $\bar{V}$  satisfies all of (??)-(??) and that equality holds in (??).

**Lemma 1.** *The roots  $\beta_i < 0 < \alpha_i$  of  $P_i$  satisfy*

$$(2.22) \quad \beta_1 < \dots < \beta_n < 0 < \alpha_n < \dots < \alpha_1.$$

**Proof.** If  $i < j$  then since  $r > 0$ , and  $0 < \mu_i < \mu_j$  by (??) and (??)

$$(2.23) \quad \frac{r}{\mu_j} < \frac{r}{\mu_i} \quad \text{or} \quad \frac{1}{2} \frac{\sigma_j^2}{\mu_j} \alpha_j^2 + \alpha_j < \frac{1}{2} \frac{\sigma_i^2}{\mu_i} \alpha_i^2 + \alpha_i < \frac{1}{2} \frac{\sigma_j^2}{\mu_j} \alpha_i^2 + \alpha_i.$$

It follows that  $\{\frac{1}{2}(\sigma_j^2/\mu_j)(\alpha_j + \alpha_i) + 1\}(\alpha_j - \alpha_i) < 0$  so  $\alpha_j < \alpha_i$  since  $\alpha_j$  and  $\alpha_i$  are positive. Similarly  $\{\frac{1}{2}(\sigma_j^2/\mu_j)(\beta_j + \beta_i) + 1\}(\beta_j - \beta_i) < 0$  but  $(\sigma_j^2/\mu_j)\beta_j = (1/\mu_j)(-\mu_j - \sqrt{\mu_j^2 + 2r\sigma_j^2}) < -1$  and  $(\sigma_j^2/\mu_j)\beta_i < (\sigma_i^2/\mu_i)\beta_i < 1$  by the fact that  $\beta_i < 0$  and the same argument, so  $\beta_i < \beta_j$ . *q.e.d.*

Because of (??) and the fact that  $P_i$  is a quadratic which is large at  $\pm\infty$  and has zeros at  $\alpha_i, \beta_i$ , it follows that  $P_i(\alpha_{i+1}) < 0$ ,  $P_i(\alpha_{i-1}) > 0$ ,  $P_i(\beta_{i+1}) < 0$ ,  $P_i(\beta_{i-1}) > 0$ . To prove (??)

holds we must show that the argument of the logarithm in (??) is  $> 1$ , or in terms of positive quantities, we must show

$$(2.24) \quad \frac{P_{i+1}(\beta_i)}{P_{i+1}(\alpha_i)} \left( \frac{-P_{i-1}(\alpha_i)}{-P_{i-1}(\beta_i)} \right) > 1 \quad \text{or} \quad P_{i+1}(\alpha_i)P_{i-1}(\beta_i) - P_{i+1}(\beta_i)P_{i-1}(\alpha_i) > 0 .$$

Writing this last term out completely, using (??), reduces to (??) and (??) follows. Thus (??) is proved.

Next we turn to determining  $A_i, B_i, i = 1, \dots, n$  and  $\xi$  in (??). For  $1 \leq i \leq n-1$  we have from (??) and the smooth-fit heuristic

$$(2.25) \quad \bar{V}(a_i^-) = \bar{V}(a_i^+) \quad \text{or} \quad A_i e^{\alpha_i a_i} - B_i e^{\beta_i a_i} = A_{i+1} e^{\alpha_{i+1} a_i} - B_{i+1} e^{\beta_{i+1} a_i}$$

$$(2.26) \quad \bar{V}'(a_i^-) = \bar{V}'(a_i^+) \quad \text{or} \quad A_i \alpha_i e^{\alpha_i a_i} - B_i \beta_i e^{\beta_i a_i} = A_{i+1} \alpha_{i+1} e^{\alpha_{i+1} a_i} - B_{i+1} \beta_{i+1} e^{\beta_{i+1} a_i}$$

$$(2.27) \quad \bar{V}''(a_i^-) = \bar{V}''(a_i^+) \quad \text{or} \quad A_i \alpha_i^2 e^{\alpha_i a_i} - B_i \beta_i^2 e^{\beta_i a_i} = A_{i+1} \alpha_{i+1}^2 e^{\alpha_{i+1} a_i} - B_{i+1} \beta_{i+1}^2 e^{\beta_{i+1} a_i} .$$

We also have

$$(2.28) \quad \bar{V}(0) = 0 \quad \text{or} \quad A_1 = B_1$$

and for  $i = n$  we have

$$(2.29) \quad \bar{V}(a_n^-) = \bar{V}(a_n^+) \quad \text{or} \quad A_n e^{\alpha_n a_n} - B_n e^{\beta_n a_n} = a_n + \xi$$

$$(2.30) \quad \bar{V}'(a_n^-) = \bar{V}'(a_n^+) \quad \text{or} \quad A_n \alpha_n e^{\alpha_n a_n} - B_n \beta_n e^{\beta_n a_n} = 1$$

$$(2.31) \quad \bar{V}''(a_n^-) = \bar{V}''(a_n^+) \quad \text{or} \quad A_n \alpha_n^2 e^{\alpha_n a_n} - B_n \beta_n^2 e^{\beta_n a_n} = 0 .$$

Now multiply (??), (??), (??) by  $-r, \mu_i, \frac{1}{2}\sigma_i^2$  respectively and add to obtain, using the fact that  $\alpha_i$  and  $\beta_i$  are roots of (??) to get for  $1 \leq i < n$ ,

$$(2.32) \quad 0 = A_{i+1} P_i(\alpha_{i+1}) e^{\alpha_{i+1} a_i} - B_{i+1} P_i(\beta_{i+1}) e^{\beta_{i+1} a_i}$$

noting that (??) also holds for  $i = 0$ , consistently with (??). Similarly multiplying (??), (??), (??) by  $-r, \mu_{i+1}, \frac{1}{2}\sigma_{i+1}^2$  and adding we get for  $1 \leq i < n$ ,

$$(2.33) \quad 0 = A_i P_{i+1}(\alpha_i) e^{\alpha_i B_i} - B_i P_{i+1}(\beta_i) e^{\beta_i a_i}$$

which is consistent with (??) and shows where (??) was obtained. From (??) and (??) we obtain  $A_n$  and  $B_n$ , both nonnegative,

$$(2.34) \quad A_n = \frac{\beta_n}{\beta_n - \alpha_n} \frac{e^{-\alpha_n a_n}}{\alpha_n}; \quad B_n = \frac{\alpha_n}{\beta_n - \alpha_n} \frac{e^{-\beta_n a_n}}{\beta_n} .$$

Now the rest of  $A_i$ ,  $B_i$  and  $\xi$  are fixed as follows:  $\xi$  is given by (??). We can still use (??) since we only used 2 combinations of (??)–(??) above so that since (??) holds for  $i = 0, \dots, n-1$  we have replacing  $B_i$  and  $B_{i+1}$  in (??) by the values from (??) and (??),

$$(2.35) \quad A_i e^{\alpha_i a_i} \left(1 - \frac{P_{i+1}(\alpha_i)}{P_{i+1}(\beta_i)}\right) = A_{i+1} e^{\alpha_{i+1} a_i} \left(1 - \frac{P_i(\alpha_{i+1})}{P_i(\beta_{i+1})}\right)$$

which determines  $A_1, \dots, A_{n-1}$ . Finally  $B_1, \dots, B_{n-1}$  are given by (??).

Now that  $\bar{V}$  is consistently defined we must show that (??)–(??) hold and that equality holds in (??). We see that  $\bar{V}'(a_n) = 1$  and so  $\bar{V}'(x) \geq 1$  will hold if we show that  $V''(x) < 0$ ,  $0 \leq x \leq a_n$ . From (??),

$$(2.36) \quad \bar{V}''(x) = A_i \alpha_i^2 e^{\alpha_i x} - B_i \beta_i^2 e^{\beta_i x} \quad \text{for } a_{i-1} \leq x \leq a_i$$

so we need to show that

$$(2.37) \quad A_i \alpha_i^2 e^{(\alpha_i - \beta_i)x} \leq B_i \beta_i^2 \quad \text{for } a_{i-1} \leq x \leq a_i$$

and it is enough to check this for  $x = a_i$  since  $\alpha_i - \beta_i > 0$ . Using (??), we need to show that  $\alpha_i^2 P_{i+1}(\beta_i) \leq \beta_i^2 P_{i+1}(\alpha_i)$ , or from (??),

$$(2.38) \quad \alpha_i^2 \left(-r + \mu_{i+1} \beta_i + \frac{1}{2} \sigma_{i+1}^2 \beta_i^2\right) \leq \beta_i^2 \left(-r \mu_{i+1} \alpha_i + \frac{1}{2} \sigma_{i+1}^2 \alpha_i^2\right).$$

Cancelling the  $\alpha_i^2 \beta_i^2$  terms we need that  $((\alpha_i + \beta_i)r - \alpha_i \beta_i \mu_{i+1})(\alpha_i - \beta_i) \geq 0$ . But  $\alpha_i + \beta_i = -2\mu_i/\sigma_i^2$ ,  $\alpha_i \beta_i = -2r/\sigma_i^2$ , and  $\alpha_i - \beta_i > 0$  so this is the same as  $\mu_{i+1} \geq \mu_i$  which is true by (??). So (??) and (??) hold. To prove (??), suppose  $x \in (a_{j-1}, a_j)$ ,  $1 \leq j \leq n$ . We must show for all  $i, j \in \{1, \dots, n\}$ ,

$$(2.39) \quad -r \bar{V}(x) + \mu_i \bar{V}'(x) + \frac{1}{2} \sigma_i^2 \bar{V}''(x) < 0.$$

Since this is a combination of two exponentials by (??), it suffices to check that (??) holds at the endpoints  $x = a_{j-1}$ ,  $x = a_j$ , *i.e.* for all  $i$  &  $j$

$$(2.40) \quad 0 \geq -r(A_j e^{\alpha_j a_j} - B_j e^{\beta_j a_j}) + \mu_i (A_j \alpha_j e^{\alpha_j a_j} - B_j \beta_j e^{\beta_j a_j}) + \frac{1}{2} \sigma_i^2 (A_j \alpha_j^2 e^{\alpha_j a_j} - B_j \beta_j^2 e^{\beta_j a_j})$$

since  $a_{j-1}$  is just another endpoint, because a sum of two exponentials negative at the end points of an interval is negative throughout the interval. We may rewrite (??) as

$$(2.41) \quad 0 \geq A_j e^{\alpha_j a_j} \left(-r + \alpha_j \mu_i + \frac{1}{2} \alpha_j^2 \sigma_i^2\right) - B_j e^{\beta_j a_j} \left(-r + \beta_j \mu_j + \frac{1}{2} \beta_j^2 \sigma_i^2\right)$$

Using (??) and (??) we may rewrite this as (since  $A_j, B_j$  are positive),

$$(2.42) \quad 0 \geq P_{j+1}(\beta_j)P_i(\alpha_j) - P_{j+1}(\alpha_j)P_i(\beta_j) \quad \text{for all } i, j \in \{1, \dots, n\} .$$

Putting in the definition of  $P_i, P_{j+1}$  in (??), and cancelling terms we may rewrite (??) as

$$(2.43) \quad 0 \geq -r \left( \mu_i \alpha_j + \frac{1}{2} \sigma_i^2 \alpha_j^2 \right) + \mu_{j+1} \beta_j \left( -r + \frac{1}{2} \sigma_i^2 \alpha_j^2 \right) + \frac{1}{2} \sigma_{j+1}^2 \beta_j^2 (-r + \mu_i \alpha_j) \\ + r \left( \mu_i \beta_j + \frac{1}{2} \sigma_i^2 \beta_j^2 \right) - \mu_{j+1} \alpha_j \left( -r + \frac{1}{2} \sigma_i^2 \beta_j^2 \right) - \frac{1}{2} \sigma_{j+1}^2 \alpha_j^2 (-r + \mu_i \beta_j) .$$

Dividing out by  $\alpha_j - \beta_j > 0$  and noting  $\alpha_j \beta_j = -2r/\sigma_j^2$ ,  $\alpha_j + \beta_j = -2\mu_j/\sigma_j^2$ , and cancelling a factor of  $r$  we see that (??) is the same as

$$(2.44) \quad 0 \geq (\mu_{j+1} - \mu_i) \sigma_j^2 + (\mu_j - \mu_{j+1}) \sigma_i^2 + (\mu_i - \mu_j) \sigma_{j+1}^2$$

which is the same as the convexity condition (??). Indeed, if  $i = j$  or  $i = j + 1$ , this vanishes, while otherwise if, say,  $i < j < j + 1$  then

$$(2.45) \quad \frac{\mu_j - \mu_i}{\mu_{j+1} - \mu_i} \sigma_i^2 + \frac{\mu_{j+1} - \mu_j}{\mu_{j+1} - \mu_i} \sigma_{j+1}^2 \leq \sigma_j^2 .$$

is the same as (??) and also convexity of  $(\mu_i, \sigma_i^2)$ . Similarly for  $j < j + 1 < i$ . Thus under (??) and (??), (??)–(??) hold for  $\bar{V}$  as defined. Now equality holds in the argument in (??) because when  $i = j$  or  $i = j + 1$ , we have seen that (??) vanishes.

If instead (??) or (??) fails, it is easy to see that some points may be omitted from  $\mathcal{A}$ , and a function  $\bar{V}$  may be constructed for which  $\bar{V}(x, \mathcal{A}') = V(x)$  where  $\mathcal{A}' \subset \mathcal{A}$ . Indeed the largest subset  $\mathcal{A}'$  of  $\mathcal{A}$  for which (??) and (??) hold will provide  $\bar{V}(\mathcal{A}', x)$  which will give an upper bound on  $V(\mathcal{A}, x)$ , *i.e.*  $V(\mathcal{A}, x) \leq \bar{V}(\mathcal{A}', x)$  so constructed and so  $V(\mathcal{A}, x) = V(\mathcal{A}', x)$ . In particular we must have  $\mu_i > 0$  for all plans in  $\mathcal{A}$ .

We see that the optimal plan then pays dividends only when  $x \geq a_n$  and that  $V(x) = x + \xi$  for  $x \geq a_n$ , where  $\xi$  is given by (??),

$$(2.46) \quad \begin{aligned} \xi &= A_n e^{\alpha_n a_n} - B_n e^{\beta_n a_n} - a_n \\ &= \frac{\alpha_n + \beta_n}{\alpha_n \beta_n} - a_n = \frac{\mu_n}{r} - a_n \end{aligned}$$

where  $a_n$  is given as the sum over  $i = 1, \dots, n$  in (??). It is a consequence of the theory that  $\xi > 0$  of course since  $V(x) \geq x$  because one strategy is to take all profits at once.

Now that we see that the firm remains always in the interval  $0 \leq X_t \leq a_n$ , we see that w.p.1,  $\tau_0 < \infty$ , *i.e.* the firm will go bankrupt in finite time, almost surely. This is a consequence

of the fact that every process with bounded drift and reflecting barrier at  $a_n$  will eventually hit 0, w.p.1.

We have shown that the integral with respect to  $W$  implicit in the above proof is a martingale rather than a local martingale because  $\bar{V}$  has a bounded derivative and this makes our proof complete. The fact that  $\bar{V}$  grows at most linearly as stated in (??b) was of course also used to obtain that  $\bar{V}(X_t)e^{-rt}$  tends to zero as  $t \rightarrow \infty$ .

### 3. Staying alive; maximizing the time until bankruptcy

Is the criterion (??) cogent, or are firms rather trying to maximize not profits but perhaps  $P_x\{\tau_0 = \infty\}$ , or  $E_x\tau_0$ ? It is possible to work out  $E_x\tau_0$  for the optimal policy (for (??)) of §2, but of course it will not be the policy that maximizes this (indeed this can be made infinite if  $\mu_i > 0$  for some policy). But perhaps real firms are attempting to maximize profits under a fixed level,  $P_x\{\tau_0 = \infty\} \geq \epsilon$ , of survival, where  $\epsilon > 0$  is given. Such problems are inherently very difficult because of a technical reason akin to the reason that made the treatment of Russian option [?, ?] much more amenable to analysis than the American option in Black-Scholes classic papers, [?, ?]. Namely, the model formulated in (??) is inherently one-dimensional in that the optimal strategy for the firm is stationary and depends on the value of  $X_t$ , and not also upon  $t$ . Suppose instead of (??) we want to find

$$(3.1) \quad U(x) = \sup_x \left[ E_x \int_0^\infty e^{-rt} dZ_t + cP_x\{\tau_0 = \infty\} \right]$$

where  $c > 0$  is given. This (using Lagrange multipliers) is seen to be equivalent to the  $\epsilon$ -problem stated above. But (??) is impossible to solve, it seems, because one cannot easily guess the form of the optimal policy which now depends on the solution of a partial differential equation instead of a simple ordinary differential equation as in §2. Indeed, imagine letting  $t$  run for a short interval,  $h$ . Then (??) will involve a new problem where  $c$  changes to  $c \exp(rh)$ . This problem appears to be much more difficult to solve for the reason that the value of  $c$  keeps changing as time runs. It is completely analogous to the American option analysis where the time until maturity must be tracked. The advantage of the discounting in (??) was to keep the problem invariant in time.

There is one special way to formulate a problem of the form of (??), qualitatively, which has some of the same features as (??) but allows for explicit solution. This is to find

$$(3.2) \quad U(x) = \sup_x \left[ E_x \int_0^\infty e^{-rt} dZ_t + cE_x e^{-r\tau_0} \right]$$

for fixed  $c$ . In fact, this optimization problem arises naturally in two situations of interest. First, suppose that the firm has a “salvage value” when it becomes bankrupt, which is distributed to the investors. Then the total expected discounted return to the investors is given by (??), where  $c$  is the salvage value.

Second, suppose that the manager is paid a fixed salary, say  $s$  dollars per unit time, during the life of the firm. Then his expected total discounted salary is

$$(3.3) \quad E_x \int_0^{\tau_0} e^{-rt} s dt = (s/r)(1 - E_x e^{-r\tau_0}) .$$

Suppose further that the firm wants to maximize its profit, subject to some given lower bound on the manager’s expected total discounted salary. For some value of the Lagrangian multiplier  $m$ , this is equivalent to (??), with  $c = -ms/r$ . Note that in this case  $c$  is negative which “encourages”  $\tau_0$  to be large.

Now the problem (??) remains invariant as time moves ahead because each of the terms in (??) changes by  $\exp(-rh)$  if  $t$  changes from 0 to  $h$ , and the problem remains stationary. The same methods will work to find  $U(x)$  as used in §2 to find  $V(x)$  except that  $U(0) \neq 0$ ! Instead, the boundary condition  $U'(0) = 0$  must be used. We leave it to the interested reader to pursue this direction.

#### 4. A continuum of corporate policies

We suppose here that the set  $\mathcal{A}$  of available corporate policies is a continuum, *i.e.* for each drift  $\mu$  there is a least volatility  $\sigma^2(\mu)$  in the convex set  $\mathcal{A}$  and  $\mu$  fills out an interval, which for simplicity we take to be  $0 \leq \mu \leq \beta$ . We again assume that the optimality criterion is (??). We will show how to guess the optimal corporate policy using the (heuristic) principle of smooth fit to help guess the reward function,  $V(x)$ . The least volatility in  $\mathcal{A}$  with drift  $\mu$  is given, we call it  $\sigma^2(\mu)$ . As we have seen in §1 we may as well assume that  $\sigma^2(\mu)$  is a *concave* function of  $\mu$ ,  $0 \leq \mu \leq \beta$ . In distinction to §2 where  $\sigma^2(\mu)$  had only a finite number of extreme points and was linear in-between them, here we will assume for simplicity that  $\sigma^2(\mu)$  is *strictly* concave for  $0 \leq \mu \leq \beta$  so that every  $(\mu, \sigma^2(\mu))$  is an extreme point of  $\mathcal{A}$ . Of course we will also assume  $\sigma^2(\mu)/\mu$  increases in  $\mu$  as in (??), otherwise some policies in  $\mathcal{A}$  are dominated and not needed.

Again we seek a function  $\bar{V}(x)$  with the following properties analogous to (??)-(??):

$$(4.1) \quad \begin{aligned} & \text{(a) } \bar{V}(0) = 0; \\ & \text{(b) } 0 \leq \bar{V}(x) \leq \text{const. } x, \quad 0 \leq x < \infty; \\ & \text{(c) } \bar{V} \in C^2[0, \infty) \end{aligned}$$

$$(4.2) \quad \bar{V}'(x) \geq 1, \quad 0 \leq x < \infty$$

$$(4.3) \quad -r\bar{V}(x) + \mu\bar{V}'(x) + \frac{1}{2}\sigma^2(\mu)\bar{V}''(x) \leq 0 \quad \text{for } 0 \leq \mu \leq \beta, \quad 0 \leq x < \infty .$$

We also seek a function  $u : [0, \infty) \rightarrow [0, \beta]$ ;  $(\mu, \sigma^2) = (u(x), \sigma^2(u(x)))$  will be the policy to use when  $X_t = x$ . How do we find  $\bar{V}$  (which will be  $V$  as in (??)) and  $u$ ?

Because of a fortuitous technique which we now describe it is not so hard. Indeed we should expect that for  $(\mu, \sigma^2) = (u(x), \sigma^2(x))$  the inequality (??) is an equality:

$$(4.4) \quad -r\bar{V}(x) + u(x)\bar{V}'(x) + \frac{1}{2}\sigma^2(u(x))\bar{V}''(x) = 0 .$$

But because of (??), the derivative on  $\mu$  of (??) must be zero when  $\mu = u(x)$  since the maximum of the left side of (??) over  $\mu$  is attained at  $\mu = u(x)$ , at least for  $0 < x < \mu(\beta)$ . Thus we expect

$$(4.5) \quad \bar{V}'(x) + \sigma(\mu)\sigma'(\mu)\bar{V}''(x) = 0 \quad \text{at } \mu = u(x), \quad 0 < \mu < \beta$$

where  $\sigma = \sigma(\mu)$  is the given function. We may eliminate  $\bar{V}''$  from (??) and (??) to get

$$(4.6) \quad -r\bar{V}(x) + \left( u(x) - \frac{1}{2} \frac{\sigma(u(x))}{\sigma'(u(x))} \right) \bar{V}'(x) = 0 .$$

Now we may formally differentiate on  $x$  in (??) to get

$$(4.7) \quad -r\bar{V}'(x) + \frac{1}{2}u'(x)\bar{V}'(x) + \left( u(x) - \frac{1}{2} \frac{\sigma(u(x))}{\sigma'(u(x))} \right) \bar{V}''(x) = 0 .$$

Again eliminating  $\bar{V}''$  from (??), noting  $\mu = u(x)$  in (??), gives

$$(4.8) \quad -r\bar{V}'(x) + \frac{1}{2}u'(x)\bar{V}'(x) - \left( u(x) - \frac{1}{2} \frac{\sigma(u(x))}{\sigma'(u(x))} \right) \frac{\bar{V}'(x)}{\sigma(u(x))\sigma'(u(x))} = 0 .$$

We may now divide by  $\bar{V}'(x)$  which is  $\geq 1$  by (??), and we obtain a differential equation for  $u(x)$  not involving  $V$ !

$$(4.9) \quad u'(x) = 2r + \frac{2u(x)}{\sigma(u(x))\sigma'(u(x))} - \left( \frac{1}{\sigma'(u(x))} \right)^2$$

which we can write as

$$(4.10) \quad \frac{du}{2r + \frac{2u}{\sigma(u)\sigma'(u)} - \left( \frac{1}{\sigma'(u)} \right)^2} = dx .$$



Now (??) is separable in  $u$  and  $x$  so we may integrate both sides, taking account that  $u(0) = 0$  because all points of  $\mathcal{A}$  are extreme points (again this is just intuition and heuristic) and obtain  $x = x(u)$ , and, by inversion,  $u = u(x)$ . We will have  $x(\beta) = b$  for some  $b$ , and  $u(b) = \beta$ . We will also expect that  $u(x)$  increases in  $x$  from  $x = 0$  to  $x = b$ . For  $b \leq x \leq a$ , where  $a$  is some point  $> b$  to be found we will use  $u(x) \equiv \beta$ , the most aggressive policy in  $\mathcal{A}$ , and finally for  $x > a$  we will expect

$$(4.11) \quad \bar{V}(x) = x + \xi \quad a \leq x$$

by analogy with (??) since we will set  $dZ = \infty$  for  $x > a$ . It remains to find only  $a$  and  $\xi$ . We expect  $\bar{V}(0) = 0$ , and  $\bar{V} \in C^2$  by (??) and these must determine them. Of course everything in our plan is heuristic, but once  $\bar{V}$  is guessed we can use the methods of §2 to show that  $V \equiv \bar{V}$ .

To illustrate the above we choose the simplest special case as an example,  $\sigma^2 = \sigma^2(\mu)$  given by

$$(4.12) \quad \sigma^2(\mu) = \theta^2 \mu^2, \quad 0 \leq \mu \leq \beta$$

where  $\theta$  and  $\beta$  are given parameters. We seek

$$(4.13) \quad V(x) = \sup E_x \int_0^\infty e^{-rt} dZ_t$$

where the sup is taken over all  $\mu_t \in [0, \beta]$ ,  $t \geq 0$  and  $Z_t$  with  $Z_t \geq 0$ , with both  $\mu$  and  $Z$  nonclairvoyant. We assume  $X_0 = x$  and

$$(4.14) \quad dX_t = \mu_t dt + \theta \mu_t dW_t - dZ_t, \quad 0 \leq t \leq \tau_0$$

where  $\tau_0$  is the first  $t < \infty$  for which  $X_t = 0$  (if there is such) and  $dX_t = dZ_t = 0$  for  $t \geq \tau_0$ .

From (??) we obtain simply for  $0 \leq x \leq \beta$

$$(4.15) \quad \frac{du}{2r + \frac{1}{\theta^2}} = dx \quad \text{or} \quad u(x) = kx \quad \text{with} \quad k = 2r + \frac{1}{\theta^2}.$$

Thus  $b = \beta/k$  and  $u(x) = \beta$  for  $x \geq b$ , heuristically. We will use  $\mu_t = u(X_t)$ ,  $t > 0$ ;  $dZ_t = \infty$  for  $X_t \geq b$  so that  $X_t$  will remain in  $[0, b]$ . It is simplest to use (??) to guess  $\bar{V}$  and we have, easily, since  $\bar{V}(0) = 0$ ,

$$(4.16) \quad -r\bar{V}(x) + \frac{1}{2}kx\bar{V}'(x) = 0 \quad \text{or} \quad \bar{V}(x) = A_1 x^{\frac{2r}{k}}, \quad 0 \leq x \leq b = \beta/k$$

$$(4.17) \quad \bar{V}(x) + A_2 e^{\alpha_1 2x} - B_2 e^{\beta_2 x}, \quad b \leq x \leq a; \quad \bar{V}(x) = x + \xi, \quad a \leq x.$$

where  $\alpha_2$  and  $\beta_2$  are the roots of  $-r + \beta\gamma + \frac{1}{2}\beta^2\theta^2\gamma^2 = 0$ , *i.e.*

$$(4.18) \quad \alpha_2 = \frac{-1 + \sqrt{1 + 2r\theta^2}}{\theta^2\beta} \quad \beta_2 = \frac{-1 - \sqrt{1 + 2r\theta^2}}{\theta^2\beta}.$$

It remains to find  $A_1, A_2, B_2, \xi, a$ . There are 3 conditions at each of  $b$  and  $a$  for  $\bar{V} \in C^2$ , but one of these is redundant because  $b = \beta/k$  is already guessed.

Since  $\bar{V}(b+) = \bar{V}(b-)$  and  $\bar{V}'(b+) = \bar{V}'(b-)$  we have from (??) and (??),

$$(4.19) \quad A_1 b^{\frac{2r}{k}} = A_2 e^{\alpha_2 b} - B_2 e^{\beta_2 b}$$

$$(4.20) \quad A_1 \frac{2r}{k} b^{\frac{2r}{k}-1} = A_2 \alpha_2 e^{\alpha_2 b} - B_2 \beta_2 e^{\beta_2 b} .$$

That  $\bar{V}''(b+) = \bar{V}''(b-)$  follows automatically as noted above since  $b = \beta/k$ . Since  $\bar{V}(a+) = \bar{V}(a-)$ ,  $\bar{V}'(a+) = \bar{V}'(a-)$ , and  $\bar{V}''(a+) = \bar{V}''(a-)$  we have from (??)

$$(4.21) \quad A_2 e^{\alpha_2 a} - B_2 e^{\beta_2 a} = a + \xi$$

$$(4.22) \quad A_2 \alpha_2 e^{\alpha_2 a} - B_2 \beta_2 e^{\beta_2 a} = 1$$

$$(4.23) \quad A_2 \alpha_2^2 e^{\alpha_2 a} - B_2 \beta_2^2 e^{\beta_2 a} = 0 .$$

We find

$$(4.24) \quad a = b + \frac{2\beta\theta}{\sqrt{k}} \log \frac{\sqrt{k} + 1/\theta}{\sqrt{k} - 1/\theta}, \quad b = \beta/k, \quad k = 2r + 1/\theta^2$$

$$(4.25) \quad A_2 = \frac{\sqrt{k} + 1/\theta}{\sqrt{k} - 1/\theta} \frac{e^{-\alpha_2 a}}{2\sqrt{k}} \theta \beta \quad B_2 = \frac{\sqrt{k} - 1/\theta}{\sqrt{k} + 1/\theta} \frac{e^{-\beta_2 a}}{2\sqrt{k}} \theta \beta$$

$$(4.26) \quad A_1 = b \left[ \frac{\sqrt{k} + 1/\theta}{\sqrt{k} - 1/\theta} e^{\frac{2}{\sqrt{k}\theta} L} - \frac{\sqrt{k} - 1/\theta}{\sqrt{k} + 1/\theta} e^{-\frac{2}{\sqrt{k}\theta} L} \right] \theta \sqrt{k} e^{-2\sqrt{k} L}$$

where  $L = \log[(\sqrt{k} + 1/\theta)/(\sqrt{k} - 1/\theta)]$ . From this we see that as  $\beta \rightarrow \infty$ ,  $\bar{V}(x, \beta) \rightarrow \infty$  for each  $x$ , so that if the company has the chance to make arbitrarily large risk policies  $(\mu, \theta^2 \mu^2)$  with unbounded  $\mu$  then it can make infinite profits in the sense of (??).

It is now easy to verify that  $\bar{V}$  satisfies (??)–(??) and so the method of proof of §2 shows easily that  $V$  of (??) satisfies  $V(x) \equiv \bar{V}(x)$ . It is sufficient to check that  $\bar{V}''(x) \leq 0$  for all  $x$ .

## 5. An alternative model to Bachelier; Black-Scholes-Samuelson

In Section 1 we argued that the multiplicative model (??) would be inappropriate for a model of the firm because it embodied the assumption of constant returns to scale at all scales, however large. In addition, the usual model of finance theory assumes that all assets are liquid. With this in mind, recall that, with the multiplicative model, if there were no dividends then the assets of the firm would evolve according to

$$(5.1) \quad dX_t = X_t(\mu dt + \sigma dW_t)$$

In this section we show that, with this model the criterion (??) leads to an empty problem — either  $V(x) = \infty$ , or  $V(x) = x$  and all the profits of the company are drawn at once. Note that in the case of (??),  $X_t > 0$  for all  $t > 0$  w.p.1, *i.e.*  $\tau_0 = \infty$  and the firm never goes bankrupt (not such a good model, since real firms often do!). But suppose we do use (??) in place of (??).

Let us consider for simplicity the case when  $\mathcal{A}$  is a singleton; the general case is similar. Suppose that, if dividends are paid, the assets of the firm evolve according to

$$(5.2) \quad dX_t = X_t(\mu dt + \sigma dW_t) - dZ_t, \quad 0 \leq t \leq \tau_0,$$

where  $Z_t$  is the accumulated dividends,  $dZ_t \geq 0$ , and of course now bankruptcy at  $\tau_0 < \infty$  is possible. The firm wants to determine

$$(5.3) \quad V(x) = \sup E_x \int_0^\infty e^{-rt} dZ_t,$$

where the sup is over all choice of nonclairvoyant  $Z$ .

We will now show that if  $\mu \leq r$  then  $V(x) = x$ , which may be obtained by paying the dividend  $x$  at  $t = 0$  so that  $\tau_0 = 0$ . Then we will show that if  $\mu > r$  then  $V(x) = \infty$ .

For any choice of  $Z_t$ , we set  $\bar{V}(x) \equiv x$ ,  $x > 0$ , and define

$$(5.4) \quad Y_t = \bar{V}(X_t)e^{-rt} + \int_0^t e^{-rs} dZ_s, \quad 0 \leq t.$$

We find using (??) that  $Y$  has the differential,

$$(5.5) \quad \begin{aligned} dY_t &= e^{-rt} \left\{ -r\bar{V}(X_t)dt + \bar{V}'(X_t)[X_t(\mu dt + \sigma dW_t) - dZ_t] + \bar{V}''(X_t)X_t^2 \frac{\sigma^2}{2} dt + dZ_t \right\} \\ &= e^{-rt} \{ (\mu - r)X_t dt + \sigma X_t dW_t \}, \end{aligned}$$

since  $\bar{V}'(x) \equiv 1$  and  $\bar{V}''(x) \equiv 0$ . Now if  $\mu \leq r$  we see that  $Y_t$  is a submartingale,  $EdY_t \leq 0$ , so that

$$(5.6) \quad E_x Y_\infty \leq E_x Y_0.$$

This gives from (??)

$$(5.7) \quad E_x \int_0^\infty e^{-rs} dZ_s \leq \bar{V}(x_0) = \bar{V}(x) = x.$$

Since this holds for every choice of  $Z_t$ , we conclude  $V(x) \leq x$ .

Suppose now that  $\mu > r$ . Then one simple strategy for the firm is to let the assets accumulate ( $dZ_t = 0$ ) until time  $T$ . Since  $X_t > 0$  for  $0 \leq t \leq T$ , the fortune of the firm will be

$X_T$ , which is given by the well-known exponential form,

$$(5.8) \quad X_T = x e^{\sigma W_T + \left(\mu - \frac{\sigma^2}{2}\right)T} .$$

At time  $T$  we liquidated all the assets. Since this is an allowable strategy for the firm, we have that  $V$  is certainly bigger than the profit  $X_T \exp(-rT)$  obtained in this (trivial) way, *i.e.*,

$$(5.9) \quad V(x) \geq E x e^{\sigma W_T + \left(\mu - \frac{\sigma^2}{2}\right)T} e^{-rT} .$$

But since  $W$  is a standard Wiener process,

$$(5.10) \quad E e^{\sigma W_T} = e^{\frac{\sigma^2}{2}T}$$

and so from (??)

$$(5.11) \quad V(x) \geq e^{(\mu-r)T} .$$

But  $\mu > r$  and  $T$  is arbitrary so  $V(x) \equiv \infty$ .

q.e.d.

#### **Acknowledgement of referee's comments**

A referee kindly pointed out that our formulation is not so novel and indeed several authors have considered quite closely related models although were less explicit and concrete in their analysis, supplying a kindly of authors who have dealt with these problems in the past for the reader's comparison and reference: Hotelling, Merton (1974), Cox, Ingersoll, and Ross (1985), Taksar (1985), Holmstrom and Milgrom (1987), and Sung (1995) are recent examples with a continuous-time formulation.

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