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# Complete Convergence of Short Paths and Karp's Algorithm for the TSP

## Abstract

Let  $X_i$ ,  $1 \leq i < \infty$ , be uniformly distributed in  $[0, 1]^2$  and let  $T_n$  be the length of the shortest closed path connecting  $\{X_1, X_2, \dots, X_n\}$ . It is proved that there is a constant  $0 < \beta < \infty$  such that for all  $\epsilon > 0$

$$\sum_{n=1}^{\infty} P(|T_n/n - \beta| > \epsilon) < \infty.$$

This result is essential in justifying Karp's algorithm for the traveling salesman problem under the independent model, and it settles a question posed by B. W. Weide.

## Keywords

traveling salesman problem, complete convergence, sub additive processes, subadditive Euclidean functionals, jackknife, Efron-Stein inequality

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COMPLETE CONVERGENCE OF SHORT PATHS AND  
KARP'S ALGORITHM FOR THE TSP

BY

J. MICHAEL STEELE

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DEPARTMENT OF STATISTICS  
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1. Introduction.

The main objective of the present note is to solve a problem proposed by Weide (1978) concerning the complete convergence of certain random variables associated with Karp's probabilistic analysis of the traveling salesman problem (Karp (1976), (1977)).

To set the problem precisely let  $X_i, 1 \leq i < \infty$ , be independent random variables uniformly distributed on the unit square  $[0,1]^2$ , and let  $T_n$  denote the length of the shortest path (in the usual Euclidean distance) which connects each element of  $\{X_1, X_2, \dots, X_n\}$ .

It was proved by Beardwood, Halton, and Hammersley (1959) that

$$\lim_{n \rightarrow \infty} T_n / \sqrt{n} = \beta$$

with probability one for a finite constant  $\beta$ . This fact was central to the motivation behind Karp's algorithm, but as Weide (1978) points out the Karp algorithm actually calls for the following stronger result to be proved here:

Theorem 1. There is a constant  $\beta$  such that for all  $\epsilon > 0$ , one has

$$\sum_{n=1}^{\infty} P(|T_n / \sqrt{n} - \beta| > \epsilon) < \infty .$$

This type of convergence is usually called complete convergence, and Theorem 1 stands in a similar relation to the Beardwood, Halton, Hammersley

Theorem as the Hsu-Robbins Theorem stands in relation to the strong law of large numbers (Lukacs (1968), Hsu and Robbins (1947)). The "easy-half" of the Borel-Cantelli lemma shows that Theorem 1 implies the Beardwood-Halton-Hammersley Theorem and the "hard-half" of the Borel-Cantelli lemma shows how Theorem 1 is necessary in modeling contexts where problems of increased size are generated independently of previous problems. (For a full discussion of independent versus incrementing models for random problems one should consult Weide (1978)).

The proof of Theorem 1 is given in the next section and depends upon shapening a subadditivity argument which has been useful in more general contexts (Steele (1979)). The third section discusses a generalization of Theorem 1.

## 2. Proof of Theorem 1.

Let  $N = N(t)$  denote a Poisson counting process with constant growth rate 1. Also, for any  $A \subset [0,1]^2$  let  $T_t(A)$  denote the length of the shortest path through the points  $A \cap \{X_1, X_2, \dots, X_{N(t)}\}$ . The method of proof rest upon developing recursions for the functions

$$\varphi(t) = E T_t([0,1]^2), \quad \psi(t) = (E T_t^2([0,1]^2))^{1/2}, \quad \text{and} \quad V(t) = \psi(t) - \varphi^2(t).$$

First one notes the following:

Lemma 1. There is a constant  $c$  such that for any set  $\mathcal{A} \subset [0,1]$  with  $n$  elements there is a path of length no greater than  $c\sqrt{n}$  through  $\mathcal{A}$ .

This lemma is easily proved, but for a proof which yields a good value of  $c$  one can consult Few (1955).

Now let

$$S = T_t([0, \frac{1}{2}]^2) + T_t([\frac{1}{2}, 1]^2) + T_t([\frac{1}{2}, 1] \times [0, \frac{1}{2}]) + T_t([\frac{1}{2}, 1] \times [0, \frac{1}{2}])$$

and note by elementary geometry that  $T_t([0, 1]^2) \leq S + 4$ . By well-known properties of the planar Poisson process one has that the four summands of  $S$  are independent and identically distributed. By scaling one also notes  $E T_t([0, \frac{1}{2}]^2) = \frac{1}{2} \varphi(t/4)$  and  $(E T_t^2([0, \frac{1}{2}]^2))^{1/2} = \frac{1}{2} \psi(t/4)$ . Thus, taking expectations of  $(S+4)^2$  one has

$$(2.1) \quad \psi^2(t) \leq \psi^2(t/4) + 3 \varphi^2(t/4) + 16 \varphi(t/4) + 16 .$$

To simplify (2.1) let  $\psi_1(t) = \psi(t^2)$ ,  $\varphi_1(t) = \varphi(t^2)$ , and  $V_1(t) = V(t^2)$  and note that (2.1) implies

$$\psi_1^2(2t) \leq \psi_1^2(t) + 3 \varphi_1^2(t) + 16 \varphi_1(t) + 16$$

which entails

$$V_1(2t) \leq V_1(t) + 4 \varphi_1^2(t) - \varphi_1^2(2t) + 16 \varphi_1(t) + 16$$

and dividing by  $(2t)^2$ ,

$$(2.2) \quad V_1(2t)/(2t)^2 - (1/4)V_1(t)/t^2 \leq \varphi_1^2(t)/t^2 - \varphi_1^2(2t)/(2t)^2 + 4(\varphi_1(t)+1)/t^2 .$$

Applying (2.2) successively to  $t, 2t, 2^2t, \dots, 2^{M-1}t$  and summing yields

$$\begin{aligned} & \sum_{k=1}^M V_1(2^k t)/(2^k t)^2 - \frac{1}{4} \sum_{k=0}^{M-1} V_1(2^k t)/(2^k t)^2 \\ & \leq \varphi_1^2(t)/t^2 - \varphi_1^2(2^M t)/(2^M t)^2 + 4(\varphi_1(t)+1) \left( \sum_{k=0}^{M-1} 2^{-2k} \right) t^{-2} \end{aligned}$$

and consequently

$$(2.3) \quad \sum_{k=1}^{\infty} V_1(2^k t)/(2^k t)^2 \leq \frac{4}{5}(V_1(t)/t^2 + \varphi_1^2(t)/t^2 + 8(\varphi_1(t)+1)/t^2) < \infty.$$

In terms of  $V(t)$ , (2.3) becomes for  $u = t^2$

$$(2.4) \quad \sum_{k=0}^{\infty} V(4^k u)/(4^k u) < \infty.$$

Now by Beardwood-Halton-Hammersley and dominated convergence theorems we have  $\varphi(4^k u)/(4^k u)^{1/2} \rightarrow \beta$  as  $k \rightarrow \infty$ . Hence by (2.4) and Chebychev's inequality,

$$(2.5) \quad \sum_{k=0}^{\infty} P(|T_{4^k u}([0,1]^2)/(4^k u)^{1/2} - \beta| \geq \epsilon) < \infty.$$

Now for all  $n \geq 4^p$  there is a  $4^{p+1} \geq m \geq 4^p$  such that  $4^k m \leq n \leq 4^{k+1} m$  for some  $k \geq 0$ . Hence by Boole's inequality and the monotonicity of  $T_t[0,1]$

$$\begin{aligned} (2.6) \quad & \sum_{n=0}^{\infty} P(|T_n([0,1]^2)/n^{1/2} - \beta| \geq \epsilon) \\ & \leq \sum_{k=1}^{\infty} \sum_{m=4^p}^{4^{p+1}} P(T_{4^k m} \geq (\beta + \epsilon)(4^k m)^{1/2}) \\ & \quad + \sum_{k=1}^{\infty} \sum_{m=4^p}^{4^{p+1}} P(T_{4^k m} \leq (\beta - \epsilon)(4^{k+1} m)^{1/2}) \end{aligned}$$



$$\leq \sum_{k=1}^{\infty} \sum_{m=4^p}^{4^{p+1}} P(T_{4^k(m+1)} / (4^k(m+1))^{1/2} \geq (\beta+\epsilon)(1+4^{-p})^{-1/2}) \\ + \sum_{k=1}^{\infty} \sum_{m=4^p}^{4^{p+1}} P(T_{4^k m} / (4^k m)^{1/2} \leq (\beta-\epsilon)(1+4^{-p})^{1/2}).$$

By choosing  $p$ , one can guarantee that  $(1+4^{-p})^{-1/2}(\beta+\epsilon) > \beta$  and  $(1+4^{-p})(\beta-\epsilon) < \beta$  so (2.5) insures that the last two sums in (2.6) converge, so

$$(2.7) \quad \sum_{n=0}^{\infty} P(|T_n([0,1]^2)/n^{1/2} - \beta| \geq \epsilon) < \infty.$$

To obtain Theorem 1 from (2.7) we note that

$$(2.8) \quad P(|T_n([0,1]^2)/n^{1/2} - \beta| \geq \epsilon) = \sum_{m=0}^{\infty} P(|T_m/n^{1/2} - \beta| \geq \epsilon) \frac{n^m e^{-n}}{m!} \\ \geq \sum_{m=n}^{\infty} P(T_m/n^{1/2} \geq \beta+\epsilon) \cdot \frac{n^m e^{-m}}{m!} \\ + \sum_{m=0}^n P(T_m/n^{1/2} \leq \beta-\epsilon) \cdot \frac{n^m e^{-m}}{m!} \\ \geq P(T_n/n^{1/2} \geq \beta+\epsilon) \left(\frac{1}{4}\right) + P(T_n/n^{1/2} \leq \beta-\epsilon) \left(\frac{1}{4}\right).$$

In the last inequality above one uses the fact that  $P(T_n \geq \lambda)$  is monotone increasing, and the fact that  $\sum_{m \geq n} n^m e^{-n}/m!$  and

$\sum_{m \leq n} n^m e^{-n}/m!$  each exceed  $\frac{1}{4}$ . The bounds (2.7) and (2.8) complete

the proof of Theorem 1.

### 3. Further Results.

In the previous section the aim was to give the most direct proof possible of the result conjectured by Weide (1978). The complete convergence proved there can easily be extended to the broader context of subadditive Euclidean functionals. Since the method of proving complete convergence in this context is directly parallel to the preceding argument, it is sufficient to state the more general result.

To introduce Euclidean functionals, let  $L$  denote a real value function defined on the finite subsets  $\mathbb{R}^d$ ,  $d \geq 2$ . It will be assumed that  $L(X_1, X_2, \dots, X_n)$  is measurable whenever  $X_i$ ,  $1 \leq i \leq n$ , are measurable, and that the following four assumptions are satisfied:

- A1.  $L(\alpha x_1, \alpha x_2, \dots, \alpha x_n) = \alpha L(x_1, x_2, \dots, x_n)$  for  $\alpha > 0$ .
- A2.  $L(x_1+t, x_2+t, \dots, x_n+t) = L(x_1, x_2, \dots, x_n)$  for  $t \in \mathbb{R}^d$ .
- A3.  $L(x_1, x_2, \dots, x_n, x_{n+1}) \geq L(x_1, x_2, \dots, x_n)$ .
- A4.  $\text{Var}(L(X_1, X_2, \dots, X_n)) < \infty$  when  $X_i$  are independent and uniform on the unit cube  $[0,1]^d$ .

There is one further assumption which is needed and it is the only one which is not trivial to verify in most applications.

Suppose that  $\{Q_i: 1 \leq i \leq m^d\}$  is the partition of  $[0,1]^d$  into sub-cubes of edge  $1/m$  and let  $tQ_i = \{x: x=ty, y \in Q_i\}$ . The subadditivity assumption is the following:

- A5. There is a constant  $C > 0$  such that for all positive integers  $m$  and positive reals  $t$  one has

$$L(\{x_1, x_2, \dots, x_n\} \cap [0, t]^d) \leq \sum_{i=1}^{m^d} L(\{x_1, x_2, \dots, x_n\} \cap tQ_i) + Ctm^{d-1}.$$

It is proved in Steele (1979) that if  $L$  satisfies the preceding five assumptions and  $\{X_i\}$  are independent with a uniform distribution in  $[0,1]^d$  then

$$\lim_{n \rightarrow \infty} L(X_1, X_2, \dots, X_n) / n^{(d-1)/d} = \beta$$

with probability one for some constant  $0 \leq \beta < \infty$ .

One can easily check that the preceding result implies the Beardwood, Halton, Hammersley theorem. The main observation to be made here is that this result also can be sharpened as in Section 2 to obtain the following:

Theorem 2. Suppose  $L$  is a functional which satisfies assumptions A1-A5. If  $\{X_i: 1 \leq i < \infty\}$  are independent and uniformly distributed in  $[0,1]^d$ , then there is a constant  $0 \leq \beta < \infty$  so that for all  $\epsilon > 0$

$$\sum_{n=1}^{\infty} P(|L(x_1, x_2, \dots, x_n) / n^{(d-1)/d} - \beta| \geq \epsilon) < \infty.$$

There are many functionals which arise in the theory of algorithms and which satisfy A1-A5. The easiest examples are those associated with the Steiner tree problem and the rectilinear Steiner tree problem, but in essentially any geometric problem which deals with minimized lengths one can find natural functionals which meet the conditions of Theorem 2.

As a final point one should note that both Beardwood, Halton, and Hammersley (1959) as well as Steele (1979) contain results valid for random variables with non-uniform distribution. The approximation processes used in these papers to extend the uniform case can again be applied here although to do so would require considerable space. Since the algorithmic applications and the program begun by B. W. Weide (1978) are ably served by Theorems 1 and 2, these last extentions have been omitted.

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$$\sum_{n=1}^{\infty} P(|T_n/\sqrt{n-\beta}| > \epsilon) < \infty .$$

This answers a question posed by B. W. Weide.