University of Pennsylvania
ScholarlyCommons

Department of Physics Papers

Department of Physics

10-1-1989

Cohn's Theorem for Elastic Networks

Jian Wang
University of Pennsylvania

A. Brooks Harris
University of Pennsylvania, harris@sas.upenn.edu

Follow this and additional works at: https://repository.upenn.edu/physics_papers

Part of the [Physics Commons](#)

Recommended Citation

Wang, J., & Harris, A. (1989). Cohn's Theorem for Elastic Networks. *Physical Review B*, 40 (10), 7272-7278. <http://dx.doi.org/10.1103/PhysRevB.40.7272>

This paper is posted at ScholarlyCommons. https://repository.upenn.edu/physics_papers/301
For more information, please contact repository@pobox.upenn.edu.

Cohn's Theorem for Elastic Networks

Abstract

Cohn's theorem for elastic networks is presented. The elastic noise is defined. Quantities suitable for series-expansion calculation and computer simulation are considered.

Disciplines

Physics

Cohn's theorem for elastic networks

Jian Wang* and A. Brooks Harris

Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

(Received 14 March 1988)

Cohn's theorem for elastic networks is presented. The elastic noise is defined. Quantities suitable for series-expansion calculation and computer simulation are considered.

I. INTRODUCTION

Recently there have been many studies of noise in the randomly diluted resistor networks.¹⁻⁵ Most of this work relies at least implicitly on Cohn's theorem.⁶ It is natural to look for the analog in the elastic case. Here we formulate and prove Cohn's theorem for elastic networks and discuss quantities which can be calculated numerically (by series expansion or computer simulation).

In the case of the random-resistor network (RRN) a fruitful way to investigate the macroscopic conductivity is via the two-point resistive susceptibility.⁷ Likewise the macroscopic "noise" due to resistance fluctuations introduced by Rammal and co-workers¹ can be formulated in terms of two-point resistive noise susceptibilities.³ These quantities clearly have their analogs in the randomly diluted elastic network (REN). Although the vector nature of the displacements in the REN are more complicated than the scalar voltage displacements in the RRN, we note that *force* in the REN is the analog of *current* in RRN and *displacement* in the REN is the analog of *voltage* in the RRN. The main difficulty introduced by the vector nature of the displacements in the REN is that the force is no longer the gradient of a scalar potential. As a result, whereas for the RRN one can immediately conclude that the current through any internal resistor is bounded by the current put in at one terminal and removed at another one, for the REN, as we shall see, it is *not* true that the stress in an internal spring is bounded by the magnitude of the external force applied at one point and opposed at another point.

Nevertheless, many relations for the RRN can be, and here are, generalized to the REN. Briefly this paper is organized as follows. In Sec. II we describe the models we consider for the REN. These models enable us to discuss various anharmonic systems. Here we formulate and prove the analogs of Cohn's theorem (in its weaker form) to these models. In Sec. III we use these results to discuss various susceptibilities, including those for "elastic noise" due to fluctuations in local spring constants.

Finally, in Sec. IV we discuss some extensions and summarize our conclusions.

II. MODELS AND COHN'S THEOREM

We consider an elastic Hamiltonian of the form

$$H = \frac{1}{s+1} \sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} + \frac{1}{s+1} \sum_{\langle b, b' \rangle} k_{b, b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} + H_{\text{ext}} \quad (1a)$$

$$= H_0 + H_{\text{ext}}, \quad (1b)$$

where H_{ext} includes the effect of external forces, b labels bonds, and $\langle b, b' \rangle$ indicates that the sum is over pairs of nearest-neighbor bonds. Also $\mathbf{u}_b = \mathbf{u}_s - \mathbf{u}_{s'}$, $\mathbf{u}_{b'} = \mathbf{u}_s - \mathbf{u}_{s''}$, where the s 's label sites which are at the ends of the bonds, k_b is the central-force elastic constant for bond b , $k_{b, b'}$ is bond-bending elastic constant, and $\hat{\mathbf{R}}_b$ is a unit vector along the nearest-neighbor direction. When the parameter s is not unity, Eq. (1) describes a nonlinear elastic network. Note that in the RRN the analog of H_0 is the power dissipated in the resistors and the analog of H_{ext} is $-\sum_s I_s^{\text{ext}} V_s$, where I_s^{ext} is the externally imposed current at site s . Since for the RRN one usually writes $V \sim rI^\alpha$, s is analogous to $1/\alpha$.

Generally two types of boundary conditions are specified when calculating the elastic response of the system. Either (1) the displacements of certain sites are fixed, in which case the term H_{ext} in Eq. (1a) is omitted, or (2) the external forces acting on certain sites are fixed. For instance, to calculate the elastic bulk modulus, one usually fixes the boundary sites and allows the other sites to relax. This is the way the simulations^{8,9} are done. When using the series expansion method,^{10,11} one usually fixes the external forces on a pair of sites or bonds. The versions of Cohn's theorem we will prove are the following. For case (1), fixed displacement boundary conditions,

$$M_1 \equiv \frac{1}{2} \sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s-1} \frac{d}{dk} (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b)^2 + \frac{1}{2} \sum_{\langle b, b' \rangle} k_{b, b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s-1} \frac{d}{dk} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^2 = 0. \quad (2)$$

For case (2), fixed external force boundary conditions,

$$M_2 = \frac{1}{s} \sum_b \mathbf{u}_b \cdot \hat{\mathbf{R}}_b \frac{d}{dk} [k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s-1} (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b)] \\ + \frac{1}{s} \sum_{\langle b, b' \rangle} (\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}) \cdot \frac{d}{dk} [k_{b,b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s-1} (\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'})] = 0. \quad (3)$$

In Eqs. (2) and (3) k denotes any of the spring constants k_b or $k_{b,b'}$, and the derivative with respect to k is a total derivative.

We consider first the case of fixed displacement boundary conditions and prove Eq. (2). Differentiation by the chain rule yields

$$M_1 = \frac{1}{2} \sum_{s,m} \frac{du_{sm}}{dk} \left[\sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s-1} \frac{\partial (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b)^2}{\partial u_{sm}} \right. \\ \left. + \sum_{\langle b, b' \rangle} k_{b,b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s-1} \frac{\partial}{\partial u_{sm}} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^2 \right], \quad (4)$$

where u_{sm} is the m th component of $\mathbf{u}_s$. Since the total force acting on site s is obtained by

$$-\mathbf{F}_s \equiv \frac{\partial H}{\partial \mathbf{u}_s} = \frac{1}{2} \sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s-1} \frac{\partial (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b)^2}{\partial \mathbf{u}_s} \\ + \frac{1}{2} \sum_{\langle b, b' \rangle} k_{b,b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s-1} \frac{\partial}{\partial \mathbf{u}_s} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^2, \quad (5)$$

we have

$$M_1 = \sum_s \frac{\partial \mathbf{u}_s}{\partial k} \cdot \mathbf{F}_s. \quad (6)$$

For sites where $\mathbf{u}_s$ is fixed by the boundary conditions, $d\mathbf{u}_s/dk = 0$. For all other sites $\mathbf{F}_s = 0$ in equilibrium. Thus we conclude that $M_1 = 0$ for arbitrary fixed-site boundary conditions. If $\mathbf{F}_{\text{ext}}$ were nonzero, then $\mathbf{F}_s$ would be replaced by $\mathbf{F}_s - \mathbf{F}_{\text{ext}}$ and M_1 would not vanish in the presence of external forces.

Now consider case (2), *fixed external force*. We write

$$\mathbf{u}_b \cdot \hat{\mathbf{R}}_b = \sum_{s,m} u_{sm} \frac{\partial \mathbf{u}_b \cdot \hat{\mathbf{R}}_b}{\partial u_{sm}}, \quad (7a)$$

$$\mathbf{u}_b \times \hat{\mathbf{R}}_b = \sum_{s,m} u_{sm} \frac{\partial \mathbf{u}_b \times \hat{\mathbf{R}}_b}{\partial u_{sm}}. \quad (7b)$$

Notice that $\partial \mathbf{u}_b \cdot \hat{\mathbf{R}}_b / \partial u_{sm}$ and $\partial \mathbf{u}_b \times \hat{\mathbf{R}}_b / \partial u_{sm}$ do not depend on u_{sm} . Using Eqs. (3) and (7) we have

$$M_2 = \frac{1}{s} \sum_{s,m} u_{sm} \frac{d}{dk} \left[\sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s-1} (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b) \frac{\partial \mathbf{u}_b \cdot \hat{\mathbf{R}}_b}{\partial u_{sm}} \right. \\ \left. + \sum_{\langle b, b' \rangle} k_{b,b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s-1} (\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}) \cdot \frac{\partial (\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'})}{\partial u_{sm}} \right]. \quad (8a)$$

Thus

$$M_2 = \frac{1}{s} \sum_s \mathbf{u}_s \cdot \frac{d(\mathbf{F}_{\text{ext}} - \mathbf{F}_s)}{dk} = \frac{1}{s} \sum_s \mathbf{u}_s \cdot \frac{\partial \mathbf{F}_{\text{ext}}}{\partial k}. \quad (8b)$$

Hence, for the fixed external forces, $M_2 = 0$. M_2 is also zero if some sites are fixed to have $\mathbf{u}_s = 0$.

Before discussing applications of these theorems, we will give definitions of the elastic susceptibilities obtained from the response under the two types of boundary conditions considered above.

First we consider the response of the network when a

generalized displacement is imposed on certain sites, so that the term H_{ext} is removed from Eq. (1). That is we fix

$$u_{i,m} = x_{i,m} X, \quad (9)$$

where X sets the scale of the generalized displacement in which certain sites $i_1, i_2, \dots$, are fixed. The normalization of the $x_{i,m}$'s can be defined as desired. We will refer to the generalized displacement described by Eq. (9) as $\mathbf{X}$. The elastic susceptibility associated with $\mathbf{X}$ and denoted χ_X is the inverse of the effective spring constant for this displacement:

$$H_{0,\text{eq}} = \frac{1}{s+1} \chi_X^{-s} X^{s+1}, \quad (10)$$

where eq indicates that the quantity is evaluated when all the displacements have relaxed to their equilibrium values, subject to the boundary conditions. This definition is formulated to make χ as closely as analogous to resistance R as possible.

Likewise we define a susceptibility, χ_F , with respect to a generalized force F , whose components are

$$F_{i,m} = f_{i,m} F, \quad (11)$$

where again F sets the scale of F . In this case in Eq. (1)

$$H_{\text{ext}} = -F \sum_{i,m} f_{i,m} u_{i,m}. \quad (12)$$

Since the equations of equilibrium yield

$$F f_{i,m} = \frac{\partial H_0}{\partial u_{i,m}}, \quad (13)$$

we have

$$\chi_{\text{el}}(s, i, j) = (s+1) H_{0,\text{eq}} / F^{(s+1)/s} = \left[\sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} + \sum_{(b,b')} k_{b,b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} \right] / F^{(s+1)/s}. \quad (16)$$

Obviously we ought to define these two susceptibilities so that they are equivalent. To obtain the conditions that $\chi_X = \chi_F \equiv \chi$ we compare Eq. (10) and (15), and find

$$\chi = X / F^{1/s}. \quad (17)$$

Now substitute Eq. (9) into Eq. (14), to obtain $-(s+1)H_0 = -FX \sum_{i,m} f_{i,m} x_{i,m}$. We compare this with Eq. (10), which by virtue of Eq. (17) is

$$H_{0,\text{eq}} = \frac{1}{s+1} FX \quad (18)$$

and deduce that

$$\sum_{i,m} f_{i,m} x_{i,m} = 1 \quad (19)$$

is the condition that the susceptibilities defined at fixed force and fixed displacement be identical.

We illustrate this condition for the RRN, where one can define the resistance between two nodes with either fixed-current or fixed-voltage boundary conditions. In the first case, $I_i = -I_j = I$ and these are the analogs of $F f_{i,m}$, with $f_{i,m} = \pm 1$ and $F = I$ in Eq. (15). Note that the normalization is arbitrary; the generalized current vector is not normalized by its norm in the usual vector sense. For fixed-voltage boundary conditions one would set $V_i = -V_j = \frac{1}{2}V$ and these are the analogs of $X x_{i,m}$, with $x_{i,m} = \pm \frac{1}{2}$ and $X = V$ in Eq. (10). Note that again, although in each case the normalization is arbitrary, if one wants the two definitions of the susceptibility

$$H_{\text{ext}} = - \sum_{i,m} F f_{i,m} u_{i,m} = - \sum_{i,m} \frac{\partial H_0}{\partial u_{i,m}} u_{i,m} = -(s+1) H_0, \quad (14)$$

where we used the homogeneity of H_0 in the u 's to write the last equality in Eq. (14). For fixed force boundary conditions we define the elastic susceptibility via

$$H_{\text{eq}} = - \frac{s}{s+1} \chi_F F^{(s+1)/s} = -s H_{0,\text{eq}}. \quad (15)$$

The above definitions of the susceptibilities hold for generalized forces or displacements. We now specialize to the case of two-point susceptibilities which are most useful for series and field-theoretic treatments. For instance we now define the two-point elastic susceptibility, $\chi_{\text{el}}(s, i, j)$. To do this let $\hat{\mathbf{r}}_{ij}$ be a unit vector along the line joining sites i and j . Then $\chi_{\text{el}}(s, i, j)$ is the susceptibility defined in Eq. (10) for the generalized displacement in which $\mathbf{u}_i = -\mathbf{u}_j = \frac{1}{2} X \hat{\mathbf{r}}_{ij}$. Alternatively, we may take $\chi_{\text{el}}(s, i, j)$ to be the susceptibility defined in Eq. (15) in the presence of the generalized force, $F_i = -F_j = F \hat{\mathbf{r}}_{ij}$, so that

to be equivalent, it is necessary to require that $\sum_i (I_i / I) (V_i / V) = \sum_{i,m} f_{i,m} x_{i,m} = 1$.

The dependence of the elastic susceptibility on distance is of interest. The crossover exponent $\phi_{\text{el}}(s)$ is defined by

$$\chi_{\text{el}}(s, x, x') \sim |x - x'|^{\phi_{\text{el}}(s)/\nu} \quad (20)$$

for p near the threshold value for elastic rigidity, $p_{c,\text{el}}$. More complicated response functions can, of course, be defined. For instance, we have previously¹⁰ considered a "torsional" susceptibility $\chi_{\text{el}}(b, b')$ in which a pair of bonds b and b' are subjected to equal and opposite applied torques. As we have just seen, this susceptibility could equally well be defined by imposing equal and opposite angular displacements on the bonds b and b' .

III. APPLICATIONS OF COHN'S THEOREM

Here we will discuss various applications of the above theorems to quantities of interest in the REN.

A. Derivatives of the susceptibility

For fixed displacement boundary conditions, $H_{\text{eq}} = [X^{s+1}/(s+1)] \chi_X^{-s}$, where, as before in Eq. (9), X sets the scale of the generalized displacement. (For the two-point susceptibility, X is twice the magnitude of the displacement along $\hat{\mathbf{r}}_{ij}$ of one of the sites i or j .) Thus

$$X^{s+1} \frac{d\chi_X^{-s}}{dk} = \frac{d}{dk} \left[\sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} + \sum_{\langle b, b' \rangle} k_{b, b'} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} \right] \quad (21a)$$

$$= \sum_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} \delta(k, k_b) + \sum_{\langle b, b' \rangle} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} \delta(k, k_{b, b'}) . \quad (21b)$$

We used Eq. (2) to pass from Eq. (21a) to (21b).

For fixed external forces, as in Eq. (11), we differentiate Eq. (15) with respect to k to obtain

$$-\frac{s}{s+1} F^{(s+1)/s} \frac{d\chi_F}{dk} = -\frac{s}{s+1} \frac{d}{dk} \left[\sum_b (k_b^{-1/s}) (k_b^{(s+1)/s}) |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} + \sum_{\langle b, b' \rangle} (k_{b, b'}^{-1/s}) (k_{b, b'}^{(s+1)/s}) |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} \right] \quad (22a)$$

$$= -\frac{s}{s+1} \sum_b \left[\frac{\partial}{\partial k} k_b^{-1/s} \right] (k_b^{(s+1)/s}) |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} - \frac{s}{s+1} \sum_{\langle b, b' \rangle} \left[\frac{\partial}{\partial k} k_{b, b'}^{-1/s} \right] (k_{b, b'}^{(s+1)/s}) |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} . \quad (22b)$$

We used Eq. (3) to pass from Eq. (22a) to (22b). Thus we have

$$\frac{d\chi_F}{dk} F^{(s+1)/s} = -\frac{1}{s} \left[\sum_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} \delta(k, k_b) + \sum_{\langle b, b' \rangle} |\mathbf{u}_b \times \hat{\mathbf{R}}_b - \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}|^{s+1} \delta(k, k_{b, b'}) \right] . \quad (23)$$

B. Noise on elastic networks

Similar to the resistor networks considered by Rammal *et al.*¹ and Blumenfeld *et al.* (BMAH),⁴ we can also discuss the noise problem for the elastic networks. Here we focus our attention on the case where the external force is fixed, and to start we consider the harmonic ($s = 1$) case. We consider the case when each central-force coupling constant k_b has a small random fluctuation δk_b around its average value. (The generalization to the case when the bond-bending coupling constant is also allowed to fluctuate is straightforward.) In other words, we consider two separate averaging processes. In the first, we select a set of occupied bonds with the associated probability, P , of the percolation problem: i.e., $P = p^{n_{\text{occ}}} (1-p)^{n_{\text{vac}}}$, where n_{occ} is the number of occupied bonds and n_{vac} the number of vacant bonds. The average over all configurations of occupied and vacant bonds will be denoted $[\]_{\text{av}}$. Each occupied bond, b , has a central-force spring whose spring constant k_b is governed by a probability distribution $f(k)$, which is sharply peaked near its average value k_0 . As in the formulation of Park *et al.*,³ this distribution can be characterized by the cumulants of k with respect to the distribution $f(k)$. We denote cumulant averages over the distribution $f(k)$ at fixed percolation configuration by $\langle \ \rangle_c$. Since each bond has an independent distribution, the n th cumulant of f is the same for all bonds b and we define $\langle \delta k_b^n \rangle_c / k_0^n = \Delta_n$, where k_0 is the value of k_b averaged over $f(k)$. Let $\chi_{\text{el},0}(i, j)$ denote the value of $\chi_{\text{el}}(i, j)$ when $f(k)$ has zero width where $\chi_{\text{el}}(i, j) \equiv \chi_{\text{el}}(s=1, i, j)$. Then we may write

$$\begin{aligned} \chi_{\text{el}}(i, j) &\sim \chi_{\text{el},0}(i, j) + \sum_c \frac{\partial \chi_{\text{el},0}(i, j)}{\partial k_c} \delta k_c \\ &\equiv \chi_{\text{el},0}(i, j) + \delta \chi_{\text{el}}(i, j) , \end{aligned} \quad (24)$$

where c stands for one of the coupling constants k_b . Following Ref. 3 we have

$$\langle [\delta \chi_{\text{el}}(i, j)]^n \rangle_c = \left\langle \left[\sum_c \frac{d\chi_{\text{el}}(i, j)}{dk_c} \delta k_c \right]^n \right\rangle_c \quad (25a)$$

$$= \Delta_n k_0^n \sum_c \left[\frac{d\chi_{\text{el}}(i, j)}{dk_c} \right]^n , \quad (25b)$$

since cumulant averages involving more than one site vanish. Now we use Eq. (23) in which case, we have

$$\langle [\delta \chi_{\text{el}}(i, j)]^n \rangle_c = M_n(i, j) \Delta_n (-k_0)^{-n} , \quad (26)$$

where

$$M_n(i, j) = \sum_b \left[\frac{k_0 \mathbf{u}_b \cdot \hat{\mathbf{R}}_b}{F} \right]^{2n} , \quad (27a)$$

where F is the magnitude of the equal and opposite forces applied to sites i and j . The analog of Eq. (27a) for the RRN is $M_n(i, j) = \sum_b (i_b / I)^{2n}$, where i_b / I is the fraction of the externally imposed current flowing in bond b . If we allow the coupling constants between bonds to fluctuate equally at the same time, we have

$$\begin{aligned} M_n(i, j) &= \sum_b \left[\frac{k_0 \mathbf{u}_b \cdot \hat{\mathbf{R}}_b}{F} \right]^{2n} \\ &+ \sum_{\langle b, b' \rangle} \left[\frac{k_0 \mathbf{u}_b \times \hat{\mathbf{R}}_b - k_0 \mathbf{u}_{b'} \times \hat{\mathbf{R}}_{b'}}{F} \right]^{2n} . \end{aligned} \quad (27b)$$

Therefore, we define the n th order elastic fluctuation susceptibility as

$$\chi_{\text{el}}^{(n)}(x, x') = \sum_x [\nu(x, x') M_n(x, x')]_{\text{av}}, \quad (28)$$

where x can be either a site or bond index, and $\nu(x, x')$ is unity if x and x' are rigidly connected, and is zero otherwise. In the central-force model, x should label bonds rather than sites.¹⁰ We define the critical exponent $\tilde{\psi}_{\text{el}}(n, 1)$ (the argument 1 indicates that $s = 1$) such that

$$\bar{M}_n \equiv [\nu(x, x') M_n(x, x')]_{\text{av}} / [\nu(x, x')]_{\text{av}} \quad (29a)$$

$$\sim |\mathbf{x} - \mathbf{x}'|^{\tilde{\psi}_{\text{el}}(n, 1)} \quad (29b)$$

for $|\mathbf{x} - \mathbf{x}'| \ll \xi_{\text{el}}$, where $\xi_{\text{el}} \sim |p_{\text{c,el}} - p|^{-\nu_{\text{el}}}$ is the appropriate elastic correlation length. We then have

$$\chi_{\text{el}}^{(n)} \sim |p_{\text{c,el}} - p|^{-\gamma_{\text{el}} - \psi_{\text{el}}(n, 1)}, \quad (30)$$

where $\psi_{\text{el}}(n, 1) = \tilde{\psi}_{\text{el}}(n, 1) \nu_{\text{el}}$ and γ_{el} is the exponent for the elastic susceptibility. The noise exponents can also be defined for nonlinear elastic networks. For simplicity we confine our attention to the case of central forces. Then we generalize Eqs. (25)–(27) by writing

$$\begin{aligned} M_n(s, i, j) &\equiv \sum_c \left[-k_0^{(s+1)/s} \frac{d\chi_{\text{el}}(s, i, j)}{dk_c} \right]^n \\ &= \sum_b \left[\frac{k_0 |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^s}{F} \right]^{n(s+1)/s} \end{aligned} \quad (31)$$

and the exponent associated with $M_n(s, i, j)$ is denoted $\psi_{\text{el}}(n, s)$.

Comparing Eqs. (19) and (31) one sees that

$$M_1(s, i, j) = k_0^{1/s} \chi_{\text{el}}(s, i, j), \quad (32)$$

where χ_{el} is the usual elastic susceptibility used in the series expansion calculations.¹⁰ Thus $\phi_{\text{el}}(s) = \psi_{\text{el}}(1, s)$.

We now discuss qualitatively the way these elastic noise exponents depend on n and s . We can show that the $\psi_{\text{el}}(n, 1)$ exponents are not just multiples of $\phi_{\text{el}}(1)$ by considering a fractal lattice.¹² Thus there exists a non-trivial family of noise exponents characterizing the elastic properties of elastic networks. Also, using Schwartz's inequality, one^{1,4} obtains the convexity relation for the exponent $\psi_{\text{el}}(n, s)$

$$\tilde{\psi}_{\text{el}}(n, s) + \tilde{\psi}_{\text{el}}(m, s) \geq 2 \tilde{\psi}_{\text{el}} \left[\frac{m+n}{2}, s \right]. \quad (33)$$

For the bond-bending model, we have

$$\psi_{\text{el}}(0, s) = \psi_{\text{res}}(0, s), \quad (34)$$

which is exponent for the backbone.²⁻⁴ If it were true that the stress in a single bond could not be greater than the applied external force, we would be able to conclude that $\lim_{n \rightarrow \infty} \psi_{\text{el}}(n, s)$ would be the exponent for singly connected bonds. This equality is uncertain because the stress in a single bond can be greater than the applied force. Following the arguments of BMAH we see that the crucial question is whether or not the quantity

FIG. 1. Examples of small linear ($s = 1$) systems for which the internal stress can exceed the externally applied force, F , applied at points C and D in each diagram. Cases (a) and (b) are for central forces only and give respective values of λ , where $\lambda \equiv |k_0(\mathbf{u}_A - \mathbf{u}_B) \cdot \hat{\mathbf{R}}_{AB}|/F$, and of 1.89 and 2.44. Observe that λ increases as the length of the lever arms is increased. Cases (c) and (d) are for the bond-bending model with $k_b = k_{b,b'}$. Here we see that the angular displacement θ indicated at the top also increases as the length of the lever arms is increased.

$|k_0 \mathbf{u}_b \cdot \hat{\mathbf{R}}_b|/F$ in Eq. (31) can exceed unity for a finite fraction of bonds. This can happen if long and rigid elements function like a crowbar as we show in Fig. 1. Thus the statement that $\tilde{\psi}_{\text{el}}(n, s)$ decreases monotonically with increasing n , which is true for resistor networks, may not hold for the elastic network. In fact we are presently addressing this question using the series expansion method. The behavior of $\phi_{\text{el}}(s)$ for $s \rightarrow 0$ or $s \rightarrow \infty$ is not as easy to deduce as for the RRN due to the vector nature of the displacements in the REN.

C. Crossover from linear to nonlinear elastic network

Recently, Gefen *et al.*¹³ and Aharony¹⁴ have discussed the crossover from linear to nonlinear diluted resistor networks in which each individual resistor has the relation $V = rI + aI^\alpha$, where a is small. For elastic networks, we have similar conclusions. For simplicity, we consider the central-force model and let the spring constant k be unity. We consider the response of the system when forces are applied to sites i and j with $R_{ij} \sim L$. Thus we have

$$\begin{aligned} H &= \frac{1}{2} \sum_b (\mathbf{u}_b \cdot \hat{\mathbf{R}}_b)^2 \\ &+ \frac{a}{s+1} \sum_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} - F(\mathbf{u}_i - \mathbf{u}_j) \cdot \hat{\mathbf{R}}_{ij}. \end{aligned} \quad (35)$$

One sees that because a single spring deviates from linearity when the force along the bond is of order $F^0 \sim a^{-1/(s-1)}$, the system as a whole will deviate from linearity at a critical value of F_{ext} which we denote F_c . As in resistor networks, we expect that $F_c \sim \chi_{\text{el}}(L)^{-\nu}$ for

$p \rightarrow p_c$, so that $L \ll \xi$. Using Eq. (2), we have

$$\left. \frac{dH}{da} \right|_{a=0} = \frac{1}{s+1} \sum_b |\mathbf{u}_b^0 \cdot \hat{\mathbf{R}}_b|^{s+1}, \quad (36)$$

where $\mathbf{u}_b^0$ is the displacement when $a=0$. Now we can discuss the crossover from linear to nonlinear elastic networks. To the leading order in a , Eq. (36) yields

$$H = H(a=0) + \frac{dH}{da} a \quad (37a)$$

$$= -\frac{1}{2} \sum_b (\mathbf{u}_b^0 \cdot \hat{\mathbf{R}}_b)^2 + \frac{a}{s+1} \sum_b |\mathbf{u}_b^0 \cdot \hat{\mathbf{R}}_b|^{s+1} \quad (37b)$$

$$= -\frac{1}{2} F^2 M_1 + \frac{a}{s+1} F^{s+1} M_{(s+1)/2}, \quad (37c)$$

where M_n is the n th moment elastic susceptibility for the central-force model. Thus elastic susceptibility, given by $\chi_{el} = -\partial^2 H / \partial F^2$, will show deviation from linear response for

$$\begin{aligned} F_{ext} > F(L) &\sim F^0 [M_1 / M_{(s+1)/2}]^{1/(s-1)} \\ &\sim F^0 L^{-y_{\phi_{el}(1)}} \sim \chi_{el}^{-y} \end{aligned} \quad (38)$$

with

$$y = \left[1 - \tilde{\psi}_{el} \left[\frac{s+1}{2}, 1 \right] / \tilde{\phi}_{el}(1) \right] / (s-1).$$

D. The relation between nonlinear noise exponent $\psi_{el}(q, s)$ and nonlinear crossover exponent $\phi_{el}(s)$

In the resistor networks, there exists a relation between the nonlinear noise exponent and the nonlinear crossover exponent.¹⁵ Here we derive a similar relation for the elastic case. For simplicity, let us consider the central-force model with fixed external forces applied to sites i and j . We consider the properties of a system with nonlinearity index $s + \Delta s$:

$$H_0 = \frac{1}{s+1} \sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1+\Delta s} \quad (39a)$$

$$= \frac{1}{s+1} \sum_b k_b |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} (1 + \Delta s \ln |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|). \quad (39b)$$

If we view $k_b \Delta s \ln |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|$ as a perturbation of k_b , we have

$$\chi_{el}(s + \Delta s, i, j) = \chi_{el}(s, i, j) + \sum_b k_b \frac{d\chi_{el}(s, i, j)}{dk_b} \Delta s \ln |\mathbf{u}_b \cdot \hat{\mathbf{R}}_b| \quad (40a)$$

$$= -s \sum_b \frac{d\chi_{el}(s, i, j)}{dk_b} k_b + \sum_b k_b \frac{d\chi_{el}(s, i, j)}{dk_b} \frac{\Delta s}{s+1} \ln \left| -s \frac{d\chi_{el}(s, i, j)}{dk_b} \right|. \quad (40b)$$

In the last equation, we have used the relations

$$|\mathbf{u}_b \cdot \hat{\mathbf{R}}_b|^{s+1} = -s \frac{d\chi_{el}(s, i, j)}{dk_b}, \quad (41a)$$

$$\sum_b \frac{d\chi_{el}(s, i, j)}{dk_b} k_b = -\frac{1}{s} \chi_{el}(s, i, j), \quad (41b)$$

and have set the magnitude of the external force to unity. From Eq. (40b) we obtain

$$\chi_{el}(s + \Delta s, i, j) = \sum_b k_b \frac{d\chi_{el}(s, i, j)}{dk_b} \left[-s + \frac{\Delta s}{s+1} \ln s + \frac{\Delta s}{s+1} \ln \left(-\frac{d\chi_{el}(s, i, j)}{dk_b} \right) \right] \quad (42a)$$

$$= \sum_b \left[-s \frac{d\chi_{el}(s, i, j)}{dk_b} \right]^{1 - (\Delta s)/[s(s+1)]}. \quad (42b)$$

From the definition of $\phi_{el}(s)$ and $\psi_{el}(q, s)$, we have $\phi_{el}(s + \Delta s) = \psi_{el}\{1 - \Delta s/[s(s+1)], s\}$. We get

$$s(s+1) d\phi_{el}(s)/ds = -\partial \psi_{el}(q, s) / \partial q|_{q=1}. \quad (43)$$

IV. CONCLUSION

Although we discuss only the bond-bending model, Cohn's theorem holds also for the granular disk model¹⁶ which belongs to a different universality class than that of bond-bending model in spatial dimensions higher than

two. Also we should note that the model considered here is *not* the same as that for Hookean springs for which

$$H_0 = \frac{1}{2} \sum_b k_b (|\mathbf{u}_b + \mathbf{R}_b| - R_b)^2, \quad (44)$$

where $\mathbf{R}_b$ is magnitude and direction in equilibrium of the bond b . This Hamiltonian differs from that in Eq. (1) in that transverse displacements, $\mathbf{u}$, of the type shown in Fig. 2 have a nonzero restoring force. This force is of order ku^3/R_b^2 . As discussed in Sec. III C, the presence of such anharmonicity is analogous to a resistor network with a mixture of linear and nonlinear elements. It has

FIG. 2. Example of an elastic network for which a site (labeled A) is not rigid within the model of Eq. (1) with respect to a transverse displacement u , but is rigid within the model of Eq. (44).

been asserted¹⁷ that the threshold concentration p'_c for rigidity for this anharmonic model is smaller than that, $p_{0,c}$, for the model of Eq. (1). If this is so, a novel crossover would occur for $p'_c < p < p_{0,c}$, where the linear bulk modulus would be zero but a higher-order nonlinear response coefficient would be nonzero. We are currently considering this possibility. Within the central-force model of Eq. (1), however, the threshold depends only on

which of the k_b 's are nonzero. Thus we believe that the conclusion of Ref. 18 to the contrary is clearly spurious.

We may summarize our conclusions as follows. For nonlinear elastic networks we have derived the analogs [Eqs. (2) and (3)] of Cohn's theorem for resistor networks. We have used these results to obtain expressions given in Eqs. (27) and (31) for elastic noise susceptibilities which give rise to a nontrivial family of critical exponents just as for the resistor network. These quantities should be calculated using low concentration series. We have also shown that the results of Aharony¹⁴ for the crossover from linear to nonlinear behavior and of Harris¹⁵ for the relation between the noise exponents and the elastic crossover exponent hold for the elastic network. Finally, we noted the possibility of a novel crossover for a system consisting of a randomly diluted mixture of linear and nonlinear elements.

Note added in proof. After this paper was submitted a related paper appeared: see Ref. 19.

ACKNOWLEDGMENTS

One of us (J.W.) thanks the National Science Foundation for support under Grant No. DMR 85-19059 of the Materials Research Laboratory (MRL) program. One of us (A.B.H.) thanks the National Science Foundation for partial support from Grant No. DMR 85-19216.

*Present address: Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7.

¹R. Rammal, C. Tannous, and A.-M. S. Tremblay, Phys. Rev. A **31**, 2662 (1985); R. Rammal, C. Tannous, P. Breton, and A.-M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985).

²L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985); L. de Arcangelis, A. Coniglio, and S. Redner, J. Phys. A **18**, L805 (1985).

³Y. Park, A. B. Harris, and T. C. Lubensky, Phys. Rev. B **35**, 5048 (1987).

⁴R. Blumenfeld, Y. Meir, A. Aharony, and A. B. Harris, Phys. Rev. B **35**, 3524 (1987).

⁵R. Blumenfeld, Y. Meir, A. B. Harris, and A. Aharony, J. Phys. A **19**, L791 (1986).

⁶R. M. Cohn, Am. Math. Soc. **1**, 316 (1950).

⁷A. B. Harris and R. Fisch, Phys. Rev. Lett. **38**, 796 (1977); R. Fisch and A. B. Harris, Phys. Rev. B **18**, 416 (1978).

⁸S. Feng and P. N. Sen, Phys. Rev. Lett. **52**, 216 (1984).

⁹J. G. Zabolitzky, D. J. Bergman, and D. Stauffer, J. Stat. Phys. **44**, 211 (1986).

¹⁰J. Wang and A. B. Harris, Phys. Rev. Lett. **55**, 2459 (1985).

¹¹J. Wang, A. B. Harris, and J. Adler (unpublished).

¹²J. Wang (unpublished).

¹³Y. Gefen, W. H. Shih, R. B. Laibowitz, and J. M. Viggiano, Phys. Rev. Lett. **57**, 3097 (1986).

¹⁴A. Aharony, Phys. Rev. Lett. **58**, 2726 (1987).

¹⁵A. B. Harris, Phys. Rev. B **35**, 5056 (1987).

¹⁶S. C. Feng, Phys. Rev. B **32**, 510 (1985).

¹⁷S. C. Feng, M. F. Thorpe, and E. Garboczi, Phys. Rev. B **31**, 276 (1985).

¹⁸R. Garcia-Molina, F. Guinea, and E. Louis, Phys. Rev. Lett. **60**, 124 (1988).

¹⁹A. Hansen and S. Roux, J. Stat. Phys. **53**, 759 (1988).