January 1985

The Controllability of Planar Bilinear Systems

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Keywords
planar bilinear systems

Comments

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ACKNOWLEDGMENT

The author is grateful to Prof. B. R. Barmish and the anonymous referees for their valuable comments on an earlier draft of this paper.

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The Controllability of Planar Bilinear Systems

DANIEL E. KODITSCHEK AND KUMPATI S. NARENDRA

I. INTRODUCTION

This note will summarize some recent results concerning the controllability of planar bilinear systems. We consider the homogeneous system

$$\dot{x} = Ax + uDx \tag{1}$$

Manuscript received November 8, 1982; revised April 3, 1984. This work was supported in part by the Office of Naval Research under Contract N00014-76-C-0017, and by the National Science Foundation through a Graduate Research Fellowship.

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where $x \in \mathbf{R}^2 - \{0\}$, and the general bilinear system

$$\dot{x} = Ax + u(Dx + b) \tag{2}$$

where $x \in \mathbf{R}^2$. Let u be a piecewise continuous scalar function with unconstrained magnitude, and assume A and D to be nonzero. Most significantly, we present succinct necessary and sufficient conditions for the complete controllability of both systems. All results are stated in terms of algebraic conditions on system parameters which are effectively computable.

Sufficient conditions for controllability of bilinear systems in $\mathbf{R}^n$ have been given by Jurdjevic and Kupka [5] and Jurdjevic and Sallet [4], while a general approach to the controllability of linear analytic systems has been explored by Hunt [3]. The strength of the results reported here is a consequence of insight and algebraic facility which depend heavily upon geometric properties of the plane. The extent to which such problems have equally succinct solutions in higher dimensions is not clear. However, the techniques and results afforded by such detailed attention in this special setting suggest a general approach to systems of higher dimension and degree.

II. PRELIMINARY DISCUSSION

A few definitions and preliminary results of an algebraic nature will facilitate the presentation to follow.

Define the skew symmetric matrix

$$J \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and let $x_\perp \triangleq Jx$. A close relationship between inner products, determinants, and quadratic forms in $\mathbf{R}^2$ will be used continually:

$$y^T x_\perp = y^T Jx = |x, y|$$

where the last symbol denotes the determinant of the matrix $[x \ y]$. Given a matrix A, let A_s denote its symmetric part, tr $\{A\}$ denote its trace, and $A^\# \triangleq J^T A^T J$ denote its transposed cofactor matrix. Two matrices A and D are *linearly dependent* in $\mathbf{R}^{2\times 2}$ if there exists a scalar γ such that $A = \gamma D$, and *independent* otherwise. Given a, b, and $c \in \mathbf{R}^2$, the relationship

$$a + \mu b = \lambda c$$

holds for the scalars μ and λ if and only if

$$\mu = -\frac{|a, c|}{|b, c|} \quad \lambda = -\frac{|a, b|}{|b, c|}.$$

As an immediate consequence there follows the very useful relation

$$[A + \mu(x)D]x = \lambda(x)x \tag{3}$$

when $\mu(x) \triangleq -|Ax, x|/|Dx, x|$ and $\lambda(x) \triangleq -|Ax, Dx|/|Dx, x|$ are well defined.

It will be helpful to introduce some algebraic results concerning *homogeneous quadratic transformations of the plane*,

$$Q(x) \triangleq \begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix}.$$

The notion of a singular linear transformation may be extended to arbitrary homogeneous polynomial transformations of the plane [6]. For purposes of this paper, call Q *singular* if its constituent quadratic forms have a common linear or quadratic factor—that is, if there exist a $c \in \mathbf{R}^2$ and $B \in \mathbf{R}^{2\times 2}$ such that

$$Q(x) = c^T x Bx$$

or if G or H are linearly dependent. Unlike linear transformations, not all nonsingular homogeneous quadratic transformations are surjective (evidently, none are injective).

Lemma 2-1: A homogeneous quadratic transformation of the plane Q, is surjective if and only if $|dQ_x| = |Gx, Hx|$ *is a sign definite quadratic form.*[1]

Proof: See [6].

This algebraic criterion for surjectivity has a useful geometric interpretation.

Lemma 2-2: The quadratic transformation of the plane

$$Q(x)=\begin{bmatrix} x^TGx \\ x^THx \end{bmatrix}$$

has a sign definite derivative, dQ_x, *if and only if*

i) both G and H are indefinite;

ii) their distinct zero lines alternate around the plane—i.e., $x^TGx = 0$ *has a solution both in the cone defined by* $x^THx > 0$ *and* $x^THx < 0$.

Proof: See [6].

III. Main Results

We will use the ideas from the previous section to characterize the controllability behavior of the homogeneous and general bilinear system in R^2. The behavior of interest is addressed by the following definitions. Given a control system, say that *a point y is reachable from a point x* if there exists an admissible control u and a finite time T, such that the trajectory with initial condition x of the vector field specified by u passes through y at time T. Denote the set of points reachable from x as $\mathcal{R}(x)$. If $\mathcal{R}(x)$ is equal to the state space for every point, x, in the state space, then the system is *completely controllable.* If $\mathcal{R}(x)$ is open, then the system is said to be *accessible at x*. If the system is accessible at every point, then the system is *completely accessible* or has *the accessibility property.*

Accessibility is a necessary condition for complete controllability [2], but it is certainly not sufficient, as has been long known. It has been shown [9] that the accessibility property obtains when the lie algebra generated by the family of vector fields of a control system parametrized by control input values spans the tangent space at every point of the state space. Pursuing the computational aspects of this result for bilinear systems amounts to asking which matrix pairs in $R^{n\times n}$ generate transitive lie algebras. These have been completely classified by Boothby [1], and that classification becomes simple in $R^{2\times 2}$.

Proposition 3-1 [7]: The lie algebra generated by two linearly independent matrices in $R^{2\times 2}$ *spans* R^2 *at every point of* $R^2 - \{0\}$ *if and only if the matrices do not have a real eigenvector in common.*

This condition is quite simply expressed in terms of the singularity of an appropriate quadratic transformation.

Corollary 3-1: The homogeneous bilinear system (1) fails to have the accessibility property if and only if the quadratic transformation of the plane

$$Q(x)\triangleq\begin{bmatrix} |Ax, x| \\ |Dx, x| \end{bmatrix}$$

is singular.

The common factor is a quadratic form if and only if A and D are linearly dependent, and linear form(s) if and only if A and D share an eigenvector(s). We now proceed to characterize the controllability of planar bilinear systems.

The state space of system (1), $R^2 - \{0\}$, is not simply connected; thus, complete controllability must entail an ability to transfer any ray to any other ray of R^2. If, in addition, radial control—the ability to move toward or away from the origin on a given ray—is available, then it is reasonable to expect that complete controllability holds.

Proposition 3-2: Let A and D be linearly independent. If there exist real numbers, μ, *for which* $A + \mu D$ *has nonreal eigenvalues and eigenvalues with both positive and negative real parts, then systems (2) and (1) are both completely controllable.*

Proof: Suppose it is desired to reach x_2 from x_1, for any two arbitrary points in R^2. Let $M_0 \triangleq A + \mu_0 D$ have complex conjugate eigenvalues, and consider solutions to the linear homogeneous time invariant system (2) that obtains when $u(\tau) \equiv \mu_0$. For some finite integers n and m, either a curve of n forward "logarithmic spirals" through x_1,

$$\{e^{M_0\tau}x_1|\tau\in[0, n\omega/2\pi]\}$$

(where ω is the "natural frequency" of the system), encircles x_2, or of m backward spirals,

$$\{e^{-M_0\tau}x_2|\tau\in[0, m\omega/2\pi]\},$$

through x_2, encircles x_1. Considering the former case, choose a value μ_1 for which $M_1 \triangleq A + \mu_1 D$ has an eigenvalue with negative real part. Since the backward trajectory $e^{-M_1\tau}x_2$ connects x_2 to the point at infinity, and cannot run "parallel" to the encircling spiral,[2] it must intersect that spiral at a finite point. In the latter case choose a value μ_1 for which M_1 has an eigenvalue with positive real part, and the forward trajectory through x_1 of the linear homogeneous time invariant system resulting from $u(\tau) \equiv \mu_1$ must intersect the spiral for the same reason. □

In fact, as intuition might suggest, the conditions of Proposition 3-2 are necessary as well for the complete controllability of system (1). To show this, we require an algebraic characterization of when the conditions of that proposition fail.

Simple algebra demonstrates that $A + \mu D$ fails to have eigenvalues in both the positive and negative half of the complex plane if and only if D has pure imaginary eigenvalues, and $[D^TJA]_s$ is sign definite or semidefinite [7]. In such a situation the integral curves of a linear time invariant differential equation defined by D are ellipses containing periodic solutions. It is shown in [7] that the condition on $[D^TJA]_s$ implies that either the interior of each such ellipse or the complement of its closure is a positive invariant set; hence, the system is not completely controllable.

The necessity that $A + \mu D$ have complex conjugate eigenvalues follows readily after a little more algebra. Recall that (3) expresses the range over which $A + \mu D$ has real eigenvalues by considering μ to be a scalar valued function on R^2. It follows that $A + \mu D$ has no nonreal eigenvalues if and only if μ is surjective, or, if and only if the quadratic map

$$Q(x)\triangleq\begin{bmatrix} |Ax, x| \\ |Dx, x| \end{bmatrix}$$

is surjective or is singular due to a common linear factor. In the singular case the bilinear system is "degenerate" in a sense made precise above. Otherwise, we appeal to the geometric description of nonsurjective, nonsingular quadratic transformations given by Lemma 2-2.

Proposition 3-3: If $A + \mu D$ *has no complex conjugate eigenvalues for any real* μ, *then system (1) is not completely controllable.*

Proof: Defining $G \triangleq [JA]_s$ and $H \triangleq [JD]_s$, Lemma 2-2 indicates that $A + \mu D$ fails to have any complex conjugate eigenvalues only in the case that A and D have two distinct eigenvectors which "interweave" on the plane. On the boundary of a cone defined by the eigenvectors of D, control may be affected in only a radial direction. Either this cone or the complement of its closure contains the eigenvector of A whose eigenvalue has the greater (algebraic) real value. The resultant of Ax with Dx on its boundary lines must always be oriented toward the interior of that cone, which must therefore be a positive invariant set. □.

Taken together, these results imply that the converse of Proposition 3-2 holds for the homogeneous system (1).

Theorem 1: System (1) is completely controllable on $R^2 - \{0\}$ *if and only if A and D are linearly independent, and* $A + \mu D$ *has*

[1] Thanks are due Prof. C. Byrnes for a discussion leading to this statement.

[2] M_1 yields a different linear system from M_0 under the hypothesis that A and D are linearly independent. If A and D are linearly dependent, with complex conjugate eigenvalues, then we require $b \neq 0$ to ensure two different constant values of u yield trajectories which intersect at a finite point; cf. Theorem 2.

nonreal eigenvalues and eigenvalues with both positive and negative real parts.

Necessary conditions for complete controllability of the general bilinear system (2) are very close to the sufficient conditions given in Proposition 3-2 as well. In the sequel, when referring to (2), assume that b is nonzero. The nonzero additive control term does not relieve the necessity of reaching every ray on the plane using the homogeneous portion of the field alone, as shown by the following.

Proposition 3-4: If there is no real value μ for which $A + \mu D$ has complex conjugate eigenvalues, then system (2) is not completely controllable.

Proof: If D is nonsingular, then system (2) may be written in the form

$$\dot{y} = (A + uD)y - k \tag{4}$$

where $y \triangleq x + D^{-1}b$ and $k \triangleq AD^{-1}b$. As in Proposition 3-3, on the boundary of a cone defined by the eigenvectors of D through the origin of the translated plane, the vector sum of Ay with Dy is oriented toward the interior of that cone. Since k is a constant, it cannot be oriented toward the exterior of this cone in one half plane without having an interior orientation with respect to the portion of the cone in the other half plane, which must, therefore, be a positive invariant set. Otherwise, k is an eigenvector of D, and is tangential to the boundary of the cone, which is positive invariant in its entirety.

If D is singular, then $D = de^TJ^T$ for some $d, e \in R^2$. If d is an eigenvector of A, then an entire half-plane is positive invariant. If d is in $\langle b \rangle$, then an argument identical to the previous paragraph may be given to show uncontrollability. Otherwise, an affine line can be shown to define a positive invariant half-plane [8]. □

However, the guarantee of an additive control term does afford a slight relaxation of the necessary conditions in Theorem 1. If the conditions of Theorem 1 hold with the exception that $A + \mu D$ has eigenvalues exclusively in one half of the complex plane, then (2) is still completely controllable provided that $[D^TJA]_s$ is semidefinite. This may be seen, as shown in [8], by noting that the portion of the field due to Ax is tangential to the ellipses defined by trajectories of the vector field Dx on the zero eigenvector of $[D^TJA]_s$. On this line, the additive term ub may be used to drive the state away or toward the origin.

If A and D are linearly dependent with real eigenvalues, then Proposition 3-4 precludes the possibility of complete controllability of (2). On the other hand, if $A = \delta D$ has complex conjugate eigenvalues, then the proof in Proposition 3-2 applies here (see footnote in that proof), and the system is completely controllable. These considerations permit a statement of the second major result in this note.

Theorem 2: System (2) is completely controllable if and only if either

i) A and D satisfy the conditions for complete controllability of system (1); or

ii) for all real values, μ, $A + \mu D$ has eigenvalues exclusively in one half of the complex plane but $[D^TJA]_s$ is semidefinite; or

iii) A and D are linearly dependent matrices with nonreal eigenvalues.

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A New Algorithm for the Design of Multifunctional Observers

CHIA-CHI TSUI

***Abstract*—This paper presents a general algorithm for low-order multifunctional observer design with arbitrary eigenvalues. The feature of this algorithm is that it can generate a functional observer with different orders which are no larger but usually much less than $m(\nu - 1)$, where m is the number of functionals and ν is the observability index of (A, C). Since the order needed for the observer varies with the functionals besides other system parameters, this design approach should be practical. The resulting observer system matrix is in its Jordan form. The key step of this algorithm is the generation of the basis for the transformation matrix which relates the system and observer states. The computation of this algorithm is quite reliable. It is based on the block observable lower Hessenberg form of (A, C), and all its initial and major computation involves only the orthogonal operations.**

I. INTRODUCTION

This paper deals with the problem of designing an observer for estimating several linear functions of the state variables. This is a very practical problem since the state feedback is a linear function of the states, say $Kx(t)$. Because the estimation of a function of the states does not necessarily require the estimation of all states, the order of a functional observer can be significantly less than that of a state observer [4].

The necessary and sufficient condition for the functional observer, as proposed by Fortman *et al.* [7] and restated by Kimura [5] from the geometrical point of view, is that the observer state $z(t)$ must approach a linear transformation of the states $Tx(t)$ and that K must be within the union of range spaces of T and C. In other words, it is required that the equations

$$\begin{aligned} TA - FT &= GC \\ H &= TB \end{aligned} \tag{1}$$

and

$$K = NT + MC \tag{2}$$

be satisfied if the observer equation is defined as

$$\dot{z}(t) = Fz(t) + Gy(t) + Hu(t) \tag{3}$$

$$w(t) = Nz(t) + My(t) \tag{4}$$

Manuscript received September 16, 1983; revised April 18, 1984. Paper recommended by D. P. Looze, Past Chairman of the Computational Methods and Discrete Systems Committee.

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