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Abstract

We present arguments, based on local gauge invariance, that the multicritical point of Ising spin-glasses should be located on a particular line of the phase diagram known as Nishimori's line [$\tanh(\beta J) = 2p - 1$ for the $\pm J$ distribution]. One scaling axis is along the line, and the other is along the temperature direction. This scenario is generic for any random Ising model with a Nishimori line, in any number of dimensions, if the transitions are second order. The renormalization-group fixed point located inside Nishimori's manifold is expected to control multicriticality for a wider class of models.

Disciplines

Physics

Location of the Ising Spin-Glass Multicritical Point on Nishimori's Line

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We present arguments, based on local gauge invariance, that the multicritical point of Ising spin-glasses should be located on a particular line of the phase diagram known as Nishimori's line [$\tanh(\beta J) = 2p - 1$ for the $\pm J$ distribution]. One scaling axis is along the line, and the other is along the temperature direction. This scenario is generic for any random Ising model with a Nishimori line, in any number of dimensions, if the transitions are second order. The renormalization-group fixed point located inside Nishimori's manifold is expected to control multicriticality for a wider class of models.

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The phase diagram of the Ising spin-glass in zero magnetic field, described by the nearest-neighbor Hamiltonian $H = \sum_{\langle ij \rangle} J_{ij} S_i S_j$, has been determined for asymmetric distributions of couplings $P(J_{ij})$ by the methods of Migdal-Kadanoff renormalization group^{1,2} (RG), Monte Carlo RG,³ domain-wall RG⁴, and series expansions.⁵ It is now believed that (i) in three dimensions there are three phases, paramagnetic (P), (anti)ferromagnetic (F), and spin-glass (SG), which meet at a multicritical point M ; (ii) in two dimensions there is no spin-glass phase, but there is still a multicritical point M delimiting two critical behaviors on the F-P boundary. The phase diagram is sketched in Fig. 1, for the $\pm J$ distribution:

$$P(J) = p\delta(J - J_0) + (1 - p)\delta(J + J_0). \quad (1)$$

On the other hand, Nishimori has shown⁶ that the internal energy $[\langle H \rangle]$ can be computed exactly when the distribution of couplings $P(J)$ satisfies the condition

$$P(-J) = \exp(-2\beta J)P(J) \\ \Leftrightarrow P(-t) = (1 - t)(P(t)/(1 + t)), \quad (2)$$

where $t = \tanh(\beta J)$. In the phase diagram, (2) defines the Nishimori line: $\tanh(\beta J_0) = 2p - 1$ for the $\pm J_0$ distribution (1) and $\beta \bar{J}^2 = J_0$ for the Gaussian distribution.² When more parameters are added (2) generally defines a manifold. Surprisingly, the internal energy was found to be *regular* along the line although this line crosses the boundary of the ferromagnetic phase at point N (Fig. 1). It was thus natural to ask whether M must coincide with N ^{7,8}. This is true for all infinite-dimensional models (Sherrington-Kirkpatrick⁹ or diluted models,¹⁰ Bethe lattice¹¹) and is in good agreement with approximate determinations of phase diagrams in 2D and 3D.^{7,8} Let us emphasize that there was, up to now, no compelling reason why this should be true in general, since the regularity of the internal energy at point N along the line

could be due to some accidentally vanishing critical amplitude (no result is available for the free energy).

In this Letter we give two independent arguments in favor of an exact location of the multicritical point M on Nishimori's line ($M = N$), which also allow us to identify precisely the "nongeneric" cases where the situation could be more complicated. These cases are clearly *not*

FIG. 1. "Generic scenario" proposed here for the phase diagram of the $\pm J$ spin-glass in $d=3$ [in the (p, T) plane], also supported by numerical studies (Refs. 2-5). In $d=2$ it is the same picture, without the SG phase. Nishimori's line is dashed. The RG flow on corresponding hierarchical lattices and its fixed points are indicated.

supported by present numerical or analytical results on Ising spin-glasses. They are (i) first-order P-SG or P-F transitions, (ii) the existence of more than one multicritical point in Fig. 1, or (iii) some very unlikely instances discussed below. Moreover, independently of $M=N$, one can give an exact characterization of the scaling axis at N . Explicit proofs can be found in Ref. 8.

The first argument relies on the exact equality of the spin-glass susceptibility χ_2 and the ferromagnetic susceptibility χ_1 on Nishimori's line. Here we set $\chi_k(ij) = [\langle S_i S_j \rangle^k]$ and the susceptibilities χ_k are defined by $\chi_k = \sum_j \chi_k(i, j)$ (thermal averages are denoted by angular brackets and configurational averages by square brackets). In Ref. 8 it is proven that, when the distribution $P(J)$ satisfies Nishimori's condition (2), one has for general integer k ,

$$\chi_{2k+1}(i, j) = \chi_{2k+2}(i, j) \quad (3)$$

$$\exp(-\beta H_n) \equiv \left[\prod_{(ij)} \exp \left(\beta \sum_{a=1}^n J_{ij} S_i^a S_j^a \right) \right] = \prod_{(ij)} (a_0 + a_1 \sum_a S_i^a S_j^a + a_2 \sum_{a < \beta} S_i^a S_j^a S_j^\beta + \dots), \quad (4)$$

with $a_k = [c^n t^k]$, where $c = \cosh(\beta J)$. Nishimori's condition (2) implies that $a_{2k+1} = a_{2k+2}$, for any integer k and any n . As a consequence, it is straightforward to check that on Nishimori's line the replica Hamiltonian (4) has, for any n , an *additional gauge invariance*: $S_i^1 = U_i^1$, $S_i^a = U_i^1 U_i^a$ ($a > 1$), where the U_i act as local Ising variables. The exact equality between $\chi_{2k+1}(i, j)$ and $\chi_{2k+2}(i, j)$ then follows immediately from this additional invariance, since one has

$$\chi_1(i, j) = \lim_{n \rightarrow 0} \langle S_i^1 S_j^1 \rangle_{H_n} = \lim_{n \rightarrow 0} \langle U_i^1 U_i^1 U_j^1 U_j^1 \rangle_{H_n} = \chi_2(i, j),$$

and similar relations hold for any integer k .

Summing (3) over j , we obtain that on Nishimori's line the ferromagnetic susceptibility χ_1 and the spin-glass susceptibility χ_2 are equal. The consequences are the following. In all cases where the P-F boundary is unambiguously defined by $\chi_1 \rightarrow \infty$, $\chi_2/\chi_1 \rightarrow 0$ and the P-SG boundary by $\chi_2 \rightarrow \infty$, $\chi_1/\chi_2 \rightarrow 0$, this result implies that these two boundaries must meet at point N on Nishimori's line (e.g., $M=N$), where ferromagnetic and spin-glass fluctuations simultaneously become critical ($\chi_1 = \chi_2 \rightarrow \infty$). Moreover one has $\eta_1 = \eta_2$ at $M=N$, where these exponents are defined by the asymptotic large-separation behavior $\chi_k(i, j) \sim A_k |i-j|^{-d+2-\eta_k}$. This has been obtained to order $\epsilon = 6-d$ by Chen and Lubensky¹³ but our result indicates exact equality of both exponents and amplitudes for χ_{2k+1} and χ_{2k+2} (for any k).

The argument for $M=N$ relies on the assumption that only one of the χ_i ($i=1,2$) is dominant on either the P-F or P-SG boundary (it obviously also requires second-order transitions). This is true in the $d \rightarrow \infty$ limit for Ising spin-glasses and we have checked that this remains true within a systematic $1/d$ expansion.¹⁴ The argument

[note that (3) in the particular case $k=0$ was obtained in Ref. 11 for a random-clock model].

As is the case of all results in connection with Nishimori's line, (3) is a simple consequence of the invariance of the Hamiltonian H under the local gauge transformation $J_{ij} \rightarrow \epsilon_i \epsilon_j J_{ij}$ and $S_i \rightarrow \epsilon_i S_i$ and holds for a system of any size. It is rigorous for a system with any given (disorder independent) subset of spins frozen to $S_i = +1$. From (3) we obtain, as an intermediate (new) result that *the line cannot enter the spin-glass phase*, where one expects^{1,8}:

$$\lim_{|i-j| \rightarrow \infty} \chi_{2k+1}(i, j) = 0, \quad \lim_{|i-j| \rightarrow \infty} \chi_{2k+2}(i, j) \neq 0.$$

Since for the present purpose one needs only to consider the disordered (replica symmetric¹²) phase, we give here a different derivation of (3) than in Ref. 8, based on the n -replica Hamiltonian H_n :

is certainly valid whenever mean-field exponents are exact (since then only one χ_i diverges on each boundary) which is expected for $d > 6$. Moreover, as discussed below, the $6-\epsilon$ expansion reveals no bifurcation in this scenario near $d=6$. Although it makes sense physically, this argument alone becomes less conclusive in low dimensions ($d=2,3$) since the two susceptibilities χ_1 and χ_2 could diverge similarly on the P-F boundary. The second argument, that we now introduce, clarifies the situation and allows us to identify possible alternative scenarios.

This second argument is based on the fact that Nishimori's condition (2) exhibits a remarkable property of *scale invariance* which does not seem to have been noticed previously: Namely, the distribution of disorder $P(J)$ satisfies on the line a symmetry property which is preserved by appropriate exact renormalization schemes. Physically, the symmetry (2) is reproduced at all scales in the system: For instance, the distribution of $\langle S_i S_j \rangle$ for any (i, j) satisfies (2) (in the t variable⁸). As a result, *Nishimori's line (or manifold) plays a central role in the structure of the RG flow*, as is the case for the line of symmetric disorder $p = \frac{1}{2}$, which has similar properties: They both contain RG fixed points, and this will now be precisely formulated.

These properties are most simply illustrated for spin-glasses on hierarchical lattices, where the Migdal-Kadanoff RG recursion relations are *exact*.^{2,15} These lattices are constructed iteratively by the replacement of each bond at each step by a given association of b bonds in series and $l = b^{d-1}$ in parallel. For two bonds in parallel the recursion relation is $k' = K_1 + K_2$, and is $t' = t_1 t_2$ for two bonds in series [with $K = \beta J$ and $t = \tanh(\beta J)$]. Couplings remain nearest neighbor and

uncorrelated by renormalization, but their (common) distribution is changed accordingly. Finding the phases and critical exponents is then equivalent to studying the transformation $P(K) \rightarrow P'(K')$ and its fixed distributions, starting from an initial distribution $P_0(K)$ [we choose (1) to be specific].

It is explicitly shown in Ref. 8 that, for any hierarchical lattice, if the initial distribution $P_0(K)$ satisfies Nishimori's condition (2) then *all the iterates* $P'(K')$ *satisfy the same condition (2)* [one easily checks that (2) is invariant under addition *both* in parallel and in series]. The immediate consequence is that, starting from the initial distribution $P_0(K)$ corresponding to point N of Fig. 1, one flows towards a fixed distribution $P^*(K)$ which cannot be any of the four following fixed distributions: pure critical fixed point ($K=K^*$) or low-temperature ($K \rightarrow \pm\infty$) critical fixed distribution—since they do not satisfy (2); ferromagnetic ($K=+\infty$) or paramagnetic fixed point ($K=0$), since N is on the critical surface, which is stable under renormalization. We thus obtain the rigorous result that N corresponds to a nontrivial RG fixed point. Two cases are now possible in principle: (a) $P^*(K)$ has two (or more) unstable directions and N is a multicritical point. (b) $P^*(K)$ has one unstable direction and is a “random critical fixed point” attracting a whole portion of the P-F boundary.

The fixed distribution $P^*(K)$ is computed in Ref. 8, numerically for low-dimensional hierarchical lattices, and analytically in the limit $l \rightarrow \infty$. In all cases it is found that (i) P^* has exactly two unstable directions and is thus multicritical and (ii) there is only one multicritical point in the phase diagram of Fig. 1 and $M=N$. This is presumably a generic feature:

Nishimori's subspace (2) is repulsive in one direction, but attractive in all others, so that the fixed distribution $P^*(K)$ attracts the whole multicritical manifold [a codimension 2 manifold in the space of distributions $P(K)$, to which point M of Fig. 1 belongs] or a large portion of it [no dependence of $P^*(K)$ on the starting distribution P_0 was found]. Thus we reject case (b) above, not only for hierarchical lattices, but more generally since it seems highly nongeneric [it would imply that an infinite set of equalities, like (3) for large $|i-j|$, hold on a whole portion of the P-F boundary].

On Bravais lattices the exact real-space RG generates many-spin couplings with statistical correlations. However, as shown in Ref. 8, there still exists a symmetry of the joint distribution of *all* interactions $P\{J\}$, satisfied on Nishimori's line (2), which is invariant by summation over *any* subset of spins. If we denote by J the set of all interactions $J = \{J_{ij}, J_{ijkl}, \dots\}$ the following form for $P\{J\}$ is RG invariant:

$$P\{J\} = \exp(-\beta H(J, \{S_i = +1\})) \Phi\{J\}, \quad (5)$$

where H is the Hamiltonian and $\Phi\{J\}$ an arbitrary *gauge-invariant* function of the couplings. Condition (5) is the correct generalization of Nishimori's condition (2) to arbitrary multispin couplings. The invariance of (5) has important implications for the field-theoretical renormalization-group formulation, as we now discuss. Note that the invariance of (5) [and of (2) for hierarchical lattices] again implies that Nishimori's line cannot enter the spin-glass phase.

In the $6-\epsilon$ expansions, both P-SG and P-F transitions can be described by the following Landau-Ginzburg energy density $F(M, Q)$:

$$F = \sum_{\alpha} \{(\nabla M_{\alpha})^2 + r_M M_{\alpha}^2\} + \sum_{\alpha < \beta} \{(\nabla Q_{\alpha\beta})^2 + r_Q Q_{\alpha\beta}^2 + w_1 Q_{\alpha\beta} M_{\alpha} M_{\beta}\} + w \sum_{\alpha < \beta < \gamma} Q_{\alpha\beta} Q_{\beta\gamma} Q_{\alpha\gamma} + \dots, \quad (6)$$

where the fields M_{α} and $Q_{\alpha\beta}$ are conjugate to S_i^{α} and $S_i^{\alpha} S_i^{\beta}$, respectively. This expression can be derived *exactly* from H_n by use of the Stratonovich transformation, and we find¹⁴ that, for a Gaussian distribution $P(J)$, Nishimori's line is defined by $r_M = r_Q$, $w_1 = w$. One can check that the one-loop formulas for the RG flow obtained in Ref. 13 leave this line invariant and give the multicritical fixed point $r_M = r_Q = 0$ on the line. In (6) it can be seen as an exact symmetry under the interchange of M_{α} and $Q_{1\alpha}$ ($\alpha > 1$) corresponding to the additional invariance of the replicated Hamiltonian H_n noticed above. This symmetry should hold to all orders, and no bifurcation occurs near six dimensions.

The scaling axis at the multicritical point N can be found exactly in general, from powerful topological constraints, first derived in a weaker version by Nishimori,⁶ and renormalization arguments. Let $P_N(K)$ be any distribution satisfying Nishimori's condition (2). It is proven in Ref. 8, that for any $P(K)$ obtained from $P_N(K)$ by any *odd change of variable* [e.g., $P(K) = P_N(\phi(K))$

$\times |d\phi/dK|$ with $\phi(-K) = -\phi(K)$], one has $|\langle S_i \rangle_P| \leq |\langle S_i \rangle_{P_N}|$. Thus, if $P_N(K)$ is inside the paramagnetic phase, $P(K)$ cannot be ferromagnetic. On hierarchical lattices the following arguments can be made rigorous. At the multicritical fixed point $P^*(K)$ one has that (i) the eigendirection of largest eigenvalue is inside Nishimori's surface (*even subspace*) and (ii) the second relevant eigendirection is tangent to the critical P-F surface, since the crossover exponent is different from 1 in any dimension (for $d < 6$ on Bravais lattices). The above inequality implies that this second eigendirection is in the *odd subspace* [generated by infinitesimal odd deformations of $P^*(K)$]. The tangent space at P^* splits into two locally RG-invariant subspaces, even and odd. Coming back to the plane of initial parameters of Fig. 1 this implies, in the absence of discontinuities, that one scaling axis at M is along Nishimori's line, the other being along the temperature direction, which is the only odd direction available. On Bravais lattices, the same

picture should hold, and we have checked¹⁴ from Ref. 15 that this is fully confirmed by the $6-\epsilon$ expansion. When more parameters are added, all points which are both in the multicritical manifold and inside Nishimori's manifold are such that the second scaling axis is an odd direction (not necessarily the temperature).

In conclusion, irrespective of the lattice and dimension, we expect multicriticality in Ising spin-glasses to be described by a RG fixed point with universal properties characteristic of Nishimori's disorder ($\eta_{2k+1} = \eta_{2k+2}$, and many others⁸): This should be true for a wide class of models which do not necessarily have a Nishimori line—such as, for $a \neq 1$, $P(J) = p\delta(J - J_0) + (1-p) \times \delta(J + aJ_0)$ —and in the case of model (1) one has $M = N$. Second-order P-F and P-SG transitions are crucial for this scenario. Note that in the random q -state clock model of Ref. 11, transitions become first order for $q > 6$, and $M = N$ is indeed found for $q = 3$, although not for $q = \infty$. A similar behavior is observed in the $[Z^n]$ models.¹⁶ Finally, alternative scenarios where N is a random critical fixed point, or N is multicritical but distinct from M , are highly unlikely.

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