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# The Jordan Canonical Form for a Class of Zero–One Matrices

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# The Jordan Canonical Form for a Class of Zero–One Matrices

## Abstract

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be a function. Let  $A_n = (a_{ij})$  be the  $n \times n$  matrix defined by  $a_{ij} = 1$  if  $i = f(j)$  for some  $i$  and  $j$  and  $a_{ij} = 0$  otherwise. We describe the Jordan canonical form of the matrix  $A_n$  in terms of the directed graph for which  $A_n$  is the adjacency matrix. We discuss several examples including a connection with the Collatz  $3n+1$  conjecture.

## Keywords

Jordan canonical form, directed graph, adjacency matrix

## Disciplines

Algebra | Other Mathematics

# The Jordan canonical form for a class of zero-one matrices

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## Abstract

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be a function. Let  $A_n = (a_{ij})$  be the  $n \times n$  matrix defined by  $a_{ij} = 1$  if  $i = f(j)$  for some  $i$  and  $j$  and  $a_{ij} = 0$  otherwise. We describe the Jordan canonical form of the matrix  $A_n$  in terms of the directed graph for which  $A_n$  is the adjacency matrix. We discuss several examples including a connection with the Collatz  $3n + 1$  conjecture.

*Keywords:* Jordan canonical form, directed graph, adjacency matrix

*2010 MSC:* 15A21, 05C20

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## 1. Introduction

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be any function. For each  $n \in \mathbb{N}$ , we define the  $n \times n$  matrix  $A_n = (a_{ij})$  by

$$a_{ij} = \begin{cases} 1 & \text{if } i = f(j) \text{ for some } i, j \in \{1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $A_n$  contains partial information about  $f$ . We may regard  $A_n$  as the adjacency matrix for the directed graph  $\Gamma_n$  with vertices labeled  $1, \dots, n$  having a directed edge from vertex  $j$  to vertex  $i$  if and only if  $i = f(j)$ . The main purpose of this paper is to describe the Jordan canonical form of  $A_n$  in terms of the graph  $\Gamma_n$ . This description is given in Theorem 6.

As a motivating example, let  $f$  be the function

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The Collatz conjecture states that, for each  $k \in \mathbb{N}$ , the sequence

$$k, f(k), (f \circ f)(k), (f \circ f \circ f)(k), \dots$$

contains the number 1. In §5, we will discuss this example in more detail where we develop an explicit formula for the number of Jordan blocks for the eigenvalue 0 in the Jordan decomposition of the matrix  $A_n$ .

The remainder of this paper is organized as follows: In §2, we describe how to partition the directed graph  $\Gamma_n$  into chains and cycles. These chains and cycles are related to the Jordan form of  $A_n$ . In §3, we state and prove our main theorem, Theorem 6, which describes the Jordan block structure of  $A_n$  in terms of the cycles and chains of the graph  $\Gamma_n$ . In §5, we apply Theorem 6 to several examples.

## 2. The directed graph $\Gamma_n$ associated with $A_n$

We will form a partition of the directed graph  $\Gamma_n$ , which was defined in §1, into chains and cycles. The Jordan decomposition of the adjacency matrix  $A_n$  will be related to the lengths of these chains and cycles. Recall that for the function  $f: \mathbb{N} \rightarrow \mathbb{N}$  and natural number  $n \in \mathbb{N}$ , we define  $\Gamma_n$  to be the directed graph with vertices  $1, \dots, n$  having a directed edge from  $j$  to  $i$  if and only if  $f(j) = i$ .

**Definition 1.** A *chain* in  $\Gamma_n$  is an ordered list of distinct vertices  $C = \{c_1, c_2, \dots, c_r\}$  such that  $f(c_j) = c_{j+1}$  for  $1 \leq j < r$  but  $f(c_r) \neq c_1$ . A *cycle* in  $\Gamma_n$  is an ordered list of distinct vertices  $Z = \{z_1, z_2, \dots, z_r\}$  such that  $f(z_j) = z_{j+1}$  for  $1 \leq j < r$  and  $f(z_r) = z_1$ . In either case, we call  $r$  the *length* of the chain or cycle and write  $r = \text{len } C$  or  $r = \text{len } Z$ .

Note that in Definition 1 a single vertex  $\{i\}$  is a chain or a cycle, but since either  $f(i) \neq i$  or  $f(i) = i$ , it is not both a chain and a cycle. Although an arbitrary directed graph may contain two unequal cycles that share a common vertex, this is not possible for  $\Gamma_n$ . Since  $\Gamma_n$  results from a function  $f: \mathbb{N} \rightarrow \mathbb{N}$ , if  $Z_1$  and  $Z_2$  are cycles that share a common vertex, then  $Z_1 = Z_2$ . Thus, unequal cycles in  $\Gamma_n$  are disjoint.

**Definition 2.** If  $C = \{i_1, \dots, i_s\}$  is a chain of  $\Gamma_n$ , then  $i_s$  is called the *terminal point* of the chain. A vertex  $k$  of  $\Gamma_n$  such that  $f(k) > n$  is a *terminal point* of  $\Gamma_n$ . If  $k$  is a vertex of  $\Gamma_n$  such that  $f(i) = f(j) = k$  for some  $i$  and  $j$  with  $i \neq j$ , then  $k$  is a *merge point* of  $\Gamma_n$ .

**Definition 3.** A *partition* of  $\Gamma_n$  is a collection of disjoint cycles and chains whose union is  $\Gamma_n$ . A *proper partition* of  $\Gamma_n$  is a partition

$$P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$$

where  $Z_1, \dots, Z_r$  are cycles and  $C_1, \dots, C_s$  are chains satisfying the following properties:

1. Each cycle in  $\Gamma_n$  is equal to  $Z_i$  for some  $i$ .
2. If  $\Gamma_n^{(i)}$  is the subgraph of  $\Gamma_n$  obtained by deleting the vertices in the cycles  $Z_1, \dots, Z_r$  and in the chains  $C_1, \dots, C_i$ , then  $C_{i+1}$  is a chain of maximal length in  $\Gamma_n^{(i)}$ .

**Lemma 1.** *Proper partitions of  $\Gamma_n$  exist.*

*Proof.* As noted above, the cycles of  $\Gamma_n$  are mutually disjoint. Label them as  $Z_1, \dots, Z_r$ . Let  $\Gamma_n^{(0)}$  be the subgraph of  $\Gamma_n$  obtained by removing all vertices belonging to the cycles  $Z_1, \dots, Z_r$ . The graph  $\Gamma_n^{(0)}$  is an acyclic graph. Then inductively define  $C_i$  and  $\Gamma_n^{(i)}$  for  $i \geq 1$  by choosing  $C_i$  to be a chain of maximal length in  $\Gamma_n^{(i-1)}$  and letting  $\Gamma_n^{(i)}$  be the subgraph of  $\Gamma_n^{(i-1)}$  obtained by deleting the vertices of  $C_i$ .  $\square$

If

$$P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\} \quad \text{and} \quad P' = \{Z'_1, \dots, Z'_{r'}, C'_1, \dots, C'_{s'}\}$$

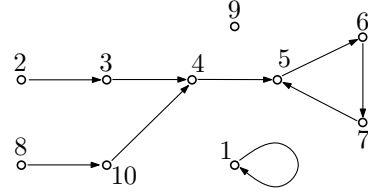
are any two proper partitions of  $\Gamma_n$ , then it is clear that  $r = r'$  and the cycles  $Z_1, \dots, Z_r$  are the same as the cycles  $Z'_1, \dots, Z'_{r'}$  up to reordering. It may be less obvious that  $s = s'$  and  $\text{len } C_i = \text{len } C'_i$  for  $1 \leq i \leq s$ . This fact is stated in Corollary 7 below.

**Example 1.** Suppose  $f: \mathbb{N} \rightarrow \mathbb{N}$  takes the values

$$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 4, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 10, 9 \mapsto 500, 10 \mapsto 4, \dots$$

The graph  $\Gamma_{10}$  and adjacency matrix  $A_{10}$  are:

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



The vertices 4 and 5 are merge points of  $\Gamma_n$ . The vertex 9 is a terminal point of  $\Gamma_{10}$ . Two proper partitions of  $\Gamma_{10}$  are:

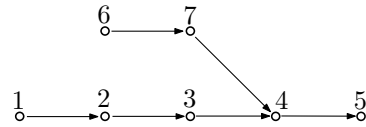
$$\begin{array}{ll} Z_1 = \{1\} & Z'_1 = \{5, 6, 7\} \\ Z_2 = \{5, 6, 7\} & Z'_2 = \{1\} \\ C_1 = \{2, 3, 4\} & C'_1 = \{8, 10, 4\} \\ C_2 = \{8, 10\} & C'_2 = \{2, 3\} \\ C_3 = \{9\} & C'_3 = \{9\} \end{array}$$

**Example 2.** Suppose  $f: \mathbb{N} \rightarrow \mathbb{N}$  takes the values

$$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 100, 6 \mapsto 7, 7 \mapsto 4, \dots$$

The graph  $\Gamma_7$  and adjacency matrix  $A_7$  are:

$$A_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Vertex 4 is a merge point, and vertex 5 is a terminal point of  $\Gamma_7$ . A proper partition of  $\Gamma_7$  is

$$C_1 = \{1, 2, 3, 4, 5\} \quad C_2 = \{6, 7\},$$

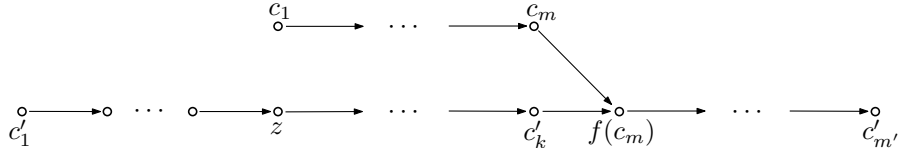


Figure 1: If the chain  $C' = \{c'_1, \dots, c'_{m'}\}$  contains the merge point  $f(c_m) = f(c'_k)$ , as in Lemma 2, then  $k \geq m$ . There exists  $z$  on the chain  $C'$  with  $f^m(z) = f^m(c_1) = f(c_m)$ .

while an improper partition of  $\Gamma_7$  is

$$C'_1 = \{6, 7, 4, 5\} \quad C'_2 = \{1, 2, 3\}.$$

Observe that in the proper partition, the chain containing the merge point 4 is longer than the other chain.

The terminal and merge points of  $\Gamma_n$  will play a crucial role in the Jordan decomposition of  $A_n$ . The next lemma makes precise the situation in Example 2 as well as the case in which the merge point belongs to a cycle.

**Lemma 2.** *Let  $C = \{c_1, \dots, c_m\}$  be any chain in a proper partition of  $\Gamma_n$ . Then exactly one of the following occurs:*

1. *The terminal point  $c_m$  of the chain is a terminal point of the graph  $\Gamma_n$ .*
2. *The point  $f(c_m)$  is a merge point of  $\Gamma_n$ .*

*Furthermore, if  $f(c_m)$  is a merge point and  $f(c_m)$  belongs to another chain  $C' = \{c'_1, \dots, c'_{m'}\}$ , then  $f(c_m) = f(c'_k)$  where  $k \geq m$ . (This is illustrated in Figure 1.) Consequently, if  $f(c_m)$  is a merge point belonging to either a cycle or a chain, then there is a unique vertex  $z$  in the cycle or chain containing  $f(c_r)$  such that  $f^m(z) = f^m(c_1) = f(c_m)$ , where  $f^m$  is the composition of  $f$  with itself  $m$  times.*

*Proof.* Suppose the terminal point  $c_m$  of the chain is not a terminal point of the graph  $\Gamma_n$ . If  $f(c_m)$  belongs to one of the cycles of  $\Gamma_n$ , then  $f(c_m)$  is a merge point since  $f(c_m)$  has both  $c_m$  and some point of the cycle as preimages. If  $f(c_m)$  belongs to another chain  $C'$ , then either  $f(c_m)$  is the initial point of the chain  $C'$  or it is not. If  $f(c_m)$  is not the initial point of  $C'$ , then  $c_m$  and a point of  $C'$  are preimages of  $f(c_m)$  making  $f(c_m)$  a merge point. If  $f(c_m)$  is the initial point of  $C'$ , then the partition is not

proper because  $C$  and  $C'$  could be joined to form a longer chain, which is a contradiction. This proves that either  $c_m$  is a terminal point of  $\Gamma_n$  or  $f(c_m)$  is a merge point of  $\Gamma_n$ .

Now, suppose  $f(c_m)$  belongs to another chain  $C' = \{c'_1, \dots, c'_{m'}\}$ . Then  $f(c_m) = f(c'_k)$  for some  $k$ . Necessarily  $k < m'$ . If, by way of contradiction,  $k < m$ , then the vertices in the two chains  $C$  and  $C'$  could be repartitioned to belong to the new chains

$$C'' = \{c_1, \dots, c_m, c'_{k+1}, \dots, c'_{m'}\} \quad \text{and} \quad C''' = \{c'_1, \dots, c'_k\}.$$

Since  $k < m'$ ,

$$\text{len } C'' = m' - k + m > m = \text{len } C.$$

Since  $k < m$ ,

$$\text{len } C''' = m' - k + m > m' = \text{len } C'.$$

Thus, the original pair of chains  $C$  and  $C'$  violate the maximality condition of a proper partition in Definition 3, a contradiction. Therefore  $k \geq m$ .  $\square$

### 3. The Jordan Structure of $A_n$

In this section we will state the main result of this paper (Theorem 6) which describes the Jordan canonical form of the adjacency matrix  $A_n$  of the graph  $\Gamma_n$ . We will need several standard facts (Propositions 3 and 4 below) about the Jordan canonical form. Good references for this material are [1, Ch. 7], [2, Ch. 3], and [3, Ch. 6].

**Definition 4.** For a complex number  $\lambda$  and natural number  $m$ ,  $J_m(\lambda)$  will denote the  $m \times m$  matrix

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

**Definition 5.** Let  $A$  be an  $n \times n$  matrix with complex entries. A nonzero vector  $v$  is a *generalized eigenvector* of  $A$  corresponding to the complex number  $\lambda$  if  $(A - \lambda I)^p v = 0$  for some positive integer  $p$ .



**Definition 6.** Let  $v$  be a generalized eigenvector of  $A$  for the eigenvalue  $\lambda$  and let  $p$  be the smallest positive integer such that  $(A - \lambda I)^p v = 0$ . Then the ordered set

$$\{(A - \lambda I)^{p-1}v, (A - \lambda I)^{p-2}v, \dots, (A - \lambda I)v, v\} \quad (1)$$

is a *chain of generalized eigenvectors* of  $A$  corresponding to  $\lambda$ . Observe that the first elements of the list,  $(A - \lambda I)^{p-1}v$ , is an ordinary eigenvector.

*Note.* In the literature, many authors refer to the list of generalized eigenvectors in Definition 6 as a *cycle* of generalized eigenvectors. In the context of this paper, it is better to call it a chain.

**Proposition 3** (Linear Independence of Generalized Eigenvectors). *Let  $\lambda$  be an eigenvalue of  $A$  and let  $\{\gamma_1, \dots, \gamma_s\}$  be chains of generalized eigenvectors of  $A$  corresponding to  $\lambda$ . If the initial vectors of the  $\gamma_i$ 's form a linearly independent set, then the  $\gamma_i$ 's are disjoint ( $\gamma_i \cap \gamma_j = \emptyset$  for  $i \neq j$ ) and the union  $\cup_{i=1}^s \gamma_i$  is linearly independent.*

**Proposition 4** (Jordan Canonical Form). *Let  $A$  be an  $n \times n$  complex matrix. Then there exists a basis  $\beta$  of  $\mathbb{C}^n$  consisting of disjoint chains  $\beta_1, \dots, \beta_r$  of generalized eigenvectors of lengths  $n_1, \dots, n_r$  for the eigenvalues  $\lambda_1, \dots, \lambda_r$  with  $n = n_1 + \dots + n_r$  such that if  $Q$  is the matrix whose columns are the members of the basis  $\beta$  then*

$$Q^{-1}AQ = J_{n_1}(\lambda_1) \oplus \dots \oplus J_{n_r}(\lambda_r).$$

As a preliminary step to determining the Jordan decomposition of the adjacency matrix  $A_n$  of the graph  $\Gamma_n$ , we begin with the following simple observation:

**Lemma 5.** *Every eigenvalue of  $A_n$  is either 0 or a root of unity.*

*Proof.* The  $j$ th column of  $A_n$  is a zero column if  $f(j) > n$ . The  $j$ th column contains a single 1 if  $1 \leq f(j) \leq n$ . Thus, for any  $k \in \mathbb{N}$ , the matrix product  $A_n^k$  also consists of either zero columns or columns containing a single 1. Consequently, the infinite sequence

$$I, A_n, A_n^2, A_n^3, \dots$$

must contain a repetition since there are only finitely many distinct  $n \times n$  matrices whose columns are zero columns or contain a single 1. Let  $0 \leq i < j$

be exponents such that  $A_n^i = A_n^j$ . Then  $A_n$  satisfies the polynomial  $x^j - x^i = x^{j-i}(x^j - 1)$ . The eigenvalues of  $A_n$  must be a subset of the roots of this polynomial. Hence, all eigenvalues are either zero or roots of unity.  $\square$

We are now ready to state the main result of this paper:

**Theorem 6.** *Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be a function. Let  $\Gamma_n$  be the directed graph associated with  $f$  for the natural number  $n$ , and let  $A_n$  be its adjacency matrix as defined in §1. Suppose*

$$P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$$

*is a proper partition of  $\Gamma_n$ , as in Definition 3, where  $Z_1, \dots, Z_s$  are the cycles and  $C_1, \dots, C_s$  are the chains. Write the lengths of the cycles and chains as*

$$\text{len } Z_j = \ell_j \quad (1 \leq j \leq r) \quad \text{and} \quad \text{len } C_j = m_j \quad (1 \leq j \leq s).$$

*Let  $\omega_j = \exp(2\pi i/\ell_j)$  be a primitive  $\ell_j$ th root of unity. The Jordan decomposition of  $A_n$  contains the following  $1 \times 1$  Jordan blocks for the eigenvalues which are roots of unity:*

$$J_1(\omega_j^k) \quad \text{for } 1 \leq j \leq r \text{ and } 1 \leq k \leq \ell_j.$$

*The Jordan decomposition contains the following blocks associated with the eigenvalue 0:*

$$J_{m_1}(0), J_{m_2}(0), \dots, J_{m_s}(0).$$

*Remark.* The proof of Theorem 6 (given in §4) will construct an explicit basis (Lemmas 8 and 10) for  $\mathbb{C}^n$  consisting of generalized eigenvectors of  $A_n$ . Letting  $Q$  be the matrix whose columns are these vectors gives  $J = Q^{-1}A_nQ$  where  $J$  is the Jordan decomposition of  $A_n$ .

**Corollary 7.** *Suppose*

$$P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\} \quad \text{and} \quad P' = \{Z'_1, \dots, Z'_{r'}, C'_1, \dots, C'_{s'}\}$$

*are any two proper partitions of  $\Gamma_n$ . Then  $s = s'$  and  $\text{len } C_j = \text{len } C'_j$  for  $1 \leq j \leq s$ .*

*Proof.* This is an immediate consequence of the uniqueness of the Jordan decomposition of  $A_n$ . By Theorem 6, the block sizes associated with the eigenvalue 0 are given by the two descending lists of numbers:

$$m_1 \geq m_2 \geq \dots \geq m_s \quad \text{and} \quad m'_1 \geq m'_2 \geq \dots \geq m'_{s'}.$$

So,  $s = s'$  and  $m_j = m'_j$  for  $1 \leq j \leq s$ .  $\square$

#### 4. Proof of Theorem 6

The proof of Theorem 6 will proceed as follows: From Lemma 5, each eigenvalue of  $A_n$  is either a root of unity or zero. In Lemma 8 below, we attached an eigenvector of  $A_n$  associated with a root of unity to each vertex of each cycle in  $\Gamma_n$ . In Lemma 9, this set of eigenvectors for the roots of unity is shown to be linearly independent. In Lemma 10, we attach chains of generalized eigenvectors of  $A_n$  for the eigenvalue 0 to chains of the graph  $\Gamma_n$ . In Lemma 11 we show that these generalized eigenvectors also form a linearly independent set and that the set of all generalized eigenvectors attached to the vertices of  $\Gamma_n$  via Lemmas 8 and 10 is a Jordan basis of  $\mathbb{C}^n$  for the matrix  $A_n$ .

Throughout the section  $P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$  will be a proper partition of  $\Gamma_n$ . Write the lengths of the cycles and chains as

$$\text{len } Z_j = \ell_j \quad (1 \leq j \leq r) \quad \text{and} \quad \text{len } C_j = m_j \quad (1 \leq j \leq s).$$

We have the relationship

$$\ell_1 + \dots + \ell_s + m_1 + \dots + m_s = n.$$

The  $i$ th standard basis vector of  $\mathbb{C}^n$  will be denoted by  $e_i$ .

**Lemma 8.** *Let  $Z = \{z_1, \dots, z_\ell\}$  be any cycle in the partition  $P$ , and let  $\omega = \exp(2\pi i/\ell)$  be a primitive  $\ell$ th root of unity. Then the vector*

$$v_k = \sum_{j=1}^{\ell} \omega^{-kj} e_{z_j} \tag{2}$$

*is an eigenvector of  $A_n$  for the eigenvalue  $\omega^k$ . Furthermore,*

$$\text{span}\{v_1, \dots, v_\ell\} = \text{span}\{e_{z_1}, \dots, e_{z_\ell}\}. \tag{3}$$

*We will say that the eigenvector  $v_k$  is attached to the vertex  $z_k$ .*

*Proof.* Since  $A_n e_{z_j} = e_{z_{j+1}}$  for  $1 \leq j < \ell$  and  $A_n e_{z_\ell} = e_{z_1}$  and since  $\omega^\ell = 1$ , we have

$$\begin{aligned} A_n v_k &= \sum_{j=1}^{\ell} \omega^{-kj} A_n e_j = \sum_{j=1}^{\ell-1} \omega^{-kj} e_{z_{j+1}} + \omega^{-k\ell} e_{z_1} \\ &= e_{z_1} + \sum_{j=2}^{\ell} \omega^{-k(j-1)} e_{z_j} = \omega^k \sum_{j=1}^{\ell} \omega^{-kj} e_{z_j} = \omega^k v_k. \end{aligned}$$

Because the eigenvectors  $v_1, \dots, v_\ell$  all belong to distinct eigenvalues they form a linearly independent set whose span has dimension  $\ell$ . But  $\text{span}\{v_1, \dots, v_\ell\}$  is a subspace of  $\text{span}\{e_{z_1}, \dots, e_{z_\ell}\}$  whose dimension is also  $\ell$ . So, the two subspaces are equal.  $\square$

**Lemma 9.** *In a proper partition  $P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$  of  $\Gamma_n$ , the set of all eigenvectors attached to the vertices in the cycles  $Z_1, \dots, Z_r$  is a linearly independent set.*

*Proof.* This follows immediately from equation (3) in Lemma 8 and the fact that any two cycles in  $\Gamma_n$  are disjoint.  $\square$

Next we determine generalized eigenvectors of  $A_n$  associated with the eigenvalue 0. Recall from Lemma 2, that if  $C = \{c_1, \dots, c_s\}$  is a chain in a proper partition of  $\Gamma_n$ , then either  $c_s$  is a terminal point of  $\Gamma_n$  or  $f(c_s)$  is a merge point.

**Lemma 10.** *Let  $C = \{c_1, \dots, c_m\}$  be any chain in a proper partition of  $\Gamma_n$ .*

1. *If  $c_m$  is a terminal point of the graph  $\Gamma_n$ , then*

$$\{e_{c_m}, e_{c_{m-1}}, \dots, e_{c_2}, e_{c_1}\}$$

*is a chain of generalized eigenvectors of  $A_n$  for the eigenvalue 0.*

2. *If  $f(c_m)$  is a merge point of  $\Gamma_n$ , let  $z$  be the vertex in the cycle or chain containing  $f(c_m)$  such that  $f^m(z) = f(c_m)$ . ( $z$  exists by Lemma 2.) Then*

$$\{e_{c_m} - e_{f^{m-1}(z)}, \dots, e_{c_3} - e_{f^2(z)}, e_{c_2} - e_{f(z)}, e_{c_1} - e_z\}$$

*is a chain of generalized eigenvectors of  $A_n$  for the eigenvalue 0.*

*In the first case, we say that the vector  $e_{c_i}$  is attached to the vertex  $c_i$ . In the second case, we say the vector  $e_{c_i} - e_{f^{i-1}(z)}$  is attached to the vertex  $c_i$ .*

*Note.* By convention, the *first* element of a chain of generalized eigenvectors, as in Equation (1) is the eigenvector, but the eigenvector corresponds to the *last* element of the chain  $\{c_1, \dots, c_m\}$  in Lemma 10. So, the order of indices in the subscripts is reversed.

*Proof.* If the first case,  $c_m$  is a terminal point of the graph  $\Gamma_n$ . This means that  $f(c_m) > n$  which implies that the  $c_m$  column of  $A_n$  is zero. Then  $A_n e_{c_m} = 0$ . So,  $e_{c_m}$  is an eigenvector of  $A_n$  for the eigenvalue 0. Because  $C = \{c_1, \dots, c_m\}$  is a chain in  $\Gamma$ ,  $A_n e_{c_i} = e_{c_{i+1}}$  for  $1 \leq i < m$ . Therefore  $\{e_{c_m}, \dots, e_{c_1}\}$  is a chain of generalized eigenvectors of  $A_n$  for the eigenvalue 0.

In the second case,  $e_{c_m} - e_{f^{m-1}(z)}$  is not the zero vector since  $f^{m-1}(z)$  does not belong to the chain  $C$ . Then

$$A_n(e_{c_m} - e_{f^{m-1}(z)}) = e_{f(c_m)} - e_{f^m(z)} = 0.$$

So,  $e_{c_m} - e_{f^{m-1}(z)}$  is an eigenvector for the eigenvalue 0. Since  $C$  is a chain,  $A_n(e_{c_i} - e_{f^{i-1}(z)}) = e_{c_{i+1}} - e_{f^i(z)}$  for  $1 \leq i < m$ . Thus,

$$\{e_{c_m} - e_{f^{m-1}(z)}, \dots, e_{c_3} - e_{f^2(z)}, e_{c_2} - e_{f(z)}, e_{c_1} - e_z\}$$

is a chain of generalized eigenvectors of  $A_n$  for the eigenvalue 0. □

With Lemmas 8 and 10 we have attached a generalized eigenvector to each of the  $n$  vertices of the graph  $\Gamma_n$ . The final step of the proof of Theorem 6 is to show that this collection of generalized eigenvectors is linearly independent. Then the chains of generalized eigenvectors that these lemmas attach to a proper partition of  $\Gamma_n$  will form a Jordan basis of  $\mathbb{C}^n$  for the matrix  $A_n$ .

**Lemma 11.** *Let  $P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$  be a proper partition of  $\Gamma_n$ . The set of all generalized vectors attached to vertices of the cycles  $Z_1, \dots, Z_r$  and to the vertices of the chains  $C_1, \dots, C_s$  is a linearly independent set of  $n$  vectors. Consequently, this set forms a Jordan basis of  $\mathbb{C}^n$  for the matrix  $A_n$ .*

*Proof.* In Lemma 9 it was shown that the set of eigenvectors attached to vertices belonging to cycles is linearly independent. All of these eigenvectors are roots of unity which are, of course, nonzero. If  $\text{len}(Z_i) = \ell_i$ , then there are

$$\ell_1 + \dots + \ell_s$$

such eigenvectors.

If  $\text{len}(C_i) = m_i$ , then there are

$$m_1 + \dots + m_s$$

generalized vectors attached to the vertices of the chains  $C_1, \dots, C_s$ . These generalized eigenvectors all belong to the generalized eigenspace of the eigenvalue 0. If these vectors are linearly independent, then then we will have a total of

$$n = \ell_1 + \dots + \ell_s + m_1 + \dots + m_s$$

linearly independent generalized eigenvectors since the union of linearly independent generalized eigenvectors from different generalized eigenspaces is linearly independent.

Thus, it remains to be shown that the generalized eigenvectors attached to the chains  $C_1, \dots, C_s$  form a linearly independent set.

Write  $\ell = \ell_1 + \dots + \ell_r$ , and let  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  be a permutation that maps the numbers  $1, \dots, \ell$  to the vertices belonging to the cycles  $Z_1, \dots, Z_r$  and such that the chains (as ordered lists) are

$$\begin{aligned} C_1 &= \{\sigma(\ell + 1), \dots, \sigma(\ell + m_1)\} \\ C_2 &= \{\sigma(\ell + m_1 + 1), \dots, \sigma(\ell + m_1 + m_2)\} \\ &\vdots \\ C_s &= \{\sigma(\ell + m_1 + \dots + m_{s-1} + 1), \dots, \sigma(\ell + m_1 + \dots + m_s)\}. \end{aligned}$$

By Lemma 10, the eigenvector attached to the *last* element of the chain  $C_j$  is one of the following:

$$e_{\sigma(\ell+m_1+\dots+m_j)} \quad \text{or} \quad e_{\sigma(\ell+m_1+\dots+m_j)} - e_{\sigma(z_j)}, \quad (4)$$

for some appropriate  $z_j$ . In the second case, since  $f(\sigma(\ell + m_1 + \dots + m_j))$  is a merge point,  $\sigma(z_j)$  belongs either to a longer chain than  $C_j$  or  $\sigma(z_j)$  belongs to a cycle. Either way, from the definition of  $\sigma$ ,

$$z_j < \ell + m_1 + \dots + m_{j-1} + 1.$$

Thus, the  $n \times s$  matrix whose  $j$ th column is

$$e_{\ell+m_1+\dots+m_j} \quad \text{or} \quad e_{\ell+m_1+\dots+m_j} - e_{z_j},$$

for  $1 \leq j \leq s$ , is upper triangular. Since no column is the zero vector, this matrix has linearly independent columns. Permuting the rows of this matrix does not alter the linear independence of the columns. Thus, the set of eigenvectors from (4) for  $1 \leq j \leq s$  is a linearly independent set. Since the vectors in (4) were the initial vectors of chains of generalized eigenvectors, Proposition 3 implies that the set of all generalized eigenvectors from those chains is a linearly independent set. Thus, the lemma has been proved.  $\square$

The proof of Theorem 6 is now complete.

## 5. Examples and Applications

We will next illustrate Theorem 6 with several examples.

**Example 3.** We will apply Theorem 6 to find the Jordan canonical form of the matrix  $A_{10}$  from Example 1. In that example,  $f: \mathbb{N} \rightarrow \mathbb{N}$  is defined by

$$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 4, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 10, 9 \mapsto 500, 10 \mapsto 4, \dots$$

and we found a proper partition of  $\Gamma_{10}$  to be

$$Z_1 = \{1\}, \quad Z_2 = \{5, 6, 7\}, \quad C_1 = \{2, 3, 4\}, \quad C_2 = \{8, 10\}, \quad C_3 = \{9\}.$$

Let  $\omega = \exp(2\pi i/3)$ . Then  $\omega$  is a primitive cube root of unity and  $\omega^3 = 1$ . By the theorem, the Jordan blocks in the Jordan canonical form of  $A_{10}$  will be

$$J_1(1), J_1(\omega), J_1(\omega^2), J_1(\omega^3), J_3(0), J_2(0), J_1(0).$$

Using Lemmas 8 and 10 we may find a basis  $\beta = \{v_1, \dots, v_{10}\}$  of generalized eigenvectors (taking care to reverse the order of indices in the chains) as follows:

vertex	generalized eigenvector
1	$v_1 = e_1$
5	$v_2 = \omega^{-1 \cdot 1} e_5 + \omega^{-1 \cdot 2} e_6 + \omega^{-1 \cdot 3} e_7 = \omega^2 e_5 + \omega e_6 + e_7$
6	$v_3 = \omega^{-2 \cdot 1} e_5 + \omega^{-2 \cdot 2} e_6 + \omega^{-2 \cdot 3} e_7 = \omega e_5 + \omega^2 e_6 + e_7$
7	$v_4 = \omega^{-3 \cdot 1} e_5 + \omega^{-3 \cdot 2} e_6 + \omega^{-3 \cdot 3} e_7 = e_5 + e_6 + e_7$
4	$v_5 = e_4 - e_7$ ( $f(4) = f(4) = 5$ is a merge point.)
3	$v_6 = e_3 - e_6$ ( $f^2(3) = f^2(3) = 5$ is a merge point.)
2	$v_7 = e_2 - e_5$ ( $f^3(2) = f^3(5) = 5$ is a merge point.)
10	$v_8 = e_{10} - e_3$ ( $f(10) = f(3) = 4$ is a merge point.)
8	$v_9 = e_8 - e_2$ ( $f^2(8) = f^2(2) = 4$ is a merge point.)
9	$v_{10} = e_9$ (The chain containing 9 has a terminal point.)

Thus setting

$$Q = (v_1 v_2 \cdots v_{10}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & \omega & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \omega & \omega^2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

gives the Jordan canonical form

$$Q^{-1}A_{10}Q = \begin{pmatrix} \boxed{1} & & & & & & & & & & \\ & \boxed{\omega^2} & & & & & & & & & \\ & & \boxed{\omega} & & & & & & & & \\ & & & \boxed{1} & & & & & & & \\ & & & & \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} & & & & & & \\ & & & & & & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & & & & \\ & & & & & & & & & \boxed{0} & \end{pmatrix}.$$

**Example 4** (The Collatz Problem). Let  $f$  be the function

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The well-known *Collatz conjecture* states that, for each  $k \in \mathbb{N}$ , the sequence

$$k, f(k), (f \circ f)(k), (f \circ f \circ f)(k), \dots$$

contains the number 1. For an extensive annotated bibliography of the literature on this problem see Lagarias [4, 5]. For any  $n \in \mathbb{N}$ , we may consider



the  $n \times n$  matrix  $A_n$  and graph  $\Gamma_n$  associated with the the Collatz function. In this case, we will call  $A_n$  the *Collatz matrix*. For example,

$$A_8 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We may apply Theorem 6 to the study of  $A_n$  and the Collatz problem. In [6], Dias et. al. also study the Collatz conjecture from the point of view of finite dimensional matrices, and they establish certain determinantal identities for these matrices.

Working several examples for small values of  $n$  (say  $n$  up to a few thousand) quickly leads to the following conjecture:

**Conjecture 12.** *For  $n \in \mathbb{N}$ , let  $\Gamma_n$  and  $A_n$  be the graph and adjacency matrix associated with the Collatz function. Then*

1. *The characteristic polynomial of  $A_n$  is  $\det(xI_n - A_n) = x^{n-2}(x^2 - 1)$ .*
2. *For  $n \geq 2$ , the only cycle in the graph  $\Gamma_n$  is the two-cycle  $\{1, 2\}$ .*
3. *For any fixed  $k \geq 3$ , if  $n$  is sufficiently large, then  $k$  belongs to the same component of graph as the cycle  $\{1, 2\}$ .*

We point out that, since this conjecture implies the Collatz conjecture, its proof would likely be quite difficult. The paper of Dias et. al. [6] has some discussion about the characteristic polynomial.

We ask the following questions:

**Open Problem 13.** *Let  $P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$  be a proper partition of  $\Gamma_n$ .*

1. *Is  $r = 1$ ?*
2. *What is the length  $\text{len}(C_1) = m_1$  of the longest chain?*
3. *How many connected components does the graph  $\Gamma_n$  have?*

As a consolation prize, we can precisely describe the number  $s$  of chains in a proper partition:

**Theorem 14.** For  $n \geq 2$ , let  $A_n$  be the  $n \times n$  Collatz matrix, let  $\Gamma_n$  be the associated graph, and let  $P = \{Z_1, \dots, Z_r, C_1, \dots, C_s\}$  be a proper partition of  $\Gamma_n$ . Then the number  $s$  of chains in the partition which, by Theorem 6, is also equal to the number of Jordan blocks for the eigenvalue 0 of the matrix  $A_n$  is

$$n - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{n-4}{6} \right\rfloor = \begin{cases} 2\lfloor n/6 \rfloor & \text{if } n \equiv 0 \pmod{6}, \\ 2\lfloor n/6 \rfloor + 1 & \text{if } n \equiv 1 \pmod{6}, \\ 2\lfloor n/6 \rfloor & \text{if } n \equiv 2 \pmod{6}, \\ 2\lfloor n/6 \rfloor + 1 & \text{if } n \equiv 3 \pmod{6}, \\ 2\lfloor n/6 \rfloor + 2 & \text{if } n \equiv 4 \pmod{6}, \\ 2\lfloor n/6 \rfloor + 2 & \text{if } n \equiv 5 \pmod{6}. \end{cases}$$

*Proof.* From the Jordan decomposition theorem, the number of Jordan blocks associated with the eigenvalue  $\lambda$  of  $A_n$  is  $n - \text{rank}(A_n - \lambda I_n)$ . In the case  $\lambda = 0$ , this is  $s = n - \text{rank}(A_n)$ . So, we need to compute  $\text{rank}(A_n)$ .

The even numbered columns of  $A_n$  consist of the standard basis vectors

$$e_1, e_2, \dots, e_{\lfloor n/2 \rfloor}.$$

The odd numbered columns of  $A_n$  which are also nonzero consist of the standard basis vectors

$$e_2, e_5, e_8, e_{11}, \dots, e_{3j+2},$$

where  $j = \lfloor \frac{n-2}{3} \rfloor$  is the largest integer such that  $3j+2 \leq n$ . The elements of the second list in common with the first list are

$$e_2, e_5, \dots, e_{3k+2}$$

where  $k = \lfloor \frac{n/2-2}{3} \rfloor = \lfloor \frac{n-4}{6} \rfloor$  is the largest integer such that  $3k+2 \leq n/2$ . So, the degree of the column space of  $A_n$  is

$$\text{rank}(A_n) = \left\lfloor \frac{n}{2} \right\rfloor + (j - k) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-2}{3} \right\rfloor - \left\lfloor \frac{n-4}{6} \right\rfloor,$$

which proves that the number of Jordan blocks for the eigenvalue 0 is

$$s = n - \text{rank}(A_n) = n - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{n-4}{6} \right\rfloor$$

We obtain the remaining portion of the formula by writing  $n = 6\lfloor n/6 \rfloor + \ell$  where  $\ell \in \{0, 1, 2, 3, 4, 5\}$  and considering each case.  $\square$

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