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Disciplines

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A Non-Linear Macroeconomic Term Structure Model

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Abstract

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published by the Fed, and one month returns for the benchmark 2, 10, and 30 year Treasury zeros calculated from the month-end yield curves from November, 1985 through March, 2013. Over the entire sample the root mean square error (RMSE) in fitting yield curves is only 4.6 basis points. I find conditional yield curve responses to changes in the state variables, which are significantly different from the unconditional factors. The model is tested out-of-sample by fitting Treasury Inflation Protected Securities. I find ample profit opportunities.
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1 Introduction

In this paper I use financial engineering based on known theory to find an accurate, practical, conditional model of the term structure of interest rates with important implications for forecasting, risk measurement, and risk management. The conditioning information is captured by three macroeconomic state variables: The annual growth rate of the Core Consumer Price Index (CPI), the Unemployment rate (UE) and the quarterly growth rate of Non-Farm Payrolls (NFP). Accurate forecasts of the three state variables results in accurate forecasts of the term structure.

I assume that the Fed sets interest rates based on a policy rule which is a modified Taylor (1993) Rule.[1] Following Black (1995), the policy rate is an affine function of the three state variables, but with a minimum lower bound:

$$r(x_t) = \max(r_\infty + (x_t - x_\infty)\gamma, r_m), \tag{1}$$

where $x_t = (CPI_t, UE_t, NFP_t)$ are the government state variables at time t, x_∞ is the Fed's equilibrium targets for the states, r_∞ is the Fed's equilibrium target short rate, $\gamma = (\gamma_{CPI}, \gamma_{UE}, \gamma_{NFP})$ are the Fed's response coefficients for the state variables and r_m is the minimum rate. The quantity

$$s_t = r_\infty + (x_t - x_\infty)\gamma \tag{2}$$

is called the target rate or the shadow rate.

This policy rule implies that there are two economic regimes: In the Normal Response Period (NP), the policy rate is set using the affine function; in the Zero Response Period (ZP) the policy rate is at the minimum, even though

[1] The Taylor Rule is a policy directive based on quarterly data using the GDP deflator and the output gap as state variables. I need to adapt the Taylor Rule to use monthly data. I substitute Core CPI as an inflation measure. I know from Okun's Law that the output gap is inversely related to the unemployment rate, which is also released monthly. So UE is a natural state variable. But Veronesi (2010) reports that the growth rate of Non-Farm Payrolls has more explanatory power for yields than UE. Since UE measures the level of the labor market and NFP measures the growth of the labor market, I add NFP as the third proposed state variable.

the FED would prefer a lower, even negative, rate.[2] While the response of the economy changes in the two periods, I assume that Fed policy does not change over the entire sample. The policy rule is the first source of non-linearity in the model.

Using a VAR, I estimate that $x_\infty = (1.52\%, 4.96\%, 0.99\%)$. These estimates are the revealed Fed state targets. The equilibrium state estimates seem to correspond well with announced FED policy: Equilibrium inflation is 1.52%, which is virtually the center of the FED's announced range; equilibrium UE is 4.96%, which is the FED's revealed level of full employment; and equilibrium NFP growth is 0.99%, which is very close to the long term growth rate of the labor force. I estimate that $\gamma = (2.06, -1.26, 0.47)$. These coefficients imply that the Fed raises Fed Funds by 206 basis points for a 100 basis point increase in Core CPI;[3] lowers Fed Funds by 126 basis points in response to a 100 basis point increase in UE; and raises Fed Funds by 47 basis points in response to a 100 basis point increase in NFP. Finally, I estimate that $r_\infty = 2.43\%$, (which is lower than Taylor's prescribed target of 4%).

I assume that r_m is one basis point for two reasons. The first reason is empirical: Actual short rates have never been below one basis point in the available data sample. Since July 1, 1954, the minimum Effective Fed Funds[4] rate has been four basis points; since November, 1985, the minimum month-end, one-month Treasury repo rate has been one basis point. The second reason is theoretical: There must be some discounting if prices for zero coupon bonds are to converge to zero, even if the risk-neutral processes for the state variables are non-stationary. If zero coupon bond prices do not converge to zero, then, in theory, a consol bond has an infinite price.

I model the state dynamics as a multivariate Ornstein-Uhlenbeck (OU) process with different coefficients in the NP and ZP.

$$dx_t = (x_\infty - x_t)\theta_p dt + dw_t\sqrt{S_p} \tag{3}$$

where θ_p is the mean reversion matrix in period $p = N, Z$ and S_p is the instantaneous covariance matrix of the states in period p. There can be only one economic equilibrium, so the equilibrium state vector is the same as the Fed targets, x_∞, in both periods. Equation (3) implies, of course, that $\{x_t\}$ is Gaussian. Not surprisingly, I show that the economy responds differently in the two periods, probably because the FED lacks it most potent policy tool in the ZP.

Market participants set prices using unobserved implicit state variables. The true states are implicit in prices, rather than explicit in the data published by the

[2] In the data, the Zero Response Period begins in December, 2008 when the FED announced that they were no longer targeting the Fed Funds rate and would allow it to drift between zero and 25 basis points.

[3] The Fed must raise rates faster than 1-to-1 in order to fight inflation.

[4] There is an active market in bank reserves trading at the Effective Fed Funds rate. This rate is usually kept close to the Fed's policy rate target through open market operations. Unsurprisingly, market participants do not lend money, even with 101% Treasury collateral, at zero rates.

government, for at least two reasons. First, the government's published state variables are really only estimates of the true states based on survey data and subject to revision. Second, the government state variable data for each month are published with a one month delay. For example, the unemployment rate and non-farm payroll are usually released on the first Friday of the next month; the inflation data are typically released in the middle of the next month. Hence even at release the data refer to time past, not the present or the future. It is unreasonable to expect market participants to use backward looking data alone in setting prices. In fact, I expect market participants to use the published data, along with all other information, to calculate the current implicit state variables. Nevertheless, the government data contains important information about the implicit state variables because both the government and the market are trying to estimate the same true states. Hence, I require that the market participants draw the implicit current state from a normal distribution with a mean equal to the forecast of the current states from the last reported government states (to compensate for the publication delay) and covariance of Σ_p in period p, where Σ_p is the monthly covariance implied by equation (3). Hence, the published state variables are observations with error of the implicit state variables.

Under rational expectations, investors are assumed to know the processes governing the evolution of both the government and implicit state variables. Equation (3) is the model for the physical processes for the government state variables in the NP and the ZP. The model for the implicit state variables follows a multivariate OU processes as well, but with different coefficients in the NP and ZP:

$$d\xi_t = (x_\infty - \xi_t)\theta_p^{\mathbb{P}} dt + dw_t^{\mathbb{P}} \sqrt{S_p}, \tag{4}$$

where $\{\xi_t\}$ are the implicit states and $\theta_p^{\mathbb{P}}$ is the mean reversion matrix in period $p = N, Z$. While the mean reversion matrix may differ between the government state process, equation (3), and the implicit state process, equation (4), I assume that the equilibrium states are the Fed targets, x_∞, for both processes. (There can only be one equilibrium.) Also, to tie the implicit states to the actual states I assume that both the government state process and the implicit state process have the same instantaneous covariance matrix in each of the economic periods.

I next model the risk neutral processes for the implicit states. The implicit state variables follow a multivariate Ornstein-Uhlenbeck process in policy period $p = n, z$:

$$d\xi_t = (\mu_p^{\mathbb{Q}} - \xi_t)\theta_p^{\mathbb{Q}} dt + dw_t^{\mathbb{Q}} \sqrt{S_p}, \tag{5}$$

where $\mu_p^{\mathbb{Q}}$ is the vector of risk-neutral equilibrium states and $\theta_p^{\mathbb{Q}}$ is a mean matrix. Girsanov's Theorem requires that the instantaneous covariance matrix be identical under the physical and risk-neutral measures.

As usual, nominal zero-coupon bond prices are the expected present value of \$1 to be delivered at maturity discounted at the compounded future policy rates using risk-neutral probabilities. Let $P(\xi_t, \tau)$[5] be the price in state ξ_t of a

[5] The model price does not depend explicitly on time, t, because none of the model parameters are functions of time.

$\tau-$year zero coupon bond:[6]

$$P(\xi_t, \tau) = E_t^{\mathbb{Q}} \exp\left(- \int\limits_{s=t}^{s=t+\tau} r(\xi_s) ds \right). \tag{6}$$

There is no known formula for the zero coupon bond prices so I calculate them numerically.

In addition to data on the state of the economy, I use month-end yield curves and zero-coupon bond returns to calibrate the model. The yield curve data are published on the Federal Reserve Research web site.[7] The data are month end continuously compounded yields on constant maturity zero coupon bonds with annual maturities of two[8] to thirty years from November, 1985 through March, 2013. The data begin in November, 1985 because that is the first month in which 30 year maturity yields are published. I also calculate one-month returns on the two, ten and thirty year maturity Treasury zero coupon bonds using the Federal Reserve yields.[9]

I use maximum likelihood estimation to find the 42 coefficients $\{\mu_p^{\mathbb{Q}}, \theta_p^{\mathbb{Q}}, \theta_p^{\mathbb{P}}\}$ and the 987 implicit state variables $\{\xi_t\}$. These are a lot of coefficients and states to estimate, but I have a tremendous amount of data: $329 \times 3 = 987$ government state variables, $329 \times 29 = 9541$ month-end yields, and $329 \times 3 = 987$ monthly returns for a total of 11515 data points. The conditional likelihood function is the joint probability of fitting the month-end yield curves, drawing the implicit state variables from the distribution of the month ahead government state forecasts, and forecasting actual one-month returns on the two, ten and thirty year maturity Treasury zeros. I find the maximum of the likelihood function by a numerical search. Notice that implicit states are not the same as latent states which are chosen only to fit yield curves.

There are at least three important uses of the model. First, it can be used to take positions on the yield curve. This requires investors to forecast the implicit state variables. Investors can take comfort in the fact that the differences

[6] Cox and Ross (1976) show that a financial asset price is equal to the expected present value, under the risk-neutral measure, of the asset's cash flows discounted at the risk-free rate.

[7] http://www.federalreserve.gov/econresdata/researchdata.htmas.
The yield curves are calculated through an extension of the Nelson-Siegel fitting technique. I use the same Nelson-Siegel spline to calculate bond yields with maturities of 23, 35, ..., 359 months which are needed to calculate one month returns. See Gürkaynak, Sack and Wright (2007). See Veronesi (2010) for a critique of fitting techniques.

[8] I eliminate the one-year bond because it is money-market eligible. The money market and the Treasury bond market are segmented by SEC regulation, which limits money market securities to 13 months or less to maturity. Furthermore, money market securities are traded on the money market desk while Treasury bonds are traded by different traders in a different market on the Treasury bond desk. This results in segmentation that is evident in market prices. [Knez, Litterman and Scheinkman (1994)]

[9] The model has only three independent zero coupon bond returns because there are only three independent state variables. Hence I select the benchmark maturities, 2, 10, and 30 years, which are the bonds auctioned regularly by the Treasury. Not surprisingly they are also the maturities with interest rate futures.

between the implicit and government state variables mean revert fairly quickly to zero.[10] Hence, investors can formulate trades to take advantage of their views on the future economy: The model translates their views on future implicit state variables into a view on the future yield curve. Conversely, by using the model to fit forward yield curves, investors can back-out the market's view of the future breakeven states of the economy. If the market's breakeven views are inconsistent or extreme, investors may want to bet against these views.

A second important use of the model is in risk measurement and risk management of bond portfolios. Traditional bond portfolio risk measures such as duration and convexity are calculated using the implausible assumption that the yield curve changes by parallel shifts. Following, the publication by Litterman and Scheinkman (1991) of a principal component analysis (PCA) of yield changes, investors began measuring risk with respect to yield curve changes implied by the three factors, level, slope and curvature. These unconditional risk measures were a big improvement over duration and convexity during the NP when interest rates were far from zero. When the economy is in the ZP, or even when Fed Funds are 2% or less, however, investors must replace the unconditional risk measures with conditional ones.

The third important use of the model is to exploit inconsistencies between the nominal Treasury market and the Treasury Inflation Protected Security (TIPS) market. TIPS values can be calculated directly from the non-linear model. The model values differ significantly from the market prices. By constructing a self-financing portfolio of a TIPS bond hedged with three nominal Treasury bonds, I can show that the model values indicate profitable trading opportunities. This can also be viewed as an out-of-sample test of the non-linear model.

This paper is organized as follows. Section 2 is a brief literature review. Section 3 contains the non-linear bond pricing model. I discuss the model implications in Section 4. In Section 5, I show that the model identifies a profitable strategy for managing a hedged TIPS portfolio. My conclusions are in Section 6. There are three Appendices. Appendix A contains a principal components analysis (PCA) of the yield curve data. Appendix B contains the development and estimation of the standard affine term structure model in continuous time, which I use as a comparison to the non-linear model. The details of the numerical solution for bond prices are found in Appendix C.

2 Literature

In this paper I weave together three strands of the term structure research literature.[11] The first strand uses an affine model with observed macroeconomic state variables to fit yield curves, such as Bernanke, Reinhart and Sack (2004)

[10] The mean reversion half lives of CPI, UE and NFP are 0.91 years, 0.32 years and 0.22 years, respectively.

[11] There is a vast literature about the term structure of interest, way too large to summarize in a literature review. Fortunately, Gurkaynak and Wright (2012) have recently written an excellent survey of the literature which focuses on the relationship between the term structure and the macroeconomy.

and Smith and Taylor (2009). Unfortunately, these models do not accurately fit the term structure.

Better fits are afforded by latent state variables, the second strand of research. Latent state variables are unidentified statistical factors which fit yield curves. The earliest latent state variable model is the PCA of Litterman and Scheinkman (1991), who identified three factors, which they called level, slope and curvature, which together explained over 99% of the variation in yield curves. A lot of subsequent research continued using latent factors, but added more structure to the model by modeling the latent variables using a VAR.[12] Armstrong and Richard (2002)[13], Ang and Piazzesi (2003), Diebold, Rudebusch and Arouba (2006) and Rudebusch and Wu (2008) have taken a fusion approach by combining observed macroeconomic state variables[14] with latent state variables in affine term structure models. Latent states have a virtue and a vice. Their virtue is that they fit yield curves very well. For example, in Appendix A, you can see that the first three principal components of the data explain 99.96% of yield curve variation. Their vice is that they lack economic content in that they are not interpreted as macroeconomic variables. It is difficult to see how market participants would be comfortable taking positions in the bond market based on their views of latent states. In this paper, I eliminate the vice but retain the virtue by using implicit state variables, which are not latent states, but a compromise between the latent states – which provide the best fit to the yield curve – and the government states – which provide economic content.

The final strand of research is non-linear term structure modeling. The great majority of structural models of the yield curve begin with a rule where the policy rate is an affine function of the state variables, either observed or latent or both, and derive an affine model of the term structure. Some of these models, following Vasicek (1977), do not restrict the state variables to be positive so that negative interest rates are possible. Others, recognizing the shortcoming of negative interest rates, follow Cox, Ingersoll and Ross (1985) by modeling the state variables using distributions that restrict the states to be non-negative.[15] Black (1995) realized the shortcoming of both of these approaches: there is no economic reason to suppose that the state variables in the policy rule are restricted to be non-negative, even though interest rates must be in the presence of currency. Policy rules based on observed state variables, such as the Taylor (1993) Rule, assume that the policy rate is an affine function of the rate of inflation, which can be negative, and some variables which measure real economic activity, such as the output gap, which can also be negative. Black theorizes that in certain states of the economy, policy makers would like to set a negative policy rate, but cannot because of currency, so they are forced to set the

[12] See Gurkaynak and Wright (2012) for a list of references.

[13] We did not publish the model because we were using it for proprietary portfolio management.

[14] To be more precise, these papers use a reduced number of economic factors to capture the effects of a larger set of nominal and real economic variables.

[15] For example, see Richard (1978) where I model both inflation and the real instantaneous risk-free rate as non-negative.

policy rate at or near zero. Hence the policy rule is a non-linear modified affine rule where the policy rate is the greater of the affine rule rate or $r_{\min}$. Recently, Feldhuter, Heyerdahl-Larsen and Illeditsch (2013) have presented an alternative to the Black model using Gaussian state variables but with a stochastic discount factor that results in a yield formula with positive interest rates.

There have been two approaches to solving Black's model. The first approach has been to solve the model exactly as possible with numerical techniques. Gorovoi and Linetsky (2004) solve Black's model with a single state variable[16] using eigenfunction techniques. They illustrate the results by fitting Japanese Government Bond (JGB) data using latent state variables.[17] Kim and Singleton (2012) use modern numerical methods to solve the differential equation for bond prices derived from Black's model with two latent state variables. Their model fits weekly JGB data from January, 1995 through March, 2008 very well with a root mean squared error (RMSE) of about 10 basis points.

Because of the curse of dimensionality, Krippner (2013) has pioneered using an approximate solution to the Black model. Krippner's approximation can be extended to more than three state variables and can be estimated quickly. Christensen and Rudebusch (2013) apply Krippner's technique to JGB data using both a two and a three state variable Black model .

This paper extends the literature in three ways: First, I introduce three *identified implicit (not latent or government)* state variables [18]; second, I use monthly *US* yield curve data with maturities from two to 30 years from November, 1985 through March, 2013; and third, I use *two economic response periods*, a new nonlinearity.

3 The Non-Linear Term Structure Model

In this section I develop and estimate the non-linear model with implicit state variables.

3.1 Two Regime State Variable VARs

My first step is to estimate the coefficients for the OU process in the NP and ZP for the government state variables given by equation (3).[19] I solve Equation (3) for a monthly VAR for the states:[20]

$$x_{t+1} = c_p + x_t M_p + \epsilon_{pt}, \tag{7}$$

[16] I am unaware of any generalization of their results to two or more state variables.

[17] They illustrate their model by successfully calibrating it to the JGB yield curve on February 3, 2002 with a respectable RMSE of 6.4 basis points.

[18] The extension from two to three state variables is not trivial. I had to use a new numerical scheme because the one used by Kim and Singleton is not unconditionally stable in three dimensions.

[19] The government state variables estimates are taken from the FRED database published by the Federal Reserve Bank of St. Louis.
http://research.stlouisfed.org/fred2/

[20] Meucci (2010)

where $M_p = e^{-\theta_p/12}$ is a 3×3 mean matrix, $c_p = x_\infty(I - M_p)$ is a 1×3 vector of constants, and $\{\epsilon_{pt} \sim N(0, \Sigma_p)\}$ are independent and identically distributed. The monthly covariance matrix, Σ_p, can be expressed in terms of the stack operator *vec* and the Kronecker sum

$$vec(\Sigma_p) = (\theta'_p \oplus \theta'_p)^{-1}(I - \exp(-\theta'_p \oplus \theta'_p/12))vec(S_p). \tag{8}$$

Hence the next step is to estimate a VAR for the Normal Rate Period, which results in estimates of c_n, M_n *and* Σ_n. I then calculate that $\theta_n = -12\ln(M_n)$. I can only estimate the Fed's implicit state targets, x_∞, using data from the NP since there is no information about x_∞ during the ZP:

$$x_\infty = c_n(I - M_n)^{-1}. \tag{9}$$

Finally I invert equation (8) to obtain an estimate of S_n.

I estimate the state variable VAR coefficients, c_n and M_n in equation (7), using ordinary least squares regression. Table 1 shows the coefficients and statistics estimated in the NP. A few comments are in order about the regression results. The CPI and UE regressions have excellent fits with R^2 greater than 98%; growth rates are more difficult to predict so the R^2 for NFP falls to about 91%. Examining the T-statistics, you can see that all but three of the regression coefficients are statistically significant at the 95% level. The Durbin-Watson statistics shows that serial correlation is not a serious problem. The three state variables all mean revert to equilibrium states, but with different rates of mean-reversion. Inflation once begun is notoriously difficult to abate; this is reflected in the long half-life for mean-reversion of 6.75 years.[21] The real economic states, UE and NFP, follow the business cycle with a mean-reversion half-life of 3.06 years.

From M_n and Σ_n I can back-out an estimate of the continuous time covariance matrix, S_n, shown in Table 2. Notice that none of the correlations are statistically different from zero.

Now I turn to the ZP during which I expect FED policy to be less effective in managing the economy. I again use ordinary least squares estimation, but I specify the equilibrium states to be the same ones I found in the NP, as discussed above. Constraining the equilibrium states to be x_∞ requires that I transform the state variables to be the difference between the current state and the equilibrium state, $\{x_t - x_\infty\}$, and then calculate the regressions. The results are shown in Table 3.

Again, a few comments are in order about the VAR. The regression fits, as measured by R^2, are worse for CPI and UE. In fact five of the nine regression coefficients are not significant at the 95% level. There are now problems with serial correlation which means that the T-statistics are biased upward although they are asymptotically consistent. Finally, the state variables mean revert in the ZP, but with shorter measured half lives.

[21] The half life is the time it takes a state varaible to return half-way to equilibrium in the absence of further random shocks.

Normal Response Period State VAR

Coefficients	CPI	UE	NFP
$\mathbf{c}_n$	0.0008	0.0017	-0.0036
	(1.67)	(3.51)	(-2.01)
$\mathbf{M}_n$	0.9989	0.0256	-0.0517
	(113.08)	(2.91)	(-1.58)
	-0.0181	0.9665	0.0898
	(-1.93)	(103.87)	(2.60)
	0.0114	-0.0405	0.9886
	(2.21)	(-7.88)	(51.83)
$\mathbf{x}_\infty$	1.52%	4.96%	0.99%
Half-Life (years)	6.75	3.06	3.06
$\mathbf{R}^2$	98.32%	98.16%	90.73%
Durbin-Watson	2.06	2.53	1.93

Table 1: Ordinary least squares regression coefficients and statistics for the state variables VAR in the Normal Response Period.

Normal Response Period Instantaneous Covariance

$\mathbf{S}_n$	CPI	UE	NFP
Volatility	0.45%	0.45%	1.66%
Correlations	0.000	0.042	-0.052
	(0.00)	(0.30)	(-0.37)

Table 2: The instantaneous volatilities and correlations in the Normal Response Period. The correlations are, in order, CPI and UE, CPI and NFP, UE and NFP.

Zero Response Period State VAR

Coefficients	**CPI**	**UE**	**NFP**
$\mathbf{M}_z$	0.9653	0.0216	-0.1162
	(26.89)	(0.53)	(-0.68)
	0.0030	0.9845	0.0325
	(0.64)	(183.69)	(1.45)
	0.0103	-0.0526	0.9585
	(1.70)	(-7.62)	(33.27)
Half Life (years)	3.34	1.52	1.52
$\mathbf{R}^2$	93.21%	96.56%	95.57%
Durbin-Watson	1.21	1.87	1.24

Table 3: Ordinary least squares regression coefficients and statistics for the state variables VAR in the Zero Response Period.

Zero Response Period Instantaneous Covariance

$\mathbf{S}_z$	**CPI**	**UE**	**NFP**
Volatility	0.44%	0.48%	2.08%
Correlations	0.061	-0.151	-0.236%
	(0.44)	(-1.10)	(-1.75)

Table 4: The instantaneous volatilities and correlations in the Zero Response Period. The correlations are, in order, CPI and UE, CPI and NFP, UE and NFP

The estimate of the continuous time covariance matrix in the ZP, S_z, is shown in Table 4. The volatilities of the real labor market state variables in the ZP are similar to the NP. The inflation volatility is greater in the ZP than the NP, perhaps reflecting less FED control. Again, none of the estimated correlation coefficients is statistically significant.

So it appears that the economy reacts differently in the two policy periods. To confirm this formally I calculated the log-likelihood for the two policy periods comprising the split regime and summed them to get 4732.1. In Appendix B, I re-calculated the log-likelihood using the restriction that the two policy periods have identical coefficients to get a log-likelihood of 4720.9. A likelihood ratio test with difference statistic $d = 2(4732.1 - 4720.9) = 22.3$ and 12 degrees of freedom shows that the probability that the economy has responded differently in the two policy periods is 96.6%. Hence, with a high degree of confidence, I conclude that a split regime model fits the data better.

3.2 The Policy Rule

I estimate the Fed's policy rule by regressing Fed Funds on the government state variables, lagged one month for publication delay, during the NP. I limit the

	Normal Response Period Policy Rule			
Coefficients	γ_0	γ_{CPI}	γ_{UE}	γ_{NFP}
	0.051	2.06	-1.26	0.47
	(7.48)	(19.28)	(-9.59)	(8.36)
$\mathbf{r}_\infty$	$\mathbf{R}^2$	**SE**	**Durbin-Watson**	**Autocorrelation**
2.43%	86.16%	0.808%	0.25	.85

Table 5: The coefficients and key statistics for a regression of Fed Funds on one-month lagged government state variables duirng the Normal Response Period. The standard errors of the coefficents are adjusted for serial correlation using the Newey-West procedure.

sample to the NP because the Fed announced that they were no longer targeting the Fed Funds rate starting in December, 2008. The regression coefficients and key statistics are shown in Table 5. The regression constant is denoted γ_0. I solve equation (1) to find that

$$r_\infty = \gamma_0 + x_\infty \gamma. \tag{10}$$

My estimate is $r_\infty = 2.43\%$, which is lower than most prescribed policy rules, (such as Taylor (1993)).

3.3 Numerical Bond Prices

Let $P(\xi_t, \tau)$ be the price in state ξ_t of a $\tau-$year zero coupon bond. Under the risk-neutral probabilities, the expected instantaneous return on this bond must equal the instantaneous risk-free rate in state ξ_t:

$$E_t^{\mathbb{Q}} \frac{dP(\xi_t, \tau)}{P(\xi_t, \tau)} = r(\xi_t)dt, \tag{11}$$

where $r(\xi_t)$ is given by equation (1). Using Ito's Formula I calculate that

$$E_t^{\mathbb{Q}} \frac{dP(\xi_t, \tau)}{P(\xi_t, \tau)} = \left[-\frac{\partial P}{\partial \tau} + \frac{1}{2} tr(S_p \frac{\partial^2 P}{\partial \xi' \partial \xi}) + (\mu_p^{\mathbb{Q}} - \xi)\theta_p^{\mathbb{Q}} \frac{\partial P}{\partial \xi} \right] dt. \tag{12}$$

Substituting equation (12) into equation (11) I get the fundamental partial differential equation for zero coupon bond prices:

$$\frac{\partial P}{\partial \tau} = \frac{1}{2} tr(S_p \frac{\partial^2 P}{\partial \xi' \partial \xi}) + (\mu_p^{\mathbb{Q}} - \xi)\theta_p^{\mathbb{Q}} \frac{\partial P}{\partial \xi} - r(\xi)P. \tag{13}$$

The Feynman-Kac Theorem guarantees that the unique solution to equation (13), subject to the boundary condition $P(\xi_t, 0) = 1$, is given by equation (6). But I can go no further with the analytical development because of the non-linearity introduced in equation (1) and because the state dynamics given by equation (5) differ by policy period. Instead I turn to numerical techniques to calculate the price by solving equation (13). The details of the numerical solution are in Appendix C.

3.4 Maximum Likelihood

Each month I want to find three implicit state variables which are rational estimates of the true current state of the economy, fit the yield curve in the sense that the pricing model using the implicit state variables has small errors, and forecast one-month benchmark bond returns as accurately as possible. How do I find rational estimates of the true current state? Normally, the government state variables are treated as observations on the true states with unknown measurement error, which is estimated as part of the calibration. Because the implicit states are endogenous, however, I cannot estimate both the implicit states and the measurement error. If I try, the maximum likelihood is infinite because setting the implicit states equal to the government states results in zero measurement error. So I have to specify the covariance matrix of the measurement errors. Hence, I assume that market participants draw the implicit states from the month-end distribution for the government state. Investors use equations (7) to forecast the month-end government states (which are not yet published) in the two policy periods. The expected states at the end of the month are

$$m_{pt} = c_p + x_{t-1} M_p. \tag{14}$$

Hence the unobserved current implicit state is drawn from the physical distribution for the month-end government states:

$$\xi_t \sim N(m_{pt}, \Sigma_p). \tag{15}$$

Thus I specify that Σ_p is the covariance matrix of state measurement errors.

In implicit state ξ_t and response period p, the expected physical one-month change in the implicit state is

$$\Delta_p^{\mathbb{P}} \xi_t = (x_\infty - \xi_t)(I - \exp(-\theta_p^{\mathbb{P}}/12)). \tag{16}$$

The covariance of the one-month state change is $\Sigma_p^{\mathbb{P}}$ is

$$vec(\Sigma_p^{\mathbb{P}}) = (\theta_p^{\mathbb{P}\prime} \oplus (\theta_p^{\mathbb{P}\prime})^{-1}(I - \exp(-(\theta_p^{\mathbb{P}\prime} \oplus \theta_p^{\mathbb{P}\prime}/12))vec(S_p). \tag{17}$$

Let $Y(\xi_t, n) = -\ln(P(\xi_t, n))/n$ be the yield on an n-year bond in state ξ_t. I can closely approximate the annualized one-month conditional expected return on an n-year bond in state ξ_t and policy period p by using a Taylor series expansion:

$$\rho(\xi_t, n) = 12[nY(\xi_t, n) - (n - \tfrac{1}{12})(Y(\xi_t, n - \tfrac{1}{12}) + (\Delta_p^{\mathbb{P}} \xi_t)\frac{\partial Y(\xi_t, n)}{\partial \xi_t} + \frac{1}{2} tr(\Sigma_p^{\mathbb{P}} \frac{\partial^2 Y(\xi_t, n)}{\partial \xi_t' \partial \xi_t})] \tag{18}$$

The joint conditional log-likelihood that the implicit states are distributed according to equation (15), that the bond prices fit the government yield curve and that the month ahead implicit state forecasts minimize the prediction errors of the one-month returns is:

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \sum_{t=1}^{T} \left[-\tfrac{3}{2}\ln(2\pi) - \tfrac{1}{2}\ln(|\Sigma_p|) - \tfrac{1}{2}(\xi_t - m_{pt})\Sigma_p^{-1}(\xi_t - m_{pt})\prime \right] \qquad (19) \\
& -\tfrac{1}{2}\left[29\ln(2\pi) + 29\ln(\sigma_{rp}^2) + (y_t - Y(\xi_t))(y_t - Y(\xi_t))\prime/\sigma_{rp}^2 \right] \\
& -\tfrac{1}{2}\left[3\ln(2\pi) + \ln(\Sigma_{\rho p}) + (R_{t+1} - \rho(\xi_t))\Sigma_{\rho p}^{-1}(R_{t+1} - \rho(\xi_t))\prime \right]
\end{aligned}
$$

where σ_{rp} is the yield curve fitting error in response period p; $y_t = [y_{2t}, ..., y_{30t}]$ is the yield curve data at time t; $Y(\xi_t) = [Y(\xi_t, 2), ..., Y(\xi_t, 30)]$ is the model yield curve in state ξ_t; $R_{t+1} = [R_{2t+1}, R_{10t+1}, R_{30t+1}]$ are the realized returns on the 2, 10 and 30 year Treasury zeros between month t and month $t+1$; and $\rho(\xi_t) = [\rho(\xi_t, 2), \rho(\xi_t, 10), \rho(\xi_t, 30)]$ are the model predicted returns on the same bonds.[22]

A numerical search finds the coefficient values and implicit states which maximize the conditional joint log-likelihood, equation (19). There are 42 coefficients , nine in each response period for the physical OU implicit state process, equation (4), and twelve coefficients in each period for the risk-neutral OU implicit state process, equation (5). Define the coefficient vector $\Psi = (M_n^{\mathbb{P}}, M_z^{\mathbb{P}}, \mu_n^{\mathbb{Q}}, \theta_n^{\mathbb{Q}}, \mu_z^{\mathbb{Q}}, \theta_z^{\mathbb{Q}})$. I use a Powell search without derivatives, which is a sequential conjugate direction search, to find improvement in Ψ. When I find improvement I use a Nelder-Mead search to update the implicit state variables, $\{\xi_t\}$. I continue the search until no further improvement is possible within numerical tolerances. The details of the search are in Appendix C.[23]

3.5 Coefficient Estimates

Turning first to the NP, I find the coefficients for both the physical and risk-neutral OU processes, shown in Tables 6 and 7, respectively. The physical VAR implies that the implicit states CPI and UE mean revert to x_∞ at about the same rate as the government states do, but implicit NFP mean reverts much faster than government NFP. The coefficient estimates imply that the states do not mean revert under the risk-neutral measure, as can be seen from the negative eigenvalue of $\theta_n^{\mathbb{Q}}$ in Table 7.

Now I turn to the Zero Response Period, where I again find the coefficients for both the physical and risk-neutral OU processes, shown in Tables 8 and 9, respectively. The largest real eigenvalue for the physical OU process in the ZP is greater than one, implying that the physical process for ξ_t is not stationary in the ZP.

[22] I use the yield curve on April 30, 2013 to compute the returns during March, 2013.

[23] In an earlier draft of this paper, I reported that the search for the optimal parameters took about one month. Since then I have improved the search so that it takes about three days. In any case, once the model parameters are estimated, the calculation of yield curves for new implicit states takes less than one second.

Normal Response Period Physical Implicit State VAR Coefficients			
NP Coefficients	**Implicit CPI**	**Implicit UE**	**Implicit NFP**
$\mathbf{M}_n^{\mathbb{P}}$	0.9787	0.0236	0.0407
	(79.57)	(1.40)	(1.70)
	0.0112	0.9485	-0.0493
	(0.77)	(48.20)	(-1.39)
	0.0218	-0.0491	0.8987
	(1.68)	(-2.35)	(31.03)
Half-Life	4.98	2.67	0.38
$\mathbf{R}^2$	97.16%	93.61%	83.12%
Durbin-Watson	1.80	1.86	2.01

Table 6: Optimal coefficients and asymptotic T-statistics for the physical implicit state VAR in the NP.

Normal Response Period Risk-Neutral OU Coefficients			
NP Coefficients	**Implicit CPI**	**Implicit UE**	**Implicit NFP**
$\boldsymbol{\mu}_n^{\mathbb{Q}}$	0.0447	0.0481	0.0163
	(232.6)	(293.2)	(64.9)
$\boldsymbol{\theta}_n^{\mathbb{Q}}$	0.0480	-0.0062	-0.0464
	(158.9)	(-3.6)	(-33.4)
	-0.0987	0.1570	-0.1377
	(-139.4)	(76.7)	(-44.1)
	0.0340	-0.1747	0.0254
	(78.6)	(-149.5)	(10.8)
Real Eigenvalues	0.248	0.070	-0.087

Table 7: Optimal coefficients and asymptotic T-statistics for the risk-neutral OU process in the NP.

Zero Response Period Physical Implicit State VAR Coefficients			
ZP Coefficients	**Implicit CPI**	**Implicit UE**	**Implicit NFP**
$M_z^{\mathbb{P}}$	0.7149	-1.4079	-0.9706
	(6.16)	(-0.72)	(-1.15)
	-0.0160	0.9211	-0.0200
	(-2.64)	(7.21)	(-0.46)
	-0.0012	0.0762	1.0204
	(-0.14)	(0.60)	(10.70)
Real Eigenvalues	1.06	0.96	0.64
$\mathbf{R}^2$	74.65%	89.67%	92.63%
Durbin-Watson	2.12	1.44	2.41

Table 8: Optimal coefficients and asymptotic T-statistics for the physical implicit state VAR in the ZP.

Zero Response Period Risk-Neutral OU Coefficients			
ZP Coefficients	**Implicit CPI**	**Implicit UE**	**Implicit NFP**
$\mu_z^{\mathbb{Q}}$	0.0062	0.0628	0.0036
	(2.8)	(5.6)	(0.2)
$\theta_z^{\mathbb{Q}}$	-0.3565	-5.9971	-1.2660
	(-7.6)	(-11.7)	(-3.1)
	-0.1223	6.2551	0.3458
	(-4.6)	(21.9)	(1.6)
	0.1251	-3.0477	0.0167
	(13.0)	(-30.4)	(0.2)
Real Eigenvalues	6.178	0.079	-0.342

Table 9: Optimal coefficients and asymptotic T-statistics for the risk-neutral OU process in the ZP.

3.5.1 Implicit State Estimates

The implicit state variables are reasonably close fits to the forecasts as can be seen in the Figure 1. You can see that the implicit CPI has trended down tracking government estimates of CPI, but there have been some notable deviations. In March 2013, for example, government Core CPI has been running about 2%, but implicit Core CPI has been about 1.4%. The non-linear model's implicit UE tracks the government's estimation of the UE rate with high fidelity. Finally, implicit NFP, like CPI, tracks government estimated NFP, but with notable deviations. For example, in March 2013, the implicit NFP growth rate is strong at 3.6%, but the government estimate from the Establishment Survey is only 2%. The fourth panel in Figure 1 shows the residual error by maturity in fitting the yield curve.

The goodness of fit statistics in Table 10 show that the non-linear model fits yield curves extremely well throughout the sample. The non-linear model's fitting error to the yield curve is only 4.6 basis points RMSE over the entire sample. In the NP the fit improves to 4.1 basis points RMSE, but deteriorates to 6.7 basis points RMSE during the ZP.

For purposes of comparison, I have also estimated the affine counterpart to the term structure model in which I eliminate both nonlinearities: The policy rule is given by equation (1), but without the lower bound at $r_{\min}$ and the physical and risk neutral OU processes for the implicit states are the same in both response periods.

In Table 10, I compare the goodness of fit measures between the non-linear model and the affine model. While the differences in σ_r between the two models is minor, the differences in the log-likelihood are major because they measure the model's ability to fit simultaneously both the yield curves and the states. Thus you can see that the non-linear model fits yields a little better than the affine model, but with a substantial improvement in the fit to the government states as shown in Figure 18 in Appendix B.

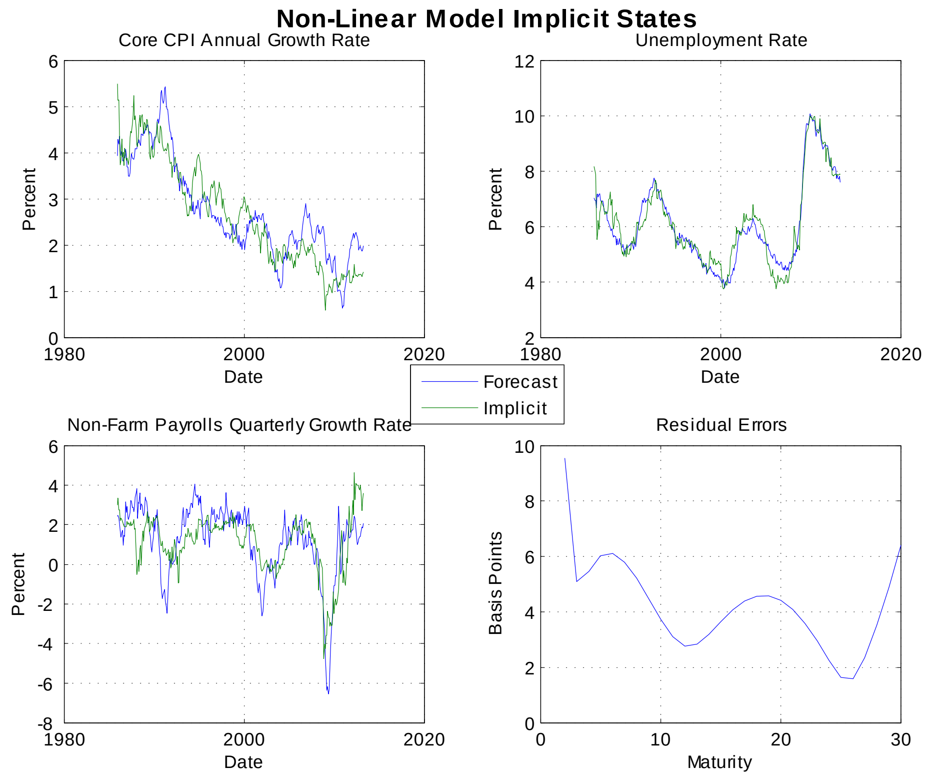

Figure 1: Implicit States and one month ahead government states VAR forecast during the entire sample period. Panel 4 is the residual fitting error by maturity.

Non-Linear and Affinel Models Goodness of Fit

	Entire Sample	Normal Period	Zero Period	Affine
$\boldsymbol{\sigma}_{rp}$	0.046%	0.041%	0.067%	0.047%
$\mathbf{R}^2$	99.946%	99.938%	99.706%	99.944%
Likelihood	59259.04	50377.22	8881.82	51381.12

Table 10: Comparison of goodness of fit measures for the non-linear model in both response periods and the affine model.

4 Model Implications

There are at least 11 important model implications.

1. The problem of forecasting the yield curve boils down to making forecasts of CPI, UE and NFP.
2. There is an equilibrium yield curve.
3. Conditional monthly yield volatilities vary substantially over the sample.
4. Conditional monthly expected excess returns vary substantially over the sample.
5. Conditional Sharpe ratios (computed by annualizing monthly data) vary substantially over the sample.
6. The market's breakeven forecasts of future states can be inferred from forward rate curves.
7. The shadow policy rate agrees closely with Fed Funds during the NP. It is, of course, negative – and sometimes substantially so – during the ZP.
8. There is little evidence that there are unspanned risks in bond returns.
9. Conditional level, slope and curvature factors are actually responses to changes in CPI, UE and NFP, respectively. They vary significantly over the sample period.
10. Conditional state duration and convexity differ significantly from the standard unconditional measures. This means that current commonly used unconditional portfolio risk measures and hedges are incorrect.[24]
11. Conditional risk premiums vary significantly over the sample.

I discuss each implication in turn.

4.1 Government States minus Implicit States

I have shown that conditioned on the implicit values of the three state variables the term structure can be forecasted very accurately. In my experience, it would increase investor comfort if the implicit state variables mean-revert to the government state variables in a usefully short time. In fact, they do mean revert fairly quickly as can be seen in Figure 2 which shows a graph of the government minus the implicit state variables and Table 11 which reports the results of a VAR of the difference between the government and implicit state variables. You can see that the implicit real variables, UE and NFP, mean-revert quickly with a half life of about four months and three months, respectively. A difference in CPI is slower to mean revert with a half-life of about 11 months.

[24] See Hansen and Richard (1987) for a discussion of the role of conditioning information on risk measures and the mean-variance frontier.

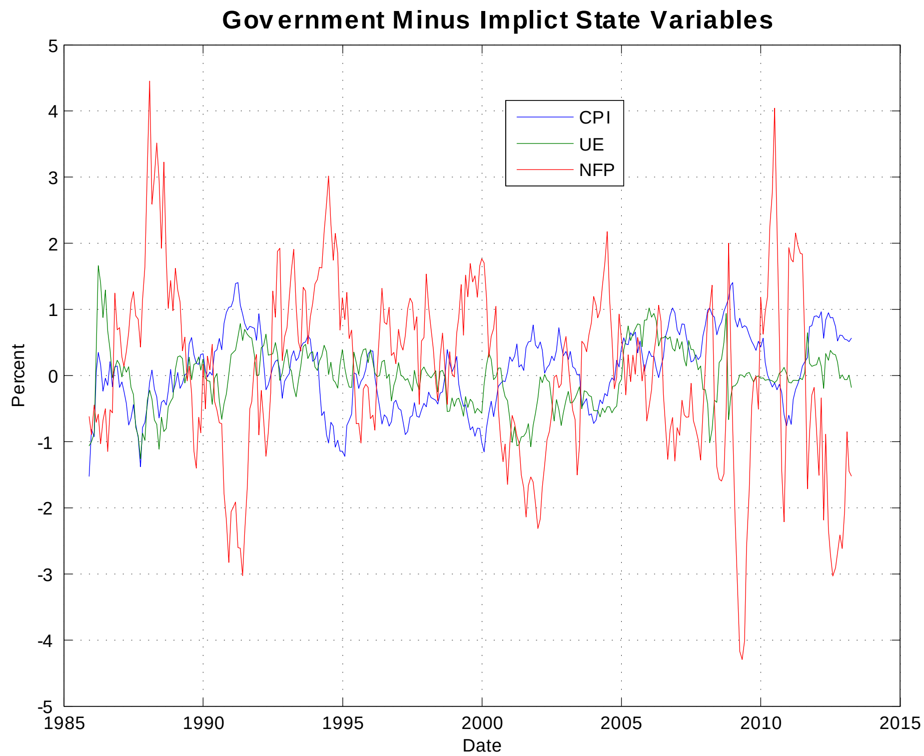

Figure 2: Government minus Implicit state variables over the entire sample period.

Government - Implicit State VAR

VAR Coefficients	Gvt-Imp CPI	Gvt-Imp UE	Gvt-Imp NFP
Gvt-Imp CPI	0.9057	-0.0003	-0.3192
	38.0	0.0	-4.1
Gvt-Imp UE	-0.0367	0.8301	0.1225
	-1.4	27.4	1.5
Gvt-Imp NFP	-0.0200	-0.0149	0.8109
	-2.1	-1.3	25.9
Half Life (Years)	0.91	0.32	0.22
Standard Error	0.20%	0.24%	0.65%
$\mathbf{R^2}$	87.29%	72.09%	77.59%
Durbin-Watson	1.83	1.93	1.80

Table 11: Coefficients and statistics for a VAR of the government minus implicit state varaibles over the entire sample period.

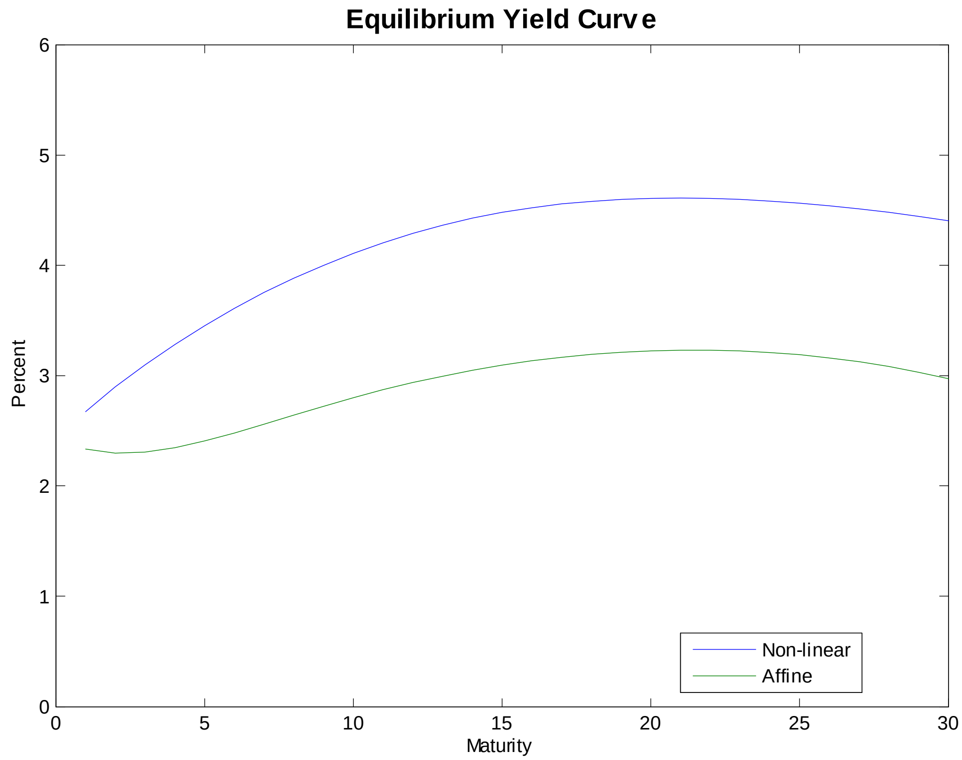

Figure 3: The non-linear model yield curve and the affine model yield curve at equilibrium, $x_\infty = [1.52\%, 4.96\%, 0.99\%]$.

4.2 Equilibrium Yield Curve

The equilibrium yield curve, which is the non-linear model yield curve when the state is x_∞, is shown in Figure 3. The shape of the equilibrium yield curve would be expected by most bond market participants because of the gains from convexity. Recall that bond convexity increases the expected rate of return of owing a bond by $0.5 \times convexity \times Yield\ Variance$. For a τ year zero coupon bonds $convexity = \tau^2$ so that the yield must fall for long maturity zero coupon bonds to offset the gains from convexity. Also shown is the equilibrium yield curve for the affine model. The affine equilibrium is about 100 basis lower than the non-linear model equilibrium because the affine model permits negative yields, hence lowering the equilibrium yield curve.

4.3 Conditional Yield Volatility

The results in this section and the following two sections are based on calculations over a one month time span for the benchmark zero coupon bonds with

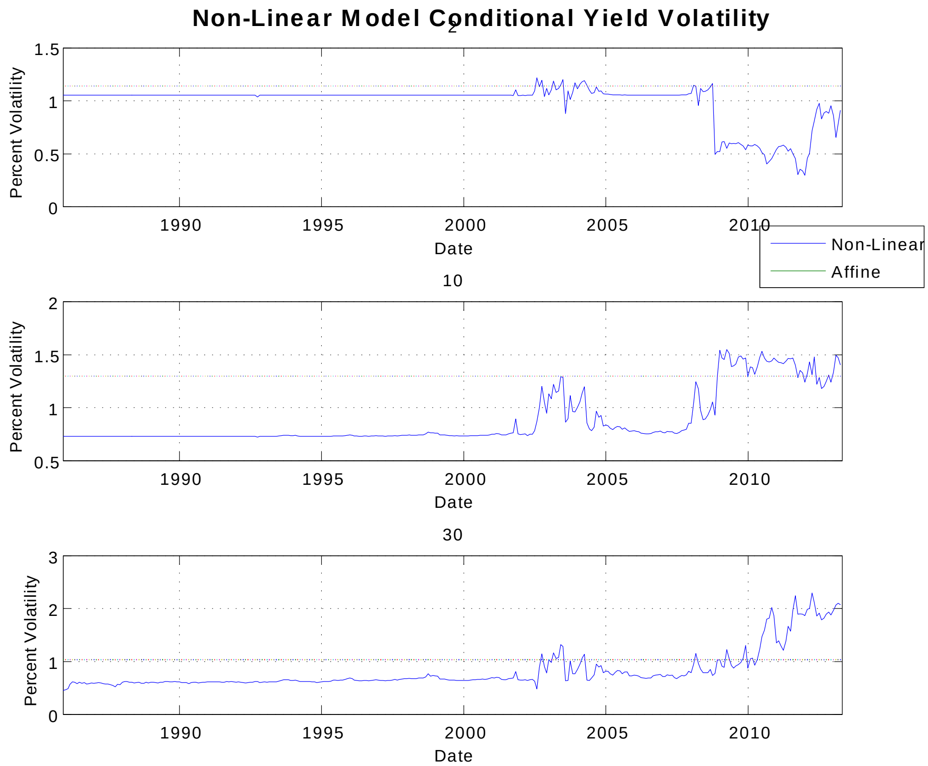

Figure 4: The conditional one-month yield volatilities for the benchmark bonds computed by both the non-linear and the affine models.

maturities of two, ten and thirty years. First, I find conditional yield volatilities shown in Figure 4. Following the crisis in 2008, conditional yield volatility on the two year bond fell dramatically as the yield on the bond neared zero, eliminating the possibility of further declines. In contrast the conditional yield volatility of the ten and thirty year maturity bonds increased dramatically due to the diminished effect of Fed policy in managing the economy. In the affine model, the conditional yield volatilities of all maturities are constant. This observation, *per se,* should eliminate affine models as serious contenders to explain the term structure.

4.4 Conditional Excess Returns

The conditional excess expected returns vary significantly during the sample as shown in Figure 5. Unlike the yield volatility the conditional expected excess returns in the affine model are not constant.

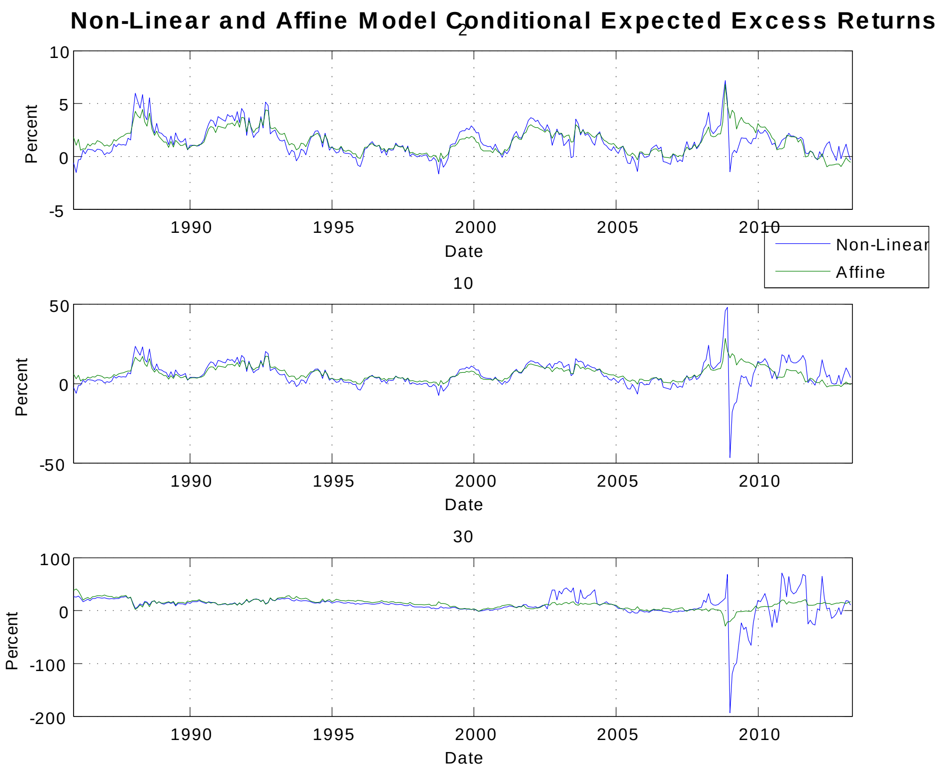

Figure 5: The conditional excess expected returns for the benchmark bonds computed by the non-linear and affine models.

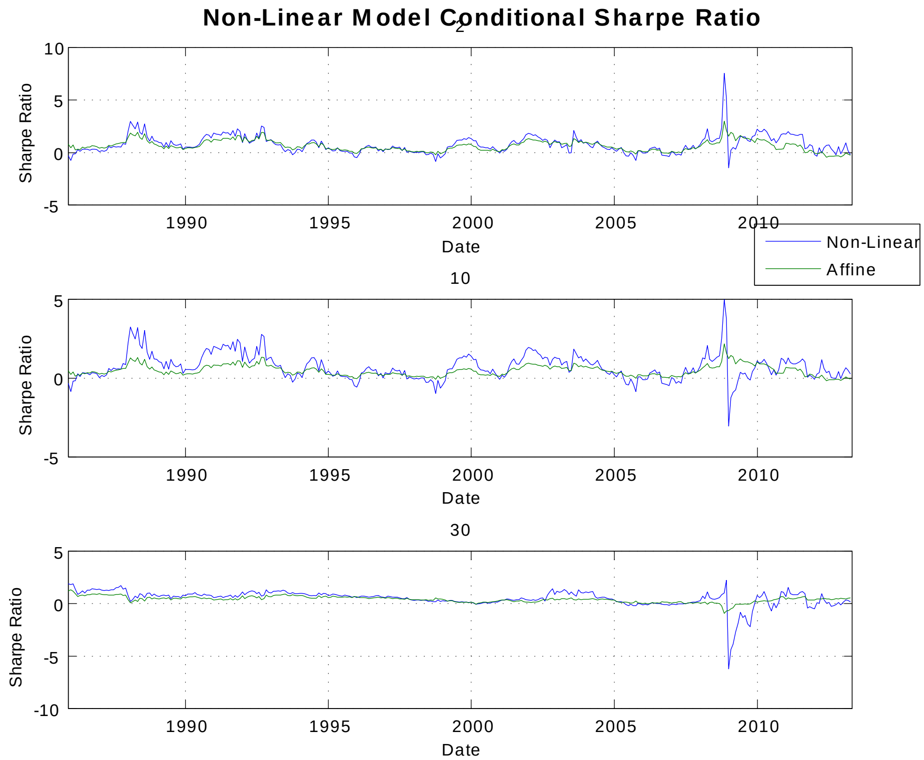

Figure 6: The conditional Sharpe ratios for the benchmark bonds for the non-linear and affine models.

4.5 Conditional Sharpe Ratio

The conditional Sharpe ratio for the non-linear and affine models are shown in Figure 6. There is noticeably less variation in the affine model Sharpe ratio over the sample period for the ten and thirty year bonds. This is due to both no variation in the conditional yield volatilities and less variation in the conditional expected excess returns.

4.6 Forward Curve Breakeven Forecasts

I can use the term structure model to back-out the forward implicit state variables from forward yield curves. The forward implicit state variables tell us how the market views inflation, unemployment and growth over the next one to three years in the sense that these are the breakeven levels built into the forward curves. They are not the market's unbiased expectations because the forward yields include a term risk premium; thus the forward yields are certainty equivalents and are not equal to expected future spot yields. They do, however, represent the breakeven levels against which investors are betting if they execute a yield curve trade. The forward implicit state variables are found by searching for those values which minimize the RMSE in fitting to the yield curves one, two and three years forward.

Figure 7 shows the breakeven forward states as of March 31, 2013. For comparison purposes I also include the VAR forecast. The forward implicit CPI breakeven is consistently about 40 - 50 basis points below the VAR projection. This is also about the difference on March 31, 2013 between government Core CPI and implicit Core CPI. My interpretation of this difference is that it reflects the current Fed policy of down-weighting inflation as a trigger for rate hikes. In fact the Fed has announced that it will not tighten before UE reaches 6.5% unless expected inflation increases to 2.5%, which it is not forecasted to do even as far out as 2016. Turning to UE, the VAR projections are similar to the forward implicit states. To finish interpreting forward UE I need also to look at forward NFP. The market is pricing forward yields as if NFP, starting from a current level of 3.6% – which is too high – is going to fall fairly rapidly starting in 2014, but that UE is also going to fall fairly rapidly. It seems that the forward implicit NFP is inconsistent with forward implicit UE. Both cannot be correct. This may be a trading opportunity since the market is making an inconsistent forecast.

4.7 Unspanned Risks

Are there unspanned risks in bond returns, i.e., are any data besides the state variables, such as yields, the latent states, or the government states, useful in forecasting returns? If so, this would indicate unspanned risks in the bond market. The answer hinges on what I use as state variables. If the model uses either latent state variables or government state variables, then there are unspanned risks. (Joslin, Priebsch, and Singleton (2013)) On-the-other-hand, if the model uses the implicit states there is little evidence of unspanned risks. I have assumed from the start that neither latent states variables nor government states variables are the market state variables. Hence in this sense I have *assumed* that there are unspanned risks in the bond market relative to the latent states and the government states. But once I replace latent and government states with implicit states, there is no longer any significant evidence of unspanned risks.

Table 12 shows the R^2 for regressions of one month benchmark returns on the beginning-of-month expected returns from the non-linear model. There is some predictive value in the returns forecasted using the implicit states. Table 13 shows the results of regressions of the residual benchmark returns on beginning-of-month government state variables and the yield curve as summarized by the level, slope and curvature factor loadings. The residuals cannot be explained by the government states or the factor loadings as shown by the feeble R^2 coefficients and the insignificant T-statistics. Only the residual regression for the two year zero has any significant coefficients at all.

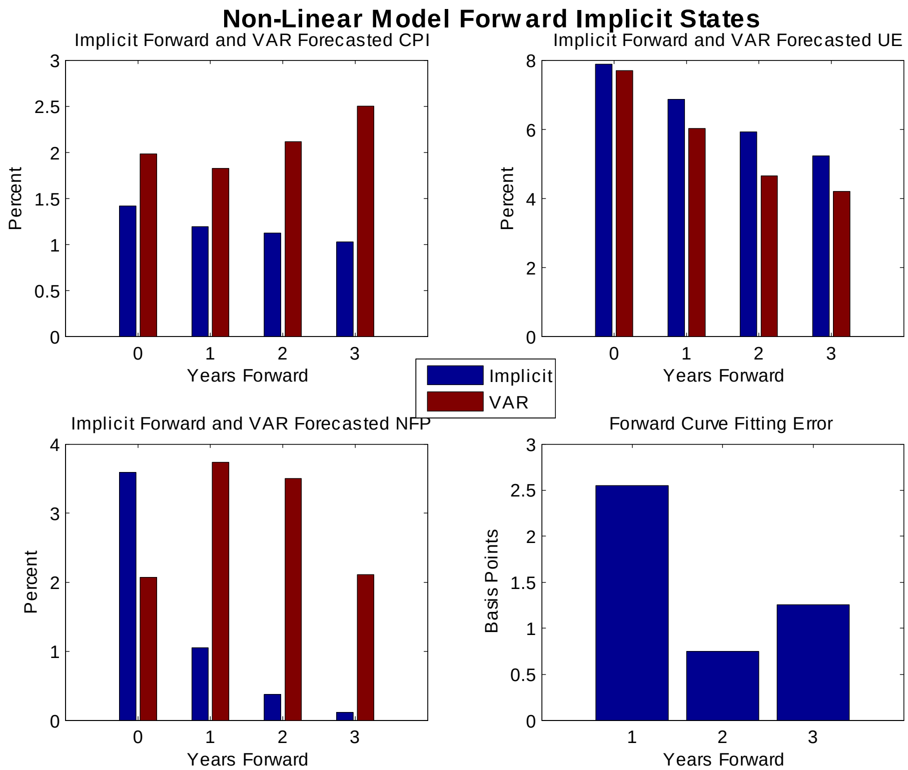

Figure 7: Panels 1 - 3 show conditional current, one, two and three year ahead breakeven implicit state forecasts from forward curves on March 31, 2013. For comparison the state VAR forecast is also shown. The fourth panel shows the RMSE error in fitting the one, two and three year forward curves.

Regression of Monthly Returns on Model Predictions

Maturity	**2**	**10**	**30**
$\mathbf{R}^2$	15.01%	4.89%	6.39%

Table 12: Regression of benchmark maturity one-month returns on non-linear model predicted returns for the entire sample period.

Regression of Residual Monthly Returns on Lagged Government States and Yields

T-statistics	2	10	30
Constant	0.22	-0.32	-0.01
CPI	-0.35	-1.19	-0.76
UE	0.53	1.33	0.69
NFP	**-2.94**	0.04	-0.49
Level	1.05	1.33	1.05
Slope	-1.32	-0.93	-0.58
Curvature	**2.24**	-0.27	0.06
$\mathbf{R}^2$	3.25%	1.06%	0.48%
Durbin-Watson	1.65	1.82	1.86

Table 13: Regression of residual benchmark returns on government CPI, UE, and NFP and level, slope and curvature loadings for the entire sample period.

4.8 Shadow Rate and Fed Funds

I can estimate the shadow rate, which is the Fed's empirical policy rule target in two ways. First I can extrapolate the regression of Fed Funds on the government state variables to the ZP. I can construct a second empirical estimate of the shadow rate using the implicit states estimated in the non-linear model. The result of both estimations are shown in Figure 8.

I interpret the Fed's use of quantitative easing as an effort to replicate the effect on the economy of setting Fed Funds equal to the shadow rate. To follow the shadow rate implicit in bond prices, the Fed would have had to purchase enough bonds to be equivalent to reducing the Fed Funds rate to about -6.00% by June, 2009. While they tried, they probably got nowhere close as can be seen from Bernanke's own testimony:

> Bernanke said the Fed's bond purchases helped the economy. Stock prices are higher and bond yields have fallen. He estimated that the effect of the program was roughly equivalent to a 40 to 120 basis-point reduction in the federal funds rate. And, the second round of bond buying lowered long-term interest rates by roughly 10 to 30 basis points.[25]

4.9 Conditional State IRS

In this section I find the conditional IRS for zero coupon bonds, which are the derivatives of zero coupon bond yields with respect to changes in the state

[25] Bloomberg, "Bernanke Says Fed 'Prepared to Respond' If Stimulus Needed", July 13, 2011. http://www.bloomberg.com/news/2011-07-13/fed-prepared-to-respond-if-needed-bernanke.html

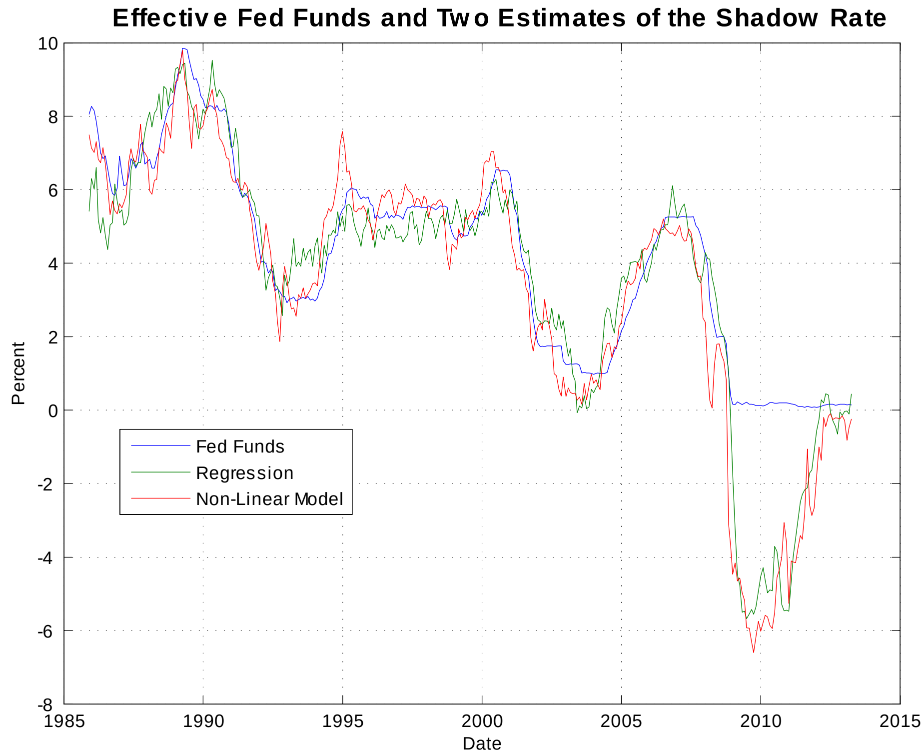

Figure 8: Effective Fed Funds and two estimates of the shadow Fed Funds rate. The first estimate, labeled Regression, is the result of an OLS regression of Effective Fed Funds on the government state variables in the Normal Response Period, which is then extrapolated to the Zero Response Period. The second estimate, labeled Non-Linear Model, is the shadow rate calculated from the implicit states estimated in the non-linear model.

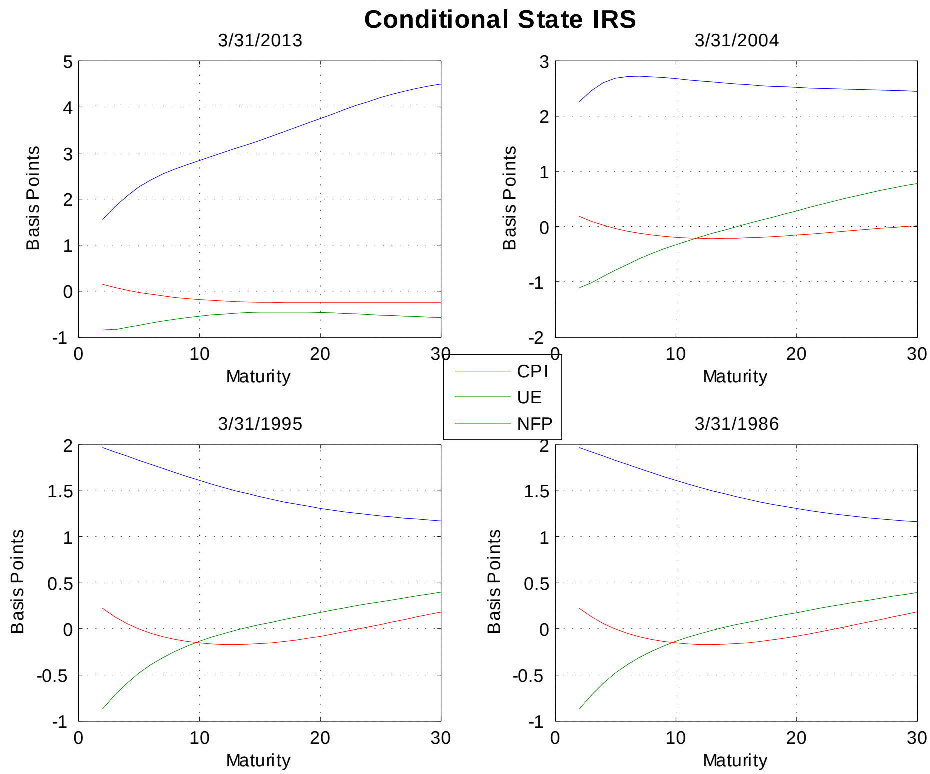

Figure 9: Conditional IRS of zero coupon bond yield with respect to a one basis point change in the states on four selected dates.

variables:

$$IRS(\xi_t, \tau) = \frac{\partial y(\xi_t, \tau)}{\partial \xi_t} \tag{20}$$

In Appendix B, I show that for the affine model the IRS for level slope and curvature do not change with the state as shown in Figure 19. If instead I use the non-linear term structure model you can see that the conditional level, slope and curvature actually differ significantly from date to date. To illustrate this difference, I selected the most recent state IRS curves and those 9, 18 and 27 years ago as shown in Figure 9. Obviously, likely yield curve reshapings are strongly conditioned by the current state of the economy and its associated yield curve. You can see that the level, slope and curvature factors are actually responses to changes in CPI, UE and NFP, respectively. Hence bond investors taking positions on the direction of rate changes are actually betting on changes in inflationary expectations. Steepening or flattening trades are actually bets on future UE. Butterfly trades are actually bets on future economic growth as measured by NFP.

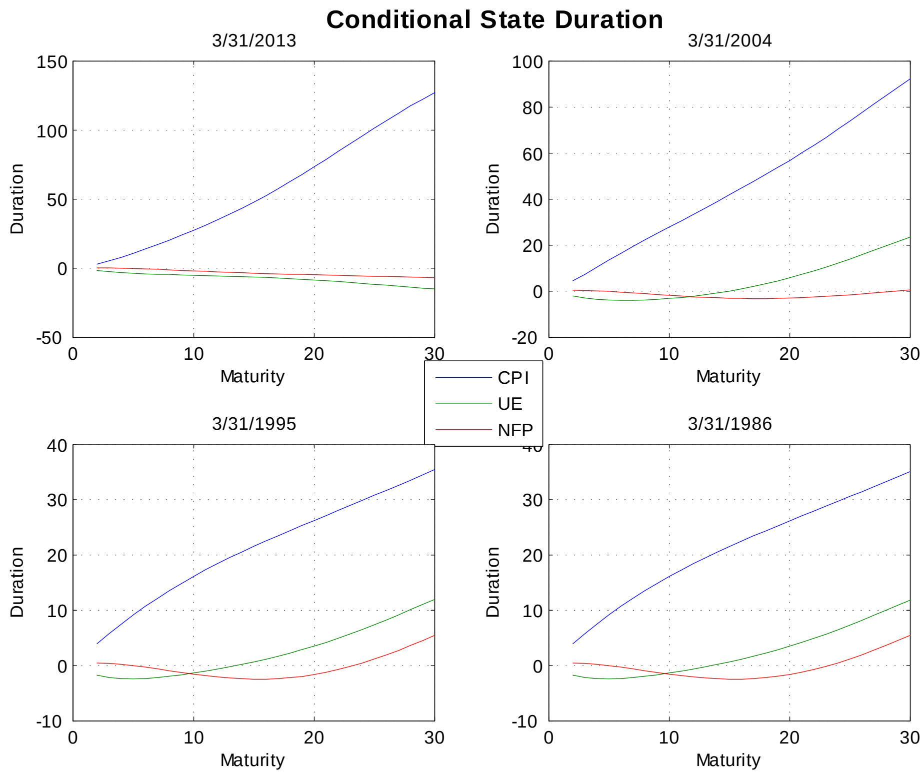

Figure 10: Conditional durations of zero coupon bond price with respect to the states on four selected dates. The conditional duration is the negative of the percentage price change caused by a one percent increase in the state.

4.10 Conditional State Duration and Convexity

Another useful measure of exposure to state changes is the conditional state durations, which are the negatives of the percentage price changes for a zero coupon bond caused by a change to the implicit state:

$$Dur(\xi_t, \tau) = -\frac{1}{P(\xi_t, \tau)} \frac{\partial P(\xi_t, \tau)}{\partial \xi_t}. \tag{21}$$

Using the state duration an investor can calculate the effect of state variable changes on her portfolio return. Alternatively, she can calculate how many futures or other bonds to sell to hedge her portfolio's exposure to a state variable. As you can see in Figure 10, the exposures or hedges vary significantly in size with the state of the economy. Hedging with unconditional durations is likely to be dangerous to your portfolio's health.

Bond investors know that duration measures the first order effect of changes in the state, but for longer maturity bonds the second order effect, called con-

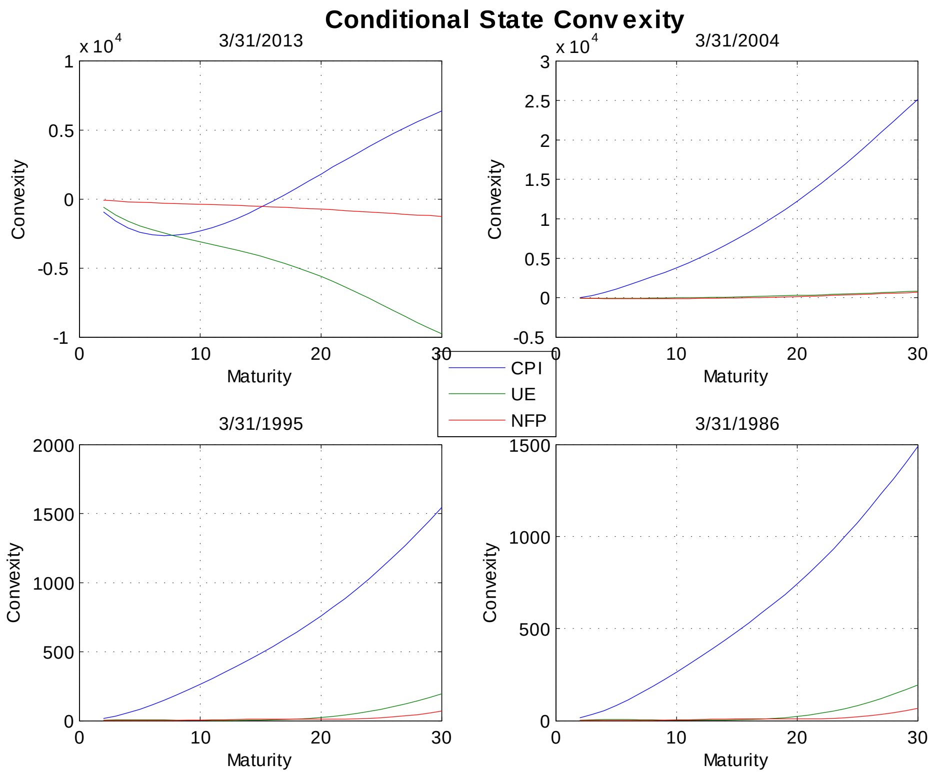

Figure 11: Conditional convexities of zero coupon bond price with respect to the states on four selected dates. The conditional convexity is the second derivative of the price change with respect to the state divided by the bond price.

vexity, cannot be ignored. Convexity is defined as

$$Con(\xi_t, \tau) = diag(\frac{1}{P(\xi_t, \tau)} \frac{\partial^2 P(\xi_t, \tau)}{\partial \xi_t' \partial \xi_t}). \tag{22}$$

Figure 11 shows the conditional convexity of zero coupon bonds on the four selected dates. On March 31, 2013, shown in Panel 1, the conditional convexities are atypical. Normally, a zero-coupon bond has significant positive convexity with respect to all three state variable as shown in Panels 2 - 4. The conditional convexities in the ZP are negative because yield changes in a rally are limited by the zero lower boundary. An investor needs to realize that holding intermediate Treasury bonds actually decreases the convexity of her portfolio during much of the ZP.

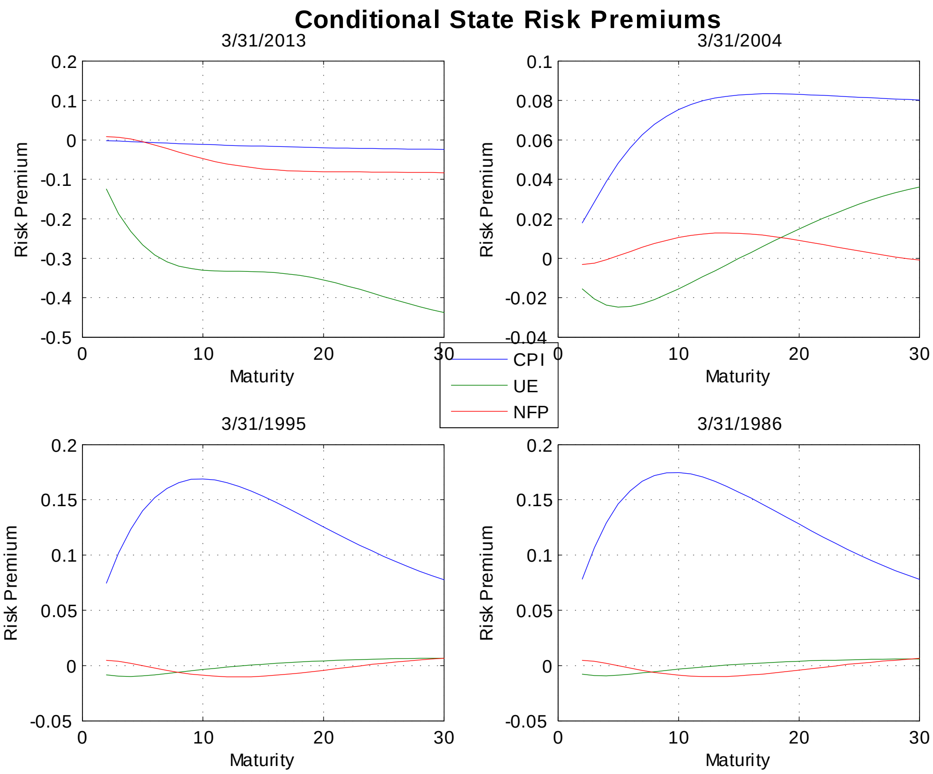

Figure 12: The state risk premiums on selected dates

4.11 State Risk Premiums

The state risk-premiums are given by equation (23):

$$\rho(\xi_t, \tau) = -[(x_\infty - \xi_t)\theta_p^{\mathbb{P}} - (\mu_p^{\mathbb{Q}} - \xi_t)\theta_p^{\mathbb{Q}}]Dur(\xi_t, \tau). \qquad (23)$$

Figure 12 shows the state risk premiums on zero coupon bonds on the four select dates. In the NP the main risk in bonds is inflation, which normally earns the largest risk premium. In contrast, in the ZP the main risks are real risks, especially UE.

4.12 Unconditional Yield Volatility

Investors and risk managers will miscalculate portfolio risk if they rely on unconditional volatility, or even rolling unconditional estimates of volatility. Figure 13 shows the unconditional volatility of monthly yield changes for the entire sample. There are some peculiar aspects to this graph. For example, why does zero coupon yield volatility rise after 26 years? The answer, I think, is a problem with the data. Recall that the zero coupon yield data are fitted to coupon bond market data. The actual longest Treasury bond is 30 years only in the month the bond is auctioned, but declines until the next auction. The auction schedule has changed throughout the history of the 30-year bond. Sometimes it was auctioned annually, sometimes semi-annually. From February 2002 through February 2006 the auction was suspended altogether. The result is that beyond 26 years, the fitting often requires extrapolating shorter yield curve data rather than interpolating actual yields.[26] What about the decline in volatility at the short end of the curve? I think that is due to the fact that the Federal Reserve controls the overnight rate as a policy instrument. This dampens the observed volatility at the front end of the yield curve. Whatever the reason for the peculiar shape of unconditional yield volatility, conditional volatilities vary significantly from date to date as can be seen in Figure 14 and are rarely similar to unconditional yield volatility.

5 TIPS

The available yield curve data have a relatively short and recent Zero Response Period, especially considering my reliance on asymptotic statistics. As a result I have used all the data in estimating the model and do not have a hold-out sample for out-of-sample testing. Luckily, the TIPS market provides ideal out-of-sample data because it was never used in the estimation and because TIPS prices can be calculated directly from the term structure model without estimating further coefficients.

Before examining the model output let me digress to review the mechanism for TIPS cash flow calculations. A TIPS bond, like a nominal Treasury coupon bond, is issued with a fixed coupon and a fixed maturity date. The TIPS also has a Reference Index (RI) which is the U.S. City Average All Items Consumer Price Index for All Urban Consumers (CPI-U), not seasonally adjusted, and using unrevised data. Principal is adjusted for inflation monthly. The principal adjustment is the current RI divided by the RI at issuance. On the first of each month the RI is CPI-U lagged three months. For example, on April 1 the RI is the level of the January CPI-U (reported in mid-February). Notice that on April 1, I can also calculate the RI for May 1, which is the February CPI-U (reported in mid-March). After the first day of the month the RI is the linear interpolation of the RI at the beginning of the month and the beginning of the next month. For example, on April 16, the RI is the average of the RI on April 1 and May 1. The TIPS bond owner receives a semi-annual coupon cash flow which equals the coupon rate times adjusted principal, including a coupon at maturity. Principal repayment at maturity is the greater of 100 or the adjusted principal at maturity, so investors implicitly own a put option at par.

[26]The data supplied on the Federal Reserve website may be a little funky because it is derived only from Treasury coupon bond prices. I also have similar zero coupon Treasury data supplied by an anonymous very large bank which is fitted directly to prices in the zero coupon bond market. The bank data show markedly less increase in unconditional yield volatilities for long maturities. I use the Federal Reserve data because it is publicly available to all researchers.

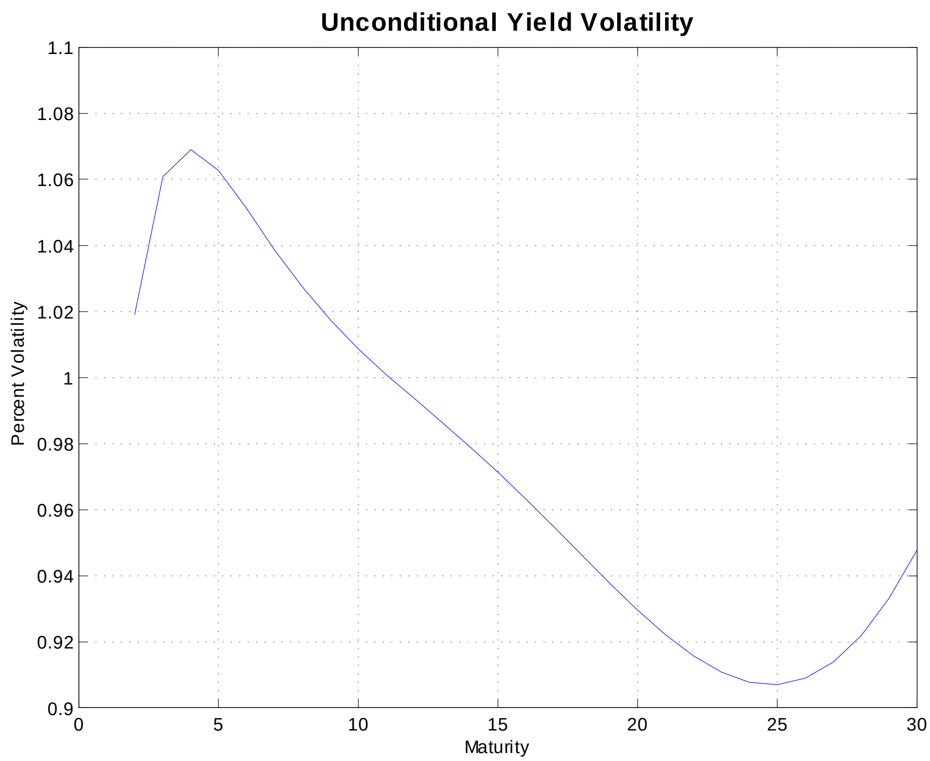

Figure 13: The unconditional annualized volatility of monthly changes in zero coupon bond yields from November, 1985 - March, 2013.

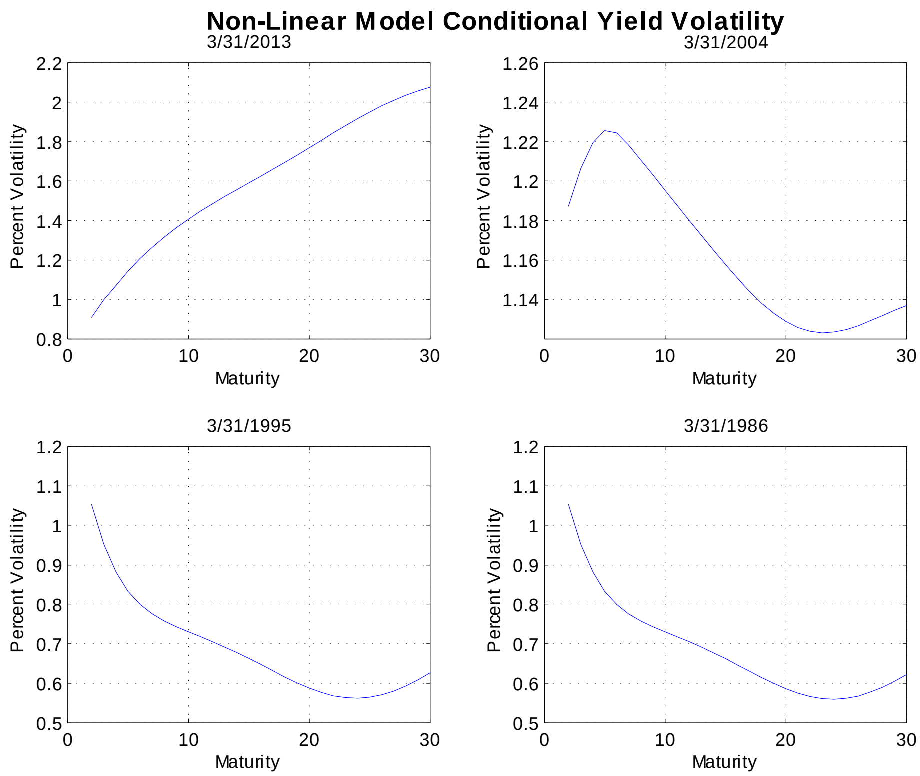

Figure 14: Conditional yield volatility of zero coupon bonds on four selected dates.

To use the model for TIPS valuation, I need only modify equation (1) for the policy rule to equation (24) which is the real policy rule:

$$R_t(\xi_t) = r(\xi_t) - \xi_{1t} = \max(\gamma_0 + \xi_t\gamma, 0) - \xi_{1t}. \tag{24}$$

I then compute real zero prices instead of nominal zero prices using the non-linear term structure model. I use the same risk-neutral coefficients and implicit states I found using nominal data. Nothing else needs to change to value TIPS. This model is a close approximation to TIPS valuation, but not exact for three reasons: The inflation adjustment is based on Core CPI, not CPI-U; the RI is adjusted with only a two month lag and without interpolation between months; and I ignore the put at maturity.

The data again come from the Federal Reserve Research web site.[27] The data are month end continuously compounded yields on constant maturity TIPS zero coupon bonds with annual maturities of five to twenty years from January, 1999 through December, 2003 and with maturities of two to twenty years from January, 2004 through March, 2013.

There are significant differences between the model's calculated TIPS values and the actual market prices. The fitting error across all yield curves in the sample is a dismal 65.3 basis points with an r-squared of only 76.8%. I can draw one of two conclusions:

1. The model does a poor job out-of-sample, or
2. There are significant mispricings and profit opportunities in the TIPS market relative to nominal Treasury bonds.

Let me show you why the model is accurate so that there are profitable trading strategies. I consider two self-financing portfolio strategies each consisting of a long ten year TIPS zero hedged with three nominal zeros chosen to eliminate all systematic state risk in the portfolio. The first strategy invests a constant \$1 each month. Specifically, at the beginning of each month invest \$1 in a 10 year maturity TIPS and hedge it by selling enough 5, 10 and 15 year nominal zeros to completely hedge out all exposure to CPI, UE and NFP, where the hedge ratios are calculated using the non-linear model. If the hedge costs less than \$1, invest the excess at one-month repo; if the hedge costs more than \$1, finance the portfolio at repo. At the end of each month liquidate and start over. This unweighted, zero-investment, strategy over the entire sample period makes 101 basis points per year with a standard deviation of 5.95% giving an information ratio of 0.17, which is below buying and holding the broad bond market index. The modest profits from this strategy indicate that the model is accurately calculating hedge ratios for TIPS.

Now consider a second strategy in which I use the indicated richness or cheapness of the 10 year TIPS to scale the investment. Specifically, if the

[27] http://www.federalreserve.gov/econresdata/researchdata.htmas.
The yield curves are again calculated through an extension of the Nelson-Siegel fitting technique. See Gürkaynak, Sack and Wright (2010).

TIPS is $c\%$ cheap I invest $c/0.025$ in the strategy; if the TIPS is $r\%$ rich, I short $r/0.025$ of the portfolio. I call 2.5% the scale factor; in my experience 2.5% is a significant mispricing, but I will circle back to see if it makes sense in terms of the standard deviation of the difference between TIPS price and value. The value-weighted strategy makes 10.32% per annum with a standard deviation of 6.92% and an information ratio of 0.61. The value-added from the model signal is the difference between the the signal-weighted returns and unweighted strategy returns. The value-added from the signal has a mean of 9.31%, a standard deviation of 16.54% and an information ratio of 0.56. The two strategies are illustrated in Figure 15. Higher scale factors reduce the profits, but leave the information ratio unchanged; lower scale factors increase the profits. What might be reasonable scale factor? The standard deviation of the TIPS percentage pricing error, model value divided by actual price, is 5.55%. So the assumed scale factor is fairly close to a one-half standard deviation signal. (But there was no way of knowing this *a priori.*) Evidently the model adds significant value to out-of-sample TIPS investments.

6 Conclusions

The model works. It shows that two essential nonlinearities, viz., interest rates are bounded below at zero and the economy behaves differently in the NP than it does in the ZP, must be included to value bonds accurately. It fits yield curves over the entire sample with a RMSE of 4.6 basis points. The implicit state variables mirror the government lagged state variables and are rational forecasts of the current states and one-month returns for benchmark bonds.[28] The model reduces the problem of forecasting the yield curve to that of forecasting the three states, CPI, UE and NFP . So investors do not have to guess how their macroeconomic views translate into yield curves. If an investor can accurately forecast the macroeconomy, an admittedly difficult task, she can make money. Hedging and risk management should be greatly improved using the conditional state durations, state convexities and yield volatilities. The model adds significant value in pricing TIPS out-of-sample.

What are model's shortcomings? First, it requires several days on a fast PC to estimate. Second, unlike affine models, the non-linear numerical solution for a bond price is not transparent. Third, the model is reduced-form assuming that the FED follows a consistent policy rule and that the economy responds consistently, but differently, in the two policy periods. A dynamic stochastic general equilibrium model would be an improvement.

[28] The model's predictions of one-month returns do not have a high R^2, but I have no reason to believe any other model can do better.

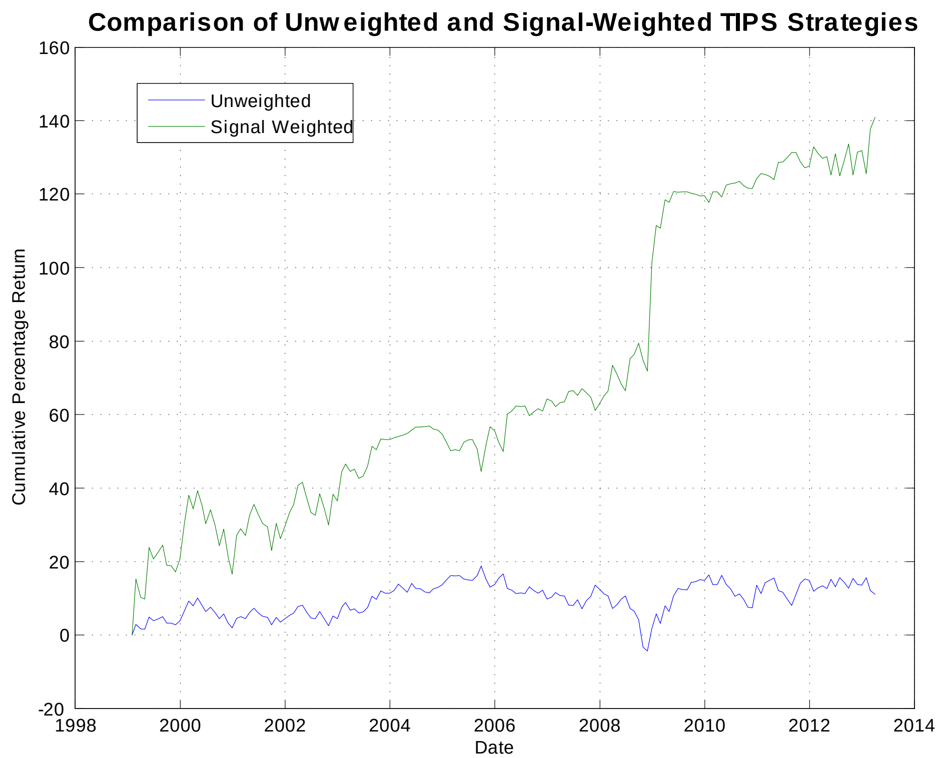

Figure 15: The cumulative returns from two TIPS investment strategies. Strategy one is rolling monthly a $1 investment in the 10 year TIPS zero hedged with the 5, 10 and 15 year Treasury zeros and financed at repo. The second strategy uses the model's indicated richness or cheapness of the TIPS as a signal to size the monthly investment.

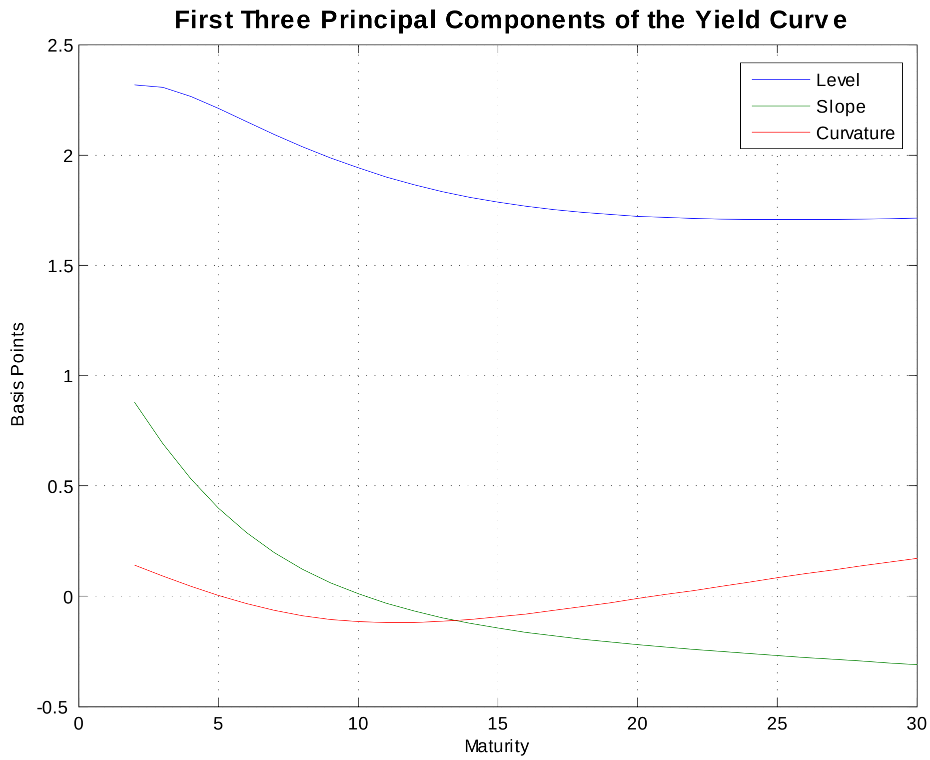

Figure 16: The first three principal components of the covariance matrix of month end yield curves from November, 1985 through March, 2013. Each line is the response to a one basis point change in the factor.

A Principal Components Analysis

In this Appendix I accomplish two tasks: First, I use principal component analysis (PCA) to motivate a three state variable model of the yield curve. The PCA analysis finds that the first three principal components explain 99.96% of the variation in yield curves with a root mean squared error (RMSE) of only 3.9 basis points over the entire sample period from November, 1985 through March, 2013. It turns out that the factor loadings are similar to three macroeconomic time series, CPI, UE and NFP.

Preliminary to the PCA analysis, I construct the 29×29 covariance matrix of yield levels. The first three principal components, commonly called the level, slope and curvature "factors", are shown in Figure 16. The level, slope, and curvature factors explain 97.06%, 2.65%, and 0.24%, respectively, of yield curve variation. No three state variable affine model can improve on this fit in-sample.

There are two important interpretations of the factors. First, the factors can be thought of as unconditional interest rate sensitivities (IRSs) to the three most common random shocks to the yield curve.[29] Hence an investor wanting to hedge her portfolio against interest rate risk would have to short three Treasury bonds (or futures) in sizes such that the hedged portfolio is insensitive to a change in the yield curve caused by any of the factor shocks.[30]

The factors can also be thought of as the building blocks necessary to closely approximate *any* yield curve by taking a weighted combination of the three curves. The weights needed to reproduce closely all the yield curves in the sample are called the factor loadings[31] and are shown in Figure 17. You can see for yourself that the normalized, (mean zero and variance one) state variables are very similar to the factors loadings as shown in Figure 17. This observation informed my choice of state variables.

B The Affine Term Structure Model

In this Appendix, I estimate the standard continuous time affine term structure model, but based on implicit state variables. The affine model does well enough, fitting the month end yield curves with a RMSE of only 4.7 basis points. But in this case well enough is not good enough for three reasons: Recently model yields are negative for shorter maturity bonds; the implicit states, especially UE, differ too much from the government states; and conditional yield volatility is constant, which is not even remotely consistent with the data.

B.1 State Variable VAR

As a preliminary to estimating the affine term structure model I first define the government state variable OU process for the entire data set by equation (25).[32]

$$dx_t = (x_\infty - x_t)\theta_A dt + dw_t\sqrt{S_A}. \tag{25}$$

Equation (25) implies that the government states mean revert to the same equilibrium, x_∞, as in the case of separate economic response periods. The VAR solution to equation (25) is

$$x_{t+1} = c_A + x_t M_A + \epsilon_t, \tag{26}$$

where $M_A = \exp(-\theta_A/12)$, $c_A = x_\infty (I - M_A)$, and $\{\epsilon_t \sim N(0, \Sigma_A)\}$ are independent and identically distributed. The monthly covariance matrix, Σ_A, can be expressed in terms of the stack operator *vec* and the Kronecker sum

$$vec(\Sigma_A) = (\theta'_A \oplus \theta'_A)^{-1}(I - \exp(-\theta'_A \oplus \theta'_A/12))vec(S_A). \tag{27}$$

[29] The IRS measures the change in the bond's yield for a one basis point change in the factor.

[30] I refer to the level, slope, and curvature as factors or as IRSs depending on the context.

[31] By construction the loadings have mean zero, variance one and are mutually orthogonal.

[32] I must use the entire data set since I have not found an affine solution to the non-linear problem with different economic response periods.

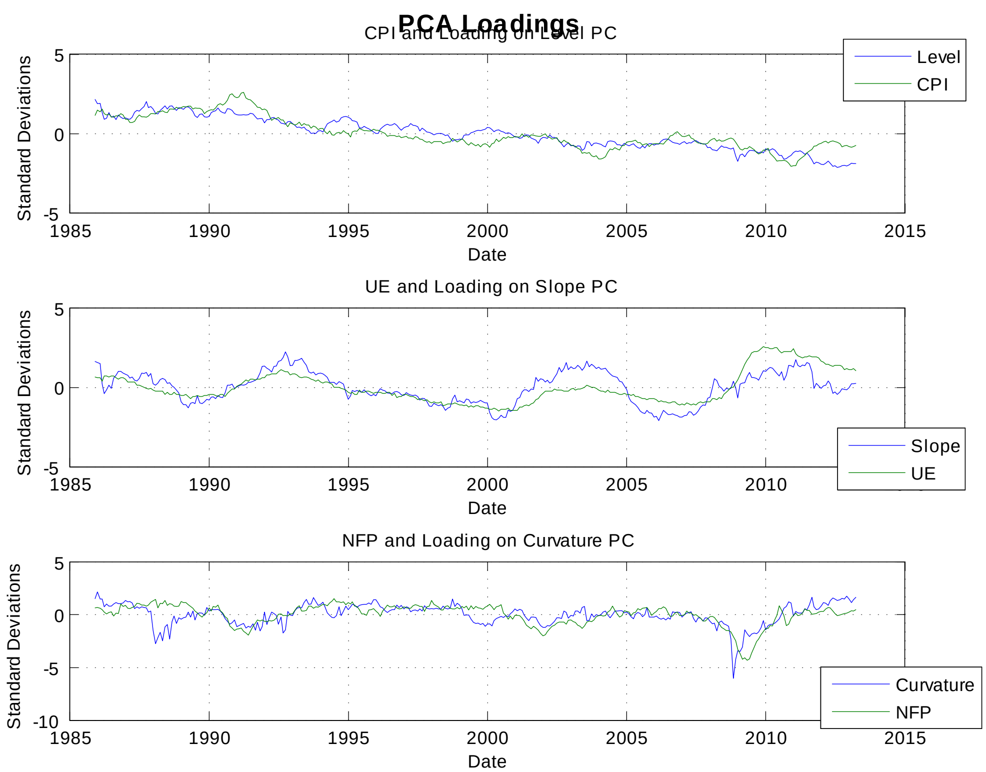

Figure 17: A comparison of the PCA factors and the normalized government state variables. The state variables are normalized to have mean zero and variance one.

Government State Variable VAR for the Entire Sample

Coefficients	CPI	UE	NFP
$\mathbf{M}_A$	0.99144	0.02003	-0.03259
	(203.13)	(4.02)	(-1.73)
	-0.00088	0.98254	0.04743
	(-0.21)	(228.15)	(2.91)
	0.01110	-0.04548	0.97767
	(2.81)	(-11.26)	(64.08)
Half-Life (years)	9.0	2.7	2.7
Volatility	0.129%	0.132%	0.497%
$\mathbf{R}^2$	98.51%	99.25%	93.16%
Durbin-Watson	1.90	2.40	1.76

Table 14: Ordinary least squares regression coefficients and statistics for the government state variables VAR over the entire sample.

Government State Varaible Instantaneous Covariance

	CPI	UE	NFP
Volatility	0.448%	0.454%	1.74%
Correlation	0.019	-0.004	-0.081
	(0.34)	(-0.07)	(-1.48)

Table 15: The instantaneous volatilities, correlations and T-statistics for the government state variables.

I now calibrate the state variable VAR using ordinary least squares estimation to find the coefficients c_A, M_A, and Σ_A in equation (26). Table 14 shows the estimated coefficients and statistics. The VAR appears to fit the data well as can be seen from the R^2 coefficients. The VAR indicates that the government states mean revert to x_∞ with a half-life of 9.0 years for CPI and a half-life of 2.7 years for UE and NFP. The instantaneous government state covariance matrix, S_A, is shown in Table 15.

B.2 Affine Bond Prices

I now turn to pricing bonds. The physical and risk-neutral OU processes for the implicit state variables are

$$d\xi_t = (x_\infty - \xi_t)\theta_A^{\mathbb{P}} dt + dw_t^{\mathbb{P}} \sqrt{S_A} \tag{28}$$

and

$$d\xi_t = (\mu_A^{\mathbb{Q}} - \xi_t)\theta_A^{\mathbb{Q}} dt + dw_t^{\mathbb{Q}} \sqrt{S_A}, \tag{29}$$

respectively.

The solution to equation (28) is

$$\xi_{t+s} = c_A^{\mathbb{P}}(s) + \xi_t M_A^{\mathbb{P}}(s) + \epsilon_{t,s}^{\mathbb{P}}, \tag{30}$$

where $M_A^{\mathbb{P}}(s) = \exp(-\theta_A^{\mathbb{P}} s)$, $c_A^{\mathbb{P}}(s) = x_\infty(I - M_A^{\mathbb{P}}(s))$ and $\epsilon_{t,\tau}^{\mathbb{P}} \sim N(0, V_s^{\mathbb{P}})$, where

$$vec(V_s^{\mathbb{P}}) = (\theta_A^{\mathbb{P}\prime} \oplus \theta_A^{\mathbb{P}\prime})^{-1}(I - \exp(-\theta_A^{\mathbb{P}\prime} \oplus \theta_A^{\mathbb{P}\prime} s)vec(S_A). \tag{31}$$

Similarly, the solution to equation (29) is

$$\xi_{t+s} = \mu_A^{\mathbb{Q}}(I - \exp(-\theta_A^{\mathbb{Q}} s)) + \xi_t \exp(-\theta_A^{\mathbb{Q}} s) + \epsilon_{t,s}^{\mathbb{Q}}, \tag{32}$$

where $\epsilon_{t,\tau}^{\mathbb{Q}} \sim N(0, V_s^{\mathbb{Q}})$, and

$$vec(V_s^{\mathbb{Q}}) = (\theta_A^{\mathbb{Q}\prime} \oplus \theta_A^{\mathbb{Q}\prime})^{-1}(I - \exp(-\theta_A^{\mathbb{Q}\prime} \oplus \theta_A^{\mathbb{Q}\prime} s)vec(S_A). \tag{33}$$

The policy rule is still given by equation (1), but without the minimum at r_m :

$$r(\xi_t) = r_\infty + (\xi_t - x_\infty)\gamma. \tag{34}$$

Substituting equation (32) into equation (34) for the instantaneous risk-free rate, I get that

$$r_{t+s}(\xi_t) = \gamma_0 + \mu_A^{\mathbb{Q}}\gamma + (\xi_t - \mu_A^{\mathbb{Q}})\exp(-\theta_A^{\mathbb{Q}} s))\gamma + \epsilon_{t,s}^{\mathbb{Q}}\gamma. \tag{35}$$

Define the cumulated policy rate

$$J(\xi_t, \tau) = \int_t^{t+\tau} r_{t+s}(\xi_t)ds \sim N(a(\xi_t, \tau), b(\tau)), \tag{36}$$

where $a(\xi_t, \tau)$ is the conditional mean and $b(\tau)$ is the conditional variance under the risk neutral distribution. Substituting equation (35) into equation (36) and integrating I find that

$$a(\xi_t, \tau) = (\gamma_0 + \mu_A^{\mathbb{Q}}\gamma)\tau + (\xi_t - \mu_A^{\mathbb{Q}})(I - \exp(-\theta_A^{\mathbb{Q}}\tau))\Gamma, \tag{37}$$

where $\Gamma = (\theta_A^{\mathbb{Q}})^{-1}\gamma$ and

$$b(\tau) = \Gamma' S_A \Gamma\tau - 2\Gamma\prime S_A(\theta_A^{\mathbb{Q}})^{-1}(I - \exp(-\theta_A^{\mathbb{Q}}\tau))\Gamma + \Gamma\prime V_\tau \Gamma. \tag{38}$$

Since $J(\xi_t, \tau)$ is normally distributed, the price of a zero coupon bond is

$$P(\xi_t, \tau) = \exp(-a(\xi_t, \tau) + \frac{1}{2}b(\tau)). \tag{39}$$

Hence the yield on the bond is

$$Y(\xi_t, \tau) = \alpha(\tau) + \xi_t\beta(\tau), \tag{40}$$

where

$$\beta(\tau) = \frac{(I - \exp(-\tau\theta_A^{\mathbb{Q}}))\Gamma}{\tau}, \tag{41}$$

and

$$\alpha(\tau) = \gamma_0 + \mu_A^{\mathbb{Q}}\gamma - \mu_A^{\mathbb{Q}}\beta(\tau) - \frac{1}{2}\Gamma' S_A \Gamma + \Gamma\prime S_A(\theta_A^{\mathbb{Q}})^{-1}\beta(\tau) - \frac{1}{2}\Gamma' V_\tau \Gamma/\tau. \tag{42}$$

B.2.1 Maximizing the Conditional Joint Likelihood

The annualized one-month conditional expected return on a n-year bond in state ξ_t, $\rho_A(\xi_t, n)$, is

$$\rho_A(\xi_t, n) = \widehat{\alpha}(n) + \xi_t \widehat{\beta}(n), \tag{43}$$

where

$$\widehat{\alpha}(n) = 12[n\alpha(n) - (n - \tfrac{1}{12})(\alpha(n - \tfrac{1}{12}) + c_A^{\mathbb{P}} \beta(n - \tfrac{1}{12}))] \tag{44}$$

and

$$\widehat{\beta}(n) = 12(n\beta(n) - (n - \tfrac{1}{12}) M_A^{\mathbb{P}} \beta(n - \tfrac{1}{12})) \tag{45}$$

Define the expected conditional return on the benchmark bonds by $\rho_A(\xi_t) = [\rho_A(\xi_t, 2), \rho_A(\xi_t, 10), \rho_A(\xi_t, 30)]$.

The joint conditional log-likelihood is

$$\begin{aligned} \mathcal{L} \quad = \quad & \sum_{t=1}^{T} \left[-\tfrac{3}{2} \ln(2\pi) - \tfrac{1}{2} \ln(|\Sigma_A|) - \tfrac{1}{2} (\xi_t - m_{At}) \Sigma_A^{-1} (\xi_t - m_{At})\prime \right] \\ & -\tfrac{1}{2} \left[29 \ln(2\pi) + 29 \ln(\sigma_r^2) + (y_t - \alpha - \xi_t \beta)(y_t - \alpha - \xi_t \beta)\prime / \sigma_{Ar}^2 \right] \\ & -\tfrac{1}{2} \left[3 \ln(2\pi) + \ln(\Sigma_{\rho A}) + (R_{t+1} - \widehat{\alpha} - \xi_t \widehat{\beta}) \Sigma_{A\rho}^{-1} (R_{t+1} - \widehat{\alpha} - \xi_t \widehat{\beta})\prime \right] \end{aligned} \tag{46}$$

where $m_{At} = c_A + x_{t-1} M_A$ is the expected month-end government state, σ_{Ar} is the volatility of the yield curve fitting residuals, $\alpha = [\alpha(2), ..., \alpha(30)]$, $\beta = [\beta(2), ..., \beta(30)]$, and $\Sigma_{A\rho}$ is the covariance matrix of the return fitting residuals. Taking the derivative of $\mathcal{L}$ with respect to ξ_t I find that

$$\xi_t = (m_{At} \Sigma_A^{-1} + (y_t - \alpha)\beta\prime / \sigma_{Ar}^2 + (R_{t+1} - \widehat{\alpha}) \Sigma_{A\rho}^{-1} \widehat{\beta}_A^{\prime}) \Omega^{-1}, \tag{47}$$

where

$$\widehat{\beta}_A = [12(n\beta(n) - (n - \tfrac{1}{12}) M_A^{\mathbb{P}} \beta(n - \tfrac{1}{12})); n = 2, 10, 30], \tag{48}$$

and

$$\Omega = \Sigma_A^{-1} + \beta\beta' / \sigma_{Ar}^2 + \widehat{\beta}_A \Sigma_{A\rho}^{-1} \widehat{\beta}_A^{\prime}. \tag{49}$$

A Nelder-Mead search finds the 21 coefficients , $\Psi_A = (M_A^{\mathbb{P}}, \mu_A^{\mathbb{Q}}, \theta_A^{\mathbb{Q}})$, which maximize the conditional joint log-likelihood, equation (46). The estimated coefficients for the physical implicit state process is shown in Table 16. The implicit states mean revert in the physical VAR. The risk-neutral coefficients are shown in Table 17. The smallest eigenvalue is negative which means that the risk-neutral state dynamics are non -stationary. This in turn implies that in the absence of a lower bound on the policy rate, yields will eventually become negative for long enough maturities.

The implicit state variables are compared to the forecasts in Figure 18. You can see that Implicit CPI has trended down with government CPI, but there have been some notable deviations. Recently, government Core CPI has been running about 2%, but implicit Core CPI has been about 1.5%. I think that this is the market's way of incorporating Bernanke's forecast that the FED will

Affine Model Physical VAR			
Coefficients	**CPI**	**UE**	**NFP**
$\mathbf{M}_A^{\mathbb{P}}$	0.9934	0.0320	0.0415
	179.42	2.21	1.95
	-0.0018	0.9760	-0.0109
	-0.35	82.99	-0.77
	-0.0046	-0.0585	0.8941
	-0.81	-2.91	36.05
Half-Life (Years)	5.93	3.57	0.49

Table 16: Coefficients and t-statistics for the physical VAR estimated using the affine term structure model.

AffineModel Risk Neutral Coefficients			
Coefficients	**CPI**	**UE**	**NFP**
$\boldsymbol{\mu}_A^{\mathbb{Q}}$	0.0366	0.0260	-0.0011
	(193.4)	(192.8)	(-3.3)
$\boldsymbol{\theta}_A^{\mathbb{Q}}$	-0.0316	0.1958	-0.1977
	(-59.8)	(61.3)	(-110.0)
	-0.0553	0.2665	-0.0536
	(-100.3)	(129.6)	(-33.8)
	0.0296	-0.2960	0.0632
	(68.1)	(-149.2)	(28.8)
Real Eigenvalues	0.25	0.10	-0.05

Table 17: Optimal coefficients and t-statistics for the risk neutral OU process estimated using the affine term structure model.

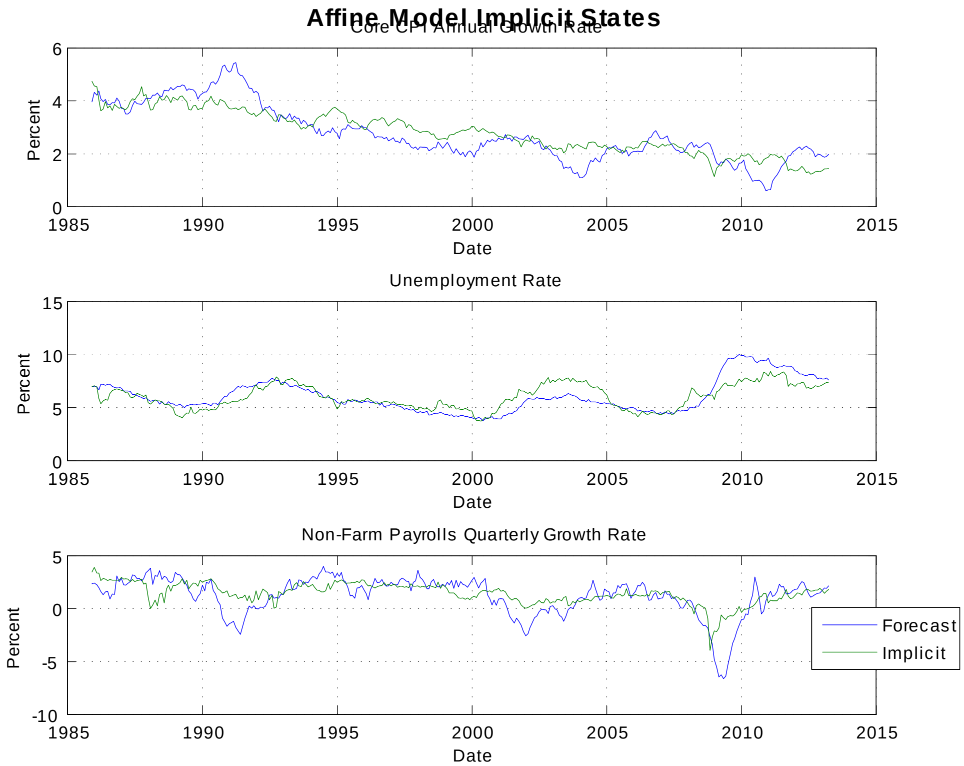

Figure 18: Affine model implicit states and one-month ahead VAR forecasts of the government states.

not tighten before 2015. You can also see that the market's implicit UE rate has exaggerated the cycles in government UE rate so the fit is not very tight. Finally, the market's view of NFP has also followed the data, but at the present, the market is pricing bonds as if future growth of NFP will be higher than the recent data.

Figure 19 shows the conditional yield curve changes in response to a one basis point change in each of the state variables. Unlike the conditional IRS in the non-linear model, these responses are always the same in every state at every time.

The goodness of fit statistics shown in Table 18 indicate that the affine model fits yield curves extremely well throughout the sample with only a 4.7 basis point RMSE. In the text I showed that the non-linear model's RMSE is only a little less, 4.6 basis points, over the entire sample. While the difference in fitting error is minor, the difference in the log-likelihood is major. This is important because the log-likelihood measure the model's ability to fit simultaneously the yield curves, the government states, and the benchmark returns.

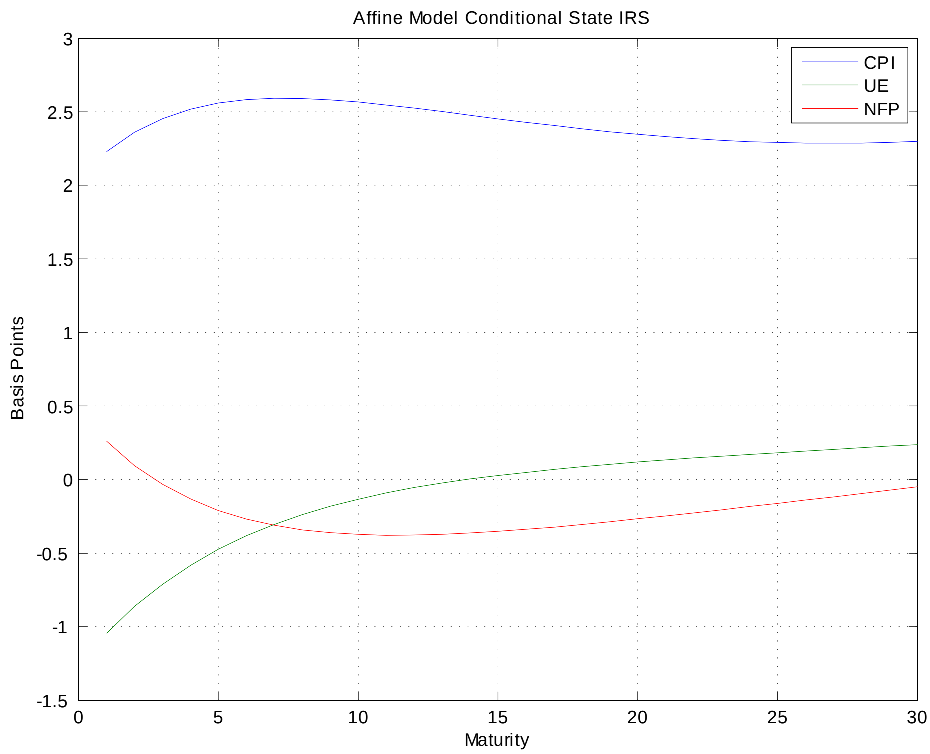

Figure 19: Affine model conditional state IRS, the change in bond yields caused by a one basis point change in the state. The conditional IRS never changes from state to state.

Affine Model Goodness of Fit

	Entire Sample	NP	ZP
RMSE	0.047%	0.044%	0.059%
R^2	99.94%	99.93%	99.77%
Log Likelihood	51381.12	46486.82	4894.31

Table 18: Goodness of fit measures for the affine model in both response periods.

C Numerical Solution Technique

In this Appendix I show how to calculate prices and implicit state variables which simultaneously maximize the joint conditional likelihood of fitting the government states, the actual yield curves, and the benchmark returns. There are only two steps. The first step is to specify the numerical scheme for approximating the solution to equation(13). Then for any vector of coefficients, Ψ, I can calculate prices. Then the final step is to search for the optimal Ψ and $\{\xi_t\}$ which maximizes the conditional likelihood function, equation (19).

C.1 The Implicit Numerical Scheme

I first specify the numerical scheme I use to solve equation(13). Equation(13) is a type of parabolic partial differential equation known as a convection-diffusion equation. It turns out that numerical solutions of convection-diffusion equations are tricky because *a priori* you do not know whether the convection or the diffusion term will dominate, and doubly tricky in this model because the coefficients in equation(13) are state dependent. It turns out for this problem that the convection term dominates in most states, but the solution scheme I use, due to Douglas (1962), works in either case. Douglas's scheme is an alternating direction implicit (ADI) solution scheme that is second order accurate in both the spacial and time steps and unconditionally stable.[33] Finally, I ignore the instantaneous state correlations in solving equation (13) because the estimated correlation coefficients are not statistically significant[34]; this greatly reduces the computational complexity of solving equation (13).

The implicit scheme begins with a lattice which approximates the state variables in space and time. Each state variable is approximated with a grid of values spaced at intervals $\delta_i, i = 1, 2, 3$. CPI start at $\xi_{1L} = -4\%$ and then increments by $\delta_1 = 0.5\%$ until it reaches $\xi_{1U} = 10\%$ for a total of $I = 29$ grid points. I choose 50 basis points as the step size because it is about equal to the annual standard deviation of CPI. I also use 50 basis point steps to approximate UE for the same reason; UE begins at a minimum of $\xi_{2L} = -4\%$ and is incremented by $\delta_2 = 0.5\%$ until it reaches a maximum of $\xi_{2U} = 15\%$ for a total of $J = 39$ grid points.[35] NFP has substantially more volatility than either CPI or UE, so I use 100 basis point steps. The grid for NFP starts at $\xi_{3L} = -12\%$ and is incremented by $\delta_3 = 1.0\%$ until it reaches a maximum of $\xi_{3U} = 12\%$ for a total of $K = 25$ grid points. Spatially the grid is $29 \times 39 \times 25 = 28,275$ lattice points, which is large. For convenience I will call the spatial domain a "cube" even though the sides are of different lengths. The time increment, denoted h, is monthly so that $h = 1/12$; hence there are a total of 361 time stages in 30 years,

[33] Unconditional stability is vital for a search because I do not know *a priori* whether the final parameters found by the search will satisfy any conditional stability criterion.

[34] Only the instantaneous covariance matrix is diagonal. Over any finite interval the multivariate OU process solution has correlation among the state variables.

[35] Reported UE cannot be negative, but we are using UE as a surrogate for the output gap, which is unbounded above and below.

$n = 0, 1, ..., 360$. Hence the total grid has a total of $28,275 \times 361 = 10,207,275$ nodes.

The implicit scheme starts at maturity where $P^0_{ijk} = 1$. The scheme then steps backward in time to calculate the value of a zero coupon bond, P^n_{ijk}, in node i, j, k with n months until maturity for every trio $i = 1, ..., I,\ j = 1, ..., J,\ k = 1, ..., K$ and every $n = 1, 2, ..., 360$. The next step in the numerical solution is to calculate the risk-free rate, r_{ijk}, at each node of the grid using equation (1). If $r_{ijk} = r_m$, then the node is in the ZP; if $r_{ijk} > r_m$, then the node is in the NP.

At each node of the grid there are three means and variances, one for each state direction, which differ by policy period. So the next step is to assign means, m_{ijkl}, and variances, v_{ijkl}, in direction $l = 1, 2, 3$ at every node, ijk, in the grid.

$$m_{ijkl} = \{ \begin{array}{c} (\mu^{\mathbb{Q}}_{nl} - \xi_{ijk})\theta^{\mathbb{Q}}_{nl}\ if\ r_{ijk} > 0, \\ (\mu^{\mathbb{Q}}_{zl} - \xi_{ijk})\theta^{\mathbb{Q}}_{zl}\ if\ r_{ijk} = r_m. \end{array} \tag{50}$$

$$v_{ijkl} = \{ \begin{array}{c} \frac{1}{2}\sigma^2_{nl}\ if\ r_{ijk} > 0, \\ \frac{1}{2}\sigma^2_{zl}\ if\ r_{ijk} = r_m. \end{array} \tag{51}$$

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C.1.1 Derivative Approximations at Interior Points

At interior points in the grid I use centered approximations for the first derivatives in equation (13). For example the first derivative for CPI in direction 1 is approximated as:

$$\Delta_1 P^n_{ijk} = \frac{P^n_{i+1jk} - P^n_{i-1jk}}{2\delta_1}. \tag{52}$$

The first derivative approximations for UE and NFP are defined similarly. The second derivative term in equation(13) for CPI is also approximated using centered differences:

$$\Delta^2_1 P^n_{ijk} = \frac{P^n_{i+1jk} + P^n_{i-1jk} - 2P^n_{ijk}}{\delta^2_1}. \tag{53}$$

The second derivative approximations for UE and NFP are defined similarly. These centered approximations are second-order accurate.

C.1.2 Derivative Approximations at Boundary Points

The boundary conditions are an essential part of the model. The boundary conditions are a variation of a Neumann problem. At the cube boundary I assume that the second derivative of the price is zero.

$$\Delta^2_1 P^n_{1jk} = \Delta^2_1 P^n_{Ijk} = \Delta^2_2 P^n_{i1k} = \Delta^2_2 P^n_{iJk} = \Delta^2_3 P^n_{ij1} = \Delta^2_3 P^n_{ijK} = 0. \tag{54}$$

Also, at the boundary I cannot use a centered approximation for the first derivative. Instead I use an upwind approximation, provided the upwind direction points into the grid; if the upwind direction points out of the grid, I set the first derivative to zero. For example, the first derivative in direction 1 at the lower boundary for CPI is

$$\Delta_1 P^n_{1jk} = \{ \begin{array}{c} (P^n_{2jk} - P^n_{1jk})/\delta_1 \; if \; m_{1jk1} \geq 0 \\ 0 \; otherwise \end{array} . \tag{55}$$

The first derivatives at the other boundary points are defined similarly. The boundary point approximations are only first order accurate. I try to minimize the effect of boundary conditions by setting the upper boundaries far above and the lower boundaries far below the actual data range.

C.1.3 Time Stepping

The time stepping is an extension of a stable, second order accurate scheme proposed by Douglas (1962).[36] This scheme is a version of ADI in which prices are updated in each of the three spacial dimensions in turn. In the first of three steps bond prices are first updated from stage n to a fictitious fractional stage $n+1/3$ by updating only CPI; then prices are updated from stage $n+1/3$ to the fictitious fractional stage $n+2/3$ by updating only UE; and finally prices are updated from stage $n+2/3$ to $n+1$ by updating only NFP. For each grid node, ijk, and each time step, n, and each direction, l, define the difference operator

$$\Lambda_{ijkl} P^n_{ijk} = v_{ijkl} \Delta_1^2 P^n_{ijk} + m_{ijkl} \Delta_1 P^n_{ijk} - \tfrac{1}{3} r_{ijk} P^n_{ijk} \tag{56}$$

Suppressing the ijk subscripts for clarity, the first time step is implicit in CPI:

$$\frac{P^{n+1/3} - P^n}{h} = \frac{1}{2}\Lambda_1(P^{n+1/3} + P^n) + \Lambda_2 P^n + \Lambda_3 P^n. \tag{57}$$

(The scheme is called implicit because the price at time $n+1/3$ appears on both sides of the equation.) The second time step is implicit in UE:

$$\frac{P^{n+2/3} - P^{n+1/3}}{h} = \frac{1}{2}\Lambda_2(P^{n+2/3} - P^n). \tag{58}$$

The final time step is implicit in NFP:

$$\frac{P^{n+1} - P^{n+2/3}}{h} = \frac{1}{2}\Lambda_3(P^{n+1} - P^n). \tag{59}$$

By mimicking the derivation in Yanenko (1971) pages 28-30 I can show that the scheme given by equations (57) - (59) is unconditionally stable, second order accurate in time, and second order accurate in space, except at the boundary points. Although the grid is necessarily large, the approximate prices can be computed in less than one minute because solving equations (57) - (59) requires only the inversion of tridiagonal matrixes.

[36] The classical ADI scheme used by Kim and Singleton (2012) is not unconditionally stable in three spatial dimensions.

C.2 Maximum Likelihood Search

I now find the implicit state variables and risk-neutral coefficients which maximize the joint conditional log-likelihood function. The data has $T = 329$ month end yield curves, of which the first 277 observations are the NP (where $p = n$) and the last 52 are the ZP (where $p = z$). The joint conditional log-likelihood is given by equation (19). To calculate $Y(\xi_t, i)$ for arbitrary ξ_t I use trilinear interpolation among the eight proximal lattice points. If I have T months of yield curves, then there are $3T$ implicit state variables.

I use a search to maximize the log-likelihood. I use a Powell search without derivatives, which is a sequential conjugate direction search in each of the 28 coefficients in Ψ, to find improvement. [Press, Flannery, Teukolsky and Vetterling (1992)]. When I find improvement I use a Nelder-Mead search to update the implicit state variables. The search continues until no further improvement is possible within numerical tolerances. The standard errors of the coefficients are estimated using the outer product of gradients estimator, which is asymptotically consistent. [Greene (2012), page 522.]

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