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Mean-Field Theory and ϵ Expansion for Anderson Localization

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Recommended Citation

Harris, A., & Lubensky, T. C. (1981). Mean-Field Theory and ϵ Expansion for Anderson Localization. *Physical Review B*, 23 (6), 2640-2673. <http://dx.doi.org/10.1103/PhysRevB.23.2640>

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Mean-Field Theory and ϵ Expansion for Anderson Localization

Abstract

A general field-theoretic formulation of the Anderson model for the localization of wave functions in a random potential is given in terms of n -component replicated fields in the limit $n \rightarrow 0$, and is analyzed primarily for spatial dimension $d \geq 4$. Lengths ξ_1 and ξ_2 associated with the spatial decay of correlations in the single-particle and two-particle Green's functions, respectively, are introduced. Two different regimes, the weak coupling and strong coupling, are distinguished depending on whether ξ_1^{-1} or ξ_2^{-1} , respectively, vanishes as the mobility energy, E_c , is approached. The weak-coupling regime vanishes as $d \rightarrow 4+$. Mean-field theory is developed from the uniform minimum of the Lagrangian for both the strong- and weak-coupling cases. For the strong-coupling case it gives the exponents $\nu_a = 1/4$, $\gamma_a = \beta_a = 1/2$, $\eta = 0$, and $\mu = 1$, where β_a is the exponent associated with the density of extended states and μ is that associated with the conductivity. Simple heuristic arguments are used to verify the correctness of these unusual mean-field values. Infrared divergences in perturbation theory for the strong-coupling case occur for $d < 8$, and an ϵ expansion ($\epsilon = 8 - d$) is developed which is found to be identical to that previously analyzed for the statistics of lattice animals and which gives $\beta_a = 1/2 - \epsilon/12$, $\eta = -\epsilon/9$, $\nu_a = 1/4 + \epsilon/36$, and $\mu = 1 - 5\epsilon/36$. The results are consistent with the Ward identity, which in combination with scaling arguments requires that $\beta_a + \gamma_a = 1$. The treatment takes account of the fact that the average of the on-site Green's function $[G(\vec{x}, \vec{x}; E)]_{av}$ is nonzero and is predicated on this quantity being real, i.e., on the density of states vanishing at the mobility edge. We also show that localized states emerge naturally from local minima of finite action in the Lagrangian. These instanton solutions are analyzed on a lattice where the cutoff produced by the lattice constant leads to *lattice instantons* which exist for all d , in contrast to the case for the continuum model where instanton solutions seem not to occur for $d > 4$. This analysis leads to a density of localized states ρ_{loc} satisfying $1/\rho_{loc} \sim -E^2$ at large E and $1/\rho_{loc} \sim -|E - E_c|^{-\zeta}$ at the mobility edge, where for the weak-coupling case $\zeta = (1/2)(d - 4)$ and for the strong-coupling case $\zeta = (d - 2 + \eta)\nu_a - 2\beta_a = 1/2 + \epsilon/18$ for $d < 8$ and $\zeta = (1/4)(d - 6)$ for $d > 8$. A brief discussion of the relationship between this work and the theories of localization below four dimensions is presented.

Disciplines

Physics

Mean-field theory and ϵ expansion for Anderson localization

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(Received 4 August 1980)

A general field-theoretic formulation of the Anderson model for the localization of wave functions in a random potential is given in terms of n -component replicated fields in the limit $n \rightarrow 0$, and is analyzed primarily for spatial dimension $d \geq 4$. Lengths ξ_1 and ξ_2 , associated with the spatial decay of correlations in the single-particle and two-particle Green's functions, respectively, are introduced. Two different regimes, the weak coupling and strong coupling, are distinguished depending on whether ξ_1^{-1} or ξ_2^{-1} , respectively, vanishes as the mobility energy, E_c , is approached. The weak-coupling regime vanishes as $d \rightarrow 4 +$. Mean-field theory is developed from the uniform minimum of the Lagrangian for both the strong- and weak-coupling cases. For the strong-coupling case it gives the exponents $\nu_a = 1/4$, $\gamma_a = \beta_a = 1/2$, $\eta = 0$, and $\mu = 1$, where β_a is the exponent associated with the density of extended states and μ is that associated with the conductivity. Simple heuristic arguments are used to verify the correctness of these unusual mean-field values. Infrared divergences in perturbation theory for the strong-coupling case occur for $d < 8$, and an ϵ expansion ($\epsilon = 8 - d$) is developed which is found to be identical to that previously analyzed for the statistics of lattice animals and which gives $\beta_a = 1/2 - \epsilon/12$, $\eta = -\epsilon/9$, $\nu_a = 1/4 + \epsilon/36$, and $\mu = 1 - 5\epsilon/36$. The results are consistent with the Ward identity, which in combination with scaling arguments requires that $\beta_a + \gamma_a = 1$. The treatment takes account of the fact that the average of the on-site Green's function $[G(\vec{x}, \vec{x}; E)]_{av}$ is nonzero and is predicated on this quantity being real, i.e., on the density of states vanishing at the mobility edge. We also show that localized states emerge naturally from local minima of finite action in the Lagrangian. These instanton solutions are analyzed on a lattice where the cutoff produced by the lattice constant leads to *lattice instantons* which exist for all d , in contrast to the case for the continuum model where instanton solutions seem not to occur for $d > 4$. This analysis leads to a density of localized states ρ_{loc} satisfying $\ln \rho_{loc} \sim -E^2$ at large E and $\ln \rho_{loc} \sim -|E - E_c|^{-\zeta}$ at the mobility edge, where for the weak-coupling case $\zeta = (1/2)(d - 4)$ and for the strong-coupling case $\zeta = (d - 2 + \eta)\nu_a - 2\beta_a = 1/2 + \epsilon/18$ for $d < 8$ and $\zeta = (1/4)(d - 6)$ for $d > 8$. A brief discussion of the relationship between this work and the theories of localization below four dimensions is presented.

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I. INTRODUCTION

An electron in a spatially uniform potential in a volume Ω has eigenstates with wave functions proportional to $\Omega^{-1/2}$. If the potential is spatially random, as first pointed out by Anderson,¹ there are eigenstates with wave functions dying off exponentially away from particular regions of space. Such eigenstates are localized and do not conduct, whereas those with wave functions proportional to $\Omega^{-1/2}$ are extended and do conduct. A schematic diagram of the density of states for an electron in a random three-dimensional potential is shown in Fig. 1(a). There is a critical energy E_c , called the mobility edge, separating localized states with $E < E_c$ from conducting states with $E > E_c$. For electrons on a three-dimensional lattice with sites $\vec{x}$ of random-site potential $V(\vec{x})$ and/or hopping $t(\vec{x}, \vec{x}')$, there can exist mobility edges, as shown in Fig. 1(c), outside of which the states are localized and inside of which the states are extended. Since Anderson's original paper,

FIG. 1. The density of states for various Gaussian random models with mobility edge(s) at E_c (and $-E_c$) separating extended states (E) from localized states (L). Figures (a) and (b) are for the continuum model of Eq. (2.27) for $d < 4$ and $d > 4$, respectively, where d is the spatial dimensionality. For $d > 4$ there are no localized states for this model. Figures (c) and (d) are for the discrete lattice model of Eq. (2.1) for $d < d^*$ and $d > d^*$, respectively. From a study of the weakly random case, we argue that d^* is probably 4. For $d < d^*$ the density of states is nonzero and smooth at the mobility edges at $\pm E_c$. For $d > d^*$, the density of extended states vanishes at $\pm E_c$ as $\rho_{\text{ext}}(E) \sim (E_c - E)^\beta$ for $|E| < E_c$ where $\beta = \beta_a$ for the strong-coupling case and $\beta = \frac{1}{2}d - 1$ for the weak-coupling case. (The distinction between these cases is discussed in detail in Sec. IV C 1.) The density of localized states vanishes at $\pm E_c$ as $\rho_{\text{loc}}(E) \sim \exp[-\text{const} \times (\Delta E)^{-\xi}]$, where $\xi = (d - 2 + \eta)\gamma_a - 2\beta_a$ [$\xi = \frac{1}{4}(d - 6)$] for the strong-coupling case for $d < 8$ [$d > 8$], and $\xi = \frac{1}{2}d - 2$ for the weak-coupling case.

considerable attention has been given to the study of electronic properties in the vicinity of the mobility edge.²

The mobility transition from nonconducting to conducting behavior as a function of E is analogous to a thermodynamic phase transition from a disordered to an ordered state as a function of temperature. After the introduction of the renormalization group³ and its successful application to the study of polymer statistics⁴ as well as to the critical properties of second-order phase transitions, it was natural to consider its application to the mobility transition. Thus, Wegner⁵ discussed this transition in terms of scaling based on renormalization-group ideas and formulated a real-space renormalization group for studying electronic properties of random systems. Since then, several authors⁶ have applied similar techniques to this problem.

One of the great successes of the renormalization group was to provide a systematic way to calculate critical exponents in a series in $\epsilon = d_c - d$, where d is the spatial dimension and d_c the dimension below which mean-field theory breaks down.^{7,8} Recently, the authors⁸ developed a mean-field theory and an ϵ expansion for the mobility edge for electrons on a lattice with Gaussian random hopping with zero mean. This treatment as well as previous ones⁹ was based on a field-

theoretic representation of the averaged Green's function $G(\vec{x}, \vec{x}')$ in terms of integrals over a replicated field $\Psi_i(\vec{x})$, where $i = 1, \dots, 2n$ and $n \rightarrow 0$. However, since information about the conductivity is contained in the two-particle Green's function $\mathcal{G}(\vec{x}, \vec{x}')$, another set of variables $Q_{ij}(\vec{x})$ was used to describe the mobility transition. Our study allowed us to (1) identify the dominant fluctuating order parameter at the mobility edge in mean-field theory to be all components of Q_{ij} and (2) determine d_c to be eight (not four or six) by considering fluctuations about the mean-field solution. Previous attempts⁹ to obtain an ϵ expansion for the mobility transition failed to produce either of these results. In this paper, we will study in detail a more general lattice model in which the site potential $V(\vec{x})$ and/or the hopping $t(\vec{x}, \vec{x}')$ have Gaussian random distributions without the restriction that the hopping have zero mean. We will present new results and calculational details not contained in Ref. 8.

Though properties near the mobility edge itself have until recently eluded systematic study, there have been many calculations of the density of localized states far from the mobility edge.¹⁰ These are based on variational or self-consistent-field calculations and lead to an exponential decaying density of states at large $(-E)$ for Gaussian random potentials. The early field-theoretic

and ϵ -expansion treatments⁹ of the mobility transition ignored localized states altogether. Cardy,¹¹ however, showed that the localized states for large $(-E)$ in a Gaussian random potential can be obtained from instantons¹² of finite action in the field-theoretic formulation for the electron Green's function in a random potential. We regard this as an important advance since it emphasized that field-theoretic formulations for the problem of an electron in a random potential are formally exact. In this paper, we will study instantons in order to obtain the density of localized states from our field theory. Our treatment however, differs in two respects from that of Cardy. First, we study instantons in terms of both the variables $\Psi_i(\vec{x})$ and $Q_{ij}(\vec{x})$, whereas Cardy considered the variable $\Psi_i(\vec{x})$. Second, we study instantons on a lattice rather than in a continuum with a random potential having no spatial correlations, i.e., a white-noise potential. The latter distinction is important since there are no instanton solutions and no localized states for $d > 4$ for white-noise potentials.^{11,12} If, however, a lattice or spatial correlations in the random potential were introduced, there are localized states¹³ for $d > 4$.

The principal results of this paper are as follows. (1) For $d > d^*$ where d^* is almost certainly 4, the density of localized and extended states goes to zero at the mobility edge as shown in Fig. 1(d). (2) When there is a nonvanishing average hopping, the universality class of the mobility transition depends on the strength of the random potential for $d > d^*$. For weakly random potentials, the field Ψ_i is critical, whereas for strongly random potentials all components Q_{ij} are critical. The weak-coupling case is in the same universality class as the band edge of a pure system and is described by the Gaussian renormalization-group fixed point. The strong-coupling case is described by the Gaussian renormalization-group fixed point for $d > d_c = 8$ and is in the same universality class as lattice animals¹⁴ for $d = 8 - \epsilon$ to first order in ϵ . The strong coupling case for $d < 8$ does not have scaling properties consistent with Wegner's⁵ conjectures because of the presence of a dangerous irrelevant variable¹⁵ leading to the violation of hyperscaling. We calculate the behavior of both the localized and extended densities of states near E_c and that of the conductivity in region of extended states.

Though the upper critical dimension for the mobility edge is not four, it has been clear for some time that dimension four plays some special role in this problem.^{13,16} This study shows that four is special in several respects. (1) The weak-coupling regime cannot exist for $d < 4$. (2) There are strong indications that for $d > 4$, the density

of states goes to zero at the mobility edge whereas for $d < 4$ it does not. When the density of states at the mobility edge is nonzero, not all components of Q_{ij} can be critical because the density of states is proportional to $\text{Im}Q$. Thus below four dimensions the mobility transition is in yet another universality class. Wegner¹⁷ and others¹⁸ have studied the mobility edge in $2 + \epsilon$ dimensions starting from a formalism very similar to ours. They do indeed find a different critical behavior than we do and one that agrees with the early scaling conjectures of Wegner.⁵ In two dimensions, these theories as well as calculations based on more conventional formalisms¹⁹ predict that all states are localized.

The model of an electron on a lattice with random hopping can be generalized to have n_0 orbitals per site. Wegner and collaborators have studied this model in the limit²⁰ $n_0 \rightarrow \infty$ and in a $1/n_0$ expansion.²¹ The $n_0 \rightarrow \infty$ limit corresponds identically to our mean-field theory for the same problem with $n_0 = 1$. The $1/n_0$ expansion predicts behavior for $2 < d < 4$ consistent with Wegner's scaling conjectures.⁵ We have verified that the same expansion yields non-mean-field behavior²² for $d < 8$ as does our ϵ expansion.

As this paper is quite lengthy, we have provided a Table of Contents. Section II provides an overview of the localization problem and attempts to make some of the results obtained by more sophisticated means seem plausible. Section III sets up the analytical language to be used in succeeding sections. Section IV contains an analysis of mean-field theory and is essential to an understanding of the behavior we find for $d < 8$. Section IV B contains the most important mean-field results. Section V presents the ϵ expansion. It is quite formal and may be skipped by those not interested in calculational details. Section VI deals with instantons and localized states. It is more heuristic than rigorous but is essential to an understanding of why we feel the density of both localized and extended states goes to zero at the mobility edge. Finally, Sec. VII discusses the relationship of this work to that in lower dimension and presents our conclusions.

II. HEURISTIC PICTURE OF LOCALIZATION AS A CRITICAL PHENOMENA PROBLEM

In this section we will first, in part A, define the various Green's functions needed to describe the transition between localized and extended behavior. In part B we will give a heuristic derivation of the mean-field values for the critical exponents for this transition. In part C we will discuss qualitatively the density of localized

states. Finally in part D we will discuss briefly the different models of randomness.

A. Definitions

The problem we consider is to describe the response of noninteracting excitations in a random lattice. We treat the Hamiltonian

$$\mathcal{H}(\theta) = \sum_{\vec{x}, \vec{x}'} [V(\vec{x})\delta_{\vec{x}, \vec{x}'} + t(\vec{x}, \vec{x}')]c_{\vec{x}}^{\dagger}c_{\vec{x}'}, \quad (2.1a)$$

$$\equiv \sum_{\vec{x}, \vec{x}'} \mathcal{H}(\vec{x}, \vec{x}'; \theta) c_{\vec{x}}^{\dagger}c_{\vec{x}'}, \quad (2.1b)$$

where δ is the Kronecker delta and $c_{\vec{x}}^{\dagger}$ ($c_{\vec{x}}$) creates (destroys) an excitation at the lattice site $\vec{x}$. Here $V(\vec{x})$ and $t(\vec{x}, \vec{x}')$ are random variables with $t(\vec{x}, \vec{x}')$ constrained to be a real symmetric matrix, and the notation $\mathcal{H}(\theta)$ indicates that $\mathcal{H}$ depends on the configuration θ of the random variables $V(\vec{x})$ and $t(\vec{x}, \vec{x}')$.

For any θ we define the Green's functions $g_{\pm}(\vec{x}, \vec{x}'; E; \theta)$ by^{2(a), 23}

$$g_{\pm}(\vec{x}, \vec{x}'; E; \theta) = \sum_n \frac{\psi_n(\vec{x}; \theta) \psi_n(\vec{x}'; \theta)}{E_n(\theta) - (E \pm i\eta)}, \quad (2.2)$$

where $\psi_n(\vec{x}; \theta)$ and $E_n(\theta)$ are the eigenvectors and eigenvalues of

$$\mathcal{H}(\theta)\psi = E\psi \quad (2.3)$$

and η is a positive infinitesimal. Unless otherwise specified, the limit $\eta \rightarrow 0$ is understood for expressions having a finite limit. For the real valued potentials considered here the $\psi_n(\vec{x}; \theta)$ are chosen to be real. We have

$$\text{Im} g_{\pm}(\vec{x}, \vec{x}; E; \theta) = \pm \pi \sum_n |\psi_n(\vec{x}; \theta)|^2 \delta(E - E_n(\theta)) \quad (2.4a)$$

$$\equiv \pm \pi \rho(E; \vec{x}; \theta), \quad (2.4b)$$

where $\rho(E; \vec{x}; \theta)$ is the local density of states at site $\vec{x}$ in the configuration θ .

In a random system we are interested in positional averages, since they are usually the observed quantities. However, positional averages are usually equivalent to configurational averages.²⁴ Thus, for example, one may write

$$\frac{1}{N} \sum_{\vec{x}} g_{\pm}(\vec{x}, \vec{x} + \vec{y}; E; \theta) = [g_{\pm}(0, \vec{y}; E; \theta)]_{\text{av}}, \quad (2.5)$$

where $[\]_{\text{av}}$ indicates an average over θ and N is the number of sites in the lattice. Equalities of the averages over space with those over θ can easily be proved for short-range interactions of the type we consider.²⁴

We are interested in the following averaged properties:

$$G_o(\vec{x}, \vec{x}'; E) \equiv [g_o(\vec{x}, \vec{x}'; E; \theta)]_{\text{av}}, \quad (2.6)$$

$$\rho(E) = \sigma \pi^{-1} \text{Im} G_o(\vec{x}, \vec{x}; E), \quad (2.7)$$

where σ assumes the values $+$ and $-$. As will become clear later on, we will also wish to consider the "two-particle" Green's functions

$$g_{\sigma\sigma'}(\vec{x}, \vec{x}'; E_{\sigma}, E_{\sigma'}) = [g_{\sigma}(\vec{x}, \vec{x}'; E_{\sigma}; \theta) g_{\sigma'}(\vec{x}, \vec{x}'; E_{\sigma'}; \theta)]_{\text{av}} \quad (2.8)$$

and

$$\begin{aligned} g_{\sigma\sigma}(\vec{x}, \vec{x}'; E) &= 2g_{\sigma\sigma}(\vec{x}, \vec{x}'; E, E) - G_o(\vec{x}, \vec{x}'; E)^2 \\ &\quad + [g_{\sigma}(\vec{x}, \vec{x}; E; \theta) g_{\sigma}(\vec{x}', \vec{x}'; E; \theta)]_{\text{av}}. \end{aligned} \quad (2.9)$$

These configurationally averaged Green's functions are obviously translationally invariant in the sense that $f(\vec{x}, \vec{x} + \vec{y})$ is independent of $\vec{x}$. One can introduce two correlation lengths ξ_1 and ξ_2 associated with the spatial extent of phase coherence and amplitude coherence, respectively, of wave functions in the random medium. The single-particle properties, e.g., G and ρ , reflect anomalies in ξ_1 , whereas the two-particle functions g reflect anomalies in ξ_2 . This concept will be explored in more detail in the next section. The spatial Fourier transforms are defined generally by the generic relation

$$f(\vec{q}) = \sum_{\vec{x}, \vec{x}'} e^{i\vec{q} \cdot (\vec{x} - \vec{x}')} f(\vec{x}, \vec{x}') \quad (2.10)$$

as, for example,

$$g_{\sigma\sigma'}(\vec{q}; E_{\sigma}, E_{\sigma'}) = \sum_{\vec{x}, \vec{x}'} e^{i\vec{q} \cdot (\vec{x} - \vec{x}')} g_{\sigma\sigma'}(\vec{x}, \vec{x}'; E_{\sigma}, E_{\sigma'}), \quad (2.11)$$

and so forth.

The one-particle and two-particle Green's functions are related by a Ward identity.²⁵ We temporarily continue the energy variables to complex values, z and z' , and write

$$g_{\sigma\sigma'}(0; z, z') = \sum_{\vec{x}} g_{\sigma\sigma'}(\vec{x}, \vec{x}'; z, z') \quad (2.12a)$$

$$= \left[\sum_{\vec{x}, n, m} \frac{\psi_n(\vec{x}; \theta) \psi_n(\vec{x}'; \theta) \psi_m(\vec{x}; \theta) \psi_m(\vec{x}'; \theta)}{[E_n(\theta) - z][E_m(\theta) - z']} \right]_{\text{av}} \quad (2.12b)$$

$$\begin{aligned} &= N^{-1} (z - z')^{-1} \\ &\quad \times \sum_n \left[[E_n(\theta) - z]^{-1} - [E_n(\theta) - z']^{-1} \right]_{\text{av}} \end{aligned} \quad (2.12c)$$

$$= (z - z')^{-1} [G_o(\vec{x}, \vec{x}; z) - G_o(\vec{x}, \vec{x}; z')]. \quad (2.12d)$$

Two cases of interest of this relation are found by setting $z = E_1 + i\sigma\eta$ and $z' = E_2 + i\sigma'\eta$, and allowing $E_1 - E_2$ to go to zero. Then

$$g_{\sigma,\sigma}(0; E, E) = \frac{\partial G_{\sigma}(\vec{x}, \vec{x}; E)}{\partial E} \quad (2.13)$$

and

$$g_{\sigma,-\sigma}(0; E, E) = \frac{\sigma\pi\rho(E)}{\eta} + \frac{\partial G'_{\sigma}(\vec{x}, \vec{x}; E)}{\partial E} \quad (2.14)$$

where G'_{σ} is the real part of G_{σ} . Thus, where $\rho(E) \neq 0$, one sees that $\eta g_{\sigma,-\sigma}(0; E, E)$ is nonzero as $\eta \rightarrow 0$, whereas when $\rho(E) = 0$, $g_{\sigma,-\sigma}(0; E, E)$ remains finite as $\eta \rightarrow 0$. This behavior is reminiscent of the transverse susceptibility of a Heisenberg system: In that case the transverse susceptibility and the longitudinal susceptibility are indistinguishable and finite for $T > T_c$ where the order parameter M van-

ishes. For $T < T_c$ the transverse susceptibility $\chi_{\perp}$ is of order M/q^2 , where q is the wave vector. Thus for $q \rightarrow 0$, $\chi_{\perp}$ diverges. The analogy with Eq. (2.14) is even closer if one investigates the frequency dependence of g , because then $\chi_{\perp}(\omega, q=0) = M/\omega$, so that $\rho(E)$ is seen to be analogous to M and η to ω . From this discussion it is clear that it is essential to treat $\rho(E)$ correctly. In the work of Aharony and Imry^{9(b)} the first term of Eq. (2.14) is not correctly treated.

At this point we record the formula for the macroscopic conductivity Σ . The Kubo-Greenwood formula for Σ is

$$\Sigma_{ij}(\omega) = \delta_{ij} \Sigma(\omega) = \frac{\hbar\pi\omega^2 e^2}{2\Omega} \sum_{\vec{x}, \vec{x}'} (\vec{x} - \vec{x}')_i (\vec{x} - \vec{x}')_j [\langle \vec{x} | \delta(\mathcal{H} - E - \omega) | \vec{x}' \rangle \langle \vec{x}' | \delta(\mathcal{H} - E) | \vec{x} \rangle]_{av}. \quad (2.15)$$

We can express this result in terms of Green's functions as

$$\Sigma(\omega) = \frac{\hbar\omega^2 e^2}{4\pi} \left(\frac{N}{\Omega} \right) \lim_{q \rightarrow 0} \frac{\partial}{\partial q^2} \sum_{\sigma\sigma'} \sigma\sigma' g_{\sigma\sigma'}(q; E + \omega, E). \quad (2.16)$$

B. Heuristic discussion of critical exponents

In this part we will give some simple arguments from which one can deduce the mean-field values of the critical exponents for localization.

For simplicity we start by discussing the random hopping model, i.e., we set $V(\vec{x}) = 0$. For any configuration θ the eigenfunctions satisfy

$$\sum_{\delta} t(\vec{x}, \vec{x} + \vec{\delta}) \psi_n(\vec{x} + \vec{\delta}; \theta) = E_n(\theta) \psi_n(\vec{x}; \theta), \quad (2.17)$$

FIG. 2. Two configurations, θ_1 and θ_2 , of the nearest-neighbor random bond model used to calculate $G_{\sigma}(\vec{x}, \vec{x}'; E)$. The values of $t(\vec{x}, \vec{x}')$ are $+t_0$ and $-t_0$ as indicated by + or -, respectively. The t 's enclosed by the solid curve are those cut by a plane separating $\vec{x}$ from $\vec{x}'$, and are the ones whose signs are changed to obtain θ_2 from θ_1 .

where $\vec{\delta}$ is a nearest-neighbor vector. Suppose now that each t independently assumes the values $+t_0$ and $-t_0$. One can easily see that $G_{\sigma}(\vec{x}, \vec{x}'; E) = 0$ for $\vec{x} \neq \vec{x}'$ as follows. Consider the two configurations θ_1 and θ_2 shown in Fig. 2. The configuration θ_2 is obtained from θ_1 by reversing the signs of the t 's along some plane separating $\vec{x}$ from $\vec{x}'$. It is clear from the form of Eq. (2.17) that

$$g_{\sigma}(\vec{x}, \vec{x}'; E; \theta_1) = -g_{\sigma}(\vec{x}, \vec{x}'; E; \theta_2), \quad \vec{x} \neq \vec{x}'. \quad (2.18)$$

Since any configuration θ_1 has such a partner θ_2 , it is clear that

$$G_{\sigma}(\vec{x}, \vec{x}'; E) = 0, \quad \vec{x} \neq \vec{x}'. \quad (2.19)$$

On the other hand, in general

$$G_{\sigma}(\vec{x}, \vec{x}; E) \neq 0. \quad (2.20)$$

This result is analogous to that for an Ising spin glass, where each coupling $J(\vec{x}, \vec{x}')$ can assume the values $+J$ and $-J$. There one has²⁶

$$[\langle \sigma(\vec{x})\sigma(\vec{x}') \rangle_T]_{av} = 0, \quad \vec{x} \neq \vec{x}' \quad (2.21a)$$

$$[\langle \sigma^2(\vec{x}) \rangle_T]_{av} = 1, \quad (2.21b)$$

where $\langle \rangle_T$ denotes a thermal average for fixed θ and $[\]_{av}$ again denotes an average over θ . For the spin glass, it is known that the correlation function which becomes long ranged at the ordering temperature is^{26,27} $[\langle \sigma(\vec{x})\sigma(\vec{x}') \rangle_T^2]_{av}$. Clearly, the analogous quantity here which plays the role of the order-parameter susceptibility is

$$\chi(\vec{x}, \vec{x}') = g_{\sigma,\sigma}(\vec{x}, \vec{x}'; E). \quad (2.22)$$

Since we are considering the case with $\rho = 0$, we can ignore the σ, σ' subscripts [cf. Eqs. (2.13) and (2.14)]. We expect

$$\chi(0) = \sum_{\vec{x}'} \chi(\vec{x}, \vec{x}') \quad (2.23)$$

to diverge as E approaches the mobility edge where the wave functions begin to become extended.

One standard way to find the mean-field form for $\chi(0)$ is to consider the uniform coupling model.²⁸ Accordingly, we study the model for which each i independently obeys

$$[t(\vec{x}, \vec{x}')]_{av} = 0, \quad (2.24a)$$

$$[t(\vec{x}, \vec{x}')^2]_{av} = \sigma_0^2/N, \quad (2.24b)$$

where the factor N^{-1} in Eq. (2.24b) is inserted to maintain a finite bandwidth in the limit $N \rightarrow \infty$. As a result of the infinite-range couplings in this model, all states are extended, and as is well known, the density of states is given by the semicircular distribution²⁹

$$\rho_0(E) = \begin{cases} (2/\pi E_c^2)(E_c^2 - E^2)^{1/2}, & |E| < E_c \\ 0, & |E| > E_c \end{cases} \quad (2.25a)$$

$$(2.25b)$$

where $E_c = 2\sigma_0$. We have, from Eqs. (2.13) and (2.14), that for $E^2 > E_c^2$

$$\chi(0) = \frac{\partial G}{\partial E} = \left[\frac{1}{N} \sum_n \frac{1}{(E_n - E)^2} \right]_{av} \quad (2.26a)$$

$$= \int_{-E_c}^{E_c} \frac{\rho_0(E')}{(E' - E)^2} dE'. \quad (2.26b)$$

Inserting Eq. (2.25) for $\rho_0(E')$ yields

$$\chi(0) = |E^2 - E_c^2|^{-1/2}, \quad (2.27)$$

so that $\gamma = \frac{1}{2}$ for localization within mean-field theory. The discussion at the end of part A of this section indicates that $\rho(E)$ should be interpreted as the order-parameter density. Thus Eq. (2.25) corresponds to $\beta = \frac{1}{2}$. An alternative way to obtain

FIG. 3. Schematic plot of the wave function, $\psi(x)$, for a one-dimensional sequence of random potential wells. It is clear that the phase of $\psi(x_2)$ relative to that of $\psi(x_1)$ depends on the number of oscillations in $\psi(x)$ and hence on the details of $V(x)$ for $x_1 < x < x_2$. Thus there is no long-range phase coherence in $\psi(x)$. In higher spatial dimension one can change the phase of $\psi(\vec{x}_2)$ by changing the potentials along a plane separating $\vec{x}_1$ and $\vec{x}_2$ in analogy with Fig. 1.

the mean-field values of γ and β is to analyze a nearest-neighbor model on the Cayley tree. This procedure is carried out in Appendix A and confirms the values $\gamma = \beta = \frac{1}{2}$.

In analogy with the spin glass one might expect a possibility of competition between ordering in $\mathcal{G}$ and ordering in G , the former being analogous to spin-glass order, the latter to ferromagnetic ordering.^{27,30} In fact, as we shall see, this type of competition can indeed occur. When the fluctuations in $t(\vec{x}, \vec{x}')$ are much larger than its average value, then one has the strong-coupling case discussed above. In the reverse limit one has single-particle ordering, i.e., $G_o(\vec{x}, \vec{x}'; E)$ becomes long ranged at the band edge. In addition, there will also be a higher-order critical point where both $G_o(\vec{x}, \vec{x}'; E)$ and $\mathcal{G}_{o,o'}(\vec{x}, \vec{x}'; E_o, E_{o'})$ show critical fluctuations.

We have carried out the discussion in terms of the random bond model because this model is the easiest one to treat. In particular, one can easily argue that $G_o(\vec{x}, \vec{x}'; E)$ might be short ranged for a random site potential, e.g., for the model

$$\mathcal{H}(\theta) = - \sum_{\vec{x}, \vec{x}'} t_0 \gamma_{\vec{x}, \vec{x}'} c_{\vec{x}}^\dagger c_{\vec{x}'} + \sum_{\vec{x}} V(\vec{x}) c_{\vec{x}}^\dagger c_{\vec{x}}, \quad (2.28)$$

where $\gamma_{\vec{x}, \vec{x}'} = 1$ if $\vec{x}$ and $\vec{x}'$ are nearest-neighboring sites and is zero otherwise. If one replaces this model by a continuum model with random square-well potentials, viz.,

$$\mathcal{H}(\vec{x}, \vec{x}'; \theta) = \left(- \frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}) \right) \delta(\vec{x} - \vec{x}'), \quad (2.29)$$

then one sees that the phase difference between widely separated points depends on the details of the intervening $V(\vec{x})$, as is illustrated in Fig. 3. This averaging over $V(\vec{x})$ introduces a phase incoherence that prevents long-range correlations in $G_o(\vec{x}, \vec{x}'; E)$. This argument is in fact not conclusive. If the variation in the potential is small enough, and if the spatial dimensionality, d , is large enough, this phase incoherence does not take place. This regime is the weak-coupling case mentioned above. As we shall see, this regime exists only for $d > 4$.

C. Localized states

Here we present some variational arguments and estimates for the density of localized states. First we consider the random site model of Eq. (2.28) where $V(\vec{x})$ obeys a Gaussian probability distribution

$$P(V) = (2\pi\sigma_s^2)^{-1/2} \exp(-V^2/2\sigma_s^2). \quad (2.30)$$

An exact variational result can be obtained by writing the Hamiltonian in the form

$$\mathcal{H} = \sum_{\vec{x}} [V(\vec{x}) + z t_0] c_{\vec{x}}^\dagger c_{\vec{x}} + W \quad (2.31)$$

$$\equiv \mathcal{H}^0 + W, \quad (2.32)$$

where z is the coordination number of the lattice and

$$W = -\frac{1}{2} t_0 \sum_{\vec{x}, \vec{x}'} \gamma_{\vec{x}, \vec{x}'} (c_{\vec{x}}^\dagger - c_{\vec{x}'}^\dagger) (c_{\vec{x}} - c_{\vec{x}'}). \quad (2.33)$$

In this decomposition it is clear that W is a non-positive operator. As a result, if the eigenvalues of $\mathcal{H}^0$ and $\mathcal{H}$ are labeled $E_1^0 \leq E_2^0 \leq E_3^0 \leq \dots$ and $E_1 \leq E_2 \leq E_3 \leq \dots$, respectively, then

$$E_k \leq E_k^0 \quad (2.34)$$

and consequently

$$\int_{-\infty}^E \rho(E') dE' \geq \int_{-\infty}^{E-zt_0} P(V) dV. \quad (2.35)$$

This inequality tells us that even for arbitrarily high spatial dimensionality there is a finite fraction of states below any finite energy. Thus, the mean-field distribution for $\rho(E)$ can not properly describe the case of finite coordination number z with a Gaussian distribution of site energies.

We can determine the asymptotic form of $\rho(E)$ for large $|E|$ by a very simple construction. For large $|E|$ the existence of an energy level depends on a single site $\vec{x}$ having $V(\vec{x}) \sim E$. The probability for this to happen is² clearly of order $e^{-E^2/2\sigma_s^2}$. To this order we may assume that the neighbors of site $\vec{x}$ have $V(\vec{x}) \sim 0$. Thus, the localized state at $\vec{x}$ has energy E approximately given by perturbation theory as

$$E = V(\vec{x}) \left(1 + \frac{z t_0^2}{V^2(\vec{x})} \right). \quad (2.36)$$

To obtain a given value of E it is clear that $V(\vec{x})$ must assume the value

$$V(\vec{x}) = E \left(1 - \frac{z t_0^2}{E^2} \right), \quad (2.37)$$

and the probability for this to happen is

$$P(V(\vec{x})) = (2\pi\sigma_s^2)^{-1/2} \exp \left[-\frac{E^2}{2\sigma_s^2} \left(1 - \frac{z t_0^2}{E^2} \right)^2 \right] \quad (2.38)$$

so that

$$P(E) \sim (2\pi\sigma_s^2)^{-1/2} \exp \left(-\frac{E^2}{2\sigma_s^2} + \frac{z t_0^2}{\sigma_s^2} \right). \quad (2.39)$$

The fact that neighboring sites $\vec{x}'$ have $V(\vec{x})$ of order σ_s^2 would induce corrections in which E^2 in Eq. (2.37) is replaced by $E^2[1 + O(\sigma_s^2/E^2)]$, which does not affect the result of Eq. (2.39). Likewise, the contributions to $\rho(E)$ from clusters of s sites having energy near E would probably be of order $e^{-sE^2/2\sigma_s^2}$, but at any rate would be

dominated by the contribution in Eq. (2.39). The result, Eq. (2.39), was obtained in three dimensions by Halperin and Lax.^{10(b)} We stress here that it is valid in all dimensions.

For the bond case we assume each nearest neighbor t to obey a probability distribution

$$P(t) = (2\pi\sigma_b^2)^{-1/2} \exp(-t^2/2\sigma_b^2). \quad (2.40)$$

If one bond has an anomalously large value of t , denoted T , and is surrounded by neighboring t 's, denoted $t_{\vec{g}}$, assumed to be much smaller, then to leading order in T the resulting energy levels are

$$E = \pm T. \quad (2.41)$$

In analogy with Eq. (2.36) this leads to a contribution to $P(E)$ of the form

$$P(E) = (2\pi\sigma_b^2)^{-1/2} A \exp(-E^2/2\sigma_b^2), \quad (2.42)$$

where A is a prefactor independent of E . However, unlike the random-site case, other configurations must be considered. In particular, those in which a central site is surrounded by b occupied bonds with $2 \leq b \leq z$ also contribute significantly to $P(E)$. [These configurations are shown in Fig. 8(b) below.] If the b bonds have t 's equal to $T_1, T_2, \dots, T_b$, then one has to leading order in T_k

$$E = \pm \left(\sum_{k=1}^b T_k^2 \right)^{1/2} \equiv \pm T, \quad (2.43)$$

which leads to the result

$$P(E) = \int \prod_{k=1}^b [P(T_k) dT_k] \delta(E - T) \quad (2.44a)$$

$$= K_b (2\pi/\sigma_b^2)^{b/2} E^{b-1} \exp(-E^2/2\sigma_b^2), \quad (2.44b)$$

where $K_b^{-1} = 2^{b-1} \pi^{b/2} \Gamma(b/2)$. Owing to the prefactor of order E^{b-1} , the dominant contribution to $P(E)$ comes from the configuration in which a central site is surrounded by z bonds. For this configuration the unperturbed eigenfunction of interest is

$$\Psi = \frac{1}{\sqrt{2}} \left(|0\rangle + \sum_{\delta} (T_{\delta}/T) |\delta\rangle \right), \quad (2.45)$$

where $|0\rangle$ and $|\delta\rangle$ are excitations on the central site and on the neighboring site δ , respectively. For this configuration the perturbation result analogous to Eq. (2.36) is

$$E = \pm T \left(1 + \frac{1}{2T^4} \sum_{\vec{\delta}_1, \vec{\delta}_2} \sum_{\vec{x}} T_{\vec{\delta}_1} T_{\vec{\delta}_2} t_0(\vec{\delta}_1, \vec{x}) t_0(\vec{\delta}_2, \vec{x}) \right). \quad (2.46)$$

It is easy to see that this leads to a result of the form

$$P(E) = C_0 E^{z-1} \exp(-E^2/2\sigma_b^2). \quad (2.47)$$

However, since the constant C_0 is algebraically quite complicated and does not depend on E , we will not consider it further.

The above states are well localized in the vicinity of a single site and can be discussed perturbatively as we have done. There is another class of localized states which exist on a length scale larger than that of the lattice and which are the only ones that exist in the continuum model with no spatial correlations in $V(x)$. These states have been studied by Zittartz and Langer, Halperin and Lax, and others.¹⁰ For large $(-E)$ and $d < 4$ they find a density of states of the form

$$\ln \rho \sim -(-E)^{2-d/2}. \quad (2.48)$$

The crossover from the behavior of Eq. (2.39) to that of Eq. (2.48) in three dimensions was studied by Halperin and Lax.^{10(b)} We will see in Sec. VI that both types of localized states emerge from an analysis of instantons of finite action in the field theories presented in the next section.

D. Overview of various models

In this part we will discuss qualitatively the density of states of various models. First we make some observations about the nonrandom models, in particular the tight-binding model. For this model the band energies are given by

$$E(\vec{k}) = 2t_0 \sum_{i=1}^d \cos k_i, \quad (2.49)$$

and we will set $t_0 = \frac{1}{2}$ for simplicity. The associated pure-system Green's functions are

$$G_{\pm}(0, 0; E) = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{\sum_{i=1}^d \cos k_i - E \mp i\eta}. \quad (2.50)$$

Using the standard relations for the Bessel function $J_0(x)$, one writes Eq. (2.50) as

$$G_{\pm}(0, 0; E) = i \int_0^{\infty} e^{iEt - \eta t} [J_0(t)]^d dt. \quad (2.51)$$

The familiar result for $\rho(E)$ for $d=3$ is shown in Fig. 4(a). Here we note the asymptotic form for $\rho(E)$ as $d \rightarrow \infty$. A steepest-descent evaluation for the hypercubic lattice yields the density of states $\rho_{HC}(E)$:

$$\sqrt{z} \rho_{HC}(E) = (2/\pi)^{1/2} e^{-2(E/\sqrt{z})^2}. \quad (2.52)$$

Corrections to this result are of relative order E^4/z^3 , so that this form is valid for $E \ll z^{3/4}$. [Of course, $\rho_{HC}(E) = 0$ for $|E| > \frac{1}{2}z$.] It is interesting to note that asymptotically as $z \rightarrow \infty$ all states for the hypercubic lattice have energies bounded by

FIG. 4. Density of states for various non random models. (a) The tight-binding model ($\sigma_s = \sigma_b = 0$) with $t_0 = \frac{1}{2}$ for $d=3$. (b) The tight-binding model in the asymptotic limit $2d = z \rightarrow \infty$. The curve labeled HC is that for the hypercubic lattice, Eq. (2.52), and that labeled C is that for the Cayley tree, Eq. (2.53).

$\pm \text{const} \times z^{1/2}$. In the limit $z \rightarrow \infty$ the density of states for the nonrandom (or bond random) Cayley tree, denoted ρ_c , obtained in Appendix A becomes

$$\sqrt{z} \rho_c(E) = \frac{2}{\pi} \left[1 - \left(\frac{E}{\sqrt{z}} \right)^2 \right]^{1/2} \quad (2.53)$$

and these results are displayed in Fig. 4(b).

The density of states and mobility edges for some of the random models we have introduced are shown in Fig. 1, and we will discuss these results briefly here.

First consider the nearest-neighbor bond model obtained from Eq. (2.1) by setting $V(\vec{x}) = 0$ and taking $t(\vec{x}, \vec{x}')$ to be nonzero only if $\vec{x}'$ is a nearest neighbor of $\vec{x}$. The density of states for this model can be seen to be an even function of E as follows. Divide the lattice into two sublattices a and b such that all nearest neighboring sites to sites of sublattice a are in sublattice b and vice versa. Let $\psi_n(x; \theta)$ and $E_n(\theta)$ be a solution to Eq. (2.17). Then consider $\Omega_n(x; \theta)$ obtained as

$$\Omega_n(\vec{x}; \theta) = \psi_n(\vec{x}; \theta), \quad \vec{x} \in a \quad (2.54)$$

$$\Omega_n(\tilde{x}; \theta) = -\psi_n(\tilde{x}; \theta), \quad \tilde{x} \in \underline{b}. \quad (2.55)$$

Clearly, $\Omega_n(x; \theta)$ satisfies

$$\sum_{\tilde{\delta}} t(\tilde{x}, \tilde{x} + \tilde{\delta}) \Omega_n(\tilde{x} + \tilde{\delta}; \theta) = -E_n(\theta) \Omega_n(\tilde{x}; \theta). \quad (2.56)$$

Thus $\rho(E) = \rho(-E)$. This argument can be extended to show that

$$g_{\sigma, \sigma'}(\tilde{x}, \tilde{x}'; E, E') = g_{\sigma, \sigma'}(\tilde{x}, \tilde{x}'; -E, -E')^*. \quad (2.57)$$

Thus, if there is a mobility edge at $E = E_c$, there is also one at $E = -E_c$.

A similar argument shows that for the nearest-neighbor site model for which $t(\tilde{x}, \tilde{x}') = -t_0 \gamma_{\tilde{x}, \tilde{x}'}$ and $V(\tilde{x})$ obeys a probability distribution which is an even function of E , the above result also holds: $\rho(E) = \rho(-E)$ and a mobility edge at $E = E_c$ implies the existence of one at $E = -E_c$.

The commonly held view is that $\rho(E)$ is smooth at the mobility edge. This situation is shown in Fig. 1(c). Our results indicate that for high d , $\rho(E) = 0$ at the mobility edge as shown in Fig. 1(d).

Finally, one can consider the continuum-site model of Eq. (2.29). For this model there are extended states for all energies above a critical value. The cases of low and high d are shown in Figs. 1(a) and 1(b), respectively.

For the symmetric models any treatment of $E < 0$ will, of course, hold with trivial modifications for the case $E > 0$, and we shall often display this symmetry explicitly. However, to fix our ideas we will concentrate on the mobility edge at the lowest (most negative) energy. Since the renormalization group is most naturally phrased in terms of long-wavelength excitations, we will adopt the convention that

$$[t(\tilde{x}, \tilde{x}')]_{av} = -t_0 \gamma_{\tilde{x}, \tilde{x}'} \equiv -t_0(\tilde{x}, \tilde{x}'), \quad (2.58)$$

so that the mobility edge at $-E_c$ corresponds to zero wave vector. Energies $E \leq -E_c$ will be referred to as "below the mobility edge," or as "in the localized regime."

III. DEVELOPMENT OF A FIELD THEORY

A. Replicated generating functions

In this section, we will develop a field-theoretic³¹ generating function for the ensemble-averaged Green's functions introduced in the preceding section. Our starting point is the representation of a matrix inverse in terms of a Gaussian integral:

$$\left\langle \tilde{x} \left| \frac{1}{-i(E - \mathcal{H})} \right| \tilde{x}' \right\rangle = \frac{1}{Z_+} \int \mathcal{D}\psi_+ e^{-L_+^0[\psi_+(\tilde{x})]} \psi_+(\tilde{x}) \psi_+(\tilde{x}'), \quad (3.1a)$$

$$\left\langle \tilde{x} \left| \frac{1}{i(E - \mathcal{H})} \right| \tilde{x}' \right\rangle = \frac{1}{Z_-} \int \mathcal{D}\psi_- e^{-L_-^0[\psi_-(\tilde{x})]} \psi_-(\tilde{x}) \psi_-(\tilde{x}'), \quad (3.1b)$$

where

$$L_{\pm}^0[\psi(\tilde{x})] = \mp \frac{1}{2} i \sum_{\tilde{x}, \tilde{x}'} \psi(\tilde{x}) [E \delta_{\tilde{x}, \tilde{x}'} - \mathcal{H}(\tilde{x}, \tilde{x}')] \psi(\tilde{x}') \quad (3.2a)$$

$$\equiv L_{\pm}^0[\psi(\tilde{x}), E], \quad (3.2b)$$

$$Z_{\pm} = \int \mathcal{D}\psi_{\pm} e^{L_{\pm}^0[\psi_{\pm}(\tilde{x})]} \quad (3.3)$$

and the integration over L_+ (L_-) is defined for E in the upper (lower) half complex E plane where the integrals converge. Here and below $\int \mathcal{D}\psi$ denotes a functional integral. For example, $\int \mathcal{D}\psi_+ = \prod_{\tilde{x}} \int d\psi_+(\tilde{x})$, where the integrations run along the real axis as shown in Fig. 5(a).

It is convenient to rotate the paths of integration for $\psi_+(\tilde{x})$ and $\psi_-(\tilde{x})$ into the complex plane via the transformation

$$\psi_{\pm}(\tilde{x}) \rightarrow e^{\mp i\pi/4} \psi_{\pm}(\tilde{x}) \quad (3.4)$$

as is shown in Fig. 5(b). This transformation eliminates the sign difference between L_+^0 and L_-^0 and the factor of i from both. The Green's func-

FIG. 5. Contours of integration $\psi_{\pm}$. (a) The original contours in Eqs. (3.1)–(3.3) along the real axis. (b) The rotated contours obtained using Eq. (3.4). (c) The deformed contour for ψ_+ . The deformed contour for ψ_- is obtained by reflection about the real axis. The part of the contour along the real axis gives the dominant contribution to $\text{Re } \Xi$ and its evaluation leads to perturbation theory and the ϵ expansion treated in Sec. V. The curved contours give the dominant contribution to $\text{Im } \Xi$ and are evaluated in Sec. VI where we treat localized states.

tions are then expressed as expectation values of products of the fields $\psi_+(\vec{x})$ and $\psi_-(\vec{x})$ with respect to the Lagrangian

$$L_{+-}^0[\psi_+, \psi_-; E_+, E_-] = L_{+-}^0[\psi_+, E_+] + L_{+-}^0[\psi_-, E_-] \quad (3.5a)$$

$$\equiv L_{+-}^0[\psi_+, E_+]. \quad (3.5b)$$

Thus we have

$$g_{\pm}(\vec{x}, \vec{x}'; E_{\pm}; \theta) = \left\langle \vec{x} \left| \frac{1}{\mathcal{H}(\theta) - E_{\pm}} \right| \vec{x}' \right\rangle \quad (3.6a)$$

$$= \frac{1}{Z} \int \mathcal{D}\psi_+ \mathcal{D}\psi_- \psi_{\pm}(\vec{x}) \psi_{\pm}(\vec{x}') e^{-L_{+-}^0[\psi_{\pm}, E_{\pm}]}, \quad (3.6b)$$

where

$$Z = \int \mathcal{D}\psi_+ \mathcal{D}\psi_- e^{-L_{+-}^0[\psi_{\pm}, E_{\pm}]}. \quad (3.7)$$

In order to calculate averages of quantities over the ensemble of random potentials, we use the familiar replica trick.^{26,28,31(c),32} Let $i = (\alpha, \sigma)$, $\alpha = 1, 2, \dots, n$, and $\sigma = \pm 1$ be a $2n$ -component replica index and introduce the replicated field $\Psi_i(\vec{x})$ by

$$\Psi_i(\vec{x}) = \psi_{\sigma}^{\alpha}(\vec{x}). \quad (3.8)$$

Then we have

$$Z^n = \int \mathcal{D}\tilde{\Psi} e^{-L_n^0[\tilde{\Psi}; E_+, E_-]}, \quad (3.9)$$

where now $\mathcal{D}\tilde{\Psi}$ denotes integration over all $\Psi_i(\vec{x})$ and

$$\begin{aligned} L_n^0[\tilde{\Psi}; E_+, E_-] &= \sum_{\alpha=1}^n L_{+-}^0[\psi_{\pm}^{\alpha}; E_+, E_-] \\ &= -\frac{1}{2} \sum_{\vec{x}, \vec{x}'} \tilde{\Psi}^T(\vec{x}) [\underline{E} \delta_{\vec{x}, \vec{x}'} - \mathcal{H}(\vec{x}, \vec{x}') \underline{1}] \tilde{\Psi}(\vec{x}'), \end{aligned} \quad (3.10)$$

where matrix multiplication in the $2n$ -dimensional space defined by the index i is understood where applicable, $\underline{1}$ is the $2n \times 2n$ unit matrix, and $\underline{E}$ is the $2n \times 2n$ diagonal energy matrix

$$E_{ij} = E_{\sigma} \delta_{\alpha\beta} \delta_{\sigma\sigma'}, \quad i = (\alpha, \sigma), \quad j = (\beta, \sigma'). \quad (3.11)$$

Quantities of interest are obtained from $[Z^n]_{av}$ in the limit $n \rightarrow 0$. Since bond and site potentials are independent random variables, we have

$$\begin{aligned} &\left[\exp \left(-\frac{1}{2} \sum_{\vec{x}, \vec{x}'} \tilde{\Psi}^T(\vec{x}) \mathcal{H}(\vec{x}, \vec{x}') \tilde{\Psi}(\vec{x}') \right) \right]_{av} \\ &= \prod_{\vec{x}} \left[\exp \left(-\frac{1}{2} \tilde{\Psi}^T(\vec{x}) V(\vec{x}) \tilde{\Psi}(\vec{x}) \right) \right]_{av} \\ &\quad \times \prod_{(\vec{x}, \vec{x}')} \left[\exp \left(-t(\vec{x}, \vec{x}') \tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x}') \right) \right]_{av}, \end{aligned} \quad (3.12)$$

where $\langle \vec{x}, \vec{x}' \rangle$ denotes the bond connecting sites $\vec{x}$ and $\vec{x}'$. Equation (3.12) can be expressed in terms

of cumulants of the random variables $V(x)$ and $t(\vec{x}, \vec{x}')$ via

$$[e^{\lambda y}]_{av} = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n!} y^n C_n(\lambda) \right), \quad (3.13)$$

where $C_n(\lambda)$ is the n th cumulant of λ . We will always choose $[V(\vec{x})]_{av}$ to be zero. For a Gaussian random $V(\vec{x})$ with variance σ_s^2 we have

$$[\exp \left(-\frac{1}{2} \tilde{\Psi}^T(\vec{x}) V(\vec{x}) \tilde{\Psi}(\vec{x}) \right)]_{av} = \exp \left\{ \frac{1}{8} \sigma_s^2 [\tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x})]^2 \right\}. \quad (3.14)$$

We wish to consider cases in which the average value of the hopping $t(\vec{x}, \vec{x}')$ is nonzero. As noted in Sec. II D, we will set the average value of $t(\vec{x}, \vec{x}')$ to be $-t_0 \gamma_{\vec{x}, \vec{x}'}$. For $t(\vec{x}, \vec{x}')$ having a Gaussian distribution about this average value we have

$$\begin{aligned} &[\exp \left(-t(\vec{x}, \vec{x}') \tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x}') \right)]_{av} \\ &= \exp [t_0(\vec{x}, \vec{x}') \tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x}')] \\ &\quad \times \exp \left[\frac{1}{2} \sigma_b^2 \gamma_{\vec{x}, \vec{x}'} [\tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x}')]^2 \right]. \end{aligned} \quad (3.15)$$

We will now concentrate on Gaussian randomness in both $V(\vec{x})$ and $t(\vec{x}, \vec{x}')$ and express the average generating function in three different ways. The first follows directly from Eqs. (3.10) through (3.15):

$$\Xi \equiv [Z^n]_{av} = \int \mathcal{D}\tilde{\Psi} e^{-L[\tilde{\Psi}; E]}, \quad (3.16)$$

where

$$\begin{aligned} L[\tilde{\Psi}; E] &= -\frac{1}{2} \sum_{\vec{x}, \vec{x}'} \tilde{\Psi}^T(\vec{x}) [\underline{E} \delta_{\vec{x}, \vec{x}'} + \underline{t}_0(\vec{x}, \vec{x}') \underline{1}] \tilde{\Psi}(\vec{x}') \\ &\quad - \frac{1}{8} \sigma_s^2 \sum_{\vec{x}} [\tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x})]^2 \\ &\quad - \frac{1}{4} \sigma_b^2 \sum_{\vec{x}, \vec{x}'} \gamma_{\vec{x}, \vec{x}'} [\tilde{\Psi}^T(\vec{x}) \tilde{\Psi}(\vec{x}')]^2 \end{aligned} \quad (3.17)$$

and we set $\underline{t}_0(x, x') = t_0(x, x') \underline{1}$. As is well known, the averaging process produces effective fourth-order potentials.³² The averaged Green's functions introduced in Sec. II can easily be expressed in terms of expectation values of $\Psi_i(\vec{x})$. In particular, we have

$$G_{\pm}(\vec{x}, \vec{x}'; E) = \langle \psi_{\pm}^{\alpha}(\vec{x}) \psi_{\pm}^{\alpha}(\vec{x}') \rangle, \quad (3.18)$$

$$g_{\alpha, \sigma}(\vec{x}, \vec{x}'; E_{\sigma}, E_{\sigma'}) = \langle \psi_{\sigma}^{\alpha}(\vec{x}) \psi_{\sigma}^{\beta}(\vec{x}) \psi_{\sigma'}^{\alpha}(\vec{x}') \psi_{\sigma'}^{\beta}(\vec{x}') \rangle \quad (i \neq j), \quad (3.19)$$

$$\begin{aligned} g_{\sigma, 0}(\vec{x}, \vec{x}'; E) &= \langle [\psi_{\sigma}^{\alpha}(\vec{x}) \psi_{\sigma}^{\alpha}(\vec{x}')]^2 \rangle \\ &\quad - \langle [\psi_{\sigma}^{\alpha}(\vec{x})]^2 \rangle \langle [\psi_{\sigma}^{\alpha}(\vec{x}')]^2 \rangle, \end{aligned} \quad (3.20)$$

where $i \neq j$ indicates that Eq. (3.19) does not apply if $\alpha = \beta$ and $\sigma = \sigma'$.

As argued in Sec. II, one expects singularities associated with the mobility edge to appear in g

but not necessarily in G . This would indicate that $\Psi_i(\vec{x})\Psi_j(\vec{x})$ rather than $\Psi_i(\vec{x})$ is the "order parameter" associated with the mobility transition. Therefore we introduce a change of variables that allows us to focus on this quantity. The Hubbard-Stratanovich³³ identity leads to the relation

$$\exp\left(\frac{1}{4} \sum_{\vec{x}, \vec{x}'} \sum_{i,j} A(\vec{x}, \vec{x}') \Psi_i(\vec{x}) \Psi_j(\vec{x}) \Psi_i(\vec{x}') \Psi_j(\vec{x}')\right) = \int \mathcal{D}\underline{Q} \exp\left(-\frac{1}{4} \sum_{\vec{x}, \vec{x}'} A^{-1}(\vec{x}, \vec{x}') \text{Tr} \underline{Q}(\vec{x}) \underline{Q}(\vec{x}') + \frac{1}{2} \sum_{\vec{x}} \tilde{\Psi}^T(\vec{x}) \underline{Q}(\vec{x}) \tilde{\Psi}(\vec{x})\right), \quad (3.21)$$

where $\underline{Q}(\vec{x}) = Q_{i,j}(\vec{x})$ is an $2n \times 2n$ component symmetric matrix, and the form factor for the cumulant of the random potential $A(\vec{x}, \vec{x}')$ in our case is given by

$$A(\vec{x}, \vec{x}') = \frac{1}{2} \sigma_s^2 \delta_{\vec{x}, \vec{x}'} + \sigma_b^2 \gamma_{\vec{x}, \vec{x}'}. \quad (3.22)$$

Using Eqs. (3.16) and (3.21) we obtain the second form for the generating function, viz.:

$$\Xi = \int \mathcal{D}\tilde{\Psi} \mathcal{D}\underline{Q} e^{-L[\tilde{\Psi}; \underline{Q}; \underline{E}]}, \quad (3.23)$$

where

$$L[\tilde{\Psi}; \underline{Q}; \underline{E}] = \frac{1}{4} \sum_{\vec{x}, \vec{x}'} A^{-1}(\vec{x}, \vec{x}') \text{Tr} [\underline{Q}(\vec{x}) \underline{Q}(\vec{x}')] - \frac{1}{2} \sum_{\vec{x}, \vec{x}'} \tilde{\Psi}^T(\vec{x}) \{[\underline{E} + \underline{Q}(\vec{x})] \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}')\} \tilde{\Psi}(\vec{x}'). \quad (3.24)$$

Correlation functions involving $\Psi_i(\vec{x})\Psi_j(\vec{x}')$ can be expressed in terms of $Q_{ij}(\vec{x})$. Using

$$\frac{\partial}{\partial Q_{ij}(\vec{x})} \left[\exp\left(\frac{1}{2} \sum_{\vec{x}} \tilde{\Psi}^T(\vec{x}) \underline{Q}(\vec{x}) \tilde{\Psi}(\vec{x})\right) \right] = \begin{cases} \Psi_i(\vec{x}) \Psi_j(\vec{x}) \exp\left(\frac{1}{2} \sum_{\vec{x}} \tilde{\Psi}^T(\vec{x}) \underline{Q}(\vec{x}) \tilde{\Psi}(\vec{x})\right), & i \neq j \\ \frac{1}{2} \Psi_i(\vec{x}) \Psi_j(\vec{x}) \exp\left(\frac{1}{2} \sum_{\vec{x}} \tilde{\Psi}^T(\vec{x}) \underline{Q}(\vec{x}) \tilde{\Psi}(\vec{x})\right), & i = j \end{cases} \quad (3.25)$$

and integrating by parts we obtain

$$G_{\pm}(\vec{x}, \vec{x}'; E) = [2A(q=0)]^{-1} \langle Q_{\pm\pm}^{\alpha\alpha}(\vec{x}) \rangle \quad (3.26)$$

and

$$\begin{aligned} \mathcal{G}_{+-}(\vec{x}, \vec{x}'; E_+, E_-) &= -\frac{1}{2} A^{-1}(\vec{x}, \vec{x}') \\ &+ \frac{1}{4} \sum_{\vec{x}_1, \vec{x}_2} A^{-1}(\vec{x}, \vec{x}_1) \langle Q_{+-}^{\alpha\alpha}(\vec{x}_1) Q_{+-}^{\alpha\alpha}(\vec{x}_2) \rangle \\ &\times A^{-1}(\vec{x}_2, \vec{x}'), \end{aligned} \quad (3.27)$$

where

$$A(\vec{q}) = \frac{1}{2} \sigma_s^2 + \sigma_b^2 \gamma(\vec{q}) \quad (3.28)$$

is the spatial Fourier transform of $A(\vec{x}, \vec{x}')$. Here and below all averages indicated by $\langle \rangle$ will be taken with respect to L in order to generate the averaged Green's functions of interest. Since we will be mainly interested in long-wavelength disturbances, the numerical factor by which $\Psi_i(\vec{x})\Psi_j(\vec{x})$ and $Q_{ij}(\vec{x})$ differ will often be ignored.

To obtain a Lagrangian describing the development of ordering in Q , we now consider integrating out the $\tilde{\Psi}$ variables. Since $L[\tilde{\Psi}; \underline{Q}; \underline{E}]$ is quadratic in $\tilde{\Psi}$, this integration can be done analytically. This is a consequence of the use of a Gaussian distribution for the random potentials. Other distributions would give rise to sixth and higher order terms in $\tilde{\Psi}$ making an analytic integration over $\tilde{\Psi}$ impossible. In that case integration over $\tilde{\Psi}(\vec{x})$

leads to an infinite series for $L(\underline{Q}; \underline{E})$ which differs from the form we obtain below by terms of higher order in $\underline{Q}$ which are irrelevant for the high spatial dimensionality we consider. Thus we claim to capture the essence of the problem with the Gaussian distribution. Performing the integration over $\tilde{\Psi}$, we find the third form for the generating function:

$$\Xi = \int \mathcal{D}\underline{Q} e^{-L[\underline{Q}; \underline{E}]}, \quad (3.29)$$

where

$$\begin{aligned} L[\underline{Q}; \underline{E}] &= \frac{1}{2} \text{Tr}_{\vec{x}, \vec{x}'} \{ \ln \{ [\underline{E} + \underline{Q}(\vec{x})] \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}') \} \} \\ &+ \frac{1}{4} \text{Tr}_{\vec{x}, \vec{x}'} [\underline{Q}(\vec{x}) A^{-1}(\vec{x}, \vec{x}') \underline{Q}(\vec{x}')]. \end{aligned} \quad (3.30)$$

In this equation the trace operation indicated by the notation $\text{Tr}_{\vec{x}, \vec{x}'}$ is over both the replica and spatial indices. Special cases of L that are of some interest are those of site randomness ($\sigma_b^2 = 0$, $L = L_s$), where

$$\begin{aligned} L_s[\underline{Q}; \underline{E}] &= \frac{1}{2} \text{Tr}_{\vec{x}, \vec{x}'} \{ \ln \{ [\underline{E} + \underline{Q}(\vec{x})] \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}') \} \} \\ &+ \frac{1}{2\sigma_s^2} \sum_{\vec{x}} \text{Tr} \underline{Q}^2(\vec{x}), \end{aligned} \quad (3.31)$$

and bond randomness ($\sigma_s^2 = t_0 = 0$, $L = L_b$), where

$$\begin{aligned} L_b[\underline{Q}; \underline{E}] &= \frac{1}{2} \sum_{\vec{x}} \text{Tr} \ln [\underline{E} + \underline{Q}(\vec{x})] \\ &+ \frac{1}{4\sigma_b^2} \text{Tr}_{\vec{x}, \vec{x}'} [\underline{Q}(\vec{x}) \gamma_{\vec{x}, \vec{x}'}^{-1} \underline{Q}(\vec{x}')]. \end{aligned} \quad (3.32)$$

Note that in the case of pure site randomness each term in L in an expansion of $Q(\vec{x})$ is spatially nonlocal, whereas in the case of pure bond randomness only the quadratic term is spatially nonlocal.

B. Complete set of $2n \times 2n$ matrices

It is somewhat confusing to work with symmetric tensor order parameters, since not all components of such tensors are independent. To circumvent this problem, we introduce a complete set of $(2n)(2n+1)/2$ symmetric matrices I_λ satisfying

$$\text{Tr } I_\lambda I_{\lambda'} = \delta_{\lambda\lambda'}. \quad (3.33)$$

Then Q and E can be decomposed as

$$Q = \sum_\lambda Q_\lambda I_\lambda, \quad (3.34a)$$

$$E = \sum_\lambda E_\lambda I_\lambda, \quad (3.34b)$$

and we have

$$\text{Tr } Q(\vec{x}) Q(\vec{x}') = \sum_\lambda Q_\lambda(\vec{x}) Q_\lambda(\vec{x}'). \quad (3.35)$$

Since the symmetry between $\sigma = +$ and $\sigma = -$ is explicitly broken, it is convenient to decompose the set $\{\lambda\}$ into three subsets (μ, p) for $p = \sigma\sigma' = ++, --, + -$ with respective associated matrices $\{I_\mu^{++}\}$ with $n(n+1)/2$ members of the form

$$I_\mu^{++} = \begin{pmatrix} K_\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.36)$$

$\{I_\mu^{--}\}$ with $n(n+1)/2$ members of the form

$$I_\mu^{--} = \begin{pmatrix} 0 & 0 \\ 0 & K_\mu \end{pmatrix}, \quad (3.37)$$

and $\{I_\mu^{+-}\}$ with n^2 members of the form

$$I_\mu^{+-} = \begin{pmatrix} 0 & M_\mu \\ M_\mu & 0 \end{pmatrix}, \quad (3.38)$$

where 0 is the $n \times n$ zero matrix and M_μ and K_μ are, respectively, $(n \times n)$ and $(n \times n)$ symmetric matrices satisfying

$$\text{Tr } K_\mu K_{\mu'} = \delta_{\mu\mu'}, \quad (3.39a)$$

$$\text{Tr } M_\mu M_{\mu'} = \frac{1}{2} \delta_{\mu\mu'}. \quad (3.39b)$$

Since $\langle Q_{\pm\pm}(\vec{x}) \rangle$ is proportional to $G_\pm$ and is thus always nonzero, we expect Q to order along the particular directions $(0, ++)$ and $(0, --)$ defined via

$$I_0^{++} = \begin{pmatrix} I_0 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.40a)$$

$$I_0^{--} = \begin{pmatrix} 0 & 0 \\ 0 & I_0 \end{pmatrix}, \quad (3.40b)$$

where

$$I_0 = n^{-1/2} \mathbf{1}^{(n)}, \quad (3.41)$$

where $\mathbf{1}^{(n)}$ is the $n \times n$ unit matrix. Note that E only has nonzero components along $(0, ++)$ and $(0, --)$:

$$E = \sqrt{n} (E_+ I_0^{++} + E_- I_0^{--}). \quad (3.42)$$

Similarly, in equilibrium we have

$$\langle Q \rangle \equiv Q^0 = \sqrt{n} (Q_+ I_0^{++} + Q_- I_0^{--}). \quad (3.43)$$

C. External fields and Legendre transforms

Though we are only interested in situations where E has the form of Eq. (3.42), it is clear that the Lagrangians of Eqs. (3.17) and (3.24) admit an $E(\vec{x})$ that is an arbitrary symmetric matrix and a function of position. In this context, $-\frac{1}{4} E_{ij}(\vec{x})$ is the field conjugate to $\Psi_i(\vec{x}) \Psi_j(\vec{x})$ for $i \neq j$ and $-\frac{1}{2} E_{ii}(\vec{x})$ is that conjugate to $\Psi_i(\vec{x})^2$. Using Eq. (3.26) we see that this implies that $-\frac{1}{2} E_\lambda(\vec{x})$ is the field conjugate to $\frac{1}{2} \sum_{\vec{x}'} A^{-1}(\vec{x}, \vec{x}') \times Q_\lambda(\vec{x}')$, i.e.,

$$\frac{\partial \ln \Xi}{\frac{1}{2} \partial E_\lambda(\vec{x})} = -\frac{1}{2} \sum_{\vec{x}'} A^{-1}(\vec{x}, \vec{x}') \langle Q_\lambda(\vec{x}') \rangle. \quad (3.44)$$

It is convenient for the purpose of generating correlation functions of $Q(\vec{x})$ to consider deviations $\delta E(\vec{x})$ of E from the diagonal form of Eq. (3.42) and introduce

$$H(\vec{x}) = -\frac{1}{4} \sum_{\vec{x}'} A^{-1}(\vec{x}, \vec{x}') \delta E(\vec{x}') \quad (3.45)$$

as the field conjugate to $Q(\vec{x})$. Similarly, to generate correlation functions involving $\Psi_i(\vec{x})$, we introduce a conjugate field $h_i(\vec{x})$. $H(\vec{x})$ and $h(\vec{x})$ appear linearly in an additive term in the Lagrangian:

$$L_{\text{ext}} = - \sum_{\vec{x}, \lambda} H_\lambda(\vec{x}) Q_\lambda(\vec{x}) - \sum_{\vec{x}, i} h_i(\vec{x}) \Psi_i(\vec{x}). \quad (3.46)$$

In the presence of L_{ext} , Ξ becomes a function of $H(\vec{x})$ and $h_i(\vec{x})$ as well as of E and we define

$$F(E, H(\vec{x}), \vec{h}(\vec{x})) = -\ln \Xi. \quad (3.47)$$

We may use F to generate all correlation functions involving $\tilde{\Psi}(\vec{x})$ and $Q(\vec{x})$. In particular, we have

$$\langle Q_\lambda(\vec{x}) \rangle = \frac{\partial F}{\partial H_\lambda(\vec{x})}, \quad (3.48a)$$

$$\begin{aligned} D_{\lambda\lambda'}(\vec{x}, \vec{x}') &= \langle Q_\lambda(\vec{x}) Q_{\lambda'}(\vec{x}') \rangle - \langle Q_\lambda(\vec{x}) \rangle \langle Q_{\lambda'}(\vec{x}') \rangle \\ &= \frac{\partial^2 F}{\partial H_\lambda(\vec{x}) \partial H_{\lambda'}(\vec{x}')}, \end{aligned} \quad (3.48b)$$

$$\begin{aligned} G_{i,i'}(\vec{x}, \vec{x}') &= \frac{\partial^2 F}{\partial h_i(\vec{x}) \partial h_{i'}(\vec{x}')} \\ &= \delta_{\alpha\alpha'} \delta_{\sigma\sigma'} [\delta_{\sigma+} G_+(\vec{x}, \vec{x}'; E) + \delta_{\sigma-} G_-(\vec{x}, \vec{x}'; E)]. \end{aligned} \quad (3.48c)$$

In the mean-field and ϵ -expansion analyses of succeeding sections, we will employ extensively the Legendre transforms, Γ , of F with associated vertex functions. Γ is a function of the expectation values $\langle Q \rangle$ and $\langle \tilde{\Psi} \rangle$ of Q and $\tilde{\Psi}$. To avoid writing too many angular brackets, however, we will write $\Gamma(Q, \tilde{\Psi})$ with the understanding that Q and $\tilde{\Psi}$ signify $\langle Q \rangle$ and $\langle \tilde{\Psi} \rangle$. There are three forms of Γ associated with the three forms of L . They are

$$\Gamma(\tilde{\Psi}) = F(\underline{E}, 0, \tilde{h}) + \sum_{\vec{x}, i} h_i(\vec{x}) \Psi_i(\vec{x}), \quad (3.49a)$$

$$\Gamma(Q, \tilde{\Psi}) = F(\underline{E}, H, \tilde{h}) + \sum_{\vec{x}, \lambda} H_\lambda(\vec{x}) Q_\lambda(\vec{x}) + \sum_{\vec{x}, i} h_i(\vec{x}) \Psi_i(\vec{x}), \quad (3.49b)$$

$$\Gamma(Q) = F(\underline{E}, H, 0) + \sum_{\vec{x}, \lambda} H_\lambda(\vec{x}) Q_\lambda(\vec{x}). \quad (3.49c)$$

We will be most concerned with the last form which can be expanded in terms of its vertex functions as

$$\Gamma(Q) = \Gamma(Q^0) + \sum_{n=1}^{\infty} \sum_{\{\lambda\}} \frac{1}{n!} \Gamma_{\lambda_1, \dots, \lambda_n}^{(n)}(\vec{x}_1, \dots, \vec{x}_n) \times \delta Q_{\lambda_1}(\vec{x}_1) \times \dots \times \delta Q_{\lambda_n}(\vec{x}_n), \quad (3.50)$$

where $\delta Q_\lambda(\vec{x}) = Q_\lambda(\vec{x}) - Q_\lambda^0(\vec{x})$, where $Q_\lambda^0(\vec{x})$ satisfies the equation of state

$$\Gamma_\lambda^{(1)}(Q_\lambda^0(\vec{x})) = H_\lambda(\vec{x}) \quad (3.51)$$

and reduces to Q_λ^0 of Eq. (3.43) when $H(\vec{x}) = 0$. The two-point vertex function is the inverse $Q_\lambda(\vec{x})$ correlation function:

$$\Gamma_{\lambda\lambda'}^{(2)}(\vec{x}, \vec{x}') = [D^{-1}]_{\lambda\vec{x}, \lambda'\vec{x}'} \quad (3.52)$$

We will also on occasion be interested in vertex functions involving the field Ψ_i . Denoting derivatives with respect to Ψ_i by subscripts Ψ_i , we write, for example,

$$\Gamma_{\Psi_i \Psi_j}^{(2)}(\vec{x}, \vec{x}') = \left(\frac{\partial^2 F}{\partial h_i(\vec{x}) \partial h_j(\vec{x}')} \right)^{-1}, \quad (3.53)$$

and $\Gamma_{\Psi_i \Psi_j}^{(2)}$ can be viewed either as a function of $\underline{Q}$ or $\underline{H} = 0$.

D. Ward identity

It is clear from the definition of Ξ as an integral over fields $\Psi_i(\vec{x})$ that F is invariant with respect to changes in $S = E + t_0$ of a particular type. For simplicity, we will treat only the case with $t_0 = 0$, though the final result, Eq. (3.62), applies with minor modifications even when $t_0 \neq 0$. Let U be any $2n \times 2n$ rotation matrix. Then the transformation

$$\Psi_i \rightarrow \sum_j U_{ij} \Psi_j \quad (3.54)$$

leaves Ξ unchanged. Thus, we have that

$$\Xi(E) = \Xi(E'), \quad (3.55)$$

where

$$\underline{E}' = U^{-1} \underline{E} U. \quad (3.56)$$

Now, chose U to have components

$$U_{\alpha\sigma, \beta\sigma'} = \delta_{\alpha\gamma} \delta_{\alpha\beta} V_{\sigma\sigma'} + \delta_{\alpha\beta} \delta_{\sigma\sigma'} (1 - \delta_{\alpha\gamma}), \quad (3.57)$$

where γ' can have any value from 1 to n and

$$V_{\sigma\sigma'} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}. \quad (3.58)$$

Setting $E_{\alpha, \sigma; \alpha', \sigma'} = E_{\sigma, \sigma'} \delta_{\alpha, \alpha'}$, we have for small φ

$$\delta E_{++} = 2E_+ \varphi, \quad (3.59a)$$

$$\delta E_{--} = -2E_+ \varphi, \quad (3.59b)$$

$$\delta E_{+-} = (E_{--} - E_{++}) \varphi, \quad (3.59c)$$

and

$$\frac{\partial F}{\partial \varphi} = 2E_+ - \frac{\partial F}{\partial E_{++}} - 2E_+ - \frac{\partial F}{\partial E_{--}} - \frac{\partial F}{\partial E_{+-}} (E_{++} - E_{--}) = 0. \quad (3.60)$$

Differentiating this expression with respect to E_{+-} , using

$$N^{-1} \frac{\partial F}{\partial E_{++}} = \frac{1}{2} G_+(\vec{x}, \vec{x}; E_+), \quad (3.61a)$$

$$N^{-1} \frac{\partial^2 F}{\partial E_{+-}^2} = g_{+-}(\vec{q} = 0; E_+, E_-), \quad (3.61b)$$

and setting $E_{++} = E_+$, we obtain

$$\lim_{q \rightarrow 0} g_{+-}(\vec{q}; E_+, E_-) = \frac{G_+(\vec{x}, \vec{x}; E_+) - G_-(\vec{x}, \vec{x}; E_-)}{E_+ - E_-}, \quad (3.62)$$

which is precisely the identity derived by Velicky²⁵ and presented in Sec. II. Expressed in terms of functions of Q_λ , this says that

$$\lim_{q \rightarrow 0} D_{\mu+-, \mu+-}(\vec{q}) = 2A(\vec{q} = 0) \left(\frac{Q_+ - Q_-}{E_+ - E_-} \right) + 1. \quad (3.63)$$

In particular, when the density of states is non-zero, $Q_+ \neq Q_-$, so that $g \sim \eta^{-1}$ and $\partial^2 \Gamma / \partial Q_+ \partial Q_- \sim \eta$ for $\eta \rightarrow 0$.

IV. MEAN-FIELD THEORY

A. Preliminaries

The simplest approximation for calculating quantities such as G , Q , etc., is mean-field theory. In this approximation, spatial fluctuations of any order parameters are ignored, and Ξ is evaluated

at the minimum value for L .³¹ In most applications there is no ambiguity in the mean-field approximation. Here, however, we must exercise some care because we have three different forms for L , and mean-field theory for one form will not correspond to that for another. In particular, $L(\underline{Q})$ and thus the mean-field expression for $\Gamma(\underline{Q})$ contain all possible loop corrections arising from the field $\underline{\Psi}$, whereas $L(\underline{\Psi})$ does not. This observation is important because one can in principle calculate the density of states either from

$$\pi\rho(E) = \pm \text{Im} G_{\pm}(\vec{x}, \vec{x}; E) = \pm \text{Im} \langle \psi_{\pm}^{\alpha}(\vec{x}) \psi_{\pm}^{\alpha}(\vec{x}) \rangle \quad (4.1a)$$

or via

$$\pi\rho(E) = \pm \text{Im} G_{\pm}(\vec{x}, \vec{x}; E) = \pm \text{Im} [2A(\vec{q}=0)]^{-1} \langle Q_{\pm\pm}^{\alpha\alpha}(\vec{x}) \rangle. \quad (4.1b)$$

In addition, when discussing competition between fluctuations in $\underline{Q}$ and fluctuations in $\underline{\Psi}$ one wishes to be able to treat these variables on an equal footing.

We will first consider the single-particle Green's function and then consider functions of $\underline{Q}$. Mean-field theory applied to $L(\underline{\Psi})$ yields the trivial result

$$\Gamma_{\Psi_i \Psi_j}^{(2)}(\vec{x}, \vec{x}') = -[E_{ij} \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}') \delta_{ij}],$$

which is just the inverse Gaussian propagator with no contributions due to scattering from the random potential. On the other hand, using Eq. (3.24), we obtain from $L(\underline{\Psi}, \underline{Q})$ that

$$\Gamma_{\Psi_i \Psi_j}^{(2)}(\vec{x}, \vec{x}') = -\{[E_{ij} + Q_{ij}(\vec{x})] \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}') \delta_{ij}\} \quad (4.2a)$$

$$= [\bar{G}^{-1}(\vec{x}, \vec{x}')]_{ij}. \quad (4.2b)$$

Equation (4.2) defines the Green's function $\bar{G}(\vec{x}, \vec{x}')$ in the presence of a spatially varying $\underline{Q}(\vec{x})$. In equilibrium, $\underline{Q}$ will have the form of Eq. (3.43), and the homogeneous mean-field Green's function is

$$G_{\pm}(\vec{q}; E_{\pm}) = -[E_{\pm} + Q_{\pm} + t_0(\vec{q})]^{-1}. \quad (4.3)$$

In mean-field theory, $\Gamma(\underline{Q})$ is identical to $L(\underline{Q})$. The equation of state is then

$$\begin{aligned} \Gamma_{\lambda}^{(1)}(\underline{Q}) &= \frac{\partial L(\underline{Q})}{\partial Q_{\lambda}(\vec{x})} \\ &= \frac{1}{2} \sum_{\vec{x}'} A^{-1}(\vec{x}, \vec{x}') Q_{\lambda}(\vec{x}') - \frac{1}{2} \text{Tr} \bar{G}(\vec{x}, \vec{x}) \underline{I}_{\lambda} = H_{\lambda}. \end{aligned} \quad (4.4)$$

For $H_{\lambda}=0$ this equation has uniform solutions with $Q_{\lambda}=0$ for $\lambda \neq 0, ++$ or $0, --$ in which case it may be written as

$$\begin{aligned} H_{\pm} &= \Gamma_{\pm}^{(1)} \\ &= \frac{1}{2} A^{-1}(\vec{q}=0) Q_{\pm} + \frac{1}{2N} \sum_{\vec{q}} [E_{\pm} + Q_{\pm} + t_0(\vec{q})]^{-1} = 0 \end{aligned} \quad (4.5)$$

for $\lambda=0, ++$ or $0, --$, where $Q_{\pm}$ is defined in Eq. (3.43). We expect these solutions to be the physical solutions though other solutions will be discussed in Sec. VI. The inverse susceptibility matrix is

$$\begin{aligned} \Gamma_{\lambda\lambda}^{(2)}(\vec{x}, \vec{x}') &= \frac{1}{2} A^{-1}(\vec{x}, \vec{x}') \delta_{\lambda\lambda} \\ &\quad - \frac{1}{2} \text{Tr}_{\vec{x}, \vec{x}'} [\bar{G}(\vec{x}, \vec{x}') \underline{I}_{\lambda} \bar{G}(\vec{x}', \vec{x}) \underline{I}_{\lambda}]. \end{aligned} \quad (4.6)$$

Assuming uniform solutions for $\underline{Q}$ of the form of Eqs. (3.43) and (4.5), we find

$$\Gamma_{\mu\sigma_1\sigma_2, \mu'\sigma_1'\sigma_2'}^{(2)}(\vec{q}) = \delta_{\mu\mu'} \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'} \Gamma_{\sigma_1\sigma_2}^{(2)}(\vec{q}) \quad (4.7)$$

with

$$\begin{aligned} r_{\sigma_1\sigma_2} &\equiv \Gamma_{\sigma_1\sigma_2}^{(2)}(\vec{q}) \\ &= \frac{1}{2} A^{-1}(\vec{q}) - \frac{1}{2N} \sum_{\vec{k}} [E_{\sigma_1} + Q_{\sigma_1} + t_0(\vec{k})]^{-1} \\ &\quad \times [E_{\sigma_2} + Q_{\sigma_2} + t_0(\vec{q} - \vec{k})]^{-1}. \end{aligned} \quad (4.8)$$

There are no cross correlation functions linking $(\mu, +)$ to $(\mu, ++)$ or to $(\mu, --)$. The three-point vertex

$$\begin{aligned} \Gamma_{\lambda_1\lambda_2\lambda_3}^{(3)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \\ &= -\frac{1}{2} \text{Tr} [\bar{G}(\vec{x}_1, \vec{x}_2) \underline{I}_{\lambda_2} \bar{G}(\vec{x}_2, \vec{x}_3) \underline{I}_{\lambda_3} \bar{G}(\vec{x}_3, \vec{x}_1) \underline{I}_{\lambda_1}] \\ &\quad - \frac{1}{2} \text{Tr} [\bar{G}(\vec{x}_1, \vec{x}_3) \underline{I}_{\lambda_3} \bar{G}(\vec{x}_3, \vec{x}_2) \underline{I}_{\lambda_2} \bar{G}(\vec{x}_2, \vec{x}_1) \underline{I}_{\lambda_1}] \end{aligned} \quad (4.9)$$

will be of some use in what follows. When $E_+ = E_-$ and $\underline{Q}(\vec{x})$ is uniform with $Q_+ = Q_- = Q$, this expression simplifies considerably:

$$\Gamma_{\lambda_1\lambda_2\lambda_3}^{(3)}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = w(\vec{q}_1, \vec{q}_2, \vec{q}_3) \text{Tr} \underline{I}_{\lambda_1} \underline{I}_{\lambda_2} \underline{I}_{\lambda_3} \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \quad (4.10)$$

with

$$w(0, 0, 0) \equiv w = \frac{1}{N} \sum_{\vec{k}} [E + Q + t_0(\vec{k})]^{-3}. \quad (4.11)$$

We will see shortly that at the mobility edge $E_+ = E_-$ and $Q_+ = Q_- = Q$ so that deviations of $\Gamma_{\lambda_1\lambda_2\lambda_3}^{(3)}$ from the form of Eq. (4.10) are small in the vicinity of the mobility edge.

B. The strong-coupling limit: Gaussian random hopping ($t_0=0, \sigma_x^2=0, \sigma_b^2 \neq 0$)

When the average hopping is zero, the analysis of mean-field theory⁸ simplifies considerably. First we recall the discussion in Sec. II which showed that $G_{\pm}(\vec{x}, \vec{x}'; E)$ is proportional to $\delta_{\vec{x}, \vec{x}'}$, so that no long-range correlations can develop in $\Psi_i(\vec{x})$. This is contrasted to the case of a pure band where $G_{\pm}(\vec{x}, \vec{x}'; E)$ develops an infinite corre-

lation length at the band edge. Thus, we know that any singularities in the conductivity must arise from fluctuations in the fields $Q(\vec{x})$. The equation of state for Q_+ and Q_- becomes purely algebraic,

$$\frac{1}{z\sigma_b^2} Q_{\pm} + \frac{1}{E_{\pm} + Q_{\pm}} = 0, \quad (4.12)$$

with solutions, for $E_{\pm} = E \pm i\eta$,

$$Q_{\pm} = \begin{cases} \frac{1}{2}E \left[\left(1 - \frac{E_c^2}{E^2}\right)^{1/2} - 1 \right], & E^2 > E_c^2 \\ \frac{1}{2}E \left[\pm i \left(\frac{E_c^2}{E^2} - 1 \right)^{1/2} - 1 \right], & E^2 < E_c^2 \end{cases} \quad (4.13a)$$

$$(4.13b)$$

where

$$E_c = 2\sigma_b z^{1/2}. \quad (4.14)$$

We have chosen the branches of $Q_{\pm}$ so that it is proportional to $-(1/E)$ for large $|E|$ as required and also $Q_+ = Q_-$ for $E^2 > E_c^2$. For $E^2 < E_c^2$, Q_+ and Q_- have imaginary parts leading to a semicircular density of states as predicted by our heuristic argument in Sec. II:

$$\rho = \frac{1}{\pi\sigma_b^2 z} \text{Im} Q_{\pm} = \begin{cases} \frac{2}{\pi E_c} \left(1 - \frac{E^2}{E_c^2}\right)^{1/2}, & E^2 < E_c^2 \\ 0, & E^2 > E_c^2. \end{cases} \quad (4.15a)$$

$$(4.15b)$$

This result also agrees with that for a generalization of the present model²⁰ to one with n_0 orbitals per site in the limit $n_0 \rightarrow \infty$. As expected, the integral of ρ over all energy is unity. Thus, within mean-field theory, all states lie in the energy range $-E_c \leq E \leq E_c$. (As noted in Sec. IIC there are states at all energies.) We shall verify shortly that all these states are extended. Note that in contrast to the pure case, the width of the region in energy containing states scales as $z^{1/2}$ rather than as z . This dependence on z agrees with the exact result for the Cayley tree discussed in Appendix A.

We now turn to the inverse susceptibility $\Gamma_{\lambda\lambda}^{(2)}$. For $E^2 > E_c^2$, $Q_+ = Q_- = Q$, and we have from Eqs. (4.7) and (4.8) that

$$\Gamma_{\lambda\lambda}^{(2)} = \delta_{\lambda\lambda} \Gamma^{(2)}, \quad (4.16)$$

where

$$\Gamma^{(2)}(\vec{q}) = \frac{1}{2\sigma_b^2 \gamma(\vec{q})} - \frac{1}{2(E_+ + Q)^2}, \quad |E| > E_c. \quad (4.17)$$

Note that this result implies that $g_{+-} = g_{++} = g_{--} = g_{+0}$ when $E^2 > E_c^2$. Using Eq. (4.13a), we obtain

$$\Gamma^{(2)}(\vec{q}; E, E) = \frac{4}{E_c^2} \left[1 - \frac{E^2}{E_c^2} + \frac{|E|}{E_c} \left(\frac{E^2}{E_c^2} - 1 \right)^{1/2} \right] + \frac{2a^2 q^2}{z E_c^2} + O(q^4), \quad (4.18)$$

where a is the lattice constant. As $|E| \rightarrow E_c$, $\Gamma^{(2)}(0; E, E)$ goes to zero:

$$\Gamma^{(2)}(0; E, E) \sim 4E_c^{-3} (E^2 - E_c^2)^{1/2}, \quad (4.19)$$

indicating that E_c marks both the critical divergence of g and the appearance of a nonzero density of states. Proceeding in analogy with phase transitions, we introduce critical exponents via the relations

$$\Gamma^{(2)}(\vec{q}; E, E) \sim (|E| - E_c)^{\gamma_a} [1 + \text{const} \times q^2 (|E| - E_c)^{-2\nu}], \quad (4.20a)$$

$$Q_+ - Q_- \sim |E_c - |E||^{\beta_a}, \quad (4.20b)$$

where in mean-field theory we have

$$\gamma_a = \beta_a = \frac{1}{2}, \quad (4.21a)$$

$$\nu_a = \frac{1}{4}. \quad (4.21b)$$

These exponents differ from the mean-field results $\gamma = 1$, $\beta = \nu = \frac{1}{2}$, for common thermodynamic transitions³ or for percolation,³⁴ where $\gamma = \beta = 1$ and $\nu = \frac{1}{2}$. However, these values do agree with the heuristic arguments of Sec. II. They also correspond to the mean-field exponents for the statistics of lattice animals,¹⁴ and in the next section we will see that these two problems remain in the same universality class in $8 - \epsilon$ dimensions. As in the case of lattice animals, these modified mean-field exponents occur because we are studying properties for fixed fields $\underline{H} = 0$. The more usual exponents, $\gamma = 1$, $\nu = \frac{1}{2}$, result if one considers properties for fixed $\underline{Q}$. Since fixing H corresponds to allowing $G_{\pm}(\vec{x}, \vec{x}; E)$ to vary, whereas fixing $\underline{Q}$ corresponds to the rather unphysical situation where $G_{\pm}(\vec{x}, \vec{x}; E)$ is fixed, the case of interest is that of fixed $\underline{H}$. We will return to this point later.

We now turn to the regime $|E| < E_c$. In this case $Q_+ \neq Q_-$ and $\Gamma_{\lambda\lambda}^{(2)}$ is not proportional to $\delta_{\lambda\lambda}$. From Eq. (4.8) we have

$$\Gamma_{\pm\pm}^{(2)} = \frac{1}{2} \frac{1}{\sigma_b^2} \frac{1}{\gamma(q)} - \frac{1}{2} \left(\frac{1}{E_{\pm} + Q_{\pm}} \right)^2 = \frac{4}{E_c^2} \left[1 - \left(\frac{E}{E_c} \right)^2 \pm i \frac{E}{E_c} \left(1 - \frac{E^2}{E_c^2} \right)^{1/2} \right]. \quad (4.22)$$

$\Gamma_{\pm\pm}^{(2)}(0; E, E)$ is finite and complex for $|E| < E_c$ and tends to zero as $|E| \rightarrow E_c$. Because of the Ward identity, Eq. (3.62), we expect $\Gamma_{+-}^{(2)}(q=0, E, E)$ to be zero for $E^2 < E_c^2$, i.e., for the region of extended states. From Eq. (4.8), we find

$$\Gamma_{+-}^{(2)}(\vec{q}) = \frac{1}{2\sigma_b^2 \gamma(\vec{q})} - \frac{1}{2(E_+ + Q_+)(E_- + Q_-)} \quad (4.23a)$$

$$= \frac{2a^2 q^2}{z E_c^2} + \frac{E_+ - E_-}{2z\sigma_b^2 (E_+ - E_- + Q_+ - Q_-)}, \quad (4.23b)$$

which agrees with Eq. (3.63). Setting $E_+ = E + \omega$ and $E_- = E$, we obtain

$$E_c^2 \Gamma_{+-}^{(2)}(\vec{q}) = \frac{2a^2 q^2}{z} - \frac{2i\omega}{(E_c^2 - E^2)^{1/2}} \quad (4.24)$$

in the small- ω limit. Using Eq. (2.16) this implies that the conductivity Σ is

$$\Sigma \sim (E_c^2 - E^2)^\mu, \quad (4.25)$$

where (see Fig. 6)

$$\mu = 1. \quad (4.26)$$

Since $\Sigma(E) \neq 0$ for $E^2 < E_c^2$, all states are conducting in the mean-field approximation. This shortcoming of mean-field theory is expected in view of the discussion of Sec. II.

C. Nonzero average hopping ($t_0 \neq 0$)

1. Strong versus weak coupling

In a pure system, $t_0 \neq 0$ and $\sigma_b^2 = \sigma_s^2 = 0$. In this case, c.f. Eq. (4.3), $G_0(\vec{x}, \vec{x}')$ has a divergent correlation length at the upper and lower band edges, $E_{b\pm} = \pm t_0(\vec{q}=0) = \pm z t_0$. Since $L(\vec{\Psi})$ for this case is quadratic in $\vec{\Psi}$, the scaling properties of these band edge critical points are described by the Gaussian renormalization-group fixed point.³ Now consider a random system, and for simplicity set $\sigma_b = 0$. In this case there are corrections to G arising from scattering off the random potential. On the other hand, we learned in the previous sections that when $t_0 = 0$ and $\sigma_s^2 = 0$, the field $\underline{Q}$ is critical at the mobility edge. Thus, as σ_b^2 and/or σ_s^2 is increased from zero when t_0

$\neq 0$, we have the possibility of either $\vec{\Psi}$, $\underline{Q}$, or both being critical at the mobility edge. The cases when $\vec{\Psi}$ and $\underline{Q}$ are critical are the weak- and strong-coupling cases, respectively.

The weak-coupling mobility edges, $E_{b\pm}$, are determined by the condition $G(\vec{q}=0)^{-1} = 0$. To one-loop order, we have from Eq. (4.3) that

$$E_{b\pm} + Q_{b\pm} = \pm t_0(0) = \pm z t_0, \quad (4.27)$$

where $Q_{b\pm}$ is the value of Q at the upper (lower) band edge. Note that in order for solutions to Eq. (4.27) to exist for real $E_{b\pm}$ and t_0 , $Q_{b\pm}$ must be real, and consequently the density of states at the shifted band edge must be zero. We shall see that this is in fact the case for $d > 4$, and a shifted band edge singularity of the random system is possible. Of course, $Q_{b\pm}$ must satisfy the equation of state which we write as

$$Q_{b\pm} = \mp \frac{A}{N} \sum_{\vec{k}} \frac{1}{t_0(0) - t_0(\vec{k})} = \mp \frac{A}{N t_0} \sum_{\vec{k}} \frac{1}{z - \gamma(\vec{k})} \quad (4.28a)$$

or as

$$Q_{b\pm} = \mp \frac{A}{N} \sum_{\vec{k}} \frac{1}{\epsilon(\vec{k})}, \quad (4.28b)$$

where we have defined the excitation energy $\epsilon(\vec{k})$ and A denotes $A(\vec{q}=0)$. Thus we have

$$E_{b\pm} = \pm \left[t_0(0) + \frac{A}{N} \sum_{\vec{k}} \left(\frac{1}{t_0(0) - t_0(\vec{k})} \right) \right]. \quad (4.29)$$

These equations locate the mobility edge only in the weak-coupling case.

We may now study the competition between fluctuations in $\vec{\Psi}$ and those in $\underline{Q}$. The equation of state determining $Q_{\pm}$ is given by Eq. (4.5), which we rewrite here in the interest of clarity:

$$Q_{\pm} + \frac{A}{N} \sum_{\vec{k}} [E_{\pm} + Q_{\pm} + \epsilon_0(\vec{k})]^{-1} = 0, \quad (4.30)$$

and the inverse $\underline{Q}$ susceptibility in the disordered regime ($Q_+ = Q_- = Q$) is $\Gamma_{\lambda\lambda}^{(2)} = \delta_{\lambda\lambda} \Gamma^{(2)}$ with

$$\Gamma_{(q=0)}^{(2)} = \frac{1}{2A} \left(1 - \frac{A}{N} \sum_{\vec{k}} [E + Q + t_0(\vec{k})]^{-2} \right). \quad (4.31)$$

As E approaches the mobility edge in the disordered phase either of two results can occur.

(a) In the strong-coupling limit the solutions to Eq. (4.30) for $Q_{\pm}$ remain real until $\Gamma^{(2)}$ vanishes at $E = E_{c-}$ or $E = E_{c+}$ or (b) the solutions to Eq. (4.30) for $Q_{\pm}$ become complex at $E = E_{b-}$ or $E = E_{b+}$ before $\Gamma^{(2)}$ becomes zero. The second possibility corresponds to the weak-coupling case. Hence we see that we have the weak-coupling case if

FIG. 6. Behavior of the macroscopic conductivity Σ as E approaches the mobility edges: $\Sigma \sim |E_c - E|^\mu$. For $d > 8$, $\mu = 1$ as shown in (a). For $d < 8$, $\mu < 1$ as shown in (b). For d near 8, $\mu = 1 - 5\epsilon/36$.

$$\frac{A}{N} \sum_{\mathbf{k}} [t_0(0) - t_0(\mathbf{k})]^{-2} = \frac{A}{N} \sum_{\mathbf{k}} \epsilon(\mathbf{k})^{-2} < 1 \quad (4.32a)$$

or

$$\frac{A}{(zt_0)^2} \frac{1}{N} \sum_{\mathbf{k}} \left(1 - \frac{\gamma(\mathbf{k})}{z}\right)^{-2} < 1 \quad (4.32b)$$

and the strong-coupling case if the inequality is reversed. Thus when $A/(zt_0)^2 = (\frac{1}{2}\sigma_s^2 + z\sigma_b^2)/(zt_0)^2$ is large enough, Q fluctuations dominate and one has the strong-coupling mobility edge. Note also that the sum on the left-hand side of Eq. (4.32) diverges as $(d-4)^{-1}$ as $d \rightarrow 4+$, so that the weak-coupling mobility edge can only occur for $d > 4$. These considerations give rise to the phase diagram for the Gaussian bond model shown in Fig. 7.

It is clear that the critical properties of the weak-coupling problem are described by the Gaussian fixed point of the field theory of Eq. (3.17). The strong-coupling problem is described by another universality class which will be studied in the next section and which becomes non-mean-field-like for $d < 8$. There is also a multicritical point which we will not study in this paper at which both Ψ and Q are critical.

FIG. 7. Phase diagram for the Gaussian nearest-neighbor bond model ($\sigma_s = 0$) as a function of E and σ_b , as defined by Eq. (3.15). The negative energy part of the diagram (not shown) is obtained by symmetry from the positive energy part. The full line separates the localized states which exist for $E > E_c$ (except for $\sigma_b = 0$) from the extended states which exist for $E < E_c$. For small σ_b the transition is a weak-coupling (w) one where $G_0(\vec{x}, \vec{x}'; E)$ becomes critical. For large σ_b the transition is a strong-coupling (s) one where $G_{\infty}(\vec{x}, \vec{x}'; E)$ becomes critical. The multicritical point where both functions have critical behavior is indicated by $\times$. Inside the extended state regime there is no obvious way to distinguish between the weak- and strong-coupling phases. If these phases could be distinguished, then one would have a phase boundary connecting the multicritical point at positive energy with that at negative energy. The dashed line is the asymptote $E = 2\sqrt{z} \sigma_b$.

2. Strong coupling

In the strong-coupling regime Q is critical at the mobility edge. Therefore we expect critical properties at the mobility edge when $t_0 \neq 0$ to be very similar to those for $t_0 = 0$ discussed in the preceding section. We will now show that this is the case in mean-field theory and, in the process, display some general properties of the field theory that will be useful in the next section.

The localization problem is naturally studied as a function of the energy E . The field theory we have derived, however, is naturally studied as a function of Q . As in the case of lattice animals,¹⁴ the values of critical exponents depend on whether Q is held constant or not, and it is of interest to study both cases. We begin by studying the case of constant Q . In this case as E varies, H varies in accord with the equation of state (4.4). If Q_+ and Q_- are fixed and are different, there will be separate critical points as a function of E for r_{++} , r_{--} , and r_{+-} [cf. Eq. (4.8)] within mean-field theory. Since we are interested in the critical point at which all are simultaneously critical, we take $Q_+ = Q_- = Q$. In this case, the critical energy, $E_c(Q)$, is determined by the equation

$$r_{\sigma\sigma'}(E_c(Q), Q) \equiv \Gamma_{\sigma\sigma'}^{(2)}(E_c(Q), Q; \vec{q} = 0) = 0 \quad (4.33)$$

which has two solutions corresponding to the top and the bottom of the conduction band. Expanding for $E_g = E_c(Q) + \Delta E_g$ near $E_c(Q)$, we have

$$\Gamma_{\sigma\sigma'}^{(2)}(Q; E_g; \vec{q}) = \frac{1}{2} w(E_c, Q) (\Delta E_g + \Delta E_{g'}) + K q^2, \quad (4.34)$$

where w is defined in Eq. (4.11) and K is a non-zero constant which can be calculated from Eq. (4.8). Note that $w(E_c, Q)$ is positive at the top of the band and negative at the bottom. Also ΔE_g is positive (negative) if E_g is in the region of localized states above (below) the conduction band. Thus $w \Delta E_g$ is positive in the region of localized states. One can define critical exponents from Eq. (4.34) in the usual way. The mean-field susceptibility, correlation length, and critical point exponents at constant Q are

$$\gamma_Q = 1, \quad (4.35a)$$

$$\nu_Q = \frac{1}{2}, \quad (4.35b)$$

$$\eta_Q = 0, \quad (4.35c)$$

in agreement with common mean-field theories. Notice, however, that γ_Q and ν_Q are twice as large as γ_a and ν_a of Eq. (4.21) that result when Q satisfies the equation of state and varies in response to changes in E .

We now consider what happens when Q satisfies the equation of state. As noted above, when Q is fixed, the critical value of E is located by setting $\Gamma_{\sigma\sigma'}^{(2)}(E, Q) = 0$ and the associated value of H is obtained from the equation of state. Thus in the constant- Q case, there exists a critical line in the EH plane. Likewise, in the constant- H case ($H_+ = H_- = H$) there is a critical line in the EQ plane determined by

$$\Gamma_{\sigma}^{(1)}(E_c, Q_c) = H, \quad (4.36a)$$

$$\Gamma_{\sigma, \sigma'}^{(2)}(E_c, Q_c) = 0. \quad (4.36b)$$

One can carry out a general analysis for the behavior near an arbitrary point on this critical line.^{14,35} For simplicity we confine our attention to the point of physical interest, i.e., to $H=0$, for which

$$\Gamma_{\sigma}^{(1)}(E_c, Q_c) = 0, \quad (4.36c)$$

$$\Gamma_{\sigma, \sigma'}^{(2)}(E_c, Q_c) = 0. \quad (4.36d)$$

To study behavior in the vicinity of the critical point, we expand in both $\Delta E_{\sigma} = E_{\sigma} - E_c$ and $\Delta Q_{\sigma} = Q_{\sigma} - Q_c$:

$$\Gamma_{\sigma\sigma'}^{(2)} = \frac{1}{2A^{1/2}} [(w_c \Delta E_{\sigma})^{1/2} + (w_c \Delta E_{\sigma'})^{1/2}] + Kq^2, \quad w_c \Delta E_{\sigma} > 0, \quad w_c \Delta E_{\sigma'} > 0 \quad (4.39)$$

$$\Gamma_{\sigma\sigma'}^{(2)} = \frac{iw_c}{2A^{1/2}} \left[\sigma \left(\frac{-\Delta E_{\sigma}}{w_c} \right)^{1/2} + \sigma' \left(\frac{-\Delta E_{\sigma'}}{w_c} \right)^{1/2} \right] + Kq^2, \quad w_c \Delta E_{\sigma} < 0, \quad w_c \Delta E_{\sigma'} < 0. \quad (4.40)$$

The latter equation implies that $\gamma_a = \frac{1}{2}$, $\nu_a = \frac{1}{4}$, and $\eta = 0$ in agreement with Eq. (4.21a). Setting $E_+ = E + \omega$ and $E_- = E$, Eq. (4.40) yields

$$\Gamma_{+-}^{(2)} = -\frac{1}{4} i \omega \left(A^{-1} \left| \frac{w_c}{\Delta E} \right| \right)^{1/2} + Kq^2, \quad (4.41)$$

implying $\mu = 1$ in agreement with the results of Sec. IV B.

3. Weak-coupling regime

In this section we analyze the mean-field equations for the weak-coupling regime. Our principal result will be that the conductivity exponent μ is unity, just as it is for the strong-coupling case.

To study the weak-coupling regime in the simplest way we take $\sigma_b = 0$ and we will work in the vicinity of the band edge at negative energy. Then Eq. (4.5) becomes

$$Q_{\pm} = -\frac{1}{2} \frac{\sigma_s^2}{N} \sum_{\mathbf{q}} [E_{\pm} + Q_{\pm} + t_0(\mathbf{q})]^{-1}. \quad (4.42)$$

To obtain the conductivity we set

$$E_{\pm} = E \pm \frac{1}{2}\omega, \quad (4.43)$$

$$\left(\frac{\partial \Gamma_{\sigma}^{(1)}}{\partial E_{\sigma}} \right)_{E_c, Q_c} (\Delta E_{\sigma}) + \frac{1}{2} w_c (\Delta Q_{\sigma})^2 = 0, \quad (4.37a)$$

$$\Gamma_{\sigma, \sigma'}^{(2)}(E_{\pm}, Q_{\pm}) = \frac{w_c}{2} (\Delta E_{\sigma} + \Delta E_{\sigma'} + \Delta Q_{\sigma} + \Delta Q_{\sigma'}), \quad (4.37b)$$

where $w_c = w(E_c, Q_c)$. In writing these equations we omitted derivatives which vanish according to Eqs. (4.36c) and (4.36d). Also we only kept terms which are needed to determine the dominant behavior of ΔQ_{σ} . From Eqs. (4.5) and (4.36d) we see that $(\partial \Gamma_{\sigma}^{(1)} / \partial E_{\sigma})_{E_c, Q_c} = -(2A)^{-1}$. Solving Eq. (4.37a) for ΔQ_{σ} , we find

$$\Delta Q_{\sigma} = \begin{cases} \frac{1}{w_c} \left(\frac{w_c \Delta E_{\sigma}}{A} \right)^{1/2}, & w_c \Delta E_{\sigma} > 0 \\ i\sigma \left(\frac{-\Delta E_{\sigma}}{w_c A} \right)^{1/2}, & w_c \Delta E_{\sigma} < 0. \end{cases} \quad (4.38a)$$

$$(4.38b)$$

As noted below Eq. (4.34), $w_c \Delta E_{\sigma}$ is positive for $|E| > E_c$ and negative for $|E| < E_c$. We have selected the branches of the square roots in Eq. (4.38) so that (a) for $|E| > E_c$, $\partial Q_{\sigma} / \partial E_{\sigma} > 0$, and (b) for $|E| < E_c$, $\sigma \text{Im} Q_{\sigma} > 0$. Substituting these solutions into Eq. (4.37b) we find that

so that

$$Q_{\pm}(E \pm \frac{1}{2}\omega) = -\frac{1}{2} \frac{\sigma_s^2}{N} \sum_{\mathbf{q}} [E \pm \frac{1}{2}\omega + Q_{\pm}(E \pm \frac{1}{2}\omega) + t_0(\mathbf{q}) \pm i\eta]^{-1} \quad (4.44)$$

and the branch of Q is chosen so that

$$\text{Im} Q_{+} \geq 0, \quad (4.45a)$$

$$\text{Im} Q_{-} \leq 0. \quad (4.45b)$$

We shall be interested in the small- k and $-\omega$ behavior of $\Gamma_{+-}^{(2)}(k, \omega)$ given by

$$\begin{aligned} \Gamma_{+-}^{(2)}(\vec{k}, \omega) &\equiv \Gamma(\vec{k}, \omega) \\ &= \frac{1}{\sigma_s^2} - \frac{1}{2N} \sum_{\mathbf{q}} \prod_{\sigma=\pm} [E + \frac{1}{2}\sigma\omega + Q_{\sigma}(E + \frac{1}{2}\sigma\omega) \\ &\quad + t_0(\vec{q} + \frac{1}{2}\sigma\vec{k})]^{-1}. \end{aligned} \quad (4.46)$$

We will study the band edge corresponding to $\vec{q} = 0$, near which we have

$$t_0(\vec{q}) \sim t_0(0) - \alpha q^2. \quad (4.47)$$

For $\vec{k} = \omega = 0$, the weak-coupling transition at

which Q becomes complex occurs for $E = E_c$, $Q = Q_c$, where

$$E_c = -Q_c - t_0(0), \quad (4.48a)$$

$$Q_c = \frac{\sigma_s^2}{2N} \sum_{\vec{q}} [t_0(0) - t_0(\vec{q})]^{-1} = \frac{\sigma_s^2}{2N} \sum_{\vec{q}} [\epsilon(\vec{q})]^{-1}, \quad (4.48b)$$

and at criticality $\Gamma = \Gamma_c(0, 0)$, where

$$\Gamma_c(0, 0) = \frac{1}{\sigma_s^2} - \frac{1}{2N} \sum_{\vec{q}} [\epsilon(\vec{q})]^{-2}. \quad (4.49)$$

As noted above, the weak-coupling regime is the one for which $\Gamma_c(0, 0) > 0$, which occurs (for $d > 4$) for sufficiently small σ_s . We now study $\Gamma(\vec{k}, \omega)$ for small $\vec{k}$ and ω for $E - E_c \rightarrow 0$, for energies just inside the band. Expanding $\Gamma(\vec{k}, \omega)$ in powers of k and ω we obtain

$$\Gamma(\vec{k}, \omega) = \Gamma(0, 0) + iA\omega + Bk^2, \quad (4.50)$$

where

$$\Gamma(0, 0) = \frac{1}{\sigma_s^2} - \frac{1}{2N} \sum_{\vec{q}} (D_+ D_-)^{-1}, \quad (4.51)$$

$$iA = -\frac{1}{4N} \sum_{\vec{q}} (D_+ D_-)^{-1} \left(\frac{1 + dQ_-/dE}{D_-} - \frac{1 + dQ_+/dE}{D_+} \right), \quad (4.52)$$

$$B = \frac{1}{8dN} \sum_{\vec{q}} (D_+ D_-)^{-1} \left[|\nabla_{\vec{q}} t_0(q)|^2 \left(\frac{1}{D_+ D_-} - \frac{1}{D_+^2} - \frac{1}{D_-^2} \right) + \frac{1}{2} [\nabla_{\vec{q}}^2 t_0(q)] \left(\frac{1}{D_+} + \frac{1}{D_-} \right) \right], \quad (4.53)$$

where

$$D_{\pm} = E + Q'(E) \pm iQ''(E) + t_0(\vec{q}), \quad (4.54)$$

where Q' and Q'' are respectively the real and imaginary parts of $Q_+(E)$. Integrals involving only negative powers of D_+ and D_- can be put into the form considered in Appendix B using

$$\frac{1}{D_+ D_-} = \frac{1}{2iQ''(E)} \left(\frac{1}{D_-} - \frac{1}{D_+} \right). \quad (4.55)$$

This relation immediately indicates that $\Gamma(0, 0) = 0$ in agreement with the Ward identity. Also, it is sufficient to obtain the dominant behavior as $E \rightarrow E_c$ to set

$$\nabla_{\vec{q}}^2 t_0(q) = -2d\alpha, \quad (4.56a)$$

$$|\nabla_{\vec{q}} t_0(\vec{q})|^2 = 4\alpha^2 q^2 = 4\alpha \epsilon(\vec{q}) \quad (4.56b)$$

$$= 2\alpha [2E + 2t_0(0) + 2Q'(E) - D_+ - D_-], \quad (4.56c)$$

in which case the expansion for B can be written

in terms of integrals involving only negative powers of D_+ or D_- . In the notation of Appendix B we have

$$A = -\frac{i\dot{\rho}}{8\rho} I_2^- + \frac{1+\beta}{8\rho} I_2^+ + \frac{i}{8\rho^2} (1+\beta) I_1^-, \quad (4.57)$$

and for $E \rightarrow E_c$ we find

$$A = -\frac{1}{2\sigma_s^2 Q''(E)}. \quad (4.58)$$

Using the approximations, Eq. (4.56), we obtain

$$B = \frac{\alpha(E) + t_0(0) + Q'(E)}{4id\rho} \left(\frac{I_1^-}{\rho^2} + \frac{I_2^+}{i\rho} - I_3^- \right) + \frac{\alpha}{4d} I_3^+ - \frac{\alpha}{16i\rho} I_2^-. \quad (4.59)$$

The evaluations of Appendix B yield

$$B = \frac{\alpha[E + Q' + t_0(0)]}{d\sigma_s^2 [Q''(E)]^2 \left(1 + \frac{dQ'}{dE} \right)} \quad (4.60a)$$

$$\sim \frac{\alpha(\Delta E)}{d\sigma_s^2 [Q''(E)]^2} \quad (4.60b)$$

for $E - E_c \rightarrow 0$.

Using Eq. (2.16) we see that the conductivity Σ is thus

$$\Sigma = B/A^2 = \frac{4\sigma_s^2 \alpha}{d} \Delta E \quad (4.61a)$$

$$\sim (\Delta E)^\mu \quad (4.61b)$$

with $\mu = 1$ as in the strong-coupling mean-field case.

We close this section by recording some explicit results for Q for $E \rightarrow E_c$. We write

$$E = E_c + \Delta E, \quad (4.62a)$$

$$Q_{\pm} = Q_c + \Delta Q' \pm iQ''. \quad (4.62b)$$

To obtain $\Delta Q'$ we may ignore corrections from Q'' and write the real part of Eq. (4.42) as

$$\Delta Q' = \frac{\sigma_s^2}{2N} \sum_{\vec{q}} \frac{\Delta E + \Delta Q'}{\epsilon(q) [\epsilon(q) - \Delta E - \Delta Q']}, \quad (4.63)$$

where we used Eq. (4.48b) for Q_c . This relation gives

$$\left(\frac{\partial Q'}{\partial E} \right)_{E_c} = \beta = \frac{\frac{1}{2}\sigma_s^2 f}{1 - \frac{1}{2}\sigma_s^2 f}, \quad (4.64)$$

where

$$f = \frac{1}{N} \sum_{\vec{q}} [\epsilon(\vec{q})]^{-2}. \quad (4.65)$$

Note that β diverges as one approaches the strong-coupling regime defined by $\sigma_s^2 f/2 > 1$.

To obtain Q'' for E near E_c we write

$$Q'' = -\frac{\sigma_s^2}{2N} \text{Im} \lim_{\eta \rightarrow 0} \sum_{\mathbf{q}} \{ [\Delta E + \Delta Q' - \epsilon(\tilde{\mathbf{q}}) + i\eta]^{-1} + \{ [\Delta E + \Delta Q' - \epsilon(\tilde{\mathbf{q}}) + iQ'']^{-1} - [\Delta E + \Delta Q' - \epsilon(\tilde{\mathbf{q}}) + i\eta]^{-1} \} \} \quad (4.66)$$

or

$$Q'' = \frac{\pi\sigma_s^2}{2N} \sum_{\mathbf{q}} \delta(\Delta E + \Delta Q' - \epsilon(\tilde{\mathbf{q}})) + \lim_{\eta \rightarrow 0} \frac{Q''\sigma_s^2}{2N} \text{Re} \sum_{\mathbf{q}} [\Delta E + \Delta Q' - \epsilon(\tilde{\mathbf{q}}) + i\eta]^{-2}. \quad (4.67)$$

If we define $\rho_0(E)$ to be the density of states corresponding to $t(q)$,

$$\rho_0(x) = N^{-1} \sum_{\mathbf{q}} \delta(x - \epsilon(\tilde{\mathbf{q}})), \quad (4.68)$$

then we may write the solution for Q'' as

$$Q''(E) = \frac{\frac{1}{2}\sigma_s^2 \pi \rho_0[\Delta E(1+\beta)]}{1 - \frac{1}{2}\sigma_s^2 f}. \quad (4.69)$$

For $\epsilon(q) = \alpha q^2$ one has

$$\rho_0(x) = K_d x^{d/2-1} / 2\alpha^{d/2}. \quad (4.70)$$

Thus the power law for the density of states is the same as for the pure system. This is as one would expect, since both models are described by the Gaussian fixed point.

V. PERTURBATION THEORY AND ϵ EXPANSION

A. Shifted field theory

As discussed in the preceding section, there can be a weak- or a strong-coupling transition.

For the weak-coupling case, perturbation theory converges for all $d > 4$ and the transition is described by the Gaussian renormalization-group fixed point. We will not consider this case further at this point. For the strong-coupling case, on the other hand, we will show in this section that perturbation theory breaks down for $d < 8$. We will then use the renormalization group to study the strong-coupling mobility transition in $8 - \epsilon$ dimensions.

We begin, as usual, by rewriting the functional integrals in Ξ in terms of deviations from the average value of Q :

$$Q_\lambda = \sqrt{n} (\delta_{\lambda,0++} Q_+ + \delta_{\lambda,0--} Q_-) + \varphi_\lambda. \quad (5.1)$$

We also express the discrete lattice sum in terms of a continuous integral, and keep terms only up to second order in $\nabla\varphi_\lambda$. To keep algebra to a minimum, we will take the coefficient of $(\nabla\varphi_\lambda)^2$ to be unity even though a field rescaling is in general necessary to produce this result.

Therefore we have

$$\Xi = \int \mathcal{D}\varphi e^{-\mathcal{L}[\varphi]}, \quad (5.2)$$

where

$$L = L_0(Q) + \frac{1}{2} \int d^d x \sum_{\mu p} \{ r_p \varphi_{\mu p}^2(\tilde{\mathbf{x}}) + [\nabla \varphi_{\mu p}(\tilde{\mathbf{x}})]^2 \} + \frac{1}{2} \sum_{pp'=++,--} \int d^d x n \tau_{pp'}^0 \varphi_{0p}(\tilde{\mathbf{x}}) \varphi_{0p'}(\tilde{\mathbf{x}}) + \sqrt{n} \sum_{\sigma=+,-} \int d^d x H_\sigma \varphi_{0\sigma}(\tilde{\mathbf{x}}) + \frac{1}{3!} w \text{Tr} I_{\lambda_1} I_{\lambda_2} I_{\lambda_3} \int d^d x \varphi_{\lambda_1}(\tilde{\mathbf{x}}) \varphi_{\lambda_2}(\tilde{\mathbf{x}}) \varphi_{\lambda_3}(\tilde{\mathbf{x}}), \quad (5.3)$$

where p runs over $++$, $--$, and $+-$, $L_0(Q)$ is the mean-field expression for $\Gamma(Q)$, $H_\pm$ is the mean-field expression for $\Gamma_\pm^{(1)}$ in Eq. (4.5), $r_{0\sigma}$ is defined in Eq. (4.8), and w in Eq. (4.11). τ_{pp}^0 is zero, but a potential $\tau_{pp'}$ of this symmetry is generated at one-loop order in perturbation theory and is essential in determining the correct upper critical dimension and universality class for the strong-coupling mobility transition. We have ignored fourth- and higher-order terms and third-order terms of lower symmetry in Eq. (5.3) because they are irrelevant to the critical point in question. We note, at this point, one very important property of r_{++} , r_{--} , and r_{+-} . Near the mean-field critical point, all are zero, and

$$r_{+-} = \frac{1}{2}(r_{++} + r_{--}) + O(r_{++}^2, r_{--}^2). \quad (5.4)$$

This relation remains valid for fluctuation renormalized variables as well.

B. Perturbation expansion

Perturbation theory to one-loop order is easily carried out. We obtain in the limit $n \rightarrow 0$,

$$\Gamma_\pm^{(1)} = H_\pm + \frac{1}{4} w \int \frac{d^d q}{(2\pi)^d} \frac{1}{t_\pm + q^2} + \frac{1}{2} w \tau_{\pm\pm} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{t_{\pm\pm} + q^2} \right)^2, \quad (5.5a)$$

$$\Gamma_{++}^{(2)}(\tilde{q}=0) \equiv t_{++} = r_{++} - \frac{1}{4}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)^2} + w^2 \tau_{++} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)^3}, \quad (5.5b)$$

$$\Gamma_{--}^{(2)}(\tilde{q}=0) \equiv t_{--} = r_{--} - \frac{1}{4}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{--}+q^2)^2} + w^2 \tau_{--} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{--}+q^2)^3}, \quad (5.5c)$$

$$\begin{aligned} \Gamma_{+-}^{(2)}(\tilde{q}=0) \equiv t_{+-} = r_{+-} - \frac{1}{8}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{+-}+q^2)} & \left(\frac{1}{t_{++}+q^2} + \frac{1}{t_{--}+q^2} \right) \\ & + \frac{1}{4}w^2 \left(\tau_{++} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{+-}+q^2)(t_{++}+q^2)^2} + \tau_{--} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{+-}+q^2)(t_{--}+q^2)^2} \right. \\ & \left. + 2\tau_{+-} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)(t_{--}+q^2)(t_{+-}+q^2)} \right), \end{aligned} \quad (5.5d)$$

$$\tau_{++} = \tau_{++}^0 - \frac{1}{8}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)^2} - \frac{1}{2}w^2 \tau_{++}^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)^4}, \quad (5.5e)$$

$$\tau_{--} = \tau_{--}^0 - \frac{1}{8}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{--}+q^2)^2} - \frac{1}{2}w^2 \tau_{--}^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{--}+q^2)^4}, \quad (5.5f)$$

$$\tau_{+-} = \tau_{+-}^0 - \frac{1}{8}w^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{+-}+q^2)^2} - \frac{1}{2}w^2 \tau_{+-}^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(t_{++}+q^2)(t_{--}+q^2)^2}. \quad (5.5g)$$

We note several important properties of Eq. (5.5). First, $\Gamma_+^{(1)}$ ($\Gamma_-^{(1)}$), $\Gamma_{++}^{(2)}$ ($\Gamma_{--}^{(2)}$), and τ_{++} (τ_{--}) are functions only of potentials in the + (−) subspace. Thus correlation and vertex functions involving only fields in the + (or −) subspace are completely independent of the existence of the other subspace. This is clearly as it should be, because G_+ (G_-) can be calculated without reference to G_- (G_+). Also as one would expect, $\Gamma_{+-}^{(2)}$ and τ_{+-} , which explicitly involve fields in both subspaces, are functions of all potentials. Second, it is clear from Eqs. (5.5a)–(5.5c) that perturbation theory for $\Gamma_{oo}^{(2)}$, at small t breaks down below eight dimensions. (That is, $\partial \Gamma^{(2)}/\partial t$ diverges for $d < 8$.) Third, the equations for $\Gamma_{++}^{(2)}$ and $\Gamma_{--}^{(2)}$ are identical in form to the equations for the two-point vertex in the lattice animals problem. We might, therefore, expect the localization problem and the lattice animals problem to be in the same universality class just below eight dimensions.

C. Recursion relations

Since perturbation theory breaks down at $d=8$, we develop renormalization-group recursion relations^{14,36} in the vicinity of eight dimensions to obtain critical exponents and scaling functions in $d=8-\epsilon$ dimensions. We find

$$\frac{dH_{\pm}}{dl} = \frac{1}{2}(d+2-\eta)H_{\pm} + \frac{1}{4}wK_d \frac{1}{1+t_{\pm\pm}} - \frac{1}{2}w\tau_{\pm\pm}K_d \frac{1}{(1+t_{\pm\pm})^2}, \quad (5.6a)$$

$$\frac{dr_{\pm\pm}}{dl} = (2-\eta)r_{\pm\pm} - \frac{1}{4}K_d w^2 \left(\frac{1}{1+r_{\pm\pm}} \right)^2 + K_d w^2 \tau_{\pm\pm} \left(\frac{1}{1+r_{\pm\pm}} \right)^3, \quad (5.6b)$$

$$\frac{dr_{+-}}{dl} = (2-\eta)r_{+-} - \frac{1}{8}K_d w^2 \left(\frac{1}{1+r_{+-}} \right) \left(\frac{1}{1+r_{++}} + \frac{1}{1+r_{--}} \right) + \frac{1}{4}K_d w^2 \left(\frac{1}{1+r_{+-}} \right) \left(\frac{\tau_{++}}{(1+r_{++})^2} + \frac{\tau_{--}}{(1+r_{--})^2} + \frac{2\tau_{+-}}{(1+r_{++})(1+r_{--})} \right), \quad (5.6c)$$

$$\frac{d\tau_{\pm\pm}}{dl} = (2-\eta)\tau_{\pm\pm} - \frac{1}{8}K_d w^2 \left(\frac{1}{1+r_{\pm\pm}} \right)^2 - \frac{1}{2}K_d w^2 \tau_{\pm\pm}^2 \left(\frac{1}{1+r_{\pm\pm}} \right)^4, \quad (5.6d)$$

$$\frac{d\tau_{+-}}{dl} = (2-\eta)\tau_{+-} - \frac{1}{8}K_d w^2 \left(\frac{1}{1+r_{+-}} \right)^2 - \frac{1}{2}K_d w^2 \tau_{+-}^2 \left(\frac{1}{1+r_{+-}} \right)^2 \left(\frac{1}{1+r_{++}} \right)^2, \quad (5.6e)$$

where $K_d^{-1} = 2^{d-1} \pi^{d/2} \Gamma(d/2)$. It is clear that $r_{++} = r_{--} = r_{+-} = r$, and $\tau_{++} = \tau_{--} = \tau_{+-} = \tau$ at the fixed point. Thus to minimize algebra, we will ignore the inequality of these variables in the equations for w and η :

$$\frac{dw}{dl} = \frac{1}{2}(6-d-3\eta)w + \frac{1}{2}K_d w^3 \left(\frac{1}{1+r}\right)^3 - 3K_d w^2 \tau \left(\frac{1}{1+r}\right)^4, \quad (5.7a)$$

$$\eta = -\frac{1}{2}K_d w^2 \tau. \quad (5.7b)$$

From Eqs. (5.6) and (5.7) we obtain the recursion relation for $g = K_d w^2 \tau$:

$$\frac{dg}{dl} = (\epsilon - 4\eta)g - \frac{13}{2}g^2. \quad (5.8)$$

This is identical to the equation for the analogous potential in the animals problem¹⁴ and is easily integrated to give

$$g(l) = e^{\epsilon l} g(0)/R(l), \quad (5.9)$$

where

$$R(l) = \left(1 + \frac{9}{2} \frac{g(0)}{\epsilon} (e^{\epsilon l} - 1)\right) - \frac{9}{2} \frac{g(0)}{\epsilon} e^{\epsilon l} \text{ as } l \rightarrow \infty. \quad (5.10)$$

Thus $g(l)$ reaches a fixed point with

$$g^* = \frac{2}{9}\epsilon \quad (5.11)$$

and

$$\eta = -\frac{1}{9}\epsilon. \quad (5.12)$$

The equations for $H_{\pm}$, $r_{\pm\pm}$, and r_{+-} in terms of g are

$$\frac{dH_{\pm}}{dl} = (5 - \frac{1}{2}\epsilon - \frac{1}{2}\eta)H_{\pm} + \frac{1}{2} \frac{g}{w} \frac{1}{1+r_{\pm\pm}}, \quad (5.13a)$$

$$\frac{dr_{\pm\pm}}{dl} = (2-\eta)r_{\pm\pm} + g \left(\frac{1}{1+r_{\pm\pm}}\right)^3, \quad (5.13b)$$

$$\frac{dr_{+-}}{dl} = (2-\eta)r_{+-} + \frac{1}{4}g \left(\frac{1}{1+r_{+-}}\right) \left(\frac{1}{1+r_{++}} + \frac{1}{1+r_{--}}\right). \quad (5.13c)$$

D. ++ and -- subspaces

The equations for r_{++} and H_{+} are again identical to those for the animals problem and decouple from the r_{+-} equation. Integrating Eq. (5.13b), we find

$$t_{++}(l) = e^{2l} [R(l)]^{-5/9} t_{++}(0) \cong e^{-\nu_Q^{-1} l} t_{++}(0), \quad (5.14)$$

where ν_Q is the constant- Q correlation length exponent satisfying

$$\nu_Q^{-1} = 2 - \frac{5}{9}\epsilon \quad (5.15)$$

and

$$r_{\pm\pm}(l) = t_{\pm\pm}(l) - \frac{1}{2}g \left(\frac{1}{2} \frac{r_{\pm\pm}}{(1+r_{\pm\pm})^2} + \frac{1+3r_{\pm\pm}}{1+r_{\pm\pm}} - 3r_{\pm\pm} \ln(1+r_{\pm\pm}) \right). \quad (5.16)$$

As usual, these rescaling relations are used until $|t_{++}(l^*)| \sim 1$ and correlation functions are matched to the one-loop expressions obtained from perturbation theory. The two-point vertices satisfy

$$\Gamma_{\lambda\lambda'}^{(2)}(r_{\lambda\lambda'}) = \exp\left(-\int_0^{l^*} [2-\eta(l)]dl\right) \Gamma_{\lambda\lambda'}^{(2)}(r_{\lambda\lambda'}(l^*)), \quad (5.17)$$

where

$$\Gamma_{\pm\pm}^{(2)}(l^*) = t_{\pm\pm}(l^*) + \frac{3}{2}g(l^*)r_{\pm\pm}(l^*) \ln r_{\pm\pm}(l^*). \quad (5.18)$$

If the matching is performed at constant Q , one easily obtains from Eqs. (5.14), (5.17), and (5.18) and $t_{\pm\pm}(l^*) = 1$, that

$$\Gamma_{\pm\pm}^{(2)}(t_{\pm\pm}) \sim t_{\pm\pm}^{\gamma_Q}, \quad (5.19)$$

where

$$\gamma_Q = (2-\eta)\nu_Q = 1 + \frac{1}{3}\epsilon. \quad (5.20)$$

However, we wish to find $\Gamma_{\lambda\lambda}^{(2)}$ as a function of E . To do this, we need to determine l^* in terms of E (not $t_{\pm\pm}$) from the equation of state. (Again the reader may wish to study the parallel analysis for lattice animals given in Ref. 14.) In this analysis, it is useful to know the behavior of the three-point function

$$\Gamma^{(3)}(r) = \exp\left(\frac{1}{2} \int_0^{l^*} (6-d-3\eta)dl\right) \Gamma^{(3)}(r(l^*)) \\ = [R(l^*)]^{2/3} w(0) \cong e^{-[\mu_3/\nu_Q]l^*} w(0), \quad (5.21)$$

where

$$\mu_3 = \frac{1}{3}\epsilon. \quad (5.22)$$

$\Gamma_{\pm}^{(1)}$ has a regular and a singular part. The singular part arises from the term proportional to $g r_{\pm\pm}^2/w$ in Eq. (5.13a) and is given by

$$\Gamma_{\pm, \text{sing}}^{(1)} = \frac{1}{2w(l^*)} \exp\left(-\int_0^{l^*} \frac{1}{2}(d+2-\eta)dl\right) t_{\pm\pm}^2(l^*) \\ = \frac{1}{2w(0)} e^{-4l^*} [R(l^*)]^{4/9} t_{\pm\pm}^2(l^*) \\ \sim \frac{1}{2w(0)} \exp\left(-\frac{2\gamma_Q - \mu_3}{\nu_Q} l^*\right) t_{\pm\pm}^2(l^*). \quad (5.23)$$

Notice that $\Gamma_{\pm, \text{sing}}^{(1)}$ is explicitly proportional to $1/w(l^*)$. Since w is an irrelevant variable, $\Gamma_{\pm, \text{sing}}^{(1)}$ does not scale like $e^{(d-2+\eta)l^*/2}$, i.e., hyperscaling is violated, and w is a *dangerous* irrelevant variable.¹⁵ The regular part of $\Gamma_{\pm}^{(1)}$ satisfies

$$\Gamma_{\pm, \text{reg}}^{(1)} = H_{\pm} - \frac{1}{2} \frac{r_{\pm\pm}^2}{w} - \frac{1}{8} \frac{g}{w} + \frac{1}{8} \frac{gr_{\pm\pm}}{w} - \frac{1}{4} \frac{g}{w} r_{\pm\pm}^2 \left(3 \ln(1+r_{\pm\pm}) - \frac{1}{1+r_{\pm\pm}} \right), \quad (5.24)$$

where w is a function of $Q_{\pm}$.

The equation of state now reads

$$\Gamma_{\pm, \text{sing}}^{(1)}(r_{\pm\pm}, Q_{\pm}) + \Gamma_{\pm, \text{reg}}^{(1)}(r_{\pm\pm}, Q_{\pm}) = 0 \equiv \Gamma_{\pm}^{(1)}(r_c, Q_c), \quad (5.25)$$

where the last part follows since the equation of state must also be satisfied at the critical point.

Thus, we have

$$e^{-4t^*} [R(l^*)]^{4/9} t_{\pm\pm}^{t^2}(l^*) = 2w(0) [\Gamma_{\pm, \text{reg}}^{(1)}(r_{\pm\pm}, Q_{\pm}) - \Gamma_{\pm, \text{reg}}^{(1)}(r_c, Q_c)] \quad (5.26a)$$

or

$$\exp \left[- \left(\frac{2\gamma_Q - \mu_3}{\nu_Q} \right) l^* \right] t_{\pm\pm}^{t^2}(l^*) = \frac{w \Delta E_{\pm}}{A}. \quad (5.26b)$$

If $w \Delta E_{+} = w \Delta E_{-}$ is positive (outside the band), we can choose $t_{++}(l^*) = t_{--}(l^*) = 1$ and we have

$$\Gamma_{++}^{(2)} = \Gamma_{--}^{(2)} = \left(\frac{w \Delta E_{\pm}}{A} \right)^{\gamma_a}, \quad (5.27)$$

where

$$\gamma_a = \frac{\gamma_Q}{2\gamma_Q - \mu_3} = \frac{1}{2} + \frac{\epsilon}{12}. \quad (5.28)$$

Similarly, the correlation length satisfies

$$\xi \sim e^{\nu_Q^{-1} l^*} \sim \left(\frac{w \Delta E_{\pm}}{A} \right)^{-\nu_a}, \quad (5.29)$$

where³⁷

$$\nu_a = \frac{\nu_Q}{2\gamma_Q - \mu_3} = \frac{1}{4} + \frac{\epsilon}{36}. \quad (5.30)$$

Inside the band, i.e., when $w \Delta E_{+}$ and $w \Delta E_{-}$ are negative, $\Gamma_{++}^{(2)}$ and $\Gamma_{--}^{(2)}$ are complex and, when $\Delta E_{+} = \Delta E_{-}$, they are complex conjugates of each other. Thus either l^* or $t_{++}(l^*)$ and $t_{--}(l^*)$ must have imaginary parts. It is much more convenient to work with real l^* . We therefore choose $t_{--}(l^*) = -i w / |w|$, and $t_{++}(l^*) = +i w / |w|$. In this case,

$$\Gamma_{\pm\pm}^{(2)} = \pm i (1 \pm \frac{3}{2} i g^*) (-w \Delta E_{\pm} / A)^{\gamma_a}. \quad (5.31)$$

As required $\Gamma_{\pm\pm}^{(2)} = (\Gamma_{\pm\pm}^{(2)})^*$ when $\Delta E_{+} = \Delta E_{-}$. Notice, however, that $\Gamma_{\pm\pm}^{(2)}$ is not purely imaginary as it was in mean-field theory. We are uncertain at this point whether the complex prefactor should be exponentiated in some way in lower dimensions.

In order to find the density of states, we need to find $\Delta Q_{\pm}$. This is done by observing that

$$t_{\pm\pm}(l) = \bar{t}_{\pm\pm}(l) + w(l) \Delta Q_{\pm}(l), \quad (5.32)$$

where $\bar{t}_{\pm\pm}(l)$ is $t_{\pm\pm}$ evaluated at Q_0 and

$$\Delta Q_{\pm}(l) = \exp \left(\frac{1}{2} \int_0^l [d - 2 + \eta(l)] dl \right) \Delta Q_{\pm}(0) \quad (5.33)$$

satisfies the recursion relations. We therefore have for $w \Delta E_{\pm} < 0$

$$\bar{t}_{\pm\pm}(l^*) = \mp i \frac{w}{|w|}, \quad (5.34)$$

implying

$$\rho \sim \pm \text{Im} \Delta Q_{\pm} \sim \frac{1}{|w|} \left(- \frac{w \Delta E_{\pm}}{A} \right)^{\beta_a}, \quad (5.35)$$

where

$$\beta_a = \frac{\gamma_Q - \mu_3}{2\gamma_Q - \mu_3} = \frac{1}{2} - \frac{1}{12} \epsilon. \quad (5.36)$$

Note that Eqs. (5.31) and (5.35) agree with the mean-field equations (4.40) and (4.38b) when $\epsilon \rightarrow 0$.

E. +- terms

So far, we have considered only variables in the ++ or -- subspaces that can be treated independently of the other subspace. Things are not so simple if we wish to calculate $\Gamma_{+-}^{(2)}$ and the conductivity. Equations (5.13b) and (5.13c) can be written in the form

$$\begin{bmatrix} \frac{dr_{++}}{dl} \\ \frac{dr_{--}}{dl} \\ \frac{dr_{+-}}{dl} \end{bmatrix} = \begin{bmatrix} 2 - \eta - 3g & 0 & 0 \\ 0 & 2 - \eta - 3g & 0 \\ -g & -g & 2 - \eta - g \end{bmatrix} \begin{bmatrix} r_{++} \\ r_{--} \\ r_{+-} \end{bmatrix} + \begin{bmatrix} f_{++}(l) \\ f_{--}(l) \\ f_{+-}(l) \end{bmatrix} \quad (5.37a)$$

or as

$$\frac{dr_e}{dl} = \sum_{p'} M_{pp'} r_{p'} + f_p, \quad (5.37b)$$

where $f_{\sigma\sigma'}(l)$ contains terms from Eqs. (5.13) not linear in $r_{\sigma\sigma'}$. The matrix $M_{pp'}$ is non-Hermitian, but is diagonalizable in terms of right and left eigenvectors

$$M_{pp'} = \sum_k \lambda^{(k)} e^{(k)}_p d^{(k)}_{p'}, \quad (5.38)$$

where $k = 1, 2, 3$.

$$\lambda^{(1)} = \lambda^{(2)} = 2 - \eta - 3g = \nu_Q^{-1} \quad (5.39a)$$

$$\lambda^{(3)} = 2 - \eta - g = 2 - \frac{1}{9} \epsilon > \nu_Q^{-1} \quad (5.39b)$$

The right eigenvectors

$$e^{(1)} = (1, 0, \frac{1}{2}), \quad e^{(2)} = (0, 1, \frac{1}{2}), \quad e^{(3)} = (0, 0, 1) \quad (5.40a)$$

and the left eigenvectors

$$d^{(1)} = (1, 0, 0), \quad d^{(2)} = (0, 1, 0), \quad d^{(3)} = (-\frac{1}{2}, -\frac{1}{2}, 1) \quad (5.40b)$$

form an orthonormal set:

$$\sum_p d_p^{(k)} e_p^{(k')} = \delta_{kk'}, \quad \sum_k d_p^{(k)} e_{p'}^{(k)} = \delta_{pp'}. \quad (5.41)$$

The scaling variables are

$$r^{(k)}(l) = \sum_p d_p^{(k)} r_p(l) \sim e^{\lambda^{(k)} l} r^{(k)}(0). \quad (5.42)$$

Thus $r^{(1)} = r_{++}$ and $r^{(2)} = r_{--}$ are scaling variables as required. The new scaling variable is

$$r^{(3)} = r_{+-} - \frac{1}{2}(r_{++} + r_{--}), \quad (5.43)$$

which is zero to the order we are considering by Eq. (5.4). When the inhomogeneous terms in Eq. (5.37) are included, the scaling variable is shifted so that

$$t^{(3)}(l) \sim e^{\lambda^{(3)} l} t^{(3)}(0), \quad (5.44)$$

where $t^{(3)}(l)$ satisfies an equation similar to Eqs. (5.16) and is zero when $r^{(3)} = 0$. $\lambda^{(3)}$ is the largest eigenvalue of the matrix $M_{pp'}$ and would normally dominate the behavior near the critical point. However, $t^{(3)} = 0$ to linear order. We believe that it remains zero when all nonlinear terms are included provided no external fields coupling to $t^{(3)}$ are introduced into the problem.

At the matching point $\Gamma_{++}^{(2)}$ is determined by

$$\begin{aligned} \Gamma_{++}^{(2)}(l^*, g) &= \frac{1}{2} [t_{++}(l^*) + t_{--}(l^*)] + t^{(3)}(l^*) \\ &+ \frac{1}{2} \left[\frac{3}{2} g r_{++}(l^*) \ln r_{++}(l^*) + \frac{3}{2} g r_{--}(l^*) \ln r_{--}(l^*) \right] \\ &+ \frac{1}{2} g t^{(3)}(l^*) \ln t^{(3)}(l^*) + e^{2l^*} g^2. \end{aligned} \quad (5.45)$$

In the nonconducting regime t_{++} and t_{--} have the same sign so that $\Gamma_{++}^{(2)}$ has essentially the same behavior as $\Gamma_{++}^{(2)}$ and $\Gamma_{--}^{(2)}$. In the conducting regime, however, t_{++} and t_{--} have opposite signs and are equal in magnitude when $E_+ = E_-$, yielding $\Gamma_{++}^{(2)}(E, E, q=0) = 0$ as required by the Ward identity. When $E_+ = E + \omega$ and $E_- = E$, we cannot choose both $t_{++}(l^*)$ and $t_{--}(l^*)$ to be unity. If we choose $t_{--}(l^*)$ to be $-i$, then from Eq. (5.26), we have $e^{-4l^*} R(l^*)^{4/9} = w \Delta E / A$. We have

$$t_{++}(l^*) = i \left(1 + \frac{w}{\Delta E} \right)^{1/2} \quad (5.46)$$

and

$$\begin{aligned} \Gamma_{++}^{(2)} &= \exp \left(- \int_0^{l^*} (2 - \eta) dl \right) \left(i \frac{1}{2} \frac{w}{\Delta E} + e^{2l^*} q^2 \right) \\ &\sim \left(- \frac{w \Delta E}{A} \right)^{\gamma_a} \left[i \frac{1}{2} \frac{w}{\Delta E} + \left(- \frac{w \Delta E}{A} \right)^{-2\nu_a} q^2 \right] \end{aligned} \quad (5.47)$$

which implies that

$$\Sigma \sim |\Delta E|^\mu \quad (5.48)$$

with

$$\mu = 2 - \gamma_a - 2\nu_a = 2\beta_a - \eta\nu_a, \quad (5.49a)$$

$$\mu = 1 - \frac{5}{36} \epsilon. \quad (5.49b)$$

Note that this implies that μ does not obey the Josephson relation $\mu = (d-2)\nu_a$. (Setting $\nu = \nu_Q$ does not work either.) The origin of the breakdown of the Josephson relation is the violation of hyperscaling in the equation of state. The ϵ -expansion results for the various exponents are summarized in Table I.

VI. LOCALIZED STATES

A. Introduction

The mean-field theory and ϵ -expansion investigation of the localization problem studied in the previous sections have failed to produce any localized states. We know from variational arguments discussed in Sec. II that localized states must exist even in high dimension. We also know that the field theory for the electron Green's functions developed in Sec. II is formally exact and should therefore produce a nonzero density of localized states.

What then have we left out of our analysis of the field theory? The answer is that there are locally stable and spatially nonuniform extrema of finite action corresponding to localized states. Cardy¹¹ has shown that these instantons of finite action reproduce the tail states for a Gaussian white-noise potential. In this case, the field theory is identical to ours except the field $\tilde{\Psi}(\vec{x})$ exists in continuous space (rather than on a lattice) and the hopping, $t_0(\vec{x}, \vec{x}')$, is the kinetic energy operator $-\hbar^2 \nabla^2 / (2m)$. Solutions of finite action for the continuum problem do not exist¹² for $d > 4$, i.e., there are no localized states for the Gaussian white-noise potential for $d > 4$. This is

TABLE I. ϵ -expansion results for various exponents for the strong-coupling regime ($\epsilon = 8 - d$). Values are for the usual case in which $G(\vec{x}, \vec{x}'; E)$ is allowed to vary. For lattice animals this would be the case of constant H .

β_a	$\frac{1}{2} - \epsilon/12$
γ_a	$\frac{1}{2} + \epsilon/12$
ν_a	$\frac{1}{4} + \epsilon/36$
η	$-\epsilon/9$
μ	$1 - 5\epsilon/36$
ζ^a	$\frac{1}{2} + \epsilon/18$

^a ζ is defined by $\ln \rho_{loc} \sim -|E - E_c|^{-\zeta}$.

presumably because there are regions of unbounded variation of the potential that prevent localized states from forming.

On a lattice or when there are spatial correlations in the random potential, there is a length scale or wave-number cutoff, Λ , associated with the lattice spacing or correlation length of the potential. This makes it possible for localized solutions of finite action to exist even above four dimensions. These solutions can be broken up into a core and a far-field part. The character of the core is determined by the nature of the cutoff whereas that of the far field is very similar to that of the Gaussian white-noise solutions. We call these solutions *lattice instantons*.

The goal of this section is, thus, to develop solutions of finite action to the field equations describing the localization problem. Although our treatment is not rigorous, we feel that our results are correct inasmuch as they agree with the variational calculations of Sec. II. Since we are interested in both strong- and weak-coupling localization, we will seek instantons in both the $\tilde{\Psi}$ and Q variables. The instantons in terms of the Q variables are essentially identical to the metastable droplets for an Ising model in a negative field studied by Langer.³⁸

B. Instantons in ψ : Site randomness

In this section, we will study localized states for Gaussian site randomness. Since we will be interested in states not only near the band edge but also for large E , we begin by adding and subtracting counter terms³⁹ to L so that $G(q=0)$ will diverge at the band edge in the dilute limit. We therefore write

$$L[\tilde{\Psi}; E, E] = L[\tilde{\Psi}; E', E'] - \frac{1}{2} \sum_{\vec{x}} \tilde{\Psi}^T(\vec{x}) (E - E') \tilde{\Psi}(\vec{x}). \quad (6.1)$$

To one-loop order we set $E'_0 = E_0 + \sigma_s^2 G_0(\vec{x}, \vec{x}; E'_0)$. In this approximation the band edges occur at $E_0 = \pm zt_0 - \sigma_s^2 G_0(\vec{x}, \vec{x}; zt_0)$. [Note that our definition of the Green's function via Eq. (2.2) differs in sign from the usual²³ one.] We seek solutions which are extrema of $L(E')$ and treat the remainder perturbatively. The condition for $L(E')$ to be an extremum is

$$\begin{aligned} \frac{\partial L}{\partial \psi_{\vec{x}}^{\alpha}(\vec{x})} &= - \sum_{\vec{x}'} [E'_0 \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}')] \psi_{\vec{x}'}^{\alpha}(\vec{x}') \\ &\quad - \frac{1}{2} \sigma_s^2 \sum_{\vec{x}'} [\psi_{\vec{x}'}^{\alpha}(\vec{x}')^2 \psi_{\vec{x}}^{\alpha}(\vec{x})] = 0. \end{aligned} \quad (6.2)$$

It is clear that solutions to this equation can be expressed in the form

$$\psi_{\vec{x}}^{\alpha}(\vec{x}) = e^{\alpha} \psi_{\vec{x}}(\vec{x}), \quad (6.3)$$

where e^{α} is any n -component unit vector and where

$$\sum_{\vec{x}'} [E'_0 \delta_{\vec{x}, \vec{x}'} + t_0(\vec{x}, \vec{x}')] \psi_{\vec{x}'}(\vec{x}') + \frac{1}{2} \sigma_s^2 \psi_{\vec{x}}^3(\vec{x}) = 0. \quad (6.4)$$

If the spatial variations of $\psi_{\vec{x}}(\vec{x})$ are slow, as they will be far from the core of a localized solution, we can expand $t_0(\vec{x}, \vec{x}')$ in powers of gradients and treat $\psi_{\vec{x}}(\vec{x})$ as a continuous field. If $E < -zt_0$, Eq. (6.4) reads

$$(\Delta E + C_0 \nabla^2) \psi_{\vec{x}}(\vec{x}) + \frac{1}{2} \sigma_s^2 \psi_{\vec{x}}^3(\vec{x}) = 0, \quad (6.5)$$

where $(\Delta E) = E' + zt_0$ measures the distance from the band edge and C_0 is $a^2 t_0$ or $\hbar^2/(2m)$ for the discrete model or the continuum model, respectively. If $E' < -zt_0$, Eq. (6.5) has scaled solutions of the form

$$\psi_{\vec{x}}(\vec{x}) \sim \frac{1}{\sigma_s} (-\Delta E)^{1/2} f_{\pm}(-\Delta E(\vec{x} - \vec{x}_0)^2/C_0), \quad (6.6)$$

where

$$\nabla^2 f_{\pm} - f_{\pm} + f_{\pm}^3 = 0. \quad (6.7)$$

In Eq. (6.6) $\vec{x}_0$ locates the position of the instanton. The above equations only apply as long as $(x - x_0) \gg \Lambda^{-1}$, and constitute the far-field part of the instanton. It contributes a part L_f to the action. Substituting Eq. (6.6) into the continuum version of Eq. (3.17) for $|\vec{x} - \vec{x}_0| > \Lambda^{-1}$, we find

$$L_f \sim \frac{C_f}{\sigma_s^2} (-\Delta E)^{2-(d/2)}, \quad (6.8)$$

which is the same form as the total action obtained by Cardy¹¹ and others⁴⁰ for the Gaussian white-noise potential for $d < 4$. Equation (6.8) is, however, valid for all dimensions. Note that $L_f \rightarrow \infty$ as $E'_0 \rightarrow -zt_0$ (i.e., as E_0 approaches the band edge) for $d > 4$ but as $E'_0 \rightarrow -\infty$ for $d < 4$.

The core part of the instanton is difficult to obtain in general. We can, however, obtain results for large $|E'|$ by solving $\psi_{\vec{x}}(\vec{x})$ at sites successively further from the center at $\vec{x} = 0$ in terms of $\psi_{\vec{x}}(0)$. Let $\psi(1)$ be the amplitude of $\psi(\vec{x})$ at sites $\vec{x}$ nearest neighbor to the origin, $\psi(2)$ be $\psi(\vec{x})$ at sites next nearest neighbor to the origin, and so on. Then we have

$$E'_0 \psi(0) + zt_0 \psi(1) + \frac{1}{2} \sigma_s^2 \psi^3(0) = 0, \quad (6.9a)$$

$$E'_0 \psi(1) + t_0 \psi(0) + t_0 \psi(3) + (z - 2) t_0 \psi(2) + \frac{1}{2} \sigma_s^2 \psi^3(1) = 0,$$

$$\dots \quad (6.9b)$$

It is clear that for k not too large, $\psi(k) \sim (t_0/E) \psi(k-1)$, so that for $|E| \gg zt_0$.

$$\psi_{\vec{x}}^2(0) = -\frac{2E'_0}{\sigma_s^2} \left[1 + O\left(\frac{zt_0^2}{E^2}\right) \right]. \quad (6.10)$$

This when substituted into Eq. (3.17) yields the core part of L ,

$$L_c = \frac{1}{2} \frac{(E'_\pm)^2}{\sigma_s^2}. \quad (6.11)$$

For $|E| \gg zt_0$, E' can be replaced by $E_\pm$. The total action is the sum of the core and far field parts:

$$L_I = L_f + L_c. \quad (6.12)$$

The leading contributions to Ξ and $G(\vec{x}, \vec{x}')$ from single-instanton states are proportional to $\exp(-L_I)$ with prefactors which may be obtained by integrating over Gaussian fluctuations about solutions to Eq. (6.2) using the method of collective coordinates.³⁸ In this paper, we do not propose to calculate prefactors exactly. We do, however, need the contributions to the prefactors coming from degrees of freedom associated with uniform translation and uniform rotation of the vector e^α that leave the energy of the instanton invariant. With these, we find that

$$\delta\Xi = ianNe^{-L_I}, \quad (6.13a)$$

$$\delta G(\vec{x}, \vec{x}') = ia'e^{-L_I} \sum_{\vec{x}_0} \psi_\pm(\vec{x} - \vec{x}_0) \psi_\pm(\vec{x}' - \vec{x}_0), \quad (6.13b)$$

where a and a' are prefactors which may depend on E . The factor of N in Eq. (6.13a) comes from the sum over all possible positions of $\vec{x}_0$ and the factor of n comes from the integral over solid angles associated with the unit vector e^α . In both Eqs. (6.13) there is a factor i because one eigenvalue of the stability matrix describing Gaussian fluctuations about the instanton solution is negative. The original contours of integration over $\psi_\pm$ are complex, as shown in Fig. 5(b), so that the integral over the wave function associated with the negative eigenvalue converges. The contour in the vicinity of the instanton solution is deformed as shown in Fig. 5(c) yielding a prefactor that is almost completely imaginary.

Equation (6.13) represents the first term in an expansion for $\delta\Xi$ and δG in the density of instantons, $\sim e^{-L_I}$. Higher-order terms can be neglected provided $e^{-L_I} \ll 1$, i.e., when $L_I \gg 1$. L_I is much greater than unity in two limits. First, for large E , L is dominated by the quadratically divergent core part. Thus, we have

$$\rho \sim \text{Im} \delta G(\vec{x}, \vec{x}) \sim e^{-(1/2)E^2/\sigma_s^2} \ll 1. \quad (6.14)$$

This result agrees with the variational calculation of Sec. II. Notice that for all $d > 0$, L_c always dominates L_f for large E . Thus on a lattice, with Gaussian site random potentials the lowest energy states are always those localized on a single site which accidentally has a very large energy. Sec-

ond, when $\Delta E \rightarrow 0$ for $d > 4$, L diverges and is dominated by L_f yielding

$$\rho \sim \text{Im} \delta G(\vec{x}, \vec{x}) \sim \exp\left(-\frac{C_f}{\sigma_s^2} |\Delta E|^{-(d/2-2)}\right). \quad (6.15)$$

Thus for $d > 4$, the density of localized states falls exponentially to zero at the band edge in the weak-coupling regime.

We close this section with an observation about the n dependence of Eq. (6.13a). We know that since $\Xi = [Z^n]_{av}$, the uniform solutions treated in Secs. VI and V, all have $F \equiv nN\mathcal{F}$ proportional to n . The contributions to Ξ from instantons are additive, implying

$$\Xi = e^{-nN\mathcal{F}} \pm i n N a e^{-L_I} \quad (6.16a)$$

or

$$\frac{1}{Nn} \ln \Xi = -\mathcal{F} \pm i a e^{-L_I} \equiv \frac{F}{Nn}. \quad (6.16b)$$

Thus instanton contributions are additive not only to Ξ but also to F and to correlation functions obtained there from.

C. Instantons in ψ : Bond randomness

When there is pure Gaussian bond randomness, there is no possibility of a weak coupling mobility edge. It is thus clear that in order to study the density of states near the mobility edge, it is more productive to study localized solutions to $L(Q)$ rather than $L(\Psi)$. It is nonetheless interesting, for states far from the mobility edge in particular, to study localized solutions to $L(\Psi)$. From Eq. (3.17), we have

$$\frac{\partial L}{\partial \psi_\pm^\alpha(\vec{x})} = -E_\pm \psi_\pm^\alpha(\vec{x}) - \sigma_b^2 \sum_{\vec{x}', \beta} \gamma_{\vec{x}, \vec{x}'} \psi_\pm^\beta(\vec{x}') \psi_\pm^\alpha(\vec{x}') = 0, \quad (6.17)$$

where we do not distinguish between E and E' . Setting $\psi_\pm^\alpha(\vec{x}) = e^\alpha \psi_\pm(\vec{x})$ as before, we find

$$E_\pm \psi_\pm(\vec{x}) + \sigma_b^2 \psi_\pm(\vec{x}) \sum_{\vec{x}'} \gamma_{\vec{x}, \vec{x}'} \psi_\pm(\vec{x}') = 0. \quad (6.18)$$

$\psi_\pm(\vec{x}) = 0$ is always a solution to this equation. For those sites $\vec{x}$ for which $\psi_\pm(\vec{x}) \neq 0$, which we call occupied sites, we have

$$\sigma_b^2 \sum_{\vec{x}'} \gamma_{\vec{x}, \vec{x}'} \psi_\pm(\vec{x}') = -E_\pm. \quad (6.19)$$

There are many solutions to Eqs. (6.18) and (6.19). All can be represented diagrammatically as clusters on a lattice in which bonds connect sites with $\psi_\pm(\vec{x}) \neq 0$. Here we will illustrate only a few. The simplest solution corresponds to a cluster of two sites as shown in Fig. 8(a). Denoting the two sites in the cluster by 1 and 2, we have

FIG. 8. Clusters for bond instantons. Bonds connecting occupied sites are indicated by solid lines. The dashed lines serve to define the lattice structure. (a) A one-bond cluster treated in Eq. (6.20). (b) A cluster in which one site is connected to several of its neighbors. (c) A cluster formed by a loop of $2b$ bonds for $b=4$.

$$\psi^2(1) = \psi^2(2) = -E/\sigma_b^2, \quad (6.20)$$

which leads to the result

$$L_I = \frac{1}{2} E^2 / \sigma_b^2. \quad (6.21)$$

This yields a density of states proportional to $\exp(-E^2/2\sigma_b^2)$ in agreement with variational arguments. There are, however, several other solutions with the same L_I . All diagrams with an occupied central site surrounded by from two up to z occupied sites, an example of which is illustrated in Fig. 8(b), also have $L_I = \frac{1}{2} E^2 / \sigma_b^2$. Another class of solutions are closed loops with $2b$ bonds with $b \geq 2$ [see Fig. 8(c)]. One can show that

$$L_I = \frac{1}{4} b E^2 / \sigma_b^2 \quad (6.22)$$

$$L[\varphi_\sigma, Q_\sigma] = n \left(\frac{1}{2} \sum_{\vec{x}} \ln[E + Q_\sigma(\vec{x})] + \frac{1}{4\sigma_b^2} \sum_{\vec{x}, \vec{x}'} Q_\sigma(\vec{x}) \gamma_{\vec{x}, \vec{x}'}^{-1} Q_\sigma(\vec{x}') \right) + \frac{1}{2} \sum_{\vec{x}} \{ \ln[E + Q_\sigma(\vec{x}) + \varphi_\sigma(\vec{x})] - \ln[E + Q_\sigma(\vec{x})] \} + \frac{1}{4\sigma_b^2} \sum_{\vec{x}, \vec{x}'} \gamma_{\vec{x}, \vec{x}'}^{-1} [\varphi_\sigma(\vec{x}') + 2Q_\sigma(\vec{x}')] \varphi_\sigma(\vec{x}). \quad (6.24)$$

The uniform physical solution obtained from Eq. (4.12) is recovered by imposing the condition that $\lim_{n \rightarrow 0} \Xi = 1$, i.e., that L be of order n . In that case $\varphi_\sigma(\vec{x}) = 0$. However, as we have seen in Eq.

for these graphs. Thus for large $|E|$, these solutions, except for $b=2$, contribute negligibly to the density of states.

One can see that Eq. (6.18) will not produce solutions that are valid near the mobility edge. This is because it does not take into account the fact that $\langle \psi^2 \rangle$ is nonzero in the uniform phase.

D. Instantons in Q

In this section, we will discuss localized solutions of finite action in terms of the fields Q rather than $\tilde{\Psi}$. For simplicity, we will limit our discussion to the case of pure bond randomness though we expect most results to apply generally to strong-coupling systems. Unlike $\tilde{\Psi}$, Q has a nonvanishing expectation value for all values of the energy, and the problem becomes similar to that of a metastable Ising magnet in negative field. We will demonstrate that there are two classes of extremal solutions to the field equations. Of these, only the one discussed in Secs. IV and V yields a free energy such that $\lim_{n \rightarrow 0} \Xi \rightarrow 1$ as required.⁴¹ We will call this the physical solution. The other violates this relation and cannot be used as a solution throughout space. In the region of localized states, however, it has a lower free energy than the physical solution. Thus the physical solution is only metastable³⁵ in the region of localized states, and localized droplets of the second solution yielding an imaginary part to Ξ and Q are possible. We will study these localized solutions first heuristically in terms of extremal droplets and then somewhat more formally in terms of solutions to the field equations. We will be concerned primarily with situations where the far-field part of the solution dominates.

In studying $\tilde{\Psi}$ instantons, we found that solutions of the form $\psi_\pm^\alpha(\vec{x}) = e^\alpha \psi_\pm(\vec{x})$ yielded extrema to the action. Since $Q_{\pm\pm}^{\alpha\beta} \sim \langle \psi_\pm^\alpha \psi_\pm^\beta \rangle$, we will seek nonuniform solutions of the form

$$Q_{\sigma\sigma'}^{\alpha\beta}(\vec{x}) = \delta_{\sigma\sigma'} (Q_\sigma(\vec{x}) \delta_{\alpha\beta} + \varphi_\sigma(\vec{x}) e^\alpha e^\beta). \quad (6.23)$$

Inserting this form into Eq. (3.32) we find that

(6.13), the instanton is accompanied by a prefactor proportional to n , so that the extrema we seek will have L of order unity. As a first step, we will study the *uniform* metastable solution found by

setting $Q_\sigma(\vec{x}) = Q_\sigma^{\text{meta}}$ and $\varphi_\sigma(\vec{x}) = \varphi_\sigma$. The condition that L be external is then

$$\frac{\partial L}{\partial Q_\sigma^{\text{meta}}} = 0 = \frac{\varphi_\sigma}{2z\sigma_b^2} + \frac{1}{2} \left(\frac{1}{E_\sigma + Q_\sigma^{\text{meta}} + \varphi_\sigma} - \frac{1}{E_\sigma + Q_\sigma^{\text{meta}}} \right), \quad (6.25a)$$

$$\frac{\partial L}{\partial \varphi_\sigma} = 0 = \frac{1}{2z\sigma_b^2} (\varphi_\sigma + Q_\sigma^{\text{meta}}) + \frac{1}{2} \frac{1}{E_\sigma + Q_\sigma^{\text{meta}} + \varphi_\sigma}. \quad (6.25b)$$

Comparing Eqs. (6.25b) and (4.12) we see that $\varphi_\sigma + Q_\sigma^{\text{meta}} = Q_\sigma^0$, where Q_σ^0 now denotes the stable solution, i.e., the solution of Eq. (4.12). Using this determination in Eq. (6.25a) yields

$$\varphi_\sigma = E_\sigma + 2Q_\sigma^0 \quad (6.26a)$$

$$= E_\sigma (1 - 4z\sigma_b^2/E_\sigma^2)^{1/2}, \quad (6.26b)$$

$$Q_\sigma^{\text{meta}} = -(E_\sigma + Q_\sigma^0). \quad (6.26c)$$

Since the free energy of the stable solution is of order $n \rightarrow 0$, we find the difference in free energy per site ΔF between the physical solution and that of Eq. (6.26) to be

$$-\Delta F \equiv L = -\frac{x}{1-x^2} + \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \quad (6.27)$$

where $x = (1 - 4\sigma_b^2 z/E_\sigma^2)^{1/2}$. For $4\sigma_b^2 z < E_\sigma^2 < \infty$, x lies between zero and one, and $L < 0$. If $\varphi_\pm = 0$, $L = 0(n)$. Thus for all $E_\pm$ in the region of localized states, the extremum with $\varphi_\pm \neq 0$ is lower in free energy than the physical solution. The system is not, however, allowed to condense into the lower energy state because it would violate the requirement $\lim_{n \rightarrow 0} \Xi \rightarrow 1$. Thus throughout the localized region the system is metastable in exactly the same sense as the Ising model in a negative field studied by Langer.³⁸

We can estimate the contribution of large, far-field dominated, droplets of the lower energy state to $\text{Im} \Xi$ and ρ in the usual way. The energy of a droplet consists of two parts: a volume part and surface part. Let R be the radius of the droplet and σ be the surface tension, then the droplet energy is

$$E_I = -\Delta F R^d + \sigma R^{d-1}. \quad (6.28)$$

For the optimum droplet, E_I is an extremum with respect to variations in R , whence

$$R = \frac{d-1}{d} \left(\frac{\sigma}{\Delta F} \right). \quad (6.29)$$

With this radius,

$$E_I = \frac{1}{d} \left(\frac{d-1}{d} \right)^{d-1} \frac{\sigma^d}{(\Delta F)^{d-1}}. \quad (6.30)$$

Note that σ is of order $(\Delta F) \xi$, where ξ is the correlation length, so that

$$E_I \sim \Delta F \xi^d. \quad (6.31)$$

Near the mobility edge,

$$\Delta F \sim \gamma^3 \sim (1 - 4\sigma_b^2 z/E^2)^{3/2} \quad (6.32)$$

and $\xi \sim \gamma^{-1/2}$, so that

$$E_I \sim \gamma^{(6-d)/2}, \quad (6.33a)$$

$$\text{Im} \Xi \sim \exp(-c_1 \gamma^{(6-d)/2}), \quad (6.33b)$$

$$\rho \sim \exp(-c_2 \gamma^{(6-d)/2}) \quad (6.33c)$$

$$\sim \exp \left[-c_2 \left(\frac{E^2 - E_c^2}{E^2} \right)^{-\xi} \right], \quad (6.33d)$$

where c_1 and c_2 are constants and $\xi = \frac{1}{4}(d-6)$. Thus above eight dimensions where mean field is valid, ρ goes to zero at the mobility edge just as in the weak-coupling case.

The results of Eq. (6.30) become invalid below eight dimensions when critical fluctuations become important. In order to study the localized states for $d < 8$, it is convenient to derive the instantons solutions from the field theory used to study the ϵ expansion. Near the mobility edge, $\varphi_\pm$ in Eq. (6.23) is small, so that we are justified in studying the field truncated at third order in φ_λ . Using the mean-field theory as a guide, we seek localized solutions with $\varphi_\pm^{\alpha\beta} = (e^\alpha e^\beta - \delta^{\alpha\beta}) \varphi_\pm$. The Lagrangian in the localized regime with this $\varphi_\pm$ is

$$L = \int d^d x \left[\frac{1}{2} (1-n) r_\pm \varphi_\pm^2 + \frac{1}{2} (1-n) (\nabla \varphi_\pm)^2 - \frac{1}{6} (1-n) w \varphi_\pm^3 \right]. \quad (6.34)$$

The linear term in φ vanishes because $Q_\pm$ is chosen to have the correct equilibrium value. The τ_{pp} term in Eq. (5.3) is also unimportant until loop corrections are calculated. Higher-order gradient terms have also been omitted from Eq. (6.34) because we are interested in instantons near the mobility edge where far-field effects dominate. Minimizing L , we find

$$-\nabla^2 \varphi_\pm + r_\pm \varphi_\pm - \frac{1}{2} w \varphi_\pm^2 = 0. \quad (6.35)$$

Solutions to this equation have a scaling form

$$\varphi_\pm = \frac{r_\pm^2}{w} f(\sqrt{r_\pm}(\vec{x} - \vec{x}_0)) \quad (6.36)$$

with

$$-\nabla_y^2 f(y) + f(y) - \frac{1}{2} f^2(y) = 0. \quad (6.37)$$

Using Eq. (3.24) in Eq. (6.34) for $|x - x_0| \gg \Lambda^{-1}$, we find

$$L = \frac{C_d}{w^2} \gamma^{(6-d)/2} + L_c, \quad (6.38)$$

where $C_d = \int dy f^3(y)$, where the integral is over the far-field region, and L_c is the core part. This answer agrees with Eqs. (6.32) and (6.33). It

shows that for $d > 6$, the density of instantons and states goes to zero at the mobility edge. Again, however, it does not incorporate critical fluctuations for $d < 8$. To include critical fluctuations, we can appeal to scaling in the vicinity of the critical point as was done for the metastable Ising model.⁴² We therefore set $r(l^*) \sim \exp(\nu_Q^{-1} l^*) r(0)$ and $w^2(l^*) \exp(6-d-3\eta-2\mu_3/\nu_Q) l^*$. Choosing l^* according to Eq. (5.26) with $r(l^*) = 1$, we obtain

$$L \sim \frac{1}{w^2} \exp\left(-\frac{3(2-\eta)\nu_Q - 2\mu_3 - d\nu_Q}{\nu_Q} l^*\right) \quad (6.39)$$

or

$$L \sim \frac{1}{w^2} (\Delta E)^{-\zeta}, \quad (6.40)$$

where

$$\zeta = (d-2+\eta)\nu_a - 2\beta_a \quad (6.41a)$$

$$= \frac{1}{2} + \epsilon/18. \quad (6.41b)$$

Thus, as in the weak-coupling case, the density of states goes exponentially to zero at E_c , although with a different power of ΔE .

VII. DISCUSSION AND CONCLUSION

A. Discussion

In this section we wish to consider some miscellaneous topics relevant to the presentation of the previous sections.

First, we note that the relation⁸ $\beta_a + \gamma_a = 1$ follows from the fact that the density of states ρ is zero at the mobility edge and that ρ can be obtained either from G_o or from $\mathcal{G}_{+, -\sigma}$ via Eq. (2.14). First consider $\mathcal{G}_{+, -}(0; E, E)$. By definition, it is a function of $E + i\eta$ and $E - i\eta$ with η a positive infinitesimal. Recall that $\mathcal{G}_{+, -}(0; E, E)$ diverges as $E \rightarrow E_c$ and $\eta \rightarrow 0$, so that η is a field that moves $\mathcal{G}_{+, -}$ away from criticality. Assuming that η is a scaling field, we may write

$$\mathcal{G}_{+, -}(0; E, E) = |E - E_c|^{-\gamma_a} f\left(\frac{i\eta}{|E - E_c|^\varphi}\right), \quad (7.1)$$

where φ is a crossover exponent. For E outside the band, we have $f(x) \sim 1$, as $x \rightarrow 0$ by the definition of γ_a . For E inside the band, Eq. (2.14) insures us that $f(x) \sim x^{-1}$ as $x \rightarrow 0$. Furthermore, it is clear from the analysis of Sec. V [cf. Eqs. (5.26), (5.34), and (5.46)] that $\varphi = 1$. Alternatively, it is obvious that Q_o is a function only of $E - E_c + i\sigma\eta$, so that

$$Q_o = |E - E_c|^{\beta_a} g\left(\frac{i\eta}{|E - E_c|}\right) \quad (7.2)$$

which implies that $\varphi = 1$. From Eqs. (7.1) and (2.14) we therefore have

$$\rho = \lim_{\eta \rightarrow 0} \frac{\eta}{\pi} \mathcal{G}_{+, -}(0; E, E) \sim |E - E_c|^{\gamma_a} \quad (7.3a)$$

$$\sim |E - E_c|^{1-\gamma_a}, \quad (7.3b)$$

so that

$$\beta_a = 1 - \gamma_a. \quad (7.4)$$

This striking relation is satisfied by the animals' exponents for which¹⁴ $\gamma_a = \gamma_Q/(2\gamma_Q - \mu_3)$ and $\beta_a = (\gamma_Q - \mu_3)/(2\gamma_Q - \mu_3)$ for all values of γ_Q and μ_3 , and not just to first order in $\epsilon = 8 - d$. This reinforces our belief that the strong-coupling theory presented here is correct. It is also interesting to observe that $2 - d\nu_a = -23\epsilon/36 < 0$ in agreement with heuristic arguments⁴³ that this quantity should be negative in a random system having a continuous transition. (Note that we use $2 - d\nu_a$ rather than the specific-heat exponent α , since hyper-scaling is violated.)

We note that our results do not apply to the site-random model proposed by Lloyd⁴⁴ in which $t(x, x') = -t_o(x, x')$ and $P(V(x))$ is

$$P(V) = \frac{\Delta}{\pi} \frac{1}{V^2 + \Delta^2}. \quad (7.5)$$

For this model one can obtain the exact solutions

$$G_o(\vec{x}, \vec{x}'; E) = -\frac{1}{N} \sum_{\vec{k}} \frac{\exp[i\vec{k} \cdot (\vec{x} - \vec{x}')] }{E + i\sigma\Delta + t_o(\vec{k})}, \quad (7.6a)$$

$$\rho(E) = \frac{\Delta}{N\pi} \sum_{\vec{k}} \{[t_o(\vec{k}) + E]^2 + \Delta^2\}^{-1/2}, \quad (7.6b)$$

both of which are analytic for all real E for arbitrary dimension. Clearly this density of states does not vanish at the mobility edge presumed to exist for this model. It is obvious, however, that this model is in a different universality class from the ones we have studied, because the second moment of P , $\int P(V) V^2 dV$, is infinite. It was the results for the Lloyd model which led to the conventional belief that $\rho(E)$ would show no structure at the mobility edge.

B. Speculations about lower dimensions

In this paper we have studied the problem of localization of electrons in an Anderson model on a lattice with random site and/or hopping potentials principally in spatial dimension greater than four. We found that the density of states at the mobility edge goes to zero as shown in Fig. 1(d). This behavior is different from the one depicted in Fig. 1(c) and which is generally believed to be true for $2 < d < 4$ in which the density of states shows no structure in the vicinity of the mobility edge. For this case, Wegner⁵ has argued that if Σ undergoes a second-order transition at E_c , then μ must satisfy the Josephson relation $\mu = (d-2)\nu$

and η the relation $2 - \eta = d$. These relations have recently been verified²¹ in a $1/n_0$ expansion for a model with n_0 orbitals per site and in a $2 + \epsilon$ dimensional expansion^{17,18} based on the action of Eq. (3.32). It would be of some interest to see if the high-dimensional theory studied here shows any precursor of the predicted lower-dimensional behavior.

We have studied weak- and strong-coupling mobility edges. In the former case, the density of extended states, ρ_{ext} , and the conductivity grow, respectively, as $(\Delta E)^{d/2-1}$ and (ΔE) , whereas the density of localized states, ρ_{loc} , goes to zero as $\exp[-(\Delta E)^{-d/2+2}]$ near the mobility edge. It is clear that $d=4$ is a special dimension. As $d=4$ is approached from above, the region around the mobility edge where ρ_{loc} is near zero becomes smaller and smaller. We cannot, however, use the weak-coupling picture all the way down to four dimensions, so that it is necessary to look at the strong-coupling case.

In the strong-coupling case, $\rho_{\text{ext}} \sim (\Delta E)^{\beta_a}$, $\Sigma \sim (\Delta E)^\mu$ and $\rho_{\text{loc}} \sim \exp[-(\Delta E)^{-\zeta}]$, where $\zeta = \frac{1}{4}(d-6)$ for $d > 8$ and $\zeta = (d-2+\eta)\nu_a - 2\beta_a$ for $d < 8$. To first order in $\epsilon = 8-d$, the strong-coupling localization problem and that of the statistics of lattice animals¹⁴ are in the same universality class. We do not know yet if this the case to higher order in ϵ . In order to go continuously to the low- d results, β_a must go to zero and η to $2-d$ at some critical dimension d^* . Since $d=4$ is special for the weak-coupling mobility edge, it seems likely that d^* is 4 for the strong-coupling edge. Furthermore, there is numerical evidence⁴⁵ that β_a is near zero at $d=4$. Thus, it is possible that localization and lattice animals remain in the same universality class for $4 < d < 8$. This view is strengthened by the fact that $\beta_a + \gamma_a = 1$ for the animals problem. There is no information about the behavior of ν_a in high dimension other than that $\nu_a = \frac{1}{4}$ for $d > 8$, and that $\nu_a = \frac{1}{4} + \epsilon/36$ to first order in ϵ .^{14,37} In order to satisfy $d = 2 - \eta = \gamma/\nu = \gamma_a/\nu_a$ and $\beta_a = 0$ at $d=4$, ν_a would have to equal $\frac{1}{4}$, since $\gamma_a = 1 - \beta_a = 1$. This would mean that ν_a would have to be a non-monotonic function of d rising from a value $\frac{1}{4}$ at $d=8$ and falling back to the same value at $d=4$, and rising again to of order⁴⁶ 0.6 and 0.5 at $d=2$ and 3, respectively. Though the exponent ν is usually monotonic, we can see no reason why it has to be. It is interesting to note that for animals the generalized Flory approximation⁴⁷ gives $\nu_a = 5/(2d+4)$ for $d < 8$. Thus we have two possibilities consistent with $d^*=4$: (a) Localization and lattice animals are in the same universality class for $4 < d < 8$ with $\beta_a = 0$ and $\nu_a = \frac{1}{4}$ at $d=4$, and (b) localization and lattice animals are not in the same universality class to second or higher order

in ϵ and the approach to $d=4$ is different from that obtained by extrapolating the animals results. At the moment, we favor the first choice, though further numerical study and calculations to higher order in ϵ are clearly of interest.

C. Conclusions

The main conclusion to be drawn from this work are the following.

(1) We have constructed an exact field-theoretic expression to describe localization, cf. Eq. (3.32). As we show in Sec. IV, one can easily obtain a mean-field approximation to the Lagrangian which produces sensible results for the density of states, the macroscopic conductivity, and other quantities of interest.

(2) We have been led to distinguish between the regimes of weak and strong coupling, the former being the regime where the randomness does not qualitatively disturb the band and the latter when the single-particle Green's function never becomes critical. We noted the existence of, but did not study, the multicritical point where the two regimes meet.

(3) We have determined the critical dimension d_c below which fluctuations invalidate mean-field theory to be⁴⁸ $d_c = 8$. In analogy with previous work on the statistics of lattice animals¹⁴ we then developed ϵ -expansion results for the correlation functions of interest near the mobility edge. The ϵ -expansion results for various exponents are given in Table I.

(4) The ϵ -expansion results obey the non-trivial relation $\gamma_a + \beta_a = 1$, which we previously derived⁸ using scaling arguments in conjunction with the Ward identity, Eq. (2.12), which related the two-particle and one-particle Green's functions.

(5) The results do not obey the Josephson relation, $\mu = (d-2)\nu$, proposed by Wegner.⁵ This discrepancy arises from the violation of hyperscaling caused by the appearance of the dangerous irrelevant variable, w , the coefficient of a cubic potential. However, it is likely that there exists a critical dimension, d^* (probably $d^*=4$) at which our results join smoothly onto the low- d results of Wegner^{5,17} and others^{17,18}

(6) We have displayed the localized states starting from the same Lagrangian as used to produce extended states. We have pointed out the existence of *lattice instantons*, i.e., instantons which exist only by virtue of a lattice cutoff. These localized excitations are shown to describe familiar results which can be obtained by elementary means in the infinite-energy limit.

(7) Our unified picture of localized and ex-

tended states obeys several consistency requirements. The validity of the ϵ -expansion treatment of fluctuations towards extended states relies on the absence of a broken symmetry corresponding to a nonzero density of states at the mobility edge. Analysis of the instanton solutions describing the localized states conforms to this requirement. At high dimension ($d > d^*$) we find the density of localized states $\rho_{1\infty}$ near the mobility edge to be of the form $\ln \rho_{1\infty} \sim -|E - E_c|^{-\zeta}$, with $\zeta = \frac{1}{2}(d-4)$ for the weak-coupling case and for the strong-coupling case $\zeta = (d-2-\eta)\nu_a - 2\beta_a$ for $d < 8$ and $\zeta = \frac{1}{4}(d-6)$ for $d > 8$.

ACKNOWLEDGMENTS

We acknowledge the support of the National Science Foundation under Grant No. DMR 76-21703 and the Office of Naval Research under Grant No. N00014-76-C01016. We would also like to thank Dr. C. Vause for checking the renormalization-group recursion relations.

APPENDIX A: EXACT SOLUTION FOR THE CAYLEY TREE

We now study the random model for the Cayley tree, each vertex of which has coordination number $z \equiv \sigma + 1$. A section of the lattice for $\sigma = 2$ is shown in Fig. 9. The model we consider is one in which each nearest-neighbor $t(\vec{x}, \vec{x}')$ independently assume the values $+t$ and $-t$ with equal probability. Let $g_n(E_\sigma)$ denote $g_\sigma(\vec{x}, \vec{x}'; E_\sigma; \theta_0)$, when $\vec{x}$ and $\vec{x}'$ are n steps apart and θ_0 is the configuration where all the t 's are positive.

Considering the phase averaging which follows along the lines discussed above Eq. (2.18), one sees that

FIG. 9. Section of a Cayley tree with $z = \sigma + 1 = 3$.

$$G_\sigma(\vec{x}, \vec{x}'; E_\sigma) = \delta_{\vec{x}, \vec{x}'} g_0(E_\sigma) \quad (A1)$$

and

$$G_{\sigma\sigma'}(\vec{x}, \vec{x}'; E_\sigma, E_{\sigma'}) = g_n(E_\sigma) g_n(E_{\sigma'}) , \quad (A2)$$

where $\vec{x}$ and $\vec{x}'$ are n steps apart on the Cayley tree. The $\vec{q}=0$ Fourier transform of Eq. (A2) yields

$$G_{\sigma\sigma'}(0; E_\sigma, E_{\sigma'}) = g_0(E_\sigma) g_0(E_{\sigma'}) + (\sigma + 1) \sum_{n=0}^{\infty} \sigma^n g_{n+1}(E_{\sigma'}) . \quad (A3)$$

It thus remains to evaluate $g_n(E)$.

We determine $g_n(E)$ as the solution of

$$E_\sigma g_\sigma(\vec{x}, \vec{x}'; E_\sigma; \theta_0) - \sum_{\vec{x}''} g_\sigma(\vec{x}, \vec{x}''; E_\sigma; \theta_0) t(\vec{x}', \vec{x}'') = \delta_{\vec{x}, \vec{x}'} . \quad (A4)$$

In terms of $g_n(E_\sigma)$ we write this as

$$E g_0(E) - (\sigma + 1) t g_1(E) = 1 , \quad (A5a)$$

$$E g_n(E) - \sigma t g_{n+1}(E) - t g_{n-1}(E) = 0, \quad n \geq 1 . \quad (A5b)$$

These equations are formally solved by a $g_n(E)$ of the form

$$g_n(E) = A_+ (y_+ / \sqrt{\sigma})^n + A_- (y_- / \sqrt{\sigma})^n , \quad (A6)$$

where

$$y_{\pm} = \frac{E}{2\sqrt{\sigma} t} \left[1 \pm \left(1 - \frac{4\sigma t^2}{E^2} \right)^{1/2} \right] \quad (A7)$$

and

$$A_{\pm} = \pm \left[g_0(E) y_{\pm} - \frac{\sqrt{\sigma}}{t(\sigma + 1)} \left(1 + \frac{E g_0(E)}{\sigma} \right) \right] / (y_+ - y_-) . \quad (A8)$$

We need only consider the regime $E^2 > 4\sigma t^2$. To obtain a convergent solution it is necessary to require that $A_+ = 0$, since $|y_+| > 1$ for $E^2 > 4\sigma t^2$. In this way we find that

$$g_0(E) = \frac{1}{2} \left(\frac{(\sigma + 1)E(1 - 4\sigma t^2/E^2)^{1/2} - (\sigma - 1)E}{E^2 - t^2(\sigma + 1)^2} \right) \quad (A9)$$

and

$$g_n(E) = g_0(E) (y_- / \sqrt{\sigma})^n . \quad (A10)$$

Note that $g_0(E)$ is analytic except for a branch cut along the real axis from $E = -E_c$ to $E = +E_c$,

where $E_c = 2\sqrt{\sigma}t$. There is no singularity in $g_0(E)$ at $E = \pm(\sigma+1)t$, since the numerator in Eq. (A9) has simple zeros at these energies. The above results for $g_0(E)$ agree with the previous exact solutions of Brinkman and Rice⁴⁹ and of Roulet *et al.*⁵⁰

Substituting the solution of Eqs. (A9) and (A10) into (A3) we find

$$g_{\sigma\sigma'}(0; E_\sigma, E_{\sigma'}) = g_0(E_\sigma)g_0(E_{\sigma'}) \left(1 + \frac{\sigma+1}{\sigma} y_-^2 \frac{1}{1-y_-^2}\right). \quad (\text{A11})$$

This "susceptibility" has a divergence at $y_- = 1$, i.e., for $E^2 = 4\sigma t^2$. We set $E = \pm 2\sqrt{\sigma}t(1+\epsilon)$ and obtain

$$g_{\sigma\sigma'}(0; \epsilon, \epsilon) \sim \frac{\sigma+1}{(\sigma-1)^2 t^2} (8\epsilon)^{-1/2} \quad (\text{A12})$$

so that we again recover $\gamma = \frac{1}{2}$. Also from Eq. (A9) we see that for $|E| < E_c$

$$\text{Im} g_0(-E) \sim \frac{\sigma+1}{(\sigma-1)^2 t^2} (-2\sigma\epsilon)^{1/2} \quad (\text{A13})$$

which corresponds to $\beta = \frac{1}{2}$, as expected.

APPENDIX B: EVALUATION OF INTEGRALS

Here we evaluate integrals of the form

$$I_n^+ = \frac{1}{N} \sum_q [(D_-)^{-n} + (D_+)^{-n}], \quad (\text{B1a})$$

$$I_n^- = \frac{1}{N} \sum_q [(D_-)^{-n} - (D_+)^{-n}], \quad (\text{B1b})$$

where

$$D_\pm = E + Q'(E) + t_0(\vec{q}) \pm iQ''(E). \quad (\text{B2})$$

Here Q' and Q'' are the real and imaginary parts of Q_+ , which satisfies

$$Q_+ = -\frac{\sigma_+^2}{2N} \sum_q (D_+)^{-1}. \quad (\text{B3})$$

Thus we have that

$$I_1^+ = -4Q'(E)/\sigma_s^2, \quad (\text{B4a})$$

$$I_1^- = 4iQ''(E)/\sigma_s^2. \quad (\text{B4b})$$

From Eq. (B1) one obtains the recursion relations

$$\left(1 + \frac{dQ'}{dE}\right) I_n^\pm - i \frac{dQ''}{dE} I_n^\pm = -\frac{1}{(n-1)} \frac{d}{dE} I_{n-1}^\pm \quad (\text{B5})$$

whose solution yields

$$I_n^\pm = -\frac{\left[\left(1 + \frac{dQ'}{dE}\right) \frac{dI_{n-1}^\pm}{dE} + i \frac{dQ''}{dE} \frac{dI_{n-1}^\pm}{dE}\right]}{(n-1) \left[\left(1 + \frac{dQ'}{dE}\right)^2 + \left(\frac{dQ''}{dE}\right)^2\right]}. \quad (\text{B6})$$

To condense the notation we set

$$\beta = \frac{dQ'}{dE}, \quad (\text{B7a})$$

$$\rho = Q'', \quad (\text{B7b})$$

and use overdots to indicate derivatives with respect to E :

$$\dot{\rho} = \frac{dQ''}{dE}, \quad (\text{B8a})$$

$$\dot{\beta} = \frac{d^2 Q'}{dE^2}, \quad (\text{B8b})$$

etc. Then we find that

$$I_2^+ = \frac{4}{\sigma_s^2} \frac{[(1+\beta)\beta + \dot{\rho}^2]}{(1+\beta)^2 + \dot{\rho}^2} \quad (\text{B9a})$$

$$I_2^- = -\frac{4}{\sigma_s^2} \frac{i\dot{\rho}}{(1+\beta)^2 + \dot{\rho}^2}. \quad (\text{B9b})$$

Since we are interested in the case for which $\rho \sim (\Delta E)^{(d-2)/2}$, we have

$$I_2^+ \sim \frac{4}{\sigma_s^2} \frac{\beta}{1+\beta}, \quad (\text{B10a})$$

$$I_2^- \sim -\frac{4i\dot{\rho}}{\sigma_s^2(1+\beta)^2}. \quad (\text{B10b})$$

In this limit we likewise find

$$I_3^+ \sim -\frac{2\dot{\beta}}{\sigma_s^2(1+\beta)^3}, \quad (\text{B11a})$$

$$I_3^- \sim \frac{2i}{\sigma_s^2(1+\beta)^4} [\dot{\rho}(1+\beta) - 3\dot{\beta}\dot{\rho}] \quad (\text{B11b})$$

$$\sim \frac{2i\ddot{\rho}}{\sigma_s^2(1+\beta)^3}, \quad (\text{B11c})$$

since even if $\dot{\beta}$ diverges $\Delta E \rightarrow 0$, one has $\ddot{\rho}/\dot{\rho} \sim (\Delta E)^{-1} \geq \beta$.

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